Universität Hamburg
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SP
Signal Processing

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Digital Media Signal Processing

4. Frequency Analysis of Signals

Outline

1. Introduction of Basic Concepts

2. Discrete-Time Signals and Systems

3. The z-Transform and Its Applications

4. Frequency Analysis of Signals

4.1 Frequency Analysis of Continuous-Time Signals
4.2 Frequency Analysis of Discrete-Time Signals
4.3 Frequency-Domain and Time-Domain Signal Properties
4.4 Properties of the Fourier Transform for Discrete-Time Signals

Motivation

- When signals are decomposed in terms of sinusoidal (or complex exponential) components, the signals is said to be represented *in the frequency domain*
- for periodic signals this is achieved by the *Fourier series*
- for finite energy signals this is achieved by the *Fourier transform*
- a linear system only modifies amplitude and phase of a sinusoid (no sinusoids are added)

➡ Important technique to analyse signals and systems

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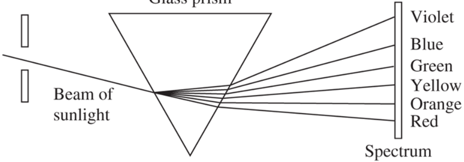
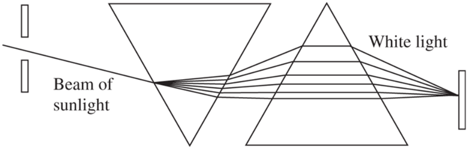
4.4 Properties of the Fourier Transform for Discrete-Time Signals

- A prism breaks up white light into the colors of the rainbow
- Newton used the term *spectrum* for the resulting bands of colors
- Each color corresponds to a specific frequency of the visible spectrum

➡ Prism: frequency decomposition or *spectral analysis* of the light



(a)



(b)

Fourier Series for Continuous-Time Periodic Signals I

- Examples of periodic signals: sinusoid, complex exponentials, square waves, triangle waves, ...
- Fourier series: decompose signal into linear weighted sum of harmonically related sinusoids (or complex exponentials)
- Jean Baptiste Joseph Fourier (1768–1830)

- Reconstruct periodic signal by combination of harmonically related complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$$

  - Fundamental period $T_p = 1/F_0$
- The complex exponentials $e^{j2\pi kF_0 t}$ are the basics “building blocks” to construct a periodic signal
- Question is: how to compute the Fourier series coefficients c_k
- Fourier coefficients c_k for a periodic signal $x(t)$ can be obtained integrating over the kth exponential (derivation in Sec. 4.1.1)

Frequency Analysis of Continuous-Time Periodic Signals: Fourier series

Synthesis equation: $$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$$

Analysis equation: $$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi kF_0 t} dt$$

➡ Periodic signals have discrete spectra

Real-valued signals

- In general Fourier coefficients c_k are complex-valued
- Show that: If the periodic signal is real-valued, c_k and c_{-k} are complex conjugates, i.e.
  - $c_k = |c_k| e^{j\theta_k}$ ➡ $c_{-k} = |c_k| e^{-j\theta_k}$
- As a consequence for real-valued signals, the Fourier expansion can be written as

$$
\begin{aligned}
x(t) &= c_0 + 2 \sum_{k=1}^{\infty} |c_k| \cos(2\pi k F_0 t + \theta_k) \\
&= a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(2\pi k F_0 t) - b_k \sin(2\pi k F_0 t) \right)
\end{aligned}
$$

  - with $a_0 = c_0,\ a_k = 2|c_k| \cos\theta_k,\ b_k = 2|c_k| \sin\theta_k$

➡ Real-valued periodic signals can be represented by a superposition of
  - cosines with different amplitudes and phases, or, alternatively
  - weighted sum of cosines and sinusoids with phase zero

Power Spectral Density of Periodic Signals I

- Recall that a periodic signal has infinite energy and a finite **average power**

$$P_x = \frac{1}{T_p} \int_{T_p} |x(t)|^2 \, dt$$

- Inserting the Fourier series with $|x(t)|^2 = x(t)x^*(t)$ we obtain

$$\begin{aligned} P_x &= \frac{1}{T_p} \int_{T_p} x(t) \sum_{k=-\infty}^{\infty} c_k^* \, e^{-j2\pi kF_0 t} \, dt \\ &= \sum_{k=-\infty}^{\infty} |c_k|^2 \end{aligned}$$

  ➡ *Parseval's relation* for power signals

- The squared Fourier coefficients $|c_k|^2$ plotted as a function of frequencies kF_0 is called **power spectral density**

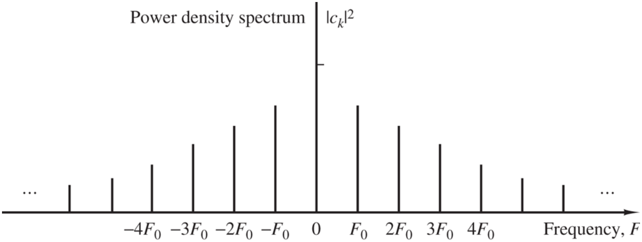


Figure 4.1.2 Power density spectrum of a continuous-time periodic signal.

- for a real-valued signal we have $c_k = c_k^*$, and
  - the **power spectral density** $|c_k|^2$ (and also the magnitude) is an even symmetric function.
  - the phase spectrum $\theta_k = \angle c_k$ is an odd function.

Fourier Transform of a Rectangular Pulse Train I

Example 4.1.1

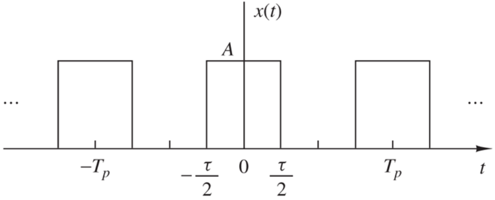


Figure 4.1.3 Continuous-time periodic train of rectangular pulses.

Example 4.1.1

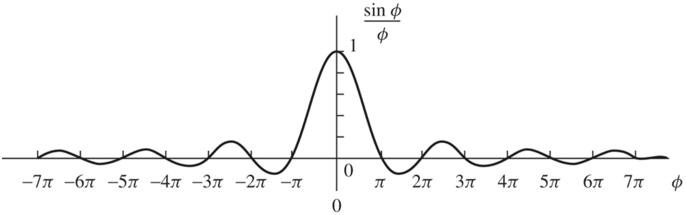


Figure 4.1.4 The function $(\sin\phi)/\phi$.

➡ The function $(\sin\phi)/\phi$ is often referred to as **Sinc function**

Example 4.1.1

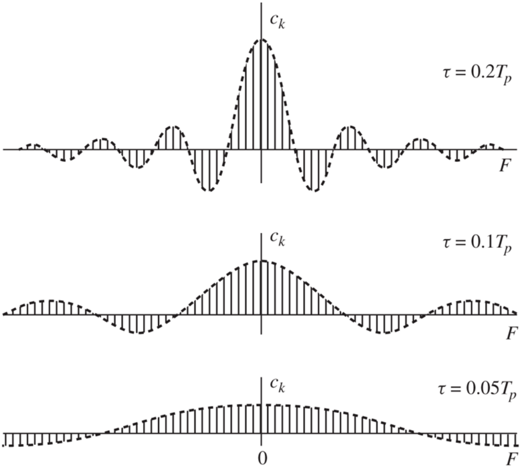


Figure 4.1.5 Fourier coefficients of the rectangular pulse train when T_p is fixed and the pulse width τ varies.

Example 4.1.1

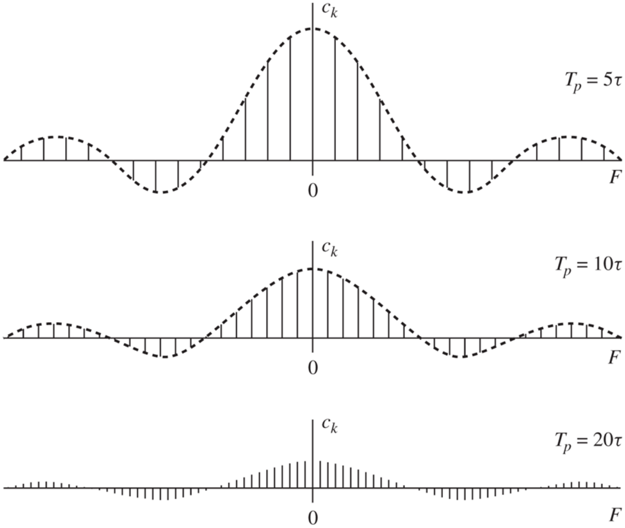


Figure 4.1.6 Fourier coefficient of a rectangular pulse train with fixed pulse width τ and varying period T_p.

Conclusions

- Periodic signals can be represented by a linear combination of harmonically related complex exponentials
- Periodic signals possess line-spectra with equidistant lines.
- The line spacing is equal to the fundamental frequency F_0
- The fundamental frequency is the inverse of the fundamental period $T_p = 1/F_0$

- The case of an aperiodic signal can be seen as a special case of a periodic signal with $T_p \to \infty$
- Then, as $T_p \to \infty$ we have $F_0 = 0$, i.e. the line line-spectra become infinitely close

➡ A continuous-time *aperiodic* signal exhibits a continuous spectrum

- $x(t) = \lim_{T_p \to \infty} x_p(t)$
- Fourier *series*: $c_k = \frac{1}{T_p} \int_{-\infty}^{\infty} x(t) e^{-2\pi k F_0 t} dt$
- Fourier *transform*: $X(F) = \int_{-\infty}^{\infty} x(t) e^{-2\pi k F t} dt$
- Main difference: F is continuous!

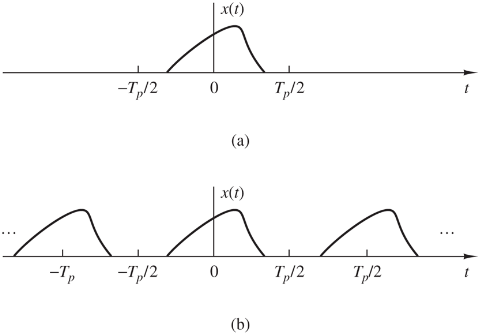


Figure 4.1.7 (a) Aperiodic signal $x(t)$ and (b) periodic signal $x_p(t)$ constructed by repeating $x(t)$ with a period T_p.

- Inverse Fourier series: $x_p(t) = \frac{1}{T_p} \sum_{k=-\infty}^{\infty} c_k e^{-j2\pi kF_0 t}$
- Inverse Fourier transform: $x(t) = \int_{-\infty}^{\infty} X(F) e^{-j2\pi Ft}$
- Interpretation: for $T_p \to \infty$, the line spacing kF_0 becomes infinitely small ➡ the sum becomes an integral

Frequency Analysis of Continuous-Time Aperiodic Signals:
Fourier transform

Synthesis equation: $x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$

Analysis equation: $X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$

➡ Aperiodic signals have continuous spectra

- The Fourier transform pair can also be represented in terms of the radian frequency variable $\Omega = 2\pi F$. Since $dF = d\Omega/2\pi$

  Synthesis equation: $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$

  Analysis equation: $X(\Omega) = \displaystyle\int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$

Energy Density Spectrum of Aperiodic Signals

- Show that: the energy of the finite energy signal $x(t)$ with Fourier transform $X(F)$ is

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\ &= \int_{-\infty}^{\infty} |X(F)|^2 \, dF \end{aligned}$$

➡ *Parseval's relation* for aperiodic finite-energy signals

- The spectrum is usually represented in polar coordinates, i.e. in terms of magnitude and phase: $X(F) = |X(F)| e^{j\theta(F)}$
- For deterministic (non-random) signals, $S_{xx}(F) = |X(F)|^2$ represents the distribution of energy as a function of frequency, referred to as the **energy density spectrum**

Example 4.1.2

- Determine the Fourier transform and the energy density spectrum of a rectangular pulse

$$x(t) = \begin{cases} A, & |t| \leq \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

Example 4.1.2

- Determine the Fourier transform and the energy density spectrum of a rectangular pulse

$$x(t) = \begin{cases} A, & |t| \leq \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

➡

$$X(F) = \int_{\tau/2}^{\tau/2} A^{-j2\pi Ft} = A\tau \frac{\sin \pi F\tau}{\pi F\tau}$$

Example 4.1.2

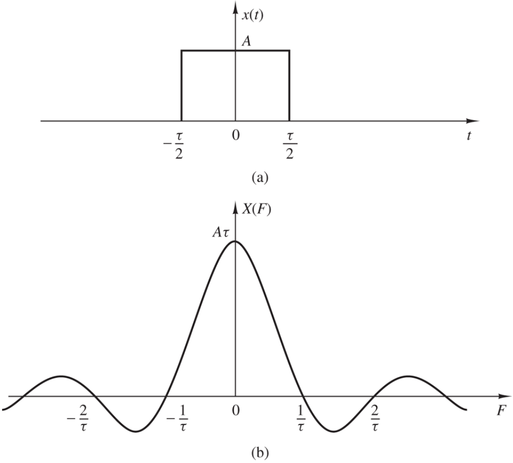


Figure 4.1.8 (a) Rectangular pulse and (b) its Fourier transform.

Example 4.1.2

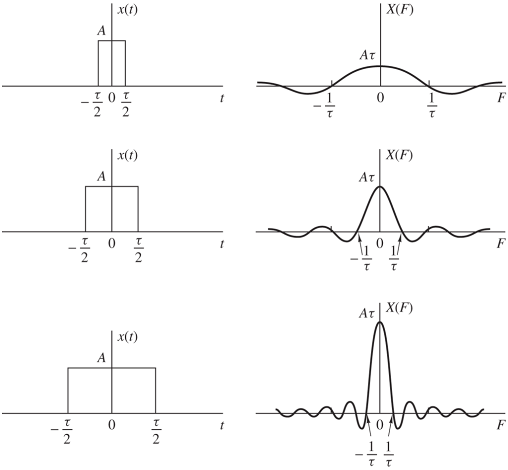


Figure 4.1.9 Fourier transform of a rectangular pulse for various width values.

➡ Uncertainty principle between time and frequency domains

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4.4 Properties of the Fourier Transform for Discrete-Time Signals

Frequency Analysis of Discrete-Time Signals

- The Fourier series can consist of a (possibly infinite) number of frequency components spaced by F_0, i.e. the spectrum may span from $-\infty$ to ∞
- In contrast, for discrete-time signals, the frequency range is unique over the interval $(-\pi, \pi)$ or $(0, 2\pi)$
- A periodic discrete signal of fundamental period N can consist of frequency components separated by $\omega = 2\pi/N$ radians or $f = 1/N$ cycles per sample
  - ➡ Fourier series of discrete-time signal will contain at most N frequency components

Frequency Analysis of Discrete-Time Periodic Signals:
Discrete-Time Fourier Series (DTFS)

Synthesis equation: $x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$

Analysis equation: $c_k = \frac{1}{N} \sum_{k=0}^{N-1} x(n) e^{-j2\pi kn/N}$

- Given: periodic sequence $x(n)$ with period N, i.e
  $x(n) = x(n+N), \quad$ for all n
- The Fourier coefficients c_k provide a description of $x(n)$ in the frequency domain, in the sense that c_k represents the amplitude and phase associated with the frequency component $e^{j2\pi kn/N} = e^{j\omega_k n}$ with $\omega_k = 2\pi k/N$

- As $e^{-j2\pi kn/N}$ is periodic in N, also the Fourier coefficients c_k are periodic

$$c_{k+N} = c_k$$

➡ *The spectrum of the periodic signal $x(n)$ is also a periodic sequence with period N*

➡ We only need to consider the coefficients $k = 0, 1, ..., N-1$

➡ Using the sampling frequency F_S, the range $0 \leq k \leq N-1$ corresponds to the frequency range $0 \leq F < F_\mathrm{S}$

Power Density Spectrum of Periodic Signals

- The average power of a discrete-time periodic signal with period N is

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

- From the definition of the Fourier series it follows that

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

  i.e. the average power in the signal is the sum of the powers of the individual frequency components

- for the energy in in single period we have

$$E_x = \sum_{n=0}^{N-1} |x(n)|^2 = N \sum_{k=0}^{N-1} |c_k|^2$$

Example 4.2.2

- Determine the Fourier series coefficients and the power density spectrum of the periodic signal shown in Fig 4.2.2

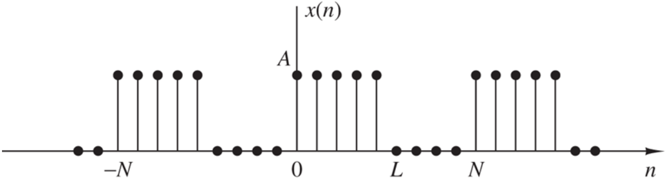

Figure 4.2.2 Discrete-time periodic square-wave signal.

Example 4.2.2

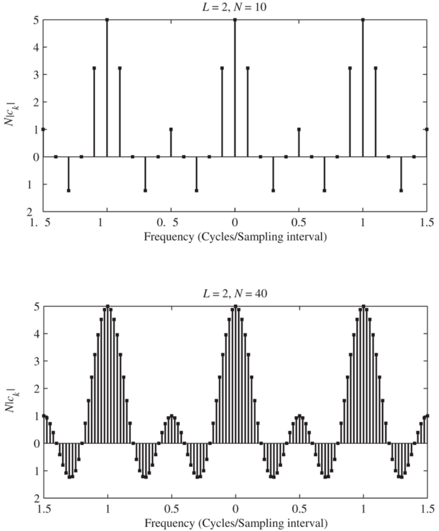

Figure 4.2.3 Plot of the power density spectrum given by (4.2.22).

Spectral Symmetries of Periodic Real-Valued Signals I

- Recall that for discrete signals, as $e^{-j2\pi kn/N}$ is periodic in N,
  - $c_{k+N} = c_k$
- Recall that, if the discrete-time signal $x(n)$ is real-valued we have
  - $x^*(n) = x(n)$
  - $c_k^* = c_{-k}$
  - $|c_k| = |c_{-k}|$ (magnitude is even symmetric)
  - $\angle c_k = -\angle c_{-k}$ (phase is odd symmetric)
- For discrete real-valued signals we have
  - $|c_k| = |c_{N-k}|$
  - $\angle c_k = -\angle c_{N-k}$
  - More specifically

$$\begin{array}{lll} |c_0| = |c_N| & \angle c_0 = -\angle c_N = 0 & \\ |c_1| = |c_{N-1}| & \angle c_1 = -\angle c_{N-1} & \\ |c_{N/2}| = |c_{N/2}| & \angle c_{N/2} = 0 & \text{if } N \text{ is even} \\ |c_{(N-1)/2}| = |c_{(N+1)/2}| & \angle c_{(N-1)/2} = -\angle c_{(N+1)/2} & \text{if } N \text{ is odd} \end{array}$$

- For a real-valued signal, **the signal is completely described** by the spectrum c_k with
  - $k = 0, 1, .., N/2$, if N is even
  - $k = 0, 1, .., (N-1)/2$, if N is odd
- Consistent: The unique spectrum is
  $0 \leq \omega_k = \frac{2\pi k}{N} \leq \pi \quad \longleftrightarrow \quad 0 \leq k \leq N/2$

Frequency Analysis of Discrete-Time Aperiodic Signals:
Discrete-Time Fourier Transform (DTFT)

Synthesis equation: $x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$

Analysis equation: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Derivation of the synthesis equation:

- For discrete-time signals, we observe that also the spectrum is periodic,

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2\pi k)n} = X(\omega)$$

- Thus, the periodic spectrum $X(\omega)$ can be analyzed by a Fourier series. This results in the synthesis equation above

Example: ideal Low-pass filter

- Ideal low-pass filter with cut-off frequency Ω_c:

$$X(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

Fourier Transform of Discrete-Time Aperiodic Signals I

Example: ideal Low-pass filter

- Ideal low-pass filter with cut-off frequency Ω_c:

$$X(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

➡

$$x(n) = \begin{cases} \frac{\omega_c}{\pi}, & n = 0 \\ \frac{\omega_c}{\pi} \frac{\sin \omega_c n}{\omega_c n}, & n \neq 0 \end{cases}$$

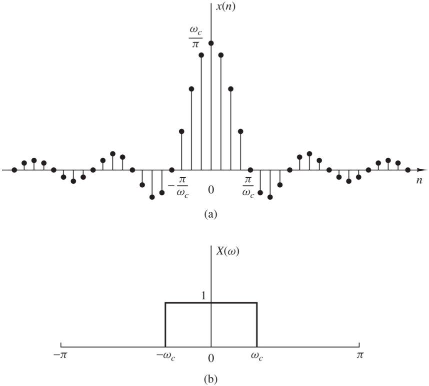

Figure 4.2.4 Fourier transform pair in (4.2.35) and (4.2.36).

Gibbs phenomenon

- Now, let's look at the spectrum of the time-domain signal $x(n) = \frac{\sin \omega_c n}{\pi n}$, $-\infty < n < \infty$, when analyzed over a finite time-window

$$X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

- Oscillatory overshoot at $\omega = \omega_c$
- for $N \to \infty$ overshoot becomes narrow, but amplitude remains

➡ **Gibbs phenomenon**

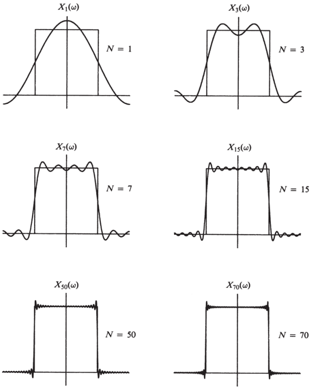

Figure 4.2.5 Illustration of convergence of the Fourier transform and the Gibbs phenomenon at the point of discontinuity.

- The energy of a discrete-time signal is

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

- From the definition of the DTFT it follows that

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{n=-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

➡ *Parseval's relation* for discrete-time aperiodic signals with finite energy

- The spectrum is usually represented in polar coordinates, i.e. in terms of magnitude and phase: $X(\omega) = |X(\omega)| e^{j\theta(\omega)}$
- For deterministic (non-random) signals, $S_{xx}(F) = |X(\omega)|^2$ represents the distribution of energy as a function of frequency, referred to as the **energy density spectrum**

- Suppose that the discrete-time signal $x(n)$ is real-valued, then
  - $x^*(n) = x(n)$
  - $X^*(\omega) = X(-\omega)$
  - $|X(-\omega)| = |X(\omega)|$ (magnitude is even symmetric)
  - $\angle X(-\omega) = -\angle X(\omega)$ (phase is odd symmetric)
  - $S_{xx}(-F) = S_{xx}(F)$ (even symmetry)

→ A real-valued signal, is completely described by half the spectrum $X(\omega)$ with $0 \leq \omega \leq \pi$, i.e. the spectrum in the range $0 \leq F \leq F_{\mathrm{S}}/2$

- Determine the Fourier transform and the energy density spectrum of the sequence

$$x(n) = \begin{cases} A, & 0 \leq n \leq L-1 \\ 0, \text{otherwise} & \end{cases}$$

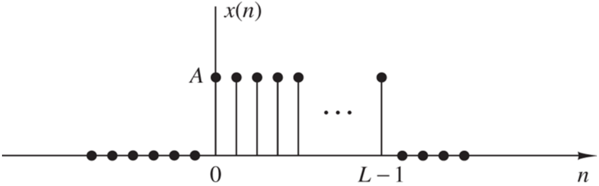

Figure 4.2.7 Discrete-time rectangular pulse.

Example 4.2.4 II

$$
\begin{aligned}
X(\omega) \\
&= \sum_{n=0}^{L-1} A e^{-j\omega n} \\
&= A e^{-j(\omega/2)(L-1)} \frac{\sin(\omega L/2)}{\sin(\omega/2)}
\end{aligned}
$$

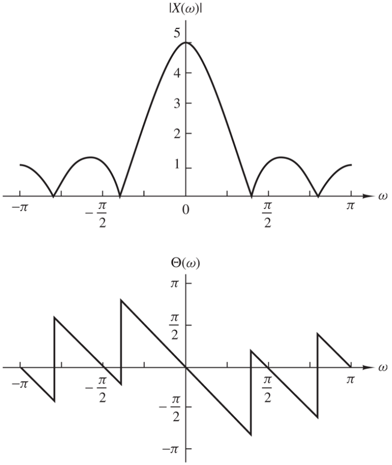

Figure 4.2.8 Magnitude and phase of Fourier transform of the discrete-time rectangular pulse in Fig 4.2.7.

- The z-Transform of a sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \quad \text{ROC: } r_2 < |z| < r_1$$

- Let us now represent the complex variable z in polar form, i.e. $z = re^{j\omega}$
- then, within the ROC we can substitute z

$$X(z)\big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} \left[x(n) r^{-n}\right] e^{-j\omega n}$$

➡ $X(z)$ can be interpreted as the Fourier transform of sequence $x(n)r^{-n}$

Relationship of the Fourier Transform to the z-Transform I

- The z-Transform of a sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \quad \text{ROC: } r_2 < |z| < r_1$$

- Let us now represent the complex variable z in polar form, i.e. $z = re^{j\omega}$
- then, within the ROC we can substitute z

$$X(z)\Big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} \left[x(n) r^{-n}\right] e^{-j\omega n}$$

➡ $X(z)$ can be interpreted as the Fourier transform of sequence $x(n)r^{-n}$

- Alternatively, if $|z| = 1$ is within the ROC, we obtain

$$X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} \left[x(n)\right] e^{-j\omega n}$$

➡ Fourier transform: z-transform evaluated on the unit circle

➡Fourier transform interpreted as a z-transform evaluated on the unit circle

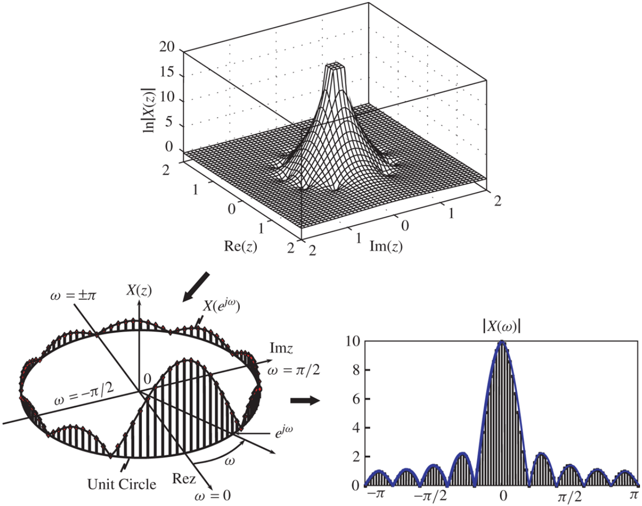

Figure 4.2.9 relationship between $X(z)$ and $X(\omega)$ for the sequence in Example 4.2.4, with $A = 1$ and $L = 10$

low-frequency, high-frequency, and bandpass signals

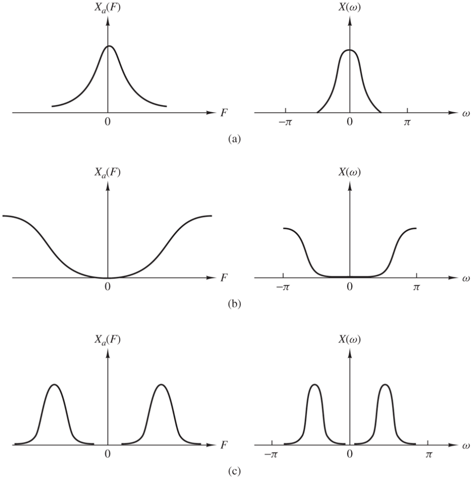

Figure 4.2.10 (a) Low-frequency, (b) high-frequency, and (c) medium-frequency signals.

➡ medium-frequency signals are also referred to as *bandpass signals* bandwitdh, narrowband, wideband, bandlimited

- 95% **bandwidth** signal has 95% of its energy (or power) in a range $F_1 \leq F \leq F_2$
- A **narrowband** signal has a bandwidth $F_2 - F_1$ which is much smaller (e.g. factor 10) than the median frequency $(F_2 + F_1)/2$. Otherwise the signal is called **wideband**
- A signal is **bandlimited** if its spectrum is zero outside the frequency range $F \geq B$
- A discrete-time finite-energy signal $x(n)$ is said to be **(periodically) bandlimited** if $|X(\omega)| = 0$ for $\omega_0 < |\omega| < \pi$
- No signal can be time-limited and bandlimited simultaneously (e.g. rect $\circ\!\!-\!\!\bullet$ Sinc)

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Introduced analysis tools

- The following frequency analysis tools have been introduced
  1. Fourier series for continuous-time periodic signals
  2. Fourier transform for continuous-time aperiodic signals
  3. Fourier series for discrete-time periodic signals
  4. Fourier transform for discrete-time aperiodic signals

A summary of the previous sections

- Continuous-time signals have aperiodic spectra
- Discrete-time signals have periodic spectra
- Periodic signals have discrete spectra
- Aperiodic finite energy signals have continuous spectra

➡ Periodicity with “period” α in one domain implies discretization with “spacing” of $1/\alpha$ in the other domain

- The **energy density spectrum** can be used to characterize finite-energy **aperiodic** signals
- The **power density spectrum** can be used to characterize **periodic** signals

Dualities between transforms

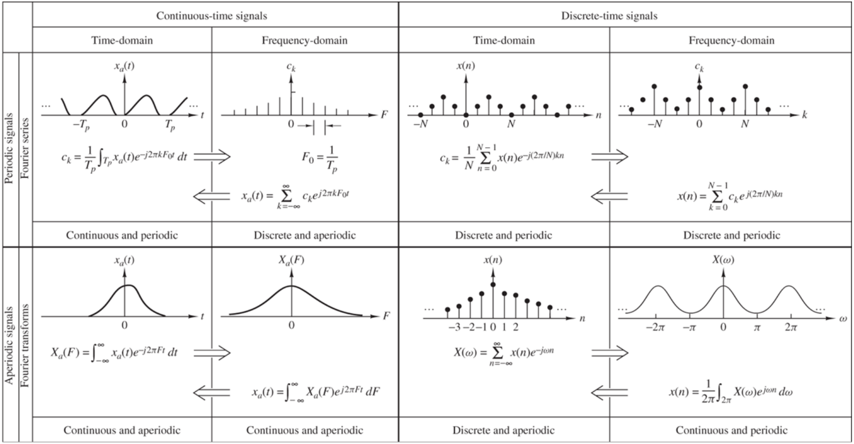


Figure 4.3.1 Summary of analysis and synthesis formulas.

Outline

1. Introduction of Basic Concepts

2. Discrete-Time Signals and Systems

3. The z-Transform and Its Applications

4. Frequency Analysis of Signals

4.1 Frequency Analysis of Continuous-Time Signals

4.2 Frequency Analysis of Discrete-Time Signals

4.3 Frequency-Domain and Time-Domain Signal Properties

4.4 Properties of the Fourier Transform for Discrete-Time Signals

SP Signal Processing

Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}\left[X(\omega) + X^*(-\omega)\right]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}\left[X(\omega) - X^*(-\omega)\right]$
$x_e(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}\left[x(n) - x^*(-n)\right]$	$jX_I(\omega)$
Real signals	
Any real-valued signal $x(n)$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\angle X(\omega) = -\angle X(-\omega)$
$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$	$X_I(\omega)$

Symmetry Properties of the DTFT II

$$x(n) = x_R^e(n) + x_R^o(n) + jx_I^e(n) = jx_I^o(n)$$

$$X(\omega) = X_R^e(\omega) + X_R^o(\omega) + jX_I^e(\omega) = jX_I^o(\omega)$$

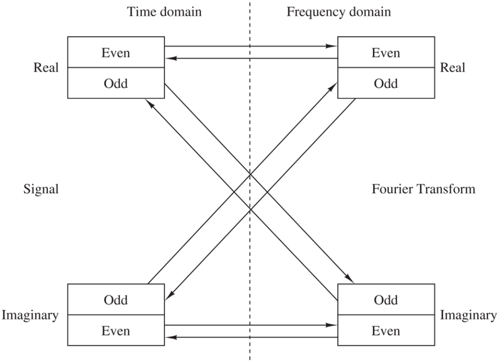


Figure 4.4.2 Summary of symmetry properties for the Fourier transform.

Example 4.4.1

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}, \quad -1 < a < 1$$

Representation of Spectra

Example 4.4.1

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}, \quad -1 < a < 1$$

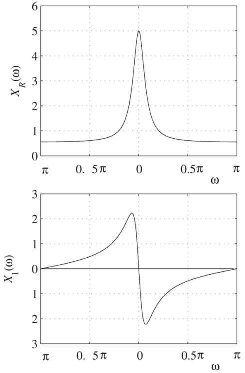


Figure 4.4.3 Graph of $X_R(\omega)$ and $X_I(\omega)$ for the transform in Example 4.4.1.

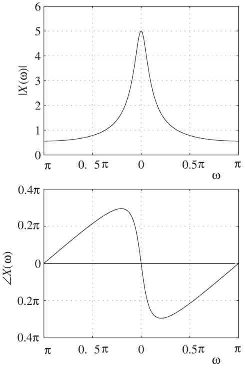


Figure 4.4.4 Magnitude and phase spectra of the transform in Example 4.4.1.

➡ The represenatation in magnitude and phase is more common.

Example 4.4.2

$$x(n) = \begin{cases} A, & -M \leq n \leq M \\ 0, & \text{elsewhere} \end{cases}$$

SP Signal Processing

Example 4.4.2

$$x(n) = \begin{cases} A, & -M \leq n \leq M \\ 0, & \text{elsewhere} \end{cases}$$

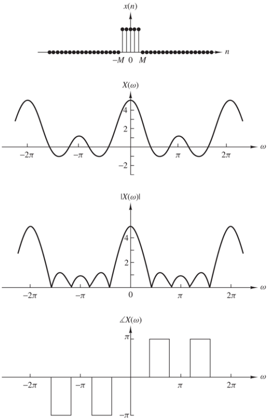


Figure 4.4.5 Spectral characteristics of rectangular pulse in Example 4.4.2.

Example 4.4.3

$$x(n) = a^{|n|}, \quad -1 < a < 1$$

Example 4.4.3

$$x(n) = a^{|n|}, \quad -1 < a < 1$$

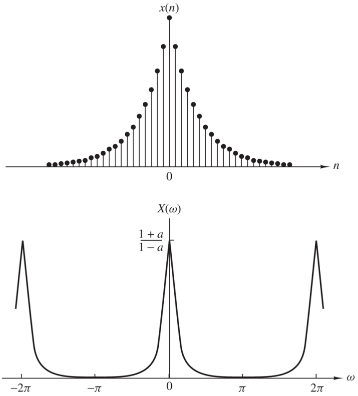


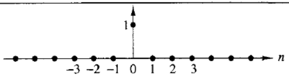
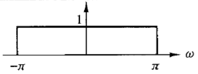
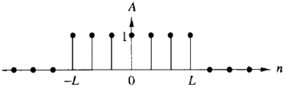
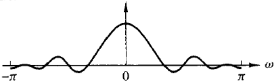
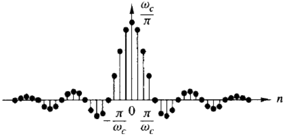
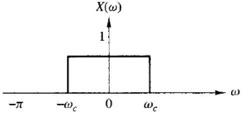
Figure 4.4.6 Sequence $x(n)$ and its Fourier transform in Example 4.4.3 with $a = 0.8$.

Properties of the DTFT

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X(\omega) + a_2 X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k} X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega) X_2(\omega)$
Correlation	$r_{x_1 x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1 x_2}(\omega) = X_1(\omega) X_2(-w)$ $= X_1(\omega) X_2^*(\omega)$ if $x_2(n)$ real
Wiener-Khintchine th.	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n} x(n)$	$S_{xx}(\omega)$
Modulation	$x(n) \cos \omega_0 n$	$\frac{1}{2}\left(X(\omega + \omega_0) + X(\omega - \omega_0)\right)$
Multiplication	$x_1(n) x_2(n)$	$\frac{1}{2\pi} \int_{-n}^{n} X_1(\lambda) X_2(\omega - \lambda) d\lambda$
Freq Differentiation	$nx(n)$	$j \frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega) X_2^*(\omega) d\omega$	

Useful Transform Pairs

TABLE 6 Some Useful Fourier Transform Pairs for Discrete-Time Aperiodic Signals

Signal $x(n)$	Spectrum $X(\omega)$
 $x(n) = \delta(n)$	 $X(\pi) = 1$
 $x(n) = \begin{cases} A, & \lvert n\rvert \le L \\ 0, & \lvert n\rvert > L \end{cases}$	 $X(\omega) = A \dfrac{\sin\left(L+\frac{1}{2}\right)\omega}{\sin\frac{\omega}{2}}$
 $x(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\sin \omega_c n}{\pi n}, & n \ne 0 \end{cases}$	 $X(\omega) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c \\ 0, & \omega_c \le \lvert\omega\rvert \le \pi \end{cases}$
$x(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n > 0 \end{cases}$	$X(\omega) = \frac{1}{1-ae^{-j\omega}}$

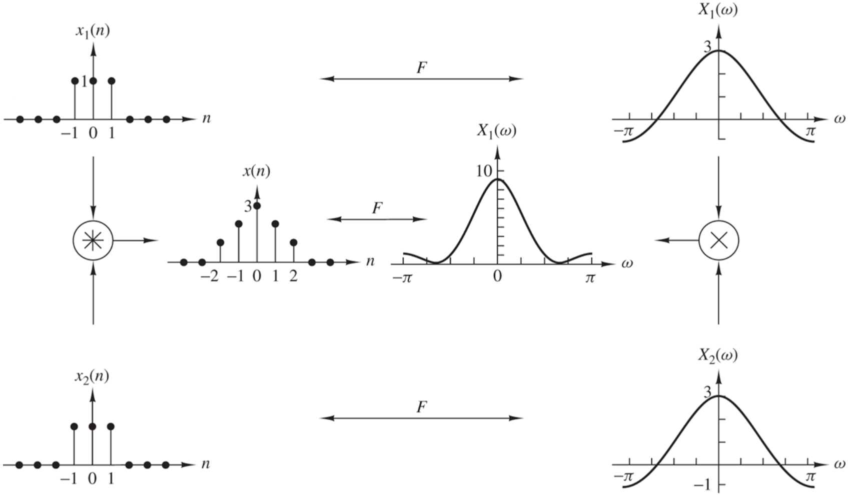


Figure 4.4.7 Graphical representation of the convolution property.

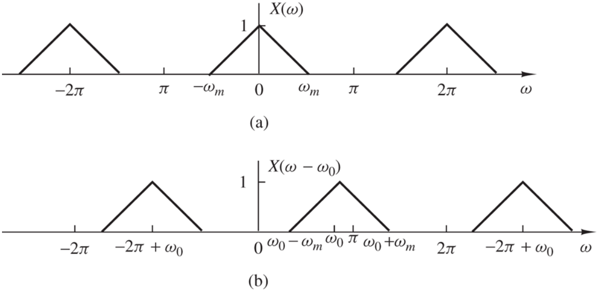


Figure 4.4.8 Illustration of the frequency-shifting property of the Fourier transform ($\omega_0 \leq 2\pi - \omega_m$).

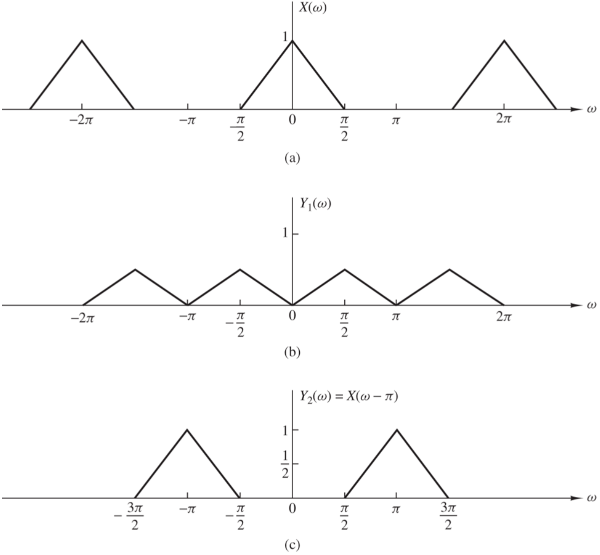


Figure 4.4.9 Graphical representation of the modulation theorem.

Content

1. Introduction of Basic Concepts

2. Discrete-Time Signals and Systems

3. The z-Transform and Its Applications

4. Frequency Analysis of Signals
4.1 Frequency Analysis of Continuous-Time Signals
4.2 Frequency Analysis of Discrete-Time Signals
4.3 Frequency-Domain and Time-Domain Signal Properties
4.4 Properties of the Fourier Transform for Discrete-Time Signals