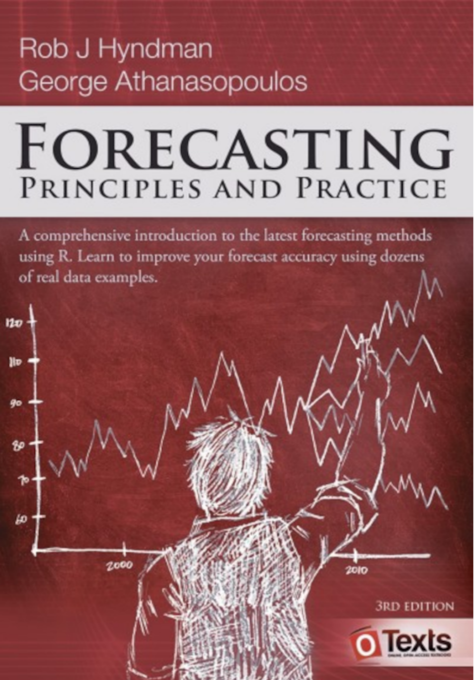

8. Exponential smoothing

8.5 Innovations state space models

OTexts.org/fpp3/

Models and methods

Methods

- Algorithms that return point forecasts.

Models

- Generate same point forecasts but can also generate forecast distributions.
- A stochastic (or random) data generating process that can generate an entire forecast distribution.
- Allow for “proper” model selection.

ETS(A,N,N): SES with additive errors

Component form

Forecast equation	$\hat{y}_{t+h\|t} = \ell_t$
Smoothing equation	$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$

ETS(A,N,N): SES with additive errors

Component form

Forecast equation	$\hat{y}_{t+h\|t} = \ell_t$
Smoothing equation	$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$

Forecast error: $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$.

ETS(A,N,N): SES with additive errors

Component form

Forecast equation	$\hat{y}_{t+h\|t} = \ell_t$
Smoothing equation	$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$

Forecast error: $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$.

Error correction form

$$\begin{aligned} y_t &= \ell_{t-1} + e_t \\ \ell_t &= \ell_{t-1} + \alpha(y_t - \ell_{t-1}) \\ &= \ell_{t-1} + \alpha e_t \end{aligned}$$

ETS(A,N,N): SES with additive errors

Component form

Forecast equation	$\hat{y}_{t+h\|t} = \ell_t$
Smoothing equation	$\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$

Forecast error: $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$.

Error correction form

$$\begin{aligned} y_t &= \ell_{t-1} + e_t \\ \ell_t &= \ell_{t-1} + \alpha(y_t - \ell_{t-1}) \\ &= \ell_{t-1} + \alpha e_t \end{aligned}$$

Specify probability distribution for e_t, we assume $e_t = \varepsilon_t \sim \text{NID}(0, \sigma^2)$.

ETS(A,N,N): SES with additive errors

Measurement equation	$y_t = \ell_{t-1} + \varepsilon_t$
State equation	$\ell_t = \ell_{t-1} + \alpha\varepsilon_t$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- "innovations" or "single source of error" because equations have the same error process, ε_t.
- Measurement equation: relationship between observations and states.
- State equation(s): evolution of the state(s) through time.

ETS(M,N,N): SES with multiplicative errors.

- Specify relative errors $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting $\hat{y}_{t|t-1} = \ell_{t-1}$ gives:
  - $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
  - $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

ETS(M,N,N): SES with multiplicative errors.

- Specify relative errors $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting $\hat{y}_{t|t-1} = \ell_{t-1}$ gives:
  - $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
  - $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

Measurement equation	$y_t = \ell_{t-1}(1 + \varepsilon_t)$
State equation	$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$

ETS(M,N,N): SES with multiplicative errors.

- Specify relative errors $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting $\hat{y}_{t|t-1} = \ell_{t-1}$ gives:
  - $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
  - $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

Measurement equation	$y_t = \ell_{t-1}(1 + \varepsilon_t)$
State equation	$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$

- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.

Specifying the model

```
ETS(y ~ error("A") + trend("N") + season("N"))
ETS(y ~ error("M") + trend("N") + season("N"))
```

By default, an optimal value for α and ℓ_0 is used.

α can be chosen manually in `trend()`.

```
trend("N", alpha = 0.5)
trend("N", alpha_range = c(0.2, 0.8))
```

ETS(A,A,N)

Holt's linear method with additive errors.

- Assume $\varepsilon_t = y_t - \ell_{t-1} - b_{t-1} \sim \text{NID}(0, \sigma^2)$.
- Substituting into the error correction equations for Holt's linear method

$$
\begin{aligned}
y_t &= \ell_{t-1} + b_{t-1} + \varepsilon_t \\
\ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
b_t &= b_{t-1} + \alpha\beta^*\varepsilon_t
\end{aligned}
$$

- For simplicity, set $\beta = \alpha\beta^*$.

ETS(M,A,N)

Holt's linear method with multiplicative errors.

- Assume $\varepsilon_t = \frac{y_t - (\ell_{t-1} + b_{t-1})}{(\ell_{t-1} + b_{t-1})}$
- Following a similar approach as above, the innovations state space model underlying Holt's linear method with multiplicative errors is specified as

$$
\begin{aligned}
y_t &= (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t) \\
b_t &= b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t
\end{aligned}
$$

where again $\beta = \alpha\beta^*$ and $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

Specifying the model

```
ETS(y ~ error("A") + trend("A") + season("N"))
ETS(y ~ error("M") + trend("A") + season("N"))
```

By default, optimal values for β and b_0 are used.

β can be chosen manually in `trend()`.

```
trend("A", beta = 0.004)
trend("A", beta_range = c(0, 0.1))
```

ETS(A,A,A)

Holt-Winters additive method with additive errors.

Forecast equation	$\hat{y}_{t+h\|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$
Observation equation	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$
State equations	$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$
	$b_t = b_{t-1} + \beta\varepsilon_t$
	$s_t = s_{t-m} + \gamma\varepsilon_t$

- Forecast errors: $\varepsilon_t = y_t - \hat{y}_{t|t-1}$
- k is integer part of $(h-1)/m$.

ETS(M,A,M)

Holt-Winters multiplicative method with multiplicative errors.

Forecast equation	$\hat{y}_{t+h\|t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$
Observation equation	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$
State equations	$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$
	$b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$
	$s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

- Forecast errors: $\varepsilon_t = (y_t - \hat{y}_{t|t-1})/\hat{y}_{t|t-1}$
- k is integer part of $(h-1)/m$.

Specifying the model

```
ETS(y ~ error("A") + trend("A") + season("A"))
ETS(y ~ error("M") + trend("A") + season("M"))
```

By default, optimal values for γ and $s_0, s_{-1}, \dots, s_{-m+1}$ are used.

γ can be chosen manually in `season()`.

```
season("A", gamma = 0.004)
season("A", gamma_range = c(0, 0.1))
```

ETS models

Additive Error

Trend Component		Seasonal Component: N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A_d	(Additive damped)	A,A_d,N	A,A_d,A	A,A_d,M

Multiplicative Error

Trend Component		Seasonal Component: N (None)	A (Additive)	M (Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A_d	(Additive damped)	M,A_d,N	M,A_d,A	M,A_d,M

Additive error models

Trend	Seasonal N	Seasonal A	Seasonal M
N	$y_t = \ell_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t$	$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/\ell_{t-1}$
A	$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $b_t = b_{t-1} + \beta\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1} + b_{t-1})$
$\mathbf{A_d}$	$y_t = \ell_{t-1} + \phi b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t$ $b_t = \phi b_{t-1} + \beta\varepsilon_t$	$y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t$ $b_t = \phi b_{t-1} + \beta\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1})s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $b_t = \phi b_{t-1} + \beta\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1} + \phi b_{t-1})$

Multiplicative error models

Trend	Seasonal N	Seasonal A	Seasonal M
N	$y_t = \ell_{t-1}(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$	$y_t = (\ell_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1} + \alpha(\ell_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1}+s_{t-m})\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m}(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$
A	$y_t = (\ell_{t-1}+b_{t-1})(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1}+b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1}+b_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+b_{t-1} + \alpha(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$ $b_t = b_{t-1} + \beta(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1}+b_{t-1})s_{t-m}(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1}+b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$
A$_\mathbf{d}$	$y_t = (\ell_{t-1}+\phi b_{t-1})(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+\phi b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1}+\phi b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1}+\phi b_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+\phi b_{t-1} + \alpha(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1}+\phi b_{t-1})s_{t-m}(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+\phi b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1}+\phi b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$