EE102

Lecture 2

EE102 Announcements

- **Syllabus link** is on BruinLearn
- **First Homework was released last Friday** and covers material from this lecture and the previous lecture. **Due 10/10 11:59 pm.**
- **Pace of the class accelerates.** The mathematics for this class will become more advanced lecture by lecture. Tip: watch for patterns. For example, proofs are often about showing LHS = RHS, even for more difficult proofs later in the course.

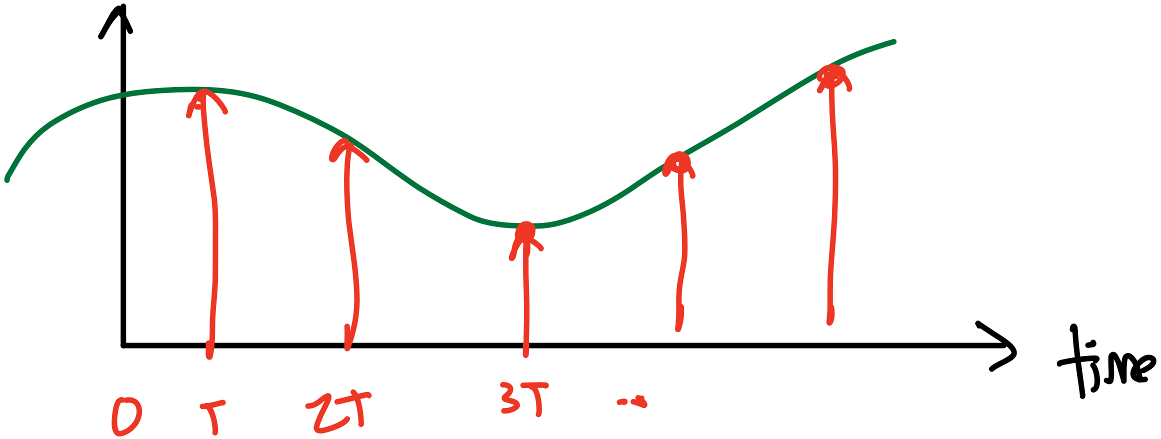
Slide Credits: This lecture adapted from materials shared by Prof. Kao (UCLA), Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA). The instructor thanks them all.

Signal operations and properties

This lecture overviews several mathematical operations and properties that will provide a foundation for the rest of the class. It jumps between various topics as we need to know all of these before moving on.

- Time scaling, reversal and shifting.
- Even and odd signals
- Periodicity
- Review of sinusoids and complex numbers
- Causality
- Energy and power signals
- Euler's formula

Discrete vs Continuous Signals



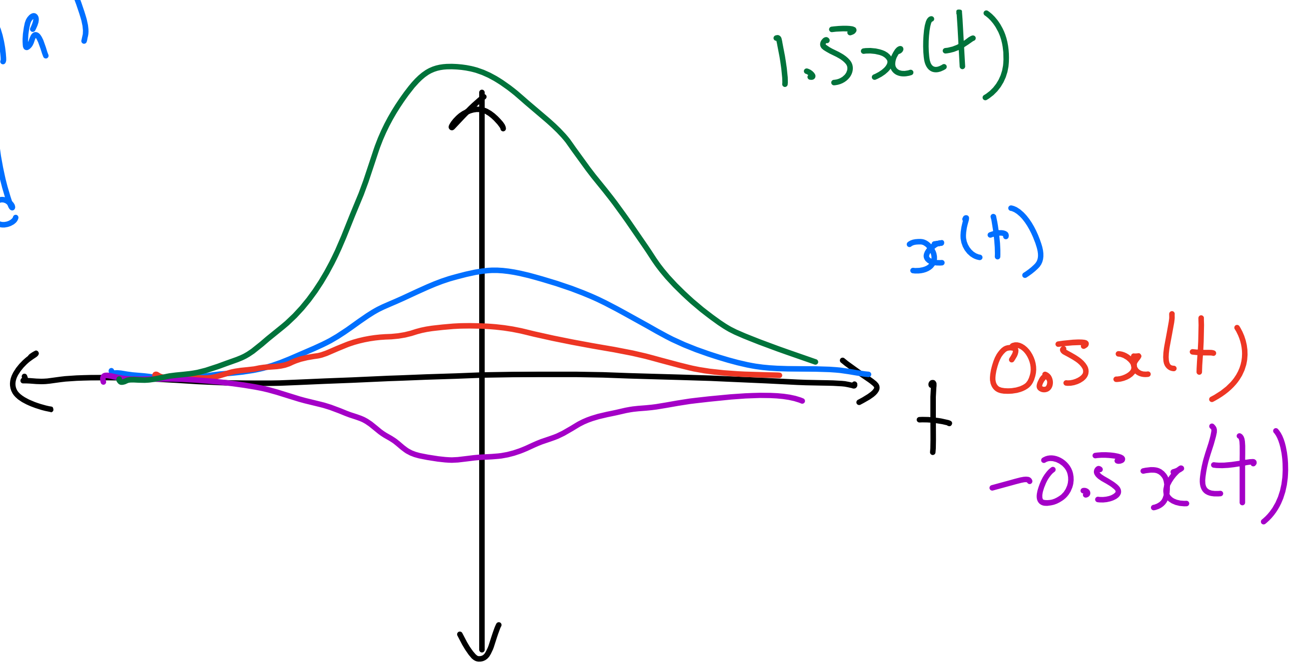
EE102 : Cts signals (time)

EE113 : Discrete Signals.

Amplitude Scaling

$x(t)$ original

$ax(t)$ scaled

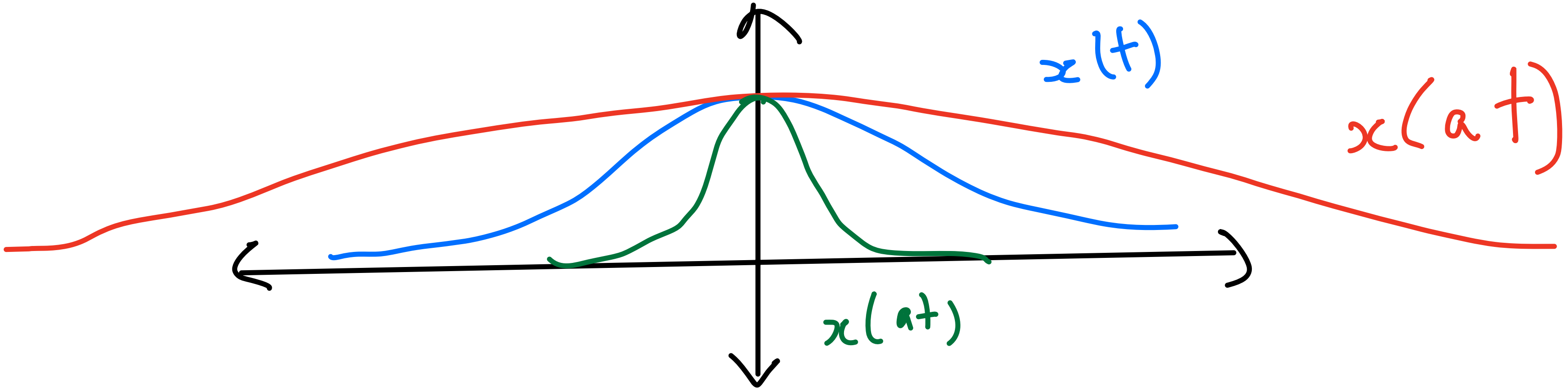


Time Scaling

$0 < a < 1$ Expansion

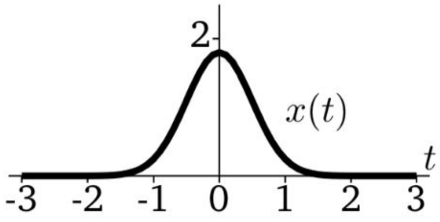
$a > 1$

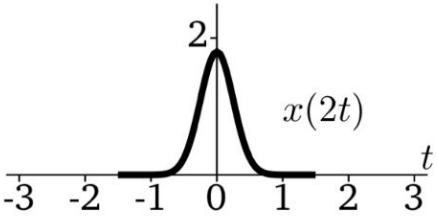
$a = 1$

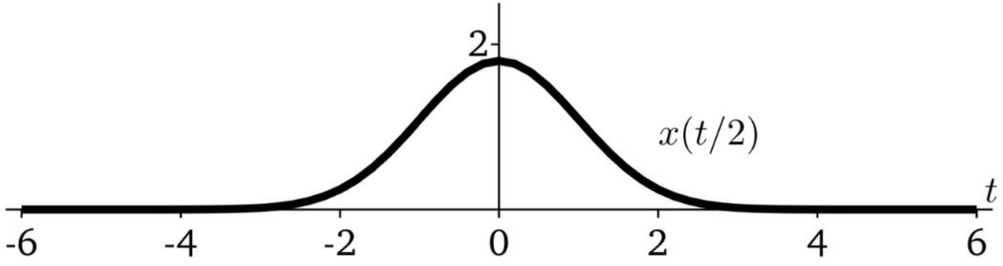


Time Scaling

- If $a > 1$ then the signal is compressed in time.
- If $0 < a < 1$ then the signal is expanded in time.



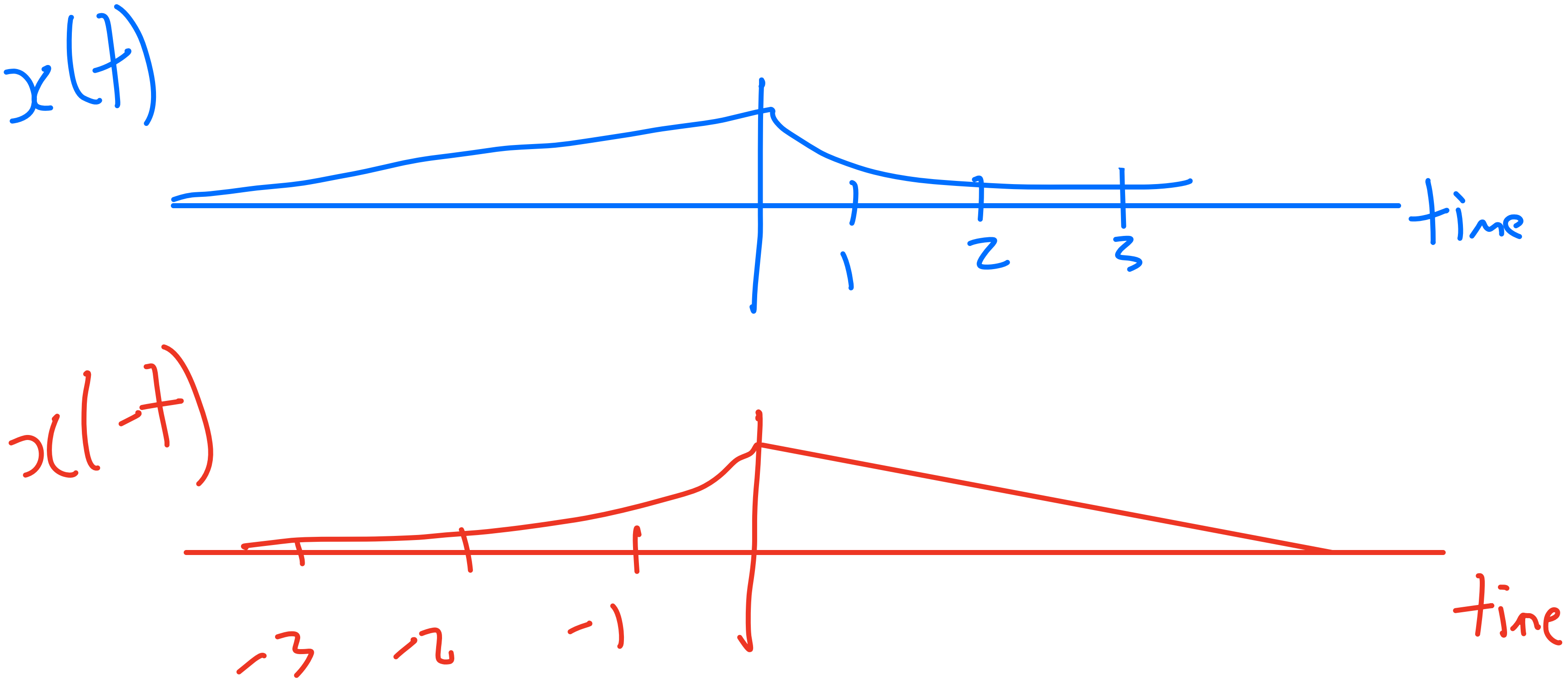




Check your Underst. (CYU)

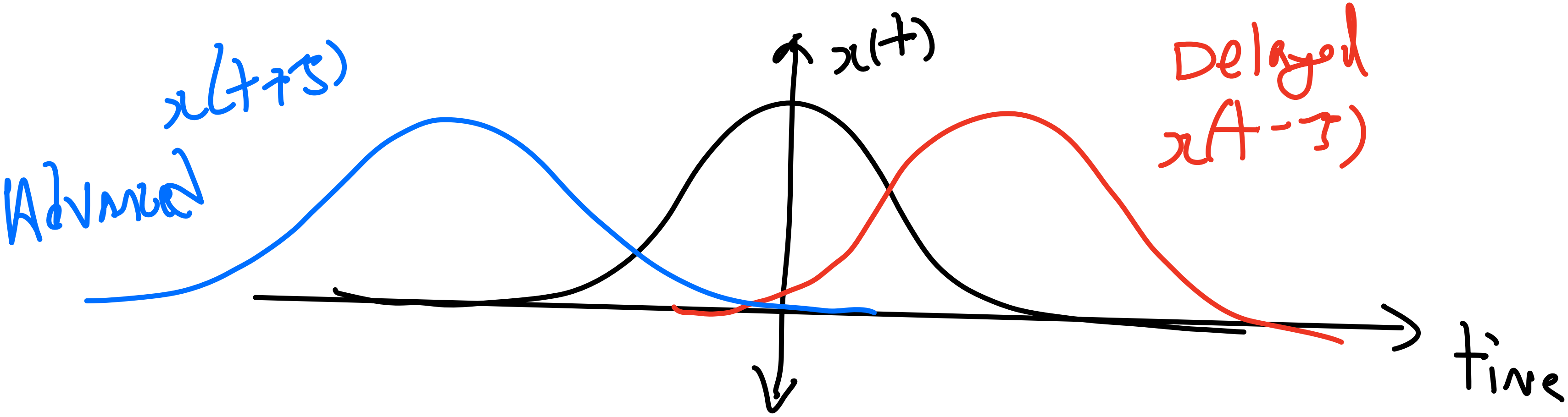
As you work on examples of this, it is sometimes helpful to plug in values of t to make sure you have compressed / expanded the values correctly.

Time Reversal



Time Shifting

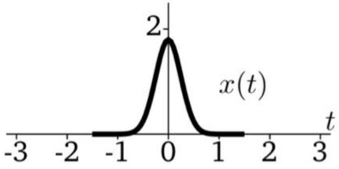
A signal $x(t)$ can be shifted in time by some amount $t_1 > 0$.



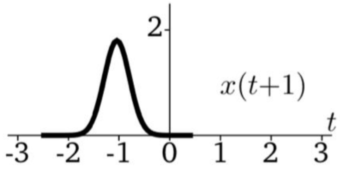
Time Shifting

- The signal $x(t - t_1)$ is delayed in time by t_1.
- The signal $x(t + t_1)$ is advanced in time by t_1.

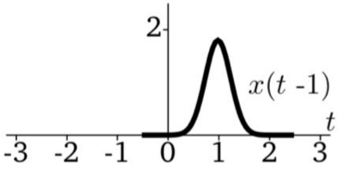
Original



Advanced



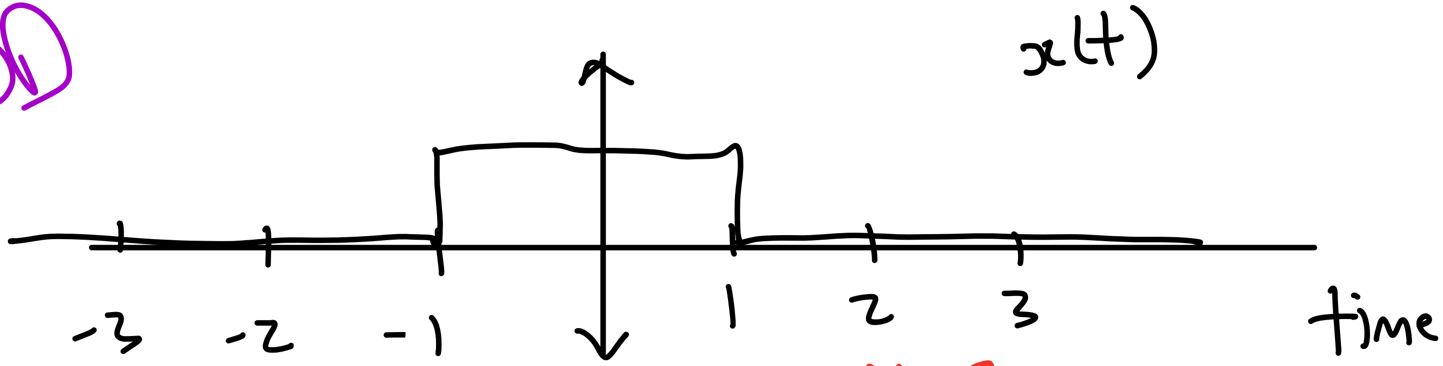
Delayed



As you work on time shift examples, it may be helpful to consider when $t - t_1 = 0$.

Combining Operations

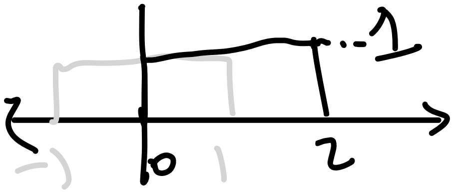
INCORRECT METHOD



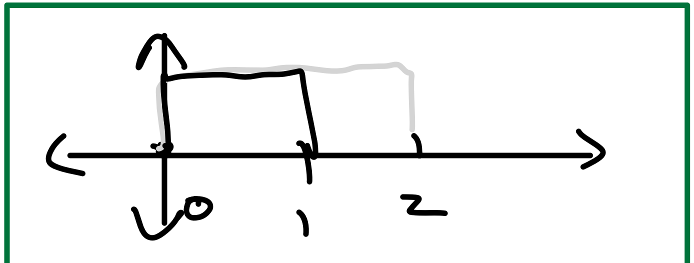
CYU: What does $x(2(t-1))$ look like?

Ans. PEMDAS

① Shift by 1 unit



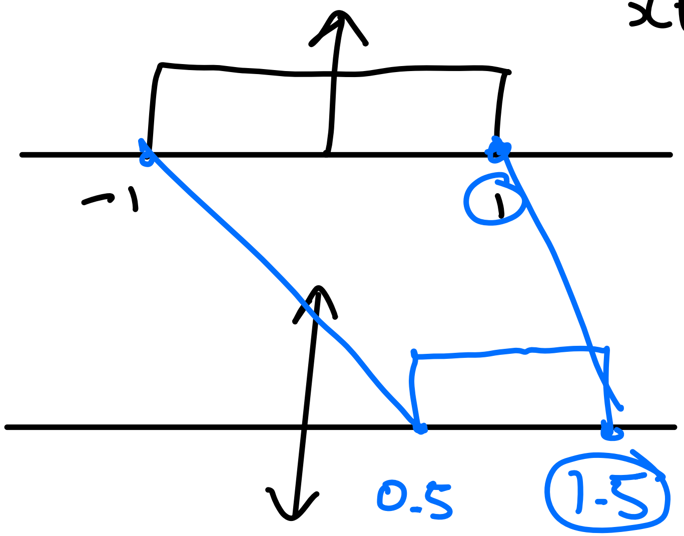
② Time Scaling by 2



Combining Operations

eg: $x(2(t-1))$

$x(2t-2)$



$2t-2=1$

$2t=3$

$t=1.5$

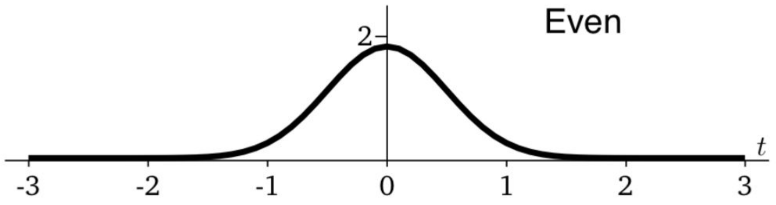
General Rule

1. Always expand parentheses : $x(2(t-1)) = x(2t-2)$
2. Next Shift
3. Apply Scaling + Rev.

Even and Odd Signals

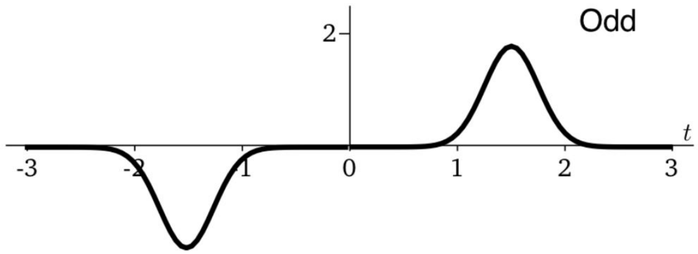
- An *even* signal is symmetric about $t = 0$,

$x(t) = x(-t)$

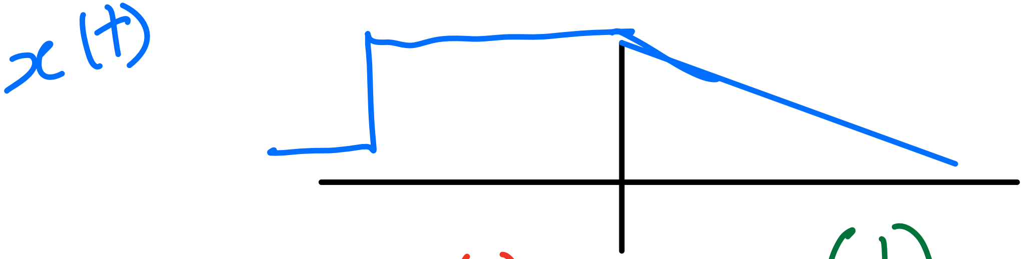


Even and Odd Signals

- An *odd* signal is antisymmetric about the origin $x(t) = -x(-t)$



Even and Odd Decomposition



CYU: Prove $\underbrace{x(t)}_{\text{LHS}} = \underbrace{x_e(t) + x_o(t)}_{\text{RHS}}$, $\forall\, x(t)$

Pf:

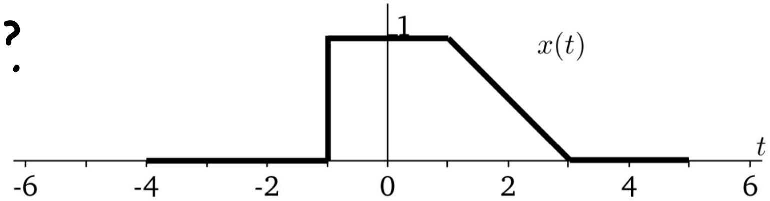
$$x_e(t) \triangleq \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$+ \quad x_o(t) \triangleq \frac{1}{2}\left(x(t) - x(-t)\right)$$

$$x_e(t) + x_o(t) = x(t)$$

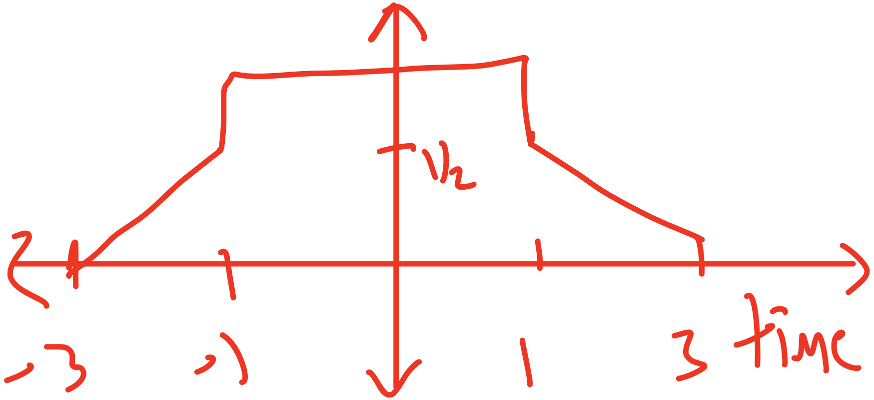
CYU Example #1

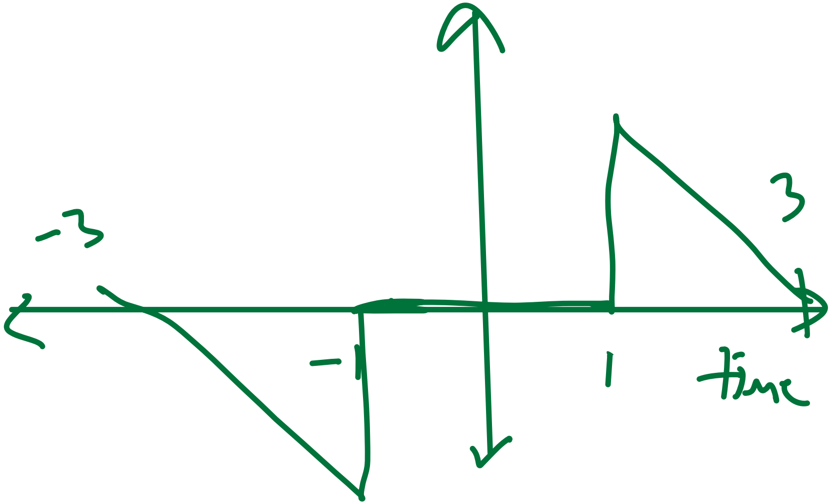
CYU: For this specific signal, what is $x_e(t)$ and $x_o(t)$?



$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$





CYU Example #2

Find the even and odd components for the following signals:

① $x(t) = 3t^2 + 2t - 2$

#1 Direct Insp Method: $x_{even} = 3t^2 - 2$ $\quad x_{odd} = 2t$

#2 Using Formula: $x_e = \frac{1}{2}\left(x(t) + x(-t)\right)$

$\vdots$

② $y(t) = cos(4\pi t) + (sin(\pi t))^3$

By direct inspection:

$x_{even} = \cos(4\pi t)$

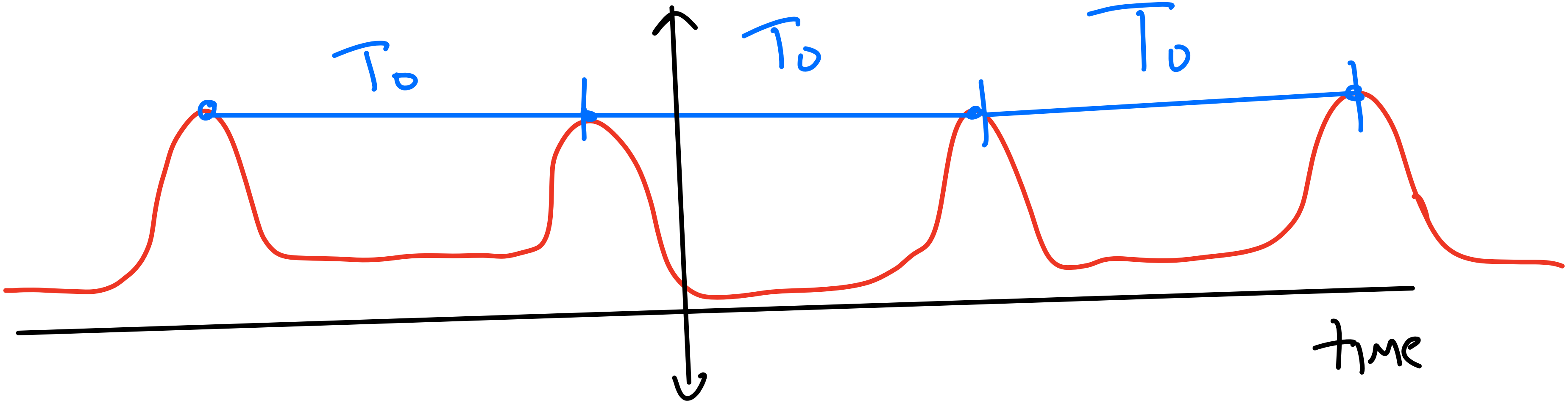
$x_{odd} = (\sin(\pi t))^3$

Periodic Signals

The concept of periodic signals is very important in this class. Colloquially, these are signals that repeat after a given interval, T_0.

A cts signal is periodic if and only if $\exists$ $T_0 > 0$

$$x(t+T_0) = x(t) \quad \forall t$$



Periodic Signal Properties

Suppose $x(t)$ periodic

$$x(t+T_0) = x(t) \quad \forall\, t$$

Is $\underbrace{x(t+2T_0)}_{LHS} = \underbrace{x(t)}_{RHS}$? How would you prove this?

$$t' \triangleq t + T_0$$

Pf: $x(\underbrace{t+T_0}_{t'} + T_0)$

$$= x(t' + T_0)$$

$$= x(t') = x(t+T_0) = x(t)$$

$$x(t+NT_0) = x(t)$$

CYU Example #3

Are the following signals periodic, if so, what is their fundamental period?

$k \in \mathbb{Z}$

if periodic $x(t) = x(t + kT_0)$

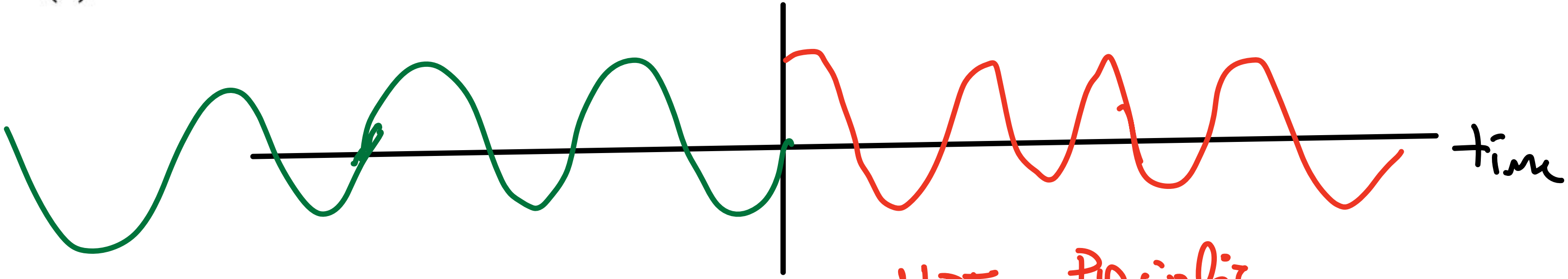
$$x(t) = 2cos\left(\frac{7}{8}t + \frac{5\pi}{6}\right)$$

$$T_0 = \frac{8}{7} \cdot 2\pi = \frac{16\pi}{7}$$

$$x\left(t + k\frac{16\pi}{7}\right) = 2\cos\left(\tfrac{7}{8}t + 2\pi k + \tfrac{5\pi}{6}\right)$$

$$= x(t)$$

$$y(t) = \begin{cases} cos(t) & \text{if } t \geq 0 \\ sin(t) & \text{if } t < 0 \end{cases}$$



NOT Periodic

Sinusoids

The most basic signal in this class is the sine or cosine wave. We'll use them *extensively* so it's worth reviewing their properties. By the end of this class, you'll be proficient at manipulating sinusoids.

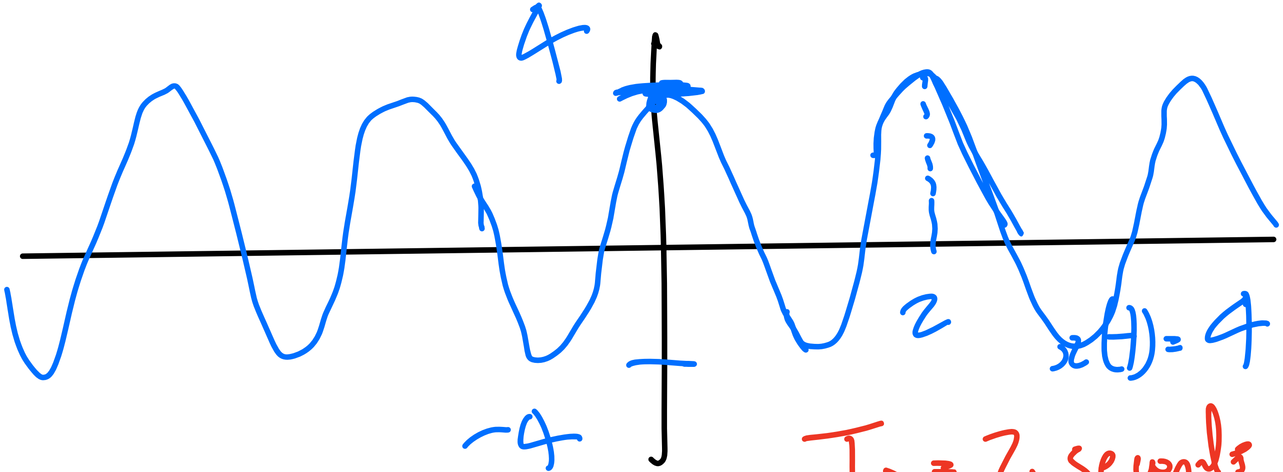
A cosine is defined by: $x(t) = A\cos(\omega t - \theta)$

$$= A\cos(2\pi f t - \theta)$$

ω: angular freq. [rad/s]

f: frequency [Hz]

$\omega = 2\pi f$



$x(t) = 4\cos\left(2\pi \frac{1}{2} t - 0\right)$

$T_0 = 2 \text{ seconds} = 1/f$

Trigonometric Rules

Some additional properties that you should be familiar with from trigonometry:

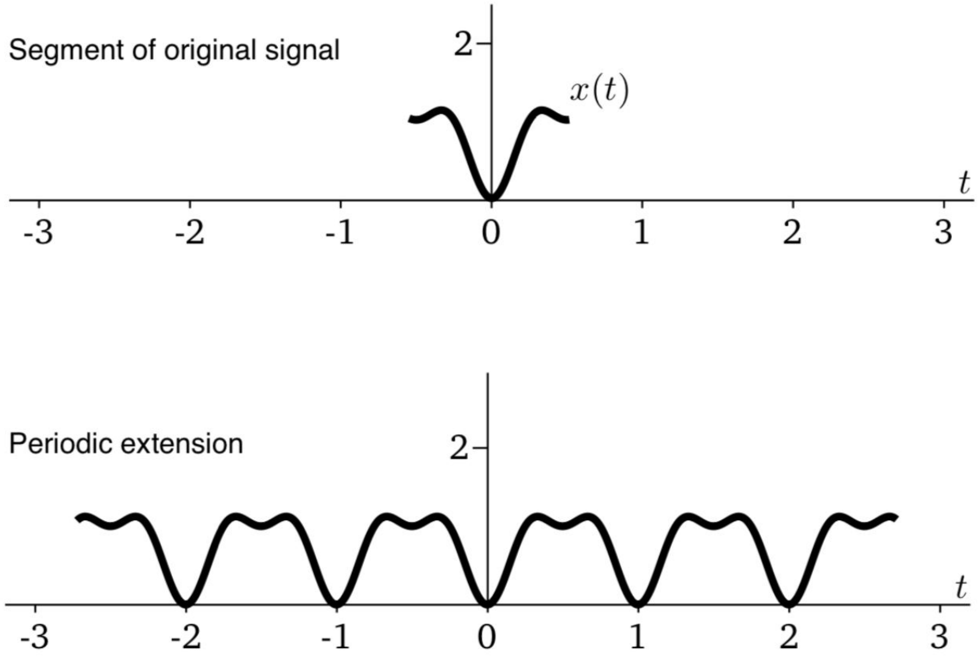
- $\sin(\theta) = \cos(\theta - \pi/2)$.
- Are either $\cos(\theta)$ or $\sin(\theta)$ even or odd?
- $\frac{d}{dt}\sin(\theta) = \cos(\theta)$ and $\frac{d}{dt}\cos\theta = -\sin(\theta)$.
- $\sin^2(\theta) + \cos^2(\theta) = 1$.

EVEN

ODD

Periodic Extension

In this class, we will sometimes be interested in taking an aperiodic signal and making its periodic extension. What this mean is that we take some interval on this signal of length T_0 and repeat it, as illustrated below:



CYU Question

Is the sum of the following two signals periodic?

$x_1(t)$, w/ period T_1

$x_2(t)$, w/ period T_2

$$z(t) = x_1(t) + x_2(t)$$

Is $z(t)$ guar. to be periodic?

Causality

Colloquially: A signal is causal if life begins at $t=0$

———— anticausal if $x(t) \neq 0$ for $t \leq 0$

A signal is non-causal if it has values everywhere

Complex Numbers Review

So far all signals we've presented are real-valued. But signals can also be complex.

- A complex signal is one that takes the form:

$$z(t) = x(t) + jy(t)$$

where $x(t)$ and $y(t)$ are real-valued signals and $j = \sqrt{-1}$.

$z = x + jy$

$x = \mathbb{R}\{z\}$ is the real part of z

$y = \Im\{z\}$ is img. part of z.

Complex Numbers Review

Because complex numbers play a large role in this class, we'll briefly review them.

- A complex number is formed from two real numbers, x and y, via:

  $$z = x + jy$$

  with $j = \sqrt{-1}$. Hence, a complex number is simply an ordered pair of real numbers, (x, y).
- $x = \Re(z)$ is called the *real* part of z. (In this class we will also write $x = \mathrm{Re}(z)$.)
- $y = \Im(z)$ is called the *imaginary* part of z. (In this class we will also write $y = \mathrm{Im}(z)$.)

- An aside: why do EE's use j as the imaginary number, while mathematicians and scientists commonly use i?

Complex Numbers Review

Phasor

Polar representation of complex numbers

The same complex number can be written in polar form,

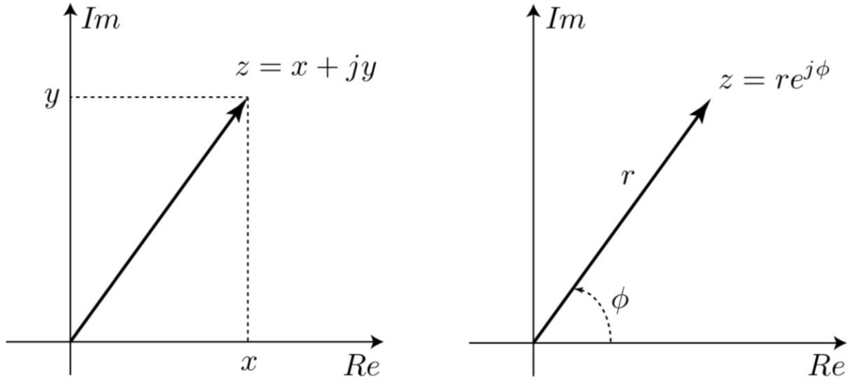
$$\begin{aligned} z &= x + jy \\ &= re^{j\phi} \end{aligned}$$

$r = \sqrt{x^2 + y^2}$

$\phi = \text{atan}(y/x)$

where

- r is the *modulus* or *magnitude* of z.
- ϕ is the *angle* or *phase* of z.
- $e^{j\phi} = \cos(\phi) + j\sin(\phi)$. We will sometimes write this as $\exp(j\phi)$. (More on this below.)



Complex Numbers Review

Cartesian vs polar coordinates

$$
\begin{aligned}
z &= x + jy \\
&= re^{j\phi}
\end{aligned}
$$

Here, the same intuitions from Cartesian and polar coordinates hold.

- $x = r\cos(\phi)$
- $y = r\sin(\phi)$
- $r = \sqrt{x^2 + y^2}$
- $\phi = \arctan y/x$

Euler's identity

Relating terms in our Cartesian and polar coordinate representation of complex numbers, we arrive at Euler's formula:

$$\begin{aligned} z &= x + jy \\ &= re^{j\phi} \end{aligned}$$

This tells us that, for $r = 1$,

$$\boxed{e^{j\phi} = \cos(\phi) + j\sin(\phi)}$$

Aside: this leads to one of the most elegant equations in mathematics:

$$e^{i\pi} + 1 = 0$$

With five terms, it incorporates Euler's constant (e), pi (π), the imaginary number (i), the multiplicative identity (1) and the additive identity (0).

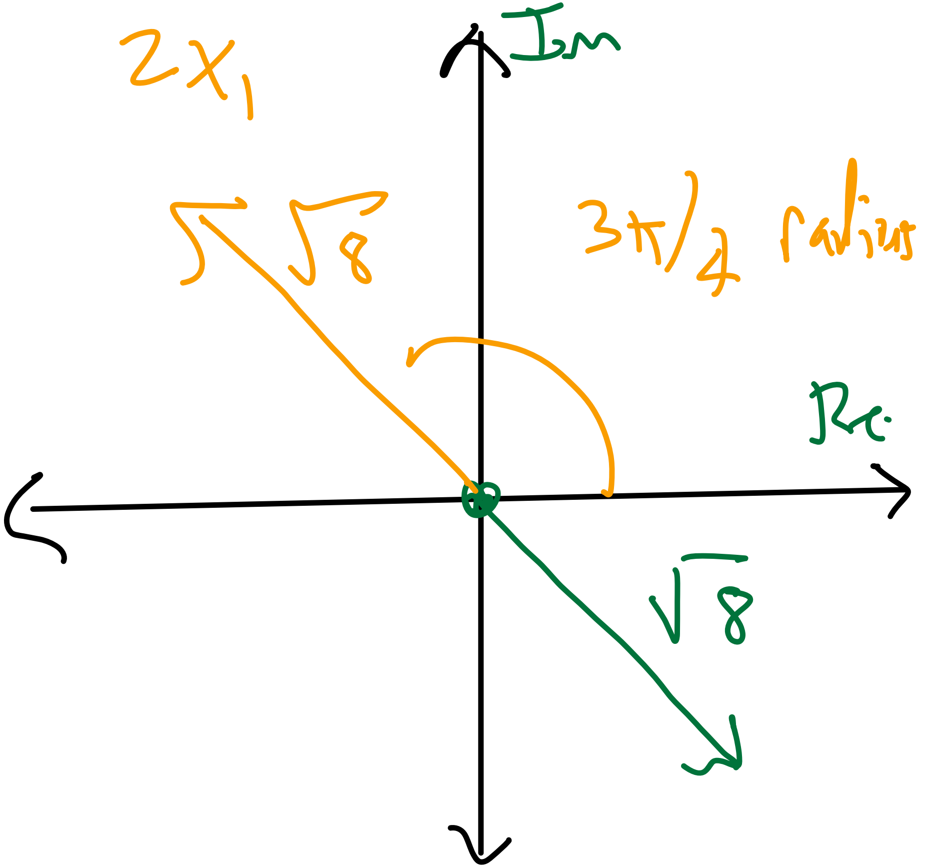
CYU Question

$x_1 = -1 + j$

$x_2 = \sqrt{8}\, e^{j(7\pi/4)}$

CYU: what is $2x_1 + x_2$?

$\boxed{0}$



Complex Conjugate

Some complex relations

Here are a few relations.

- **Complex conjugate**. If $z = x + jy$, then z^*, the complex conjugate of z, is

$$z^* = x - jy$$

- **Modulus and complex conjugate**. The following relation holds:

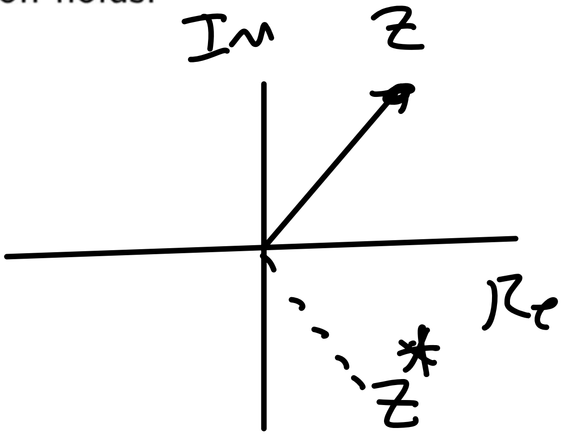
$$|z|^2 = z^* z = z z^*$$

  This is because

$$\begin{aligned} zz^* &= (x + jy)(x - jy) \\ &= x^2 + y^2 \\ &= r^2 \end{aligned}$$

  where $r = \sqrt{x^2 + y^2}$ as on the last slide.

- **Inverse of j**. Since $j^2 = -1$, we have that $-j = \frac{1}{j}$.



Euler's Formula: CYU

$$\cos(\theta) = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$$

$$\sin(\theta) = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$$

E.g. you can show

$$\cos(a+b) = \cdots$$

$$= \cos(a)\cos(b) - \sin(a)\sin(b)$$

CYU

Find the real and imaginary components of the following signal :

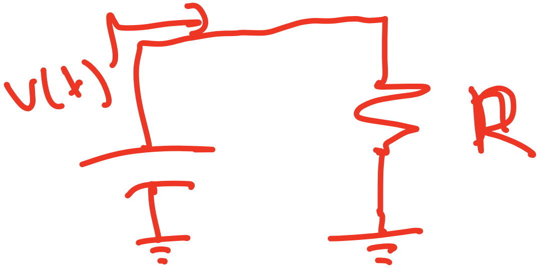
$$x(t) = j^{13} e^{2+j3t}$$

CYU (cont.)

Signal Energy and Power

"How much stuff in a signal"

$i(t)$



$v(t) = i(t)R$ Suppose $R = 1\Omega$

If $R = 1\Omega$, then $v(t) = i(t)$

Power: $p(t) = v(t)i(t) = \dfrac{v^2(t)}{R} = i^2(t)R$

$\therefore$ if $R = 1\Omega$ $p(t) = v^2(t) = i^2(t)$

Gist: squaring a signal is a measure of power.

Avg Power: $P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$

Total Energy: Power · Time $= \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$

Signal power has units of Watts (Joules per time). Hence, to get the total energy of a signal, $x(t)$, across all time, we integrate the power.

$$\boxed{E_x = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt}$$

(We incorporate the absolute value, $|\cdot|$, in case $x(t)$ is a complex signal, reviewed in the next slides.) Like signal power, signal energy is usually not a *actual* energy.

We can also calculate the *average power* of the signal by calculating:

$$\boxed{P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt}$$

Can we simplify this expression to obtain the power of a periodic signal?

$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

CYU

$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$

Finite Energy and Finite Power Signals

also known as "energy signal" and "power signal"

If $0 < E_x < \infty$, then $x(t)$ is energy signal

If $0 < P_x < \infty$, then $x(t)$ is power signal

CYU: If $E_x = C$, what can you say about P_x?

CYU

Finite Energy and Finite Power Signals