EE102

Lecture 17

ABET Learning Outcomes

- ✔ Understand the concept of a signal and a system, plot continuous-time signals, evaluate the periodicity of a signal.
- ✔ Identify properties of continuous-time systems such as linearity, time-invariance, and causality.
- ✔ Calculate with the Dirac delta function.
- ✔ Compute convolution of continuous-time functions.
- ✔ Understand the concept of the impulse response function of a linear system, and its use to describe the input/output relationship.
- ✔ Compute the Laplace transform of a continuous function, identify its domain of convergence, and be familiar with its basic properties, including the initial and final value theorems.
- ✔ Find the inverse Laplace transform by partial fractions.
- ✔ Use the Laplace transform to solve constant-coefficient differential equations with initial conditions
- ✔ Use the Laplace transform to evaluate the transfer function of linear time-invariant systems.
- ✔ Understand Parseval's relation in Fourier series, and its interpretation in terms of decomposing the signal's energy between its harmonics
- ✔ Evaluate the response of a linear time-invariant system to periodic inputs.
- ✔ Evaluate the Fourier transform of a continuous function, and be familiar with its basic properties. Relate it to the Laplace transform.
- ✔ Evaluate and plot the frequency responses (magnitude and phase) of linear time-invariant systems, and apply it to filtering of input signals.
- ✔ Understand conditions under which a band-limited function can be recovered from its samples

Inversion of Laplace Transform

Motivation

The inverse of the Laplace transform is given by

$$f(t) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} F(s) e^{st} ds$$

where σ is large enough that F(s) is defined for $\Re(s) \geq c$.

Catalog of Inverse Transforms to Keep Handy

May seem specific

But come up in Diff Eq!

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{a+s}$$

$$\begin{aligned}
\mathcal{L}[\cos(\omega t)] &= \frac{s}{s^2+\omega^2} \\
\mathcal{L}[\sin(\omega t)] &= \frac{\omega}{s^2+\omega^2} \\
\mathcal{L}[e^{-at}\cos(\omega t)] &= \frac{(s+a)}{(s+a)^2+\omega^2} \\
\mathcal{L}[e^{-at}\sin(\omega t)] &= \frac{\omega}{(s+a)^2+\omega^2}
\end{aligned}$$

$$\mathcal{L}^{-1}\left[\frac{r}{(s-\lambda)^k}\right] = \frac{r}{(k-1)!}t^{k-1}e^{\lambda t}$$

Partial Fractions

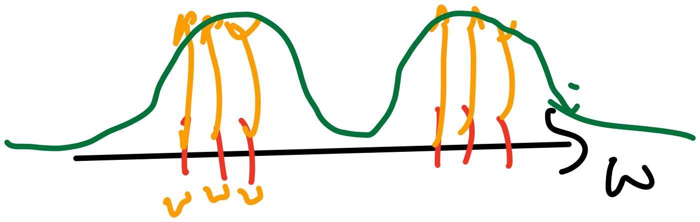
Partial fraction expansion

Let

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

b(s) has m roots

a(s) has n roots



roots of b(s) are "zeros"

roots of a(s) are "poles".

Partial Fractions

Partial fraction expansion

Let

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

Let's first assume that no poles are repeated and that $m < n$ (i.e., more poles than zeros).

No poles repeated: $\lambda_1 \neq \lambda_2 \neq \lambda_n$

Then, $F(s)$ can be written in its *partial fraction expansion*:

$$F(s) = \frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n}$$

where

- $\lambda_1, \ldots, \lambda_n$ are the poles of F.
- The numbers $r_1, \ldots, r_n$ are called residues.
- It turns out when $\lambda_k = \lambda_l^*$, then $r_k = r_l^*$.

Partial Fractions

Inversion of a partial fraction

In partial fraction form, inverting the Laplace transform is easy because

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{r_1}{s-\lambda_1} + \cdots + \frac{r_n}{s-\lambda_n}\right]$$

$$= r_1 \mathcal{L}^{-1}\left[\frac{1}{s-\lambda_1}\right] + \dots + r_n \mathcal{L}^{-1}\left[\frac{1}{s-\lambda_n}\right]$$

$$= r_1 e^{\lambda_1 t} + \dots + r_n e^{\lambda_n t}$$

Partial Fractions

How to find the partial fraction expansion

To find the partial fraction expansion, we

- Find the poles $\lambda_1, \ldots, \lambda_n$, which means we find the zeros of $a(s)$.
- Find the residues of $r_1, \ldots, r_n$.

3 Main Methods to find Partial Fractions

These are (sometimes painful) exercises in algebra.

Last Homework and Final Exam will use the least painful (cover-up method), which is sufficient to glean the concept. Supplemental lecture 17 may discuss other methods.

Partial Fractions via "Cover-up" Method

Here, we solve for each residual individually in the following way. E.g., to get r_1, we first multiply both sides by $(s - \lambda_1)$.

$$\frac{b_0 + b_1 s + b_2 s^2}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = \frac{r_1}{s-\lambda_1} + \frac{r_2}{s-\lambda_2} + \frac{r_3}{s-\lambda_3}$$

becomes

$$\frac{\cancel{(s-\lambda_1)}(b_0 + b_1 s + b_2 s^2)}{\cancel{(s-\lambda_1)}(s-\lambda_2)(s-\lambda_3)} = r_1 + \frac{r_2(s-\lambda_1)}{s-\lambda_2} + \frac{r_3(s-\lambda_1)}{s-\lambda_3}$$

Set $s = \lambda_1$

$$r_k = (s - \lambda_k) F(s)\Big|_{s=\lambda_k}$$

Concrete Example of Using the Cover-Up

eg:

Let's find the following partial fraction expansion:

$$F(s) = \frac{s^2-2}{s(s+1)(s+2)} = \frac{r_1}{s} + \frac{r_2}{s+1} + \frac{r_3}{s+2}$$

$$r_1: \; sF(s)\Big|_{s=\lambda_1} = \frac{s^2-2}{(s+1)(s+2)}\Big|_{s=0}$$

$$= \frac{-2}{2} = \boxed{-1}$$

$$r_2: \; \frac{s^2-2}{s(s+2)}\Big|_{s=-1} = \frac{1-2}{-1\cdot 1} = \boxed{1}$$

$$r_3: \; \frac{s^2-2}{s(s+1)}\Big|_{s=-2} = \frac{4-2}{-2(-2+1)} = \boxed{1}$$

$$r_k = (s-\lambda_k)F(s)\Big|_{s=\lambda_k}$$

$$\frac{-1}{s} + \frac{1}{s+1} + \frac{1}{s+2} = F(s)$$

CYU: Putting it Together

Compute the Inverse Laplace Transform

$$F(s) = \frac{s+4}{s^3+4s} = \frac{s+4}{s(s^2+4)} = \frac{r_1}{s} + \frac{r_2}{s+j2} + \frac{r_3}{s-j2}$$

Use cover-up method

$$r_1 = 1$$

$$r_2 = \frac{j-2}{4}$$

$$r_3 = \frac{-j-2}{4}$$

$$\therefore F(s) = \frac{1}{s} + \frac{1}{4}\frac{j-2}{s+2j} - \frac{1}{4}\frac{j+2}{s-2j}$$

CATALOG

ALSO IN CATALOG.

CYU: Apply it to Differential Equations

LTI system (20 points)

Assume a causal LTI system $\mathcal{S}_1$ is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = ax(t), \qquad y(0) = 0,\ y'(0) = 0$$

where a is a constant. Moreover, we know that when the input is e^t, the output of the system $\mathcal{S}_1$ is $\frac{1}{2}e^t$.

(a) (6 points) Find the transfer function $H_1(s)$ of the system. (The answer should not be in terms of a, i.e., you should find the value of a).

$$s^2Y(s) + 3sY(s) + 2Y(s) = aX(s)$$

$$H_1(s) = \frac{Y(s)}{X(s)} = \frac{a}{s^2+3s+2} = \frac{a}{(s+1)(s+2)}$$

$$e^{st} \rightarrow \boxed{} \rightarrow |H(s)|e^{st}$$

$$H_1(s)\Big|_{s=1} = \frac{1}{2} \qquad \frac{a}{(1+1)(1+2)} = \frac{1}{2} \quad \therefore \quad \boxed{a=3}$$

Recall: Laplace Helps us Solve ODEs

Let's solve the following ODE:

$$v'''(t) - v(t) = 0$$

Initial Conditions

$v(0) = 1$

$v'(0) = 0$

$v''(0) = 0$

$$\mathcal{L}[v'''(t)] = s^3V(s) - s^2v(0) - sv'(0) - v''(0)$$

$$= s^3V(s) - s^2$$

$$s^3V(s) - s^2 - V(s) = 0$$

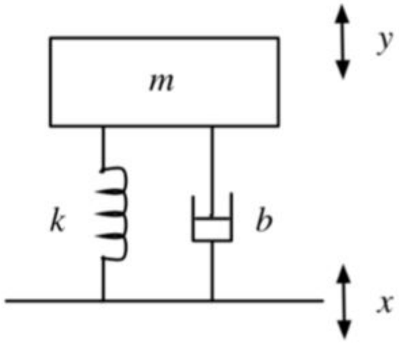
$$V(s) = \frac{s^2}{s^3 - 1} \longrightarrow \text{INV LT} \longrightarrow v(t) = \frac{1}{3}e^{t} + \frac{2}{3}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)$$

$t \geq 0$

Applying Laplace to Mass Spring Damper

Example: vehicle suspension

Consider a vehicle suspension system as illustrated below.



- input x is road height (along vehicle path);
- output y is vehicle height

(Fig acknowledgment: Prof. John Pauly)

The vehicle's dynamics, which you don't need to know how to derive, are

$$my'' + by' + ky = bx' + kx$$

Mass Spring Damper Example

Example: vehicle suspension (cont.)

Let's choose $m = 1$, $k = 2$ and $b = 3$, and say we want to solve for the step response ($x(t) = u(t)$, i.e., the car is driving onto a curb). Then,

$$\frac{Y(s)}{X(s)} = H(s) = \frac{bs + k}{ms^2 + bs + k}$$

$$= \frac{3s+2}{s^2+3s+2}$$

$$= \frac{3s+2}{(s+1)(s+2)}$$

$$\frac{Y(s)}{X(s)} = H(s)$$

$$Y(s) = H(s)\underbrace{X(s)}_{1/s}$$

Goal: Understand Step Resp.

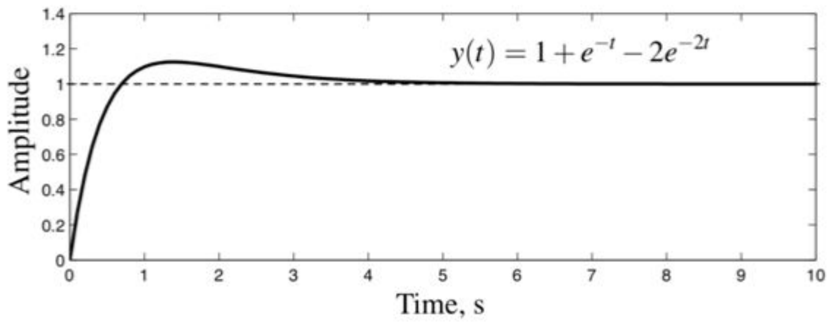
$$Y(s) = H(s)\frac{1}{s} = \frac{3s+2}{s(s+1)(s+2)} = \frac{1}{s} + \frac{1}{s+1} - \frac{2}{s+2} \Leftrightarrow y(t) = 1 + e^{-t} - 2e^{-2t}$$

This is your answer.

Mass-Spring-Damper Example

Example: vehicle suspension (cont.)

This is the step response for $m = 1, k = 2, b = 3$.



(Fig acknowledgment: Prof. John Pauly)

We can check some intuitions.

- If $x(t) = u(t)$, i.e., the level of the road changes as a step (like in a curb) then $y(t) = 1$ in steady state. This makes sense; if the road rises, the car should also rise by the same amount (i.e. 1).
- When the car first hits the curb, there is some transient. We first see finite rise time (i.e., it takes some time for the car's level to reach the curb due to the suspension system).
- After that, the car overshoots; this is a result of the spring which allows the car to displace vertically.
- Finally, there is dampening so that the vehicle's level, $y(t)$ does not keep oscillating due to the spring.

Mass-Spring-Damper Example

Example: vehicle suspension (cont.)

What if we make the spring stiffer ($k = 5$) and reduce the shock absorbance ($b = 2$)? The partial fraction expansion (please work out on your own; we used a quadratic factor) is

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \left(\frac{2s+5}{s^2+2s+5}\right)\left(\frac{1}{s}\right) \\ &= \frac{1}{s} - \frac{s}{(s+1)^2+4} \end{aligned}$$

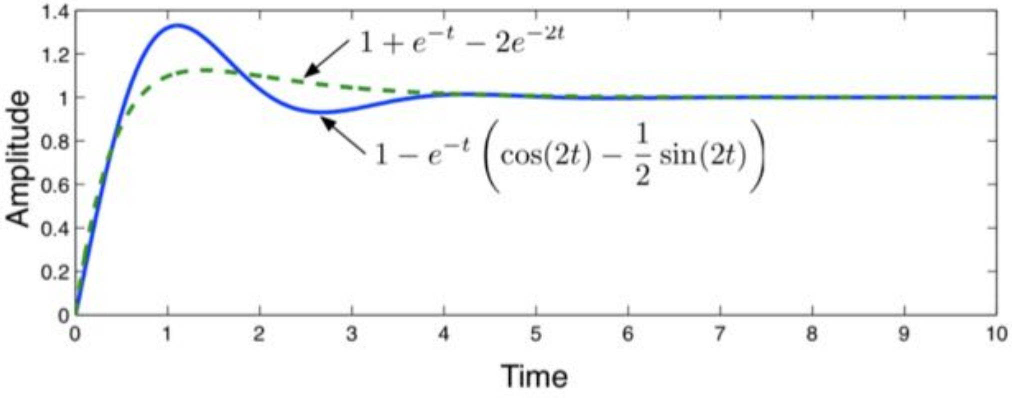
This can be further simplified (to use an inverse Laplace transform table)

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s}{(s+1)^2+4} \\ &= \frac{1}{s} - \frac{(s+1)-1}{(s+1)^2+4} \\ &= \frac{1}{s} - \frac{(s+1)}{(s+1)^2+2^2} + \left(\frac{1}{2}\right)\frac{2}{(s+1)^2+2^2} \end{aligned}$$

Vehicle Suspension (continued)

By look up table, the inverse Laplace transform is:

$$
\begin{aligned}
y(t) &= 1 - e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) \\
&= 1 - e^{-t}\left(\cos(2t) - \frac{1}{2}\sin(2t)\right)
\end{aligned}
$$



(Fig acknowledgment: Prof. John Pauly)

Here we see that due to less shock absorbance, the car vibrates more (it has greater over and undershoot) and because of the spring's stiffness increasing, it has a higher frequency of oscillation. This matches intuition.

Filtering

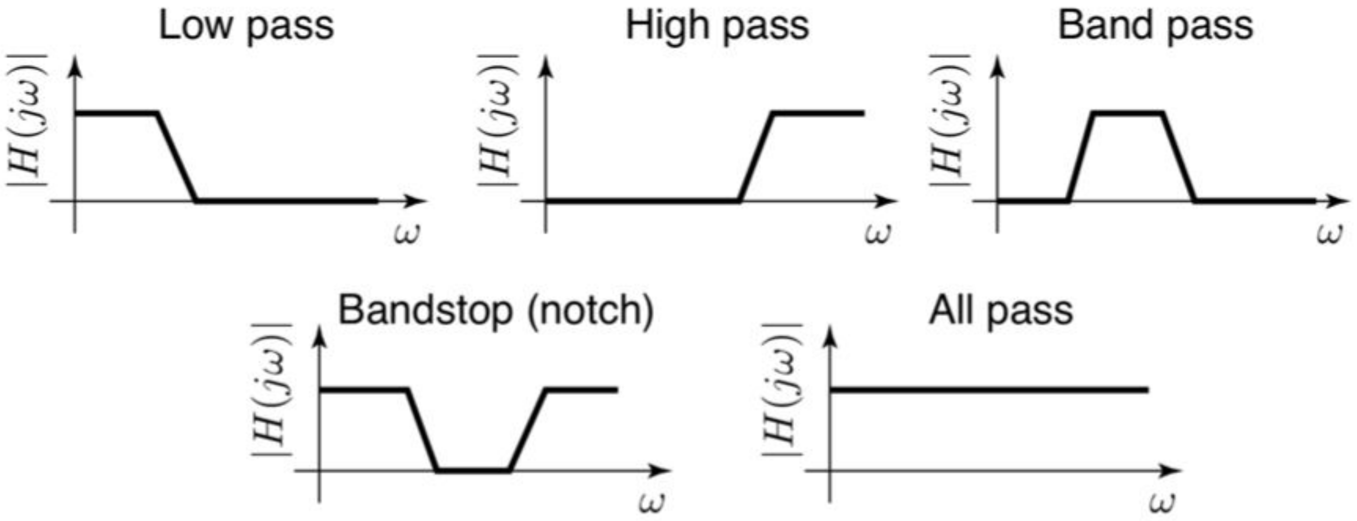
We prior talked about the *frequency response*, $H(j\omega)$. This is a special case of the Laplace transform when $\sigma = 0$ and $s = j\omega$. The frequency response completely describes an LTI system.

However, sometimes, the $j\omega$ axis may not be in the region of convergence, and so the frequency response may not exist. Further, we may be interested in *filter* design: how do we actually physically realize implementable filters?

For this, we turn to the Laplace transform. We will see that we can design filters to achieve certain qualities by choosing its poles and zeros.

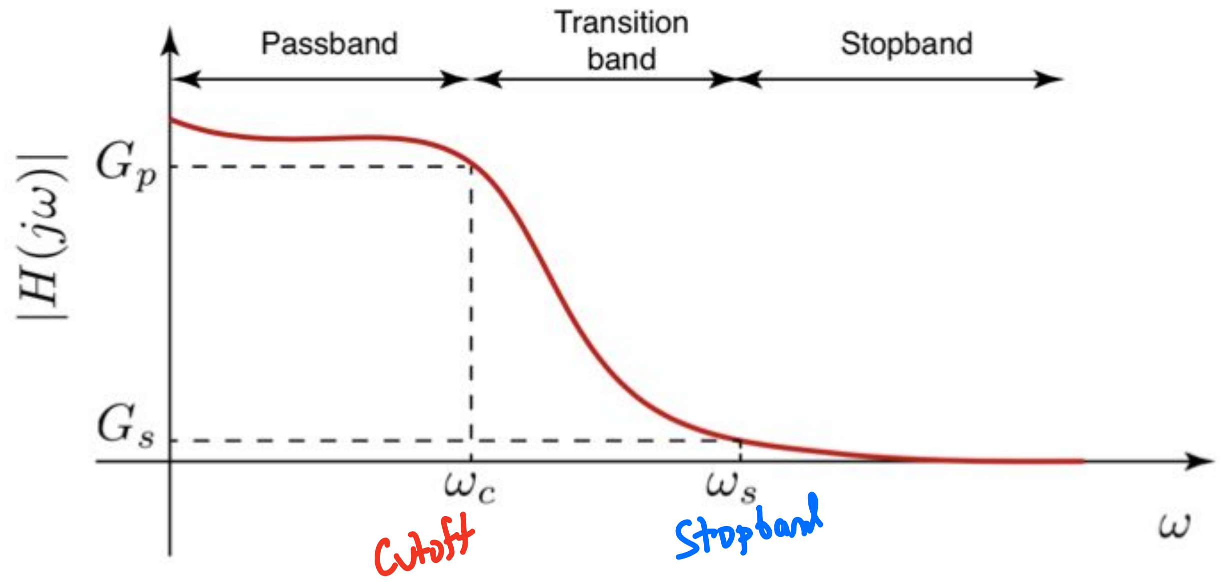
Types of Filters

We previously talked about lowpass, highpass and bandpass filters. There are a few more that you may use.



- What might a notch filter be used for?
- What might an all pass filter be used for?

Non-Ideal Filters



- G_p is the minimum passband gain. The passband is defined where the gain of the filter (i.e., $|H(j\omega)|$) is greater than G_p.
- G_s is the maximum stopband gain. The stopband gain is defined where the gain of the filter is less than G_s.
- ω_c is called the cut-off frequency, where the passband ends.
- ω_c is called the stopband frequency, where the stopband begins.

Filter Design

In prior lectures, we talked about filters in terms of their frequency response, and they were e.g., a rect or a convolution of rect's. These were ideal filters that were not implementable.

- Practically, real-time filters need to be implemented as an analog or discrete circuit.
- Circuits implement a rational function.
- Therefore, we need to analyze and design rational functions to arrive at a certain frequency response.

A rational function H can be factorized, as in the partial fraction lecture.

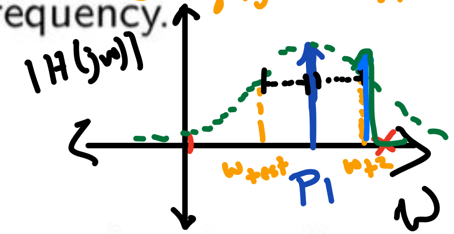
$$H(s) = \frac{b(s)}{a(s)} = k\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

where z_i are the zeros of $H(s)$ and p_i are the poles of $H(s)$.

Filter Design (cont)

With filters, we are concerned about how certain frequencies are attenuated. To do this, a common tool are Bode plots. Bode plots quantify how the magnitude and phase of the transfer function, H change with frequency.

$\Sigma \log |j\omega - p_i|$



Bode magnitude: The Bode magnitude plots $20\log_{10}|H(j\omega)|$ as a function of ω. This is therefore a log-scale view of $|H(j\omega)|$, where a change in value of 20 corresponds to $H(j\omega)$ changing by one order of magnitude. Note:

$$20\log_{10}|H(j\omega)| \quad = \quad 20\log_{10}|k| + \sum_{i=1}^{m} 20\log_{10}|j\omega - z_i| - \sum_{i=1}^{n} 20\log_{10}|j\omega - p_i|$$

and so the magnitude response decays as we move away from poles, i.e., $|j\omega - p_i|$ large, and increases as we move away from zeros, i.e., $|j\omega - z_i|$ large. Is this what we intuitively expect?

Filter Design (Cont'd)

Bode phase: The Bode phase plots $\angle H(j\omega)$ as a function of ω. Note:

$$\angle H(j\omega) = \angle k + \sum_{i=1}^{m} \angle(j\omega - z_i) - \sum_{i=1}^{n} \angle(j\omega - p_i)$$

As $\omega \gg p_i$ and $\omega \gg z_i$, each pole (with negative real part) shifts the signal by -90° while each zero (with negative real part) shifts the signal by $+90^\circ$. These signs are reversed for poles and zeros with positive real parts.

Filter Design (cont'd)

Taking the magnitude of $H(s)$, and plugging in $s = j\omega$ (to evaluate the frequency response given poles and zeros at certain locations), we arrive at

$$|H(j\omega)| = |k| \frac{|j\omega - z_1| \cdots |j\omega - z_m|}{|j\omega - p_1| \cdots |j\omega - p_n|}$$

As before, we see that

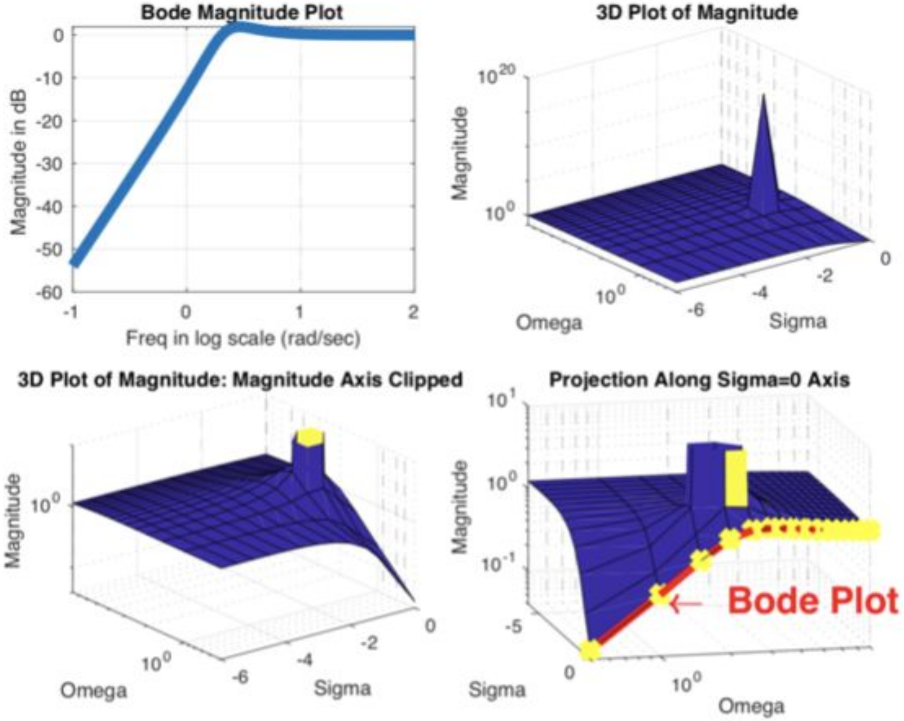
- $|H(j\omega)|$ gets big when $j\omega$ is near a pole.
- $|H(j\omega)|$ gets small when $j\omega$ is near a zero.

Filter Design (cont'd)

(This example is thanks to: `https://tinyurl.com/y75dpem5`) Consider the transfer function

$$H(s) = \frac{s^2}{s^2 + 2s + 5}$$

This has two zeros at $s = 0$ and two poles at $s = 1 \pm 2j$.



Filter Design (cont'd)

Single pole low pass filter

Consider the transfer function

$$H(s) = \frac{1}{s+a}$$

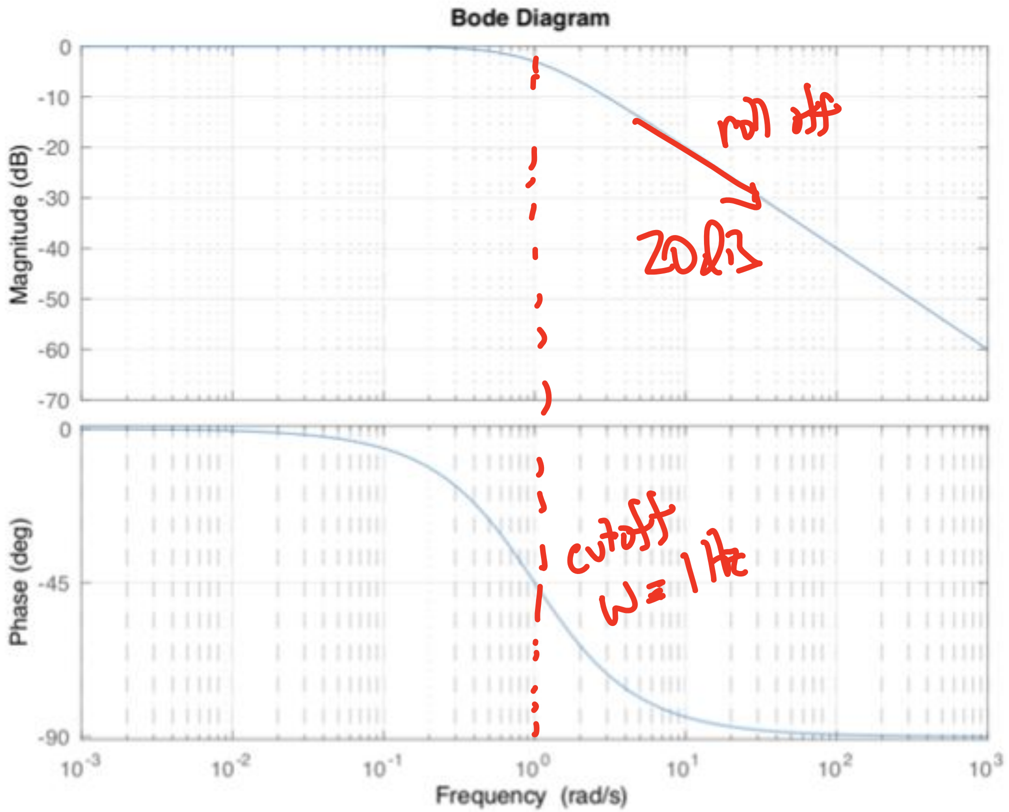
with $a > 0$. What, intuitively, should its Bode plot look like? Below is a Bode plot for $a = 1$.

Filter Design (cont'd)

Single pole low pass filter

Consider the transfer function

$$H(s) = \frac{1}{s+a}$$



Filtering

Single pole low pass filter (cont.)

What we see is that:

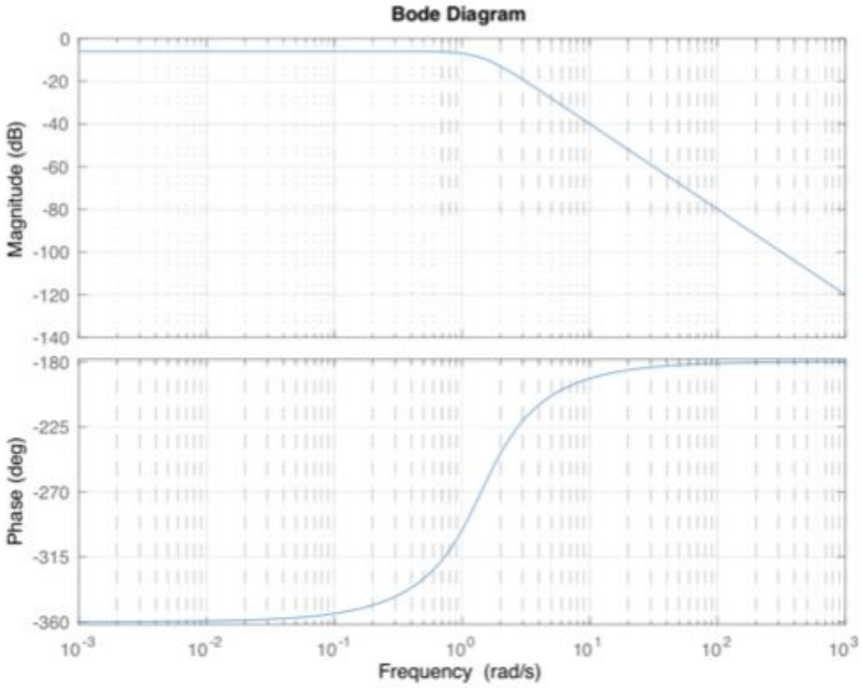
- The single pole low pass filter is far from ideal, but it does attenuate high frequencies.
- At its cut off frequency, which is $\omega_c = a$, it starts to attenuate higher frequencies.
- When $\omega_c > a$, attenuation occurs at 20 dB/decade. dB are the units decibals and decade refers to an order of magnitude of frequency. Recall each pole contributes a 20 dB attenuation every increase in order of magnitude of the frequency. Since there's only one pole, this is why the slope is -20 dB/decade.
- Also centered around $\omega_c = a$ is a frequency-dependent phase shift.

Two pole low pass filter

Consider two complex conjugate poles. Let's say our poles are at $-1 \pm j$. This has transfer function

$$H(s) = \frac{1}{s^2 - 2s + 2}$$

Its Bode plot is:



As there are two poles, the attenuation is at -40 dB/decade.

Filtering

Two pole low pass filter (cont.)

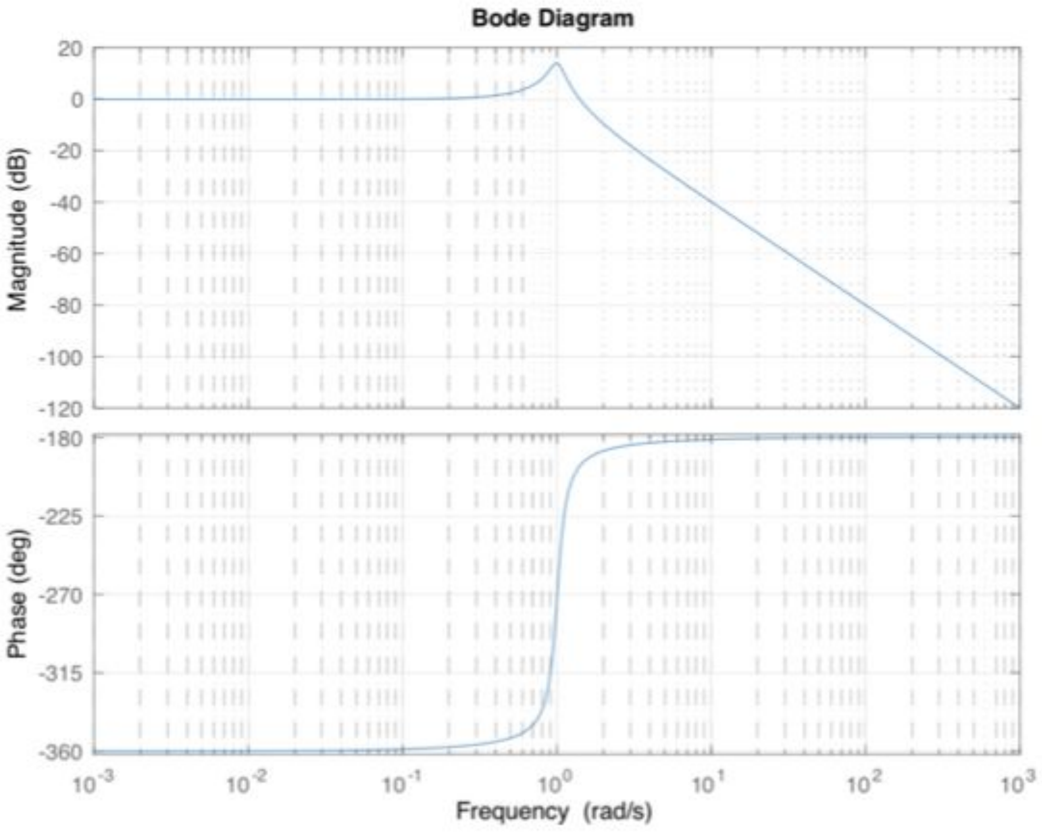
What happens if we move the poles closer to the $j\omega$ axis? Let's say our poles are at $-0.1 \pm j$. This has transfer function

$$H(s) = \frac{1}{s^2 - 0.2s + 1.01}$$

What, intuitively, should happen to the Bode plot?

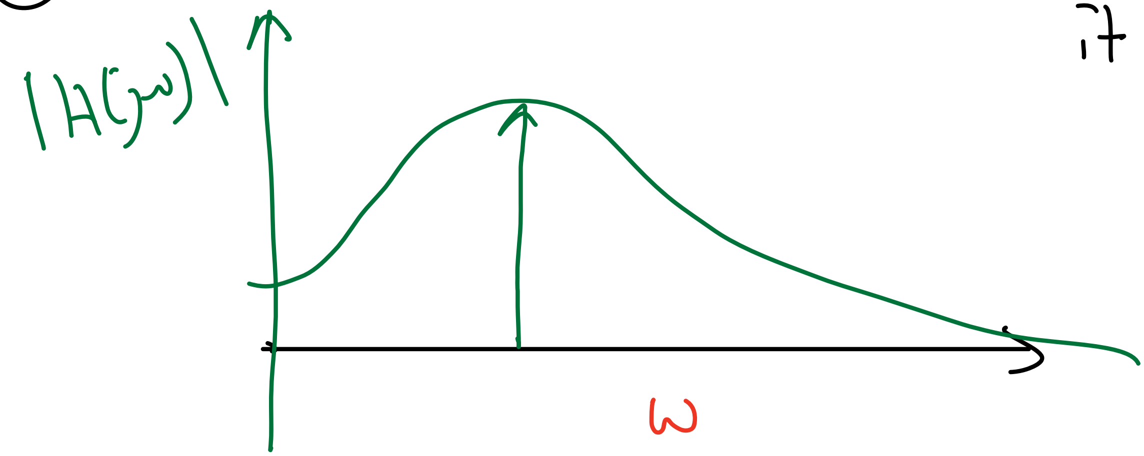
$$H(s) = \frac{1}{s^2 - 0.2s + 1.01}$$

What, intuitively, should happen to the Bode plot?

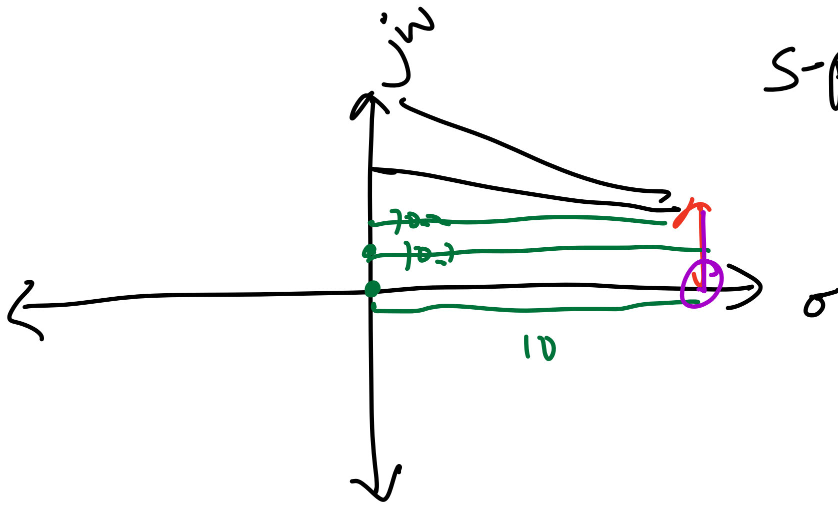


TLDR

(1) Poles raise your surface, zeros bring it down



(2)



s-plane