EE102

Lecture 14

EE102 Announcements

- **HW #5 is due Friday 5/21.**
- **HW #6 is due Friday 5/28.**

- **Midterm Exams have been released.**
    - **Deadline for submitting regrade requests through gradescope is December 1st.**

ABET Learning Outcomes

✔ Understand the concept of a signal and a system, plot continuous-time signals, evaluate the periodicity of a signal.

✔ Identify properties of continuous-time systems such as linearity, time-invariance, and causality.

✔ Calculate with the Dirac delta function.

✔ Compute convolution of continuous-time functions.

✔ Understand the concept of the impulse response function of a linear system, and its use to describe the input/output relationship.

Compute the Laplace transform of a continuous function, identify its domain of convergence, and be familiar with its basic properties, including the initial and final value theorems.

Find the inverse Laplace transform by partial fractions.

Use the Laplace transform to solve constant-coefficient differential equations with initial conditions

Use the Laplace transform to evaluate the transfer function of linear time-invariant systems.

✔ Understand Parseval's relation in Fourier series, and its interpretation in terms of decomposing the signal's energy between its harmonics

✔ Evaluate the response of a linear time-invariant system to periodic inputs.

✔ **Evaluate the Fourier transform of a continuous function, and be familiar with its basic properties.** Relate it to the Laplace transform.

✔ **Evaluate and plot the frequency responses (magnitude and phase) of linear time-invariant systems, and apply it to filtering of input signals.**

Understand conditions under which a band-limited function can be recovered from its samples

CYU

$$\boxed{\mathcal{F}[\ \int_{-\infty}^{t} y(\tau)\, d\tau\ ] = \mathcal{F}[\ y(t)\ *\ u(t)\ ]}$$

$$\boxed{u(t)\ \Longleftrightarrow\ \pi\delta(\omega) + \frac{1}{j\omega}}$$

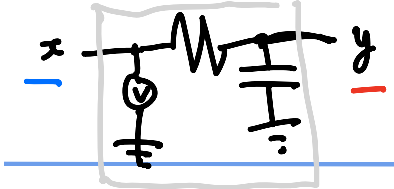
$$\boxed{\mathcal{F}[f(t)e^{j\omega_0 t}] \quad = \quad F(j(\omega - \omega_0))}$$

Lots of info .. so summarizing last class

1. (last class) various properties of F.T
2. Applied F.T. to assess RC circuit output.

Today:

3. Analyzing frequency response of RC circuit
4. Sampling a signal and its effects



Example: RC Circuit

$x \to \boxed{h} \to y$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

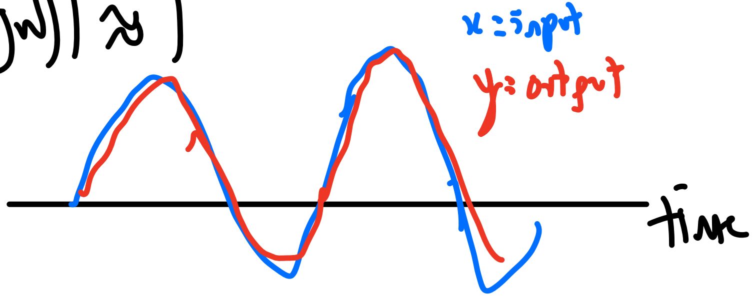
Transfer Fn. Magn.

$$|Y(j\omega)| = |H(j\omega)|\,|X(j\omega)|$$

Low Freq. Regime

$\omega << \frac{1}{RC}$

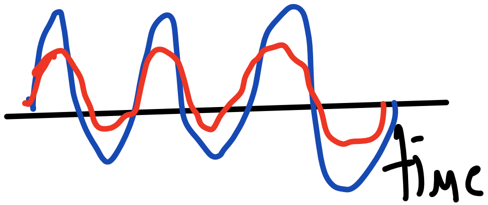
$|H(j\omega)| \approx 1$



Moderate Freq.

$\omega = 1/RC$

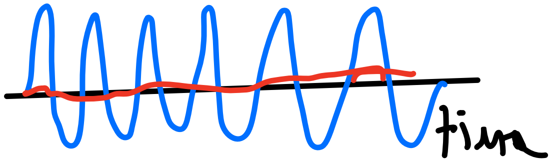
$|H(j\omega)| = 1/\sqrt{2}$



High Freq.

$\omega >> 1/RC$

$|H(j\omega)| \to 0$



Frequency Response

- In addition to *frequency response*, $H(j\omega)$ is sometimes called the *transfer function* of the system.
- The reason its called frequency response is that $H(j\omega)$ describes how the input is changed at every single frequency.
- In particular, the frequency response scales the amplitude response by $|H(j\omega)|$, i.e.,

Ampl. Spectrum

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

- The frequency response shifts the phase response by $\angle H(j\omega)$, i.e.,

Phase Spectrum

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Frequency Response

To see this, note that if the input to a system is a complex exponential, $e^{j\omega_0 t}$ (recall, these are the eigenfunctions of an LTI system), then

$$\begin{aligned} X(j\omega) &= \mathcal{F}\left[e^{j\omega_0 t}\right] \\ &= 2\pi\delta(\omega - \omega_0) \end{aligned}$$

Therefore, the output is

$$\begin{aligned} Y(j\omega) &= H(j\omega)(2\pi\delta(\omega - \omega_0)) \\ &= H(j\omega_0)(2\pi\delta(\omega - \omega_0)) \end{aligned}$$

Now, Y is transfer fn.

Frequency Response

This means that

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}[Y(j\omega)] \\ &= \mathcal{F}^{-1}[H(j\omega_0)(2\pi\delta(\omega-\omega_0))] \\ &= H(j\omega_0)e^{j\omega_0 t} \\ &= |H(j\omega_0)|\, e^{j(\omega_0 t+\angle H(j\omega_0))} \end{aligned}$$

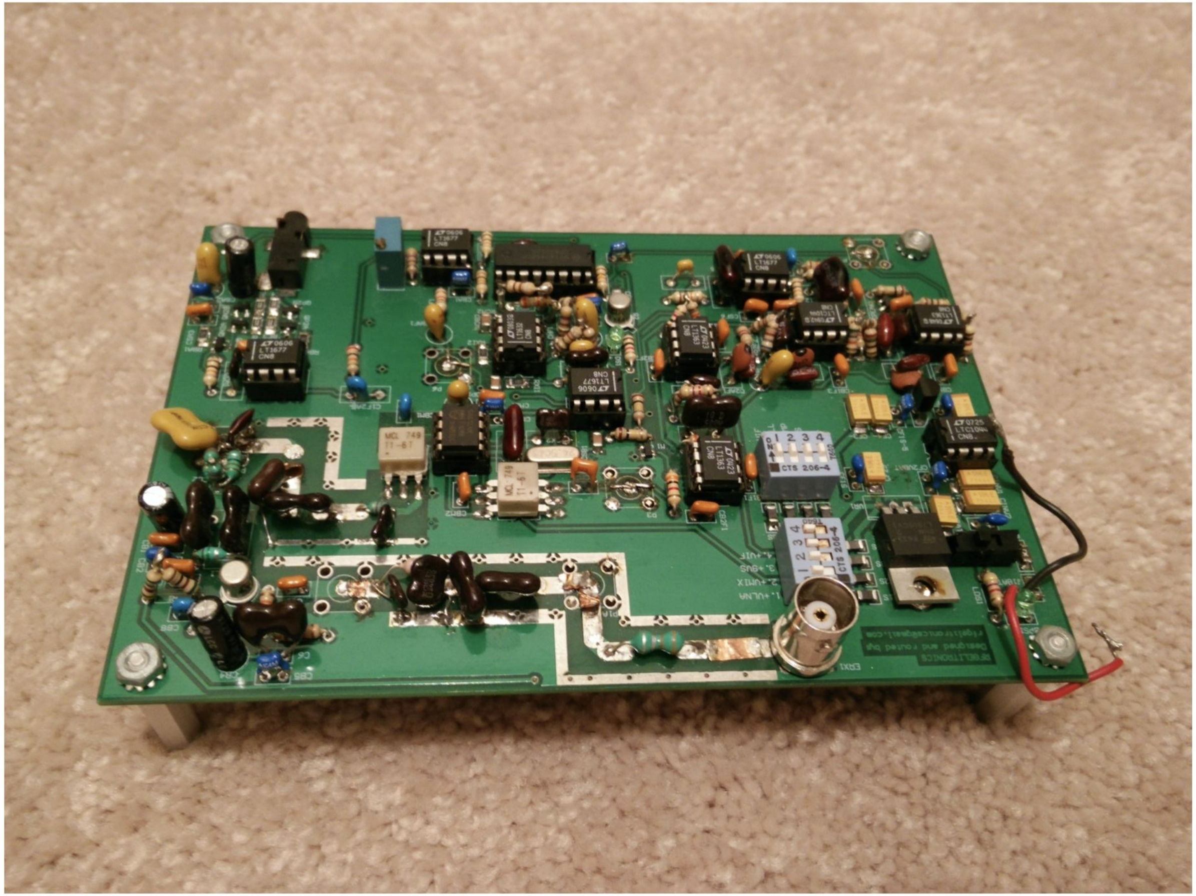
To summarize here, we input a sinusoidal input, $x(t) = e^{j\omega_0 t}$ to a LTI system, and saw that the output was

$$y(t) = |H(j\omega_0)|\, e^{j(\omega t+\angle H(j\omega_0))}$$

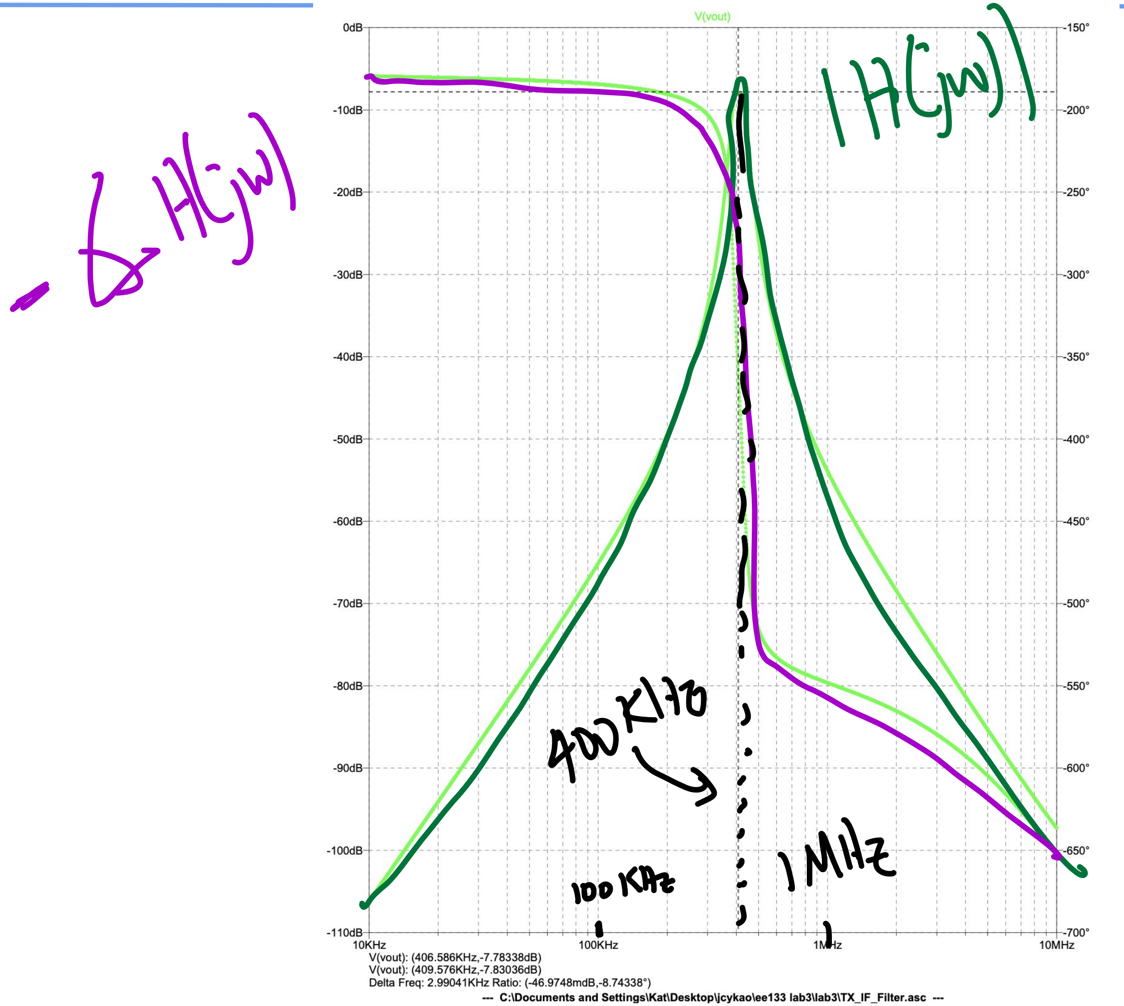
i.e., inputting a complex sinusoid to an LTI system produces an output that:

- is at the same frequency, ω_0.
- is scaled in amplitude by $|H(j\omega_0)|$.
- is phase shifted by $\angle H(j\omega_0)$.

Application: tuning circuits



Application: tuning circuits



Spectrum Analysis

$f(at) \Leftrightarrow \frac{1}{|a|} F(j\omega \cdot \frac{1}{a})$

Frequency Response Example

Consider the input:

$$x(t) = 2\cos(t) + 3\cos(3t/2) + \cos(2t)$$

and system with impulse response

$$h(t) = \frac{2}{\pi}\,\mathrm{sinc}^2(t/\pi)$$

Find $y(t) = (x * h)(t)$.

$H(j\omega) = \mathcal{F}[h(t)]$

$\mathrm{sinc}^2(t) \Leftrightarrow \Delta(\omega/2\pi)$

$\mathrm{sinc}^2(t/\pi) \Leftrightarrow \pi\Delta(\omega/2\pi \cdot 1/\tfrac{1}{\pi})$

$= \pi\Delta(\omega/2)$

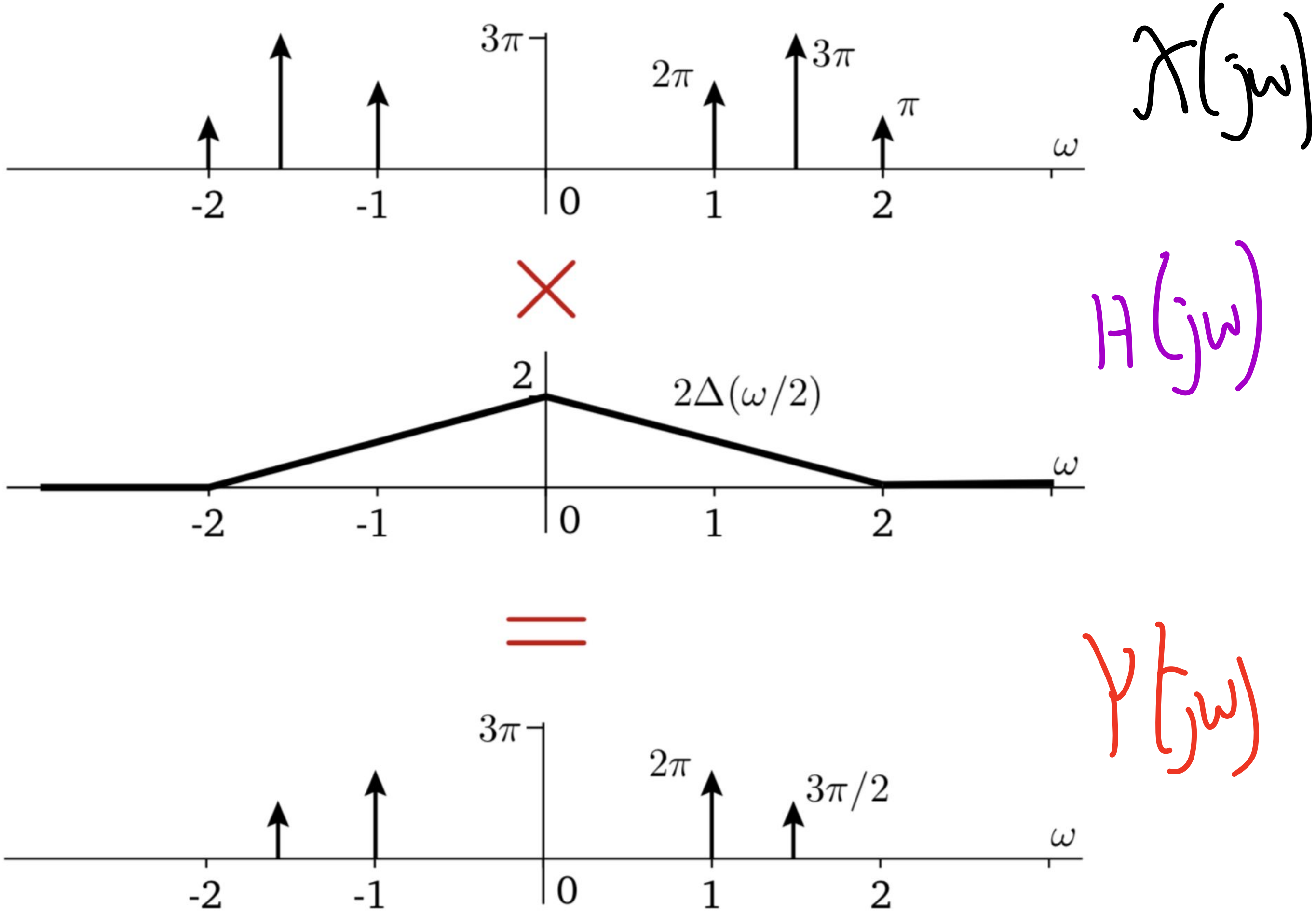
$H(j\omega) = 2\Delta(\omega/2)$

$Y(j\omega) = H(j\omega)X(j\omega)$

$= 2\Delta(\omega/2)\Big[2\pi[\delta(\omega-1)+\delta(\omega+1)] + 3\pi[\delta(\omega-1.5)+\delta(\omega+1.5)] +$

$\pi[\delta(\omega-2)+\delta(\omega+2)]\Big]$

Frequency Response Example



Frequency Response Example

This gives that

$$Y(j\omega) = 2\pi\left[\delta(\omega - 1) + \delta(\omega + 1)\right] + \frac{3\pi}{2}\left[\delta(\omega - 3/2) + \delta(\omega + 3/2)\right]$$

Taking the inverse Fourier transform, we get that

$$y(t) = 2\cos(t) + \frac{3}{2}\cos(3t/2)$$

Frequency Response Example 2 $e^{-at}u(t) \Leftrightarrow \frac{1}{a+j\omega}$

Let $x(t) = e^{-t}u(t)$. We input this signal into a system with impulse response:

$\Updownarrow$

$X(j\omega) = \frac{1}{1+j\omega}$

$$h(t) = 2e^{-2t}u(t) \Leftrightarrow \frac{2}{2+j\omega}$$

What are $Y(j\omega)$ and $y(t)$?

$$Y(j\omega) = X(j\omega)\,H(j\omega)$$

$$= \frac{2}{(1+j\omega)(2+j\omega)}$$

Frequency Reponse Example 2

$$Y(j\omega) = \left[\frac{2}{(1+j\omega)(2+j\omega)}\right] \cdot (1+j\omega)(2+j\omega)$$

LHS

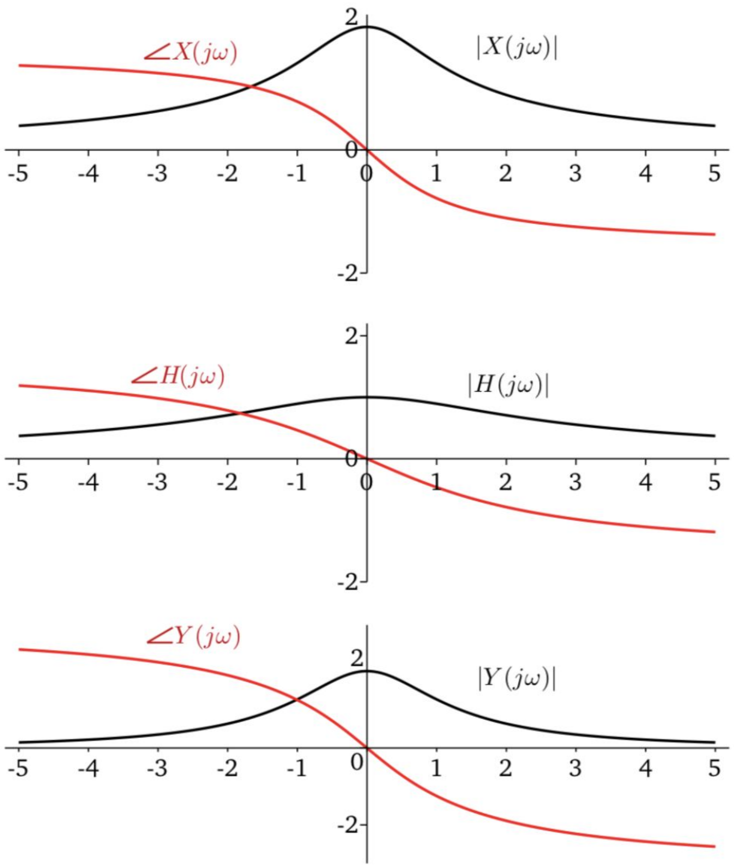
$$= \left[\frac{A}{1+j\omega} + \frac{B}{2+j\omega}\right](1+2j\omega)(2+j\omega)$$

RHS

Exercise for HW: take IFT to get $y(t)$

Frequency Response Example 2

Recall that when we multiply two complex numbers, their magnitudes multiply and their phases add. This is shown below.



Filters

Filters are designed to extract or attenuate certain desired frequencies from a signal. For example, consider a recording of music where the microphones accidentally recorded the sopranos too loudly. It would be possible to rebalance the audio by attenuating higher frequencies in the signal.

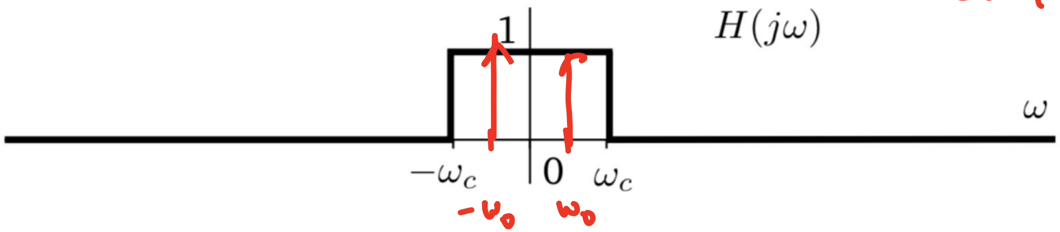
We'll first discuss ideal filters, which only pass through certain frequencies. There are three main types of filters we'll discuss:

- Low pass filter: suppresses all frequencies that are higher than a specified frequency, ω_c. Its name comes from the fact that it lets frequencies less than ω_c through (i.e., low frequencies).
- High pass filter: supresses all frequencies that are lower than a specified frequency, ω_c.
- Band pass filter: supresses all frequencies outside of a range $\pm\omega_c$ around a chosen frequency ω_0.
- These three filters are illustrated on the next page.

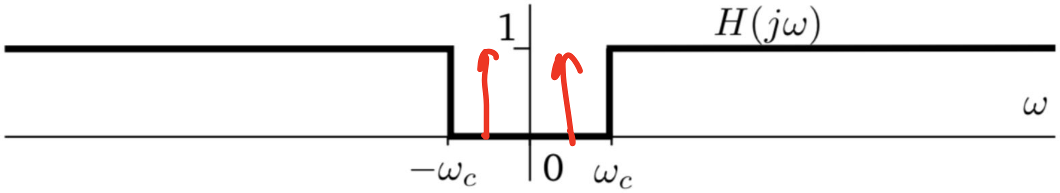
Filter Illustration

$$\cos(\omega_0 t) = \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{j\omega_0 t}$$

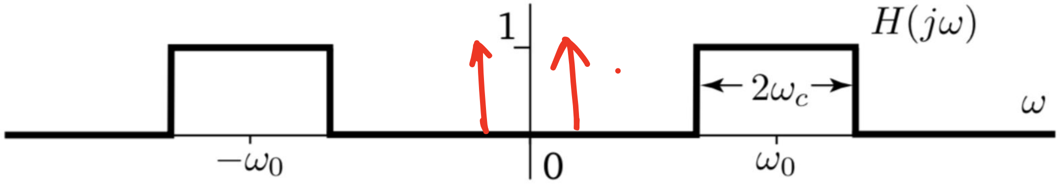
- Low pass filter:



- High pass filter:



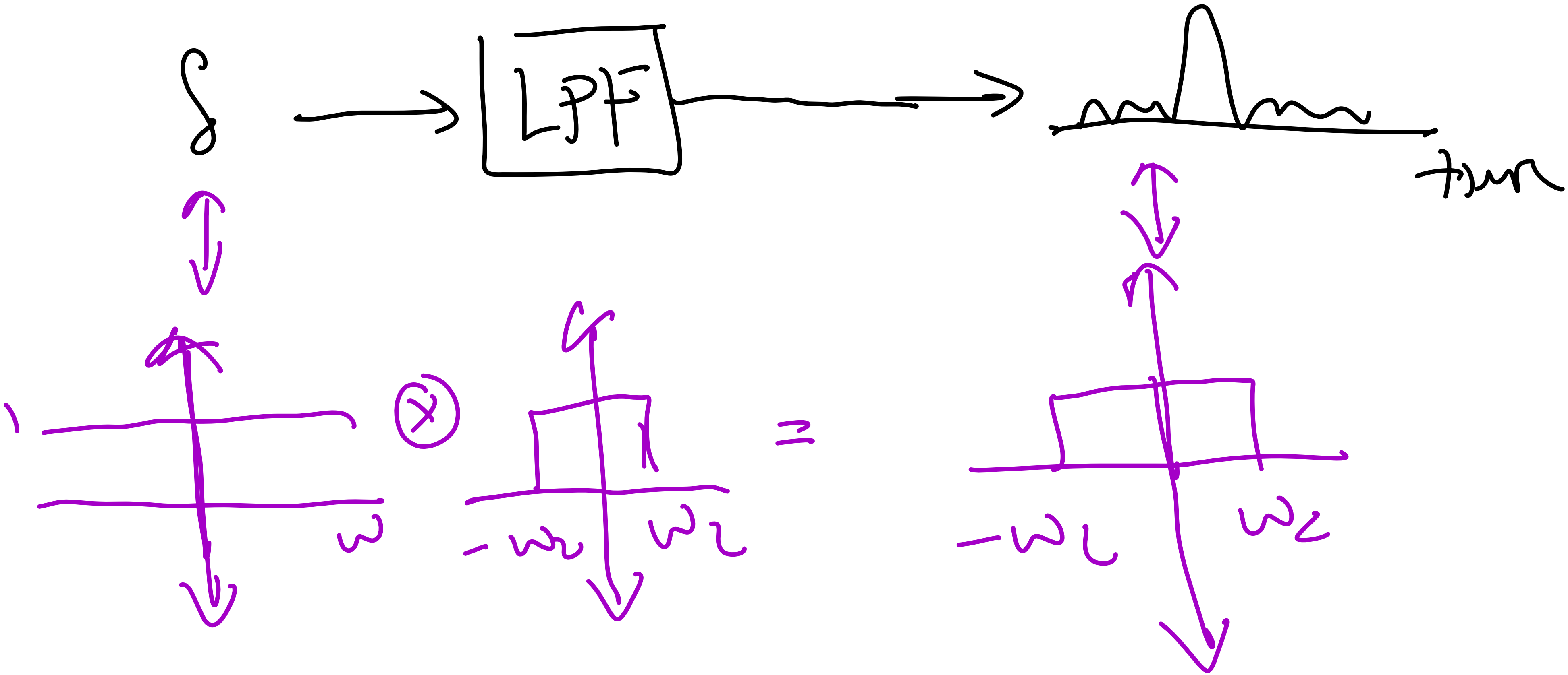
- Band pass filter:



These may alternatively be written as “lowpass” or “low-pass” filter, etc.

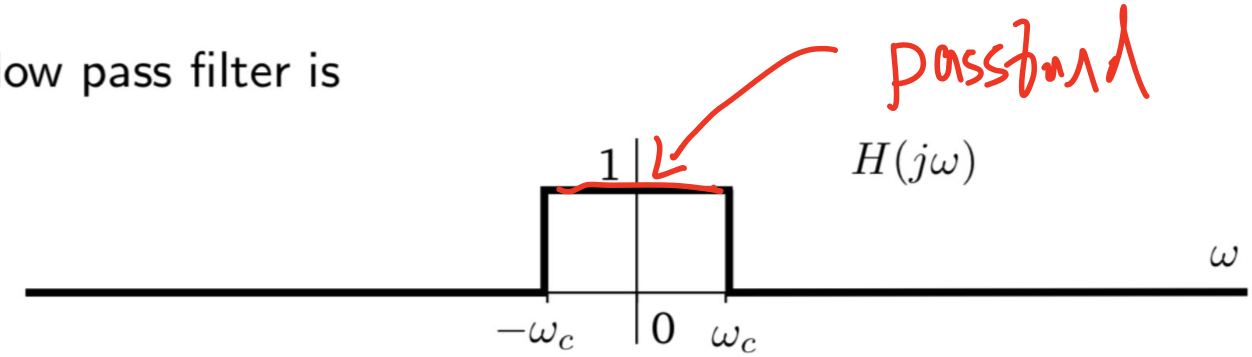
CYU on Filters $\delta \rightarrow \boxed{\text{LPF}} \rightarrow ?$

Imagine I pass the canonical delta function through a low-pass filter. What is the output?



Ideal Low Pass Filter

The ideal low pass filter is



We call the region where frequencies are not suppressed (i.e., up to frequency ω_c for this ideal low pass filter) the "passband." This filter can be represented as

$$H(j\omega) = \mathrm{rect}(\omega/(2\omega_c))$$

Ideal Low Pass Filter

$$H(j\omega) = \mathrm{rect}\left(\frac{\omega}{2\omega_c}\right)$$

What is $h(t)$?

$$h(t) = \frac{2\omega_c}{2\pi}\,\mathrm{sinc}\left(\frac{t\,2\omega_c}{2\pi}\right)$$

$$= \frac{\omega_c}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c}{2\pi}t\right)$$

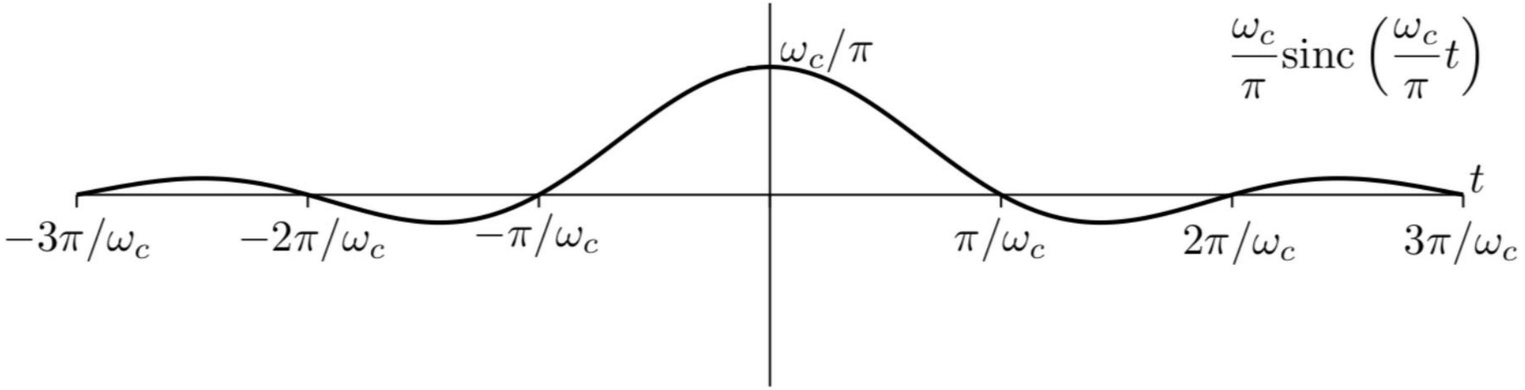
$$y(t) = x(t) * h(t)$$

$$\mathrm{rect}(t/T) \Leftrightarrow T\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$$

Duality $T\,\mathrm{sinc}\left(\frac{tT}{2\pi}\right) \Leftrightarrow 2\pi\,\mathrm{rect}\left(\frac{\omega}{T}\right)$

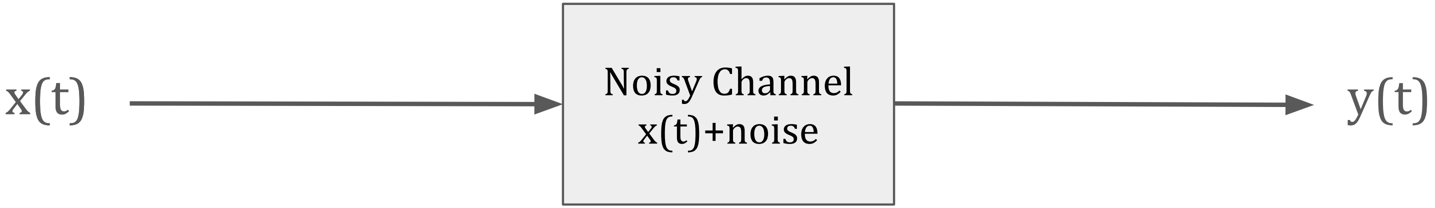
Ideal Low Pass Filter

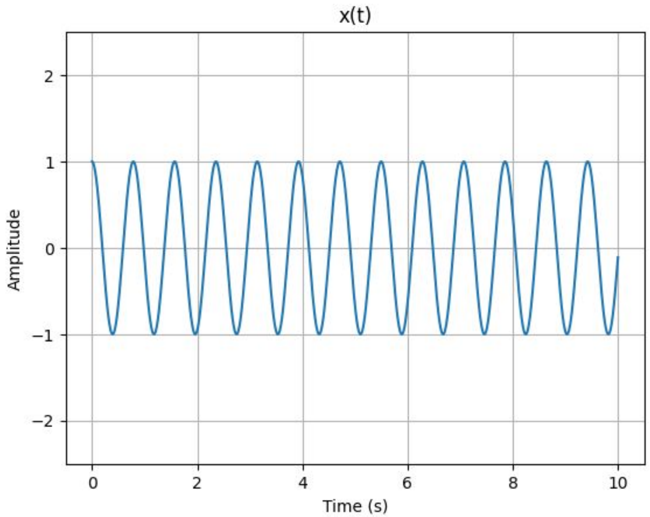
Thus, the ideal low pass filter's impulse response is:

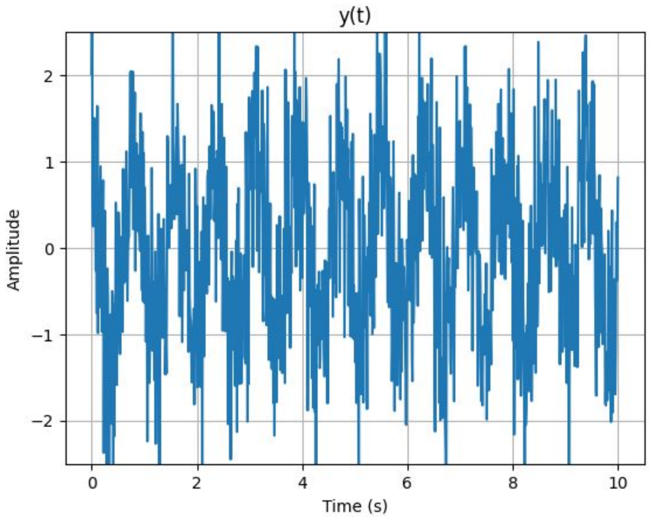


Note that we've only shown a small interval here, the sinc function is nonzero for t outside of the plotted window.

CYU: Noise and Filtering

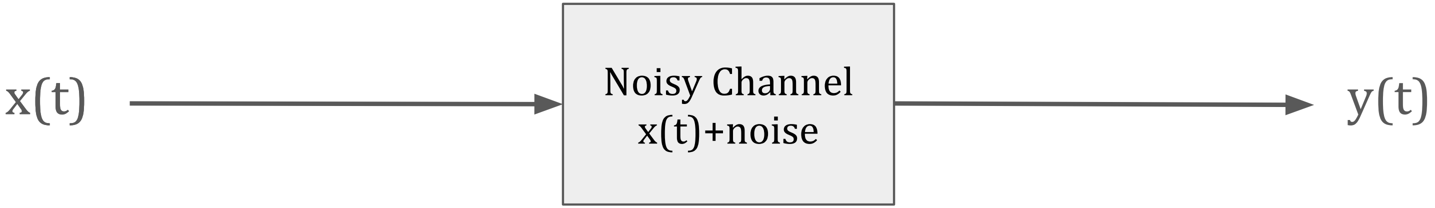


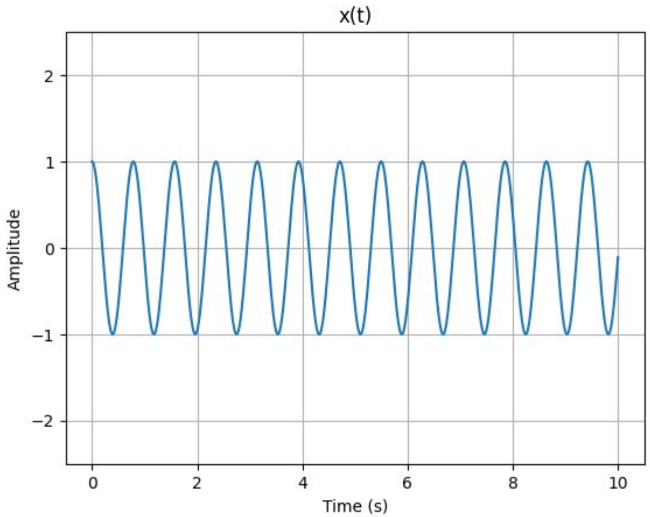


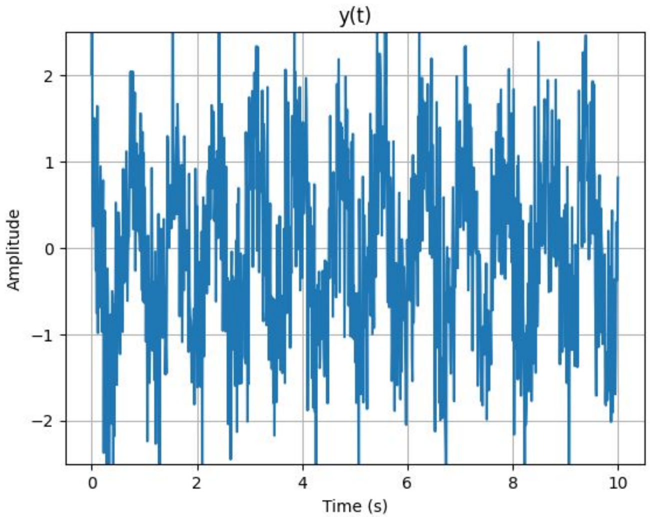


CYU: Noise and Filtering

How will adding Gaussian noise to x(t) affect its spectrum?







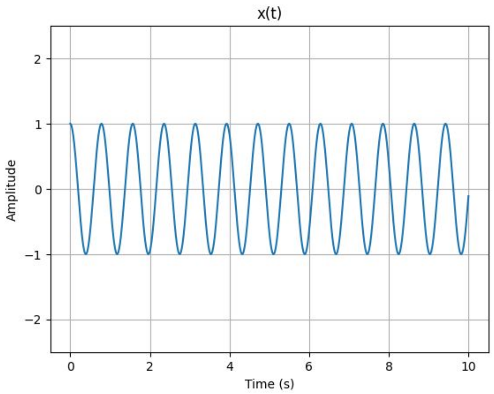
CYU: Noise and Filtering

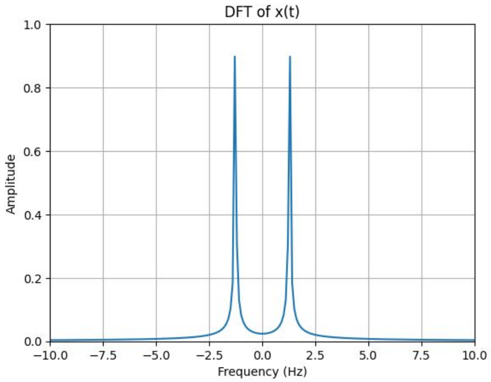
How will adding Gaussian noise to x(t) affect its spectrum?

CYU: Noise and Filtering

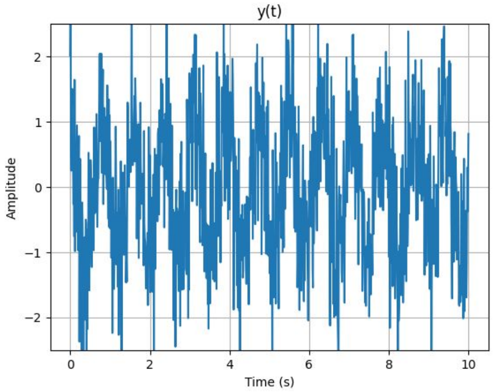
How will adding Gaussian noise to x(t) affect its spectrum?

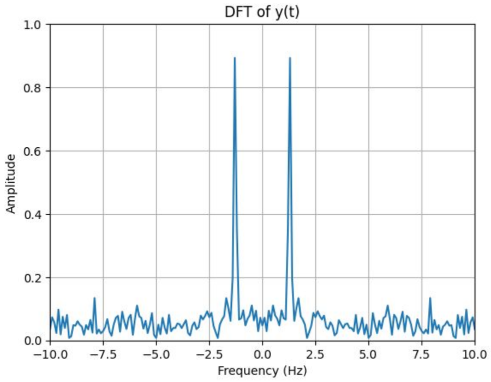
x(t)





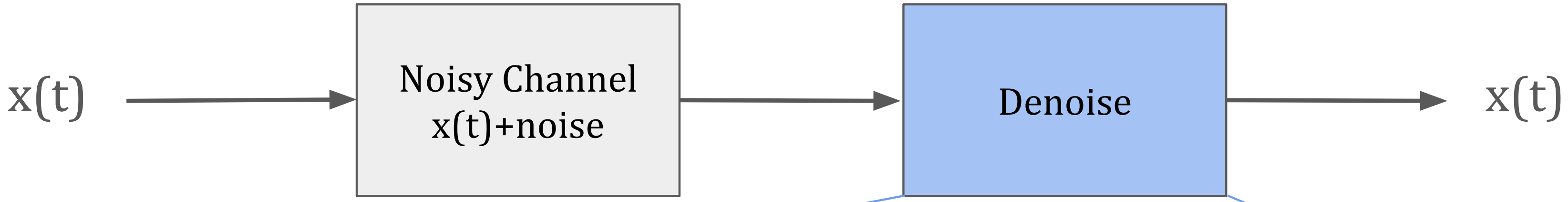
y(t)



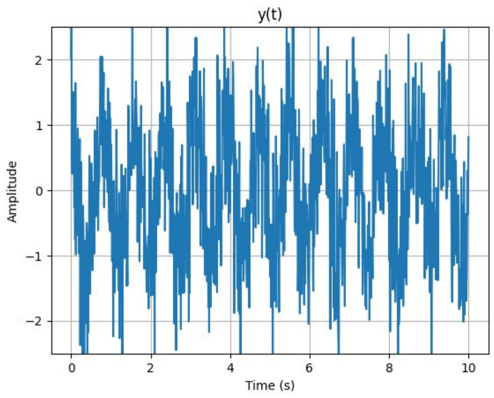


CYU: Noise and Filtering

What is the strategy to remove this noise?

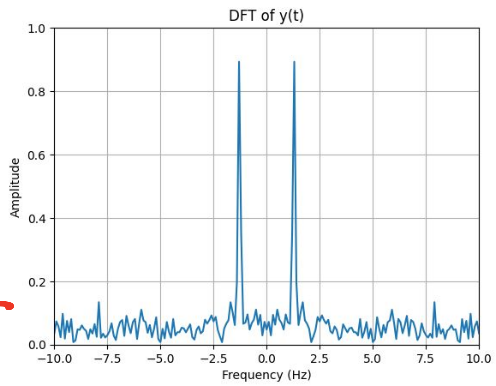


y(t)



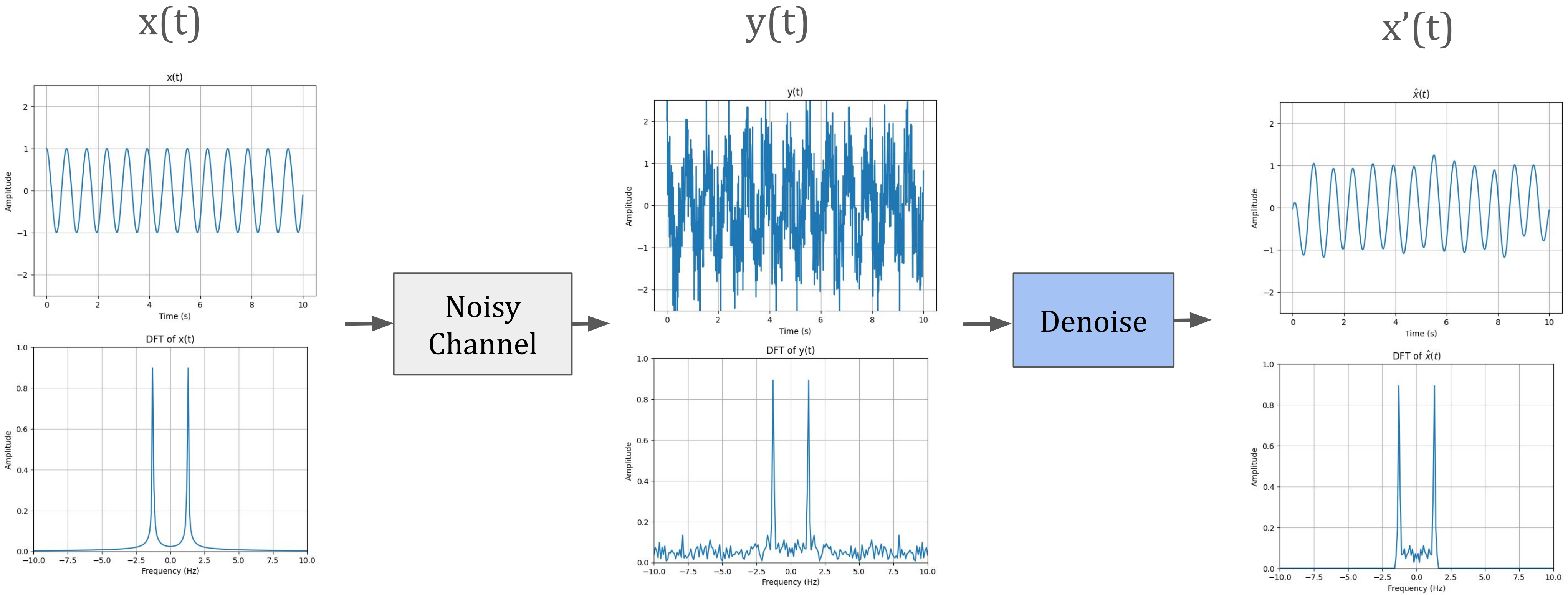
$* \ h(t) =$

$\otimes \ H(j\omega) =$



CYU: Noise and Filtering

What is the strategy to remove this noise?



CYU: Noise and Filtering

Why will the approach that we took not work in the real world?

ideal LPF: (1) Sinc with infinite values in time domain

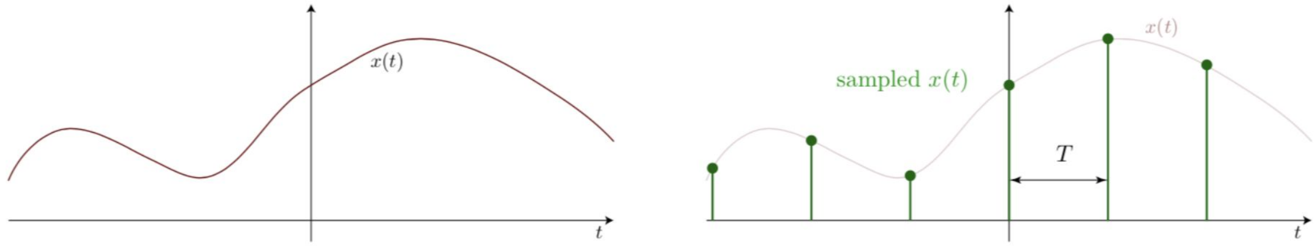
(2)

The Ideal LPF is infinite length and not causal

Sampling

Motivation

In reality, we could never store a continuous time signal. Instead, as we see in `MATLAB`, we store the signal's value at various times. This is called sampling, as illustrated below.



A key variable of interest is the sampling frequency, i.e., the time in between our samples, denoted T in the above diagram.

This is related to discrete signals, i.e., $x[n] = x(nT)$.

Sampling

How to sample a continuous signal?

How do we sample a continuous signal? You may have several intuitions to do so already using the $\delta(t)$ signal and its property that $f(t)\delta(t) = f(0)\delta(t)$.

- We will arrive at sampling by first studying a related problem: the Fourier transform of periodic signals.
- The reason we approach this is that Fourier series are discrete coefficients, c_k, while the Fourier transform is typically some continuous signal. i.e., it seems like there may be a relationship whereby the Fourier series is like a sampled Fourier transform.
- So we ask: what is the relationship between the Fourier series and the Fourier transform?
- To see this, we can begin by identifying the relationship between the Fourier series and the Fourier transform.

$\mathcal{F}$ [Fourier Series]

F.T. of Periodic Signal

We cannot directly take the Fourier transform of a periodic signal, since they do not have finite energy. However, we can use a few tricks (like in the Generalized Fourier Transform lecture) to calculate the FT of a periodic signal.

Let $f(t)$ have a Fourier series (with period $T_0 = \omega_0/2\pi$)

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with

$$c_k = \frac{1}{T_0}\int_0^{T_0} f(t) e^{-jk\omega_0 t} dt$$

There's a close relationship between the two, as the Fourier series equation looks like the Fourier transform equation but with a $\sum$ instead of an $\int$.

F.T. of Periodic Signal

Fourier transform of the Fourier series representation

Let's take the Fourier transform of the Fourier series representation.

$$\mathcal{F}[f(t)] = \mathcal{F}\left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right]$$

$$= \sum_k c_k \mathcal{F}\left[e^{jk\omega_0 t}\right]$$

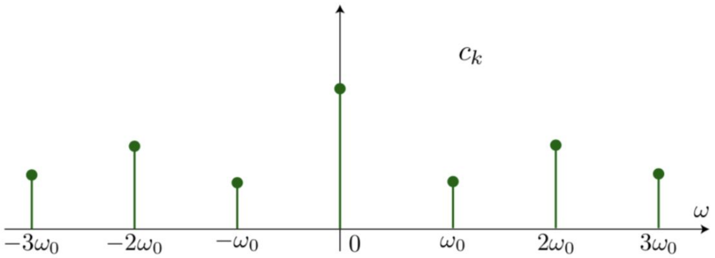
$$= \sum_k c_k 2\pi\, \delta(\omega - k\omega_0)$$

F.T. of periodic signal

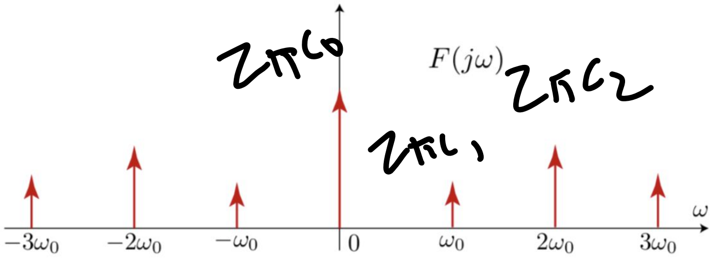
Fourier transform of the Fourier series (cont.)

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \Longleftrightarrow \sum_{k=-\infty}^{\infty} c_k 2\pi\delta(\omega - k\omega_0) = F(j\omega)$$

Fourier series



Fourier transform



F.T. of periodic signal

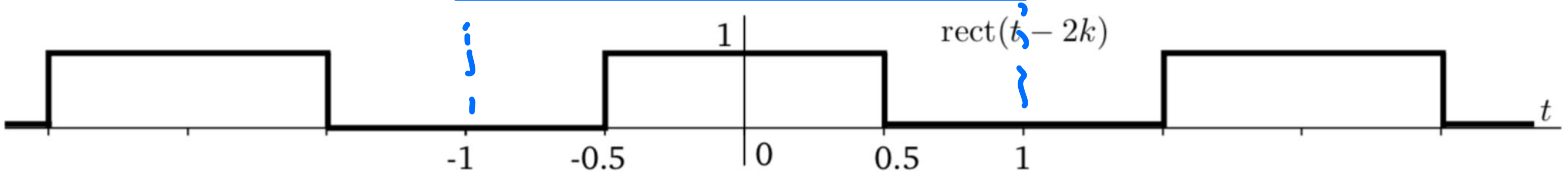
Example: square wave

Consider the square wave below:

$$f(t) = \sum_{k=-\infty}^{\infty} \mathrm{rect}(t - 2k)$$

$T=2$

This is illustrated below.



$T=2$

$\omega_0=\pi$

F.T. of Periodic Signal

In the Fourier series lecture (slide 8-32), we calculated that the Fourier series of this signal is

$$c_k = \frac{1}{2}\,\mathrm{sinc}(k/2)$$

$T = 2$

$\omega_0 = \pi$

What is its Fourier transform?

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \iff \sum_{k=-\infty}^{\infty} c_k 2\pi \underline{\delta(\omega - k\omega_0)}$$

$$F(j\omega) = \sum_k \frac{1}{2}\,\mathrm{sinc}\left(\frac{k}{2}\right) 2\pi\, \delta(\omega - k\omega_0)$$

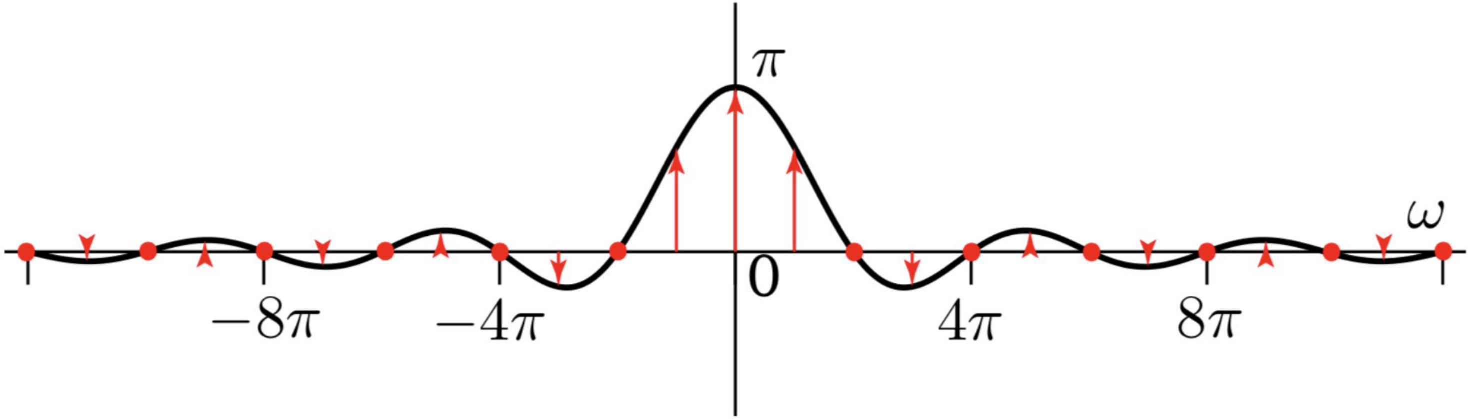
$$= \sum_k \pi\, \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) \delta(\omega - k\omega_0)$$

$$\delta(\omega - k\omega_0) = \delta(\omega - k\pi)$$

F.T. of Periodic Signal

Hence, the Fourier transform of the square wave is the Fourier transform of a rect multiplied by evenly spaced δ's, i.e.,

$$F(j\omega) = \pi \sum_{k=-\infty}^{\infty} \mathrm{sinc}(\omega/2\pi)\delta(\omega - k\pi)$$



Impulse Train

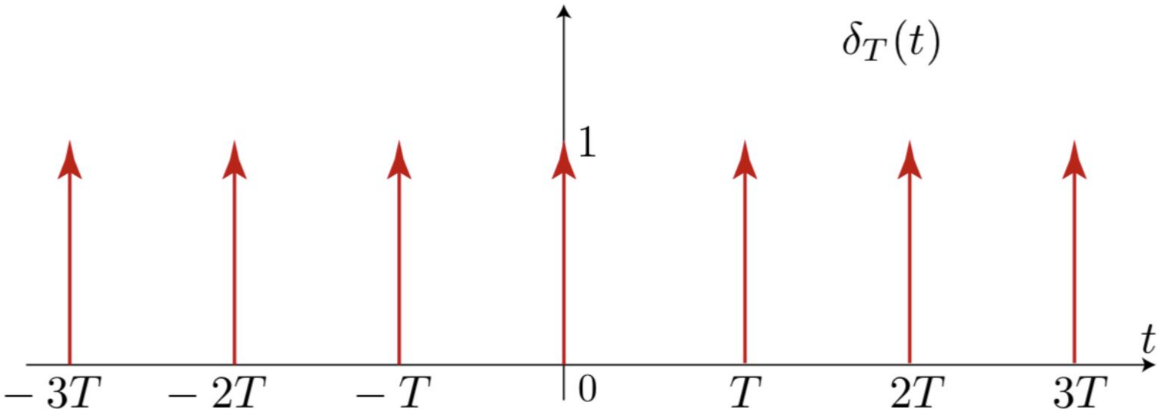
To simplify notation here, we can define an *impulse train*, which ends up being our sampling function. We let $\delta_T(t)$ be a sequence of unit δ functions spaced by T.

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Impulse Train

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

This is illustrated below.



With this, we can write the Fourier transform of the square wave as

$$F(j\omega) = \pi \operatorname{sinc}(\omega/2\pi)\delta_\pi(\omega)$$

Impulse Train

The impulse train may have been your first thought when thinking of how to sample a signal every T.

Indeed, this signal has very important qualities. Let's start off with a simple question: intuitively, what is the Fourier transform of a impulse train?

Guess CYU: What's the Fourier Transform of Impulse train

No need to calculate - please think about square wave example and hazard a guess.

Impulse Train F.T.

Let's think through this using our square wave example.

- We know that the Fourier transform of the square wave is a sinc multiplied by $\delta_\pi(\omega)$.
- From the convolution theorem, this means that the inverse Fourier transform (i.e., the square wave) is the inverse Fourier transform of a sinc (i.e., a rect) convolved with the inverse Fourier transform of a impulse train.
- We know that a square wave is simply a rect repeated over and over again, i.e., convolved with a impulse train.
- So intuitively, by duality, the Fourier transform of a impulse train should be a impulse train.

Note, we will sometimes use the term 'delta train' to describe an impulse train.

F.T. of Impulse Train

Let's check our intuition and compute the Fourier transform of an impulse train. To do so, we'll use our trick of finding the Fourier series of the (periodic) impulse train, and then multiplying by $2\pi\delta(\cdot)$.

F.T. of Impulse Train

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