How many physical qubits are needed exactly for fault-tolerant quantum computing?

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Abstract

Physical implementation of quantum computing is a continuing challenge. The main obstacle is understanding how to map abstract qubits and quantum operations into physical hardware without decoherence, that is, losing quantum information. A solution is proposed here.

To manufacture quantum computers, abstract *logical* qubits require realization as *physical* qubits in the classical world. But this mapping has to be done in a way that the resulting physical qubits are *fault-tolerant* [1, 2], that is, they stay coherent when a quantum computation is performed on them.

The idea that one can perform a quantum computation on physical qubits in the classical world requires elaboration. *Logical quantum computing* occurs in the quantum physical realm, a notion described mathematically by the pair $\left\{\mathbb{C}P^{n-1}, Q\right\}$ where the $\mathbb{C}P^{n-1}$ is the quantum register of n qubits, the state of which is transformed by the unitary operation Q. To implement the logical quantum computation $\left\{\mathbb{C}P^{n-1}, Q\right\}$ in physical hardware in the classical world requires mapping into *physical quantum computation*, a notion captured mathematically by the pair $\left\{\mathbb{R}^d, R\right\}$ where $\mathbb{R}^d$ is the physical register for the n qubits, the state of which is transformed by the operation R.

For the physical quantum computation to be fault-tolerant, the mapping from $\left\{\mathbb{C}P^{n-1}, Q\right\}$ to $\left\{\mathbb{R}^d, R\right\}$ should be robust against fundamental notions of faults that arise when traversing the classical-quantum physical divide. From a general mathematical perspective, faults in the traversal of the classical-quantum physical divide are due to changes in the topology, geometry, or differential structure. If the traversal function from $\left\{\mathbb{C}P^{n-1}, Q\right\}$

$$
\begin{tikzcd}
\mathbb{C}P^{n-1} \arrow[r, leftrightarrow, "Q"] \arrow[d, leftrightarrow, "e"'] & \mathbb{C}P^{n-1} \arrow[d, leftrightarrow, "e"] \\
S \subset \mathbb{R}^d \arrow[r, leftrightarrow, "R"] & S \subset \mathbb{R}^d
\end{tikzcd}
$$

Figure 1: Nash embedding e of the initial and final state of the quantum register, together with unitarity of quantum computations Q, requires that the physical quantum computation R is in fact an orthogonal, reversible computation

to $\left\{\mathbb{R}^d, R\right\}$ preserves topology, geometry, and differential structure, than the resulting physical quantum computation will be considered fault-tolerant.

The class of functions $e : \left\{\mathbb{C}P^{n-1}, Q\right\} \longrightarrow \left\{\mathbb{R}^d, R\right\}$ preserving topology, geometry, and differential structure are Nash embeddings [3]. A Nash embedding is a one-to-one map that is a homeomorphism (preserves topological features), diffeomorphism (preserves differential structures), and an isometry (preserves distances). These properties of e imply, as shown in Figure 1, that the physical quantum computation R emulating the logical quantum computation Q is necessarily a reversible computation, that is, an orthogonal transformation.

Nash embedding gives the exact number of fault-tolerant physical qubits needed to implement coherent quantum computation in hardware in the form of the number d. As per Nash's theorem

$$d = \max\left\{\frac{k(k+5)}{2}, \frac{k(k+3)}{2} + 5\right\}. \tag{1}$$

where k is the dimension of the quantum register $\mathbb{C}P^{n-1}$ *as a Riemannian manifold* [4]. This number is $k = 2^{n+1} - 2$ (where n is the number of qubits). This means that one qubit register $\mathbb{C}P^1$ maps to the fault-tolerant physical register $\mathbb{R}^{10}$; therefore, 1 logical qubit maps into 10 fault-tolerant physical qubits. Similarly, 2 logical qubits map into 19 fault-tolerant physical qubits, 3 logical qubits map into 52 fault-tolerant physical qubits, and 4 logical qubits map into 168 fault-tolerant physical qubits, ...

Any meaningful effort in fault-tolerant physical qubit design and manufacturing should account for these *Nash values*, despite how large they get for even a small number of qubits. For example, 20 logical qubits give $k = 2,097,150$, and hence map into $2,199,024,304,125$ fault-tolerant physical qubits!

References

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