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SP Signal Processing

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Digital Media Signal Processing

5. Frequency Analysis of Linear Time-Invariant (LTI) Systems

Outline

1. Introduction of Basic Concepts
2. Discrete-Time Signals and Systems
3. The z-Transform and Its Applications
4. Frequency Analysis of Signals
5. Frequency Analysis of linear time-invariant (LTI) Systems

5.1 Frequency-Domain Characteristics of LTI Systems
5.2 Frequency Response of LTI Systems
5.3 Correlation Functions and Spectra at the Output of LTI Systems
5.4 LTI Systems as Frequency-Selective Filters
5.5 Inverse Systems and Deconvolution

Summary

- Characterization of linear time-invariant (LTI) systems in the frequency domain.
- Basic excitations are complex exponentials and sinusoidal functions
- LTI systems perform a filtering (attenuation/amplification) on the various frequency components.
- Each frequency is filtered independently of other frequencies. No frequencies are added.

➡ Simple description by input-output function (the transfer function / frequency response) possible

- This transfer function is the frequency transform of the response of the LTI system to an impulse in the time-domain (the impulse response)

Outline

- The frequency response $H(\omega)$
  - is the Fourier transform of the impulse response $h(n)$ of the system
  - completely characterizes an LTI system in the frequency domain
  - allows us to determine the steady-state response of the system to any arbitrary weighted linear combination of sinusoids or complex exponentials
- As, using a Fourier transform, signals can be seen as a superposition of complex exponentials, the response of an LTI system to arbitrary signals can be determined using the frequency response

The Frequency Response Function I

- Recall that the response of any relaxed linear time-invariant system to an arbitrary input signal $x(n)$ is given by the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

  ➡ time-domain: system characterized by unit sample response $h(n)$

- To develop a frequency-domain characterization, let us excite by a complex exponential

$$x(n) = Ae^{j\omega n}$$

  where A: amplitude; $\omega \in [-\pi, \pi]$ an arbitrary frequency

The Frequency Response Function II

- Then

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} h(k)\left[Ae^{j\omega(n-k)}\right] \\
&= A\underbrace{\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]}_{H(\omega)} e^{j\omega n} \\
&= H(\omega)Ae^{j\omega n}
\end{aligned}
$$

- The system response is again given by a complex exponential at the same frequency ω, but multiplicatively altered by $H(\omega)$
- Thus, the exponential signal $x(n) = Ae^{j\omega n}$ is called an *eigenfunction* of the system
- The multiplicative factor $H(\omega)$ is called an *eigenvalue* of the system

Example 5.1.1

- Determine the output sequence of the system with impulse response

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

when the input is the complex exponential sequence

$$x(n) = Ae^{j\pi n/2}, \quad -\infty < n < \infty$$

The Frequency Response Function III

- In general, the frequency response is a complex-valued function and is typically expressed in polar form

$$H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$$

  i.e. the system modifies

  - the magnitude multiplicatively
  - the phase additively (results in a phase-shift)

- $H(\omega)$: *frequency response*
  $|H(\omega)|$: *magnitude response*
  $|\Theta(\omega)|$: *phase response*
- For a many real-world systems, the impulse response $h(n)$ is real-valued. Consequently
  - $|H(\omega)|$ is an even function in ω
  - $\Theta(\omega)$ is an odd function in ω
  - ➜ if we know $H(\omega)$ for $0 \leq \omega \leq \pi$, we also know $H(\omega)$ for $-\pi \leq \omega \leq 0$

Example 5.1.2

- Determine magnitude and phase of $H(\omega)$ for the tree-point moving average (MA) system

$$y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$$

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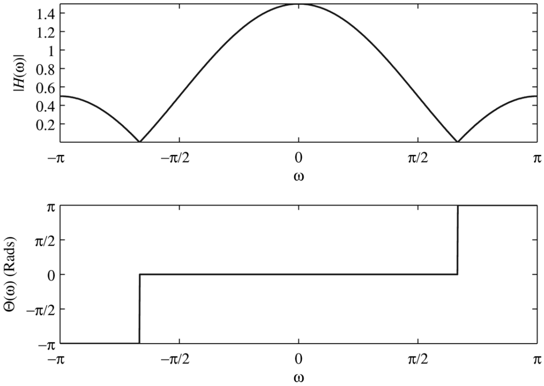


Figure 5.1.1 Magnitude and phase responses for the MA system in Example 5.1.2.

- An LTI system is described by the difference equation

$$y(n) = ay(n-1) + bx(n), \quad 0 < a < 1$$

a) Determine the magnitude and phase of the frequency response $H(\omega)$
b) Choose b so that the maximum value of $|H(\omega)|$ is unity
c) Sketch $|H(\omega)|$ and $\Theta(\omega)$ for $a = 0.9$
d) Determine the output of the system to the input signal

$$x(n) = 5 + 12\sin\left(\frac{\pi}{2}n\right) - 20\cos\left(\pi n + \frac{\pi}{4}\right)$$

Example 5.1.4 II

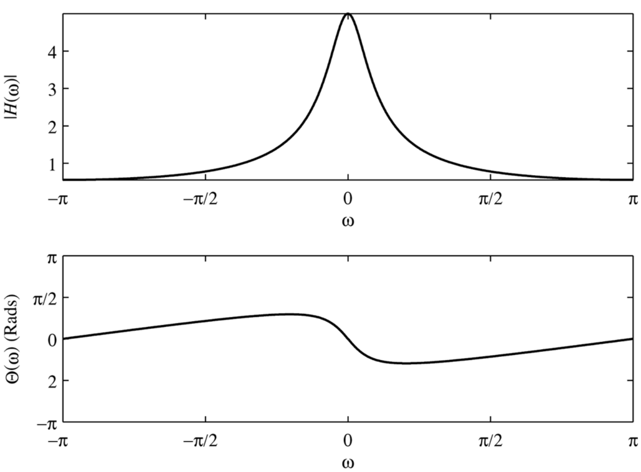


Figure 5.1.2 Magnitude and phase responses for the system in Example 5.1.4 with $a = 0.9$.

The Frequency Response Function

Sum of sinusoids

- If the input signal consists of an arbitrary linear combination of sinusoids

$$x(n) = \sum_{i=1}^{L} A_i \cos(\omega_i n + \phi_i)\,, \quad -\infty < n < \infty$$

The response of the system is simply

$$y(n) = \sum_{i=1}^{L} A_i |H(\omega_i)| \cos(\omega_i n + \phi_i + \Theta(\omega_i))\,, \quad -\infty < n < \infty$$

- Different frequencies are affected differently by the system
  - Some frequencies may be set to zero,
  - others may not be modified at all

Steady-State and Transient Response I

To Sinusoidal Input Signals

- So far, we considered eternal sinusoids/exponentials, i.e. those applied at time $n = -\infty$. The observed response is the *steady-state response*
- To demonstrate the behavior of a system when the signal is applied at, say, $n = 0$, let us consider the system described by the first-order difference equation

$$y(n) = ay(n-1) + x(n)$$

- The system response to any $x(n)$ applied at $n = 0$ is

$$y(n) = a^{n+1} \underbrace{y(-1)}_{\text{initial condition}} + \sum_{k=0}^{n} a^k x(n-k), \quad n \geq 0$$

- Given the complex-exponential input $x(n) = Ae^{j\omega n}$, $n \geq 0$,

$$\begin{aligned}
y(n) &= a^{n+1} y(-1) + A \sum_{k=0}^{n} a^k e^{j\omega(n-k)} \\
&= a^{n+1} y(-1) + A \left[\sum_{k=0}^{n} a^k e^{-j\omega k} \right] e^{j\omega n} \\
&= a^{n+1} y(-1) + A \left[\sum_{k=0}^{n} \left(a e^{-j\omega} \right)^k \right] e^{j\omega n} \\
&= a^{n+1} y(-1) + A \frac{1 - a^{n+1} e^{-j\omega(n+1)}}{1 - a e^{-j\omega}} e^{j\omega n}, \quad n \geq 0 \\
&= a^{n+1} y(-1) - \frac{A a^{n+1} e^{-j\omega(n+1)}}{1 - a e^{-j\omega}} + \frac{A}{1 - a e^{-j\omega}} e^{j\omega n}, \quad n \geq 0
\end{aligned}$$

Steady-State and Transient Response III

To Sinusoidal Input Signals

- Steady-state response

$$\begin{aligned} y_{\mathrm{ss}}(n) &= \lim_{n\to\infty} y(n) = \frac{A}{1-ae^{-j\omega}}e^{j\omega n} \\ &= AH(\omega)e^{j\omega n} \end{aligned}$$

- Transient response

$$y_{\mathrm{tr}}(n) = a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}}, \quad n \geq 0$$

- For BIBO stable systems (here $|a| < 1$), the transient part goes to zero for $n \to \infty$

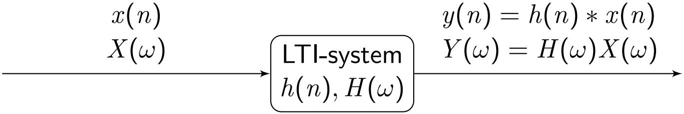
➡ In many practical applications the transient response is unimportant and is usually ingnored in dealing with the response to sinusoidal inputs

Response to Aperiodic Input Signals

- From the convolution theorem it follows that

$$Y(\omega) = H(\omega)X(\omega)$$

- $H(\omega)$ acts as a filter to the frequency components
- $|H(\omega)|$ determines which amplitudes are attenuated or amplified
- $\Theta(\omega) = \angle H(\omega)$ determines the phase shift
- *The output of an LTI-system cannot contain frequency components that are not contained in the input signal*



Outline

Systems with a Rational System Function

- Recall that LTI-systems with rational system functions are described by constant-coefficient difference equations in time-domain.
- If the system function $H(z)$ converges on the unit circle we obtain the frequency response as

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

- If $H(z)$ is a rational function, i.e. of the form $H(z) = B(z)/A(z)$

$$\begin{aligned} H(\omega) &= \frac{B(\omega)}{A(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}} \\ &= b_0 \frac{\prod_{k=1}^{M}(1 - z_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - p_k e^{-j\omega})} \end{aligned}$$

  - where a_k, b_k are real-valued, but the zeros z_k and poles p_k are generally complex-valued.

Magnitude of the Frequency Response

- The magnitude can be obtained by $|H(\omega)|^2 = H(\omega)H^*(\omega)$ with

$$H^*(\omega) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k^* e^{j\omega})}{\prod_{k=1}^{N}(1 - p_k^* e^{j\omega})}$$

- $H^*(\omega)$ is obtained by evaluating $H^*(1/z^*)$ on the unit circle

$$H^*(1/z^*) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k^* z)}{\prod_{k=1}^{N}(1 - p_k^* z)}$$

- If $h(n)$ is real-valued, poles and zeros occur in complex-conjugate pairs, and $H^*(1/z^*) = H(z^{-1})$, such that

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = H(\omega)H(-\omega) = H(z)H(z^{-1})\Big|_{z=e^{j\omega}}$$

  - $H(z)H(z^{-1}) \overset{z}{\circ\!\!-\!\!\bullet} r_{hh}(m)$ (autocorrelation sequence of $h(n)$)
  - $|H(w)|^2 \overset{F}{\circ\!\!-\!\!\bullet} r_{hh}(m)$ (Wiener-Khintchine theorem))

Computation of the Frequency Response

- To determine the frequency response it is convenient to use a representation by means of poles and zeros, i.e.

$$H(\omega) \quad = \quad b_0\, e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

- Let us now represent the complex-valued factors in polar form as

$$\begin{aligned} e^{j\omega} - z_k &\equiv V_k(\omega)\, e^{j\Theta_k(\omega)} \\ e^{j\omega} - p_k &\equiv U_k(\omega)\, e^{j\Phi_k(\omega)} \end{aligned}$$

- Then, since $|e^{j\omega(N-M)}| = 1$, the magnitude of $H(\omega)$ is

$$|H(\omega)| = |b_0| \frac{\prod_{k=1}^{M} V_k}{\prod_{k=1}^{N} U_k}$$

- The phase is given by

$$\angle H(\omega) = \angle b_0 + \omega(N-M) + \left(\sum_{k=1}^{M} \Theta_k(\omega)\right) - \left(\sum_{k=1}^{M} \Phi_k(\omega)\right)$$

➡ magnitude and phase of $H(\omega)$ can be computed for given z_k, p_k

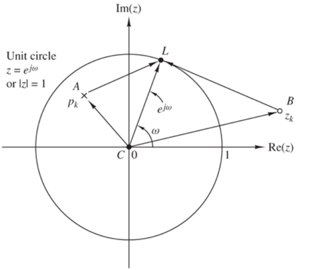
Geometric Interpretation

- let $\mathbf{CL} = e^{j\omega}, \mathbf{CA} = p_k$, and $\mathbf{CB} = z_k$ denote vectors. Then
- $\mathbf{CL} = \mathbf{CA} + \mathbf{AL}$,
  $\mathbf{CL} = \mathbf{CB} + \mathbf{BL}$,

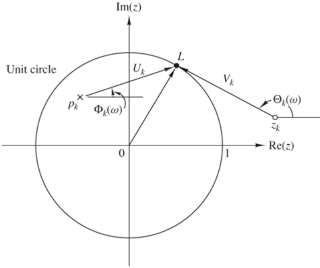
➡ $\mathbf{AL} = e^{j\omega} - p_k = U_k(\omega)e^{j\Phi_k(\omega)}$,
$\mathbf{BL} = e^{j\omega} - z_k = V_k(\omega)e^{j\Theta_k(\omega)}$

Thus,

- distance between pole p_k and $e^{j\omega}$ corresponds to $U_k(\omega)$
- distance between zero z_k and $e^{j\omega}$ corresponds to $V_k(\omega)$



(a)



Computation of the Frequency Response II

Geometric Interpretation

- Presence of a zero close to the unit circle causes $|H(\omega)|$ to be small close to that zero
- Presence of a pole close to the unit circle causes $|H(\omega)|$ to be large close to that pole

➡ Poles have the opposite effect of zeros

➡ If both poles and zeros are present a greater variety of shapes of $H(\omega)$ can be realized

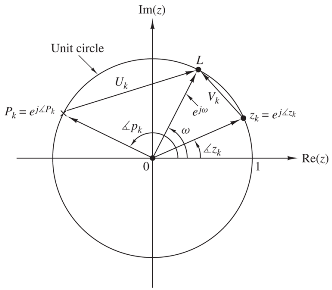


Figure 5.2.2 A zero on the unit circle causes $|H(\omega)| = 0$ and $\omega = \measuredangle z_k$. In contrast, a pole on the unit circle results in $|H(\omega)| = \infty$ at $\omega = \measuredangle p_k$.

Outline

Recall from Chapter 2

- Correlation measures the *similarity* between two signals
- Correlation is mathematically similar to convolution, the difference is that for correlation, the signals are not folded

Convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n)$$

Correlation

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) = x(l) * y(-l)$$

- l is called the signal **lag**

- Consider the following relationships between the input and output sequences of an LTI system

$$\begin{aligned} r_{yx}(l) &= y(l) * x(-l) = h(l) * r_{xx}(l) \\ r_{yy}(l) &= y(l) * y(-l) = h(l) * h(-l) * r_{xx}(l) \end{aligned}$$

  - $r_{xx}(l)$: autocorrelation sequence of input signal $x(n)$
  - $r_{yy}(l)$: autocorrelation sequence of output signal $y(n)$
  - $r_{yx}(l)$: crosscorrelation sequence of input and output signal
- The z-transform results in

$$\begin{aligned} S_{yx}(z) &= H(z)S_{xx}(z) \\ S_{yy}(z) &= H(z)H(z^{-1})S_{xx}(z) \end{aligned}$$

  - recall: $h(-n) \overset{z}{\circ\!\!-\!\!\bullet} H(z^{-1})$; $\quad r_{hh} = h(n) * h(-n) \overset{z}{\circ\!\!-\!\!\bullet} H(z)H(z^{-1})$

Determining the Frequency Response

- Substituting $z = e^{j\omega}$

$$S_{yx}(\omega) = H(\omega)S_{xx}(\omega)$$
$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

- For a flat (white) spectrum $S_{xx}(w) = S_x =$ constant for all ω thus

$$H(\omega) = S_{yx}(\omega)/S_x$$

or equivalently

$$h(n) = r_{yx}(l)/S_x$$

➡ The impulse response $h(n)$ can be determined by exciting the system by a spectrally flat input signal and cross-correlating the input and the output of the system

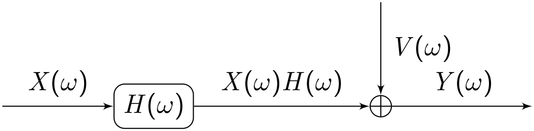
➡ Very useful and practically relevant in measuring the impulse response!

Determining the Frequency Response

- Uncorrelated measurement noise (e.g. microphone noise) will cancel out:

$$Y(\omega) = X(\omega)H(\omega) + V(\omega)$$

$$S_{yx}(\omega) = H(\omega)S_x + \underbrace{S_{xv}(\omega)}_{\to 0}$$



Outline

LTI Systems as Frequency-Selective Filters

- A *filter* is a device that discriminates what passes through it.
- An LTI-system discriminates different of various frequencies, described by $H(\omega)$, or ak, b_k
- $H(\omega)$ acts as a *spectral weighting function* to different frequency components
- "LTI-system" and "filter" can be used interchangeably
- Examples of filtering
  - removal of undesired noise from target signals
  - spectral shaping, equalization
  - signal detection
  - spectral analysis filters

Ideal Filter Characteristics

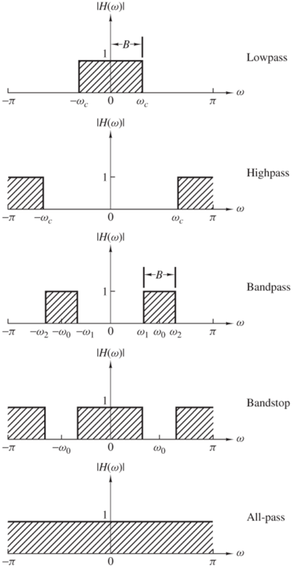


Figure 5.4.1 Magnitude responses for some ideal frequency-selective discrete-time filters.

Ideal Filter Characteristics

Phase

- Ideal filters
  - have a constant (unity-)gain in in passband
  - a linear phase in passband $\Theta(\omega) = -\omega n_0$

$$H(\omega) = \begin{cases} Ce^{-j\omega n_0}, & \omega_1 < \omega < \omega_2 \\ 0, & \text{otherwise} \end{cases}$$

- Due to the time-shifting property, a linear phase results in a constant delay

$$y(n) = Cx(n - n_0)$$

  This pure delay is usually tolerable an not considered a distortion of the signal
- The derivative of the phase w.r.t frequency is envelope delay or **group delay**

$$\tau_g(\omega) = \frac{d\Theta(\omega)}{d\omega}$$

- The group delay is the time-delay frequency component ω undergoes when passing the filter
- For a linear phase, the group delay is constant $\tau_g = n_0$

Limitations

- Ideal filters are not physically realizable (but serve as a mathematical idealization of practical filters)
- Example: ideal low-pass filter:

$$h_{\mathrm{lp}} = \frac{\sin \omega_c \pi n}{\pi n}, \quad -\infty < n < \infty$$

✗ not causal
✗ infinitely long
➡ physically unrealizable

Design of Filters by Placing Zeros and Poles

- Locate poles near the unit circle to emphasize frequencies
- Place zeros near the unit circle to deemphasize frequencies
- The following constraints must be imposed:

Design of Filters by Placing Zeros and Poles

1. All poles should be placed inside the unit circle to generate a stable and causal filter.
   Zeros can be placed anywhere in the z-plane
2. All complex zeros and poles must occur in complex-conjugate pairs in order for the filter coefficients to be real-valued

- Recall that for a given pole-zero pattern, the system function $H(z)$ can be expressed as

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

- b_0 often chosen such that $H(\omega_0) = 1$ in the passband
- often $N \geq M$, filter has more (non-trivial) poles than zeros

Lowpass and Highpass Filters

- In the design of lowpass filters,
  - poles: near unit circle at low frequencies (near $\omega = 0$)
  - zeros: near or on the unit circle at high frequencies (near $\omega = \pi$)
- The opposite holds for highpass filters

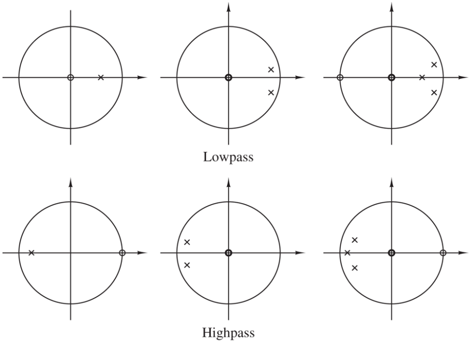


Figure 5.4.2 Pole–zero patterns for several lowpass and highpass filters.

Magnitude and Phase of Lowpass Filter

$$H_1(\omega) = \frac{1-a}{1-az^{-1}}$$

- Additional zero at $z=-1$

$$H_2(z) = \frac{1-a}{2}\,\frac{1+z^{-1}}{1-az^{-1}}$$

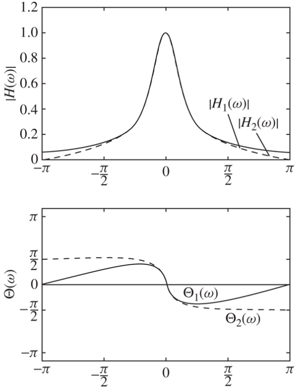


Figure 5.4.3 Magnitude and phase response of (1) a single-pole filter and (2) a one-pole, one-zero filter; $H_1(z) = (1-a)/(1-az^{-1})$, $H_2(z) = [(1-a)/2][(1+z^{-1})/(1-az^{-1})]$ and $a = 0.9$.

Magnitude and Phase of Highpass Filter

- Obtained by reflecting (folding) the pole-zero locations of previous lowpass about the imaginary axis in the z-plane

$$H_3(z) = \frac{1-a}{2}\frac{1-z^{-1}}{1+az^{-1}}$$

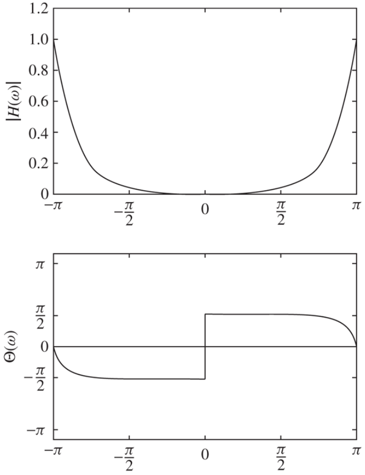


Figure 5.4.4 Magnitude and phase response of a simple highpass filter; $H(z) = [(1-a)/2][(1-z^{-1})/(1+az^{-1})]$ with $a = 0.9$.

Example 5.4.1

- A two-pole lowpass filter has the system function

$$H(z) = \frac{b_0}{(1 - pz^{-1})^2}$$

- Determine the values of B_0 and p such that the frequency response of $H(\omega)$ satisfies the conditions
  - $H(0) = 1$
  - $|H(\frac{\pi}{4})|^2 = \frac{1}{2}$

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- Determine the values of B_0 and p such that the frequency response of $H(\omega)$ satisfies the conditions
  - $H(0) = 1$
  - $|H(\frac{\pi}{4})|^2 = \frac{1}{2}$
- Solution:

$$H(z) = \frac{0.46}{(1 - 0.32z^{-1})^2}$$

Example 5.4.2

- Design a two-pole bandpass filter that has the center of its passband at $\omega = \pi/2$, zero in its frequency response characteristic at $\omega = 0$ and $\omega = \pi$, and a magnitude repsonse of $1/\sqrt{2}$ at $\omega = 4\pi/9$

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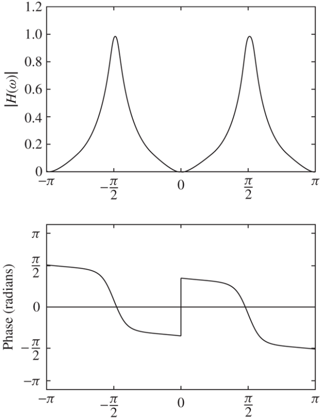


Figure 5.4.5 Magnitude and phase response of a simple bandpass filter in Example 5.4.2; $H(z) = 0.15[(1 - z^{-2})/(1 + 0.7z^{-2})]$.

- Given a prototype lowpass, a highpass can be obtained by translation

$$H_{\mathrm{hp}}(\omega) = H_{\mathrm{lp}}(\omega - \pi)$$

- Or due to the frequency shifting property of the Fourier transform

$$h_{\mathrm{hp}}(n) = e^{j\pi n} h_{\mathrm{lp}}(n) = (-1)^n h_{\mathrm{lp}}(n)$$

and conversely

$$h_{\mathrm{lp}}(n) = (-1)^n h_{\mathrm{hp}}(n)$$

i.e. sign-change for every other sample

Example 5.4.3

- Convert the following lowpass filter into a highpass

$$y(n) = 0.9y(n-1) + 0.1x(n)$$

- Convert the following lowpass filter into a highpass

$$y(n) = 0.9y(n-1) + 0.1x(n)$$

- Solution: The difference equation for the HP filter is

$$y(n) = -0.9y(n-1) + 0.1x(n)$$

with frequency response

$$H_{\mathrm{hp}}(\omega) = \frac{0.1}{1 + 0.9e^{-j\omega}}$$

Digital Resonators I

- A *digital resonator* is a special two-pole bandpass filter with the pair of complex-conjugate poles located near the unit circle
- The name *resonator* refers ti the fact the the filter has a large magnitude response (i.e. it resonates) in the vicinity of the poles
- Useful in many applications including bandpass filtering and speech generation
- For a resonant peak at $\omega = \omega_0$ we select the complex-conjugate poles at

$$p_{1,2} = re^{\pm j\omega_0}, \quad 0 < r < 1$$

- Two placings of zeros are of special interest
  - Two zeros at the origin
  - One zero at $z = 1$ and one at $z = -1$

Two zeros at the origin

$$H(z) = \frac{b_0}{(1 - re^{j\omega_0 z^{-1}})(1 - re^{-j\omega_0 z^{-1}})}$$

$$= \frac{b_0}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}}$$

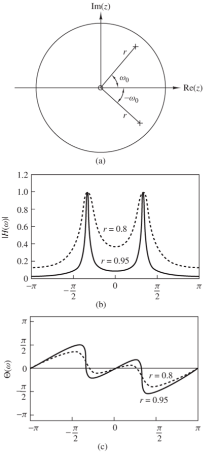


Figure 5.4.6 (a) Pole–zero pattern and (b) the corresponding magnitude and phase response of a digital resonator with (1) $r = 0.8$ and (2) $r = 0.95$.

Digital Resonators III

One zero at $z = 1$ and one at $z = -1$

- Slightly smaller bandwidth
- Very small shift in the resonant frequency due to the presence of the zeros

$$H(z) = \frac{G(1 - z^{-1})(1 + z^{-1})}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}$$

$$= \frac{G(1 - z^{-2})}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}}$$

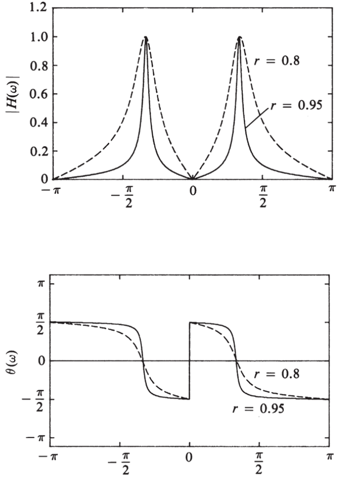


Figure 5.4.7 Magnitude and phase response of digital resonator with zeros at $\omega = 0$ and $\omega = \pi$ and (1) $r = 0.8$ and (2) $r = 0.95$.

- A filter that contains one or more deep notches (or ideally perfect nulls) in its frequency response.

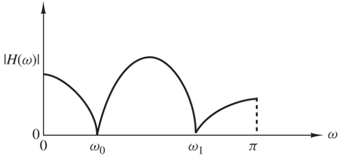


Figure 5.4.8 Frequency response characteristic of a notch filter.

- Useful in many applications where specific frequency components must be eliminated
  - Example: Instrumentation and recording systems require the elimination of the 50Hz power-line frequency
- Simply introduce a pair of complex-conjugate zeros on the unit circle at an angle ω_0

$$z_{1,2} = e^{\pm j w_0}$$

$$H(z) = b_0(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$$

Notch Filters II

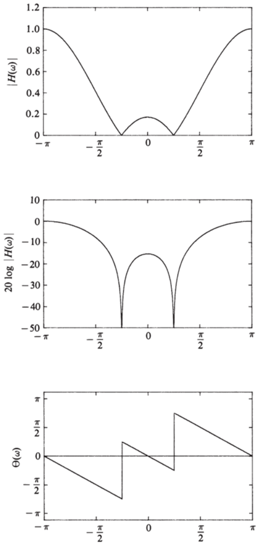


Figure 5.4.9 Frequency response characteristics of a notch filter with a notch at $\omega = \pi/4$ or $f = 1/8$; $H(z) = G[1 - 2\cos\omega_0 z^{-1} + z^{-2}]$.

Notch Filters

With additional poles

- Problem of an all-zero notch filter: broad bandwidth of notches
- idea: place poles in vicinity of zeros to reduce bandwidth of notches

$$p_{1,2} = re^{\pm j\omega_0}$$

- Problem: may result in ripples in the passband. Reduction possible by introducing additional poles/zeros (trial-and-error)

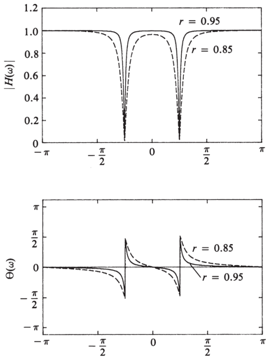


Figure 5.4.10 Frequency response characteristics of two notch filters with poles at (1) $r = 0.85$ and (2) $r = 0.95$;

Comb Filters I

- In its simplest form, a notch filter with periodically occurring notches

➡ E.g. suppression of power-line harmonics

- Taking any FIR filter $h(n)$ with system function

$$H(z) = \sum_{k=0}^{M} h(k) z^{-k}$$

an Lth order repetition can be obtained by replacing z by z^L

$$H_L(z) = \sum_{k=0}^{M} h(k) z^{-kL}$$

$$H_L(\omega) = \sum_{k=0}^{M} h(k) e^{-jkL\omega} = H(L\omega)$$

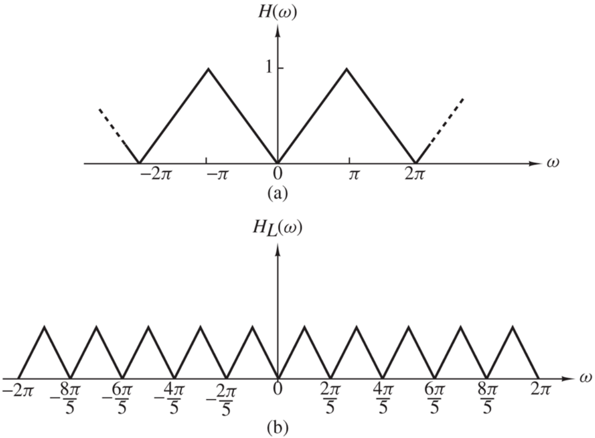


Figure 5.4.12 Comb filter with frequency response $H_L(\omega)$ obtained from $H(\omega)$.

All-Pass Filters I

- An all-pass filter is defined as a system that has a constant magnitude response for all frequencies

$$|H(\omega)| = 1, \quad 0 \leq \omega \leq \pi$$

- simplest examples: pure delay system $H(z) = z^{-k}$
- All pass filter obtained when zeros and poles are reciprocal to each other (if z_0 is a pole of $H(z)$, then $1/z_0$ is a zero)

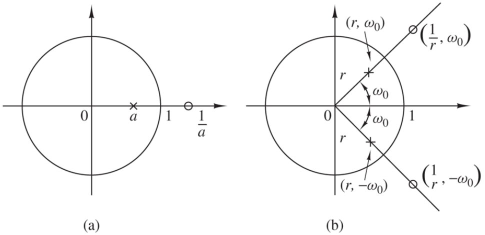
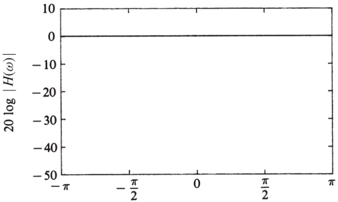


Figure 5.4.16 Pole–zero patterns of (a) a first-order and (b) a second-order all-pass filter.



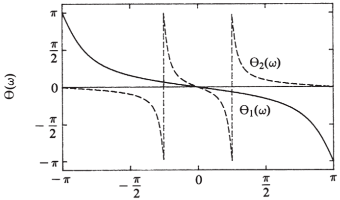


Figure 5.4.17 Frequency response characteristics of an all-pass filter with system functions (1) $H(z) = (0.6 + z^{-1})/(1 + 0.6z^{-1})$, (2) $H(z) = (r^2 - 2r\cos\omega_0 z^{-1} + z^{-2})/\ (1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2})$, $r = 0.9$, $\omega_0 = \pi/4$.

➡ All-pass filters find application as phase equalizers

- Two-pole resonator with complex-conjugate poles *on* the unit circle

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

with $a_1 = -2\cos\omega_0$ and $a_2 = 1$

- for $b_0 = A\sin\omega_0$ the unit sample response is a *sinusoid*

$$h(n) = A\sin((n+1)\omega_0)\, u(n)$$

- Basic component of a digital frequency synthesizer

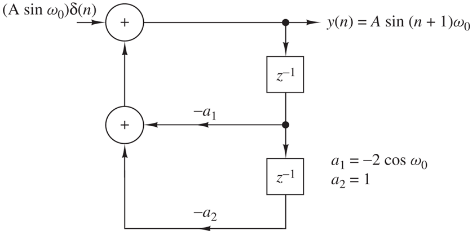


Figure 5.4.18 Digital sinusoidal generator.

- $y(n) = -a_1 y(n-1) - y(n-2) + b_0\delta(n)$

Outline

SP Signal Processing

Invertibility of LTI Systems I

- Often we are interested in inverting the effect of LTI systems, e.g.
  - remove the distortion introduced by a telephone channel
  - remove the reverberance introduced by a room
- However, not every system is invertible.
- To be invertible, the inverse of the tranfer function has to be stable.
- Think of an ideal low-pass that sets higher frequency components to zero. These frequencies are lost and cannot be reconstructed

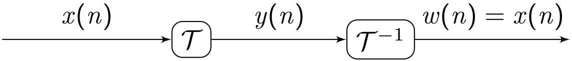
Invertibility of LTI Systems II

- A system is said to be **invertible** if there is a one-to-one correspondence between its input and output signals
  - If we know the output sequence $y(n), -\infty < n < \infty$ of an invertible system $\mathcal{T}$, we can uniquely determine its input $x(n), -\infty < n < \infty$
  - The **inverse system** with input $y(n)$ and output $x(n)$ is denoted by $\mathcal{T}^{-1}$
  - The cascade connection of a system and its inverse is equivalent to the identity system

$$w(n) = \mathcal{T}^{-1}\{y(n)\} = \mathcal{T}^{-1}\{\mathcal{T}\{x(n)\}\} = x(n)$$

- Examples for invertible systems: $y(n) = ax(n)$, $y(n) = x(n-5)$
- Examples for non-invertible systems: $y(n) = x^2(n)$, $y(n) = 0$

Invertibility of LTI Systems III



- The cascade can be written as the convolution

$$w(n) = h_I(n) * h(n) * x(n) = x(n)$$

- Implies that $h(n) * h_I(n) = \delta(n)$
- Solution in time-domain usually difficult. Easier in z-domain, where

$$H(z)H_I(z) = 1$$

and therefore

$$H_I(z) = \frac{1}{H(z)}$$

Invertibility of LTI Systems IV

- Thus, if $H(z)$ has a rational system function

  $$H(z) = \frac{B(z)}{A(z)}$$

  then

  $$H_I(z) = \frac{A(z)}{B(z)}$$

  i.e. the poles and zeros switch!
- if $H(z)$ is an FIR system, its inverse $H_I(z)$ is an all-pole system
- if $H(z)$ is an all-pole system, its inverse $H_I(z)$ is an FIR system

Example 5.5.1

- Determine the inverse of the system with impulse response

$$\left(\frac{1}{2}\right)^{n} u(n)$$

Example 5.5.1

- Determine the inverse of the system with impulse response

$$\left(\frac{1}{2}\right)^n u(n)$$

Solution:

- The system function corresponding to $h(n)$ is

$$\frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

The system is both causal and stable. Since $H(z)$ is an all-pole system, ints inverse is FIR and given by the system function

$$H_I(z) = 1 - \frac{1}{2}z^{-1}$$

- The impulse response is given by

$$h_I(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Example 5.5.2 I

- Determine the inverse of the system with the impulse response

$$h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

- Determine the inverse of the system with the impulse response

$$h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Solution:

- This is an FIR system with system function
  $H(z) = 1 - \frac{1}{2}z^{-1}$, ROC: $|z| > 0$
- The inverse system has the system function

$$H_I(z) = \frac{1}{H(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z}{z - \frac{1}{2}}$$

  i.e. it has a zero at the origin and a pole at $z = \frac{1}{2}$
- Two possible solutions:
  - If we take ROC as $|z| > \frac{1}{2}$: causal and stable system
  - If we take ROC as $|z| < \frac{1}{2}$: anticausal and unstable system

- Determine the inverse of the system with the impulse response

$$h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Solution:

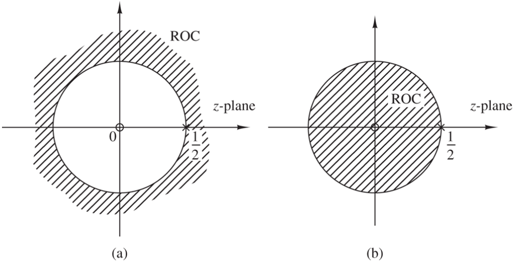


Figure 5.5.2 Two possible regions of convergence for $H(z) = z/(z - \frac{1}{2})$.

- For causal systems we have

$$\sum_{k=0}^{n} h(k)h_I(n-k) = \delta(n)$$

- By assumption,, $h_I(n) = 0$ for $n < 0$.
- For $n = 0$ we obtain

$$h_I(0) = \frac{1}{h(0)}$$

- for $n \geq 1$ the inverse can be obtained recursively as

$$h_I(n) = -\sum_{k=1}^{n} \frac{h(n)h_I(n-k)}{h(0)}, \quad n \geq 1$$

- Practical problem: numerical accuracy deteriorates for large n

Example 5.5.3

- Determine the causal inverse of the FIR system with impulse response

$$h(n) = \delta(n) - \alpha\delta(n-1)$$

- Determine the causal inverse of the FIR system with impulse response

$$h(n) = \delta(n) - \alpha\delta(n-1)$$

Solution:

- Since $h(0) = 1, h(1) = -\alpha$, and $h(n) = 0$ for $n \geq \alpha$, we have

$$h_I(0) = 1/h(0) = 1$$

and

$$h_I(n) = \alpha h_I(n-1), \quad n \geq 1.$$

Consequently

$$h_I(1) = \alpha, h_I(2) = \alpha^2, ..., h_I(n) = \alpha^n$$

$\rightarrow$ the inverse is an IIR system

Motivation

- We will show that for the invertibility of an LTI system is intimately related to the characteristics of its spectral phase function.
- Illustration: Consider the following two systems

$$H_1(z) = 1 + \frac{1}{2}z^{-1} = z^{-1}(z + \frac{1}{2})$$

$$H_2(z) = \frac{1}{2} + z^{-1} = z^{-1}(\frac{1}{2}z + 1)$$

  - $H_1(z)$ has a zero at $z = -\frac{1}{2}$ and impulse response $h = [1, \frac{1}{2}]$
  - $H_2(z)$ has a zero at $z = -2$ and impulse response $h = [\frac{1}{2}, 1]$

➡ reciprocal zeros result in time reversal ($x(-n) \overset{z}{\circ\!\!-\!\!\bullet} X(z^{-1})$)

Motivation

- In the frequency domain the two systems differ only in their phase

$$|H_1(\omega)| = |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos(\omega)}$$

$$\Theta_1(\omega) = -\omega + \tan^{-1}\left(\frac{\sin(\omega)}{\frac{1}{2} + \cos(\omega)}\right)$$

$$\Theta_2(\omega) = -\omega + \tan^{-1}\left(\frac{\sin(\omega)}{2 + \cos(\omega)}\right)$$

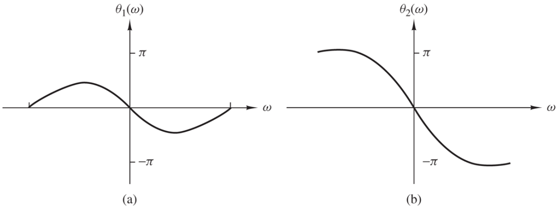


Figure 5.5.3 Phase response characteristics for the systems in (5.5.10). and (5.5.11).

Motivation

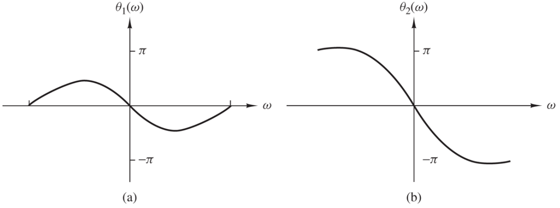


Figure 5.5.3 Phase response characteristics for the systems in (5.5.10). and (5.5.11).

- System one starts with a phase of zero at $\omega = 0$ and ends with a phase of zero at $\omega = \pi$. As the net phase change $\Theta(\pi) - \Theta(0) = 0$, we refer to this system as **minimum-phase system**
- System one starts with a phase of zero at $\omega = 0$ and ends with a phase of π at $\omega = \pi$. As the net phase change $\Theta(\pi) - \Theta(0) = \pi$, we refer to this system as **maximum-phase system**

FIR filters of arbitrary lengths

- Extension to FIR filters of arbitrary lengths is straight forward:
  - An FIR system of length $M+1$ has M zeros.

$$
\begin{array}{lllllll}
H(w) & = b_0 & (1 - z_1 e^{-j\omega}) & (1 - z_2 e^{-j\omega}) & \dots & (1 - z_M e^{-j\omega}) \\
 & = b_0 & H_1(\omega) & H_2(\omega) & \dots & H_M(\omega) \\
 & = b_0 & |H_1(\omega)| e^{j\Theta_1(\omega)} & |H_2(\omega)| e^{j\Theta_1(\omega)} & \dots & |H_M(\omega)| e^{j\Theta_1(\omega)} \\
\Theta(w) & = \angle b_0 & \Theta_1(\omega) & +\Theta_2(\omega) & +\dots+ & \Theta_M(\omega)
\end{array}
$$

  - If all zeros are inside the unit circle, each zero will contribute a net phase change of zero ➡ **minimum phase system**
  - If all zeros are outside the unit circle, each zero will contribute a net phase change of π and $\Theta(\pi) - \Theta(0) = M\pi$
    ➡ **maximum-phase system**
  - FIR systems with some zeros inside and outside the unit circle are called **mixed-phase systems**

Implications

- For an FIR system with real-valued coefficients we have

$$|H(\omega)|^2 = H(z)H(z^{-1})\big|_{z=e^{j\omega}}$$

- This implies that replacing the a zero z_k by its inverse $1/z_k$ does not change the magnitude response

➡ For real-valued FIR systems, we can make a system minimum-phase without changing $|H(\omega)|$ by replace $z_k \to 1/z_k$ such that all zeros are inside the unit circle

- A minimum phase system has the lowest possible delay for a given magnitude!
- The inverse of a minium phase system is an all-pole system with all zeros inside the unit circle ➡ stable causal inverse
- The inverse of mixed-phase system and maximum-phase systems have poles outside the unit circle ➡ not a stable causal inverse

Rational IIR systems

- The minimum-phase property carries over to rational IIR systems described by

$$H(z) = \frac{B(z)}{A(z)}$$

➡ $H(z)$ is **minimum-phase** if all poles and zeros are inside the unit circle

➡ For a stable and causal system (i.e. all roots of $A(z)$ are within the unit circle), the system is **maximum-phase** if all zeros are outside the unit circle

➡ For a stable and causal system, the system is **mixed-phase** if some but not all zeros are outside the unit circle

- A stable pole-zero system that is minimum phase has a stable inverse which is also minimum phase given by

$$H^{-1}(z) = \frac{A(z)}{B(z)}$$

- Mixed-phase and Maximum-phase systems result in unstable inverse systems

Decomposition into minimum-phase and all-pass systems

Goal: make a rational IIR system $H(z) = B(z)/A(z)$ minimum phase!

- Express numerator as $B(z) = B_1(z)B_2(z)$
  - $B_1(z)$ consists of all zeros inside the unit circle
  - $B_2(z)$ consists of all zeros outside the unit circle
- A minium phase system with $|H_{\mathrm{min}}(\omega)| = |H(\omega)|$ is obtained by mirroring the zeros outside the unit circle to be inside as

$$H_{\mathrm{min}}(z) = \frac{B_1(z)B_2(z^{-1})}{A(z)}$$

- The original system is obtained by multiplying an all-pass system $H_{\mathrm{ap}}(z)$ with $|H_{\mathrm{ap}}(\omega)| = 1$

$$H(z) = H_{\mathrm{min}}(z)H_{\mathrm{ap}}(z)$$

with $H_{\mathrm{ap}}(z) = \frac{B_2(z)}{B_2(z^{-1})}$, i.e. a maximum phase system with reciprocal poles and zeros

Content