EE102

Lecture 4

EE102 Announcements

- **Syllabus link** is on BruinLearn

Slide Credits: This lecture adapted from Prof. Jonathan Kao (UCLA) and recursive credits apply to Prof. Cabric (UCLA), Prof. Yu (Carnegie Mellon), Prof. Pauly (Stanford), and Prof. Fragouli (UCLA)

$f(\vec{x}) \to \vec{y}$

What is a system?

A system transforms an *input signal*, $x(t)$, into an output signal $y(t)$.

Input signal

$$x \longrightarrow \boxed{S} \longrightarrow y$$

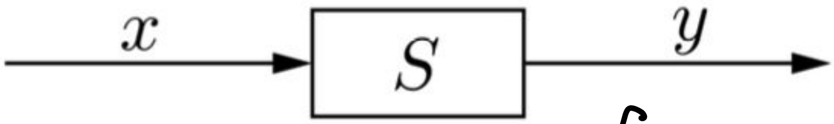
Output Signal

System

Very technically, this is called a functional

- Systems, like signals, are also *functions*. However, their inputs and outputs are signals.
- Systems can have either single or multiple inputs (SI or MI, respectively) and single or multiple outputs (SO and MO). In this class, we focus on *single input, single output* systems (SISO).

Systems have Properties



$[0\ \ 3\ \ 1] \to \boxed{S} \to [\infty\ \ \infty\ \ \infty]$

Stability

English ||

A system is *bounded-input, bounded-ouput* (BIBO) stable if every bounded input leads to a bounded output.

Math ||

$$|x(t)| < \infty \implies |y(t)| < \infty \quad \forall t$$

Causality

English ||

A system is causal if its output only depends on past and present values of the input.

Math || S is causal if $y(t)$ depends $x(t), x(t-1), x(t-2), \ldots \ \forall t$

Systems have Properties

Time-invariance

English

A system is *time invariant* if a time shift in the input only produces an identical time shift of the output.

Math

Mathematically, a system S is time-invariant if

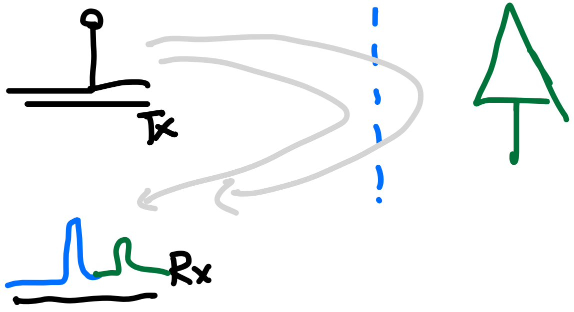
$$y(t) = S(x(t))$$

implies that

$$y(t-\tau) = S(x(t-\tau))$$

$$x(t) \rightarrow \boxed{\text{Delay}} \rightarrow x(t-\tau) \rightarrow \boxed{S} \rightarrow y(t-\tau)$$

$$x(t) \rightarrow \boxed{S} \rightarrow y(t) \rightarrow \boxed{\text{Delay}} \rightarrow y(t-\tau)$$



t_1: $\$100 \rightarrow \boxed{\text{SPY}} \rightarrow \$110$

t_2: $\$100 \rightarrow \boxed{\text{SPY}} \rightarrow \$90$

Assessing Time Invariance (CYU #1)

Is the following system time-invariant?

$$y(t) = x(t)^2$$

shift input

shift output

$\stackrel{?}{=}$

$$x(t-\tau) \rightarrow \boxed{S_1} = [x(t-\tau)]^2$$

$$y(t-\tau) = x(t-\tau)^2$$

Assessing Time Invariance (CYU #2)

Is the following system time-invariant?

$$y(t) = x(t)t$$

Shift Input $= x(t-5)t$

Shift Output $= x(t-5)(t-5)$

Not T.I.

Assessing Time Invariance (CYU # ~~2~~ 3)

Is the following system (AM-radio) time-invariant?

$$y(t) = x(t)cos(\omega_c t)$$

Shift Input = $x(t-\tau)\cos(\omega_c t)$

Shift Output = $x(t-\tau)\cos(\omega_c(t-\tau))$

NOT T.I.

Linearity

A system is *linear* if the following two properties hold:

$x \to \boxed{S} \to y$

$ax \to \boxed{S} \to ay$

1. **Homogeneity**: for any signal, x, and any scalar a,

$$S(ax) = aS(x)$$

2. **Superposition**: for any two signals, x and $\tilde{x}$,

$$S(x + \tilde{x}) = S(x) + S(\tilde{x})$$

$$S(ax + b\tilde{x}) = aS(x) + bS(\tilde{x})$$

$$\forall a, b, x, \tilde{x}$$

// Combines Homogeneity & Superposition

Assessing Linearity (CYU #1)

Is the AM radio system from before linear?

$$y(t) = x(t)\cos(\omega_c t)$$

Goal: Show $\underbrace{AM\left(a x(t) + b\tilde{x}(t)\right)}_{LHS} = \underbrace{a\,AM\left(x(t)\right) + b\,AM\left(\tilde{x}(t)\right)}_{RHS}$

LHS $\left[a x(t) + b\tilde{x}(t)\right]\cos(\omega_c t)$

$a x(t)\cos(\omega_c t) + b\tilde{x}(t)\cos(\omega_c t)$

$a\,AM\left[x(t)\right] + b\,AM\left[\tilde{x}(t)\right] = RHS$

~~Signal~~ System Assessing Linearity (CYU #2)

Is the integrator ~~signal~~ linear?

$$S(x(t)) = \int_{-\infty}^{t} x(\tau)d\tau$$

Check $\underbrace{S[a x(t) + b\tilde{x}(t)]}_{\text{LHS}} = \underbrace{a S[x(t)] + b S[\tilde{x}(t)]}_{\text{RHS}}$

$$\int_{-\infty}^{t} a x(\tau) + b\tilde{x}(\tau)\, d\tau = a \int_{-\infty}^{t} x(\tau) + b \int_{-\infty}^{t} \tilde{x}(\tau)\, d\tau$$

$$= a S[x(t)] + b S[\tilde{x}(t)]$$

$$= \text{RHS}$$

Yes, system is liner

Linearity and time-invariance recap

	Linear?	Time-In
1. $y(t) = \sqrt{x(t)}$	No	Yes
2. $y(t) = x(t) \cdot z(t)$ for $z(t)$ non-zero	Yes	No
3. $y(t) = x(at)$	Yes	No
4. $y(t) = x(t-5)$	Yes	Yes
5. $y(t) = x(5-t)$	Yes	No

Memory

A system has *memory* if its output depends on past or future values of the input. If the output depends only on present values of the input, the system is called *memoryless*.

Memory?

AM Radio: $y(t) = x(t)\cos(\omega_c t)$ Memoryless

Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ Memory

Invertibility

$x \rightarrow \boxed{S} \rightarrow y \rightarrow \boxed{S^{inv}} \rightarrow x$

A system is called *invertible* if an input can always be exactly recovered from the output. That is, a system S is invertible if there exists an S^{inv} such that

$$x = S^{\text{inv}}(S(x))$$

$$= S^{inv}[y]$$

Sq. $y(t) = [x(t)]^2$ Not Invertible b/c we can't get x.

Diff. $y(t) = \frac{dx(t)}{dt}$ Not Invertible b/c of "$+C$" constant

Scalar $y(t) = ax(t)$ for $a \neq 0$ Invertible $x(t) = \frac{y(t)}{a}$

Invertibility (CYU)

Suppose there exists a system S, which is linear and invertible with inverse S_{inv}. Is S_{inv} also linear?

(1) $$S(ax_1 + bx_2) = aS(x_1) + bS(x_2)$$

(2) $$S^{-1}[S[x]] = S[S^{-1}[x]] = x$$

Goal: $$\underbrace{aS^{-1}(x_1) + bS^{-1}(x_2)}_{\text{LHS}} = \underbrace{S^{-1}(ax_1 + bx_2)}_{\text{RHS}}$$

$$aS^{-1}(x_1) + bS^{-1}(x_2) = S^{-1}[S[aS^{-1}(x_1) + bS^{-1}(x_2)]]$$ Using Eq. 1

$$= S^{-1}[aS[S^{-1}(x_1)] + bS[S^{-1}(x_2)]]$$ Using Eq. 2

$$= S^{-1}[ax_1 + bx_2] = \text{RHS}$$

Impulse Response

System impulse response

This lecture introduces time-domain analysis of systems, including the impulse response. It also discusses linear time-invariant systems. Topics include:

- Impulse response definition
- Impulse response of LTI systems
- The impulse response as a sufficient characterization of an LTI system
- Impulse response and the convolution integral

Impulse Response (cont.)

Why do we need the impulse response?

Types of Responses

Impulse Response Definition

$$h(t, \tau) = H(\delta(t - \tau))$$

There are important things to be careful of when looking at this equation.

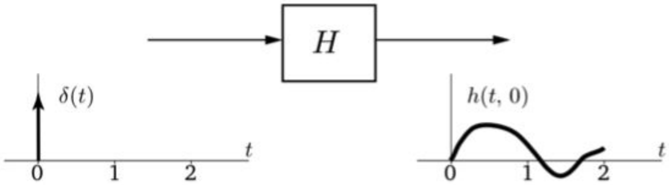
- The t on the left and right hand side of these equations *are not the same*!
- The t on the left hand side is the impulse response at a specific value of time.
- The t on the right hand side varies across all time.
- The output at the specific time t on the left will depend on the input at several times t on the right.

Notation on t

$$h(t, \tau) = H(\delta(t - \tau))$$

There are important things to be careful of when looking at this equation.

An example of these t's not being the same is shown below. In this example, let $\tau = 0$.



It may be tempting to write:

$$h(1, 0) = H(\delta(1))$$

This is wrong.

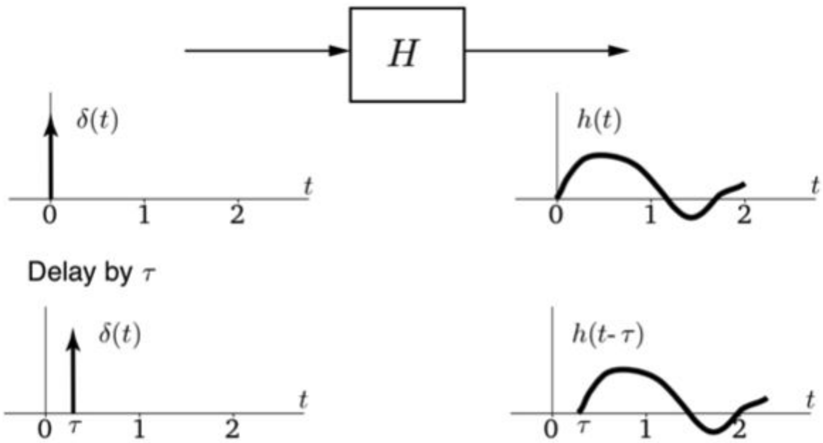
- On the left, $\delta(1) = 0$. We know if H is linear, then $H(0) = 0$, implying that $h(1, 0) = 0$.
- But in general, the impulse response can be non-zero, i.e., $h(1, 0) \neq 0$ in the above diagram, if the impulse response produces some non-zero response.

Time invariant Impulse Response

Time Invariant Impulse Response

Impulse response of a time-invariant system (cont.)

This property of the impulse response for a time-invariant system is drawn below:



Important Fact about the Impulse Response

FACT: If H is an LTI (linear time-invariant system) with impulse response

$$h(t) = H(\delta(t))$$

then we can calculate $H(x(t))$ for ANY $x(t)$ **IF** we know $h(t)$.

This is a ***very important* result.**