EE102

Lecture 16

EE102 Announcements

- **Homework 6 is due today.**
- **Homework 7 is due Friday.**

- **MyUCLA Course Evaluations close on Saturday December 6 @ 8AM.**

1% Extra Credit on Final If 80% response rate by Friday morning.

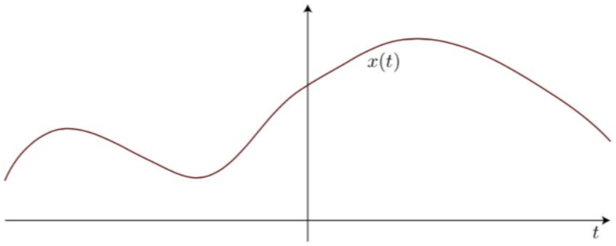
ABET Learning Outcomes

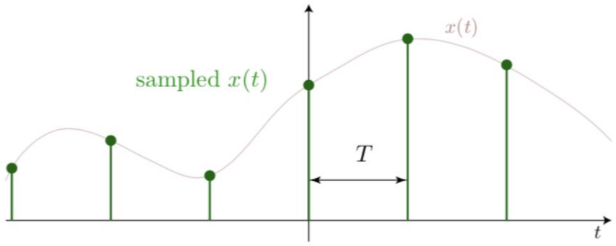
- ✔ Understand the concept of a signal and a system, plot continuous-time signals, evaluate the periodicity of a signal.
- ✔ Identify properties of continuous-time systems such as linearity, time-invariance, and causality.
- ✔ Calculate with the Dirac delta function.
- ✔ Compute convolution of continuous-time functions.
- ✔ Understand the concept of the impulse response function of a linear system, and its use to describe the input/output relationship.
- ✔ Compute the Laplace transform of a continuous function, identify its domain of convergence, and be familiar with its basic properties, including the initial and final value theorems.
- Find the inverse Laplace transform by partial fractions.
- Use the Laplace transform to solve constant-coefficient differential equations with initial conditions
- Use the Laplace transform to evaluate the transfer function of linear time-invariant systems.
- ✔ Understand Parseval's relation in Fourier series, and its interpretation in terms of decomposing the signal's energy between its harmonics
- ✔ Evaluate the response of a linear time-invariant system to periodic inputs.
- ✔ Evaluate the Fourier transform of a continuous function, and be familiar with its basic properties. Relate it to the Laplace transform.
- ✔ Evaluate and plot the frequency responses (magnitude and phase) of linear time-invariant systems, and apply it to filtering of input signals.
- ✔ Understand conditions under which a band-limited function can be recovered from its samples

Review: Sampling

Motivation

In reality, we could never store a continuous time signal. Instead, as we see in `MATLAB`, we store the signal's value at various times. This is called sampling, as illustrated below.





A key variable of interest is the sampling frequency, i.e., the time in between our samples, denoted T in the above diagram.

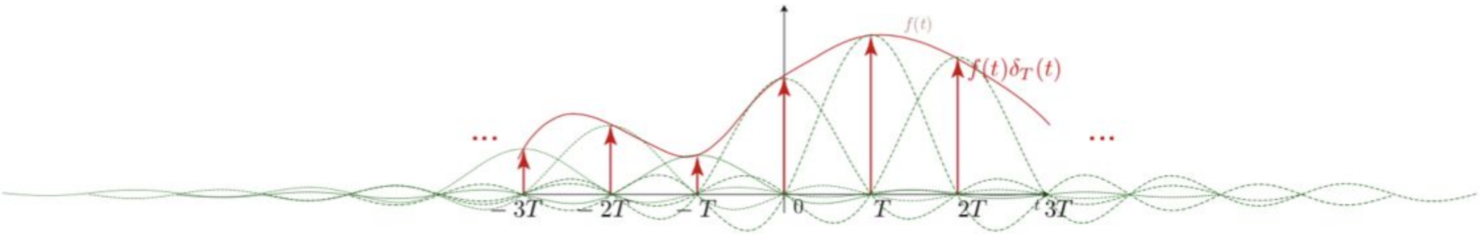
This is related to discrete signals, i.e., $x[n] = x(nT)$.

Review: Recovering Signal through Interpolation

This reconstruction,

$$\tilde{f}(t) * h(t) \quad = \quad \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt - k))$$

is called the Whittaker-Shannon interpolation formula. Intuitively, it does the following:

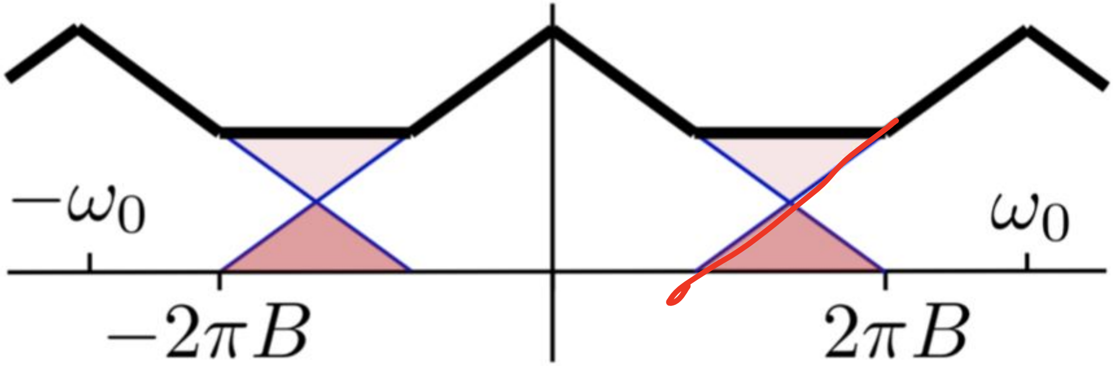


The sum of the green sinc functions will equal the red function, $f(t)$.

Review: Aliasing in Frequency

We see that as ω_0 decreases, the bands start to overlap. When the replicas overlap, even with ideal low pass filtering, we cannot recover the original $F(j\omega)$.

This overlap is called *aliasing* because low frequencies of one spectral replica appear (or alias) as high frequencies in the next spectral replica. The vice versa is true as well; high frequencies of one spectral replica alias as low frequencies in an adjacent spectral replica. The alias'd sections are shown in the darker red.



Aliasing (Look at it in Time)

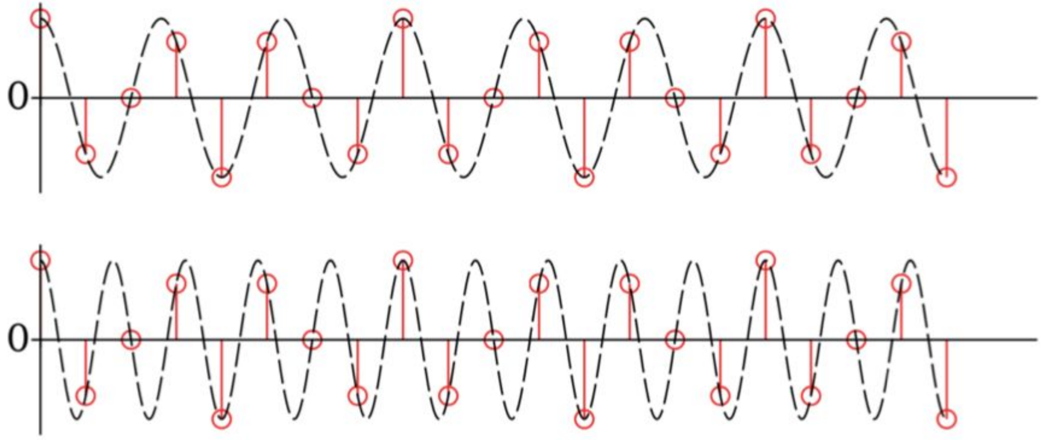
We can intuitively understand aliasing in time domain for sinusoidal signals

Here's an example of aliasing. Below are two sinusoids. The upper one is at a frequency of $f = 0.75$ Hz and the one below is at $f = 1.25$ Hz.

Sampling both signals at $f_s = 2$ Hz,

0.75Hz

1.25Hz

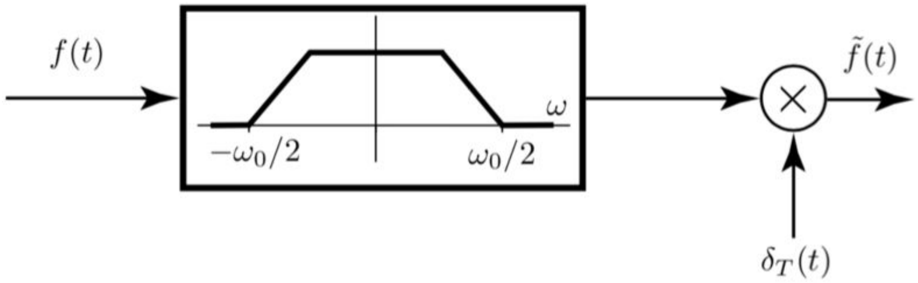


As you can see, the samples are the same for both sinusoids..

How to Ameliorate Aliasing

If you sample below the Nyquist rate, there will be aliasing. Once aliasing happens, there is no way to eliminate aliasing without having additional information about the signal.

One way we can ameliorate aliasing is to first low pass filter the signal, then sample:



- In reality, our low pass filters are not perfect, and so the bandwidth will be larger than $\omega_0/2$, however, we'll attenuate frequencies outside of range.
- Low pass filtering will distort the signal.
- However, the point is that when sampling, frequencies beyond $\omega_0/2$ would cause artifacts. Low pass filtering ameliorates this.
- It also supresses noise outside of $\omega_0/2$.

Laplace Transform

We're in our last major topic of the class: the Laplace transform.

The Laplace transform will extend much of the intuition that you've developed thus far. Informally, this part of the class is more algebraic.

We will see that one major application of the Laplace transform is that it gives us a simple framework to solve differential equations.

Laplace Transform

This lecture introduces the Laplace Transform and its properties. Topics include:

- s spectrum and region of convergence
- Bilateral Laplace transform
- Unilateral Laplace transform
- Relationship between Fourier and Laplace transforms
- Laplace transforms of e^{at}, $u(t)$, t^n, $\delta(t)$, and $\cos(\omega t)$
- Laplace transform properties
- Examples
- Solving differential equations

Motivation for Laplace Transform

The Fourier transform is powerful, but it doesn't exist for some signals and systems. In several applications, including image processing, communications, and circuit design, its sufficient for analysis.

However, some systems are unstable, or are power signals where the Fourier transform can not be straightforwardly generalized. Some examples of this are signals that grow with time, like (ideally) your bank account, or the S&P 500.

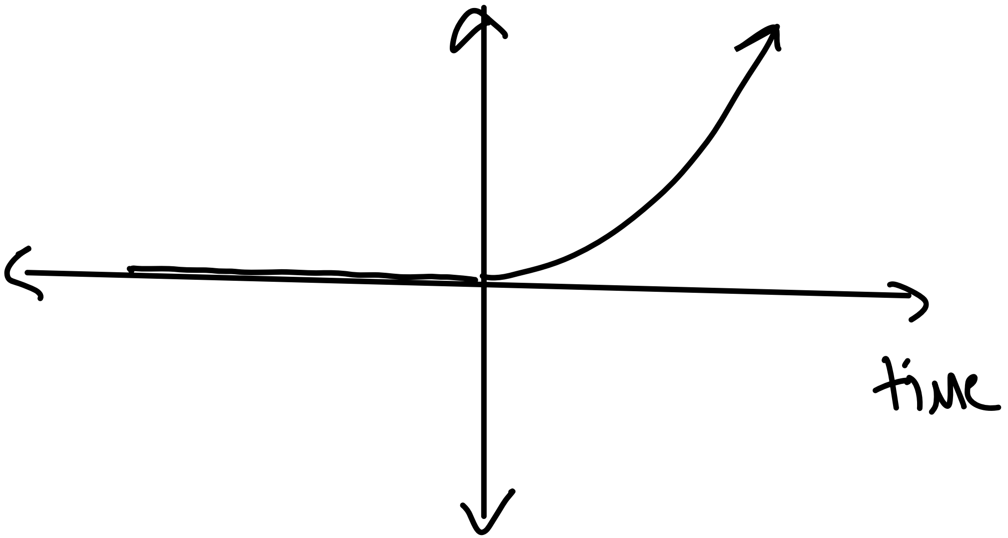
How do we analyze these systems in a similar framework to what Fourier analysis enables us to do?

Laplace Transform

Let

$$f(t) = e^{at}u(t)$$

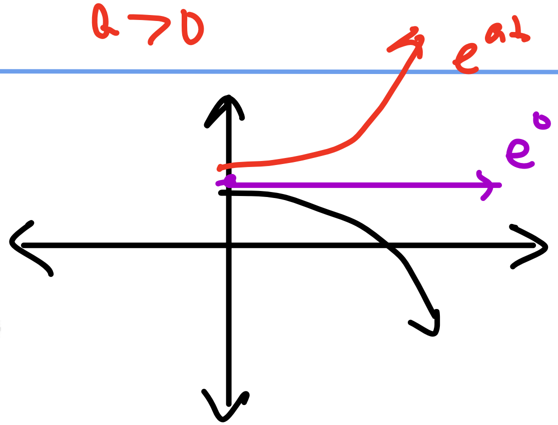
When $a > 0$ this signal does not have a Fourier transform.



$\int \quad t \to \infty \quad F(j\omega) < B.$

is not

Laplace Transform



Let

$$f(t) = e^{at}u(t)$$

When $a > 0$ this signal does not have a Fourier transform.

One approach to arrive at a Fourier transform is to define a new function

$$g(t) = f(t)e^{-\sigma t} = e^{at}u(t)e^{-\sigma t} = e^{(a-\sigma)t}u(t)$$

If $\sigma > a$, then $g(t)$ is a decreasing exponential, which has a Fourier transform.

Laplace Transform

The function $g(t) = f(t)e^{-\sigma t}$ has a Fourier transform for σ sufficiently large.
The Fourier transform of $g(t)$ comprises how to sum spectral components $e^{j\omega t}$,

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} d\omega$$

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$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)e^{j\omega t}d\omega$$

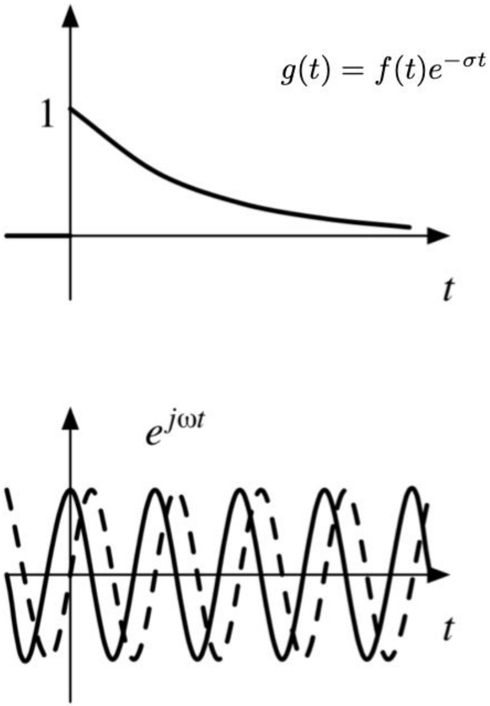
The intuition here is that because $f(t) = g(t)e^{\sigma t}$, $f(t)$ has spectral components

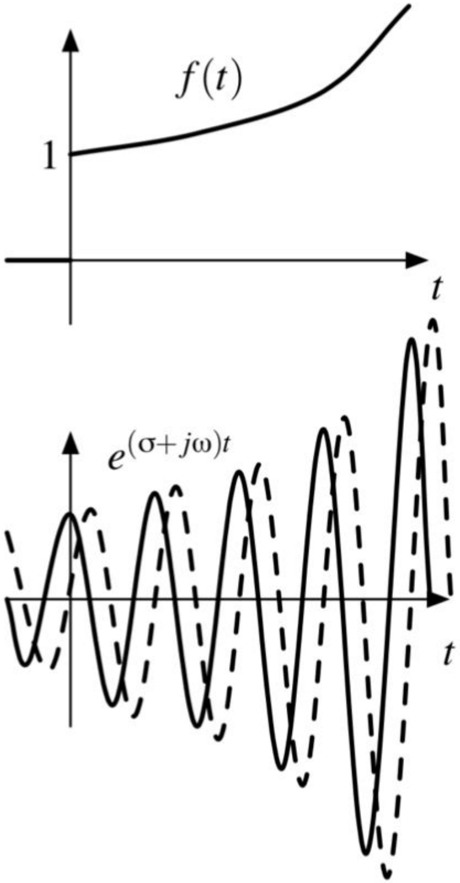
$$e^{\sigma t}e^{j\omega t} = e^{(\sigma + j\omega)t}$$

Hence, the Laplace transform gives us a spectrum of $f(t)$ in terms of a complex exponential with both real and imaginary components (where as the Fourier transform was only with imaginary components).

Laplace Transform

$\mathcal{F}$





$\mathcal{L}$

Region of Convergence

When does the s-spectrum exist?

For what values of σ does this work? In the case where $f(t) = e^{at}u(t)$, this is clear, i.e., $\sigma > a$.

In general, there is some σ_0 for which

$$f(t)e^{-\sigma_0 t}$$

goes to zero. If it does, then this $f(t)e^{-\sigma_0 t}$ is an energy signal, and its spectrum will exist.

The portion of the complex plane where $\sigma > \sigma_0$ is called the "region of convergence."

$s \triangleq \sigma + j\omega$

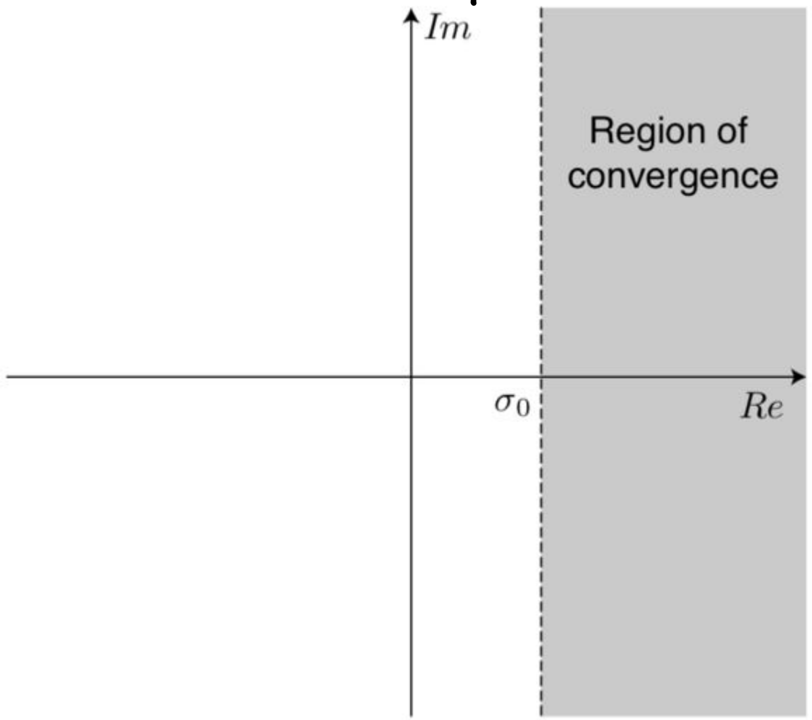
Region of Convergence

The region of convergence is illustrated below:

$e^{\underline{(\sigma+j\omega)}t} \longrightarrow e^{\underline{s}t}$

$s = \underbrace{\sigma}_{\text{Real}} + \underbrace{j\omega}_{\text{Imag.}}$

$\sim \omega$



$\sim \sigma$

Notation $\mathcal{L}\{f(t)\} \Rightarrow F(s)$

Laplace transform notation

Our notation for the Laplace transform is very similar to our prior notation. We denote

$s = \sigma + j\omega$

$$\begin{aligned} F(s) &= \mathcal{L}[f(t)] \\ f(t) &= \mathcal{L}^{-1}[F(s)] \end{aligned}$$

We will also denote this:

$$f(t) \Longleftrightarrow F(s) \quad \text{Laplace}$$

$$f(t) \Longleftrightarrow F(j\omega) \quad \text{Fourier}$$

Fourier: $e^{-j\omega t}$

Laplace: $e^{-j\omega t} e^{\sigma t} = e^{st}$

Bilateral Laplace Transform

The Laplace transform incorporates the real exponential. With $s = \sigma + j\omega$, as before,

- $j\omega$ is related to the oscillatory component of the complex exponential
- σ is related to the decay or growth of the complex exponential

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Then, the **bilateral** Laplace transform is:

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt} = \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}\,dt$$

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$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt}$$

To invert the bilateral Laplace transform, we calculate:

$$\boxed{f(t) = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} F(s)e^{st}ds}$$

for $c > \sigma_0$.

This slide is for your reference.

We will not use the bilateral transform in class examples.

Bilateral Laplace Transform

We won't use the bilateral Laplace transform, but it's worth mentioning this for completeness.

Unilateral Laplace Transform

Usually, we are interested in analyzing causal signals. In this case, we can simplify the bilateral Laplace transform. A causal signal can be written as $f(t)u(t)$, and its Laplace transform is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t)u(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} f(t)e^{st}dt \end{aligned}$$

Like a bilateral but with causal signals.

When we write 0^-, this indicates that impulses at the origin are included (e.g., $\delta(t)$ would have a contribution to this integral).

The Laplace transform is (essentially) unique. From now on, we'll use $\mathcal{L}[f(t)]$ to denote the unilateral Laplace transform of $f(t)$.

Relationship between Fourier and Laplace Transforms

The Fourier transform is a special case of the Laplace transform, i.e.,

$$\boxed{F(j\omega) = F(s)\big|_{s=j\omega}}$$

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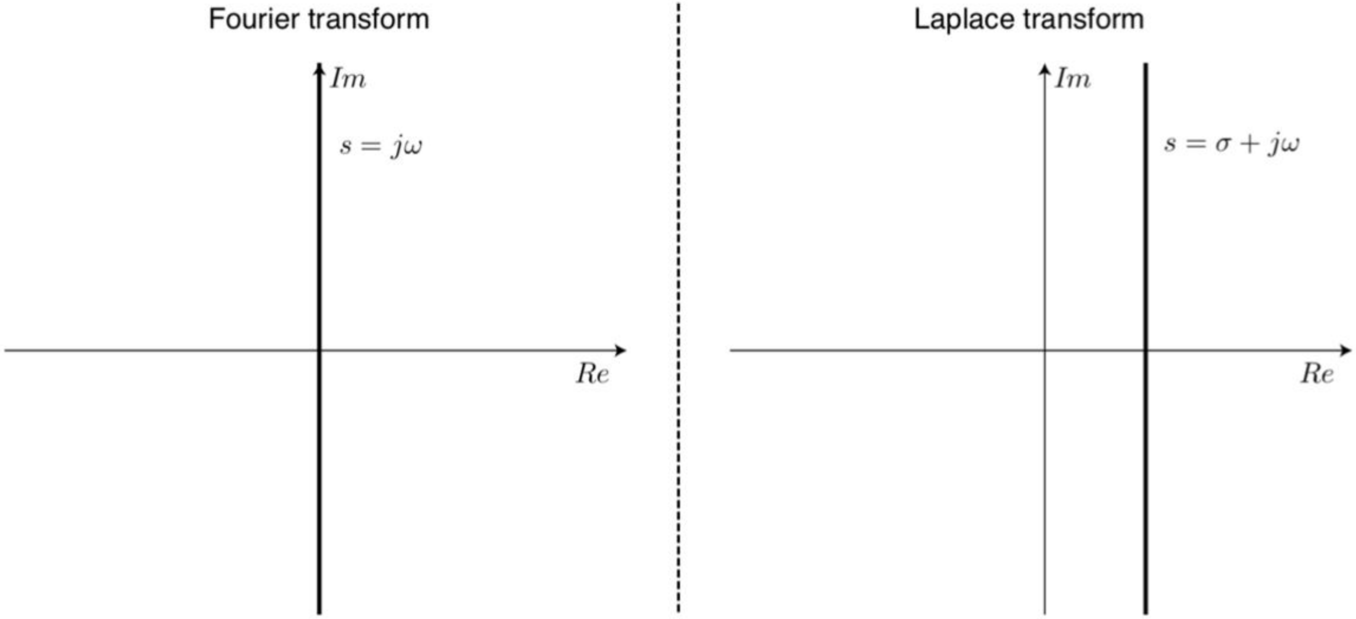
$$\boxed{F(j\omega) = F(s)|_{s=j\omega}}$$

The Fourier transform is evaluated at $s = j\omega$ and the Laplace transform is evaluated at a particular $s = \sigma + j\omega$.



Relationship between Fourier and Laplace Transforms

You may imagine that for signals where we know the Fourier transform, the Laplace transform merely replaces $j\omega$ with s. This is sometimes the case. Let's consider $f(t) = e^{-at}u(t)$.

$f(t) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{a+j\omega}$

Recall: Fourier of exp decay

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-at}e^{-st}dt \\
&= \int_0^\infty e^{-(a+s)t}dt \\
&= -\frac{1}{a+s}e^{-(a+s)t}\Big|_0^\infty \\
&= \frac{1}{a+s}
\end{aligned}
$$

Conditional

as long as $e^{-(a+s)t} \to 0$ as $t \to \infty$. When does this happen?

Relationship b/w Fourier and Laplace (cont'd)

If $e^{-(a+s)t}$ goes to zero, then so does $\left|e^{-(a+s)t}\right|$.

$e^{-(a+\sigma)t} \to 0$ when $\sigma > -a$

$$\left|e^{-(a+s)t}\right| = \left|e^{-(a+\sigma+j\omega)t}\right|$$

$$= \left|e^{-(a+\sigma)t}\right| \cdot \underbrace{\left|e^{-j\omega t}\right|}_{=1}$$

$$= e^{-(a+\sigma)t}$$

$\therefore$ If $\sigma > -a$, then L.T. exists

$$\boxed{\text{R.O.C. } \mathrm{Re}\{s\} > -a}$$

Relationship b/w Fourier and Laplace (cont'd)

Hence, we have that

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

for R.O.C. $Re\{s\} > -a$

and we know prior, for $a > 0$,

$$\mathcal{F}[e^{-at}u(t)] = \frac{1}{a+j\omega}$$

Here, the Laplace transform is the Fourier transform with $j\omega$ replaced with s.

Relationship b/w Fourier and Laplace (cont'd)

A key thing to note is that with

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

holds for all a, positive or negative, as long as $\sigma > -a$.

This means that, for $a > 0$,

$$\mathcal{L}[e^{at}u(t)] = \frac{1}{s-a}$$

Of course, this signal does not have a Fourier transform.

CYU: Comparing Fourier and Laplace

Let's take the Laplace transform of the unit step

$$f(t) = u(t)$$

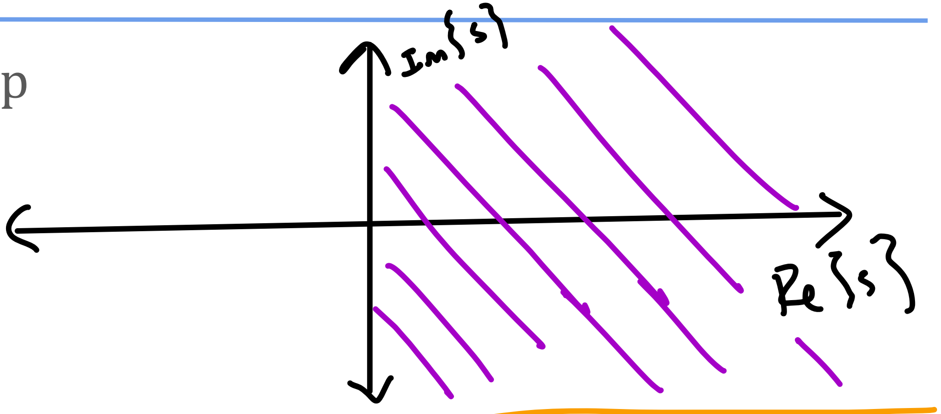
$$F(s) = \int_0^\infty u(t)\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^\infty$$

$$= \frac{1}{s}$$

R.O.C. $Re\{s\} > 0$



$\boxed{\text{Note: } \mathcal{F}[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)}$

To find ROC, we need $e^{-st} \to 0$ at $t \to \infty$.

Relationship between Fourier and Laplace

Recall the Fourier transform of the unit step is:

$$\mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

This resembles the Laplace transform with $s = j\omega$, but there is an additional $\pi\delta(\omega)$ term.

Relationship between Fourier and Laplace

We will see this tends to be the case for some of our generalized Fourier transforms. For example, consider the Laplace transform of

$$\begin{aligned} f(t) &= \cos(\omega t) \\ &= \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right] \end{aligned}$$

$$\mathcal{L}\{f(t)\} = F(s) = \frac{1}{2}\left[\mathcal{L}\left[e^{j\omega t}\right] + \mathcal{L}\left[e^{-j\omega t}\right]\right]$$

$$= \frac{s}{s^2 + \omega^2}$$

R.O.C. $\mathrm{Re}\{s\} > 0$.

Example Laplace Transforms

Laplace transform of powers of t

Laplace transforms, given all we've learned thus far, should be fairly straightforward to evaluate. We'll go over a few examples here. Let

$$f(t) = t^n u(t)$$

$u \triangleq t^n \quad dv = e^{-st}\,dt$

$du = nt^{n-1}\,dt$

$v = -\frac{1}{s}e^{-st}$

for $n \geq 1$. Then,

$$F(s) = \int_0^\infty t^n e^{-st}\,dt$$

Int by parts $\int u\,dv = uv - \int v\,du$

$$\mathcal{L}[t^n] = F(s) = \left.\frac{-t^n e^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st} n t^{n-1}\,dt$$

$$= 0 - 0 + \frac{n}{s}\boxed{\int_0^\infty t^{n-1} e^{-st}\,dt} = \frac{n}{s}\mathcal{L}\left[t^{n-1}\right]$$

Example Laplace Transforms

Power Rule

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}]$$

$$u(t)\,t^0 = \begin{cases} 1, & t \geq 0 \\ 0, & \text{o.w.} \end{cases} = u(t)$$

cyu: Please calculate $\mathcal{L}[t^1]$

$$u(t) \overset{\mathcal{L}}{\Longleftrightarrow} 1/s$$

$$\therefore \mathcal{L}[t^1] = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

General

$$\mathcal{L}[t^0] = 1/s$$

$$\mathcal{L}[t^1] = 1/s^2$$

$$\left.\begin{array}{l}\mathcal{L}[t^2] = 2/s^3 \\ \vdots \end{array}\right\} \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

Example Laplace Transforms

Laplace transform of impulse

Let

$$f(t) = \delta(t)$$

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty \delta(t) e^{-st}\, dt$$

$$= \int_0^\infty \delta(t) e^{0}\, dt$$

$$= \int_0^\infty \delta(t)\, dt$$

$$= 1$$

$$\therefore \mathcal{L}[\delta(t)] = 1$$

A Trend Emerging ..

Pattern for integration and differentiation?

$x'''(t) + 5\cancel{x}'(t) + x(t) = 0$

Notice the following trends:

Key: L.T. transform takes a diff eqn. and converts it to a algebraic equation

$$
\begin{aligned}
\delta(t) &\overset{\mathcal{L}}{\Longleftrightarrow} 1 \\
u(t) &\Longleftrightarrow \frac{1}{s} \\
tu(t) &\Longleftrightarrow \frac{1}{s^2} \\
\frac{1}{2}t^2u(t) &\Longleftrightarrow \frac{1}{s^3} \\
\frac{1}{6}t^3u(t) &\Longleftrightarrow \frac{1}{s^4} \\
&\vdots
\end{aligned}
$$

$x(t) \underset{\mathcal{L}}{\Longleftrightarrow} X(s)$

$s^3X(s) + 5sX(s) + X(s) = 0$

We see a clear pattern: differentiating a signal is equivalent to multiplying the Laplace transform by s while integrating is equivalent to multiplying the Laplace transform by $1/s$.

Laplace Transform Properties

1. **Linearity:** $\boxed{\mathcal{L}[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)}$

2. **Time scaling:** $\boxed{\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)}$

3. **Time shift:** $\boxed{\mathcal{L}[f(t-T)] = e^{-sT}F(s)}$

4. **Frequency shift:** $\boxed{\mathcal{L}\left[f(t)e^{s_0 t}\right] = F(s - s_0)}$

5. **Convolution:** $\boxed{\mathcal{L}[f_1(t) * f_2(t)] = F_1(s)F_2(s)}$

6. **Integration:** $\boxed{\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)}$

7. **Derivative:** $\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)}$

8. **Multiplication by *t*:** $\boxed{\mathcal{L}[tf(t)] = -F'(s)}$

Differentiation and Integration Property

Key reason to use Laplace Transforms! Turns differential equations into algebraic equations.

$g(t)$	$G(s)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

CYU: Apply the Property to get L.T. of Step and Ramp

Example: unit step and ramp functions

We were able to calculate the Fourier transform of the unit step; however, this required generalizing the Fourier transform. How can we find the Laplace transforms of the unit step and unit ramp function using the integral properties of the Laplace transform?

If $f(t) = \delta(t)$, then $F(s) = 1$. Then,

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[u(t)] = \frac{1}{s}\mathcal{L}[\delta(t)] = 1/s$$

$$\mathcal{L}[r(t)] = \frac{1}{s}\mathcal{L}[u(t)] = 1/s^2$$

CYU: Laplace Transform and Differential Equations

Find the Laplace transform of $f(t)$ given the following differential equation:

$$f''(t) + 3f'(t) + 2f(t) = 0 \; where \; f(0) = 1 \; and \; f'(0) = 0.$$

$$\mathcal{L}[f''(t)] + 3\mathcal{L}[f'(t)] + 2\mathcal{L}[f(t)] = 0$$

$$s^2\mathcal{L}[f(t)] - s f(0) - f'(0) + 3s\mathcal{L}[f(t)] - 3f(0) + 2\mathcal{L}[f(t)] = 0$$

$$(s^2 + 3s + 2)\mathcal{L}[f(t)] = s + 3$$

$$\mathcal{L}[f(t)] = \frac{s+3}{(s^2 + 3s + 2)}$$