EE102

Lecture 8

EE102 Announcements

- **Third Homework due this Friday**
- **Midterm is 10/30**

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

Complex Numbers

CYU: Midterm Review

Find the real and imaginary parts of the following signal:

$$x(t) = (1 - \sqrt{3}j)(\underline{e^{j(t+2)}})$$

$$e^{j(t+2)} = \cos(t+2) + j\sin(t+2)$$

Ans. $(1 - \sqrt{3}\,j)\left[\cos(t+2) + j\sin(t+2)\right]$

$= \cos(t+2) + j\sin(t+2) - \sqrt{3}j\cos(t+2) + \sqrt{3}\sin(t+2)$

$= \cos(t+2) + \sqrt{3}\sin(t+2) + \sin(t+2)j - \sqrt{3}\cos(t+2)j$

Re: $\cos(t+2) + \sqrt{3}\sin(t+2)$ Im: $\sin(t+2) - \sqrt{3}\cos(t+2)$

CYU: Midterm Review

System Properties CYU: Midterm Review

Is the following system linear? Time-invariant? Stable? Causal?

$$y(t) = \pi + tx(t+3)$$

Linearity: No. Violates Homogeneity $S[0] \neq 0$

Time-Invariance: No. Direct Inspection of $+$ term.

Stable: No. [E.g Counter-e.g. if $x=1$, at $t=\infty \to y$ is ∞.]

Causal: No [E.g. $y(0)$ depends on $x(3)$]

CYU: Midterm Review

Convolution

CYU: Convolution with Dirac Delta

Compute the convolution of the two signals below:

$$f(t) = \delta(t+5) + 2\delta(t) \text{ and } g(t) = e^t$$

$$y(t) = f(t) * g(t)$$

$$= [\delta(t+5) + 2\delta(t)] * e^t$$

$$= \delta(t+5) * e^t + 2\delta(t) * e^t$$

$$= \boxed{e^{t+5} + 2e^t}$$

CYU: Convolution with Dirac Delta

Signal Properties

Periodicity and Fundamental Period

Consider the following signals and determine whether they are periodic or not. If periodic, identify the fundamental period.

i. $x(t) = sin(\frac{4}{5}t + \frac{\pi}{3})$

$x(t+T_0) = sin(\frac{4}{5}t + \frac{\pi}{3} + \frac{4}{5}T_0)$

$\frac{\frac{4}{5}T_0}{2\pi}$

$\frac{4}{5}T_0 = 2\pi$

$T_0 = \frac{10\pi}{4} = \boxed{\frac{5\pi}{2}}$

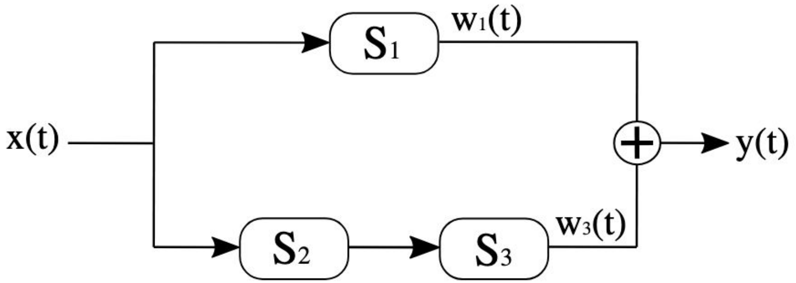
ii. $x(t) = \boxed{e^{-t}} sin(\frac{4}{5}t + \frac{\pi}{3})$

Not periodic,

By Direct Inspection, we observe the e^{-t} term.

Periodicity and Fundamental Period

System Response of LTI System



We know that Systems 1, 2 and 3 are all LTI systems, which are used in parts a through c. $w_1(t)$ and $w_3(t)$ are the outputs of Systems 1 and 3, respectively. Let $h_1(t)$, $h_2(t)$ and $h_3(t)$ represent the impulse response for System 1, 2 and 3, respectively.
For parts (a) through (c), we have prior knowledge of Subsystem 3

$$h_3(t) = \int_{-\infty}^{t-3} \delta(\tau - 4) d\tau.$$

For parts (a) through (c), we also have prior knowledge of the input/output mapping of the entire system

$$y(t) = \int_{-\infty}^{t} e^{-3(t-\tau)} x(\tau) d\tau + \int_{-\infty}^{t+1} x(\tau) d\tau.$$

(a) (10 points) What is the impulse response of the entire system (i.e. S_{eq})?

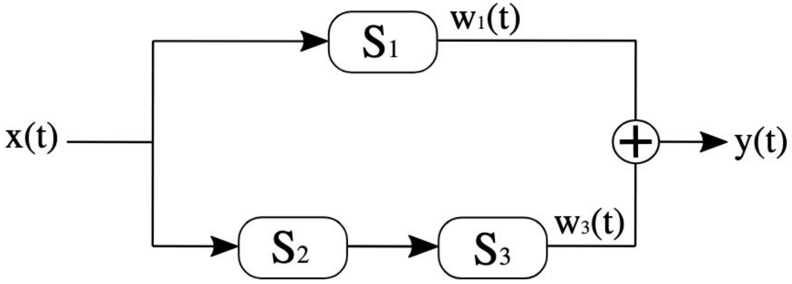
(b) (10 points) Find $h_1(t)$ and $h_2(t)$ that satisfies the input and output relationship that is given. It might be useful to determine the values of $w_1(t)$ and $w_3(t)$ first.

(c) (6 points) Using the solution from parts a and b, is System 1 Causal/Stable? Is the other subsystem (Cascaded Systems 2 and 3) Causal/Stable? Finally, is S_{eq} Causal/Stable?

Sifting + Integration

System Response of LTI System

$$y(t) = \int_{-\infty}^{t} e^{-3(t-\tau)} x(\tau) d\tau + \int_{-\infty}^{t+1} x(\tau) d\tau.$$



(a) What is the impulse response of the entire system?

$$h(t) = \int_{-\infty}^{t} e^{-3t} e^{3\tau} \delta(\tau) d\tau + \int_{-\infty}^{t+1} \delta(\tau) d\tau$$

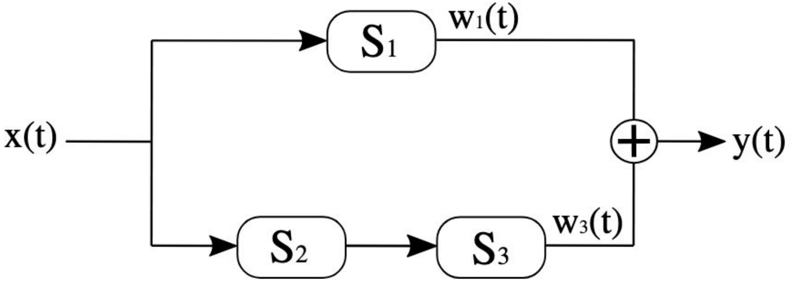
$$= e^{-3t} \int_{-\infty}^{t} e^{3\tau} \delta(\tau) d\tau + \int_{-\infty}^{t+1} \delta(\tau) d\tau$$

$$= e^{-3t} u(t) + u(t+1)$$

System Response of LTI System

System Response of LTI System

$$y(t) = \int_{-\infty}^{t} e^{-3(t-\tau)} x(\tau)d\tau + \int_{-\infty}^{t+1} x(\tau)d\tau.$$



(b) Find $h_1(t)$ and $h_2(t)$ that satisfies the input and output relationship that is given. It might be useful to determine the values of $w_1(t)$ and $w_3(t)$ first.

$$h(t) = \underbrace{e^{-3t}u(t)}_{\text{TRY this being } h_1} + \underbrace{u(t+1)}_{\text{If } h_2 \text{ and } h_3 \text{ giving me this branch}}$$

Given h_3: $h_3(t) = \int_{-\infty}^{t-3} \delta(\tau - 4)\, d\tau$

$= u(t-7)$

h_2's action on this is a time shift by 8 units.

$\therefore \quad h_2(t) = \delta(t+8)$

System Response of LTI System

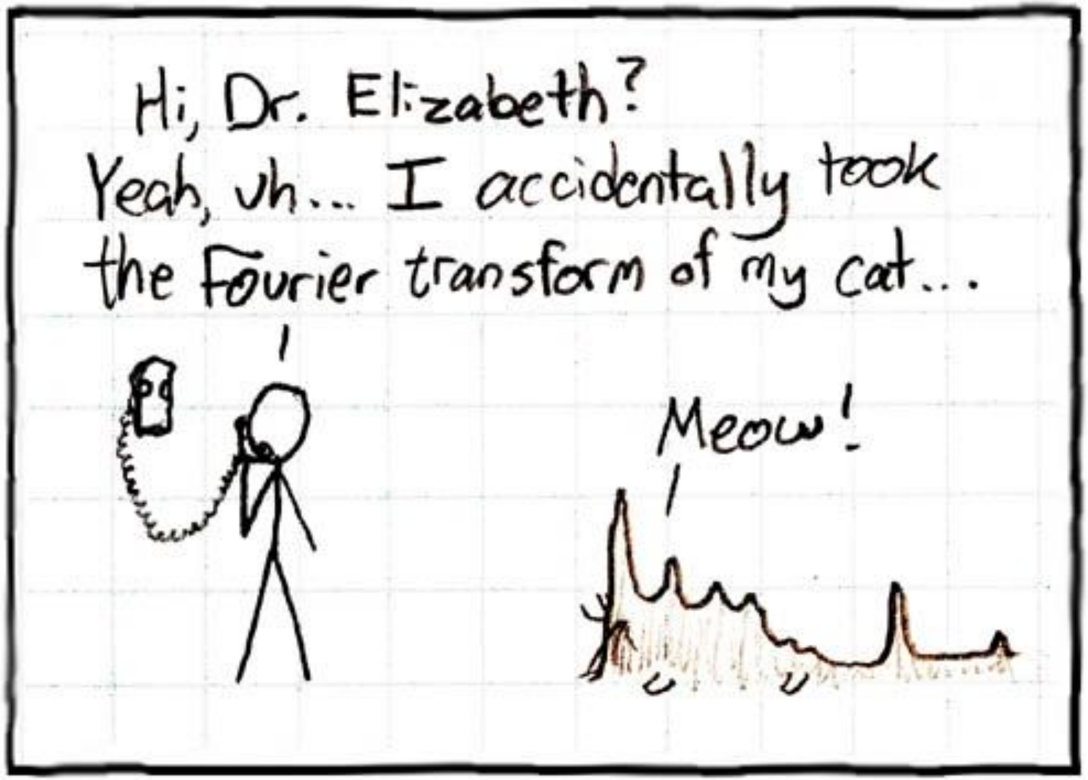
(c) Using the solution from parts a and b, is System 1 Causal/Stable? Is the other subsystem (Cascaded Systems 2 and 3) Causal/Stable? Finally, is S_{eq} Causal/Stable?

$$h_1 = e^{-3t}u(t) \Rightarrow \text{Causal, Stable}$$

$$h_2 * h_3 = u(t+1) \Rightarrow \text{Not Causal, Not Stable}$$

$$h_{eq} = e^{-3t}u(t) + u(t+1) \Rightarrow \text{Not Causal, Not Stable}$$

This Lecture Breaks a New Seal



Joseph Fourier

Fourier was a Super Interesting Guy

- Born, France.
- Son of a Tailor; Orphan at 10 years old.
- Aspires to be Newton, but at age 21 feels like a failure.
- Joins politics and goes to Jail
- Narrowly escapes guillotine in the French Revolution
- Confidante of Napoleon

French Revolution

Intuition: Why Fourier Series?

Why Fourier Series?

Bottom Line of F.S.

Any periodic signal can be expressed as a sum of sinusoids

1) $\underbrace{x(t)}_{\Downarrow} = x_1(t) + x_2(t) + \cdots$

Some periodic signal — Sinusoid — Sinusoid

△ Goal: Simplify analysis of systems

2) A different & Interesting way to think about data/info.

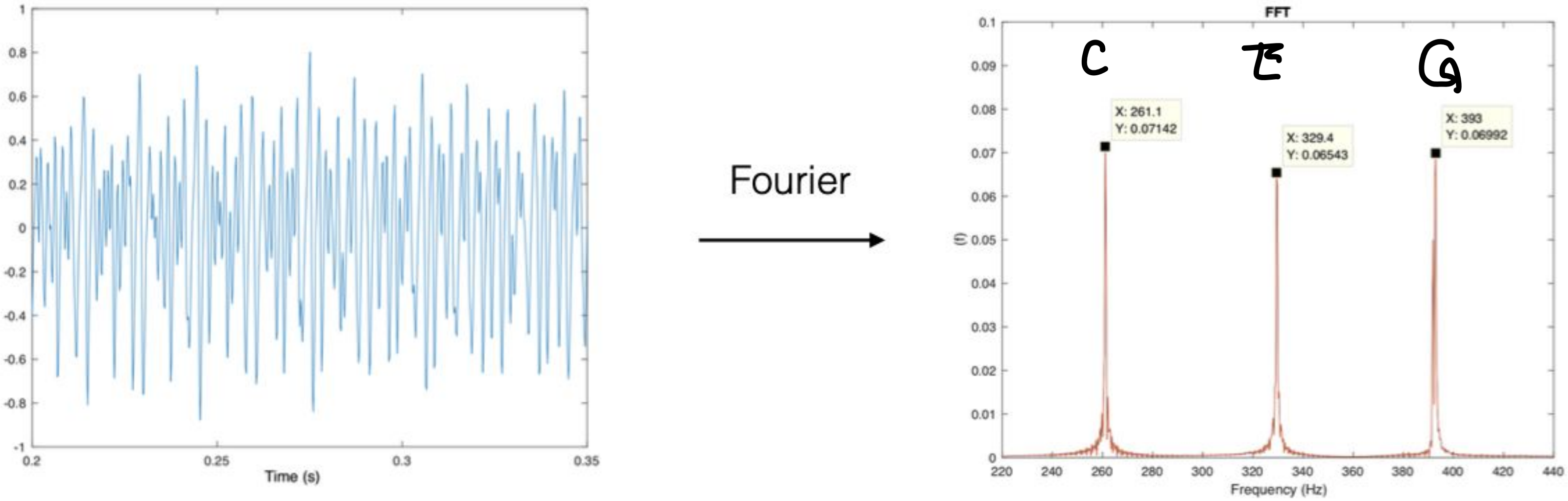
Can reveal interesting and unexpected structure.

Fourier Series

Extracts *frequency structure* from a signal.

Below: we have a C-major chord. It consists of three notes: C (261.1 Hz), E (329.4 Hz) and G (393 Hz). When we play these three notes at the same time, they create a waveform indicated by the blue line. It's hard to see structure here.

The Fourier series is able to write this as a sum of sinusoids. When we do, we find that there are only 3 frequencies, corresponding to our C-major chord.

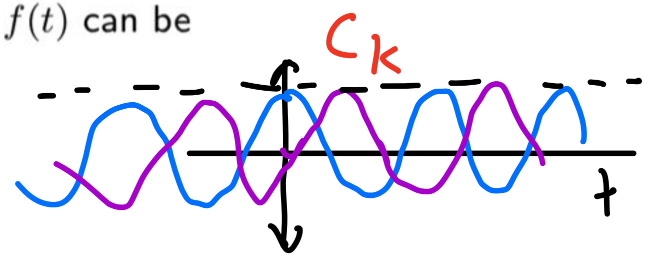


Fourier Series - Bottom Line

$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$

These are the main mathematical results of this lecture, written here for convenience.

If $f(t)$ is a well-behaved periodic signal with period T_0, then $f(t)$ can be written as a Fourier series



1) Any Periodic Signal

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

where $\omega_0 = \frac{2\pi}{T_0}$ and

2) Find the coefficients of each sinusoid.

$$\boxed{c_k = \frac{1}{T_0} \int_{\tau}^{\tau+T_0} f(t) e^{-jk\omega_0 t} dt}$$

for all integers k. The c_k are called the *Fourier coefficients* of $f(t)$.

Here, $f(t)$ is the *weighted average* of complex exponentials (which are simply complex sines and cosines).

Eigenfunctions

Eigenfunction $x(t)$ is an *eigenfunction* of a system if, when inputting $x(t)$ to the system, the output is simply a scaled version of $x(t)$, i.e., $y(t) = ax(t)$ where a is a constant (called an eigenvalue). Note that a may be a complex constant.

$$x(t) \longrightarrow \boxed{H} \longrightarrow a\,x(t)$$

~~A~~ A A Complex Exponentials are eigenfunctions of LTI systems

$$\underbrace{e^{(\sigma+j\omega)t}}_{\text{Eigenfn}} \longrightarrow \boxed{H} \longrightarrow a e^{(\sigma+j\omega)t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

Eigenfunctions of LTI Systems

Consider an LTI system with impulse response $h(t)$. If the input is a complex exponential, i.e., e^{st} where $s = \sigma + j\omega$, then

$$e^{st} \rightarrow \boxed{h(t)} \rightarrow H(s)e^{st}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$$

Algebra

$$= \int_{-\infty}^{\infty} h(\tau)e^{st}e^{-s\tau}\,d\tau$$

Pop Out Constant

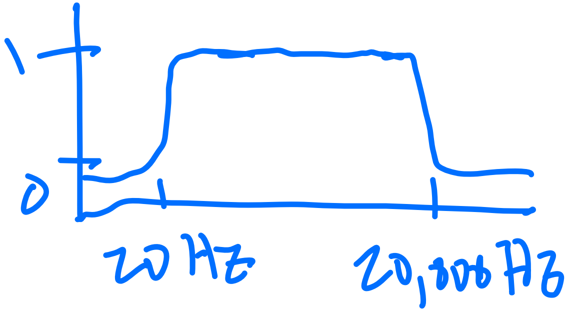
$$= e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau}_{\text{"Transfer Fn"}}$$

$$H(s) \triangleq \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau$$

$$= e^{st}H(s)$$

Human Ear

$H(s)$



Eigenfunctions of LTI Systems (Intuition)

This shows that the complex exponential is an eigenfunction of an LTI system.

- If I input a complex exponential into the LTI system, I get the same complex exponential out scaled by $H(s)$.

From here, we can see how Fourier series might help:

- First, I decompose my signal, $x(t)$, into the sum of complex exponentials.
- After this, I put each complex exponential into my LTI system. Since the system is LTI, each complex exponential comes out scaled by $H(s)$.
- Then, I can add my scaled complex exponentials at the output to get the system output.
- Conveniently, since the output is a sum of (scaled) complex exponentials, it is also a Fourier series.

Let's formalize this intuition we've stated here and get to the math.

Eigenfunctions of LTI Systems

Eigenfunctions of LTI Systems

For LTI systems, a simple way to analyze them is to:

- Decompose $x(t)$, the input, into its Fourier series, i.e., a sum of complex exponentials. This represents its decomposition into "fundamental" components, which are sinusoids at different frequencies $k\omega_0$.
- Because the complex exponential is an eigenfunction of an LTI system, if I pass in a complex exponential into my system, I get the same complex exponential at the output, scaled by $\hat{H}(jk\omega_0)$.
- Since LTI systems are distributive, if I pass in a sum of these complex exponentials into my system, I get back an output, $y(t)$, that is a sum of scaled complex exponentials (where the scale term is ______).

(Simple) Fourier Series Example

Motivation for Fourier series

With this motivation, we now need to know how to actually calculate Fourier series, i.e., how do I find the c_k so that

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

And further, when is this possible? The rest of this lecture will cover this.

(Simple) Fourier Series Example

Fourier series of a cosine (cont.)

Let's start simple. Consider the sinusoid:

$$f(t) = A\cos(\omega_0 t + \theta)$$

(Simple) Fourier Series Example

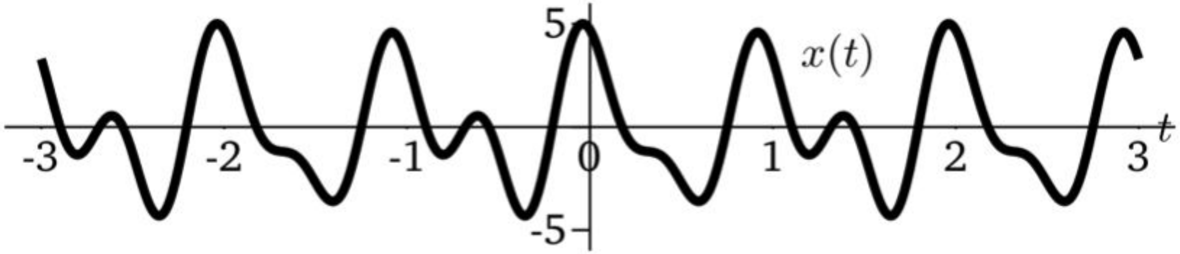
Using spectrum to find structure

For the cosine example, it doesn't look like we made our lives easier by representing it as a spectrum. But for any more complex signal, it can.

Consider the signal

$$x(t) = 3\cos(2\pi t) + \cos(3\pi t - \pi/4) + 2\cos(4\pi t + \pi/3)$$

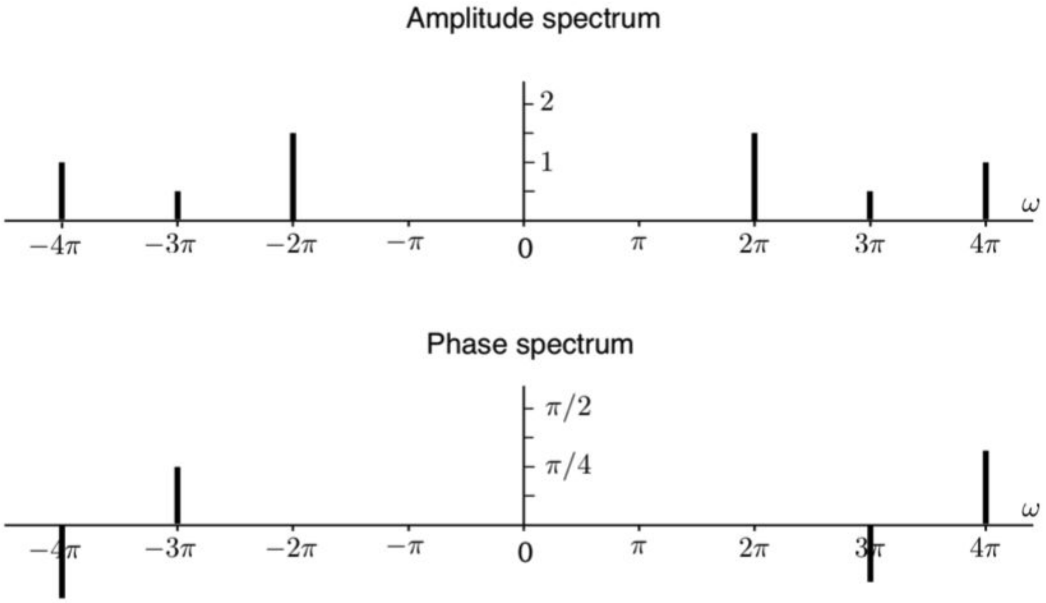
This signal is very simple. However, if I gave you a plot of its time domain representation, it'd be hard to recover exactly what $x(t)$ is.



(Simple) Fourier Series Example

Using spectrum to find structure (cont.)

However, if we plotted the spectrum of this signal, it would look like the following:



From the spectrum, we can read off that this signal is composed of sinusoids at three different frequencies (2π, 3π, and 4π) with amplitudes given by the amplitude spctrum and phases given by the phase spectrum.

Deriving the Fourier Series Coefficients

Deriving Fourier series

How do we find the c_k?

Our derivation is as follows:

- First, we *assume* that the signal $f(t)$ can be written as a sum of complex exponentials that are scaled by coefficients c_k.
- Given this assumption, we find if there are c_k such that we can represent $f(t)$ in this way.