Universität Hamburg
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SP Signal Processing

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Digital Media Signal Processing

3. The z-Transform

Outline

Motivation

- The z-transform is a specific spectral transform
- It is particularly well suited to analyze discrete-time LTI-systems
- The convolution of two time-domain signals corresponds to a their multiplication in z-domain
- The z-transform plays the same role for discrete-time signals as the Laplace transform for continous-time signals.

Outline

Signal Processing

The z-transform of a discrete-time signal $x(n)$ is defined as the power series

$$X(z) \equiv Z\{x(n)\} \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{1}$$

with $z \in \mathbb{C}$ a complex-valued variable

- z-transform exists only for those values of z where the power series converges
- The **region of convergence (ROC)** of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value
- Thus, whenever we cite a z-transform, we should also indicate the ROC.

Examples

$$X(z) \equiv Z\{x(n)\} \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

1. $x(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$
2. $x(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$
3. $x(n) = \delta(n)$
4. $x(n) = \delta(n-k), k > 0$
5. $x(n) = \delta(n+k), k > 0$

Examples

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Solution

1. $X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$, ROC: entire z-plane, except $z = 0$
2. $X(z) = 1z^{2} + 2z^{1} + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane, except $z = 0, z = \infty$
3. $\delta(n) \overset{z}{\circ\!\!-\!\!\bullet} 1$, ROC: entire z-plane
4. $\delta(n-k) \overset{z}{\circ\!\!-\!\!\bullet} z^{-k}$, ROC: entire z-plane, except $z = 0$
5. $\delta(n+k) \overset{z}{\circ\!\!-\!\!\bullet} z^{k}$, ROC: entire z-plane, except $z = \infty$

The Region of Convergence (ROC)

- The ROC of a finite-duration signal is the entire z-plane, except possibly $z = 0$ and $z = \infty$
  - z^k $(k > 0)$ becomes unbounded for $z = \infty$
  - $z^{-k} = \frac{1}{z^k}$ $(k > 0)$ becomes unbounded for $z = 0$
- Mathematically speaking, the z-transform is simply an alternative representation of a signal.
  - The coefficient of z^{-n} is the signal value at time n

$$\begin{aligned} x(n) &= \{\dots, x(0), x(1), x(2), x(3), \dots\} \\ &\updownarrow z \\ X(z) &= \dots + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \dots \end{aligned}$$

Example 3.1.2 I

- Determine the z-transform of the signal

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

Example 3.1.2 II

- Solution:

$$x(n) = \left\{1, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \ldots, \left(\frac{1}{2}\right)^n, \ldots\right\}$$

$$X(z) = 1 + \frac{1}{2}z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \ldots + \left(\frac{1}{2}\right)^n z^{-n} + \ldots$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n$$

- Note that in general the *geometric series* converges as

$$1 + A + A^2 + A^3 + \ldots = \frac{1}{1-A}, \quad \text{if } |A| < 1$$

- Consequently for $|\frac{1}{2}z^{-1}| < 1$, i.e. $|z| > \frac{1}{2}$, $X(z)$ converges as

$$x(n) = \left(\frac{1}{2}\right)^n u(n) \quad \overset{z}{\circ\!\!-\!\!\bullet} \quad X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

z in polar form

- Let us represent the complex variable $z \in \mathbb{C}$ in polar form

$$z = re^{j\theta}$$

with $r = |z|$ and $\theta = \angle z$ Then, the z-transform results in

$$X(z)\Big|_{z=re^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n}$$

- To determine the ROC, it must hold that $|X(z)| < \infty$, with

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n} \right| \leq \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} e^{-j\theta n} \right| = \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right|$$

➡ ROC only depends on $r = |z|$

Region of Convergence (ROC) with z in polar form

$$\begin{aligned}|X(z)| &\leq \sum_{n=-\infty}^{\infty} \left|x(n)r^{-n}\right| \\ &\leq \sum_{n=1}^{\infty} |x(-n)r^{n}| \\ &\quad + \sum_{n=0}^{\infty} \left|\frac{x(n)}{r^n}\right| \\ &\overset{!}{\leq} \infty\end{aligned}$$

- from the first summand it follows that $r \overset{!}{<} r_1$
- from the second summand it follows that $r \overset{!}{>} r_2$

➡ ROC: $r_2 < r < r_1$

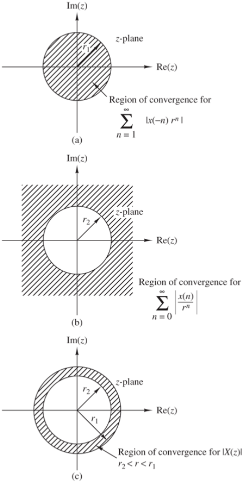

Figure 3.1.1 Region of convergence for $X(z)$ and its corresponding causa anticausal components.

Example 3.1.3

- Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Example 3.1.3

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$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- From the z-transform definition we get

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \left(\alpha z^{-1}\right)^n$$

- if $|\alpha z^{-1}| < 1$ (i.e. $|z| > |\alpha|$) this power series converges to

$$x(n) = \alpha^n u(n) \overset{z}{\circ\!\!-\!\!\bullet} X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{ROC: } |z| > |\alpha|$$

Visualization

$$x(n) = \alpha^n u(n) \overset{z}{\circ\!\!-\!\!\bullet} X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{ROC: } |z| > |\alpha|$$

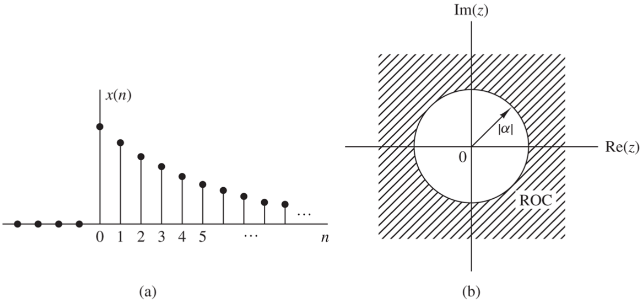


Figure 3.1.2 The exponential signal $x(n) = \alpha^n u(n)$ (a), and the ROC of its z-transform (b).

Example 3.1.4

- Determine the z-transform of the signal

$$x(n) = -\alpha^n u(-n-1)$$

- Determine the z-transform of the signal

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

$$x(n) = -\alpha^n u(-n-1) \overset{z}{\circ\!\!-\!\!\bullet} X(z) = \frac{1}{1-\alpha z^{-1}}, \quad \text{ROC: } |z| < |\alpha|$$

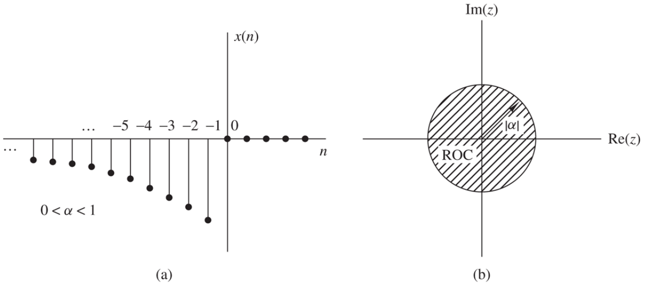


Figure 3.1.3 Anticausal signal $x(n) = -\alpha^n u(-n-1)$ (a), and the ROC of its z-transform (b).

Discussion of Examples 3.1.3 and 3.1.4

- The causal signal $\alpha^n u(n)$ and the anticausal signal $x(n) = -\alpha^n u(-n-1)$ have the exact same z-transform!

➡ The closed-form expression of the z-transform does not uniquely specify the time-domain signal

➡ The ambiguity can be resolved if the ROC is specified!

Example 3.1.5

- Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

Example 3.1.5

- Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}}$$

with ROC $|\alpha| < |z| < |b|$

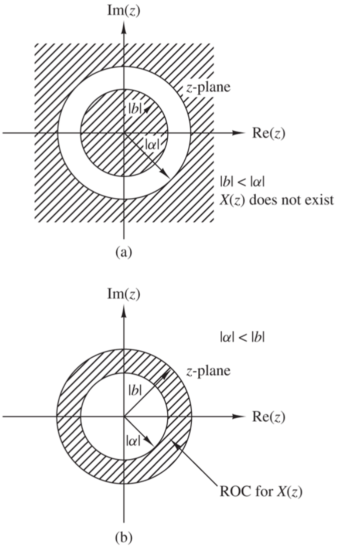
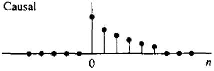
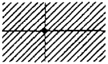
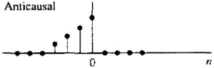
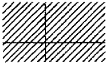
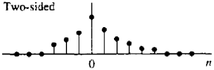
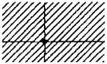
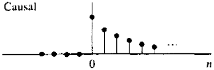
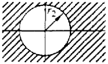
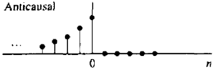
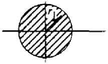
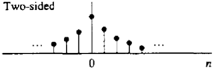
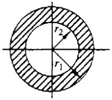


Figure 3.1.4 ROC for z-transform in Example 3.1.5.

Conclusions from the Examples

- A discrete time-domain signal is uniquely determined by its z-transform $X(z)$ and the region of convergence (ROC) of $X(z)$
- The ROC of a causal signal is the exterior of a circle of some radius r_2, while
- the ROC of an anticausal signal is the interior of a circle of some radius r_1
- If there exists a ROC for an infinite-duration two-sided signal, it is a ring in the z-plane

Signal		ROC	
Finite-Duration Signals			
Causal			Entire z-plane except $z = 0$
Anticausal			Entire z-plane except $z = \infty$
Two-sided			Entire z-plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals			
Causal			$\|z\| > r_2$
Anticausal			$\|z\| < r_1$
Two-sided			$r_2 < \|z\| < r_1$

Cauchy Integral Theorem

- The inverse z-Transform is based on the *Cauchy Integral Theorem*

$$\frac{1}{2\pi j}\oint_C z^{n-1-k}\,dz = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

The $\oint_C$ denotes an integration over a close contour within the ROC of $X(z)$

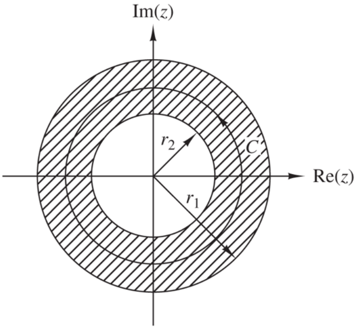


Figure 3.1.5 Contour C for integral in (3.1.13).

Derivation

see Section 3.1.2

The z-Transform and its inverse

The z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The inverse z-Transform

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- Typically, we deal with signals which have a rational z-transforms (i.e. z-transforms that are a ratio of two polynomials)
- Simpler method for inversion can be used that employs a table lookup!

Some Common z-Transform Pairs

Table 3.2

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Properties of the z-Transform

- The properties of the z-Transform can all be derived using the definition of the z-transform
- See Section 3.2 in Proakis Manolakis book

Time Shifting

$$x(n-k) \circ\!\!\stackrel{z}{-}\!\!\bullet\; z^{-k}X(z)$$

- Derivation

$$
\begin{aligned}
X(z) &= \sum_{n'=-\infty}^{\infty} x(n')z^{-n'} \\
\downarrow &\quad n' = n-k \\
&= \sum_{n=-\infty}^{\infty} x(n-k)z^{-n}z^{k} \\
X(z)z^{-k} &= \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \\
x(n-k) &\circ\!\!\stackrel{z}{-}\!\!\bullet\; z^{-k}X(z)
\end{aligned}
$$

Time Reversal

$$x(n) \circ\!\!\overset{z}{-}\!\!\bullet X(z), \qquad \text{ROC: } r_1 < |z| < r_2$$

$$x(-n) \circ\!\!\overset{z}{-}\!\!\bullet X(z^{-1}), \qquad \text{ROC: } \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

- Derivation

$$\begin{aligned} Z\{x(-n)\} &= \sum_{n=-\infty}^{\infty} x(-n) z^{-n} \\ &\quad \downarrow l = -n \\ &= \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l} \\ &= X(z^{-1}) \end{aligned}$$

Convolution of Two Sequences

$$x_1(n) * x_2(n) \quad \overset{z}{\circ\!\!-\!\!\bullet} \quad X_1(z)X_2(z)$$

- Follows from the definition of convolution
- One of the most powerful properties!

Convolution of Two Sequences

$$x_1(n) * x_2(n) \overset{z}{\circ\!\!-\!\!\bullet} X_1(z)X_2(z)$$

- Follows from the definition of convolution
- One of the most powerful properties!

Computation of the convolution $x(n) = x_1(n) * x_2(n)$

1. Compute z-transform of the signals to be convolved

$$\begin{aligned} X_1(z) &= Z\{x_1(n)\} \\ X_2(z) &= Z\{x_2(n)\} \quad \text{(time domain} \longrightarrow z\text{-domain)} \end{aligned}$$

2. Multiply the two z-transforms

$$X(z) = X_1(z)X_2(z) \quad (z\text{-domain})$$

3. Find the inverse z-transform of $X(z)$

$$x(n) = Z^{-1}\{X(z)\} \quad (z\text{-domain} \longrightarrow \text{time domain})$$

Properties of the z-Transform

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$	ROC: $r_2 < \|z\| < r_1$ ROC_1 ROC_2
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k} X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\mathrm{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\mathrm{Im}\{x(n)\}$	$\frac{1}{2}[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

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Rational z-Transforms

- As we saw from previous table, typical z-transforms are *rational functions*, i.e. ratios of polynomials in z^{-1}
- We will now show that rational z-transforms are also encountered when characterizing discrete-time LTI-systems described by difference equations

Poles and Zeros

- The **zeros** of a z-transform $X(z)$ are the values of z for which $X(z) = 0$
- The **poles** of a z-transform $X(z)$ are the values of z for which $X(z) = \infty$
- If $X(z)$ is a rational function then

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_K z^{-k}}{\sum_{k=0}^{N} a_K z^{-k}}$$

- If $a_0 \neq 0$ and $b_0 \neq 0$ we can avoid negative powers of z by factorizing as

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 z^{-M}}{a_0 z^{-N}} \frac{z^M + (b_1/b_0) z^{M-1} + \ldots + b_M/b_0}{z^N + (a_1/a_0) z^{N-1} + \ldots + a_M/a_0}$$

Rational z-Transforms II

Poles and Zeros

- Since $B(z)$ and $A(z)$ are polynomials, they can be expressed in factored form as

$$X(z) = \frac{b_0}{a_0} z^{-M+N} \frac{(z-z_1)(z-z_2)\dots(z-z_M)}{(z-p_1)(z-p_2)\dots(z-p_N)}$$
$$= \underbrace{G}_{\frac{b_0}{a_0}} z^{N-M} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

$X(z)$ has

- M finite zeros z_k (the roots of the numerator polynomial)
- N finite poles p_k (the roots of the denominator polynomial)
- $|N-M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin $z = 0$
- A zero (pole) exists at $z = \infty$ if $X(\infty) = 0$ ($X(\infty) = \infty$)
- $X(z)$ has exactly as many poles as zeros!

Rational z-Transforms

Pole-zero plot

- $X(z)$ can be represented graphically by a *pole-zero plot* in the complex plane
- poles are represented by crosses $\times$
- zeros are represented by circles $\circ$
- Obviously, by definition, the ROC should not contain any poles!

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➡ Exercise 3.3.1: Determine pole-zero plot for $x(n) = a^n u(n)$

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➡ Exercise 3.3.1: Determine pole-zero plot for $x(n) = a^n u(n)$

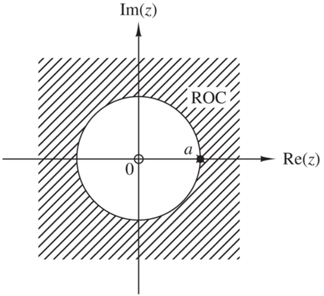

Figure 3.3.1 Pole–zero plot for the causal exponential signal $x(n) = a^n u(n)$.

Example 3.3.2

- Determine the pole-zero plot for the signal

$$x(n) = \begin{cases} a^n, & 0 \le n \le M-1 \\ 0, & \text{elsewhere} \end{cases}$$

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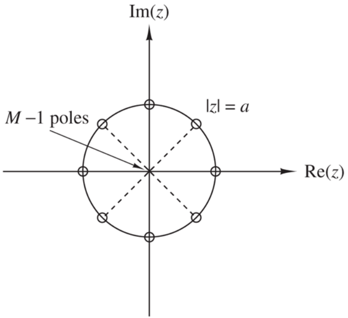

Figure 3.3.2 Pole–zero pattern for the finite-duration signal $x(n) = a^n$, $0 \le n \le M-1 (a > 0)$, for $M = 8$.

Signal Processing

Example 3.3.3

- Determine the z-transform and the signal that corresponds to the pole-zero plot of Figure 3.3.3

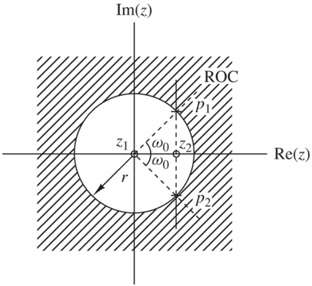

Figure 3.3.3 Pole-zero pattern for Example 3.3.3.

Preliminary conclusions from the examples

- The product $(z - p_1)(z - p_2)$ results in a polynomial with real-valued coefficients *if* p_1 *and* p_2 *are complex conjugates*
- In general, if a polynomial has real-valued coefficients, its roots are either real-valued or occur in complex-conjugate pairs

Visualization of $|X(z)|$

- Instead of the pole-zero plot, we can also represent the magnitude $|X(z)|$ as a two-dimensional surface in the complex plane, e.g.

$$X(z) = \frac{z^{-1} - z^{-2}}{1 + 1.2732z^{-1} + 0.81z^{-2}} \tag{3.3.3}$$

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➡ one zero at $z_1 = 1$, two poles at $p_1, p_2 = 0.9e^{\pm j\pi/4}$

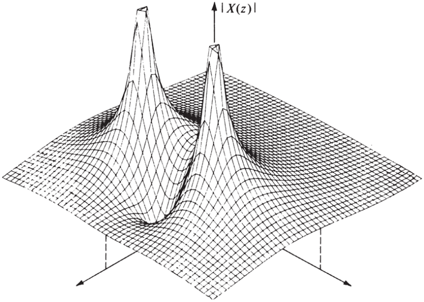

Figure 3.3.4 Graph of $|X(z)|$ for the z-transform in (3.3.3).

Single real-valued pole

- Here, we investigate the relation pole-pairs and the corresponding time-domain signal
- Here, we exclusively deal with real-valued causal signals
- We will see that the characteristic behavior of causal signals depends on whether the poles are inside $(|z| < 1)$ or outside $(|z| > 1)$, or *on* $(|z| = 1)$ the **unit circle**
- If a real-valued signal has a z-transform with one pole, this pole has to be real-valued. The only such signal is the real-valued exponential!

$$x(n) = a^n u(n) \overset{z}{\circ\!\!-\!\!\bullet} X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a|$$

having one zero at $z_1 = 0$ and one pole at $p_1 = a$ on the real axis

- see illustration on next page

Single real-valued pole

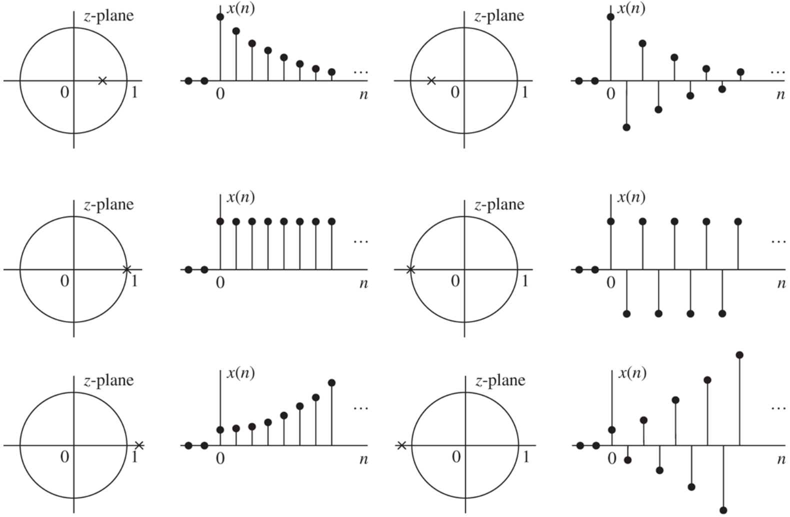

Figure 3.3.5 Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.

Pole Location and Time-Domain Behavior III

Single real-valued pole

- Signal is decaying if pole is inside the unit circle
- Signal is fixed if the pole is on the unit circle
- Signal is growing if the pole is outside the unit circle
- Negative poles results in a signal that alternates in sign.

➡ Causal signals with poles outside the unit circle become unbounded, cause overflow in digital systems and should generally be avoided

Double real-valued pole

- A causal real-valued signal with a double real-valued pole has the form (see Tables): $x(n) = na^n u(n) \;\circ\!\!\stackrel{z}{-\!\!\!-}\!\!\bullet\; \frac{az^{-1}}{(1-az^{-1})^2}$

➡ A double real pole on the unit circle results in an unbounded signal

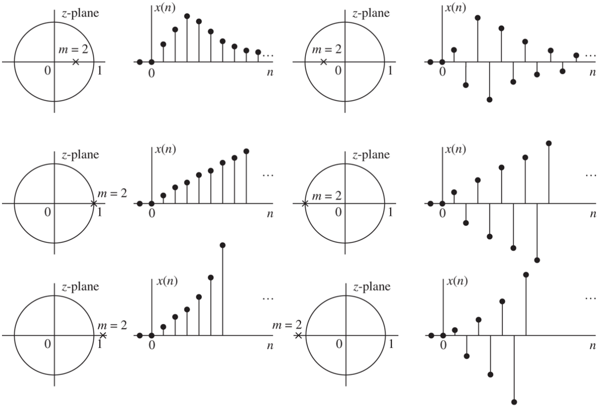

Figure 3.3.6 Time-domain behavior of causal signals corresponding to a double ($m = 2$) real pole, as a function of the pole location.

Complex conjugate poles

- Complex conjugate poles ➡ exponentially weighted sinusoidal signal

$$r^n \cos(\omega_0 n) u(n) \quad \overset{z}{\circ\!\!-\!\!\bullet} \quad \frac{1 - r z^{-1} \cos(\omega_0)}{1 - 2 r z^{-1} \cos(\omega_0) + z^{-2} r^2}$$

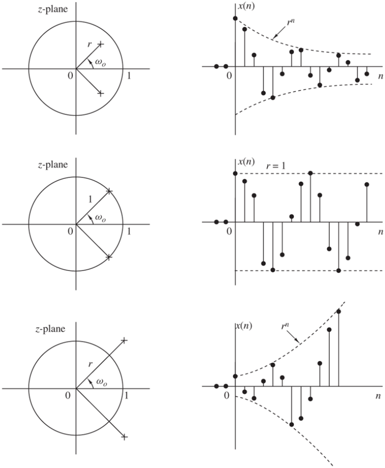

Figure 3.3.7 A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

Double conjugate poles

- Careful with double poles on the unit circle

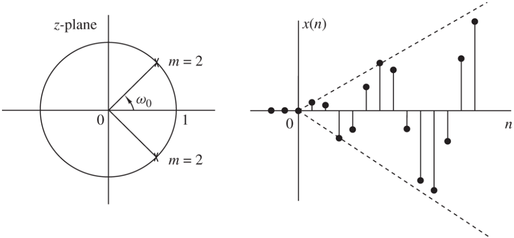

Figure 3.3.8 Causal signal corresponding to a double pair of complex-conjugate poles on the unit circle.

Summary

- Causal real-valued signals with simple real-valued poles or simple complex-conjugate pairs of poles which are inside or on the unit circle are always bounded in amplitude
- The closer poles are to the origin, the more rapidly the signal decays

➡ Time-domain behavior of a signal depends strongly on the *location of the poles relative to the unit circle*

➡ Zeros affect behavior of a signal not as strongly as poles

  - e.g. for a sinusoidal signal, the presence and location of zeros affects only its phase

- The results found for causal signals also applies to the impulse responses of causal LTI systems

  ➡ If a system has a pole outside the unit circle, the system is unstable

System Function of an LTI-System

- The output of an LTI-system to an input sequence $x(n)$ is given by

$$y(n) = x(n) * h(n) \quad \overset{z}{\circ\!\!-\!\!\bullet} \quad Y(z) = H(z)X(z)$$

- Impulse response $h(n)$ corresponds to the time-domain characterization of the system
- System function $H(z) = Z\{h(n)\}$ corresponds to the z-domain characterization of the system
- Show that: If the system is described by a linear constant-coefficient difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

the system function is a *rational function*

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the system function is a *rational function*

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

System Function of an LTI-System

All-zero systems

- $a_k = 0$ for $1 \le k \le N$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \cancel{\sum_{k=1}^{N} a_k z^{-k}}}$$

$$= \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

$$y(n) = -\sum_{k=1}^{N} \cancel{a_k y(n-k)} + \sum_{k=0}^{M} b_k x(n-k)$$

$$= \sum_{k=0}^{M} b_k x(n-k)$$

- $H(z)$ contains M zeros whose values are determined by the system parameters b_k (and Mth order pole at $z = 0$)
- called **all-zero** system or moving average (**MA**) system
- has finite-duration impulse response (**FIR**)

Signal Processing

All-pole systems

- $b_k = 0$ for $1 \leq k \leq M$

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=0}^{N} a_k z^{-k}}, \quad a_0 \equiv 1$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + b_0 x(n)$$

  - $H(z)$ contains N poles whose values are determined by the system parameters a_k (and Nth order zero at $z=0$)
  - called **all-pole** system or autoregressive (**AR**) system
  - has infinite-duration impulse response (**IIR**)

Pole-zero systems

- The general system

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

  - $H(z)$ contains N poles and M zeros b_k
  - Poles and/or zeros at $z = 0$ and $z = \infty$ are implied but not counted explicitly
  - called **pole-zero** system or autoregressive moving average (**ARMA**) system
  - has infinite-duration impulse response (**IIR**)

Signal Processing

Example 3.3.4

- Determine system function and the unit sample response of the system described by the difference equation

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

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- Determine system function and the unit sample response of the system described by the difference equation

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

- Solution:

$$H(z) = \frac{2}{1 - \frac{1}{2}z^{-1}}$$

  - pole at $z = 1/2$ and a zero at the origin.

- Using the table we obtain the inverse and thus the unit sample response

$$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

Outline

Representations of Rational z-Transforms

- Recall that if $X(z)$ is a rational function then

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The polynomials $B(z)$ and $A(z)$ can also be represented by their roots

$$X(z) = \underbrace{G}_{\frac{b_0}{a_0}} z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

  ➡ realization by a serial concatenation of sub-filters possible

- For distinct poles $p_1, p_2, \ldots, p_N$ a partial fraction expansion results in

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \ldots + \frac{A_N}{z - p_N}, \qquad A_k = \left.\frac{(z - p_k)X(z)}{z}\right|_{z=p_k}$$

  ➡ realization by a parallel sub-filters (the summands) possible
  ➡ the **inverse z-transform** can be found by using a partial fraction expansion and looking up each summand in an inversion table

Outline

- A necessary and sufficient condition for an LTI system to be BIBO stable is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

- Since

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

it follows that

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n) z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}| \overset{|z|=1}{=} \sum_{n=-\infty}^{\infty} |h(n)|$$

- in the ROC, per definition $|H(z)| < \infty$
- if unit circle $|z| = 1$ is in the ROC, then $|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)| < \infty$
  - ➡ the system is BIBO stable

Causality and Stability

Causality

An LTI system is causal, iff the ROC of the system is the exterior of a circle of radius $r < \infty$, including $z = \infty$

Stability

An LTI system is BIBO stable, iff the ROC of the system includes the unit circle

Since per definition, the ROC cannot contain any poles:

Stability for Causal Systems

A *causal* LTI system is BIBO stable, iff all poles of $H(z)$ are inside the unit circle.

Example 5.2

- An LTI system is given by

$$
\begin{aligned}
H(z) &= \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}} \\
&= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}}
\end{aligned}
$$

- Specify the ROC, determine $h(n)$, and comment on the causality/stability for the following conditions
  a) The system is stable
  b) The system is causal
  c) The system is anticausal

Stability of Second-Order Systems I

- Consider a causal two-pole system described by the 2nd order difference equation
$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n)$$
- The system function is then
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-1}} = \frac{b_0 z^2}{z^2 + a_1 z + a_2}$$
- poles are given by pq-formula
$$p_{1,2} = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}$$
- To be BIBO-stable it must hold that $p_{1,2} < 1$ (poles inside unit circle)
  - $a_1 = -(p_1 + p_2)$
    $a_2 = p_1 p_2$

- ➡ $|a_2| = |p_1 p_2| < 1$
  $|a1| < 1 + a_2$
- ➡ This defines a region in the coefficient plane (a_1, a_2) which is in the form of a triangle, the **stability triangle**

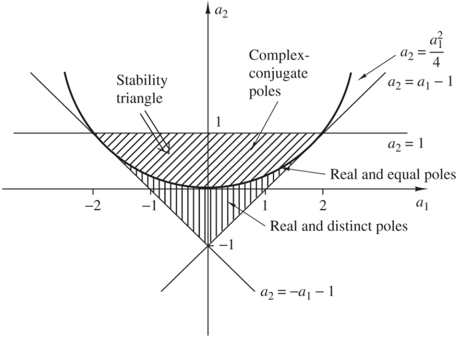

Figure 3.5.1 Region of stability (stability triangle) in the (a_1, a_2) coefficient plane for a second-order system.

Outline

Summary

- Continuous-time equivalent for z-transform is the Laplace transform
- The convolution of two time-domain signals results in the multiplication of their z-transforms
- For LTI-systems the input output relation is given by $Y(z) = H(z)X(z)$, with the system function $H(z) = Z\{h(n)\}$
- Many signals of practical interest have rational z-transforms
- LTI-systems described by constant-coefficient linear difference equations also possess rational system functions $H(z)$
- Inverses of rational z-transforms can be found by table look-ups (inverse depends on ROC!)
- Stability and causality of a system depends on position of poles!