Applications of Network Flow

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Maximum Flow and Minimum Cut

- Two rich algorithmic problems.
- Fundamental problems in combinatorial optimization.
- Beautiful mathematical duality between flows and cuts.
- Numerous non-trivial applications:
  - Bipartite matching.
  - Data mining.
  - Project selection.
  - Airline scheduling.
  - Baseball elimination.
  - Image segmentation.
  - Network connectivity.
  - Open-pit mining.
  - Network reliability.
  - Distributed computing.
  - Egalitarian stable matching.
  - Security of statistical data.
  - Network intrusion detection.
  - Multi-camera scene reconstruction.
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  - Multi-camera scene reconstruction.
  - Gene function prediction.
- We will only sketch proofs. Read details from the textbook.

Matching in Bipartite Graphs

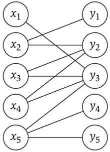

Figure 7.1 A bipartite graph.

- *Bipartite Graph*: a graph $G(V, E)$ where
  1. $V = X \cup Y$, X and Y are disjoint and
  2. $E \subseteq X \times Y$.
- Bipartite graphs model situations in which objects are matched with or assigned to other objects: e.g., marriages, residents/hospitals, jobs/machines.

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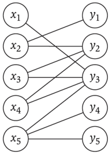

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- Bipartite graphs model situations in which objects are matched with or assigned to other objects: e.g., marriages, residents/hospitals, jobs/machines.
- A *matching* in a bipartite graph G is a set $M \subseteq E$ of edges such that each node of V is incident on at most edge of M.
- A set of edges M is a *perfect matching* if every node in V is incident on exactly one edge in M.

Bipartite Graph Matching Problem

BIPARTITE MATCHING

INSTANCE: A Bipartite graph G.

SOLUTION: The matching of largest size in G.

Algorithm for Bipartite Graph Matching

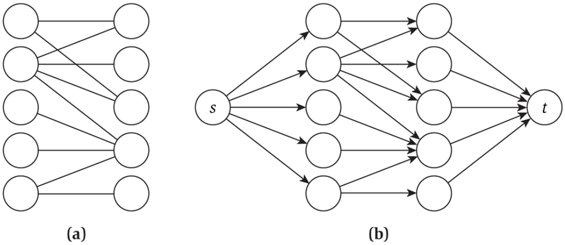

Figure 7.9 (a) A bipartite graph. (b) The corresponding flow network, with all capacities equal to 1.

- Convert G to a flow network G': direct edges from X to Y, add nodes s and t, connect s to each node in X, connect each node in Y to t, set all edge capacities to 1.
- Compute the maximum flow in G'.
- Claim: the value of the maximum flow is the size of the maximum matching.

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- Read the book on what augmenting paths mean in this context.

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- Ford-Fulkerson algorithm runs in $O(mn)$ time.

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DIRECTED EDGE-DISJOINT PATHS

INSTANCE: Directed graph $G(V, E)$ with two distinguished nodes s and t.

SOLUTION: The maximum number of edge-disjoint paths between s and t.

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  - Prove by induction on the number of edges in f that carry flow.
- We just proved: there are k edge-disjoint paths from s to t in a directed graph G iff the maximum value of an s-t flow in G is $\geq k$.

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Certificate for Edge-Disjoint Paths Algorithm

- A set $F \subseteq E$ of edge separates s and t if the graph $(V, E - F)$ contains no s-t paths.
- Menger's Theorem: In every directed graph with nodes s and t, the maximum number of edge-disjoint s-t paths is equal to the minimum number of edges whose removal disconnects s from t.

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- Can obtain an integral flow where only one of the directed counterparts of (u, v) has non-zero flow.
- We can find the maximum number of edge-disjoint paths in $O(mn)$ time.
- We can prove a version of Menger's theorem for undirected graphs: in every undirected graph with nodes s and t, the maximum number of edge-disjoint s–t paths is equal to the minimum number of edges whose removal separates s from t.

Image Segmentation

- A fundamental problem in computer vision is that of segmenting an image into coherent regions.
- A basic segmentation problem is that of partitioning an image into a foreground and a background: label each pixel in the image as belonging to the foreground or the background.

Formulating the Image Segmentation Problem

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- Let E be the set of pairs of neighbouring pixels.
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- We want the foreground/background boundary to be smooth: For each pair (i, j) of pixels, assign separation penalty $p_{ij} \geq 0$ for placing one of them in the foreground and the other in the background.

The Image Segmentation Problem

IMAGE SEGMENTATION

INSTANCE: Pixel graphs $G(V, E)$, likelihood functions $a, b : V \rightarrow \mathbb{R}^+$, penalty function $p : E \rightarrow \mathbb{R}^+$

SOLUTION: *Optimum labelling*: partition of the pixels into two sets A and B that maximises

$$q(A, B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}.$$

Developing an Algorithm for Image Segmentation

- There is a similarity between cuts and labellings.
- But there are differences:
  - We are maximising an objective function rather than minimising it.
  - There is no source or sink in the segmentation problem.
  - We have values on the nodes.
  - The graph is undirected.

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- Notice that $\sum_{i \in A} a_i + \sum_{j \in B} b_j = Q - \sum_{i \in A} b_i + \sum_{j \in B} a_j$.
- Therefore, maximising

$$\begin{aligned} q(A, B) &= \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cup \{i,j\}| = 1}} p_{ij} \\ &= Q - \sum_{i \in A} b_i - \sum_{j \in B} a_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij} \end{aligned}$$

is identical to minimising

$$q'(A, B) = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

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- Add a new "super-source" s to represent the foreground.
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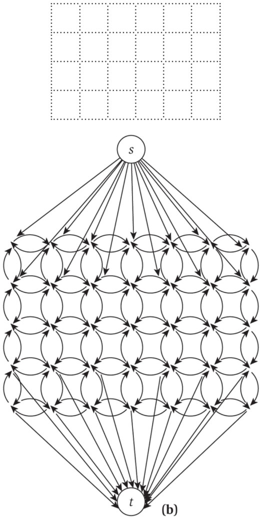
- Connect s and t to every pixel and assign capacity a_i to edge (s, i) and capacity b_i to edge (i, t).
- Direct edges away from s and into t.
- Replace each edge (i, j) in E with two directed edges of capacity p_{ij}.



(b)

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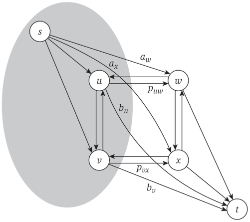

Figure 7.19 An s-t cut on a graph constructed from four pixels. Note how the three types of terms in the expression for $q'(A, B)$ are captured by the cut.

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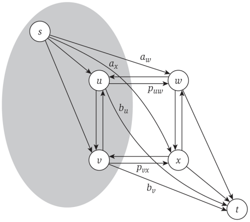

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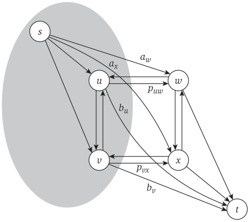

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Solving the Image Segmentation Problem

- The capacity of a s-t cut $c(A, B)$ exactly measures the quantity $q'(A, B)$.
- To maximise $q(A, B)$, we simply compute the s-t cut (A, B) of minimum capacity.
- Deleting s and t from the cut yields the desired segmentation of the image.