EE102

Lecture 13

EE102 Announcements

- **HW #4 is due today.**
- **HW #5 is due Friday 11/21.**
- **HW #6 is due Friday 11/28.**

ABET Learning Outcomes

✔ Understand the concept of a signal and a system, plot continuous-time signals, evaluate the periodicity of a signal.

✔ Identify properties of continuous-time systems such as linearity, time-invariance, and causality.

✔ Calculate with the Dirac delta function.

✔ Compute convolution of continuous-time functions.

✔ Understand the concept of the impulse response function of a linear system, and its use to describe the input/output relationship.

Compute the Laplace transform of a continuous function, identify its domain of convergence, and be familiar with its basic properties, including the initial and final value theorems.

Find the inverse Laplace transform by partial fractions.

Use the Laplace transform to solve constant-coefficient differential equations with initial conditions

Use the Laplace transform to evaluate the transfer function of linear time-invariant systems.

✔ Understand Parseval's relation in Fourier series, and its interpretation in terms of decomposing the signal's energy between its harmonics

✔ Evaluate the response of a linear time-invariant system to periodic inputs.

✔ **Evaluate the Fourier transform of a continuous function, and be familiar with its basic properties.** Relate it to the Laplace transform.

✔ **Evaluate and plot the frequency responses (magnitude and phase) of linear time-invariant systems, and apply it to filtering of input signals.**

Understand conditions under which a band-limited function can be recovered from its samples

Convolution Theorem

$\star\star\star$ The Convolution Theorem $\star\star\star$

It's probably worth taking up an entire slide here just to say:

This is one of the most important theorems of the class, and is a key reason why a lot of our technology works. (!) This theorem enables us to do convolution, and thus any LTI operation, straightforwardly. With it, we no longer have to do the impulse response integral we saw earlier.

Convolution Theorem

⋆⋆⋆ The Convolution Theorem ⋆⋆⋆

If $f_1(t)$ and $f_2(t)$ are two signals with Fourier transforms $F_1(j\omega)$ and $F_2(j\omega)$, respectively, then

$$\boxed{\mathcal{F}[(f_1 * f_2)(t)] = F_1(j\omega)F_2(j\omega)}$$

Stated simply: **convolution in the time domain is multiplication in the frequency domain.**

(And multiplication is easy.)

Duality

Duality of the Fourier transform

If $\mathcal{F}[f(t)] = F(j\omega)$, then

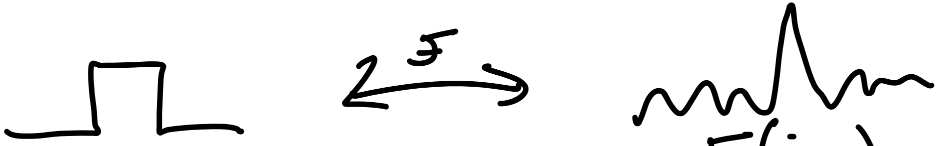
$$\boxed{F(t) \iff 2\pi f(-j\omega)}$$

This expression may be opaque at first. What this is saying is that if I take a Fourier transform pair, I can find the dual pair by replacing all the ω's with t's in $F(j\omega)$ and all the t's with $-\omega$'s in $f(t)$. After scaling by 2π, this results in another Fourier transform pair.

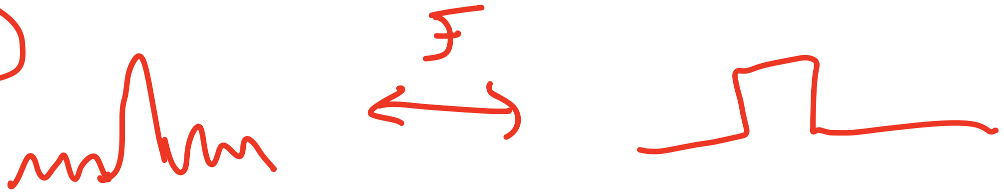
Essentially, every Fourier transform pair we derive really gives us two Fourier transform pairs.

Duality

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

① 

$f(t)$ $F(j\omega)$

② 

Inv. F.T. Eqn.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

$$2\pi f(-t) = \int F(j\omega) e^{-j\omega t} d\omega$$

Duality

Duality of the Fourier transform (cont.)

To show this, recognize that as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

then

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(j\omega)e^{-j\omega t}d\omega$$

Now, the r.h.s. of this equation is the Fourier transform of $F(j\omega)$ with the roles of ω and t reversed. Hence, $2\pi f(-t)$ is the Fourier transform of $F(j\omega)$ (!) and after we swap the ω and the t's, we arrive at the duality result.

Duality Examples

Duality examples

- Since $\mathrm{rect}(t) \iff \mathrm{sinc}(\omega/2\pi)$, then

$$\begin{aligned} \mathrm{sinc}(t/2\pi) &\iff 2\pi\mathrm{rect}(-\omega) \\ &= 2\pi\mathrm{rect}(\omega) \end{aligned}$$

Even

Thus, we have that $\mathrm{sinc}(t/2\pi) \iff 2\pi\mathrm{rect}(\omega)$.

- Since

$$e^{-at}u(t) \iff \frac{1}{a+j\omega}$$

then

$$\frac{1}{a+jt} \iff 2\pi e^{a\omega}u(-\omega)$$

Dual intuition: convolution in time domain is multiplication in frequency domain. Thus, multiplication in the time domain ought be convolution in frequency domain.

Frequency domain convolution

The frequency domain convolution theorem is that for $f_1(t) \iff F_1(j\omega)$ and $f_2(t) \iff F_2(j\omega)$, then

$$\boxed{\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\nu)F_2(j(\omega-\nu))d\nu}$$

We typically write this as:

$$\boxed{\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}(F_1 * F_2)(j\omega)}$$

but note that the convolution is w.r.t. ω, not $j\omega$.

This means that multiplication in the time domain is convolution in the frequency domain. This proof is very similar to the time domain proof.

Modulation: duality of time-shifting

Dual intuition: Time shift in the time domain is multiplication by a complex exp. in freq domain
Thus, multiplication by a complex exp. in the freq domain ought be a shift in the freq domain.

Recall that:

Modulation //

$$\boxed{\mathcal{F}[f(t-\tau)] = e^{-j\omega\tau}F(j\omega)}$$

Using duality, we arrive at:

Dual //

$$\mathcal{F}[f(t)e^{j\omega_0 t}] \quad = \quad F(j(\omega-\omega_0))$$

Using linearity, we also see that:

$$\begin{aligned}
\mathcal{F}[f(t)\cos(\omega_0 t)] &= \frac{1}{2}\left(F(j(\omega-\omega_0)) + F(j(\omega+\omega_0))\right) \\
\mathcal{F}[f(t)\sin(\omega_0 t)] &= \frac{1}{2j}\left(F(j(\omega-\omega_0)) - F(j(\omega+\omega_0))\right)
\end{aligned}$$

Modulation

Fourier transform of a modulated signal (cont.)

To prove the modulation result, note that if $\mathcal{F}[f(t)] = F(j\omega)$ then

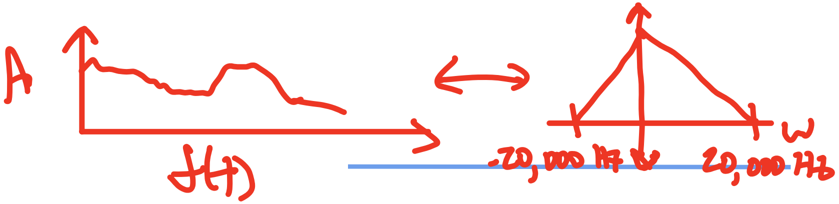
$$\begin{aligned}
\mathcal{F}[f(t)e^{j\omega_0 t}] &= \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt \\
&= F(j(\omega-\omega_0))
\end{aligned}$$

To get the cosine and sine results, we note that e.g., for cosine,

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

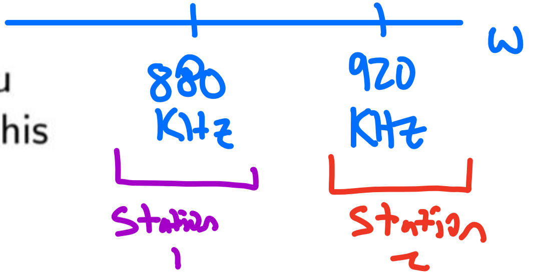
From here, we can use linearity to compute the Fourier transform.

Modulation



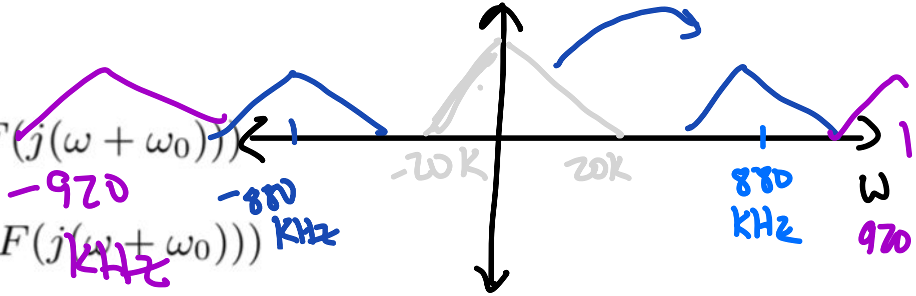
Fourier transform of a modulated signal

A major component of communications has to do with *modulation*. For example, AM and FM radio are amplitude modulation and frequency modulation respectively. AM radio involves multiplying $f(t)$, the signal you wish to transmit, with a complex exponential at a carrier frequency, ω_0. This frequency, ω_0, is the frequency you dial in your car to get AM radio.



Here are three ways to modulate a signal: If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$
\begin{aligned}
\mathcal{F}[f(t)e^{j\omega_0 t}] &= F(j(\omega - \omega_0)) \\
\mathcal{F}[f(t)\cos(\omega_0 t)] &= \frac{1}{2}\left(F(j(\omega - \omega_0)) + F(j(\omega + \omega_0))\right) \\
\mathcal{F}[f(t)\sin(\omega_0 t)] &= \frac{1}{2j}\left(F(j(\omega - \omega_0)) - F(j(\omega + \omega_0))\right)
\end{aligned}
$$



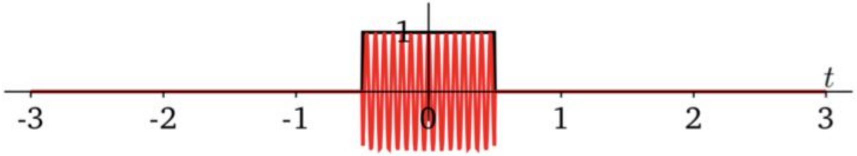
Typically, modulation is done through multiplication by $\cos(\omega_0 t)$. Modulation is dual to the time shift Fourier transform.

What modulation intuitively does is take $F(j\omega)$ and create replicas at $\pm\omega_0$.

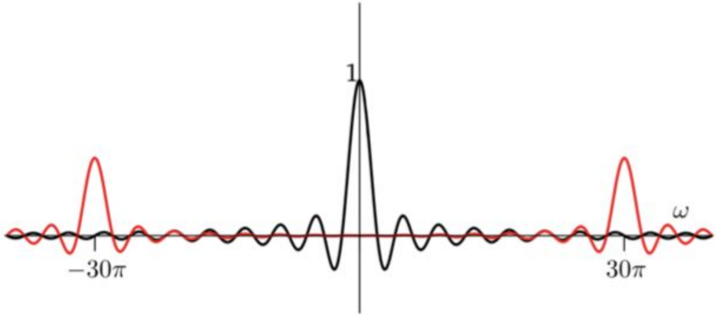
Modulation

Fourier transform of a modulated signal (cont.)

Below, we show what modulation does. We take a signal (here a rect) and multiply it by a cosine with $\omega_0 = 30\pi$. This is denoted in red in the plot below.



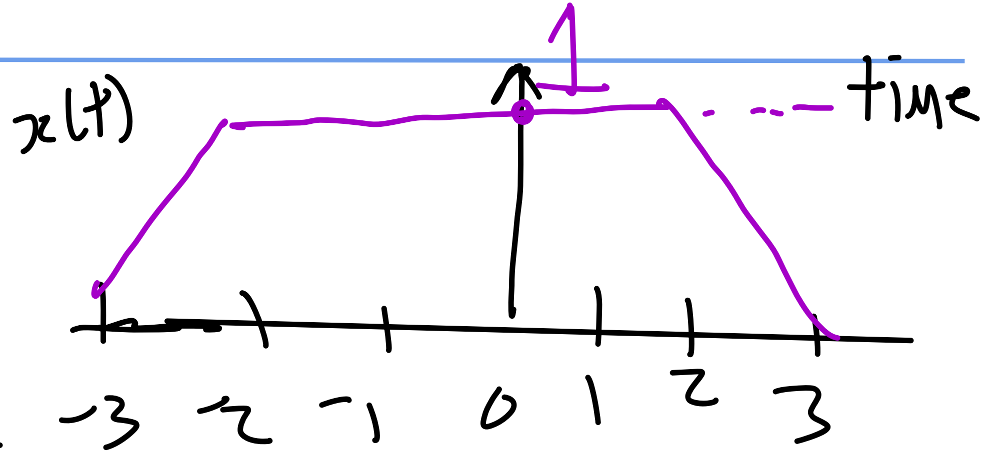
The spectrum takes the FT of our signal (i.e., a sinc) and creates replicas at $\pm 30\pi$.



From here, you can gain some intuition for why different radio stations use different frequencies. They're given these frequencies to transmit whatever signals they like; each radio station occupies a different part of the spectrum!

CYU

Compute $\int_{-\infty}^{\infty} e^{-j\omega} X(j\omega)\, d\omega$.



$$x(t) = \frac{1}{2\pi} \int X(j\omega) e^{j\omega t}\, d\omega$$

$$x(t)\Big|_{-1} = \frac{1}{2\pi} \int X(j\omega) e^{-j\omega}\, d\omega$$

$$2\pi\, x(t)\Big|_{-1} = \boxed{2\pi}$$

Time-reversal

Fourier transform of a time-reversed signal

If $\mathcal{F}[f(t)] = F(j\omega)$, then

$$\boxed{\mathcal{F}[f(-t)] = F(-j\omega)}$$

To show this, apply the time-scaling result with $a = -1$.

Time-reversal

Time reversal example

Find the Fourier transform of $f(t) = e^{-a|t|}$ (for $a > 0$) without doing integration.

We know that

$$e^{-at}u(t) \Longleftrightarrow \frac{1}{a + j\omega}$$

LYW

@ home.

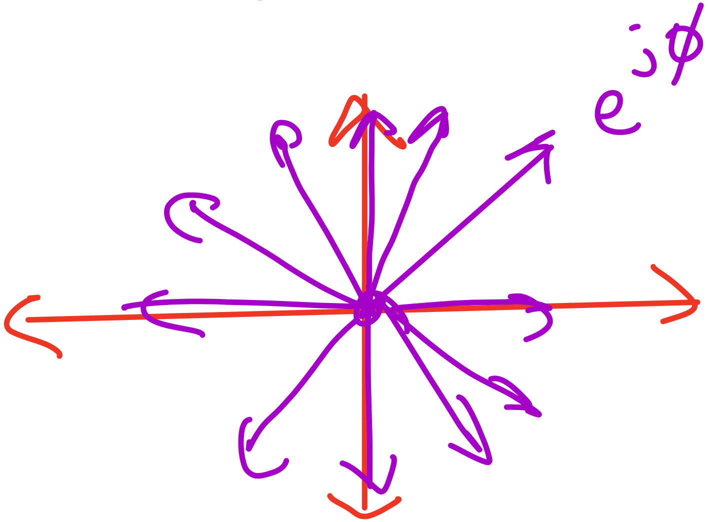
Time-reversal

CYU: What is the Fourier Transform of “1”

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}$$

$f(t) = 1$

Ans: $$\int_{-\infty}^{\infty} e^{-j\omega t}\, dt = \cancel{\left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-\infty}^{\infty}}$$



$$\hookrightarrow \int e^{0} + \int \cancel{e^{-j\omega t}}\, dt$$

$$1 \Leftrightarrow 2\pi\,\delta(\omega)$$

Other Fourier Transforms

$$\boxed{\delta(t) \iff 1}$$

$$\boxed{\delta(t-\tau) \iff e^{-j\omega\tau}}$$

$$\boxed{1 \iff 2\pi\delta(\omega)}$$

$$\boxed{u(t) \iff \pi\delta(\omega) + \frac{1}{j\omega}}$$

$$\boxed{e^{j\omega_0 t} \iff 2\pi\delta(\omega-\omega_0)}$$

$$\boxed{\cos(\omega_0 t) \iff \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))}$$

$$\boxed{\sin(\omega_0 t) \iff j\pi(\delta(\omega+\omega_0)-\delta(\omega-\omega_0))}$$

CYU: Fourier transform of Step Function

What is the FT of step?

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

Ans: $\mathcal{F}[u(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$

$$= \int_{0}^{\infty} e^{-j\omega t}\, dt$$

$$= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{0}^{\infty} \rightarrow \text{doesn't converge.}$$

Introduce "Limiting FT"

Fourier transform of Step Function

Limiting Fourier transforms

When the Fourier transform integral doesn't converge, and there's not a "trick" we can use, an alternative approach is to use limiting Fourier transforms.

In this approach, we represent the signal as a limit of a sequence of signals for which the Fourier transforms do exist. i.e., consider $f_n(t)$ which does have a Fourier transform. If

$$f(t) = \lim_{n\to\infty} f_n(t)$$

then we also have that

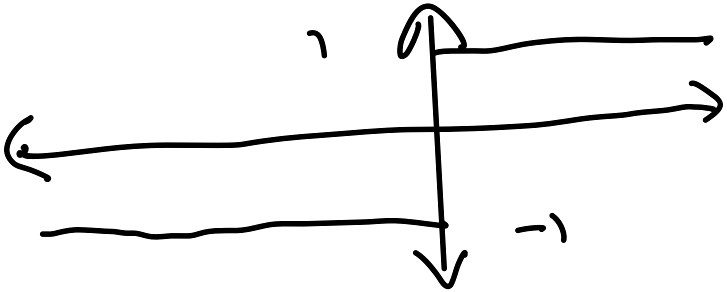
$$F(j\omega) = \lim_{n\to\infty} F_n(j\omega)$$

if the limit makes sense.

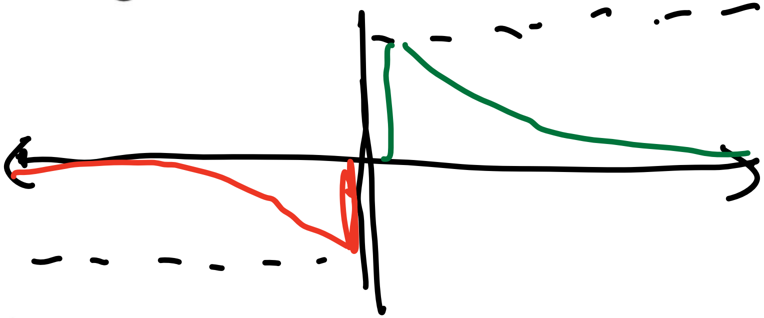
Fourier Transform of a Step Function

Limiting Fourier transform example

Consider the Fourier transform of $f(t) = \mathrm{sign}(t)$. This signal is defined as



$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$



We previously derived the Fourier transform for $e^{-at}u(t)$. We can use this signal to make a limiting approximation to $\mathrm{sign}(t)$ by setting

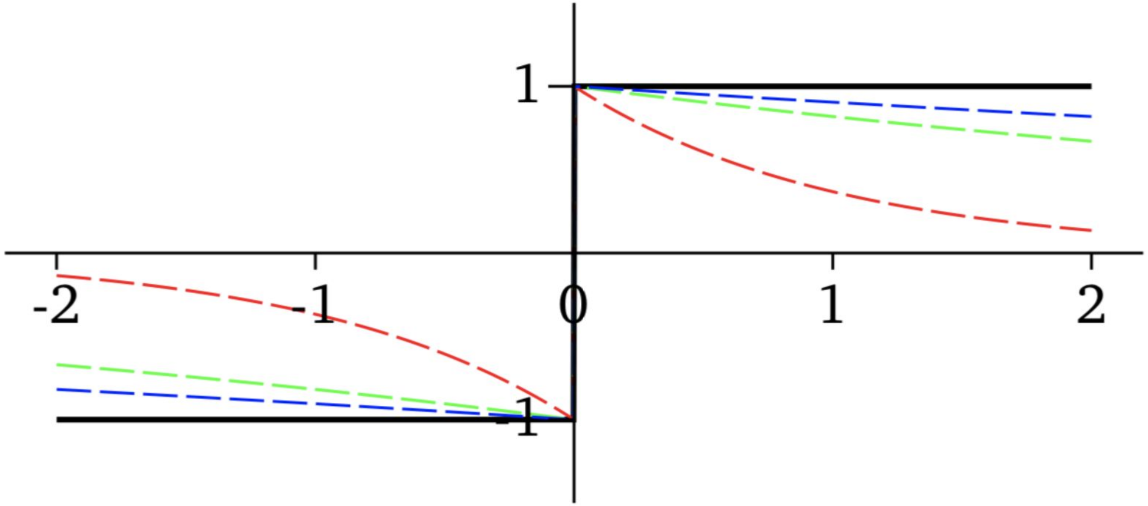
$$f_a(t) = e^{-at}u(t) - e^{at}u(-t)$$

This approximation is shown on the next slide.

Fourier Transform of a Step Function

Limiting Fourier transform example (cont.)

Below we show $f_a(t)$ for $a = 1$ (red), $a = 1/5$ (green), and $a = 1/10$ (blue).



As $a \to 0$, then, $f_a(t) \to \mathrm{sign}(t)$.

Fourier Transform of a Step Function

Hence, we can compute the Fourier transform, $F_a(j\omega) = \mathcal{F}[f_a(t)]$, and then compute the Fourier transform of $\mathrm{sign}(t)$ as the limit of $F_a(j\omega)$ as $a \to 0$.

$$\begin{aligned}
F_a(j\omega) &= \mathcal{F}[f_a(t)] \\
&= \mathcal{F}\left[e^{-at}u(t) - e^{at}u(-t)\right] \\
&= \mathcal{F}\left[e^{-at}u(t)\right] - \mathcal{F}\left[e^{at}u(-t)\right] \\
&= \frac{1}{a+j\omega} - \frac{1}{a-j\omega} \\
&= \frac{-2j\omega}{a^2+\omega^2}
\end{aligned}$$

① find $f_a(t)$

② Compute $\mathcal{F}_a(t)$

③ $\lim\limits_{a\to 0} \mathcal{F}_a[t]$

When $\omega = 0$, then $F_a(j\omega) = 0$ for any $a \neq 0$. Otherwise, if $\omega \neq 0$, then

$$\frac{1}{j} \triangleq -j$$

$$\begin{aligned}
\lim_{a\to 0} F_a(j\omega) &= \lim_{a\to 0} \frac{-2j\omega}{a^2+\omega^2} \\
&= \frac{-2j\omega}{\omega^2} \\
&= \frac{2}{j\omega}
\end{aligned}$$

Fourier Transform of a Step Function

Limiting Fourier transform example (cont.)

With this, we can state that

$$\boxed{\mathrm{sign}(t) \iff \begin{cases} \frac{2}{j\omega}, & \omega \neq 0 \\ 0, & \omega = 0 \end{cases}}$$

DC component as a sanity check.

Fourier Transform of a Step Function

The step function can be written in terms of the sign function, i.e.,

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sign}(t)$$

Therefore,

$$\begin{aligned}
\mathcal{F}[u(t)] &= \mathcal{F}\left[\frac{1}{2} + \frac{1}{2}\mathrm{sign}(t)\right] \\
&= \frac{1}{2}2\pi\delta(\omega) + \frac{1}{2}\left(\frac{2}{j\omega}\right) \\
&= \pi\delta(\omega) + \frac{1}{j\omega}
\end{aligned}$$

$-j \triangleq \frac{1}{j}$

Note that the second term is zero at $\omega = 0$, and so the spectrum of $u(t)$ is $\pi\delta(\omega)$ at $\omega = 0$. Thus,

$$\boxed{u(t) \iff \pi\delta(\omega) + \frac{1}{j\omega}}$$

Fourier Transform of Integral

$$\mathcal{F}\left[\int_{-\infty}^{t} f(t)\,dt\right] = \mathcal{F}\left[f(t) * u(t)\right]$$

$$= \mathcal{F}\left[f(t)\right] \cdot \mathcal{F}\left[u(t)\right]$$

$$= F(j\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

$$= \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega}$$

All Properties

1. **Linearity:** $$\boxed{\mathcal{F}[af_1(t)+bf_2(t)] = a\mathcal{F}[f_1(t)] + b\mathcal{F}[f_2(t)]}$$
2. **Time scaling:** $$\boxed{\mathcal{F}[f(at)] = \frac{1}{|a|}F\left(j\frac{\omega}{a}\right)}$$
3. **Time reversal:** $$\boxed{\mathcal{F}[f(-t)] = F(-j\omega)}$$
4. **Complex conjugate:** $$\boxed{f^*(t) \iff F^*(-j\omega)}$$
5. **Duality:** $$\boxed{F(t) \iff 2\pi f(-j\omega)}$$
6. **Time-shifting:** $$\boxed{\mathcal{F}[f(t-\tau)] = e^{-j\omega\tau}F(j\omega)}$$
7. **Derivative:** $$\boxed{\mathcal{F}[f'(t)] = j\omega F(j\omega)}$$
8. **Convolution:** $$\boxed{\mathcal{F}[(f_1 * f_2)(t)] = F_1(j\omega)F_2(j\omega)}$$

All Properties

9. Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega$$

10. Multiplication:

$$\mathcal{F}[f_1(t) f_2(t)] = \frac{1}{2\pi} (F_1 * F_2)(j\omega)$$

11. Modulation:

$$\mathcal{F}[f(t) e^{j\omega_0 t}] \quad = \quad F(j(\omega - \omega_0))$$

12. Integral:

$$\int_{-\infty}^{t} f(\tau) d\tau \iff \pi F(0)\delta(\omega) + \frac{F(j\omega)}{j\omega}$$

FT Pairs

$$\text{rect}(t/T) \iff T\,\text{sinc}(\omega T/2\pi)$$

$$e^{-at}u(t) \iff \frac{1}{a+j\omega}$$

$$\mathcal{F}\left[e^{a|t|}\right] = \frac{2a}{a^2+\omega^2}$$

$$\frac{2}{(2+j\omega)^2}$$

$$\text{sinc}(t/2\pi) \iff 2\pi\text{rect}(\omega)$$

$$\Delta(t) \iff \text{sinc}^2(\omega/2\pi)$$

$$\mathcal{F}[\text{sinc}^2(t)] = \Delta(\omega/2\pi)$$

$$\boxed{\delta(t) \iff 1}$$

$$\boxed{\delta(t-\tau) \iff e^{-j\omega\tau}}$$

$$\boxed{1 \iff 2\pi\delta(\omega)}$$

$$\boxed{u(t) \iff \pi\delta(\omega) + \frac{1}{j\omega}}$$

$$\boxed{e^{j\omega_0 t} \iff 2\pi\delta(\omega-\omega_0)}$$

$$\boxed{\cos(\omega_0 t) \iff \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))}$$

$$\boxed{\sin(\omega_0 t) \iff j\pi(\delta(\omega+\omega_0)-\delta(\omega-\omega_0))}$$

cyu

Find $X(\omega)$ if $x(t) = 2te^{-2t}u(t)$

$e^{-2t}u(t) \iff \frac{1}{2+j\omega}$

Hint $-jtf(t) = F'(j\omega)$

Let $f(t) = e^{-2t}u(t) \Rightarrow x(t) = \frac{-2t(-jf(t))}{j}$

$$X(j\omega) = -\frac{2}{j}F'(j\omega) = -\frac{2}{j}\left(\frac{d}{d\omega}\frac{1}{2+j\omega}\right) =$$

$x \to \boxed{h} \to y$

Frequency Response

Motivation for this lecture

We previously discussed the impulse response, $h(t)$, which is the output of a system when the input is an impulse, $\delta(t)$. We saw that $h(t)$ characterized any LTI system, as for any LTI system with input, $x(t)$, we could calculate the output as

$$\begin{aligned} y(t) &= (x * h)(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \end{aligned}$$

A complication we discussed is that computing the output this way requires evaluating a convolution integral, which can be difficult and time-consuming.

Frequency Response

Motivation for this lecture (cont.)

But now, equipped with the convolution theorem, why not just take the Fourier transform of both sides? This turns the convolution into multiplication.

$$Y(j\omega) = H(j\omega)X(j\omega)$$

where $X(j\omega)$ is the Fourier transform of the input, $Y(j\omega)$ is the Fourier transform of the output, and $H(j\omega)$ is the *frequency response*, i.e., the Fourier transform of the impulse response.

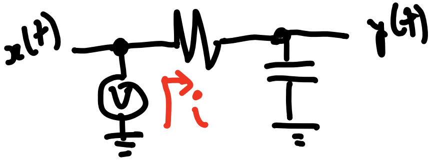
Transfer Fn

Freq Response Fn

Example: RC Circuit

$x(t) \Leftrightarrow X(j\omega)$

$y(t) \Leftrightarrow Y(j\omega)$



(1) $x - y = iR$

(2) $i = C\dfrac{dy}{dt}$

Combine to get $x - y = \dfrac{dy}{dt} RC$

Solve through diff eq OR use F.T.

$\mathcal{F}[x - y] = \mathcal{F}\left[\dfrac{dy}{dt} RC\right]$

$X(j\omega) - Y(j\omega) = j\omega Y(j\omega) RC$

$X(j\omega) = Y(j\omega)\left[1 + j\omega RC\right]$

$H(j\omega) = \dfrac{1}{1 + j\omega RC}$

$$\boxed{\begin{aligned} Y &= XH \\ X &= Y/H \end{aligned}}$$

Example: RC Circuit

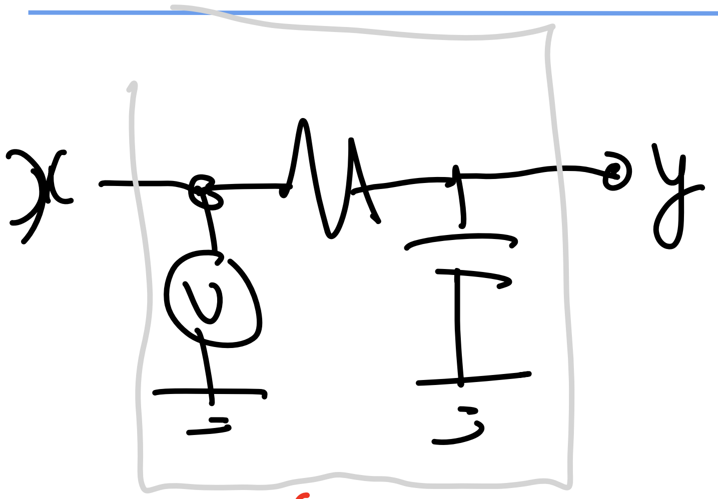
Calculate $|H(j\omega)|$

$$H(j\omega) = \frac{1}{1+j\omega RC} = \frac{1}{1+j\omega RC}\left(\frac{1-j\omega RC}{1-j\omega RC}\right) = \frac{1-j\omega RC}{1+\omega^2R^2C^2} \Rightarrow a+bj$$

$$|H(j\omega)|^2 = H(j\omega)\cdot H^*(j\omega) = \frac{1}{1+\omega^2R^2C^2}$$

$$\boxed{|H(j\omega)| = \sqrt{\frac{1}{1+\omega^2R^2C^2}}}$$

Example: RC Circuit



$$|Y(j\omega)| = |H(j\omega)|\,|X(j\omega)|$$

$$H(j\omega) = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$$