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# FORECASTING

## PRINCIPLES AND PRACTICE

A comprehensive introduction to the latest forecasting methods using R. Learn to improve your forecast accuracy using dozens of real data examples.



3RD EDITION

 **OTexts**  
Open Texts Publishing

## 9. ARIMA models

### 9.8 Forecasting

[OTexts.org/fpp3/](http://OTexts.org/fpp3/)

# Point forecasts

- 1 Rearrange ARIMA equation so  $y_t$  is on LHS.
- 2 Rewrite equation by replacing  $t$  by  $T + h$ .
- 3 On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with  $h = 1$ . Repeat for  $h = 2, 3, \dots$

# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

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## Point forecasts ( $h=1$ )

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## ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} - \phi_3y_{T-3} + \varepsilon_{T+1} + \theta_1\varepsilon_T$$

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$$\hat{y}_{T+1|T} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} - \phi_3y_{T-3} + \theta_1e_T$$

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$$\hat{y}_{T+2|T} = (1 + \phi_1)\hat{y}_{T+1|T} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} - \phi_3y_{T-2}.$$

# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

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- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

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- Other models beyond scope of this subject.

# Prediction intervals

- Prediction intervals **increase in size with forecast horizon.**
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed.**
- Prediction intervals tend to be too narrow.
  - ▶ the uncertainty in the parameter estimates has not been accounted for.
  - ▶ the ARIMA model assumes historical patterns will not change during the forecast period.
  - ▶ the ARIMA model assumes uncorrelated future errors