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SP Signal Processing

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Digital Media Signal Processing

Formulas

The z-Transform

z-Transform: $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

inverse z-Transform: $$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

Frequency Analysis of Continuous-Time Periodic Signals: **Fourier series**

Synthesis equation: $$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

Analysis equation: $$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$$

Frequency Analysis of Continuous-Time Aperiodic Signals:
Fourier transform

Synthesis equation: $x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$

Analysis equation: $X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$

Frequency Analysis of Discrete-Time Periodic Signals:
Discrete-Time Fourier Series (DTFS)

Synthesis equation: $x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$

Analysis equation: $c_k = \frac{1}{N} \sum_{k=0}^{N-1} x(n) e^{-j2\pi kn/N}$

Frequency Analysis of Discrete-Time Aperiodic Signals:
Discrete-Time Fourier Transform (DTFT)

Synthesis equation: $x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$

Analysis equation: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Properties of the z-Transform

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$	ROC: $r_2 < \lvert z\rvert < r_1$ ROC_1 ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z\rvert < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\mathrm{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\mathrm{Im}\{x(n)\}$	$\frac{1}{2}[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \lvert z\rvert < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

Properties of the DTFT

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(\omega) + a_2 X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k} X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega) X_2(\omega)$
Correlation	$r_{x_1 x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1 x_2}(\omega) = X_1(\omega) X_2(-w)$
		$= X_1(\omega) X_2^*(\omega)$ if $x_2(n)$ real
Wiener-Khintchine th.	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n} x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n) \cos \omega_0 n$	$\frac{1}{2} \left(X(\omega + \omega_0) + X(\omega - \omega_0)\right)$
Multiplication	$x_1(n) x_2(n)$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda) d\lambda$
Freq Differentiation	$n x(n)$	$j \frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega) X_2^*(\omega) d\omega$	

Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}\left[X(\omega) + X^*(-\omega)\right]]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}\left[X(\omega) - X^*(-\omega)\right]$
$x_e(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}\left[x(n) - x^*(-n)\right]$	$jX_I(\omega)$

Real signals

Any real-valued signal $x(n)$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\angle X(\omega) = -\angle X(-\omega)$
$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$	$X_I(\omega)$

Symmetry Properties of the DTFT II

$$x(n) = x_R^e(n) + x_R^o(n) + jx_I^e(n) + jx_I^o(n)$$

$$X(\omega) = X_R^e(\omega) + X_R^o(\omega) + jX_I^e(\omega) + jX_I^o(\omega)$$

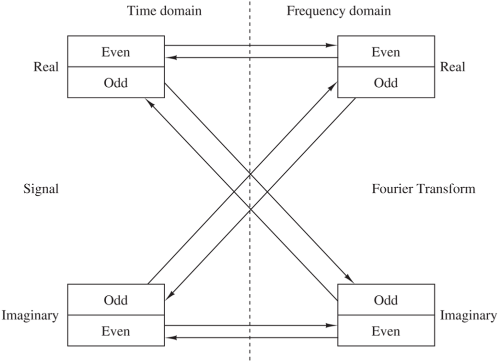


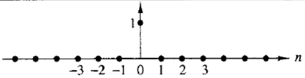
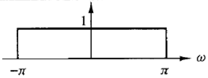
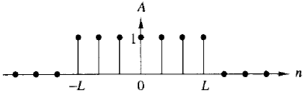
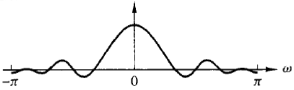
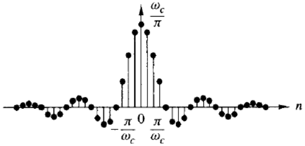
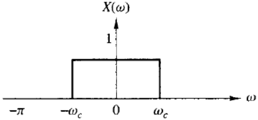
Figure 4.4.2 Summary of symmetry properties for the Fourier transform.

Some Common z-Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos \omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin \omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n \cos \omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n \sin \omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Useful Transform Pairs

TABLE 6 Some Useful Fourier Transform Pairs for Discrete-Time Aperiodic Signals

Signal $x(n)$	Spectrum $X(\omega)$
 $x(n) = \delta(n)$	 $X(\pi) = 1$
 $x(n) = \begin{cases} A, & \lvert n \rvert \le L \\ 0, & \lvert n \rvert > L \end{cases}$	 $X(\omega) = A \dfrac{\sin\left(L+\frac{1}{2}\right)\omega}{\sin\frac{\omega}{2}}$
 $x(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\sin \omega_c n}{\pi n}, & n \ne 0 \end{cases}$	 $X(\omega) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c \\ 0, & \omega_c \le \lvert\omega\rvert \le \pi \end{cases}$
$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n > 0 \end{cases}$	$X(\omega) = \frac{1}{1-ae^{-j\omega}}$