ETC3550/ETC5550
Applied forecasting

Ch9. ARIMA models

OTexts.org/fpp3/

ARIMA models

AR: autoregressive (lagged observations as inputs)
I: integrated (differencing to make series stationary)
MA: moving average (lagged errors as inputs)

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An ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns.

Stationarity

Definition

If $\{y_t\}$ is a stationary time series, then for all s, the distribution of $(y_t, \ldots, y_{t+s})$ does not depend on t.

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Transformations help to **stabilize the variance**.

For ARIMA modelling, we also need to **stabilize the mean**.

Differencing

- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series: $y'_t = y_t - y_{t-1}$.
- The differenced series will have only $T - 1$ values since it is not possible to calculate a difference y'_1 for the first observation.

Random walk model

If differenced series is white noise with zero mean:

$$y_t - y_{t-1} = \varepsilon_t \quad \text{or} \quad y_t = y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim NID(0, \sigma^2)$.

- Very widely used for non-stationary data.
- This is the model behind the **naïve method**.
- Random walks typically have:
  - long periods of apparent trends up or down
  - Sudden/unpredictable changes in direction
- Forecast are equal to the last observation
  - future movements up or down are equally likely.

Random walk with drift model

If differenced series is white noise with non-zero mean:

$$y_t - y_{t-1} = c + \varepsilon_t \quad \text{or} \quad y_t = c + y_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim NID(0, \sigma^2)$.

- c is the **average change** between consecutive observations.
- If $c > 0$, y_t will tend to drift upwards and vice versa.
- This is the model behind the **drift method**.

Unit root tests

Statistical tests to determine the required order of differencing.

1. Augmented Dickey Fuller test: null hypothesis is that the data are non-stationary and non-seasonal.
2. Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test: null hypothesis is that the data are stationary and non-seasonal.
3. Other tests available for seasonal data.

Automatically selecting differences

STL decomposition: $y_t = T_t + S_t + R_t$

Seasonal strength $F_s = \max\left(0, 1 - \frac{\text{Var}(R_t)}{\text{Var}(S_t + R_t)}\right)$

If $F_s > 0.64$, do one seasonal difference.

Backshift notation

A very useful notational device is the backward shift operator, B, which is used as follows:

$$By_t = y_{t-1}$$

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Two applications of B to y_t **shifts the data back two periods**:

$$B(By_t) = B^2y_t = y_{t-2}$$

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$$y'_t = y_t - y_{t-1} = y_t - By_t = (1 - B)y_t$$

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A first-order difference can be written as

$$y'_t = y_t - y_{t-1} = y_t - By_t = (1 - B)y_t$$

Similarly, if second-order differences (i.e., first differences of first differences) have to be computed, then:

$$y''_t = y_t - 2y_{t-1} + y_{t-2} = (1 - B)^2 y_t$$

Backshift notation

- Second-order difference is denoted $(1 - B)^2$.
- *Second-order difference* is not the same as a *second difference*, which would be denoted $1 - B^2$;
- In general, a dth-order difference can be written as

$$(1 - B)^d y_t$$

- A seasonal difference followed by a first difference can be written as

$$(1 - B)(1 - B^m) y_t$$

Backshift notation

The “backshift” notation is convenient because the terms can be multiplied together to see the combined effect.

$$
\begin{aligned}
(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\
&= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.
\end{aligned}
$$

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$$\begin{aligned}(1-B)(1-B^m)y_t &= (1-B-B^m+B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

For monthly data, $m = 12$ and we obtain the same result as earlier.

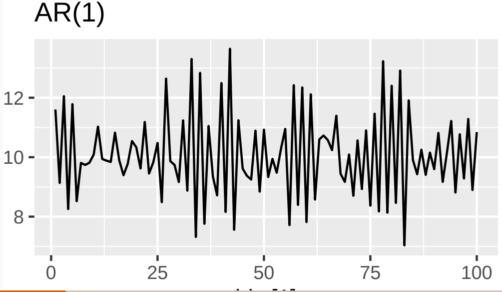
Outline

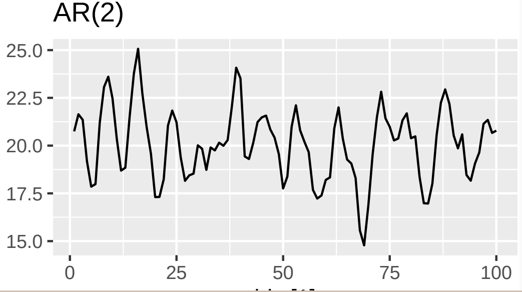
Autoregressive models

Autoregressive (AR) models:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where ε_t is white noise. This is a multiple regression with **lagged values** of y_t as predictors.

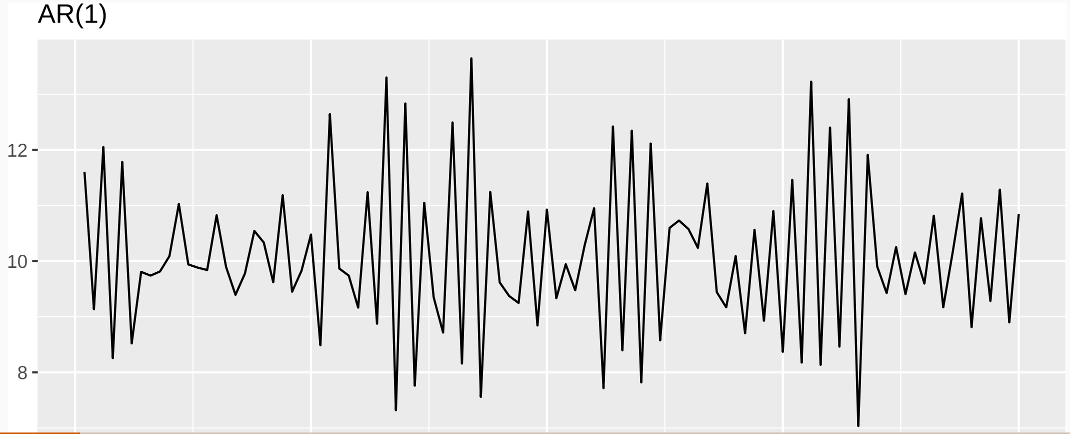




AR(1) model

$$y_t = 18 - 0.8y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0,1), \quad T = 100.$$



AR(1) model

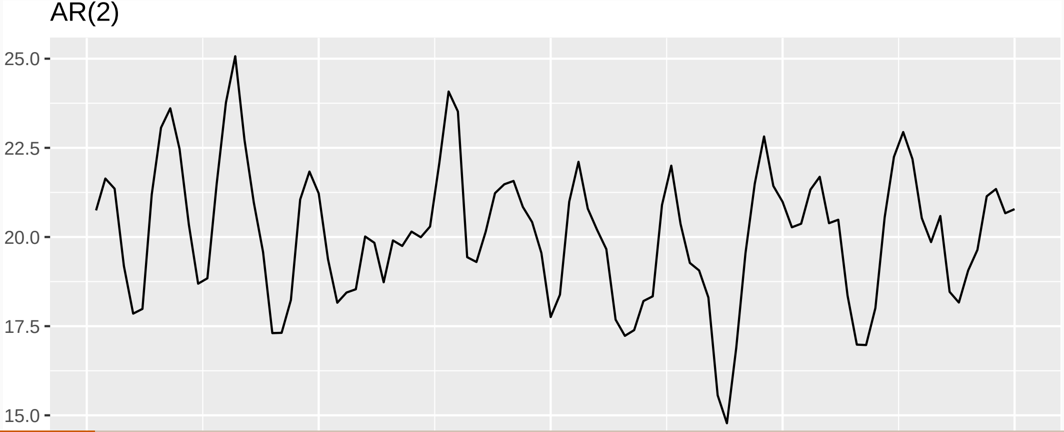
$$y_t = c + \phi_1 y_{t-1} + \varepsilon_t$$

- When $\phi_1 = 0$, y_t is **equivalent to WN**
- When $\phi_1 = 1$ and $c = 0$, y_t is **equivalent to a RW**
- When $\phi_1 = 1$ and $c \neq 0$, y_t is **equivalent to a RW with drift**
- When $\phi_1 < 0$, y_t tends to **oscillate between positive and negative values**.

AR(2) model

$$y_t = 8 + 1.3y_{t-1} - 0.7y_{t-2} + \varepsilon_t$$

$$\varepsilon_t \sim N(0,1), \qquad T = 100.$$



Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ lie outside the unit circle on the complex plane.

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General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ lie outside the unit circle on the complex plane.

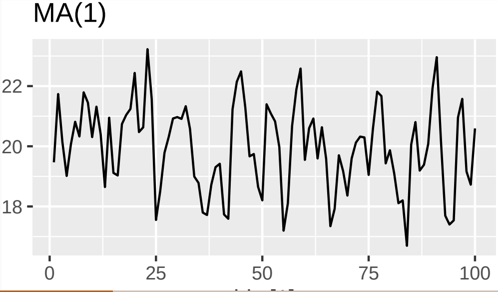
- For $p = 1$: $-1 < \phi_1 < 1$.
- For $p = 2$:
  $-1 < \phi_2 < 1 \qquad \phi_2 + \phi_1 < 1 \qquad \phi_2 - \phi_1 < 1.$
- More complicated conditions hold for $p \geq 3$.
- Estimation software takes care of this.

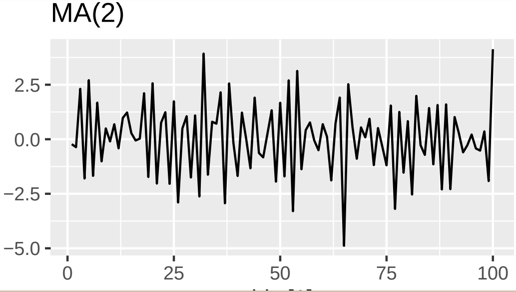
Moving Average (MA) models

Moving Average (MA) models:

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q},$$

where ε_t is white noise. This is a multiple regression with **past *errors*** as predictors. *Don't confuse this with moving average smoothing!*

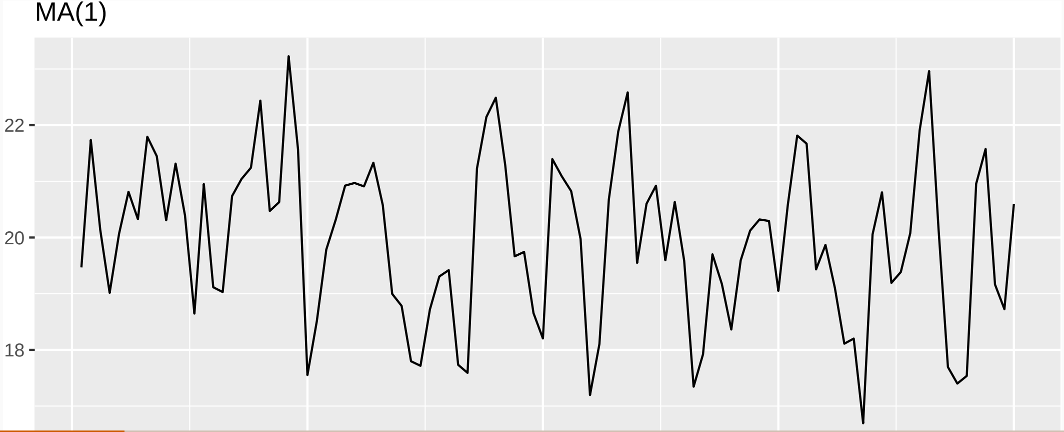




MA(1) model

$$y_t = 20 + \varepsilon_t + 0.8\varepsilon_{t-1}$$

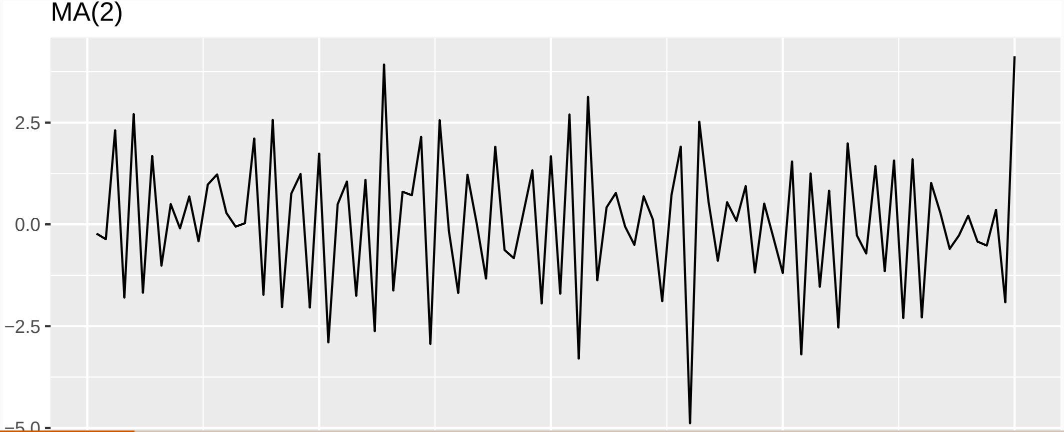
$$\varepsilon_t \sim N(0,1), \quad T = 100.$$



MA(2) model

$$y_t = \varepsilon_t - \varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

$$\varepsilon_t \sim N(0,1), \quad T = 100.$$



MA(∞) models

It is possible to write any stationary AR(p) process as an MA(∞) process.

Example: AR(1)

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \varepsilon_t \\
&= \phi_1(\phi_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= \phi_1^2 y_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\
&= \phi_1^3 y_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\
&\dots
\end{aligned}
$$

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&= \phi_1^2 y_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\
&= \phi_1^3 y_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\
&\dots
\end{aligned}
$$

Provided $-1 < \phi_1 < 1$:

Invertibility

- Any MA(q) process can be written as an AR(∞) process if we impose some constraints on the MA parameters.
- Then the MA model is called “invertible”.
- Invertible models have some mathematical properties that make them easier to use in practice.
- Invertibility of an ARIMA model is equivalent to forecastability of an ETS model.

Invertibility

General condition for invertibility

Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ lie outside the unit circle on the complex plane.

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Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ lie outside the unit circle on the complex plane.

- For $q = 1$: $-1 < \theta_1 < 1$.
- For $q = 2$:

  $$-1 < \theta_2 < 1 \qquad \theta_2 + \theta_1 > -1 \qquad \theta_1 - \theta_2 < 1.$$
- More complicated conditions hold for $q \geq 3$.
- Estimation software takes care of this.

ARIMA models

Autoregressive Moving Average models:

$$\begin{aligned} y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t. \end{aligned}$$

ARIMA models

Autoregressive Moving Average models:

$$
\begin{aligned}
y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\
+ \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.
\end{aligned}
$$

- Predictors include both **lagged values of y_t and lagged errors.**
- Conditions on AR coefficients ensure stationarity.
- Conditions on MA coefficients ensure invertibility.

ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

- Predictors include both **lagged values of y_t and lagged errors.**
- Conditions on AR coefficients ensure stationarity.
- Conditions on MA coefficients ensure invertibility.

Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing**.
- $(1-B)^d y_t$ follows an ARMA model.

ARIMA models

Autoregressive Integrated Moving Average models

ARIMA(p, d, q) model

AR:	p = order of the autoregressive part
I:	d = degree of first differencing involved
MA:	q = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR(p): ARIMA(p,0,0)
- MA(q): ARIMA(0,0,q)

Backshift notation for ARIMA

- ARMA model:

$$y_t = c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \cdots + \theta_q B^q \varepsilon_t$$

$$\text{or} \quad (1 - \phi_1 B - \cdots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \cdots + \theta_q B^q)\varepsilon_t$$

- ARIMA(1,1,1) model:

$$\underset{\substack{\uparrow \\ \text{AR(1)}}}{(1 - \phi_1 B)} \quad \underset{\substack{\uparrow \\ \text{First} \\ \text{difference}}}{(1 - B) y_t} \quad = \quad c + \underset{\substack{\uparrow \\ \text{MA(1)}}}{(1 + \theta_1 B)\varepsilon_t}$$

Backshift notation for ARIMA

- ARMA model:

$$y_t = c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \cdots + \theta_q B^q \varepsilon_t$$

$$\text{or} \quad (1 - \phi_1 B - \cdots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \cdots + \theta_q B^q)\varepsilon_t$$

- ARIMA(1,1,1) model:

$$\underset{\substack{\uparrow \\ \text{AR(1)}}}{(1 - \phi_1 B)} \quad \underset{\substack{\uparrow \\ \text{First} \\ \text{difference}}}{(1 - B)} y_t = c + \underset{\substack{\uparrow \\ \text{MA(1)}}}{(1 + \theta_1 B)} \varepsilon_t$$

Expand: $\quad y_t = c + y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$

R model

Intercept form

$$(1 - \phi_1 B - \cdots - \phi_p B^p) y_t' = c + (1 + \theta_1 B + \cdots + \theta_q B^q)\varepsilon_t$$

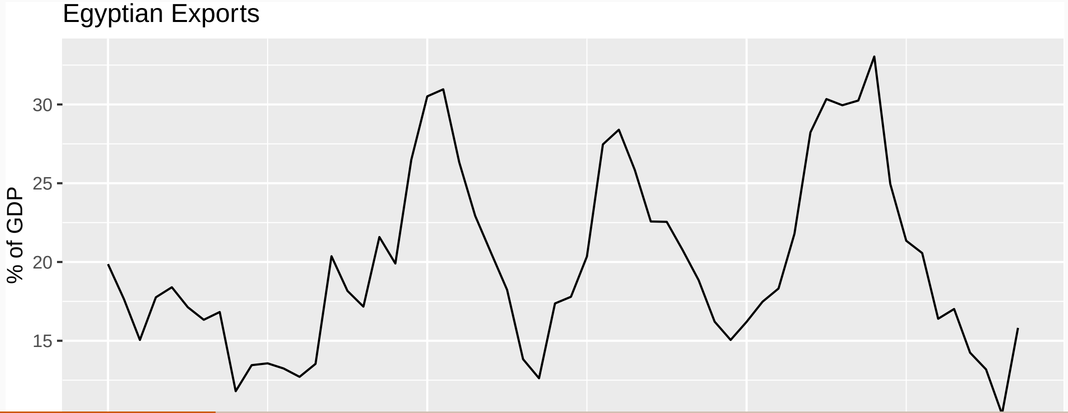
Mean form

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(y_t' - \mu) = (1 + \theta_1 B + \cdots + \theta_q B^q)\varepsilon_t$$

- $y_t' = (1 - B)^d y_t$
- μ is the mean of y_t'.
- $c = \mu(1 - \phi_1 - \cdots - \phi_p)$.
- fable uses intercept form

Egyptian exports

```
global_economy |>
  filter(Code == "EGY") |>
  autoplot(Exports) +
  labs(y = "% of GDP", title = "Egyptian Exports")
```



Egyptian exports

```
fit <- global_economy |>
  filter(Code == "EGY") |>
  model(ARIMA(Exports))
report(fit)
```

```
Series: Exports
Model: ARIMA(2,0,1) w/ mean

Coefficients:
        ar1      ar2     ma1  constant
      1.676  -0.8034  -0.690     2.562
s.e.  0.111   0.0928   0.149     0.116

sigma^2 estimated as 8.046:  log likelihood=-142
AIC=293   AICc=294   BIC=303
```

Egyptian exports

```r
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```

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sigma^2 estimated as 8.046:  log likelihood=-142
```

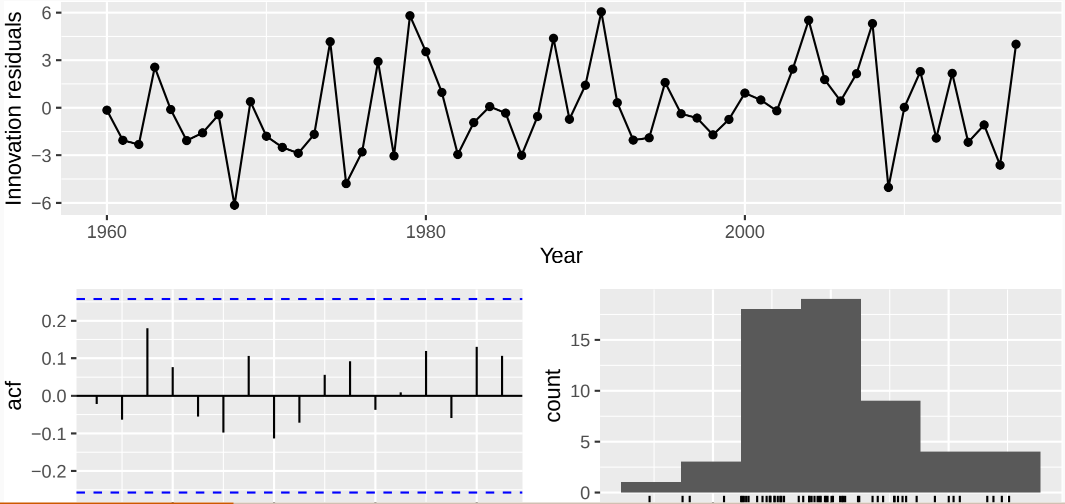
ARIMA(2,0,1) model:

$$y_t = 2.56 + 1.68y_{t-1} - 0.80y_{t-2} - 0.69\varepsilon_{t-1} + \varepsilon_t,$$

where ε_t is white noise with a standard deviation of $2.837 = \sqrt{8.046}$

Egyptian exports

```r
gg_tsresiduals(fit)
```



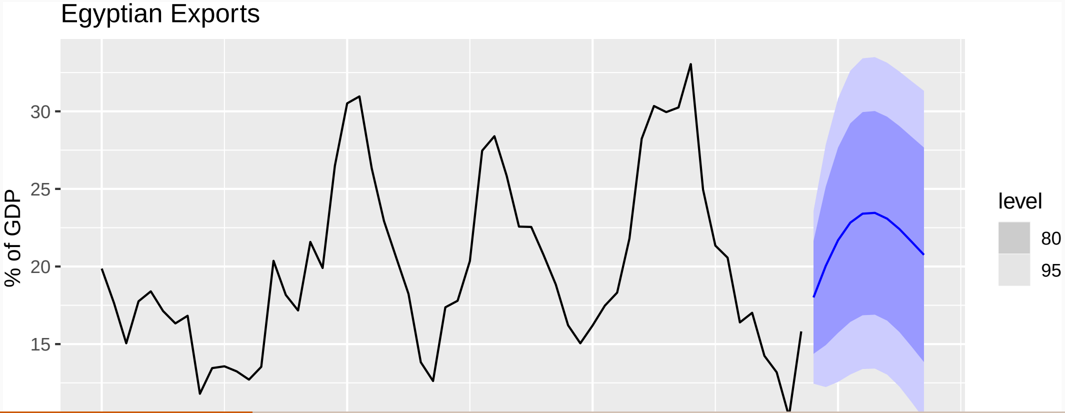
Egyptian exports

```
augment(fit) |>
  features(.innov, ljung_box, lag = 10, dof = 4)
```

```
# A tibble: 1 x 4
  Country          .model          lb_stat lb_pvalue
  <fct>            <chr>             <dbl>     <dbl>
1 Egypt, Arab Rep. ARIMA(Exports)     5.78     0.448
```

Egyptian exports

```
fit |>
  forecast(h = 10) |>
  autoplot(global_economy) +
  labs(y = "% of GDP", title = "Egyptian Exports")
```



Understanding ARIMA models

- If $c = 0$ and $d = 0$, the long-term forecasts will go to zero.
- If $c = 0$ and $d = 1$, the long-term forecasts will go to a non-zero constant.
- If $c = 0$ and $d = 2$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 0$, the long-term forecasts will go to the mean of the data.
- If $c \neq 0$ and $d = 1$, the long-term forecasts will follow a straight line.

Understanding ARIMA models

Forecast variance and d

- The higher the value of d, the more rapidly the prediction intervals increase in size.
- For $d = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

Cyclic behaviour

- For cyclic forecasts, $p \geq 2$ and some restrictions on coefficients are required.
- If $p = 2$, we need $\phi_1^2 + 4\phi_2 < 0$. Then average cycle of length

$$(2\pi)/\left[\text{arc cos}(-\phi_1(1-\phi_2)/(4\phi_2))\right].$$

Outline

Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters $c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$.

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- MLE is very similar to least squares estimation obtained by minimizing
$$\sum_{t-1}^{T} e_t^2$$
- The `ARIMA()` function allows CLS or MLE estimation.
- Non-linear optimization must be used in either case.
- Different software will give different estimates.

Partial autocorrelations

Partial autocorrelations measure relationship between y_t and y_{t-k}, when the effects of other time lags — $1, 2, 3, \dots, k-1$ — are removed.

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Partial autocorrelations measure relationship between y_t and y_{t-k}, when the effects of other time lags — $1, 2, 3, \dots, k-1$ — are removed.

$$\alpha_k = k\text{th partial autocorrelation coefficient}$$
$$= \text{equal to the estimate of } \phi_k \text{ in regression:}$$
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k} + \varepsilon_t.$$

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$$
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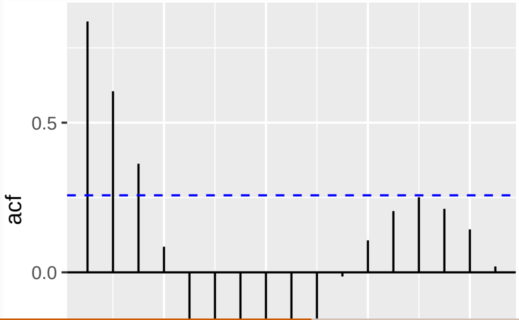
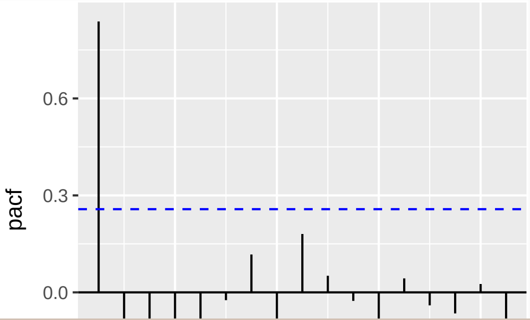
&= \text{equal to the estimate of } \phi_k \text{ in regression:} \\
&\qquad y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k} + \varepsilon_t.
\end{aligned}
$$

- Varying number of terms on RHS gives α_k for different values of k.
- $\alpha_1 = \rho_1$
- same critical values of $\pm 1.96/\sqrt{T}$ as for ACF.
- Last significant α_k indicates the order of an AR model.

Egyptian exports

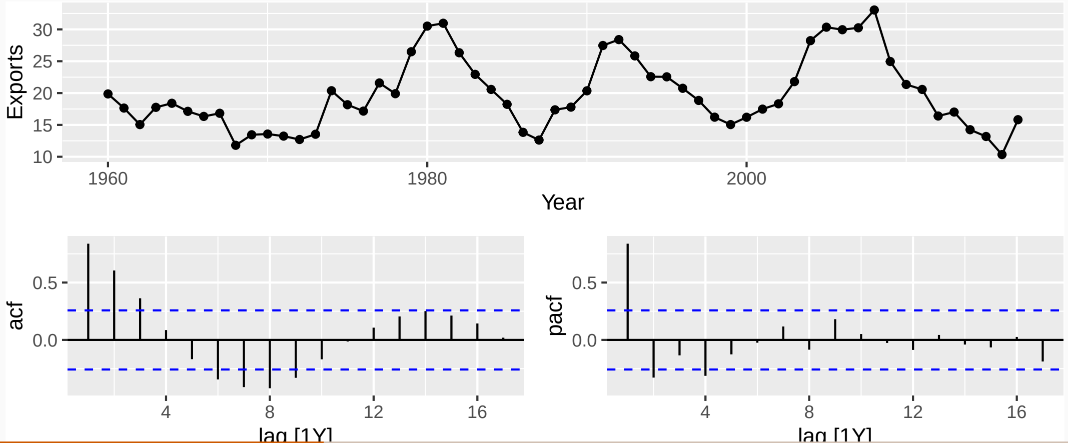
```
egypt <- global_economy |> filter(Code == "EGY")
egypt |>
  ACF(Exports) |>
  autoplot()
egypt |>
  PACF(Exports) |>
  autoplot()
```





Egyptian exports

```
global_economy |>
  filter(Code == "EGY") |>
  gg_tsdisplay(Exports, plot_type = "partial")
```



ACF and PACF interpretation

AR(1)

$$\rho_k = \phi_1^k \qquad \text{for } k = 1, 2, \dots;$$
$$\alpha_1 = \phi_1 \qquad \alpha_k = 0 \qquad \text{for } k = 2, 3, \dots.$$

So we have an AR(1) model when

- autocorrelations exponentially decay
- there is a single significant partial autocorrelation.

ACF and PACF interpretation

AR(p)

- ACF dies out in an exponential or damped sine-wave manner
- PACF has all zero spikes beyond the pth spike

So we have an AR(p) model when

- the ACF is exponentially decaying or sinusoidal
- there is a significant spike at lag p in PACF, but none beyond p

ACF and PACF interpretation

MA(1)

$$\rho_1 = \theta_1/(1+\theta_1^2) \qquad \rho_k = 0 \qquad \text{for } k = 2,3,\dots;$$
$$\alpha_k = -(-\theta_1)^k/(1+\theta_1^2+\cdots+\theta_1^{2k})$$

So we have an MA(1) model when

- the PACF is exponentially decaying and
- there is a single significant spike in ACF

ACF and PACF interpretation

MA(q)

- PACF dies out in an exponential or damped sine-wave manner
- ACF has all zero spikes beyond the qth spike

So we have an MA(q) model when

- the PACF is exponentially decaying or sinusoidal
- there is a significant spike at lag q in ACF, but none beyond q

Information criteria

Akaike's Information Criterion (AIC):

$$\text{AIC} = -2\log(L) + 2(p + q + k + 1),$$

where L is the likelihood of the data, $k = 1$ if $c \neq 0$ and $k = 0$ if $c = 0$.

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Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p+q+k+1)(p+q+k+2)}{T-p-q-k-2}.$$

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Bayesian Information Criterion:

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Information criteria

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Bayesian Information Criterion:

$$\text{BIC} = \text{AIC} + [\log(T) - 2](p+q+k+1).$$

Good models are obtained by minimizing either the AIC, AICc or BIC.
Our preference is to use the AICc.

Outline

How does ARIMA() work?

A non-seasonal ARIMA process

$$\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$$

Need to select appropriate orders: p, q, d

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d and D via KPSS test and seasonal strength measure.
- Select p, q by minimising AICc.
- Use stepwise search to traverse model space.

How does ARIMA() work?

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \frac{(p+q+k+2)}{T-p-q-k-2}\right].$$

where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

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Step1: Select current model (with smallest AICc) from:
ARIMA$(2, d, 2)$, ARIMA$(0, d, 0)$, ARIMA$(1, d, 0)$, ARIMA$(0, d, 1)$

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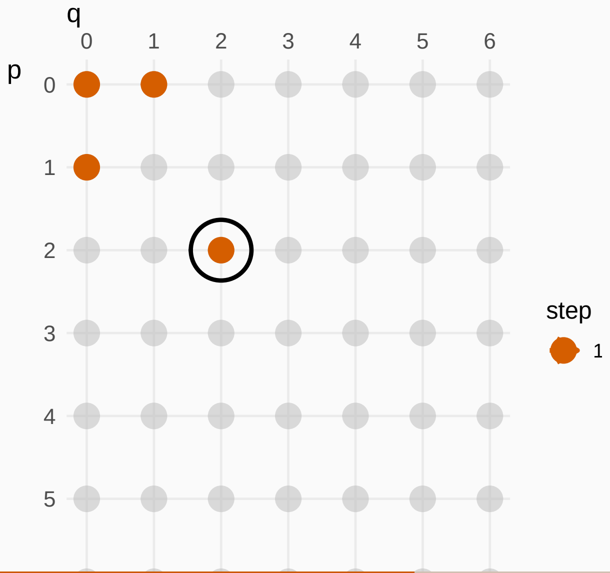
Step1: Select current model (with smallest AICc) from:
ARIMA$(2, d, 2)$, ARIMA$(0, d, 0)$, ARIMA$(1, d, 0)$, ARIMA$(0, d, 1)$

Step 2: Consider variations of current model:

- vary one of p, q, from current model by ± 1;
- p, q both vary from current model by ± 1;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

How does ARIMA() work?



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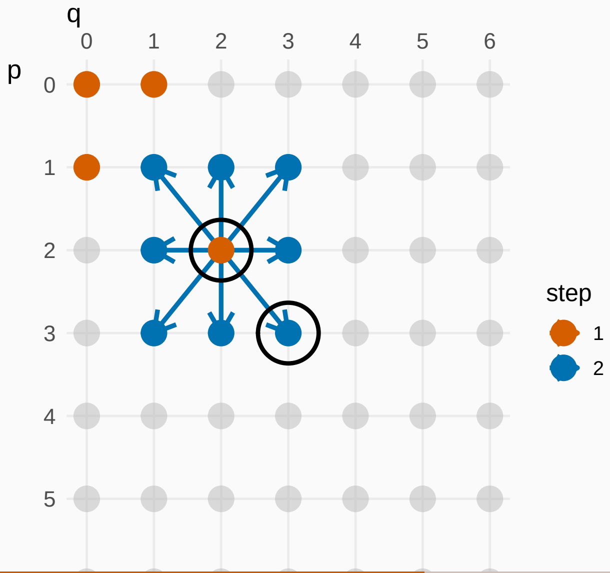

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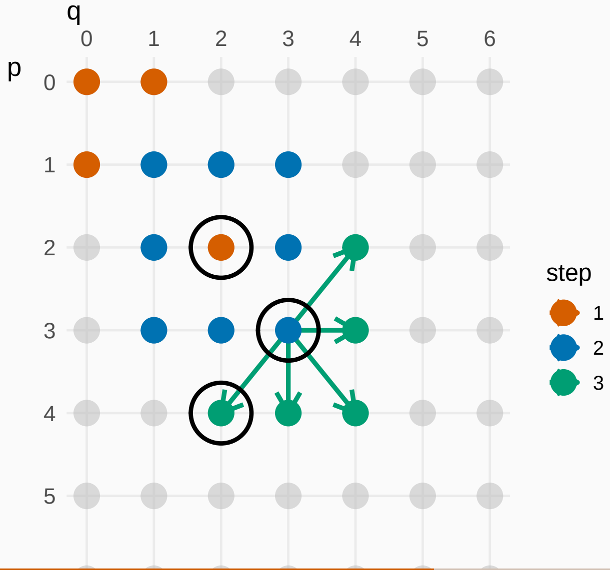
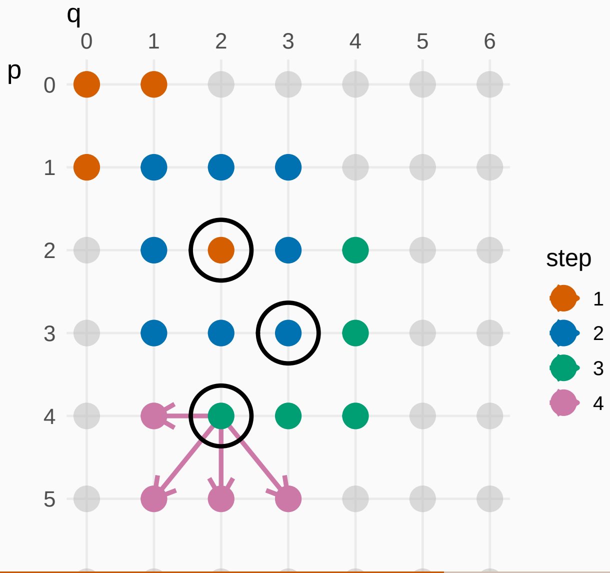
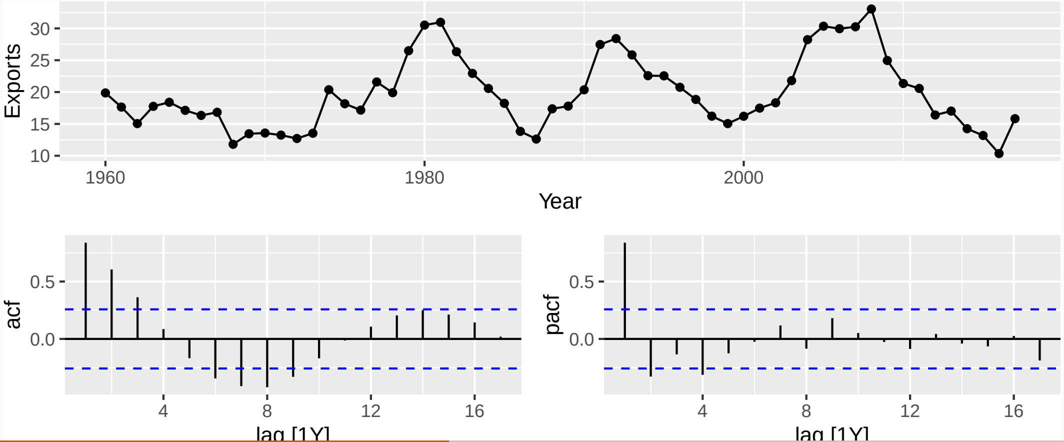


How does ARIMA() work?



Egyptian exports

```
global_economy |>
  filter(Code == "EGY") |>
  gg_tsdisplay(Exports, plot_type = "partial")
```



Egyptian exports

```r
fit1 <- global_economy |>
  filter(Code == "EGY") |>
  model(ARIMA(Exports ~ pdq(4, 0, 0)))
report(fit1)
```

```
Series: Exports
Model: ARIMA(4,0,0) w/ mean

Coefficients:
        ar1     ar2    ar3     ar4  constant
      0.986  -0.172  0.181  -0.328     6.692
s.e.  0.125   0.186  0.186   0.127     0.356

sigma^2 estimated as 7.885:  log likelihood=-141
AIC=293   AICc=295   BIC=305
```

Egyptian exports

```r
fit2 <- global_economy |>
  filter(Code == "EGY") |>
  model(ARIMA(Exports))
report(fit2)
```

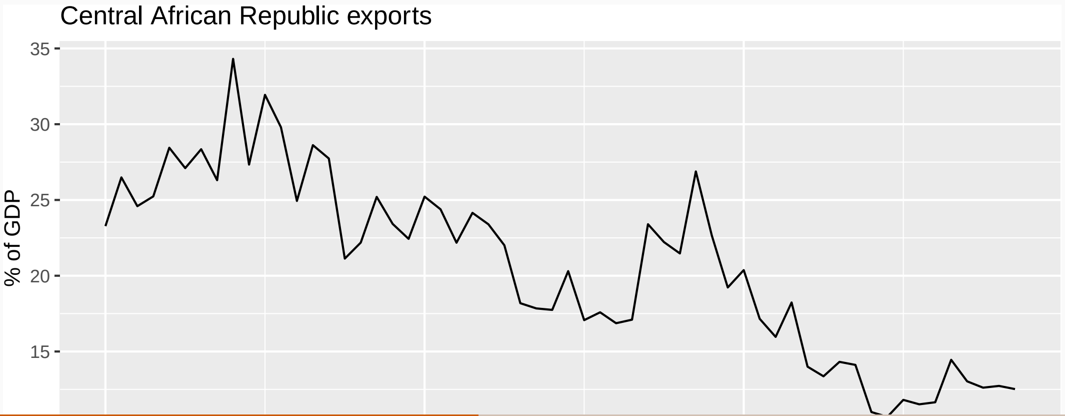
```
Series: Exports
Model: ARIMA(2,0,1) w/ mean

Coefficients:
         ar1      ar2     ma1  constant
       1.676  -0.8034  -0.690     2.562
s.e.   0.111   0.0928   0.149     0.116

sigma^2 estimated as 8.046:  log likelihood=-142
AIC=293   AICc=294   BIC=303
```

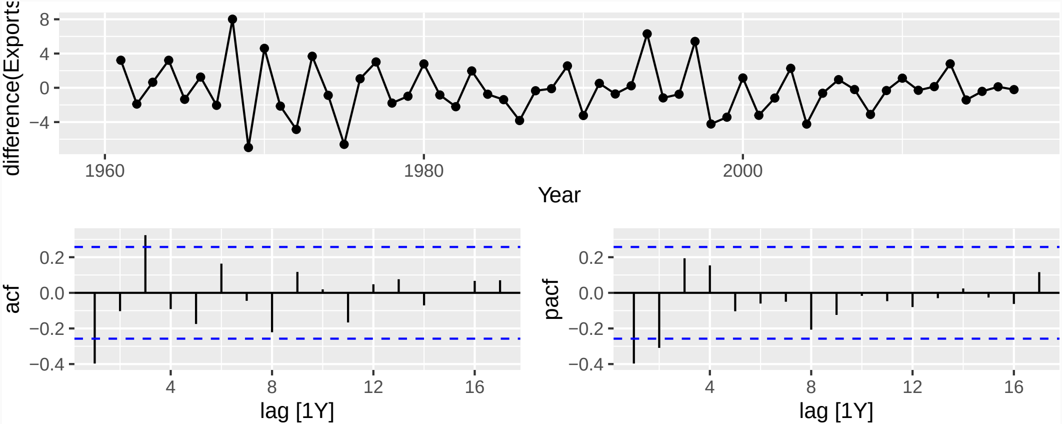
Central African Republic exports

```r
global_economy |>
  filter(Code == "CAF") |>
  autoplot(Exports) +
  labs(title = "Central African Republic exports", y = "% of GDP")
```



Central African Republic exports

```r
global_economy |>
  filter(Code == "CAF") |>
  gg_tsdisplay(difference(Exports), plot_type = "partial")
```



Central African Republic exports

```
caf_fit <- global_economy |>
  filter(Code == "CAF") |>
  model(
    arima210 = ARIMA(Exports ~ pdq(2, 1, 0)),
    arima013 = ARIMA(Exports ~ pdq(0, 1, 3)),
    stepwise = ARIMA(Exports),
    search = ARIMA(Exports, stepwise = FALSE)
  )
```

Central African Republic exports

```r
caf_fit |> pivot_longer(!Country,
  names_to = "Model name",
  values_to = "Orders"
)
```

```
# A mable: 4 x 3
# Key:     Country, Model name [4]
  Country                  `Model name`         Orders
  <fct>                    <chr>               <model>
1 Central African Republic arima210      <ARIMA(2,1,0)>
2 Central African Republic arima013      <ARIMA(0,1,3)>
3 Central African Republic stepwise      <ARIMA(2,1,2)>
4 Central African Republic search        <ARIMA(3,1,0)>
```

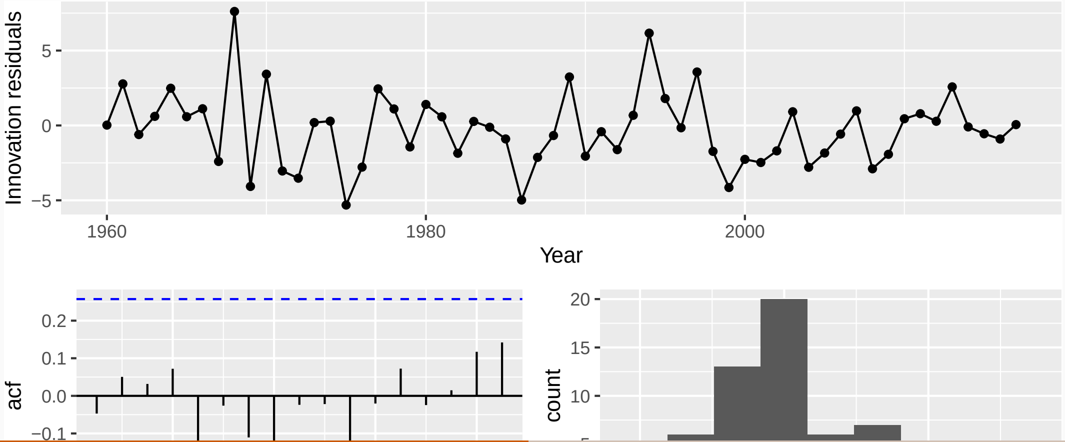
Central African Republic exports

```r
glance(caf_fit) |>
  arrange(AICc) |>
  select(.model:BIC)
```

```
# A tibble: 4 x 6
  .model   sigma2 log_lik   AIC  AICc   BIC
  <chr>     <dbl>   <dbl> <dbl> <dbl> <dbl>
1 search     6.52   -133.  274.  275.  282.
2 arima210   6.71   -134.  275.  275.  281.
3 arima013   6.54   -133.  274.  275.  282.
4 stepwise   6.42   -132.  274.  275.  284.
```

Central African Republic exports

```
caf_fit |>
  select(search) |>
  gg_tsresiduals()
```



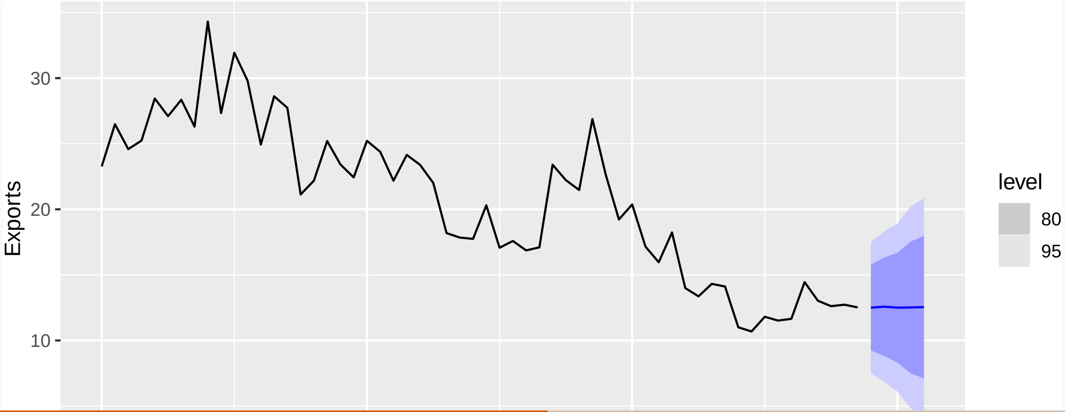
Central African Republic exports

```r
augment(caf_fit) |>
  filter(.model == "search") |>
  features(.innov, ljung_box, lag = 10, dof = 3)
```

```
# A tibble: 1 x 4
  Country                  .model lb_stat lb_pvalue
  <fct>                    <chr>    <dbl>     <dbl>
1 Central African Republic search    5.75     0.569
```

Central African Republic exports

```r
caf_fit |>
  forecast(h = 5) |>
  filter(.model == "search") |>
  autoplot(global_economy)
```



Modelling procedure with `ARIMA()`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
3. If the data are non-stationary: take first differences of the data until the data are stationary.
4. Examine the ACF/PACF: Is an AR(p) or MA(q) model appropriate?
5. Try your chosen model(s), and use the AICc to search for a better model.
6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
7. Once the residuals look like white noise, calculate forecasts.

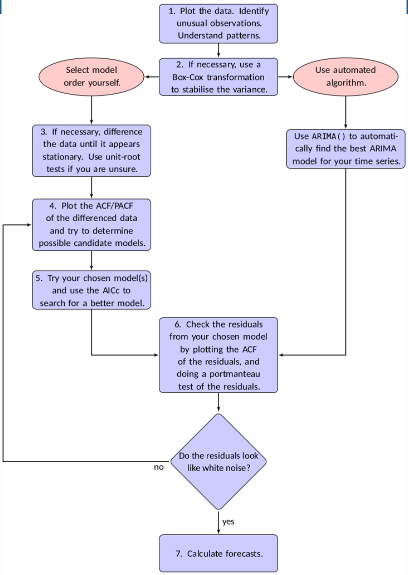
Automatic modelling procedure with `ARIMA()`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.

3. Use `ARIMA` to automatically select a model.

6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
7. Once the residuals look like white noise, calculate forecasts.

Modelling procedure



Outline

Point forecasts

1. Rearrange ARIMA equation so y_t is on LHS.
2. Rewrite equation by replacing t by $T + h$.
3. On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with $h = 1$. Repeat for $h = 2, 3, \ldots$.

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B) y_t = (1 + \theta_1 B)\varepsilon_t,$$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B) y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned}\left[1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4\right] y_t \\ = (1 + \theta_1 B)\varepsilon_t,\end{aligned}$$

Point forecasts

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$$\begin{aligned}\left[1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4\right] y_t \\ = (1 + \theta_1 B)\varepsilon_t,\end{aligned}$$

$$\begin{aligned}y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + \phi_3 y_{t-4} = \varepsilon_t + \theta_1 \varepsilon_{t-1}.\end{aligned}$$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1-\phi_1 B-\phi_2 B^2-\phi_3 B^3)(1-B)y_t=(1+\theta_1 B)\varepsilon_t,$$

$$\begin{aligned}\left[1-(1+\phi_1)B+(\phi_1-\phi_2)B^2+(\phi_2-\phi_3)B^3+\phi_3 B^4\right]y_t \\ =(1+\theta_1 B)\varepsilon_t,\end{aligned}$$

$$\begin{aligned}y_t-(1+\phi_1)y_{t-1}+(\phi_1-\phi_2)y_{t-2}+(\phi_2-\phi_3)y_{t-3} \\ +\phi_3 y_{t-4}=\varepsilon_t+\theta_1\varepsilon_{t-1}.\end{aligned}$$

Point forecasts (h=1)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

Point forecasts (h=1)

$$\begin{aligned} y_t = (1+\phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}. \end{aligned}$$

ARIMA(3,1,1) forecasts: Step 2

$$\begin{aligned} y_{T+1} = (1+\phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3 y_{T-3} + \varepsilon_{T+1} + \theta_1\varepsilon_T. \end{aligned}$$

Point forecasts (h=1)

$$\begin{aligned} y_t = (1+\phi_1)y_{t-1} - (\phi_1-\phi_2)y_{t-2} - (\phi_2-\phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}. \end{aligned}$$

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ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+1|T} = (1+\phi_1)y_T - (\phi_1-\phi_2)y_{T-1} - (\phi_2-\phi_3)y_{T-2}$$

Point forecasts (h=2)

$$
\begin{aligned}
y_t = (1+\phi_1)y_{t-1} - (\phi_1-\phi_2)y_{t-2} - (\phi_2-\phi_3)y_{t-3} \\
- \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.
\end{aligned}
$$

Point forecasts (h=2)

$$\begin{aligned} y_t = (1+\phi_1)y_{t-1} - (\phi_1-\phi_2)y_{t-2} - (\phi_2-\phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}. \end{aligned}$$

ARIMA(3,1,1) forecasts: Step 2

$$\begin{aligned} y_{T+2} = (1+\phi_1)y_{T+1} - (\phi_1-\phi_2)y_{T} - (\phi_2-\phi_3)y_{T-1} \\ - \phi_3 y_{T-2} + \varepsilon_{T+2} + \theta_1\varepsilon_{T+1}. \end{aligned}$$

Point forecasts (h=2)

$$\begin{aligned} y_t = (1+\phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}. \end{aligned}$$

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ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+2|T} = (1+\phi_1)\hat{y}_{T+1|T} - (\phi_1 - \phi_2)y_{T} - (\phi_2 - \phi_3)y_{T-1}$$

Prediction intervals

95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

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95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

- $v_{T+1|T} = \hat{\sigma}^2$ for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2\right], \qquad \text{for } h = 2, 3, \ldots.$$

Prediction intervals

95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

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$$y_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

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Prediction intervals

95% prediction interval

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$$y_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2\right], \qquad \text{for } h = 2, 3, \dots.$$

- AR(1): Rewrite as MA(∞) and use above result.

Prediction intervals

- Prediction intervals **increase in size with forecast horizon**.
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed**.
- Prediction intervals tend to be too narrow.
  - the uncertainty in the parameter estimates has not been accounted for.
  - the ARIMA model assumes historical patterns will not change during the forecast period.
  - the ARIMA model assumes uncorrelated future errors

Outline

Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	$\uparrow$ Non-seasonal part of the model	$\uparrow$ Seasonal part of of the model

where m = number of observations per year.

Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

Seasonal ARIMA models

E.g., ARIMA$(1, 1, 1)(1, 1, 1)_4$ model (without constant)

$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t \;=\; (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$

Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

$$\underbrace{(1-\phi_1 B)}_{\begin{pmatrix}\text{Non-seasonal}\\ \text{AR(1)}\end{pmatrix}}\underbrace{(1-\Phi_1 B^4)}_{\begin{pmatrix}\text{Seasonal}\\ \text{AR(1)}\end{pmatrix}}\underbrace{(1-B)}_{\begin{pmatrix}\text{Non-seasonal}\\ \text{difference}\end{pmatrix}}\underbrace{(1-B^4)}_{\begin{pmatrix}\text{Seasonal}\\ \text{difference}\end{pmatrix}}y_t = \underbrace{(1+\theta_1 B)}_{\begin{pmatrix}\text{Non-seasonal}\\ \text{MA(1)}\end{pmatrix}}\underbrace{(1+\Theta_1 B^4)}_{\begin{pmatrix}\text{Seasonal}\\ \text{MA(1)}\end{pmatrix}}\varepsilon_t.$$

Seasonal ARIMA models

E.g., ARIMA$(1, 1, 1)(1, 1, 1)_4$ model (without constant)

$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t \;=\; (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t = {} & (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + (1 + \Phi_1)y_{t-4} \\ & - (1 + \phi_1 + \Phi_1 + \phi_1\Phi_1)y_{t-5} + (\phi_1 + \phi_1\Phi_1)y_{t-6} \\ & - \Phi_1 y_{t-8} + (\Phi_1 + \phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} \end{aligned}$$

Common ARIMA models

The US Census Bureau uses the following models most often:

ARIMA(0,1,1)(0,1,1)$_m$	with log transformation
ARIMA(0,1,2)(0,1,1)$_m$	with log transformation
ARIMA(2,1,0)(0,1,1)$_m$	with log transformation
ARIMA(0,2,2)(0,1,1)$_m$	with log transformation
ARIMA(2,1,2)(0,1,1)$_m$	with no transformation

Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

ARIMA(0,0,0)(0,0,1)$_{12}$ will show:

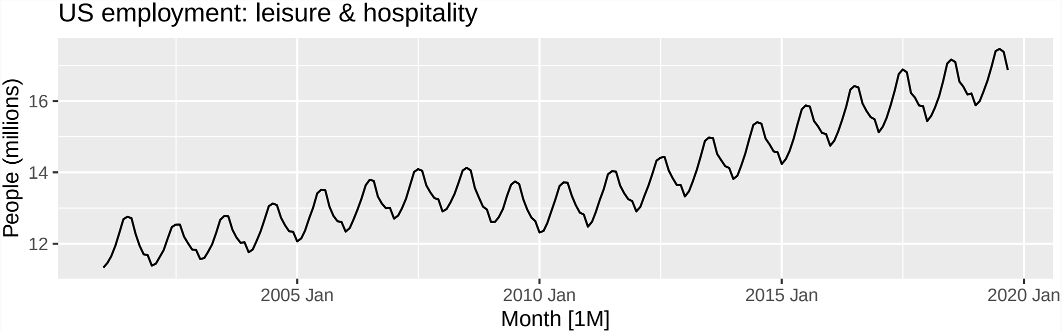
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ...

ARIMA(0,0,0)(1,0,0)$_{12}$ will show:

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

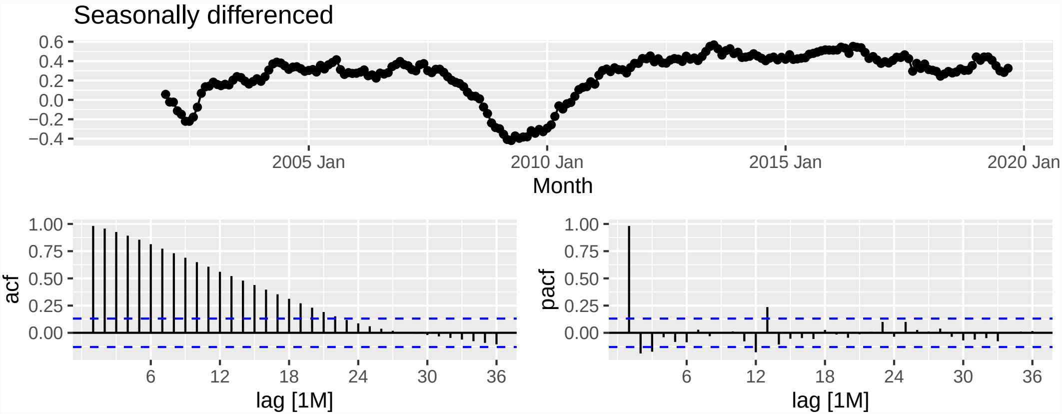
US leisure employment

```
leisure <- us_employment |>
  filter(Title == "Leisure and Hospitality", year(Month) > 2000) |>
  mutate(Employed = Employed / 1000) |>
  select(Month, Employed)
autoplot(leisure, Employed) +
  labs(title = "US employment: leisure & hospitality", y = "People (millions)")
```



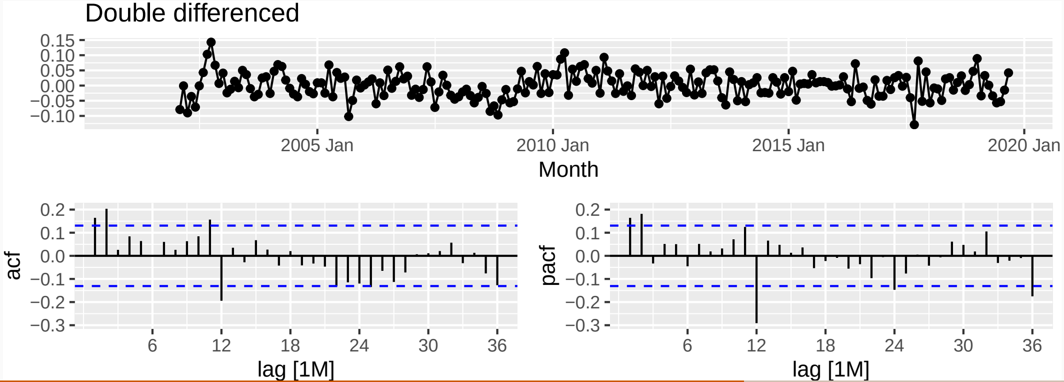
US leisure employment

```
leisure |>
  gg_tsdisplay(difference(Employed, 12), plot_type = "partial", lag = 36) +
  labs(title = "Seasonally differenced", y = "")
```



US leisure employment

```
leisure |>
  gg_tsdisplay(difference(Employed, 12) |> difference(),
    plot_type = "partial", lag = 36
  ) +
  labs(title = "Double differenced", y = "")
```



US leisure employment

```
fit <- leisure |>
  model(
    arima012011 = ARIMA(Employed ~ pdq(0, 1, 2) + PDQ(0, 1, 1)),
    arima210011 = ARIMA(Employed ~ pdq(2, 1, 0) + PDQ(0, 1, 1)),
    auto = ARIMA(Employed, stepwise = FALSE, approx = FALSE)
  )
fit |> pivot_longer(everything(),
  names_to = "Model name",
  values_to = "Orders"
)
```

```
# A mable: 3 x 2
# Key:     Model name [3]
  `Model name`                    Orders
  <chr>                          <model>
1 arima012011  <ARIMA(0,1,2)(0,1,1)[12]>
2 arima210011  <ARIMA(2,1,0)(0,1,1)[12]>
3 auto         <ARIMA(2,1,0)(1,1,1)[12]>
```

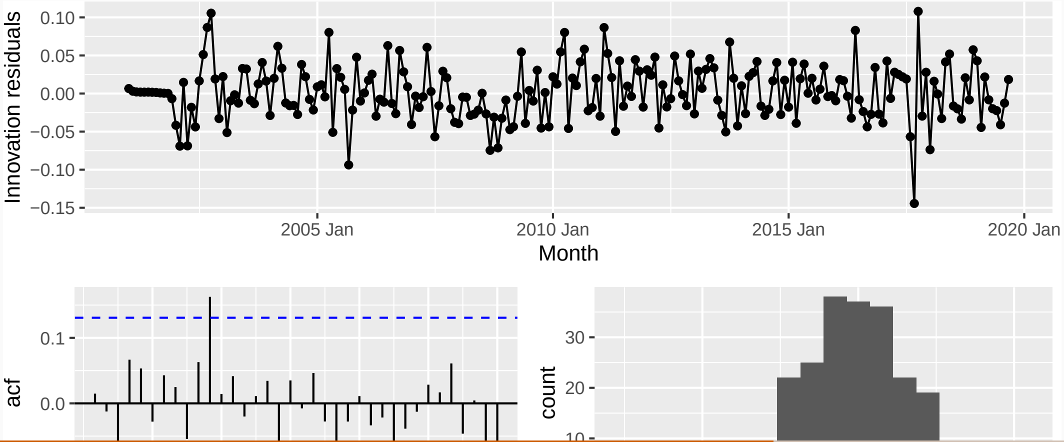
US leisure employment

```r
glance(fit) |>
  arrange(AICc) |>
  select(.model:BIC)
```

```
# A tibble: 3 x 6
  .model       sigma2 log_lik   AIC  AICc   BIC
  <chr>         <dbl>   <dbl> <dbl> <dbl> <dbl>
1 auto        0.00142    395. -780. -780. -763.
2 arima210011 0.00145    392. -776. -776. -763.
3 arima012011 0.00146    391. -775. -775. -761.
```

US leisure employment

```
fit |>
  select(auto) |>
  gg_tsresiduals(lag = 36)
```



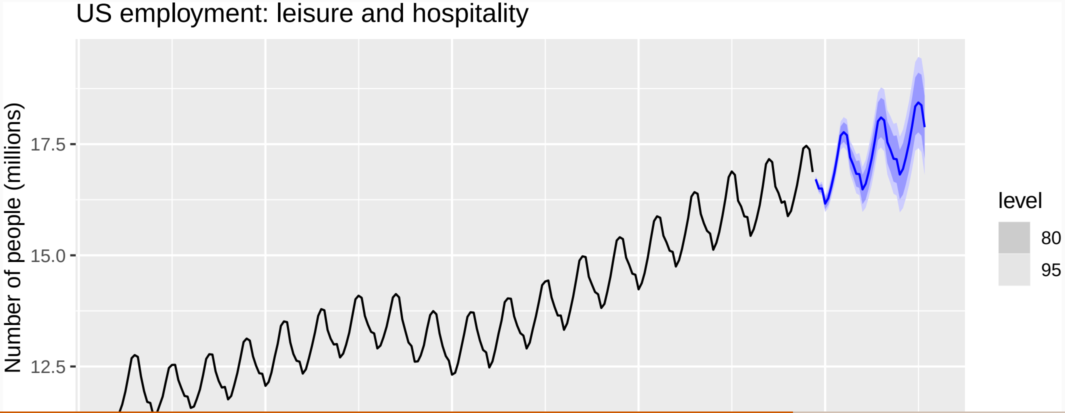
US leisure employment

```r
augment(fit) |> features(.innov, ljung_box, lag = 24, dof = 4)
```

```
# A tibble: 3 x 3
  .model       lb_stat lb_pvalue
  <chr>          <dbl>     <dbl>
1 arima012011     22.4     0.320
2 arima210011     18.9     0.527
3 auto            16.6     0.680
```

US leisure employment

```
forecast(fit, h = 36) |>
  filter(.model == "auto") |>
  autoplot(leisure) +
  labs(title = "US employment: leisure and hospitality", y = "Number of people (mill
```

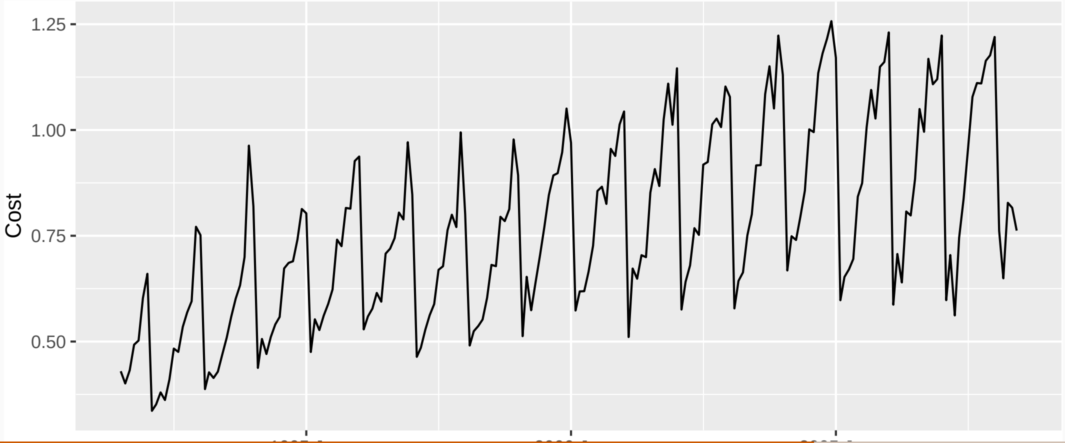


Cortecosteroid drug sales

```
h02 <- PBS |>
  filter(ATC2 == "H02") |>
  summarise(Cost = sum(Cost) / 1e6)
```

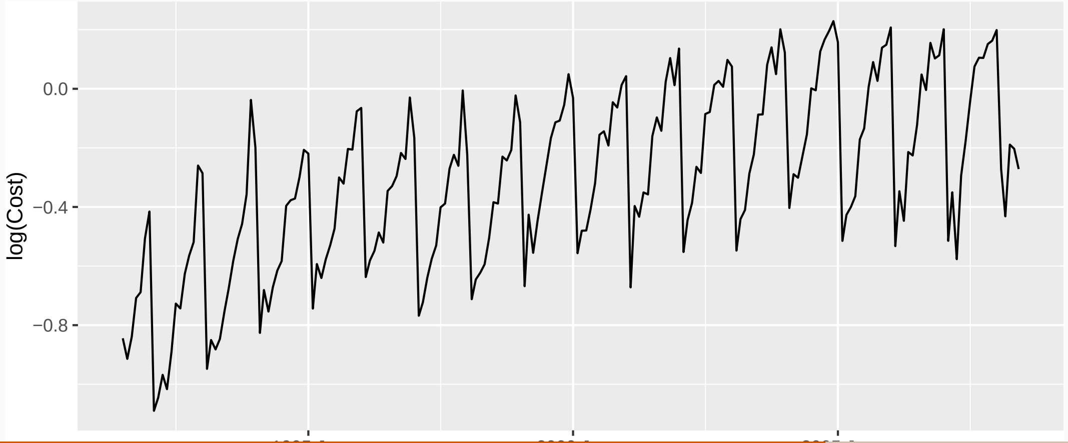
Cortecosteroid drug sales

```r
h02 |> autoplot(
  Cost
)
```



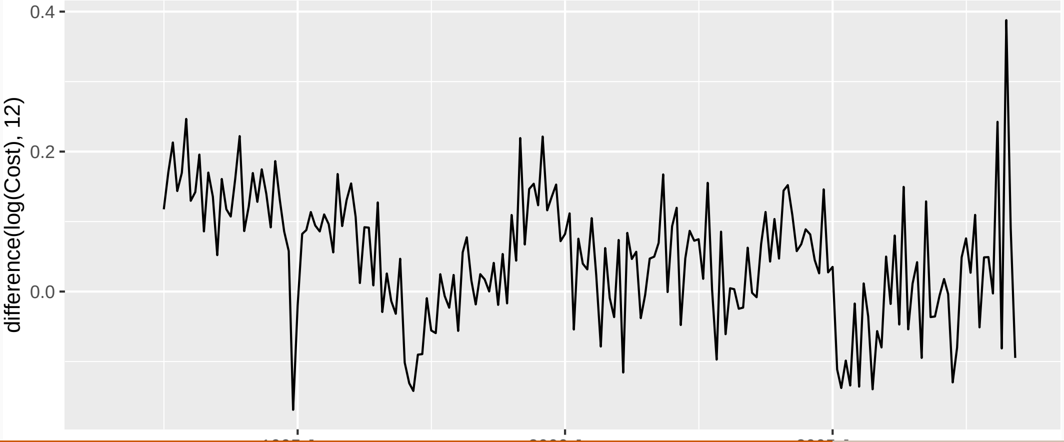
Cortecosteroid drug sales

```r
h02 |> autoplot(
  log(Cost)
)
```



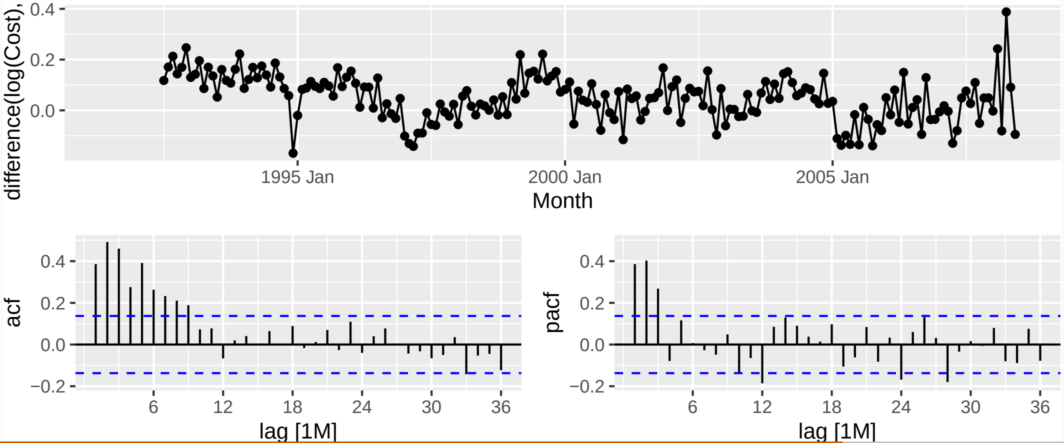
Cortecosteroid drug sales

```
h02 |> autoplot(
  log(Cost) |> difference(12)
)
```



Cortecosteroid drug sales

```
h02 |> gg_tsdisplay(difference(log(Cost), 12),
  lag_max = 36, plot_type = "partial"
)
```



Cortecosteroid drug sales

- Choose $D = 1$ and $d = 0$.
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggests possible non-seasonal AR(3) term.
- Initial candidate model: ARIMA$(3,0,0)(2,1,0)_{12}$.

Cortecosteroid drug sales

.model	AICc
ARIMA(3,0,1)(0,1,2)[12]	-485
ARIMA(3,0,1)(1,1,1)[12]	-484
ARIMA(3,0,1)(0,1,1)[12]	-484
ARIMA(3,0,1)(2,1,0)[12]	-476
ARIMA(3,0,0)(2,1,0)[12]	-475
ARIMA(3,0,2)(2,1,0)[12]	-475
ARIMA(3,0,1)(1,1,0)[12]	-463

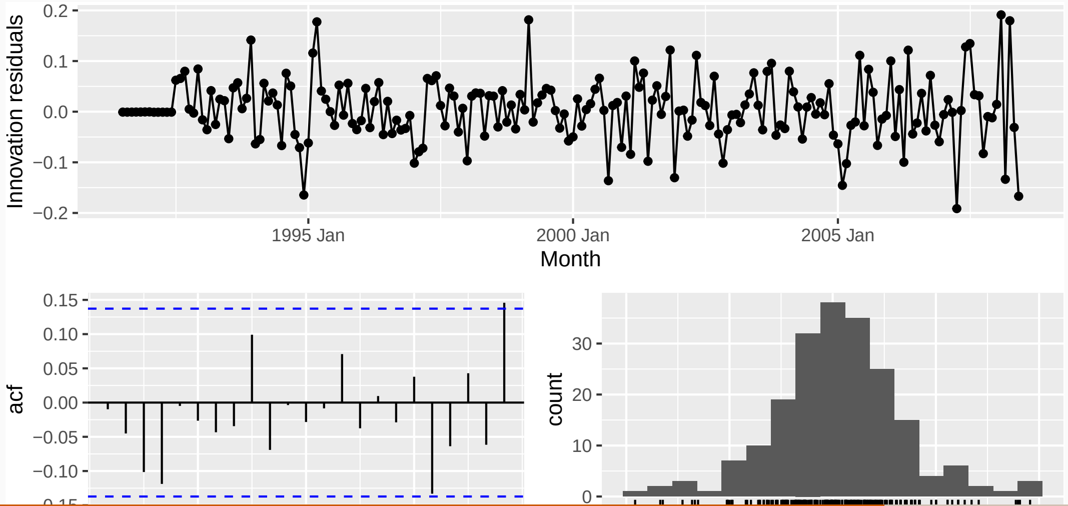
Cortecosteroid drug sales

```r
fit <- h02 |>
  model(best = ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(0, 1, 2)))
report(fit)
```

```
Series: Cost
Model: ARIMA(3,0,1)(0,1,2)[12]
Transformation: log(Cost)

Coefficients:
         ar1     ar2     ar3    ma1     sma1     sma2
      -0.160  0.5481  0.5678  0.383  -0.5222  -0.1768
s.e.   0.164  0.0878  0.0942  0.190   0.0861   0.0872

sigma^2 estimated as 0.004278:  log likelihood=250
AIC=-486   AICc=-485   BIC=-463
```

Cortecosteroid drug sales

```
gg_tsresiduals(fit)
```



Cortecosteroid drug sales

```
augment(fit) |>
  features(.innov, ljung_box, lag = 36, dof = 6)
```

```
# A tibble: 1 x 3
  .model lb_stat lb_pvalue
  <chr>    <dbl>     <dbl>
1 best      50.7    0.0104
```

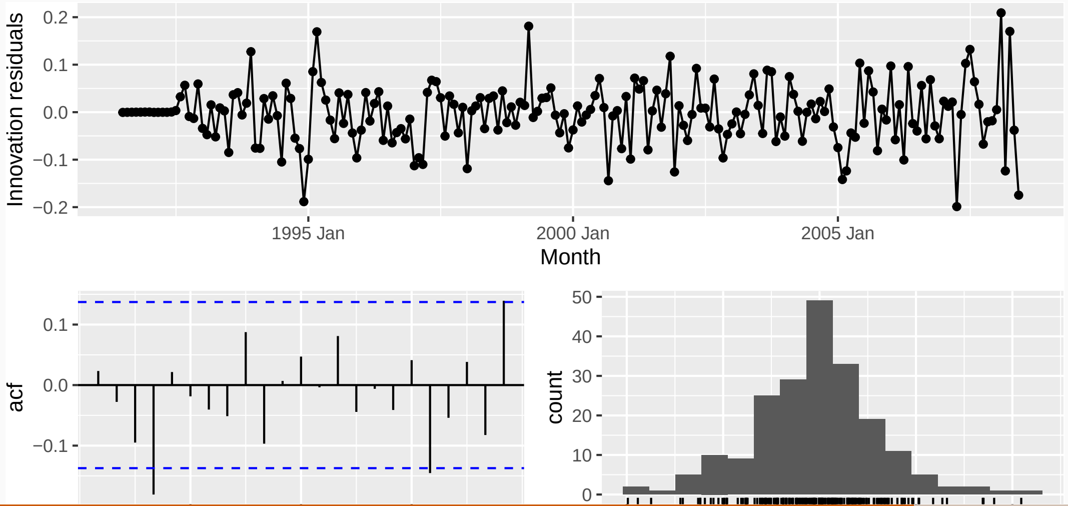
Cortecosteroid drug sales

```r
fit <- h02 |> model(auto = ARIMA(log(Cost)))
report(fit)
```

```
Series: Cost
Model: ARIMA(2,1,0)(0,1,1)[12]
Transformation: log(Cost)

Coefficients:
          ar1      ar2     sma1
      -0.8491  -0.4207  -0.6401
s.e.   0.0712   0.0714   0.0694

sigma^2 estimated as 0.004387:  log likelihood=245
AIC=-483   AICc=-483   BIC=-470
```

Cortecosteroid drug sales

```
gg_tsresiduals(fit)
```



Cortecosteroid drug sales

```
augment(fit) |>
  features(.innov, ljung_box, lag = 36, dof = 3)
```

```
# A tibble: 1 x 3
  .model lb_stat lb_pvalue
  <chr>    <dbl>     <dbl>
1 auto      59.3   0.00332
```

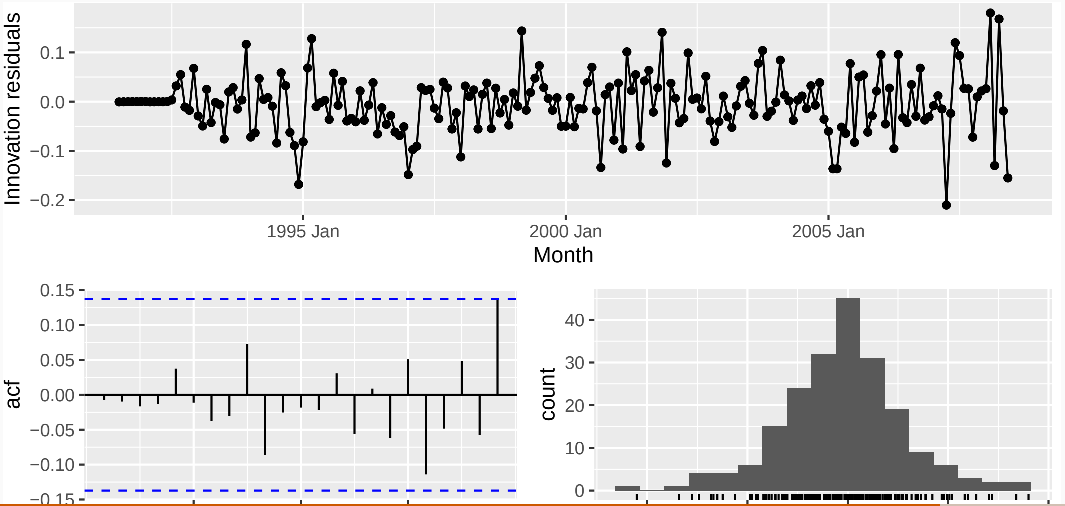
Cortecosteroid drug sales

```
fit <- h02 |>
  model(best = ARIMA(log(Cost),
    stepwise = FALSE,
    approximation = FALSE,
    order_constraint = p + q + P + Q <= 9
  ))
report(fit)
```

```
Series: Cost
Model: ARIMA(4,1,1)(2,1,2)[12]
Transformation: log(Cost)

Coefficients:
          ar1    ar2    ar3     ar4     ma1   sar1    sar2    sma1   sma2
      -0.0425  0.210  0.202  -0.227  -0.742  0.621  -0.383  -1.202  0.496
s.e.   0.2167  0.181  0.114   0.081   0.207  0.242   0.118   0.249  0.213

sigma^2 estimated as 0.004049:  log likelihood=254
```

Cortecosteroid drug sales

```
gg_tsresiduals(fit)
```



Cortecosteroid drug sales

```
augment(fit) |>
  features(.innov, ljung_box, lag = 36, dof = 9)
```

```
# A tibble: 1 x 3
  .model lb_stat lb_pvalue
  <chr>    <dbl>     <dbl>
1 best      36.5     0.106
```

Cortecosteroid drug sales

Training data: July 1991 to June 2006

Test data: July 2006–June 2008

```
fit <- h02 |>
  filter_index(~ "2006 Jun") |>
  model(
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 0) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 2) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(1, 1, 0))
    # ... #
  )

fit |>
  forecast(h = "2 years") |>
  accuracy(h02)
```

Cortecosteroid drug sales

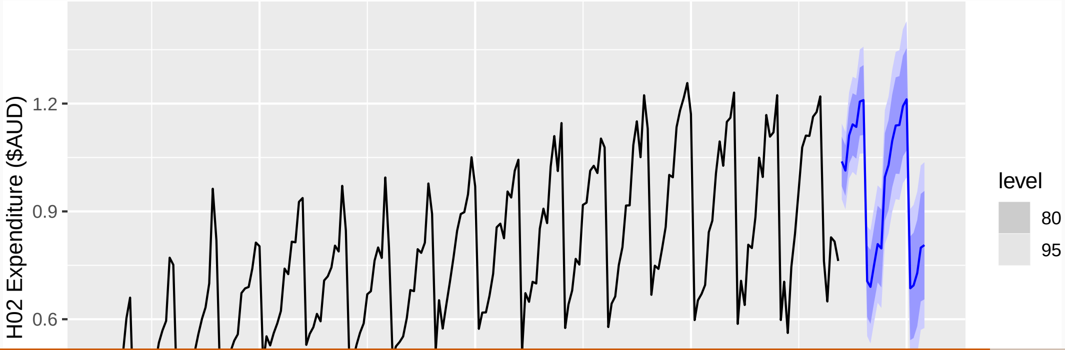
.model	RMSE
ARIMA(3,0,1)(1,1,1)[12]	0.0619
ARIMA(3,0,1)(0,1,2)[12]	0.0621
ARIMA(3,0,1)(0,1,1)[12]	0.0630
ARIMA(2,1,0)(0,1,1)[12]	0.0630
ARIMA(4,1,1)(2,1,2)[12]	0.0631
ARIMA(3,0,2)(2,1,0)[12]	0.0651
ARIMA(3,0,1)(2,1,0)[12]	0.0653
ARIMA(3,0,1)(1,1,0)[12]	0.0666
ARIMA(3,0,0)(2,1,0)[12]	0.0668

Cortecosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- Use the best model available, even if it does not pass all tests.

Cortecosteroid drug sales

```r
fit <- h02 |>
  model(ARIMA(Cost ~ 0 + pdq(3, 0, 1) + PDQ(0, 1, 2)))
fit |>
  forecast() |>
  autoplot(h02) +
  labs(y = "H02 Expenditure ($AUD)")
```



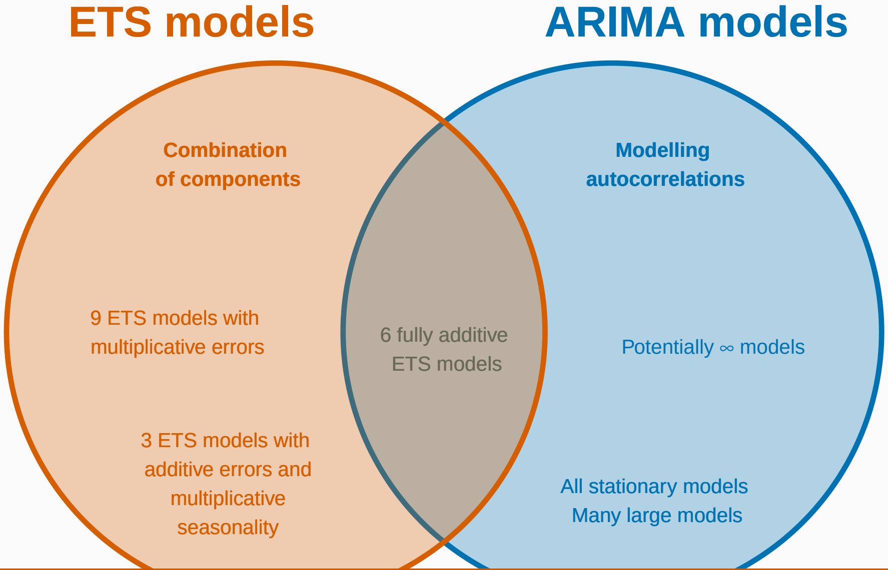
Outline

1 Non-seasonal ARIMA models
2 Estimation and order selection
3 ARIMA modelling in R
4 Forecasting
5 Seasonal ARIMA models
6 ARIMA vs ETS

ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

ARIMA vs ETS


ETS models
ARIMA models
Combination of components
Modelling autocorrelations
9 ETS models with multiplicative errors
6 fully additive ETS models
Potentially ∞ models
3 ETS models with additive errors and multiplicative seasonality
All stationary models
Many large models

Equivalences

ETS model	ARIMA model	Parameters
ETS(A,N,N)	ARIMA(0,1,1)	$\theta_1 = \alpha - 1$
ETS(A,A,N)	ARIMA(0,2,2)	$\theta_1 = \alpha + \beta - 2$
		$\theta_2 = 1 - \alpha$
ETS(A,A_d,N)	ARIMA(1,1,2)	$\phi_1 = \phi$
		$\theta_1 = \alpha + \phi\beta - 1 - \phi$
		$\theta_2 = (1 - \alpha)\phi$
ETS(A,N,A)	ARIMA$(0,0,m)(0,1,0)_m$	
ETS(A,A,A)	ARIMA$(0,1,m+1)(0,1,0)_m$	
ETS(A,A_d,A)	ARIMA$(1,0,m+1)(0,1,0)_m$	

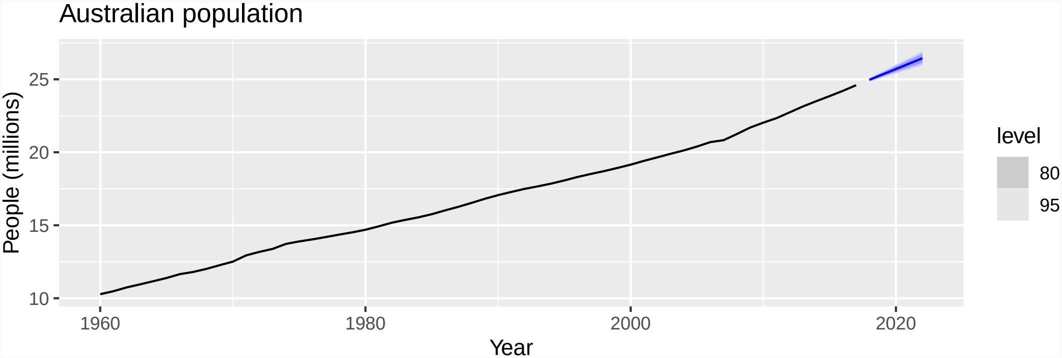
Example: Australian population

```
aus_economy <- global_economy |>
  filter(Code == "AUS") |>
  mutate(Population = Population / 1e6)
aus_economy |>
  slice(-n()) |>
  stretch_tsibble(.init = 10) |>
  model(
    ets = ETS(Population),
    arima = ARIMA(Population)
  ) |>
  forecast(h = 1) |>
  accuracy(aus_economy) |>
  select(.model, ME:RMSSE)
```

```
# A tibble: 2 x 8
  .model      ME    RMSE    MAE    MPE   MAPE   MASE  RMSSE
  <chr>    <dbl>   <dbl>  <dbl>  <dbl>  <dbl>  <dbl>  <dbl>
1 arima   0.0420 0.194   0.0789 0.277  0.509  0.317  0.746
```

Example: Australian population

```
aus_economy |>
  model(ETS(Population)) |>
  forecast(h = "5 years") |>
  autoplot(aus_economy) +
  labs(title = "Australian population", y = "People (millions)")
```



Example: Cement production

```
cement <- aus_production |>
  select(Cement) |>
  filter_index("1988 Q1" ~ .)
train <- cement |> filter_index(. ~ "2007 Q4")
fit <- train |>
  model(
    arima = ARIMA(Cement),
    ets = ETS(Cement)
  )
```

Example: Cement production

```r
fit |>
  select(arima) |>
  report()
```

```
Series: Cement
Model: ARIMA(1,0,1)(2,1,1)[4] w/ drift

Coefficients:
         ar1     ma1    sar1    sar2    sma1  constant
      0.8886  -0.237  0.081  -0.234  -0.898      5.39
s.e.  0.0842   0.133  0.157   0.139   0.178      1.48

sigma^2 estimated as 11456:  log likelihood=-464
AIC=941   AICc=943   BIC=957
```

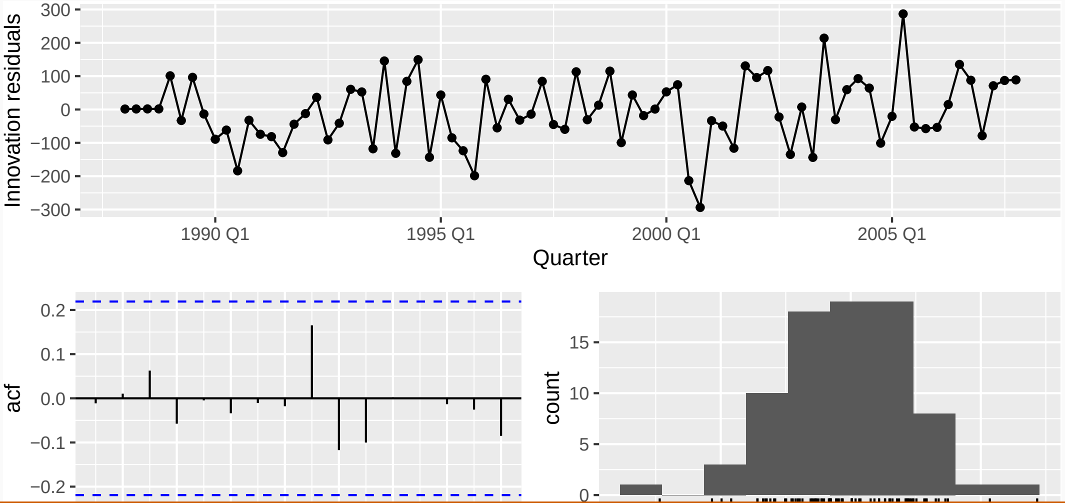
Example: Cement production

```
fit |>
  select(ets) |>
  report()
```

```
Series: Cement
Model: ETS(M,N,M)
  Smoothing parameters:
    alpha = 0.753
    gamma = 1e-04

  Initial states:
 l[0] s[0] s[-1] s[-2] s[-3]
 1695 1.03  1.05  1.01 0.912

  sigma^2:  0.0034

 AIC AICc  BIC
1104 1106 1121
```

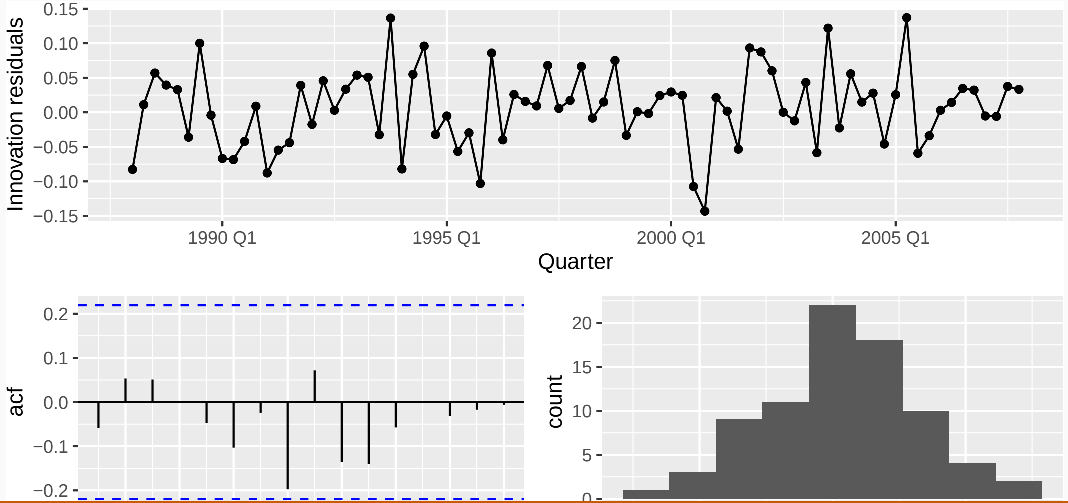
Example: Cement production

```r
gg_tsresiduals(fit |> select(arima), lag_max = 16)
```



Example: Cement production

```
gg_tsresiduals(fit |> select(ets), lag_max = 16)
```



Example: Cement production

```
fit |>
  select(arima) |>
  augment() |>
  features(.innov, ljung_box, lag = 16, dof = 6)
```

```
# A tibble: 1 x 3
  .model lb_stat lb_pvalue
  <chr>    <dbl>     <dbl>
1 arima     6.37     0.783
```

Example: Cement production

```
fit |>
  select(ets) |>
  augment() |>
  features(.innov, ljung_box, lag = 16, dof = 6)
```

```
# A tibble: 1 x 3
  .model lb_stat lb_pvalue
  <chr>    <dbl>     <dbl>
1 ets       10.0     0.438
```

Example: Cement production

```
fit |>
  forecast(h = "2 years 6 months") |>
  accuracy(cement) |>
  select(-ME, -MPE, -ACF1)
```

```
# A tibble: 2 x 7
  .model .type  RMSE   MAE  MAPE  MASE RMSSE
  <chr>  <chr> <dbl> <dbl> <dbl> <dbl> <dbl>
1 arima  Test   216.  186.  8.68  1.27  1.26
2 ets    Test   222.  191.  8.85  1.30  1.29
```

Example: Cement production

```
fit |>
  select(arima) |>
  forecast(h = "3 years") |>
  autoplot(cement) +
  labs(title = "Cement production in Australia", y = "Tonnes ('000)")
```