EE102

Lecture 7

EE102 Announcements

- **Third Homework due this Friday, 10/24**
- **Fourth Homework out, due 11/14**
- **Midterm is 10/30**

△ Midterm does not include Fourier (next Tuesday)

△ Next Tues Lecture will be Remote Recording

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

How to Compute Convolution: flip and drag

□ Learn How to Convolve + Properties

To calculate $y(t) = (x * h)(t)$,

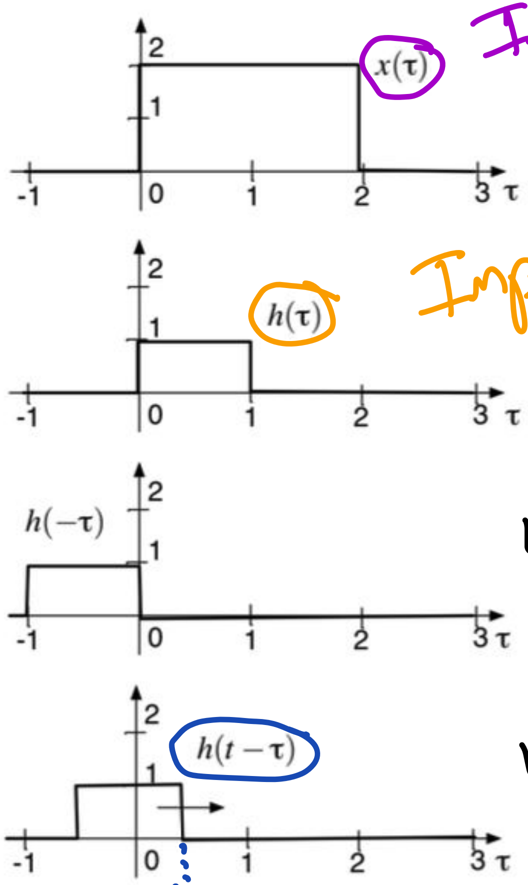
- Flip (i.e., reverse in time) the impulse response. This changes $h(\tau)$ to $h(-\tau)$.
- Begin to drag the reversed time response by some amount, t. This results in $h(t-\tau)$.
- For a given t, multiply $h(t-\tau)$ pointwise by $x(\tau)$. This produces $x(\tau)h(t-\tau)$.
- Integrate this product over τ. This produces $y(t)$ at this particular time t.

This technique is referred to as the "flip-and-drag" technique.

$h(\tau)$

How to Compute Convolution: flip and drag

$$y = x * h$$
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$



Input

Impulse Response

Flip Phase $h(-\tau)$

Drag Phase $h(t-\tau)$

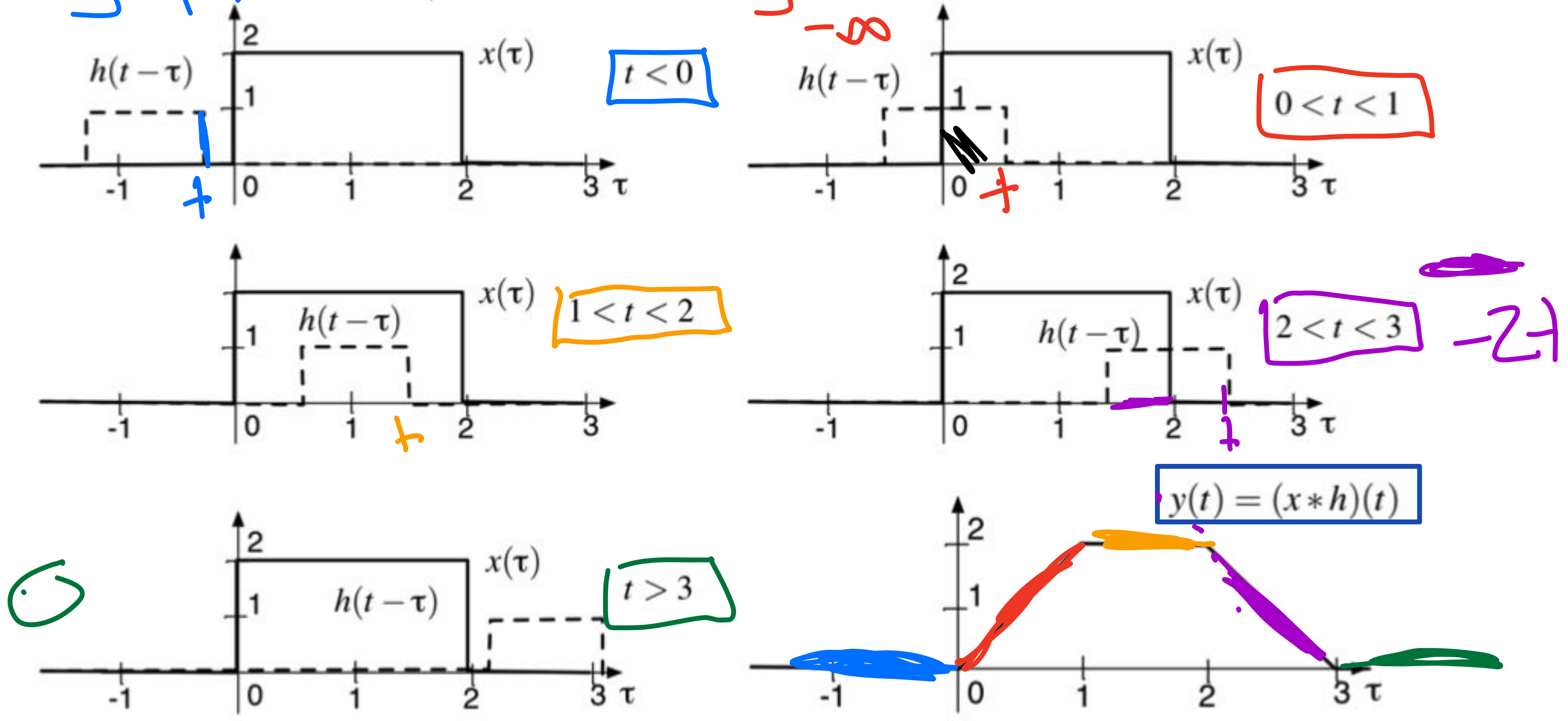
$t = 0.4$

$y(t) = \int x(\tau) h(t-\tau) d\tau$

How to Compute Convolution: flip and drag

$\int x(\tau) h(t-\tau) = 0$

$\int_{-\infty}^{\infty} x(\tau) h(t-\tau) = 2t$

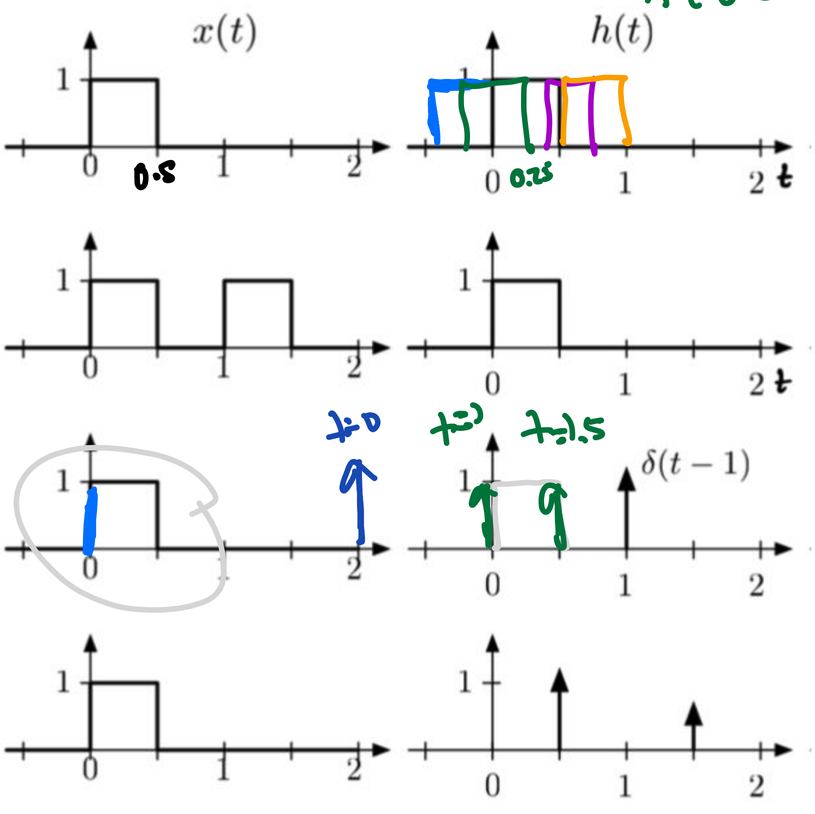


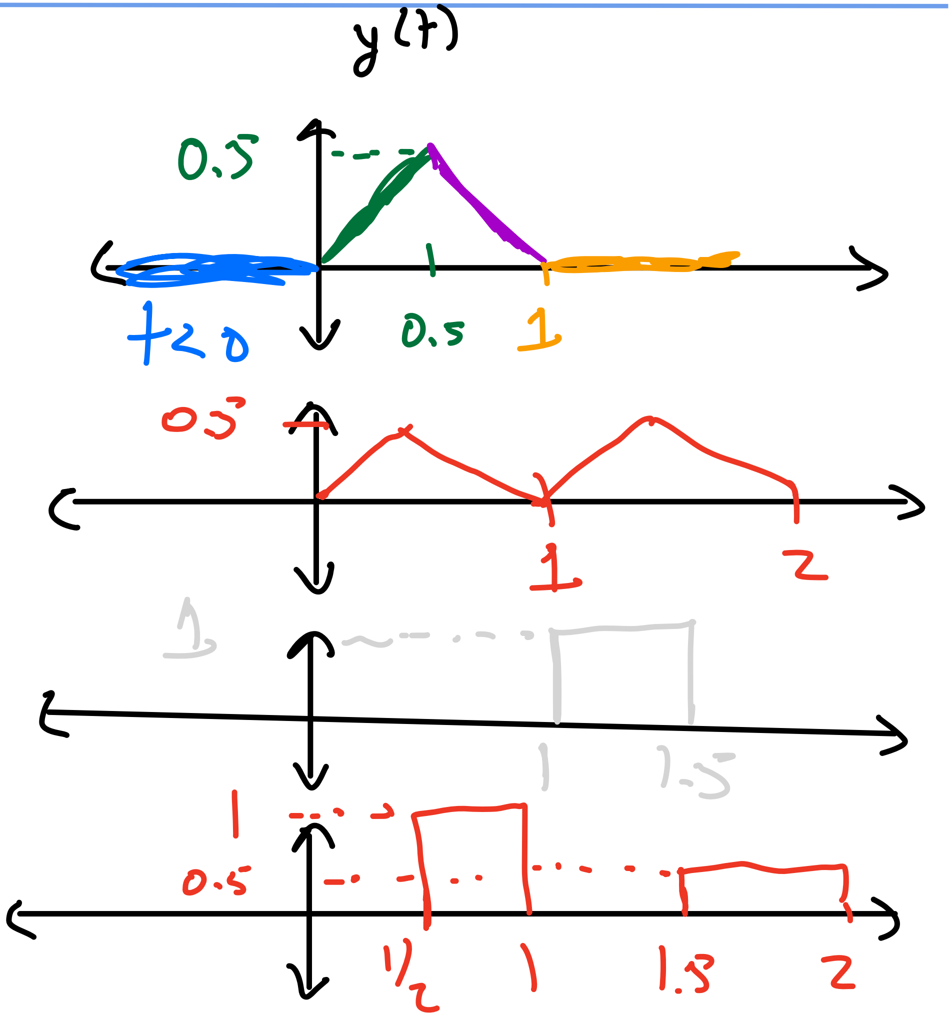
How to Compute Convolution: flip and drag

$h(t-\tau) = h(0-\tau)$

$h(0.5-\tau)$

Examples: Try these:



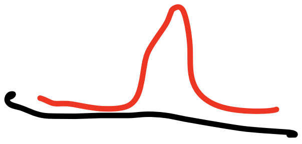


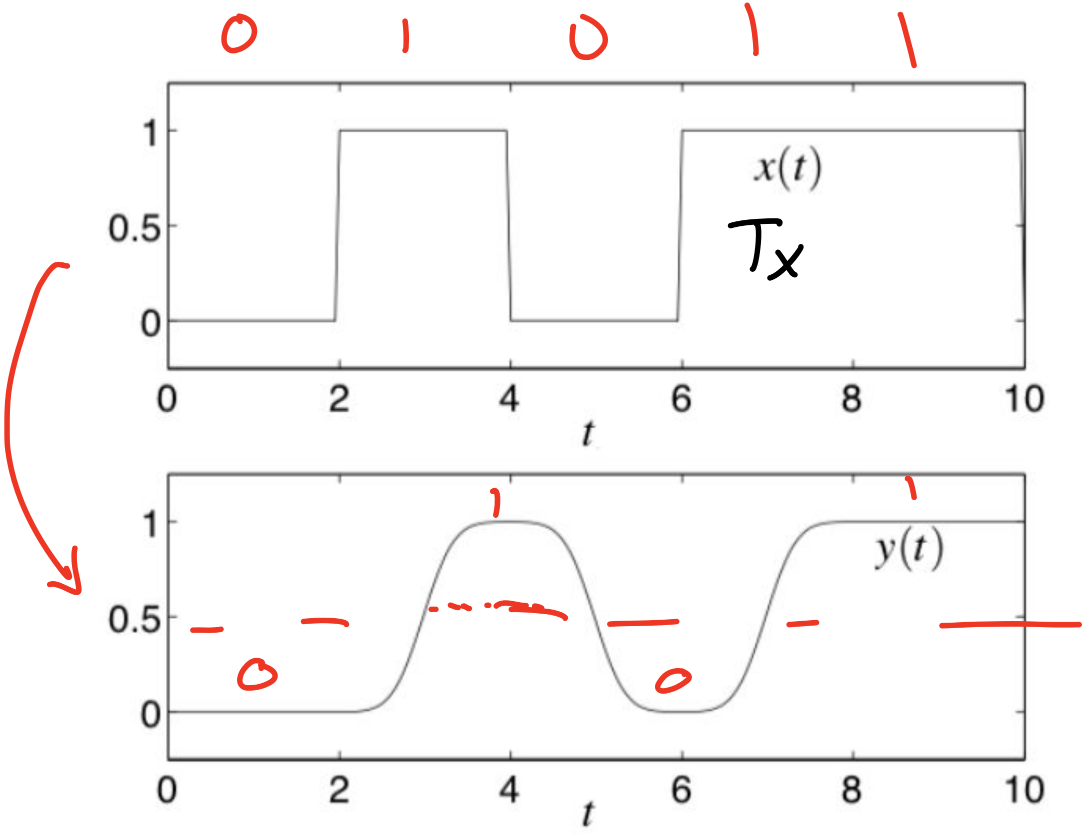
Example: Noisy Communication

LTI

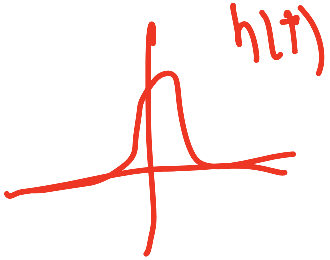
$x \rightarrow \boxed{h(t)} \rightarrow y$

Bw: 0.5 bits/second

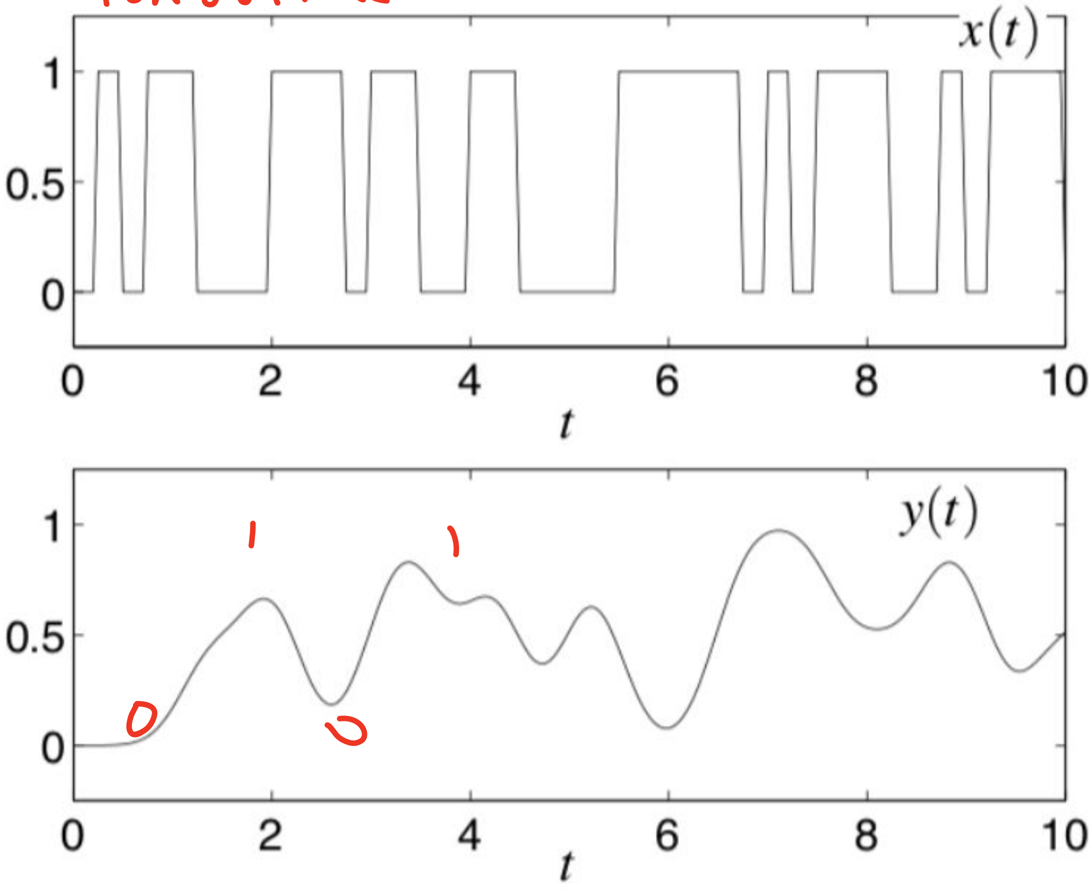




Example Noisy Communication



0100 0011 ...



4 bits/second

Convolution

Causal Convolution

Convolution for a causal system

In a causal system, $h(t) = 0$ for $t < 0$. (Why? Hint: what happens if $h(t) \neq 0$ for some $t < 0$?)

This means that $h(t - \tau) = 0$ if $\tau > t$. Hence, there is no need to integrate if τ exceeds t, since $h(t - \tau) = 0$. We can use this to simplify the convolution integral.

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau
\end{aligned}
$$

This equation tells us that only past and present values of $x(\tau)$ contribute to $y(t)$.

Properties of Convolution

Commutativity $\boxed{(x * h)(t) = (h * x)(t)}$

Associativity $\boxed{(f * (g * h))(t) = ((f * g) * h)(t)}$

Distributivity $\boxed{f * (g + h) = f * g + f * h}$

Linearity $h * (\alpha x_1 + \beta x_2) = \alpha(h * x_1) + \beta(h * x_2)$

Time-invariance

Commutativity

$x \to \boxed{H} \to y$

cyu: Given $y = x * h$ ... Show that $x * h = h * x$

LHS

RHS

$$x * h = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

$$\int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau$$

$$\int_a^b f(t)\, dt = -\int_b^a f(t)\, dt$$

Set $\tau' = t - \tau \Rightarrow d\tau' = -d\tau$

When $\tau = -\infty \Rightarrow \tau' = +\infty$

When $\tau = \infty \Rightarrow \tau' = -\infty$

$$-\int_{\infty}^{-\infty} x(t-\tau') h(\tau')\, d\tau'$$

$$\int_{-\infty}^{\infty} h(\tau') x(t-\tau')\, d\tau' = h(t) * x(t)$$

BIBO Stability

System Stability

Stability: If $|x(t)| \leq M_x < \infty$ and $y = H(x)$, then system is stable if $|y(t)| \leq M_y < \infty \ \forall t$.

CYU: **Show that if (ht) is abs. integrable then the system is BIBO stable.**

Ans. $|y(t)| = \left| \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right|$

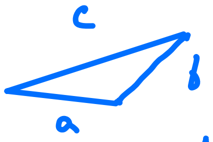
Commutativity of Conv.

$= \left| \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \right|$

$\leq \int_{-\infty}^{\infty} |h(\tau)| |x(t-\tau)| d\tau$

$\leq M_x \underbrace{\int_{-\infty}^{\infty} |h(\tau)| d\tau}_{\text{Bounding } \#} < M_y$

Triangle Ineq.



$|c| \leq |a| + |b|$

$|a+b| \leq |a| + |b|$

By the way, you can also show if system is BIBO, then h(t) is abs. integrable. You can do this by plugging in x(t) is sign function as test input.

Associativity

Set $\tau_2 = \tau_3 - \tau_1$

$\tau_3 = \tau_2 + \tau_1$

$$f * (g * h) = (f * g) * h$$

$$// f * (g * h) = \int_{-\infty}^{\infty} f(\tau_1)\left[g * h(t - \tau_2)\right] d\tau_2$$

$$\vdots$$

$$\int_{-\infty}^{\infty} (f * g)(\tau_3)\, h(t - \tau_3)\, d\tau_3$$

$$= (f * g) * h$$

Associativity and Commutativity

$$f * g * h = f * h * g$$

$$= g * f * h$$

$$\vdots$$

Distributivity

Convolution is distributive, meaning that:

$$\boxed{f * (g + h) = f * g + f * h}$$

To prove this, we write out the definition of convolution:

$$\begin{aligned}
(f * (g + h))(t) &= \int_{-\infty}^{\infty} f(\tau)\,[g(t-\tau) + h(t-\tau)]\,d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau + \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \\
&= (f * g)(t) + (f * h)(t)
\end{aligned}$$

CYU: Identity Element Proof

Here, we have something that looks like an "algebra of signals," with addition like in ordinary algebra, and multiplication is replaced by convolution. In standard algebra, the multiplicative identity is 1. In signals, the convolution identity is the Dirac delta function, $\delta(t)$.

In particular, note that:

$$\boxed{x(t) * \delta(t) = x(t)}$$

Hint: Use commutativity + sifting

Commutativity

$$x(t) * \delta(t) = \delta(t) * x(t)$$
$$= \int \delta(\tau)\, x(t-\tau)\, d\tau$$
$$= x(t)$$

Delay via Convolution

Convolution with the impulse can also be used to delay signals, i.e.,

$$\boxed{x(t) * \delta(t - t_d) = x(t - t_d)}$$

To prove this, note that:

$$x(t) * \delta(t - t_d) \quad = \quad \int_{-\infty}^{\infty} x(\tau)\delta(t - t_d - \tau)d\tau$$

i.e., $x(\tau)$ is being multiplied by an impulse that occurs at $\tau = t - t_d$. From what we know about convolution, this extracts out the value of $x(\tau)$ at $t - t_d$. So,

$$\begin{aligned} x(t) * \delta(t - t_d) &= \int_{-\infty}^{\infty} x(\tau)\delta(t - t_d - \tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t - t_d)\delta(t - t_d - \tau)d\tau \\ &= x(t - t_d)\int_{-\infty}^{\infty} \delta(t - t_d - \tau)d\tau \\ &= x(t - t_d) \end{aligned}$$

Integration with Convolution

Convolution can be used to implement integration. In particular, to integrate a signal x from $-\infty$ to t, we integrate it with a unit step.

$$\begin{aligned} x(t) * u(t) &= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau \\ &= \int_{-\infty}^{t} x(\tau)d\tau \end{aligned}$$

where we used the fact that $u(t-\tau)$ is zero for when $\tau > t$.

Properties of Convolution

Given these properties of convolution, there are now a few properties we can derive regarding convolution.

- **Linearity**: Convolution is **linear**, since for all signals x_1, x_2 and all $\alpha, \beta \in \mathbb{R}$,
$$h * (\alpha x_1 + \beta x_2) = \alpha(h * x_1) + \beta(h * x_2)$$
- **Time-invariance**: if $y(t) = x(t) * h(t)$, then if we delay the input by T, i.e., the new input is $x(t - T)$, then the output is $y(t - T)$. How would you prove this?

Additional Properties of Convolution

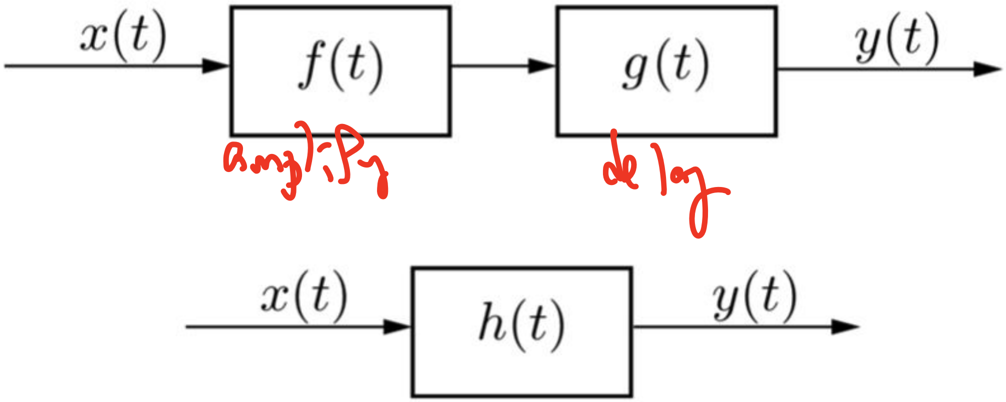
- **Cascade (composition)**: Due to the associativity of convolution, the cascade connection of two convolution systems,

$$y = (x * f) * g$$

is equivalent to a single systsem

$$y = x * h$$

where $h = f * g$. That is, the following two block diagrams are equivalent:



Additional Properties of Convolution

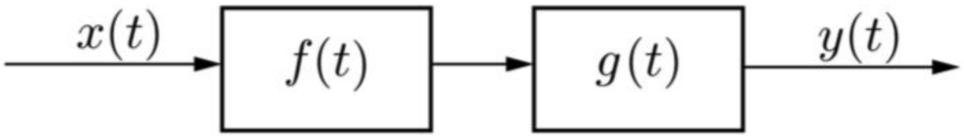
- **Swapping (composition II)**: If

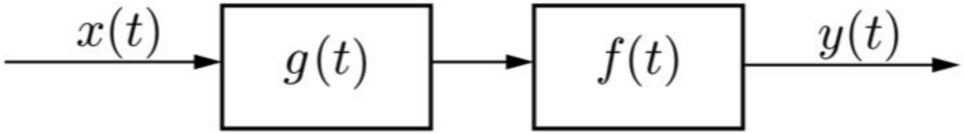
$$y = (x * f) * g$$

then, due to the commutivity of convolution, this is equivalent to

$$y = (x * g) * f$$

This means that you can swap the order of convolutions, as illustrated in the block diagram below:





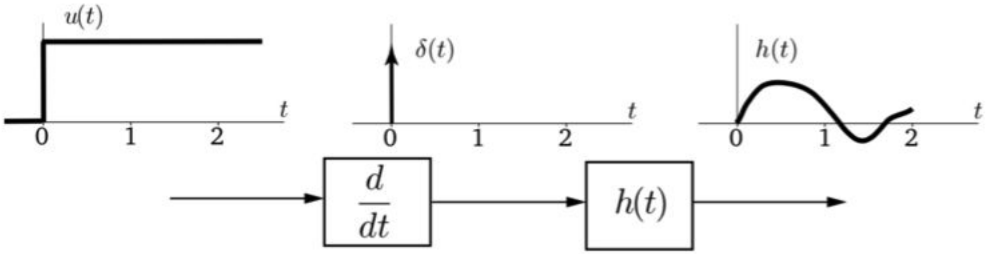
Many operations can be written as convolutions (integration, delays, differentiation, etc.) and these operations all commute.

Additional Properties of Convolution

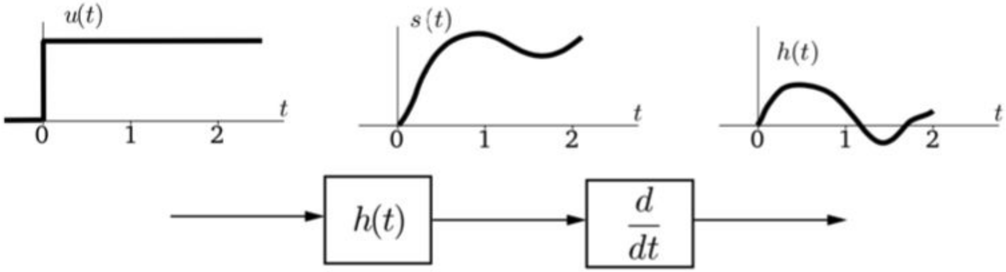
Due to commutivity, we can now find the impulse response by differentiating the step response, i.e.,

$$h(t) = \frac{ds(t)}{dt}$$

This is illustrated below.



is equivalent to



CYU: Check if a System is Linear

$$y(t) = \frac{d}{dt}\left(\frac{1}{2}x(t)^2\right)$$