ECE102, Fall 2025 **Homework #3**
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Due Friday, 24 October 2025, by 11:59pm to Gradescope.
100 points total.

1. (20 points) **Linear systems**

   Determine whether each of the following systems is linear or not. Explain your answer.

   (a) $y(t) = cos(t)x(t)$

   Solution:
   Let input $x_1(t)$ and $x_2(t)$ have outputs $y_1(t)$ and $y_2(t)$, respectively:

$$y_1(t) = cos(t)x_1(t)$$
$$y_2(t) = cos(t)x_2(t)$$

   For an input $x(t) = a_1x_1(t) + a_2x_2(t)$, we get the output:

$$\begin{aligned} y(t) &= cos(t)x_1(t) + cos(t)x_2(t) \\ &= a_1y_1(t) + a_2y_2(t) \end{aligned}$$

   So, the system is linear. We could also check homogeneity and additivity separately if we wanted.

   (b) $y(t) = \frac{d}{dt}(\frac{1}{2}x(t)^2)$

   Solution:
   We can simplify the equation as:

$$y(t) = x(t)\frac{d}{dt}x(t)$$

   We'll first check for homogeneity. Let $y_a(t)$ be the output for the input $ax(t)$. Then:

$$\begin{aligned} y_a(t) &= (ax(t))\frac{d}{dt}(ax(t)) \\ &= a^2x(t)\frac{d}{dt}x(t) = a^2y(t) \neq ay(t) \end{aligned}$$

   Then the system is not linear. To be thorough, we'll check additivity as well. Let $y_{12}(t)$

be the output for the input $x_1(t) + x_2(t)$. Then:

$$\begin{aligned}
y_{12}(t) &= (x_1(t) + x_2(t))\frac{d}{dt}(x_1(t) + x_2(t)) \\
&= (x_1(t) + x_2(t)) * (\frac{d}{dt}x_1(t) + \frac{d}{dt}x_2(t)) \\
&= x_1(t)\frac{d}{dt}x_1(t) + x_2(t)\frac{d}{dt}x_2(t) + x_2(t)\frac{d}{dt}x_1(t) + x_1(t)\frac{d}{dt}x_2(t) \\
&= y_1(t) + y_2(t) + x_2(t)\frac{d}{dt}x_1(t) + x_1(t)\frac{d}{dt}x_2(t) \\
&\neq y_1(t) + y_2(t)
\end{aligned}$$

(c) $y(t) = e^{x(t)}$

Solution: Like above, we'll define $x(t) = a_1x_1(t)+a_2x_2(t)$. Then, we'll get the output:

$$\begin{aligned}
y(t) &= e^{a_1x_1(t)+a_2x_2(t)} \\
&= e^{a_1x_1(t)}e^{a_2x_2(t)} \\
&\neq a_1e^{x_1(t)} + a_2e^{x_2(t)} \\
&= a_1y_1 + a_2y_2
\end{aligned}$$

The system is not linear.

(d) $y(t) = x(t) + 2u(t + 1)$

Solution:
Let's check homogeneity first:

$$\begin{aligned}
y_a(t) &= ax(t) + 2u(t + 1) \\
&\neq ay(t)
\end{aligned}$$

The system is non-linear.

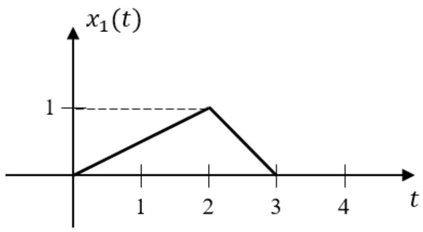
For additivity:

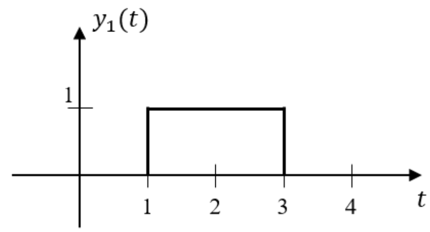
$$\begin{aligned}
y_{12}(t) &= (x_1(t) + x_2(t)) + 2u(t + 1) \\
&\neq y_1(t) + y_2(t)
\end{aligned}$$

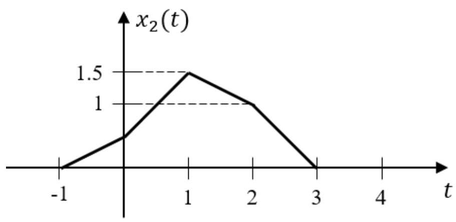
2. (13 points) **LTI systems**

(a) (7 points) Consider an LTI (linear time-invariant) system whose response to $x_1(t)$ is $y_1(t)$, where $x_1(t)$ and $y_1(t)$ are illustrated as follows:
Sketch the response of the system to the input $x_2(t)$.

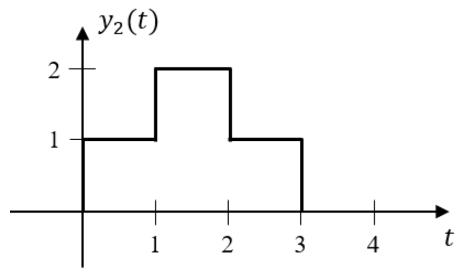
Solution:







(a) $x_2(t)$



(b) $y_2(t)$

We can express $x_2(t)$ in terms of $x_1(t)$ as follows:

$$x_2(t) = x_1(t+1) + x_1(t)$$

Since the system is LTI, the response to $x_2(t)$ is:

$$y_2(t) = y_1(t+1) + y_1(t)$$

(b) (6 points) Assume we have a linear system with the following input-output pairs:

- the output is $y_1(t) = cos(t)u(t)$ when the input is $x_1(t) = u(t)$;
- the output is $y_2(t) = cos(t)(u(t+1) - u(t))$ when the input is $x_2(t) = \mathrm{rect}(t + \frac{1}{2})$.

Is the system time-invariant?

Solution:
The signal $x_2(t)$ can be written as: $x_2(t) = u(t+1) - u(t)$. Let,

$$x_3(t) = x_1(t) + x_2(t) = u(t) + u(t+1) - u(t) = u(t+1)$$

We see that $x_3(t) = x_1(t+1)$. Let us now use the properties of linear system to get the output $y_3(t)$ to input $x_3(t)$, we then compare $y_3(t)$ to $y_1(t+1)$. Since, $x_3(t) = x_1(t) + x_2(t)$, the output is then

$$y_3(t) = y_1(t) + y_2(t) = cos(t)u(t) + cos(t)(u(t+1) - u(t)) = cos(t)u(t+1)$$

On the other hand,

$$y_1(t+1) = cos(t+1)u(t+1)$$

Since $y_3(t) \neq y_1(t+1)$, the system is not time-invariant.

3. (38 points) **Convolution**

(a) (10 points) For each pair of the signals given below, compute their convolution using the flip-and-drag technique.

i. $f(t) = \delta(t+1) + 2\delta(t-2)$, $\qquad g(t) = e^{-t}u(t)$

Solution:

$$\begin{aligned} y(t) = f(t) \star g(t) &= (\delta(t+1) + 2\delta(t-2)) \star e^{-t}u(t) \\ &= \delta(t+1) \star e^{-t}u(t) + 2\delta(t-2) \star e^{-t}u(t) \\ &= e^{-t-1}u(t+1) + 2e^{-t+2}u(t-2) \end{aligned}$$

Note that if you flip the impulse signal, you get the same impulse. Therefore, when you do a flip and drag operation when convolving impulses with the other signal, you are essentially shifting the other signal.

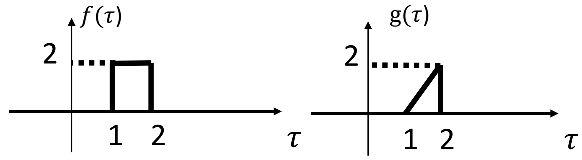
ii. $f(t) = 2\ \mathrm{rect}(t - \frac{3}{2})$, $\qquad g(t) = 2\ r(t-1)\mathrm{rect}(t - \frac{3}{2})$

Solution:
Let,

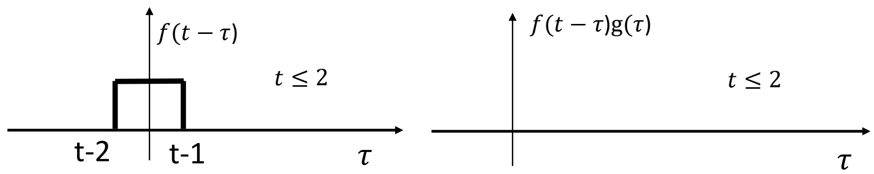
$$y(t) = f(t) \star g(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau$$

We will flip and drag the rect function. As we can see in the figure below, for $t \leq 2$, there is no overlap between the two plots, therefore $y(t) = 0$ for for $t \leq 2$. For $2 < t \leq 3$, the rect function starts to overlap with the triangle, the convolution integral in this case is equal to the overlapped area. For $3 < t \leq 4$, the rect function starts to go out from the triangle, the convolution integral is also equal to the overlapped area. For $t > 4$, there is no overlap between the two plots, then $y(t) = 0$ for $t > 4$. Therefore, $y(t)$ is given as:

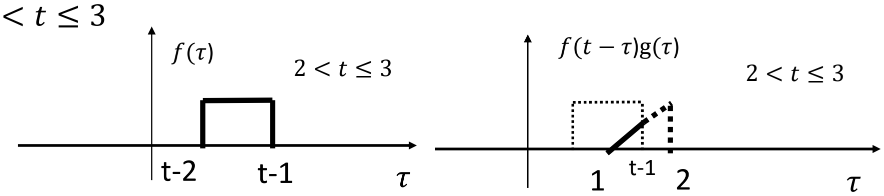
$$y(t) = \begin{cases} 0, & t \leq 2 \\ \int_1^{t-1} 4(\tau - 1)\, d\tau = 2(t-2)^2, & 2 < t \leq 3 \\ \int_{t-2}^{2} 4(\tau - 1)\, d\tau, = -2(t-4)(t-2) & 3 < t \leq 4 \\ 0, & t > 4 \end{cases}$$

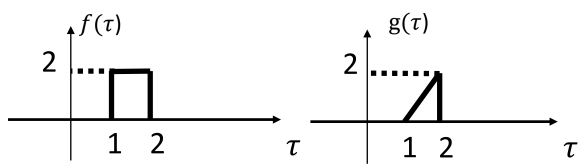


Case 1: $t \leq 2$

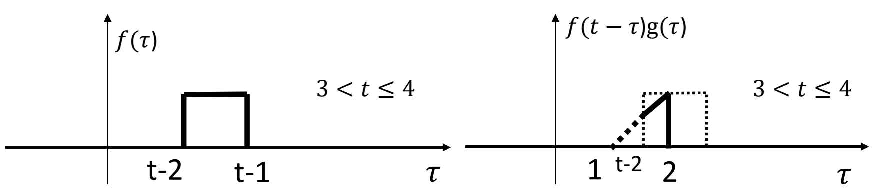


Case 2: $2 < t \leq 3$

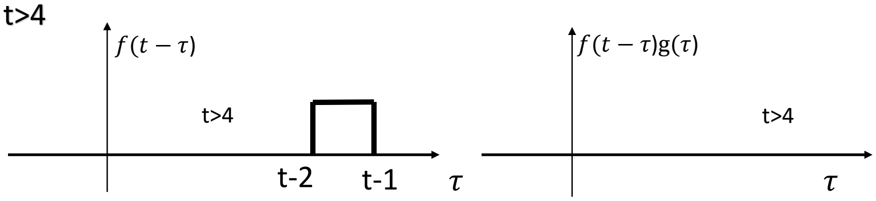




Case 3: $3 < t \leq 4$



Case 4: t>4



(b) (10 points) For each of the following, find a function $h(t)$ such that $y(t) = x(t) \star h(t)$.

i. $y(t) = \int_{t-T}^{t} x(\tau)d\tau$

Solution:
We can think about $h(t)$ as the impulse response of the give LTI system, therefore

$$h(t) = \int_{t-T}^{t} \delta(\tau)d\tau = u(t) - u(t-T)$$

ii. $y(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s)$

Solution:
Also by applying the sifting property,

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

(c) (10 points) Simplify the following expressions:

i. $[\delta(t-3) + \delta(t+2)] * \left[e^{3t}u(-t) + \delta(t+2) + 2\right]$

Solution:
We apply the sifting property:

$$\begin{aligned} e^{3(t-3)}u(-t+3) + \delta(t-1) + 2 + e^{3(t+2)}u(-t-2) + \delta(t+4) + 2 = \\ e^{3(t-3)}u(-t+3) + \delta(t-1) + e^{3(t+2)}u(-t-2) + \delta(t+4) + 4 \end{aligned}$$

ii. $\frac{d}{dt}\left[(u(t) - u(t+1)) \star u(t-2)\right]$, *Hint: Show first that* $u(t) \star u(t) = r(t)$ *where* $r(t)$ *is the ramp function.*

Solution:
We first show that $u(t) \star u(t) = r(t)$:

$$u(t) \star u(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \left(\int_{0}^{t} 1d\tau\right) u(t) = tu(t) = r(t)$$

Therefore, using the properties of convolutions,

$$(u(t) - u(t+1)) \star u(t-2) = r(t-2) - r(t-1)$$

Thus,

$$\frac{d}{dt}\left(r(t-2) - r(t-3)\right) = u(t-2) - u(t-1)$$

(d) (8 points) Explain whether each of the following statements is true or false.

i. If $x(t)$ and $h(t)$ are both odd functions, and $y(t) = x(t) \star h(t)$, then $y(t)$ is an even function.

Solution:

True: We will show this statement by applying the definition of convolution.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Let $\tau' = -\tau$, then

$$y(t) = -\int_{\infty}^{-\infty} x(-\tau')h(t+\tau')d\tau' = \int_{-\infty}^{\infty} x(-\tau')h(t+\tau')d\tau'$$

Since $x(t)$ and $h(t)$ are both odd functions, we have:

$$y(t) = \int_{-\infty}^{\infty} (-x(\tau'))(-h(-t-\tau'))d\tau' = \int_{-\infty}^{\infty} x(\tau')h(-t-\tau')d\tau' = y(-t)$$

Therefore, $y(t)$ is even.

ii. If $y(t) = x(t) \star h(t)$, then $y(2t) = h(2t) \star x(2t)$.

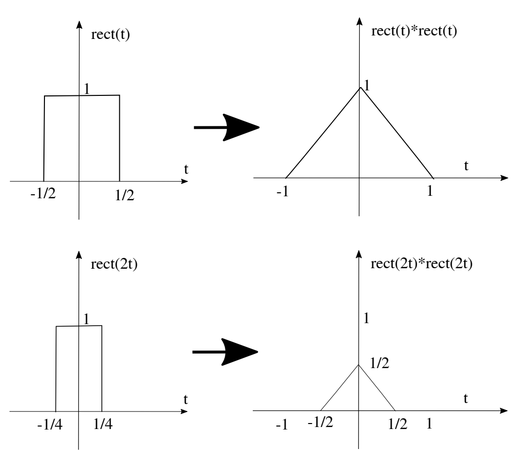
Solution:
False: Consider the following counter example; let $x(t) = \delta(t)$ and $h(t) = u(t)$, then $x(2t) = \delta(2t) = \frac{1}{2}\delta(t)$ and $h(2t) = u(2t) = u(t)$. Therefore, we have: $y(t) = x(t)\star h(t) = u(t)$, so $y(2t) = u(2t) = u(t)$. On the other hand, $x(2t)\star h(2t) = \frac{1}{2}u(t)$. Thus, $y(2t) \neq h(2t) \star x(2t)$.
What is true instead is that $y(2t) = 2\,(h(2t) \star x(2t))$. This can be shown using the definition of convolution:

$$x(2t) \star h(2t) = \int_{-\infty}^{\infty} x(2\tau)h(2t-2\tau)d\tau$$

Let $\tau' = 2\tau$, then

$$x(2t) \star h(2t) = \frac{1}{2}\int_{-\infty}^{\infty} x(\tau')h(2t-\tau')d\tau' = \frac{1}{2}y(2t)$$

For intuition, consider the case when x(t) = h(t) = rect(t).



4. (12 points) **Impulse response and LTI systems**

Consider the following three LTI systems:

- The first system $\mathcal{S}_1$ is given by its input-output relationship: $y(t) = \int_{-\infty}^{t} x(\tau - t_0)d\tau$;
- The second system $\mathcal{S}_2$ is given by its impulse response: $h_2(t) = u(t-2)$;
- The third system $\mathcal{S}_3$ is given by its impulse response: $h_3(t) = u(t+3)$.

(a) (4 points) Compute the impulse responses $h_1(t)$ of system $\mathcal{S}_1$.

Solution:

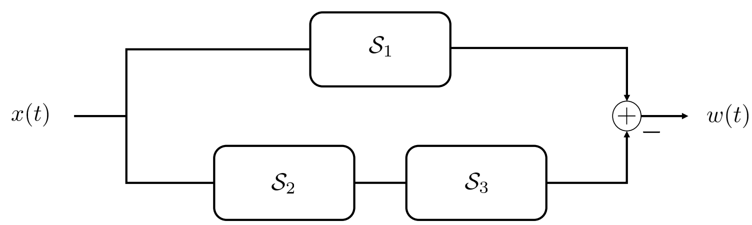
When substitute $\tau - t_0$ with τ', we simplify $y(t) = \int_{-\infty}^{t} x(\tau - t_0)d\tau$ as:

$y(t) = \int_{-\infty}^{t-t_0} x(\tau')d\tau' = \int_{-\infty}^{t-t_0} x(\tau)d\tau$

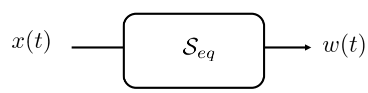
For system $\mathcal{S}_1$, the impulse response is given by:

$$h_1(t) = \int_{-\infty}^{t-t_0} \delta(\tau)d\tau = u(t-t_0)$$

(b) (4 points) The three systems are interconnected as shown below.



Determine the impulse response $h_{eq}(t)$ of the equivalent system.



Solution:

$$h_{eq}(t) = h_1(t) - (h_2(t) \star h_3(t)) = u(t-t_0) - r(t+1)$$

(c) (4 points) Determine the response of the overall system to the input $x(t) = \delta(t) + \delta(t-3)$.

Solution:

The response is: $y(t) = x(t) \star h_{eq}(t) = u(t-t_0) - r(t+1) + u(t-t_0-3) - r(t-2)$

5. (17 points) **Python tasks**

For this question, please check the provided jupyter notebook solution file.