EE102

Lecture 15

EE102 Announcements

- **Final Exam Details:**
  - **Date: Thursday, December 11th, 6:30-9:30**
  - **Location: Young Hall CS24**
  - **180 minutes and 1 page cheat sheet (double-sided)**
- **Homework 6 has been extended to Tuesday, December 2nd!**
- **Homework 7 is due Friday, December 5th:**
  - **Homework is on Lectures 15, 16, and 17**
- **Practice Final Exam is out 12/2**
- **Final Math Cheat Sheet posted on BruinLearn**
  - **Includes math identities, Fourier, Laplace Transforms/Properties, etc. that you may find helpful on the final exam.**

ABET Learning Outcomes

✔ Understand the concept of a signal and a system, plot continuous-time signals, evaluate the periodicity of a signal.

✔ Identify properties of continuous-time systems such as linearity, time-invariance, and causality.

✔ Calculate with the Dirac delta function.

✔ Compute convolution of continuous-time functions.

✔ Understand the concept of the impulse response function of a linear system, and its use to describe the input/output relationship.

Compute the Laplace transform of a continuous function, identify its domain of convergence, and be familiar with its basic properties, including the initial and final value theorems.

Find the inverse Laplace transform by partial fractions.

Use the Laplace transform to solve constant-coefficient differential equations with initial conditions

Use the Laplace transform to evaluate the transfer function of linear time-invariant systems.

✔ Understand Parseval's relation in Fourier series, and its interpretation in terms of decomposing the signal's energy between its harmonics

✔ Evaluate the response of a linear time-invariant system to periodic inputs.

✔ **Evaluate the Fourier transform of a continuous function, and be familiar with its basic properties.** Relate it to the Laplace transform.

✔ **Evaluate and plot the frequency responses (magnitude and phase) of linear time-invariant systems, and apply it to filtering of input signals.**

✔ Understand conditions under which a band-limited function can be recovered from its samples

Impulse Train

The impulse train may have been your first thought when thinking of how to sample a signal every T.

Indeed, this signal has very important qualities. Let's start off with a simple question: intuitively, what is the Fourier transform of a impulse train?

Guess CYU: What’s the Fourier Transform of Impulse train

No need to calculate - please think about square wave example and hazard a guess.

Impulse Train F.T.

Let's think through this using our square wave example.

- We know that the Fourier transform of the square wave is a sinc multiplied by $\delta_\pi(\omega)$.
- From the convolution theorem, this means that the inverse Fourier transform (i.e., the square wave) is the inverse Fourier transform of a sinc (i.e., a rect) convolved with the inverse Fourier transform of a impulse train.
- We know that a square wave is simply a rect repeated over and over again, i.e., convolved with a impulse train.
- So intuitively, by duality, the Fourier transform of a impulse train should be a impulse train.

Note, we will sometimes use the term 'delta train' to describe an impulse train.

F.T. of Impulse Train

Let's check our intuition and compute the Fourier transform of an impulse train. To do so, we'll use our trick of finding the Fourier series of the (periodic) impulse train, and then multiplying by $2\pi\delta(\cdot)$.

F.T. of Impulse Train

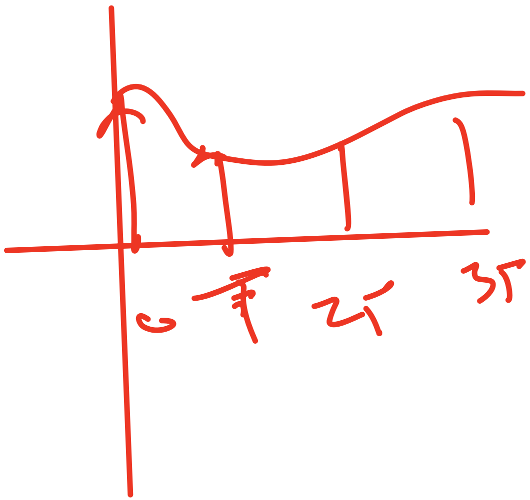
F.T. of Impulse Train

Sampling with an Impulse Train

As we saw earlier, one of the things we will use the impulse train for is to sample signals.

Given a signal $f(t)$,

$$f(t)\delta_T(t) \quad = \quad f(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$



$$= \sum_k f(t)\,\delta(t-kT)$$

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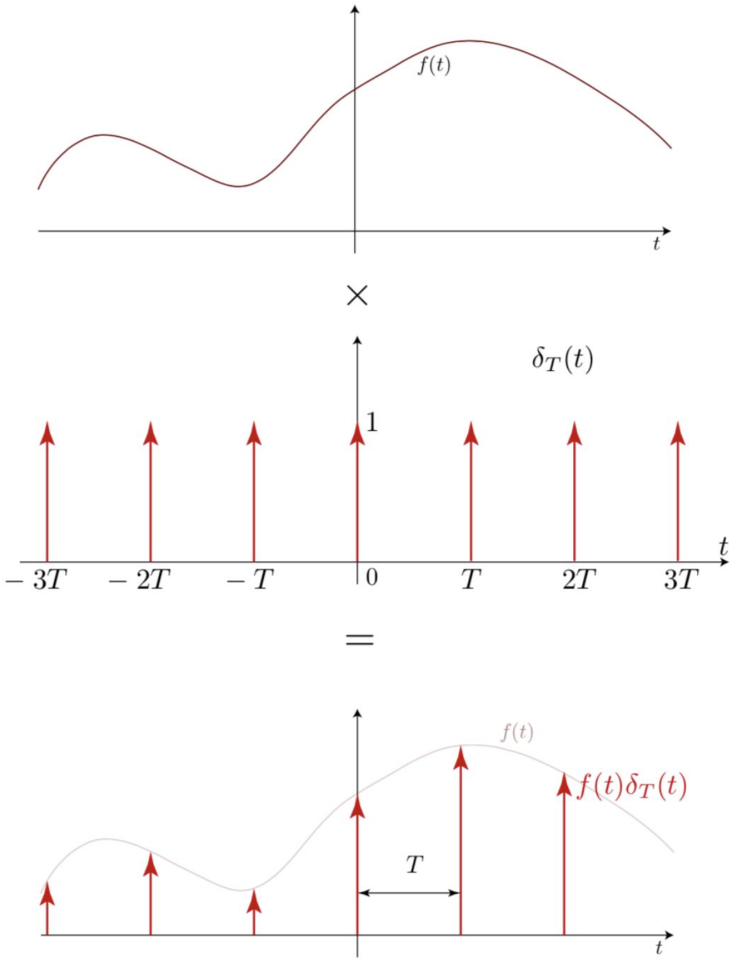
Sampling with an Impulse Train



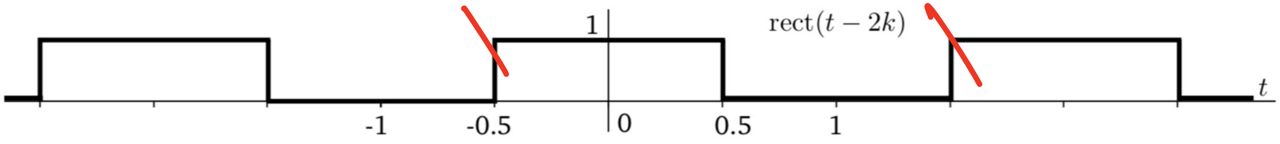
$$= \sum_k f(kT)\,\delta(t - kT)$$

Sampling and Periodicity

Square wave, part 2

Let's revisit our square wave example, where

$$f(t) = \sum_{k=-\infty}^{\infty} \mathrm{rect}(t-2k)$$



Another way to represent this square wave is as follows:

$$F(j\omega) = \underbrace{\mathcal{F}[\mathrm{rect}(t)]}_{\text{Catalog}} \cdot \underbrace{\mathcal{F}[\delta_2(t)]}_{\text{Catalog}}$$

$$f(t) = \mathrm{rect}(t) * \delta_2(t)$$

Hence, we can calculate its Fourier transform by using the convolution theorem. Recall that, for $\omega_0 = 2\pi/T$,

$$\mathrm{rect}(t) \iff \mathrm{sinc}(\omega/2\pi)$$

and

$$\delta_T(t) \iff \omega_0 \delta_{\omega_0}(\omega)$$

Sampling and Periodicity

Square wave, part 2 (cont.)

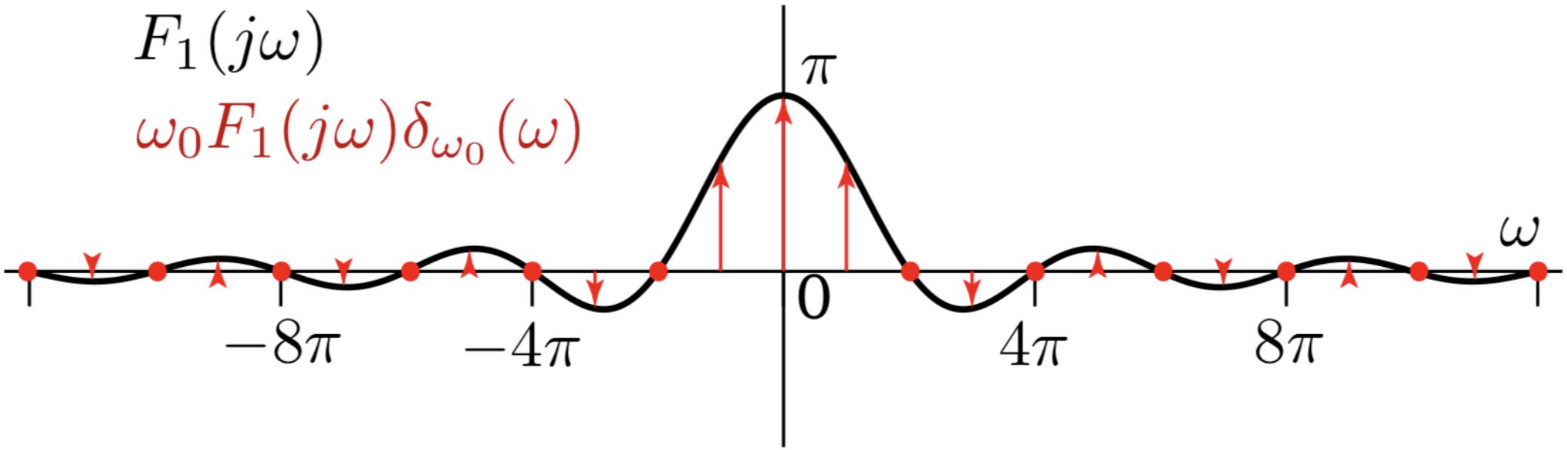
Note that when $T = 2$, then $\omega_0 = \pi$. Then, we have that,

$$\begin{aligned}
\mathcal{F}\left[f(t)\right] &= \mathcal{F}\left[\text{rect}(t) * \delta_2(t)\right] \\
&= \mathcal{F}\left[\text{rect}(t)\right] \mathcal{F}\left[\delta_2(t)\right] \\
&= \text{sinc}(\omega/2\pi)\pi\delta_\pi(\omega)
\end{aligned}$$

This is exactly the same Fourier transform we calculated earlier using the Fourier series of the square wave.

Sampling and Periodicity

Another intuition to remember here is that the Fourier transform of a periodic signal is the Fourier transform of one period of the signal (which we can denote f_1), sampled by an impulse train at multiples of ω_0.



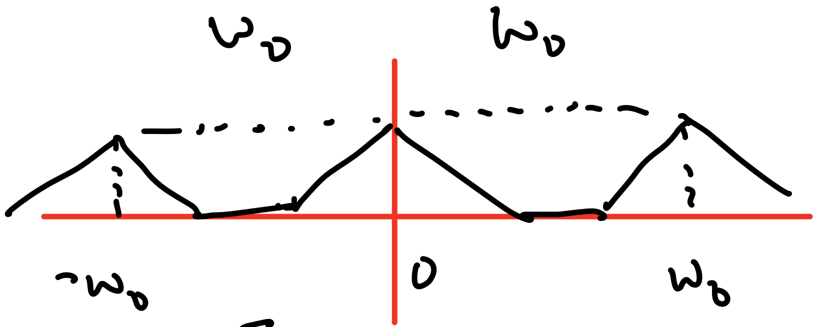
Sampling and Periodicity

Discrete - periodic duality

$\tilde{f}(t) = \mathrm{sinc}^2(t) \cdot \delta_T(t)$

We can determine the Fourier transform of a signal sampeld in the time-domain. Consider

$$\tilde{f}(t) = f(t)\delta_T(t)$$



Its Fourier transform is

$$\tilde{F}(j\omega) = \mathcal{F}\left[f(t)\delta_T(t)\right]$$

$$= \frac{1}{2\pi}\mathcal{F}\left[f(t)\right] \star \mathcal{F}\left[\delta_T(t)\right]$$

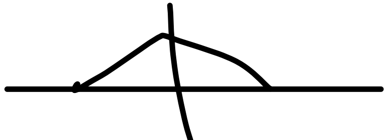
$$= \frac{1}{2\pi}F(j\omega) \star \omega_0 \delta_{\omega_0}(\omega)$$

$$= \frac{1}{T}F(j\omega) \star \delta_{\omega_0}(\omega)$$

Reminder

$f(t) = \mathrm{sinc}^2(t)$

$F(j\omega) = \Delta(\cdot)$



Sampling and Periodicity

Discrete - periodic duality (cont.)

This are merely samples of $F(j\omega)$ repeated every ω_0, since

$$\tilde{F}(j\omega) \quad = \quad \frac{1}{T}F(j\omega) * \delta_{\omega_0}(\omega)$$

$$= \frac{1}{T} F(j\omega) * \sum_{k} \delta(\omega - k\omega_0)$$

$$= \frac{1}{T} \sum_{k} F\big(j(\omega - k\omega_0)\big)$$

Sampling and Periodicity

Discrete - periodic duality (cont.)

This are merely samples of $F(j\omega)$ repeated every ω_0, since

$$
\begin{aligned}
\tilde{F}(j\omega) &= \frac{1}{T}F(j\omega) * \delta_{\omega_0}(\omega) \\
&= \frac{1}{T}F(j\omega) * \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} F(j(\omega - k\omega_0))
\end{aligned}
$$

This leads us to the realization that:

- A signal that is periodic in time is discrete in spectrum. 1st segment of lecture
- A signal that is discrete in time is periodic in spectrum. 2nd segment

There are important consequences from this result when we consider sampling signals in the time domain.

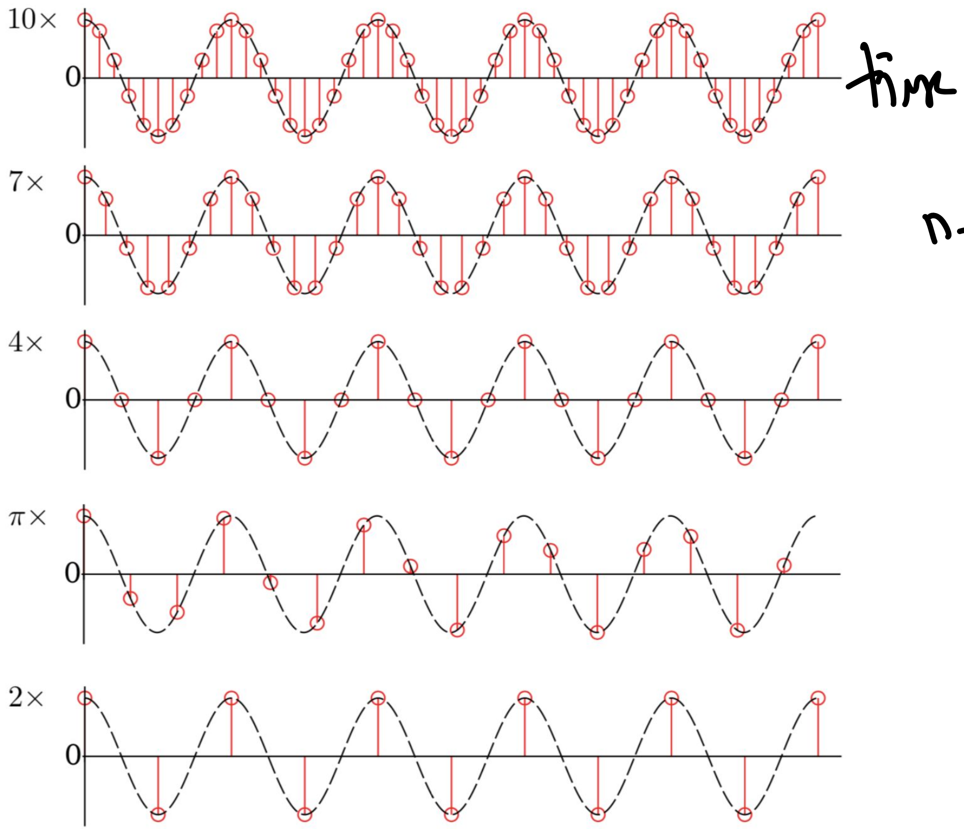
Sampling Theorem Motivation

Consider the following problem. We have a signal $f(t)$, and we need to store it. Our experimental set up is able to sample this signal at an interval T. How do we set T so that we can faithfully store $f(t)$? If T is too large, we sample infrequently and may lose information about $f(t)$. If T is too small, we waste memory and resources to store values we don't need.

The sampling theorem uses the results we've derived to tell us the minimum frequency at which we must sample $f(t)$ to not lose information. It is a very important theorem.

Sampling Theorem

Sampling example: sinusoids (cont.)



$\cos(2\pi t)$ $\omega = 2\pi$

$f = 1\,Hz$

"Bandwidth" $\triangleq$ Max Frequency of the Signal

$B = 1\,Hz$

Sampling Theorem

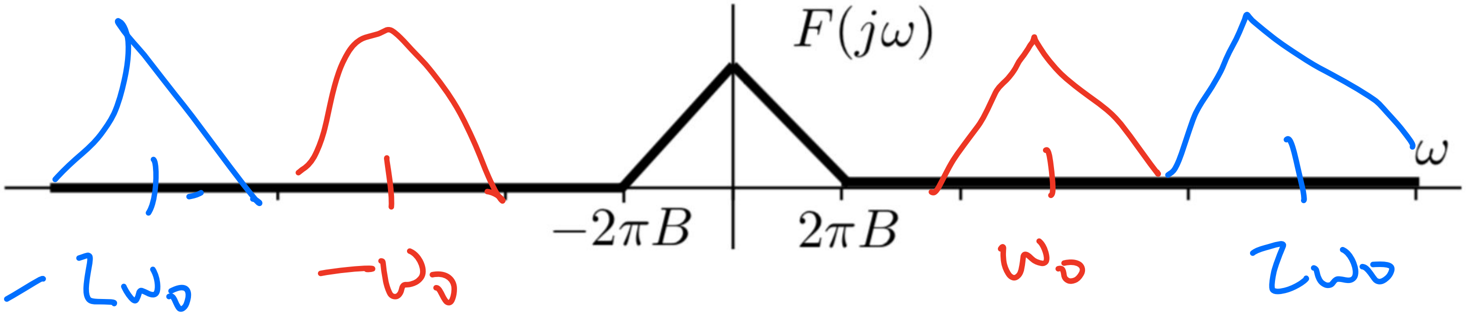
If $\tilde{f}(t) = f(t)\delta_T(t)$, then as shown on the previous slides,

$$\tilde{F}(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(j(\omega - k\omega_0))$$

$\omega_0 = \frac{2\pi}{T}$

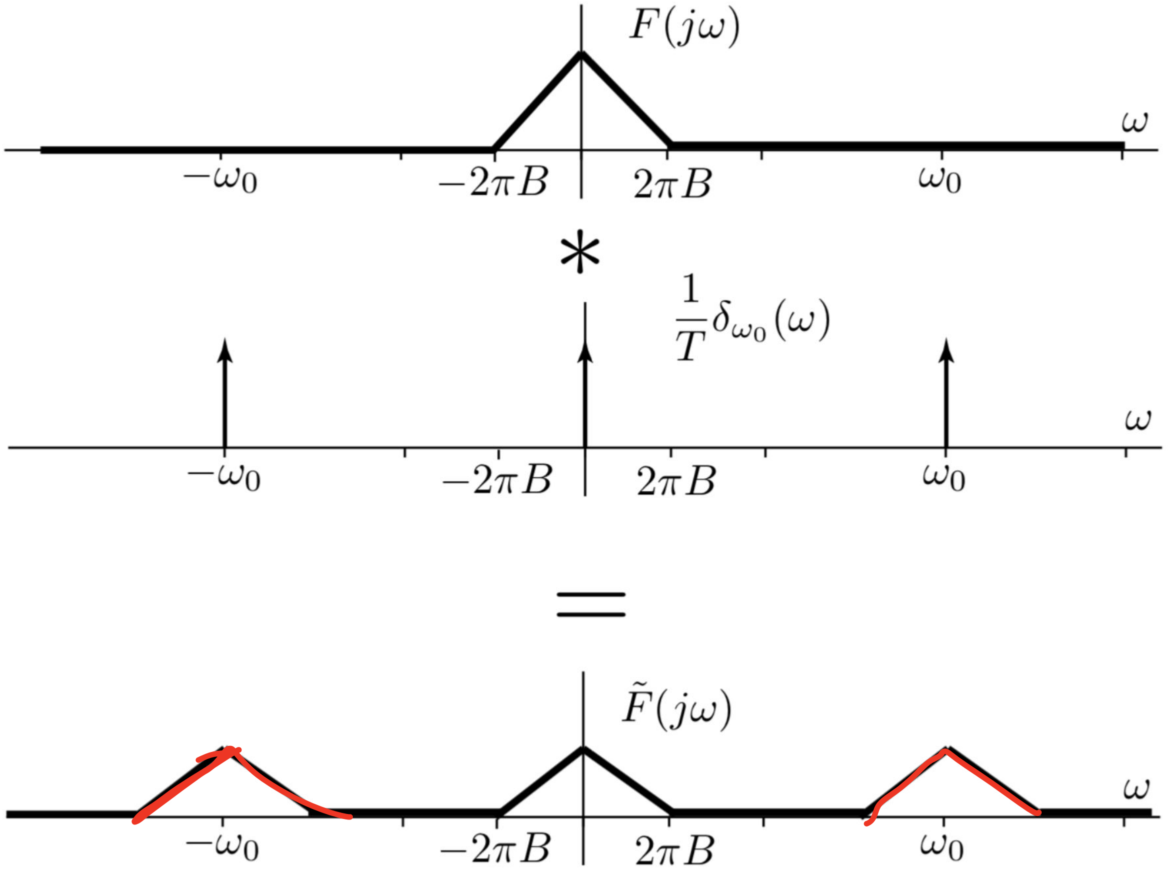
Therefore, the spectrum of $\tilde{f}(t)$ are shifted replicas of the spectrum, $F(j\omega) = \mathcal{F}[f(t)]$ spaced every ω_0 and scaled by $1/T$.

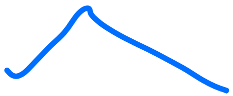
We define the bandwidth of $f(t)$ to be $\pm B$ Hz, e.g.,



Sampling Theorem

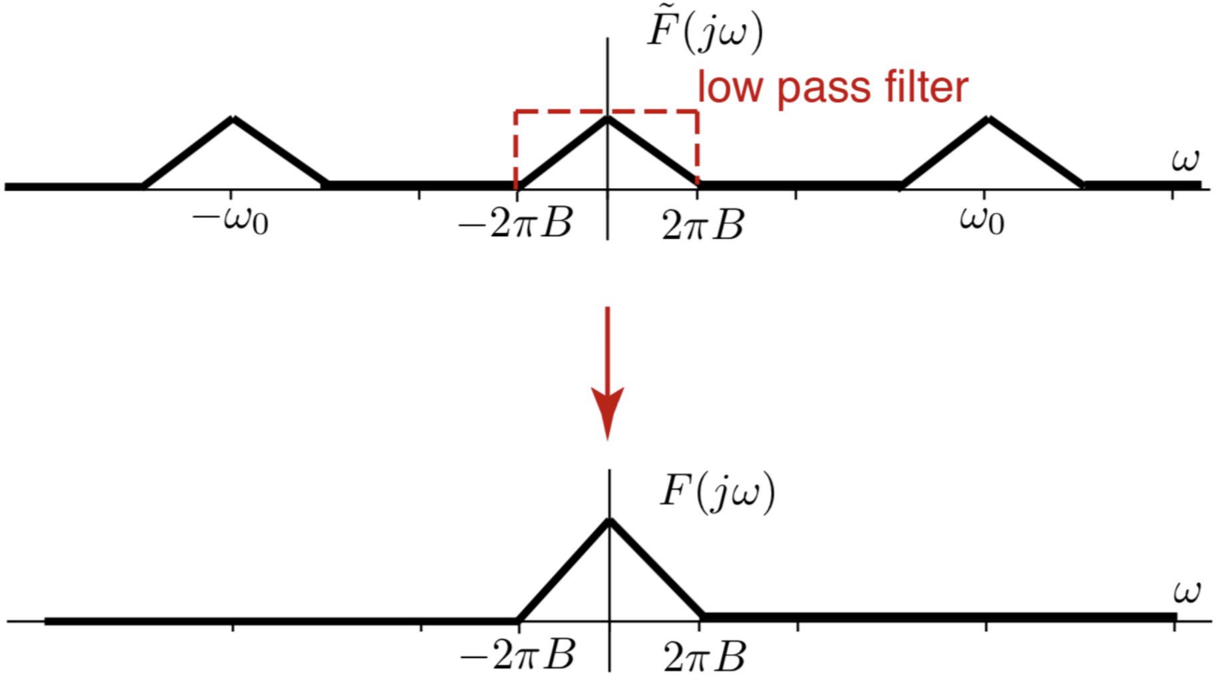
For a particular choice of ω_0, where $\omega_0 \gg 2\pi B$, we see the spectrum of $\tilde{F}(j\omega)$ looks like:





Sampling Theorem

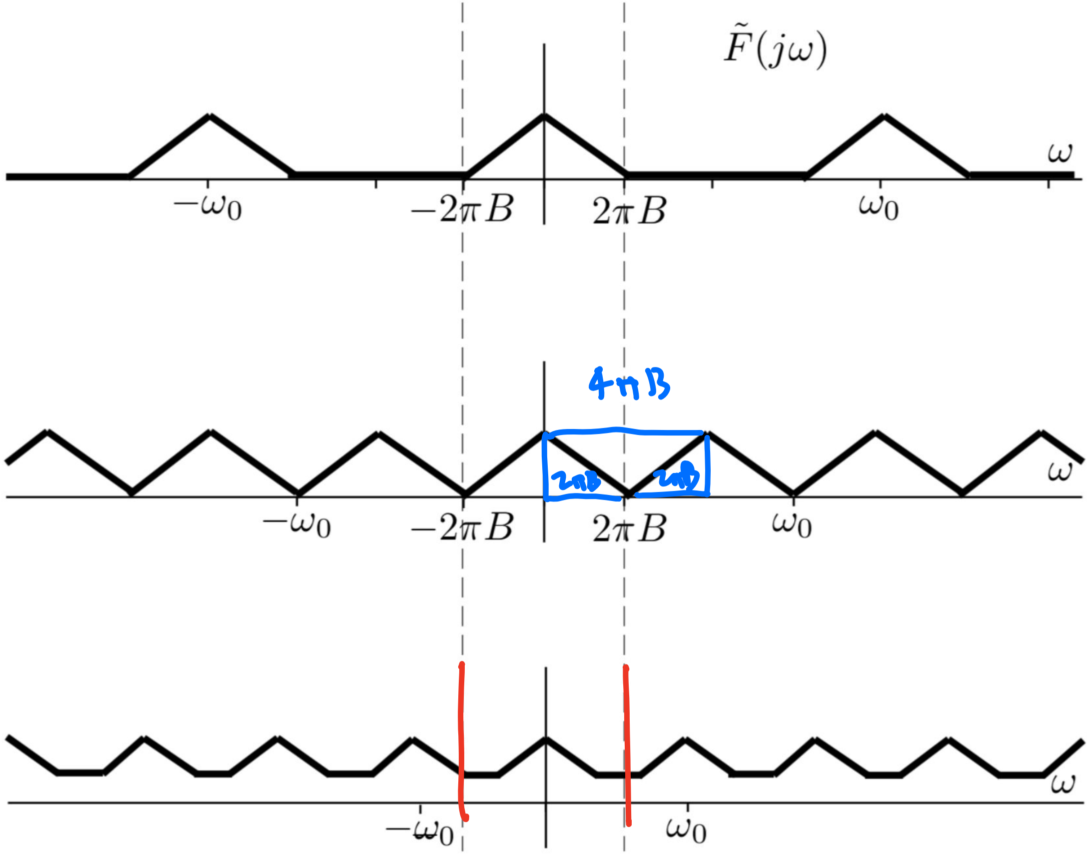
For this choice of ω_0, the original $F(j\omega)$ can be recovered through low pass filtering.



With ideal low pass filtering for the illustrated ω_0, we can *perfectly* recover $f(t)$ after sampling.

Sampling Thm

But now, as we increase the time T between samples, which decreases ω_0, the replicas of $F(j\omega)$ get closer and closer together.



$\omega_0 >> 4\pi B$

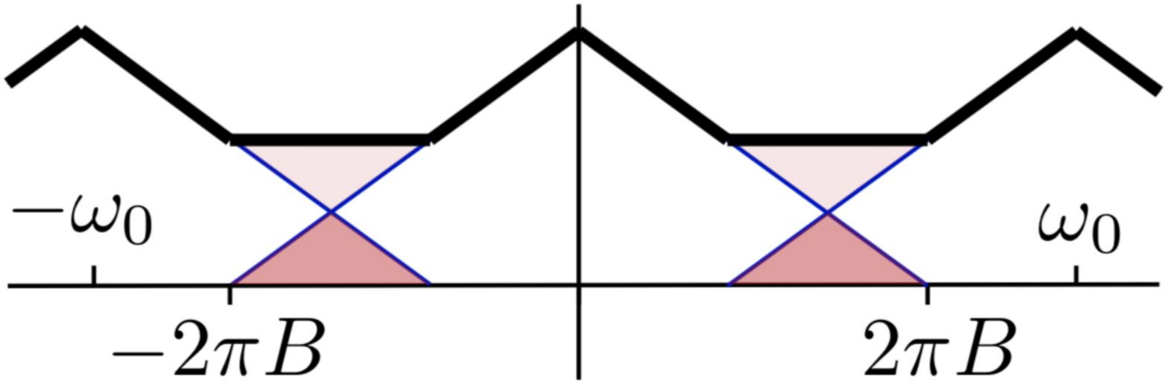
$\omega_0 = 4\pi B$ "critical sampling"

$\omega_0 < 4\pi B$ "undersampling"

Sampling Thm - Aliasing

We see that as ω_0 decreases, the bands start to overlap. When the replicas overlap, even with ideal low pass filtering, we cannot recover the original $F(j\omega)$.

This overlap is called *aliasing* because low frequencies of one spectral replica appear (or alias) as high frequencies in the next spectral replica. The vice versa is true as well; high frequencies of one spectral replica alias as low frequencies in an adjacent spectral replica. The alias'd sections are shown in the darker red.



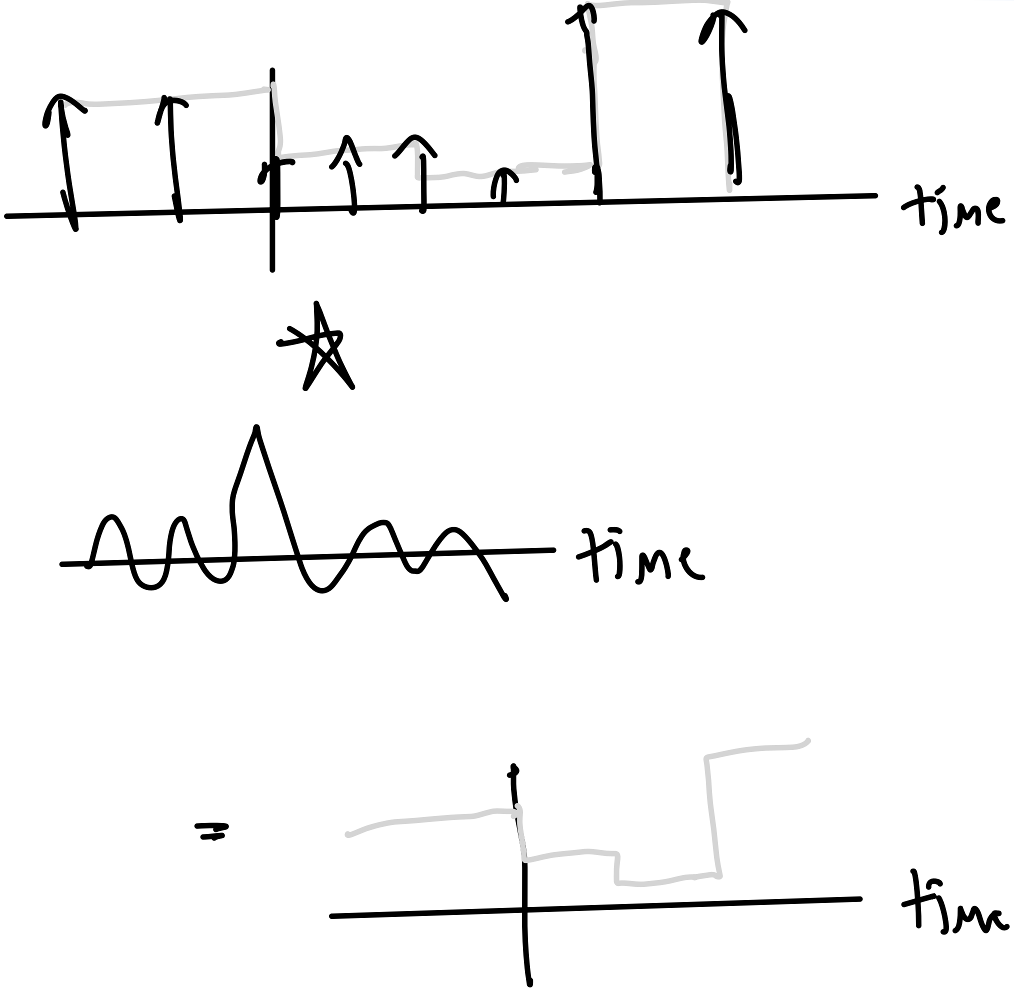
Sampling Thm - Aliasing

To be able to perfectly recover a signal, we need to sample so as to avoid aliasing. No aliasing happens if $2\pi B < \omega_0/2$. We can simplify this as

$$\begin{aligned} 2B &< \omega_0/2\pi \\ &= \frac{2\pi}{T}\frac{1}{2\pi} \\ &= \frac{1}{T} \end{aligned}$$

Therefore, the signal can only be recovered exactly if the signal bandwidth $2B$ is less than or equal to the sampling rate $1/T$. Hence, we need to sample at intervals less than or equal to $T = 2B$. This sampling rate, $2B$ is called the *Nyquist rate* for $f(t)$, and it is the lowest rate that we can sample $f(t)$ so that it can be perfectly recovered. T is called the *Nyquist interval*.

DAC: Digital to Analog Converter Interpolation
Q: Do this w/o Fourier.



① Analog Signal Exists.

② Sample It

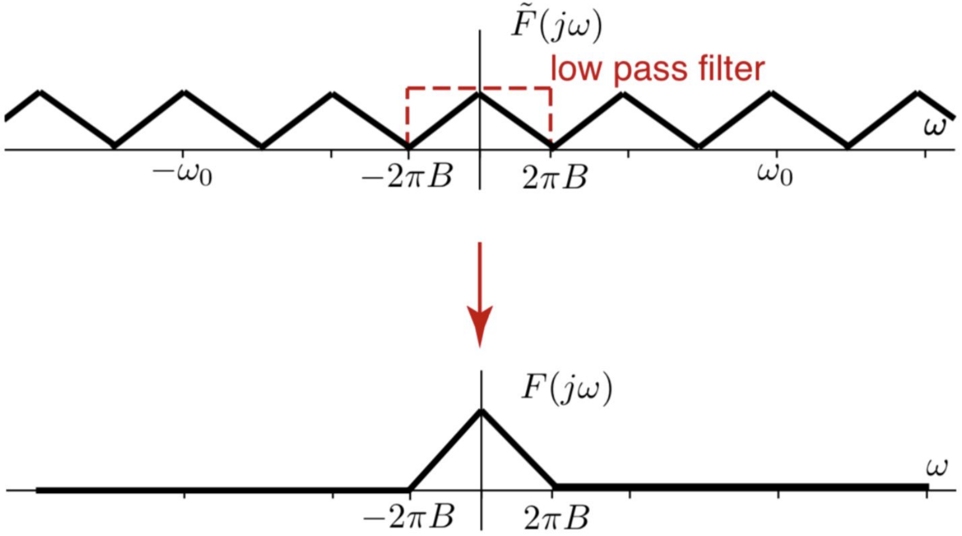
③ Sampling Creates Gaps in Time & Copies in Fourier.

④ Isolate 1 copy by multiplying w/ low pass filter

⑤ Now you take IFT.

Interpolation

With a sampled signal, $\tilde{f}(t)$, as long as we have sampled at a rate $\geq 2B$, we can perfectly recover the original signal through ideal low pass filtering. Let's formalize how this happens, using the particular instantiation that $T = 1/2B$, i.e., we sample at the Nyquist rate.



Our low pass filter has frequency response

$$H(j\omega) = T\mathrm{rect}\left(\frac{\omega}{4\pi B}\right)$$

Sinc

Interpolation

The inverse Fourier transform of $H(j\omega)$ is

$$h(t) = 2BT\,\mathrm{sinc}(2Bt)$$

Since $T = 1/2B$, we can simplify this expression to

$$h(t) = \mathrm{sinc}(2Bt)$$

Therefore, to reconstruct $f(t)$ from $\tilde{f}(t)$, we calculate:

$$\begin{aligned}
\tilde{f}(t) * h(t) &= \left(\sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT)\right) * h(t) \\
&= \sum_{k=-\infty}^{\infty} f(kT)h(t-kT) \\
&= \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2B(t-kT)) \\
&= \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt-k))
\end{aligned}$$

Interpolation

This reconstruction,

$$\tilde{f}(t) * h(t) \quad = \quad \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt - k))$$

is called the Whittaker-Shannon interpolation formula.

Intuition?

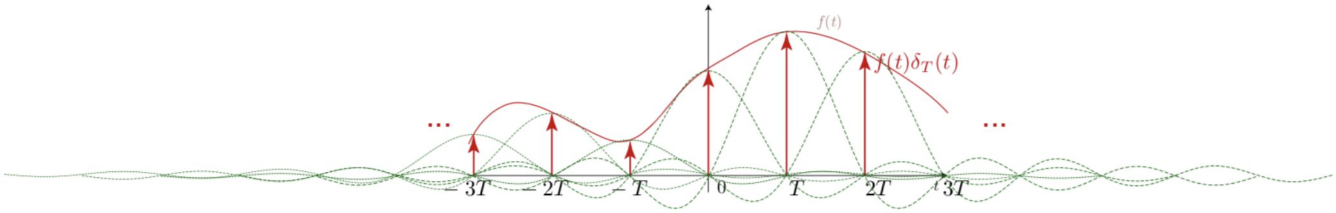
Interpolation

Recovering the original signal through interpolation (cont.)

This reconstruction,

$$\tilde{f}(t) * h(t) \quad = \quad \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt - k))$$

is called the Whittaker-Shannon interpolation formula. Intuitively, it does the following:



The sum of the green sinc functions will equal the red function, $f(t)$.

To not mince words, this result, which combines many of the things we've learned thus far, is remarkable. Through this reconstruction, we are able to *perfectly* recover an original signal from samples.

Further Reading
"Compressed Sensing".

CYU: Bandwidth of Signals

Assume $x(t)$ a real bandlimited signal where $X(j\omega)$ is non-zero for $|\omega| \leq 2\pi B$ rad/s. If F_s Hz is the Nyquist rate of $x(t)$, determine the Nyquist rate of the following signal, $y(t)$, in terms of B:

$$y(t) = x(t) + \frac{dx(t)}{dt} + x(2t)$$

Ans: $Y(\omega) = X(\omega) + j\omega X(j\omega) + \underbrace{\frac{1}{2} X\left(j \frac{1}{2}\omega\right)}_{\Downarrow}$

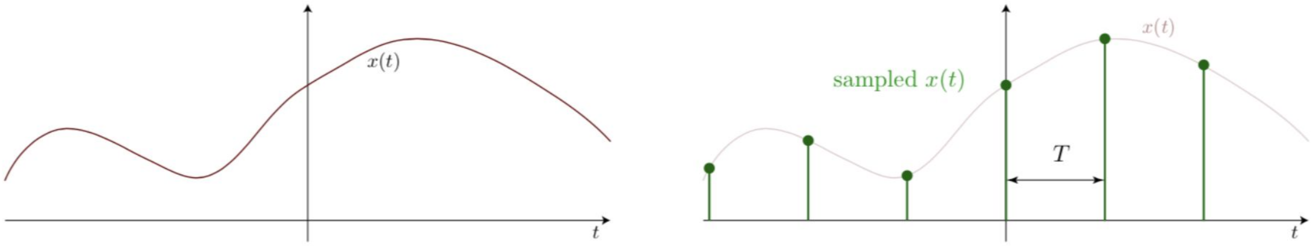
Non-zero $|\omega| \leq 4\pi B$

$\therefore$ Nyquist Rate: $4B$

Review: Sampling

Motivation

In reality, we could never store a continuous time signal. Instead, as we see in `MATLAB`, we store the signal's value at various times. This is called sampling, as illustrated below.



A key variable of interest is the sampling frequency, i.e., the time in between our samples, denoted T in the above diagram.

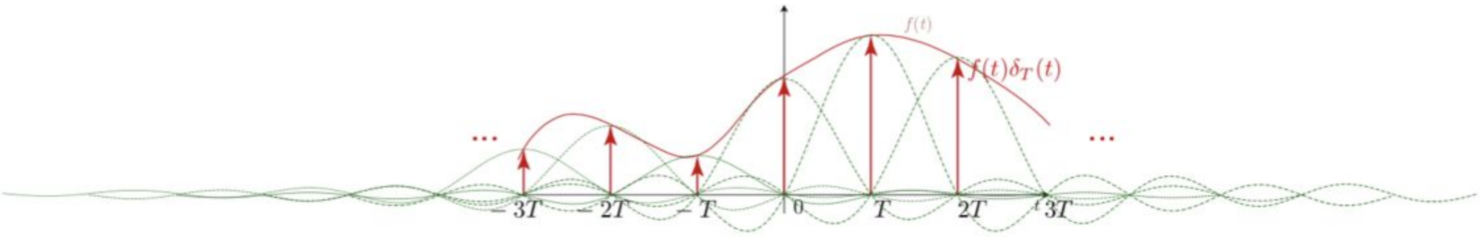
This is related to discrete signals, i.e., $x[n] = x(nT)$.

Review: Recovering Signal through Interpolation

This reconstruction,

$$\tilde{f}(t) * h(t) \quad = \quad \sum_{k=-\infty}^{\infty} f(kT)\,\mathrm{sinc}(2Bt - k))$$

is called the Whittaker-Shannon interpolation formula. Intuitively, it does the following:

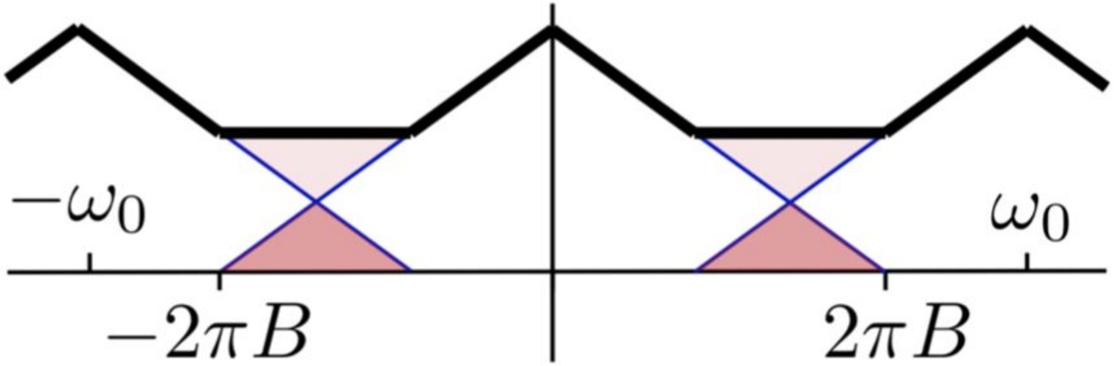


The sum of the green sinc functions will equal the red function, $f(t)$.

Review: Aliasing in Frequency

We see that as ω_0 decreases, the bands start to overlap. When the replicas overlap, even with ideal low pass filtering, we cannot recover the original $F(j\omega)$.

This overlap is called *aliasing* because low frequencies of one spectral replica appear (or alias) as high frequencies in the next spectral replica. The vice versa is true as well; high frequencies of one spectral replica alias as low frequencies in an adjacent spectral replica. The alias'd sections are shown in the darker red.

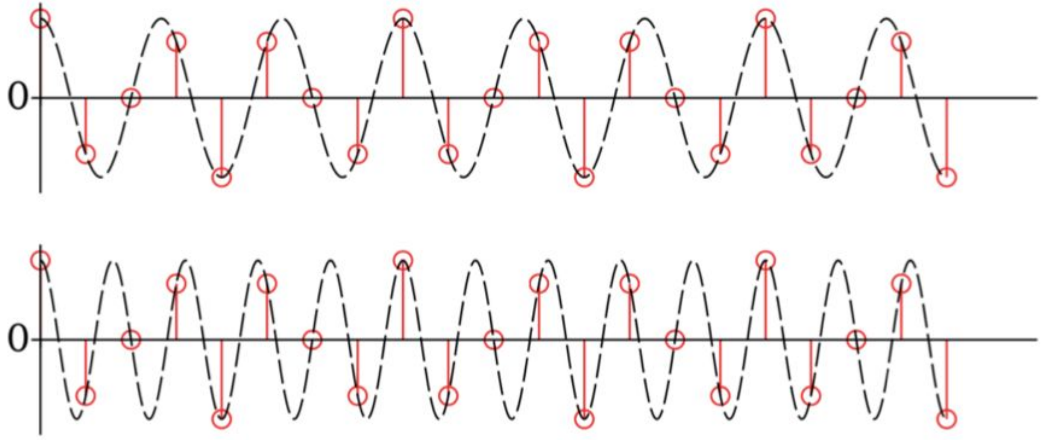


Aliasing (Look at it in Time)

We can intuitively understand aliasing in time domain for sinusoidal signals

Here's an example of aliasing. Below are two sinusoids. The upper one is at a frequency of $f = 0.75$ Hz and the one below is at $f = 1.25$ Hz.

Sampling both signals at $f_s = 2$ Hz,

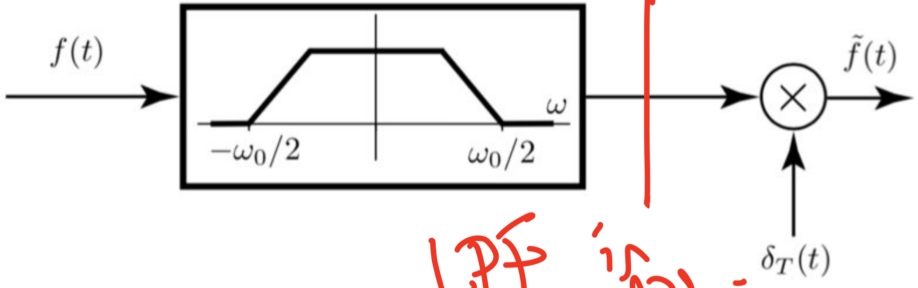


As you can see, the samples are the same for both sinusoids..

How to Ameliorate Aliasing

If you sample below the Nyquist rate, there will be aliasing. Once aliasing happens, there is no way to eliminate aliasing without having additional information about the signal.

One way we can ameliorate aliasing is to first low pass filter the signal, then sample:



LPF in analog -

- In reality, our low pass filters are not perfect, and so the bandwidth will be larger than $\omega_0/2$, however, we'll attenuate frequencies outside of range.
- Low pass filtering will distort the signal.
- However, the point is that when sampling, frequencies beyond $\omega_0/2$ would cause artifacts. Low pass filtering ameliorates this.
- It also supresses noise outside of $\omega_0/2$.