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# ETC3550/ETC5550

## Applied forecasting

Ch8. Exponential smoothing

[OTexts.org/fpp3/](https://OTexts.org/fpp3/)



# Outline

- 1 Exponential smoothing
- 2 Simple exponential smoothing
- 3 Models with trend
- 4 Models with seasonality
- 5 Innovations state space models
- 6 Forecasting with exponential smoothing

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# Historical perspective

- Developed in the 1950s and 1960s as methods (algorithms) to produce point forecasts.
- Combine a “level”, “trend” (slope) and “seasonal” component to describe a time series.
- The rate of change of the components are controlled by “smoothing parameters”:  $\alpha$ ,  $\beta$  and  $\gamma$  respectively.
- Need to choose best values for the smoothing parameters (and initial states).
- Equivalent ETS state space models developed in the 1990s and 2000s.

# Big idea: control the rate of change

$\alpha$  controls the flexibility of the **level**

- If  $\alpha = 0$ , the level never updates (mean)
- If  $\alpha = 1$ , the level updates completely (naive)

$\beta$  controls the flexibility of the **trend**

- If  $\beta = 0$ , the trend is linear
- If  $\beta = 1$ , the trend changes suddenly every observation

$\gamma$  controls the flexibility of the **seasonality**

- If  $\gamma = 0$ , the seasonality is fixed (seasonal means)
- If  $\gamma = 1$ , the seasonality updates completely (seasonal naive)

# A model for levels, trends, and seasonalities

We want a model that captures the level ( $\ell_t$ ), trend ( $b_t$ ) and seasonality ( $s_t$ ).

**How do we combine these elements?**

# A model for levels, trends, and seasonalities

We want a model that captures the level ( $\ell_t$ ), trend ( $b_t$ ) and seasonality ( $s_t$ ).

How do we combine these elements?

Additively?

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

Multiplicatively?

$$y_t = \ell_{t-1} b_{t-1} s_{t-m} (1 + \varepsilon_t)$$

Perhaps a mix of both?

$$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$$

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$$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$$

How do the level, trend and seasonal components evolve over time?



# ETS models

**General notation**     E T S : ExponenTial Smoothing



Error Trend Season

**Error:** Additive ("A") or multiplicative ("M")

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Error Trend Season

**Error:** Additive ("A") or multiplicative ("M")

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**Seasonality:** None ("N"), additive ("A") or multiplicative ("M")

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# Simple methods

Time series  $y_1, y_2, \dots, y_T$ .

## Random walk forecasts

$$\hat{y}_{T+h|T} = y_T$$

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## Average forecasts

$$\hat{y}_{T+h|T} = \frac{1}{T} \sum_{t=1}^T y_t$$

# Simple methods

Time series  $y_1, y_2, \dots, y_T$ .

## Random walk forecasts

$$\hat{y}_{T+h|T} = y_T$$

## Average forecasts

$$\hat{y}_{T+h|T} = \frac{1}{T} \sum_{t=1}^T y_t$$

- Want something in between these methods.
- Most recent data should have more weight.

# Simple Exponential Smoothing

## Forecast equation

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1 - \alpha)y_{T-1} + \alpha(1 - \alpha)^2 y_{T-2} + \cdots,$$

where  $0 \leq \alpha \leq 1$ .



# Simple Exponential Smoothing

## Forecast equation

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1 - \alpha)y_{T-1} + \alpha(1 - \alpha)^2 y_{T-2} + \cdots,$$

where  $0 \leq \alpha \leq 1$ .

Observation	Weights assigned to observations for:			
	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
$y_T$	0.2	0.4	0.6	0.8
$y_{T-1}$	0.16	0.24	0.24	0.16
$y_{T-2}$	0.128	0.144	0.096	0.032
$y_{T-3}$	0.1024	0.0864	0.0384	0.0064
$y_{T-4}$	$(0.2)(0.8)^4$	$(0.4)(0.6)^4$	$(0.6)(0.4)^4$	$(0.8)(0.2)^4$
$y_{T-5}$	$(0.2)(0.8)^5$	$(0.4)(0.6)^5$	$(0.6)(0.4)^5$	$(0.8)(0.2)^5$

# Simple Exponential Smoothing

# Simple Exponential Smoothing

## Component form

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t$$

Smoothing equation

$$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$$

- $\ell_t$  is the level (or the smoothed value) of the series at time  $t$ .
- $\hat{y}_{t+1|t} = \alpha y_t + (1 - \alpha)\hat{y}_{t|t-1}$

# Simple Exponential Smoothing

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- $\hat{y}_{t+1|t} = \alpha y_t + (1 - \alpha)\hat{y}_{t|t-1}$

Iterate to get exponentially weighted moving average form.

## Weighted average form

$$\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha(1 - \alpha)^j y_{T-j} + (1 - \alpha)^T \ell_0$$

# Optimising smoothing parameters

- Need to choose best values for  $\alpha$  and  $\ell_0$ .
- Similarly to regression, choose optimal parameters by minimising SSE:

$$\text{SSE} = \sum_{t=1}^T (y_t - \hat{y}_{t|t-1})^2.$$

- Unlike regression there is no closed form solution — use numerical optimization.

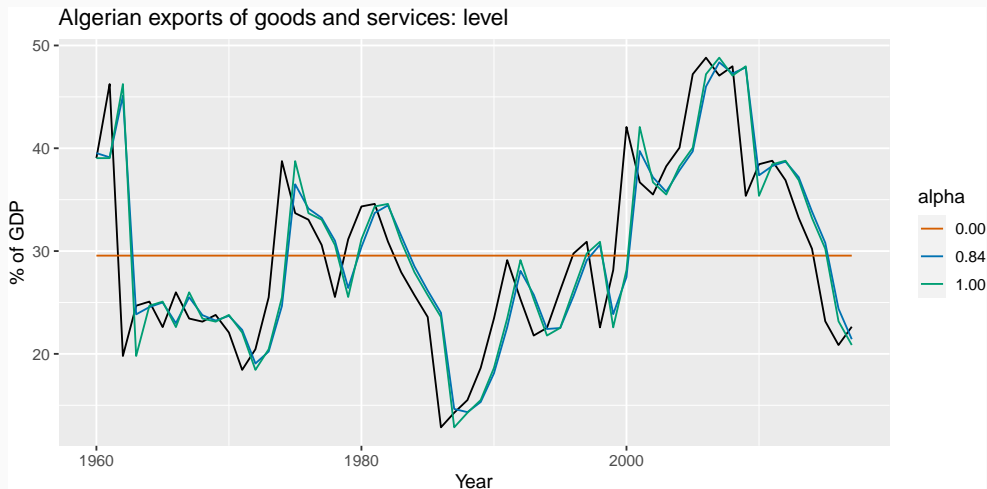
# Optimising smoothing parameters

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- Unlike regression there is no closed form solution — use numerical optimization.
- For Algerian Exports example:
  - ▶  $\hat{\alpha} = 0.8400$
  - ▶  $\hat{\ell}_0 = 39.54$

# Simple Exponential Smoothing



# Models and methods

## Methods

- Algorithms that return point forecasts.

## Models

- Generate same point forecasts but can also generate forecast distributions.
- A stochastic (or random) data generating process that can generate an entire forecast distribution.
- Allow for “proper” model selection.



# ETS(A,N,N): SES with additive errors

## Component form

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Smoothing equation

$$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$$

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Forecast equation

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Smoothing equation

$$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$$

Forecast error:  $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$ .

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$$\hat{y}_{t+h|t} = \ell_t$$

Smoothing equation

$$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$$

Forecast error:  $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$ .

## Error correction form

$$y_t = \ell_{t-1} + e_t$$

$$\ell_t = \ell_{t-1} + \alpha(y_t - \ell_{t-1})$$

$$= \ell_{t-1} + \alpha e_t$$

# ETS(A,N,N): SES with additive errors

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Forecast error:  $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$ .

## Error correction form

$$y_t = \ell_{t-1} + e_t$$

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$$= \ell_{t-1} + \alpha e_t$$

Specify probability distribution for  $e_t$ , we assume  $e_t = \varepsilon_t \sim \text{NID}(0, \sigma^2)$ . 16

## ETS(A,N,N): SES with additive errors

Measurement equation

$$y_t = l_{t-1} + \varepsilon_t$$

State equation

$$l_t = l_{t-1} + \alpha \varepsilon_t$$

where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

- “innovations” or “single source of error” because equations have the same error process,  $\varepsilon_t$ .
- Measurement equation: relationship between observations and states.
- State equation(s): evolution of the state(s) through time.

## ETS(M,N,N): SES with multiplicative errors.

- Specify relative errors  $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting  $\hat{y}_{t|t-1} = \ell_{t-1}$  gives:
  - ▶  $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
  - ▶  $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

## ETS(M,N,N): SES with multiplicative errors.

- Specify relative errors  $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting  $\hat{y}_{t|t-1} = \ell_{t-1}$  gives:
  - ▶  $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
  - ▶  $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

Measurement equation

$$y_t = \ell_{t-1}(1 + \varepsilon_t)$$

State equation

$$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$$

## ETS(M,N,N): SES with multiplicative errors.

- Specify relative errors  $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
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  - ▶  $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

Measurement equation

$$y_t = \ell_{t-1}(1 + \varepsilon_t)$$

State equation

$$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$$

- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.



# ETS(A,N,N): Specifying the model

```
ETS(y ~ error("A") + trend("N") + season("N"))
```

By default, an optimal value for  $\alpha$  and  $\ell_0$  is used.

$\alpha$  can be chosen manually in `trend()`.

```
trend("N", alpha = 0.5)  
trend("N", alpha_range = c(0.2, 0.8))
```

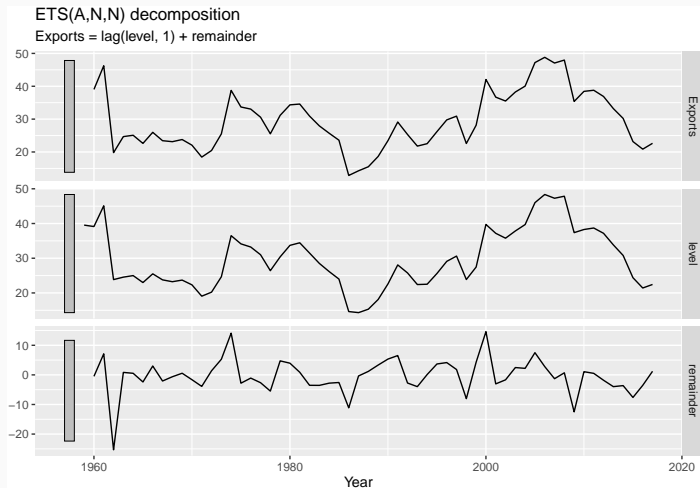
# Example: Algerian Exports

```
algeria_economy <- global_economy |>
  filter(Country == "Algeria")
fit <- algeria_economy |>
  model(ANN = ETS(Exports ~ error("A") + trend("N") + season("N")))
report(fit)
```

```
## Series: Exports
## Model: ETS(A,N,N)
##   Smoothing parameters:
##     alpha = 0.84
##
##   Initial states:
##     l[0]
##     39.5
##
##     sigma^2:  35.6
##
##   ATC ATCc BTC
```

# Example: Algerian Exports

```
components(fit) |> autoplot()
```



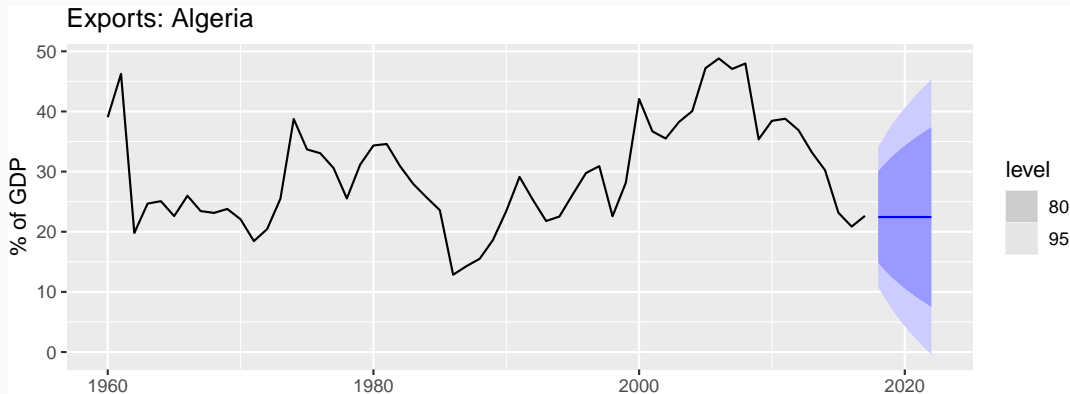
# Example: Algerian Exports

```
components(fit) |>  
  left_join(fitted(fit), by = c("Country", ".model", "Year"))
```

```
## # A dable: 59 x 7 [1Y]  
## # Key:      Country, .model [1]  
## # :        Exports = lag(level, 1) + remainder  
##   Country .model Year Exports level remainder .fitted  
##   <fct>   <chr>  <dbl>   <dbl> <dbl>      <dbl>   <dbl>  
## 1 Algeria ANN    1959    NA    39.5      NA      NA  
## 2 Algeria ANN    1960   39.0   39.1    -0.496   39.5  
## 3 Algeria ANN    1961   46.2   45.1     7.12    39.1  
## 4 Algeria ANN    1962   19.8   23.8    -25.3    45.1  
## 5 Algeria ANN    1963   24.7   24.6     0.841   23.8  
## 6 Algeria ANN    1964   25.1   25.0     0.534   24.6  
## 7 Algeria ANN    1965   22.6   23.0    -2.39    25.0  
## 8 Algeria ANN    1966   26.0   25.5     3.00    23.0  
## 9 Algeria ANN    1967   23.4   23.8    -2.07    25.5  
## 10 Algeria ANN   1968   23.1   23.2    -0.630   23.8
```

# Example: Algerian Exports

```
fit |>  
  forecast(h = 5) |>  
  autoplot(algeria_economy) +  
  labs(y = "% of GDP", title = "Exports: Algeria")
```



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# Holt's linear trend

## Component form

Forecast

$$\hat{y}_{t+h|t} = \ell_t + hb_t$$

Level

$$\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + b_{t-1})$$

Trend

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1},$$

# Holt's linear trend

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Forecast

$$\hat{y}_{t+h|t} = \ell_t + hb_t$$

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Trend

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1},$$

- Two smoothing parameters  $\alpha$  and  $\beta^*$  ( $0 \leq \alpha, \beta^* \leq 1$ ).
- $\ell_t$  level: weighted average between  $y_t$  and one-step ahead forecast for time  $t$ , ( $\ell_{t-1} + b_{t-1} = \hat{y}_{t|t-1}$ )
- $b_t$  slope: weighted average of  $(\ell_t - \ell_{t-1})$  and  $b_{t-1}$ , current and previous estimate of slope.
- Choose  $\alpha, \beta^*, \ell_0, b_0$  to minimise SSE.



Holt's linear method with additive errors.

- Assume  $\varepsilon_t = y_t - \ell_{t-1} - b_{t-1} \sim \text{NID}(0, \sigma^2)$ .
- Substituting into the error correction equations for Holt's linear method

$$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$$

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$$

$$b_t = b_{t-1} + \alpha \beta^* \varepsilon_t$$

- For simplicity, set  $\beta = \alpha \beta^*$ .

# Exponential smoothing: trend/slope

Holt's linear method with multiplicative errors.

- Assume  $\varepsilon_t = \frac{y_t - (\ell_{t-1} + b_{t-1})}{(\ell_{t-1} + b_{t-1})}$
- Following a similar approach as above, the innovations state space model underlying Holt's linear method with multiplicative errors is specified as

$$y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t)$$

$$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$$

$$b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$$

where again  $\beta = \alpha\beta^*$  and  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

# ETS(A,A,N): Specifying the model

```
ETS(y ~ error("A") + trend("A") + season("N"))
```

By default, optimal values for  $\beta$  and  $b_0$  are used.

$\beta$  can be chosen manually in `trend()`.

```
trend("A", beta = 0.004)  
trend("A", beta_range = c(0, 0.1))
```

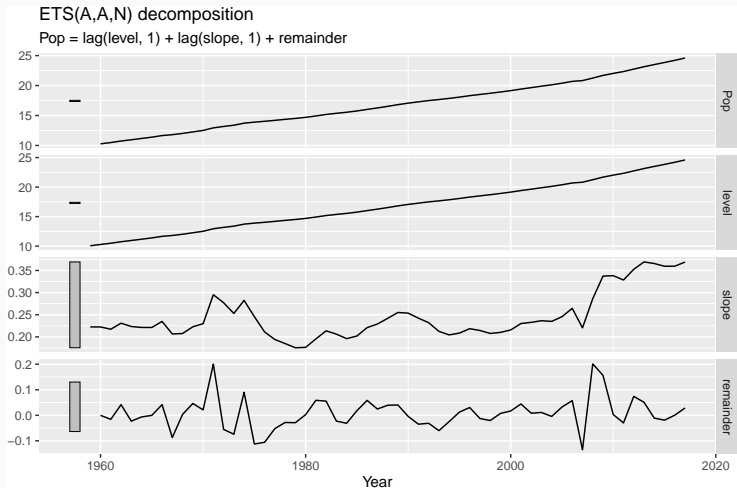
# Example: Australian population

```
aus_economy <- global_economy |>
  filter(Code == "AUS") |>
  mutate(Pop = Population / 1e6)
fit <- aus_economy |>
  model(AAN = ETS(Pop ~ error("A") + trend("A") + season("N")))
report(fit)
```

```
## Series: Pop
## Model: ETS(A,A,N)
##   Smoothing parameters:
##     alpha = 1
##     beta  = 0.327
##
##   Initial states:
##   l[0]  b[0]
##  10.1  0.222
##
##   sigma^2:  0.0041
```

# Example: Australian population

```
components(fit) |> autoplot()
```



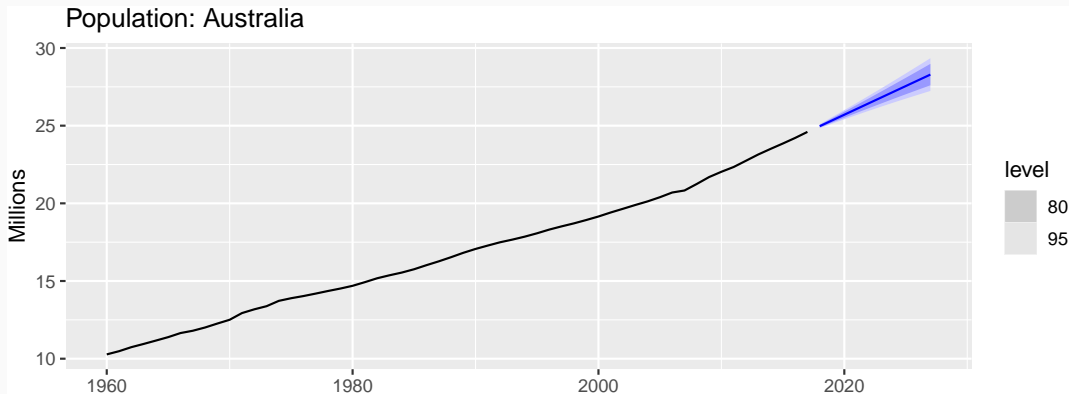
# Example: Australian population

```
components(fit) |>  
  left_join(fitted(fit), by = c("Country", ".model", "Year"))
```

```
## # A dable: 59 x 8 [1Y]  
## # Key:      Country, .model [1]  
## # :      Pop = lag(level, 1) + lag(slope, 1) + remainder  
##   Country   .model  Year   Pop level slope remainder .fitted  
##   <fct>     <chr>   <dbl> <dbl> <dbl> <dbl>      <dbl>   <dbl>  
## 1 Australia AAN     1959  NA    10.1 0.222  NA      NA  
## 2 Australia AAN     1960  10.3  10.3 0.222 -0.000145  10.3  
## 3 Australia AAN     1961  10.5  10.5 0.217 -0.0159    10.5  
## 4 Australia AAN     1962  10.7  10.7 0.231  0.0418    10.7  
## 5 Australia AAN     1963  11.0  11.0 0.223 -0.0229    11.0  
## 6 Australia AAN     1964  11.2  11.2 0.221 -0.00641   11.2  
## 7 Australia AAN     1965  11.4  11.4 0.221 -0.000314  11.4  
## 8 Australia AAN     1966  11.7  11.7 0.235  0.0418    11.6  
## 9 Australia AAN     1967  11.8  11.8 0.206 -0.0869    11.9  
## 10 Australia AAN     1968  12.0  12.0 0.208  0.00350    12.0
```

# Example: Australian population

```
fit |>  
  forecast(h = 10) |>  
  autoplot(aus_economy) +  
  labs(y = "Millions", title = "Population: Australia")
```





# Damped trend method

## Component form

$$\hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \dots + \phi^h)b_t$$

$$\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}.$$

# Damped trend method

## Component form

$$\hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \dots + \phi^h)b_t$$

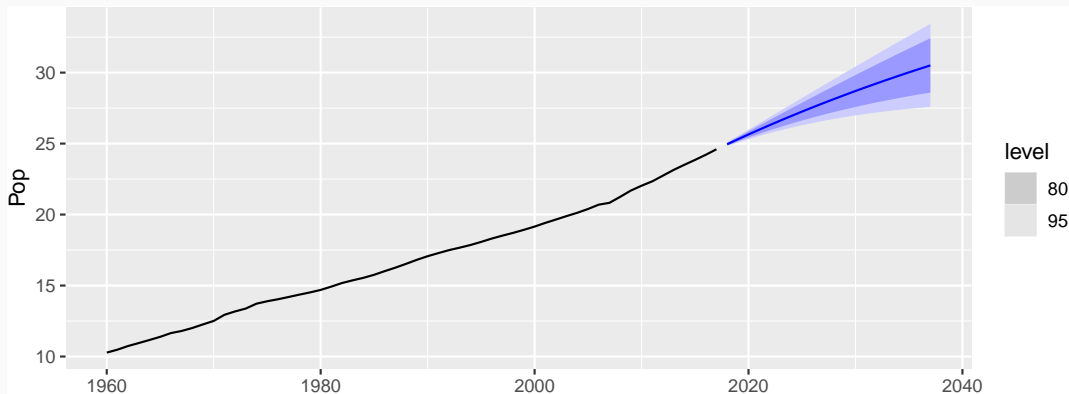
$$\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}.$$

- Damping parameter  $0 < \phi < 1$ .
- If  $\phi = 1$ , identical to Holt's linear trend.
- As  $h \rightarrow \infty$ ,  $\hat{y}_{T+h|T} \rightarrow \ell_T + \phi b_T / (1 - \phi)$ .
- Short-run forecasts trended, long-run forecasts constant.

# Example: Australian population

```
aus_economy |>  
  model(holt = ETS(Pop ~ error("A") + trend("Ad") + season("N"))) |>  
  forecast(h = 20) |>  
  autoplot(aus_economy)
```



# Example: Australian population

```
fit <- aus_economy |>
  filter(Year <= 2010) |>
  model(
    ses = ETS(Pop ~ error("A") + trend("N") + season("N")),
    holt = ETS(Pop ~ error("A") + trend("A") + season("N")),
    damped = ETS(Pop ~ error("A") + trend("Ad") + season("N"))
  )
```

```
tidy(fit)
accuracy(fit)
```

## Example: Australian population

	term	SES	Linear trend	Damped trend
	$\alpha$	1.00	1.00	1.00
	$\beta^*$		0.30	0.40
	$\phi$			0.98
	NA		0.22	0.25
	NA	10.28	10.05	10.04
Training RMSE		0.24	0.06	0.07
Test RMSE		1.63	0.15	0.21
Test MASE		6.18	0.55	0.75
Test MAPE		6.09	0.55	0.74
Test MAE		1.45	0.13	0.18

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# Holt-Winters additive method

Holt and Winters extended Holt's method to capture seasonality.

## Component form

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$$

$$\ell_t = \alpha(y_t - s_{t-m}) + (1 - \alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$$

$$s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1 - \gamma)s_{t-m}$$

- $k = \text{integer part of } (h - 1)/m$ . Ensures estimates from the final year are used for forecasting.
- Parameters:  $0 \leq \alpha \leq 1$ ,  $0 \leq \beta^* \leq 1$ ,  $0 \leq \gamma \leq 1 - \alpha$  and  $m = \text{period of seasonality}$  (e.g.  $m = 4$  for quarterly data).

# Holt-Winters additive method

- Seasonal component is usually expressed as

$$s_t = \gamma^*(y_t - \ell_t) + (1 - \gamma^*)s_{t-m}.$$

- Substitute in for  $\ell_t$ :

$$s_t = \gamma^*(1 - \alpha)(y_t - \ell_{t-1} - b_{t-1}) + [1 - \gamma^*(1 - \alpha)]s_{t-m}$$

- We set  $\gamma = \gamma^*(1 - \alpha)$ .
- The usual parameter restriction is  $0 \leq \gamma^* \leq 1$ , which translates to  $0 \leq \gamma \leq (1 - \alpha)$ .



# Exponential smoothing: seasonality

# ETS(A,A,A)

Holt-Winters additive method with additive errors.

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$$

Observation equation

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$$

$$b_t = b_{t-1} + \beta \varepsilon_t$$

$$s_t = s_{t-m} + \gamma \varepsilon_t$$

- Forecast errors:  $\varepsilon_t = y_t - \hat{y}_{t|t-1}$
- $k$  is integer part of  $(h - 1)/m$ .

# Holt-Winters multiplicative method

Seasonal variations change in proportion to the level of the series.

## Component form

$$\hat{y}_{t+h|t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$$

$$\ell_t = \alpha \frac{y_t}{s_{t-m}} + (1 - \alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$$

$$s_t = \gamma \frac{y_t}{(\ell_{t-1} + b_{t-1})} + (1 - \gamma)s_{t-m}$$

- $k$  is integer part of  $(h - 1)/m$ .
- Additive method:  $s_t$  in absolute terms — within each year  $\sum_i s_i \approx 0$ .
- Multiplicative method:  $s_t$  in relative terms — within each year  $\sum_i s_i \approx m$ .

# ETS(M,A,M)

Holt-Winters multiplicative method with multiplicative errors.

Forecast equation	$\hat{y}_{t+h t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$
Observation equation	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$
State equations	$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

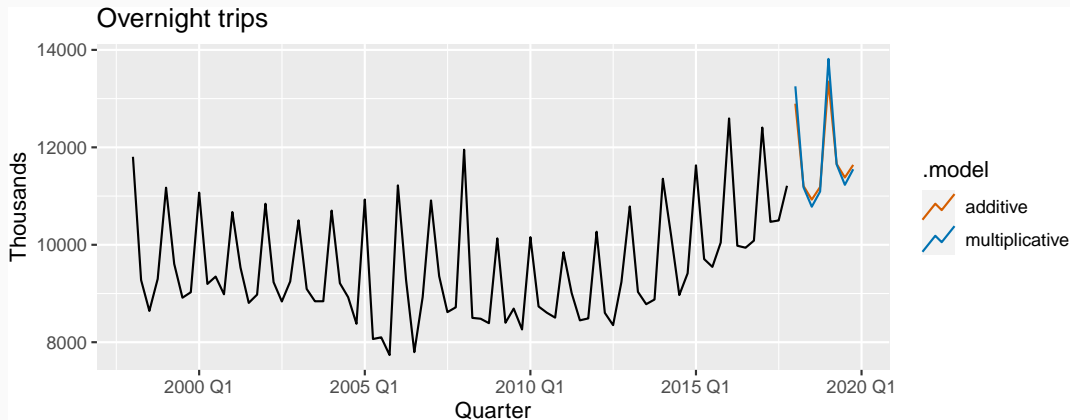
- Forecast errors:  $\varepsilon_t = (y_t - \hat{y}_{t|t-1})/\hat{y}_{t|t-1}$
- $k$  is integer part of  $(h - 1)/m$ .

# Example: Australian holiday tourism

```
aus_holidays <- tourism |>
  filter(Purpose == "Holiday") |>
  summarise(Trips = sum(Trips))
fit <- aus_holidays |>
  model(
    additive = ETS(Trips ~ error("A") + trend("A") + season("A")),
    multiplicative = ETS(Trips ~ error("M") + trend("A") + season("M"))
  )
fc <- fit |> forecast()
```

# Example: Australian holiday tourism

```
fc |>  
  autoplot(aus_holidays, level = NULL) +  
  labs(y = "Thousands", title = "Overnight trips")
```



# Estimated components

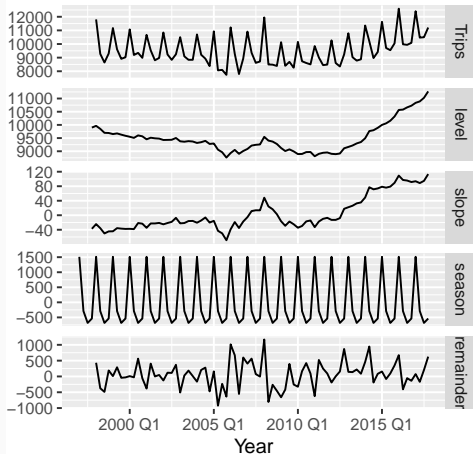
```
components(fit)
```

```
## # A dable: 168 x 7 [1Q]
## # Key:      .model [2]
## # :      Trips = lag(level, 1) + lag(slope, 1) + lag(season, 4) +
## # remainder
```

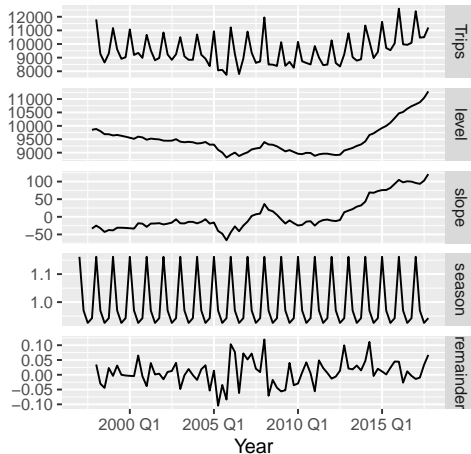
##	.model	Quarter	Trips	level	slope	season	remainder
##	<chr>	<qtr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
##	1 additive	1997 Q1	NA	NA	NA	1512.	NA
##	2 additive	1997 Q2	NA	NA	NA	-290.	NA
##	3 additive	1997 Q3	NA	NA	NA	-684.	NA
##	4 additive	1997 Q4	NA	9899.	-37.4	-538.	NA
##	5 additive	1998 Q1	11806.	9964.	-24.5	1512.	433.
##	6 additive	1998 Q2	9276.	9851.	-35.6	-290.	-374.
##	7 additive	1998 Q3	8642.	9700.	-50.2	-684.	-489.
##	8 additive	1998 Q4	9300.	9694.	-44.6	-538.	188.

# Estimated components

## Additive states



## Multiplicative states





# Holt-Winters damped method

Often the single most accurate forecasting method for seasonal data:

$$\hat{y}_{t+h|t} = [\ell_t + (\phi + \phi^2 + \dots + \phi^h)b_t]s_{t+h-m(k+1)}$$

$$\ell_t = \alpha(y_t/s_{t-m}) + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$$

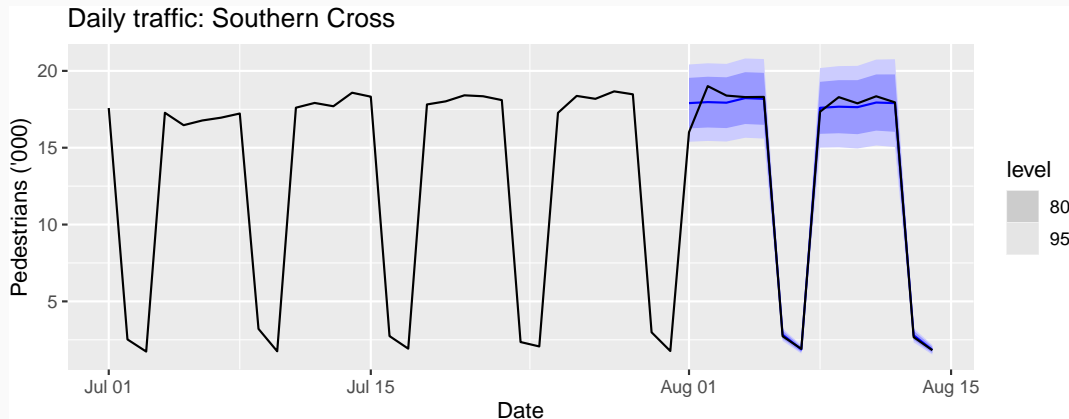
$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}$$

$$s_t = \gamma \frac{y_t}{(\ell_{t-1} + \phi b_{t-1})} + (1 - \gamma)s_{t-m}$$

# Holt-Winters with daily data

```
sth_cross_ped <- pedestrian |>
  filter(
    Date >= "2016-07-01",
    Sensor == "Southern Cross Station"
  ) |>
  index_by(Date) |>
  summarise(Count = sum(Count) / 1000)
sth_cross_ped |>
  filter(Date <= "2016-07-31") |>
  model(
    hw = ETS(Count ~ error("M") + trend("Ad") + season("M"))
  ) |>
  forecast(h = "2 weeks") |>
  autoplot(sth_cross_ped |> filter(Date <= "2016-08-14")) +
  labs(
    title = "Daily traffic: Southern Cross",
    y = "Pedestrians ('000)"
  )
```

# Holt-Winters with daily data



# Outline

- 1 Exponential smoothing
- 2 Simple exponential smoothing
- 3 Models with trend
- 4 Models with seasonality
- 5 Innovations state space models**
- 6 Forecasting with exponential smoothing

# Exponential smoothing methods

		Seasonal Component		
		N	A	M
Trend Component		(None)	(Additive)	(Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A <sub>d</sub>	(Additive damped)	(A <sub>d</sub> ,N)	(A <sub>d</sub> ,A)	(A <sub>d</sub> ,M)

**(N,N):** Simple exponential smoothing

**(A,N):** Holt's linear method

**(A<sub>d</sub>,N):** Additive damped trend method

**(A,A):** Additive Holt-Winters' method

**(A,M):** Multiplicative Holt-Winters' method

**(A<sub>d</sub>,M):** Damped multiplicative Holt-Winters' method

# Exponential smoothing methods

		Seasonal Component		
		N	A	M
Trend Component		(None)	(Additive)	(Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A <sub>d</sub>	(Additive damped)	(A <sub>d</sub> ,N)	(A <sub>d</sub> ,A)	(A <sub>d</sub> ,M)

**(N,N):** Simple exponential smoothing

**(A,N):** Holt's linear method

**(A<sub>d</sub>,N):** Additive damped trend method

**(A,A):** Additive Holt-Winters' method

**(A,M):** Multiplicative Holt-Winters' method

**(A<sub>d</sub>,M):** Damped multiplicative Holt-Winters' method

There are also multiplicative trend methods (not recommended).

# ETS models

## Additive Error

Trend Component		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A <sub>d</sub>	(Additive damped)	A,A <sub>d</sub> ,N	A,A <sub>d</sub> ,A	A,A <sub>d</sub> ,M

## Multiplicative Error

Trend Component		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A <sub>d</sub>	(Additive damped)	M,A <sub>d</sub> ,N	M,A <sub>d</sub> ,A	M,A <sub>d</sub> ,M

# Additive error models

Trend	Seasonal		
	N	A	M
N	$y_t = \ell_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t$	$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = \ell_{t-1} s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / \ell_{t-1}$
A	$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$ $b_t = b_{t-1} + \beta \varepsilon_t$	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t$ $b_t = b_{t-1} + \beta \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $b_t = b_{t-1} + \beta \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / (\ell_{t-1} + b_{t-1})$
A <sub>d</sub>	$y_t = \ell_{t-1} + \phi b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t$ $b_t = \phi b_{t-1} + \beta \varepsilon_t$	$y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t$ $b_t = \phi b_{t-1} + \beta \varepsilon_t$ $s_t = s_{t-m} + \gamma \varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1}) s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t / s_{t-m}$ $b_t = \phi b_{t-1} + \beta \varepsilon_t / s_{t-m}$ $s_t = s_{t-m} + \gamma \varepsilon_t / (\ell_{t-1} + \phi b_{t-1})$



# Multiplicative error models

Trend	Seasonal		
	N	A	M
N	$y_t = \ell_{t-1}(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$	$y_t = (\ell_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + \alpha(\ell_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + s_{t-m})\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m}(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
A	$y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + b_{t-1} + s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$
A <sub>d</sub>	$y_t = (\ell_{t-1} + \phi b_{t-1})(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1} + s_{t-m})(1 + \varepsilon_t)$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$ $s_t = s_{t-m} + \gamma(\ell_{t-1} + \phi b_{t-1} + s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1})s_{t-m}(1 + \varepsilon_t)$ $\ell_t = (\ell_{t-1} + \phi b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = \phi b_{t-1} + \beta(\ell_{t-1} + \phi b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

# Estimating ETS models

- Smoothing parameters  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\phi$ , and the initial states  $\ell_0$ ,  $b_0$ ,  $s_0$ ,  $s_{-1}$ ,  $\dots$ ,  $s_{-m+1}$  are estimated by maximising the “likelihood” = the probability of the data arising from the specified model.
- For models with additive errors equivalent to minimising SSE.
- For models with multiplicative errors, **not** equivalent to minimising SSE.

# Innovations state space models

Let  $\mathbf{x}_t = (\ell_t, b_t, s_t, s_{t-1}, \dots, s_{t-m+1})$  and  $\varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$ .

$$y_t = \underbrace{h(\mathbf{x}_{t-1})}_{\mu_t} + \underbrace{k(\mathbf{x}_{t-1})\varepsilon_t}_{e_t}$$

$$\mathbf{x}_t = f(\mathbf{x}_{t-1}) + g(\mathbf{x}_{t-1})\varepsilon_t$$

## Additive errors

$$k(x) = 1. \quad y_t = \mu_t + \varepsilon_t.$$

## Multiplicative errors

$$k(\mathbf{x}_{t-1}) = \mu_t. \quad y_t = \mu_t(1 + \varepsilon_t).$$

$$\varepsilon_t = (y_t - \mu_t) / \mu_t \text{ is relative error.}$$

# Innovations state space models

## Estimation

$$\begin{aligned} L^*(\boldsymbol{\theta}, \mathbf{x}_0) &= T \log \left( \sum_{t=1}^T \varepsilon_t^2 \right) + 2 \sum_{t=1}^T \log |k(\mathbf{x}_{t-1})| \\ &= -2 \log(\text{Likelihood}) + \text{constant} \end{aligned}$$

- Estimate parameters  $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \phi)$  and initial states  $\mathbf{x}_0 = (\ell_0, b_0, s_0, s_{-1}, \dots, s_{-m+1})$  by minimizing  $L^*$ .

# Parameter restrictions

## *Usual region*

- Traditional restrictions in the methods  $0 < \alpha, \beta^*, \gamma^*, \phi < 1$  (equations interpreted as weighted averages).
- In models we set  $\beta = \alpha\beta^*$  and  $\gamma = (1 - \alpha)\gamma^*$ .
- Therefore  $0 < \alpha < 1$ ,  $0 < \beta < \alpha$  and  $0 < \gamma < 1 - \alpha$ .
- $0.8 < \phi < 0.98$  — to prevent numerical difficulties.

# Parameter restrictions

## Usual region

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- Therefore  $0 < \alpha < 1$ ,  $0 < \beta < \alpha$  and  $0 < \gamma < 1 - \alpha$ .
- $0.8 < \phi < 0.98$  — to prevent numerical difficulties.

## Admissible region

- To prevent observations in the distant past having a continuing effect on current forecasts.
- Usually (but not always) less restrictive than *traditional* region.
- For example for ETS(A,N,N):  
*traditional*  $0 < \alpha < 1$  while *admissible*  $0 < \alpha < 2$ .

# Model selection

## Akaike's Information Criterion

$$AIC = -2 \log(L) + 2k$$

where  $L$  is the likelihood and  $k$  is the number of parameters & initial states estimated in the model.

# Model selection

## Akaike's Information Criterion

$$\text{AIC} = -2 \log(L) + 2k$$

where  $L$  is the likelihood and  $k$  is the number of parameters & initial states estimated in the model.

## Corrected AIC

$$\text{AIC}_c = \text{AIC} + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).



# Model selection

## Akaike's Information Criterion

$$\text{AIC} = -2 \log(L) + 2k$$

where  $L$  is the likelihood and  $k$  is the number of parameters & initial states estimated in the model.

## Corrected AIC

$$\text{AIC}_c = \text{AIC} + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).

## Bayesian Information Criterion

$$\text{BIC} = \text{AIC} + k[\log(T) - 2].$$

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

# Automatic forecasting

**From Hyndman et al. (IJF, 2002):**

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best method using AICc:
- Produce forecasts using best method.
- Obtain forecast intervals using underlying state space model.

Method performed very well in M3 competition.

# Example: National populations

```
fit <- global_economy |>
  mutate(Pop = Population / 1e6) |>
  model(ets = ETS(Pop))
fit
```

```
## # A mable: 263 x 2
## # Key:      Country [263]
##   Country      ets
##   <fct>        <model>
## 1 Afghanistan <ETS(A,A,N)>
## 2 Albania     <ETS(M,A,N)>
## 3 Algeria     <ETS(M,A,N)>
## 4 American Samoa <ETS(M,A,N)>
## 5 Andorra     <ETS(M,A,N)>
## 6 Angola      <ETS(M,A,N)>
## 7 Antigua and Barbuda <ETS(M,A,N)>
## 8 Arab World  <ETS(M,A,N)>
## 9 Argentina   <ETS(A,A,N)>
```

# Example: National populations

```
fit |>  
  forecast(h = 5)
```

```
## # A tibble: 1,315 x 5 [1Y]  
## # Key:      Country, .model [263]  
##   Country      .model Year      Pop .mean  
##   <fct>        <chr>  <dbl>    <dist> <dbl>  
## 1 Afghanistan ets     2018    N(36, 0.012) 36.4  
## 2 Afghanistan ets     2019    N(37, 0.059) 37.3  
## 3 Afghanistan ets     2020    N(38, 0.16) 38.2  
## 4 Afghanistan ets     2021    N(39, 0.35) 39.0  
## 5 Afghanistan ets     2022    N(40, 0.64) 39.9  
## 6 Albania      ets     2018    N(2.9, 0.00012) 2.87  
## 7 Albania      ets     2019    N(2.9, 6e-04) 2.87  
## 8 Albania      ets     2020    N(2.9, 0.0017) 2.87  
## 9 Albania      ets     2021    N(2.9, 0.0036) 2.86  
## 10 Albania     ets     2022    N(2.9, 0.0066) 2.86  
## # ... with 1,305 more rows
```

# Example: Australian holiday tourism

```
holidays <- tourism |>
  filter(Purpose == "Holiday")
fit <- holidays |> model(ets = ETS(Trips))
fit
```

```
## # A mable: 76 x 4
```

```
## # Key:      Region, State, Purpose [76]
```

##	Region	State	Purpose	ets
##	<chr>	<chr>	<chr>	<model>
## 1	Adelaide	South Australia	Holiday	<ETS(A,N,A)>
## 2	Adelaide Hills	South Australia	Holiday	<ETS(A,A,N)>
## 3	Alice Springs	Northern Territory	Holiday	<ETS(M,N,A)>
## 4	Australia's Coral Coast	Western Australia	Holiday	<ETS(M,N,A)>
## 5	Australia's Golden Outback	Western Australia	Holiday	<ETS(M,N,M)>
## 6	Australia's North West	Western Australia	Holiday	<ETS(A,N,A)>
## 7	Australia's South West	Western Australia	Holiday	<ETS(M,N,M)>
## 8	Ballarat	Victoria	Holiday	<ETS(M,N,A)>
## 9	Barkly	Northern Territory	Holiday	<ETS(A,N,A)>

# Example: Australian holiday tourism

```
fit |>  
  filter(Region == "Snowy Mountains") |>  
  report()
```

```
## Series: Trips  
## Model: ETS(M,N,A)  
## Smoothing parameters:  
##   alpha = 0.157  
##   gamma = 1e-04  
##  
## Initial states:  
## l[0] s[0] s[-1] s[-2] s[-3]  
## 142 -61 131 -42.2 -27.7  
##  
## sigma^2: 0.0388  
##  
## AIC AICc BIC  
## 852 854 869
```

# Example: Australian holiday tourism

```
fit |>
  filter(Region == "Snowy Mountains") |>
  components(fit)
```

```
## # A dable: 84 x 9 [1Q]
## # Key:      Region, State, Purpose, .model [1]
## # :        Trips = (lag(level, 1) + lag(season, 4)) * (1 + remainder)
##   Region      State Purpose .model Quarter Trips level season remain~1
##   <chr>        <chr> <chr>  <chr>    <qtr> <dbl> <dbl> <dbl> <dbl>
## 1 Snowy Mountains New S~ Holiday ets    1997 Q1  NA      NA    -27.7 NA
## 2 Snowy Mountains New S~ Holiday ets    1997 Q2  NA      NA    -42.2 NA
## 3 Snowy Mountains New S~ Holiday ets    1997 Q3  NA      NA    131.  NA
## 4 Snowy Mountains New S~ Holiday ets    1997 Q4  NA    142.   -61.0 NA
## 5 Snowy Mountains New S~ Holiday ets    1998 Q1 101.    140.   -27.7 -0.113
## 6 Snowy Mountains New S~ Holiday ets    1998 Q2 112.    142.   -42.2  0.154
## 7 Snowy Mountains New S~ Holiday ets    1998 Q3 310.    148.    131.  0.137
## 8 Snowy Mountains New S~ Holiday ets    1998 Q4  89.8   148.   -61.0  0.0335
## 9 Snowy Mountains New S~ Holiday ets    1999 Q1 112.    147.   -27.7 -0.0687
```

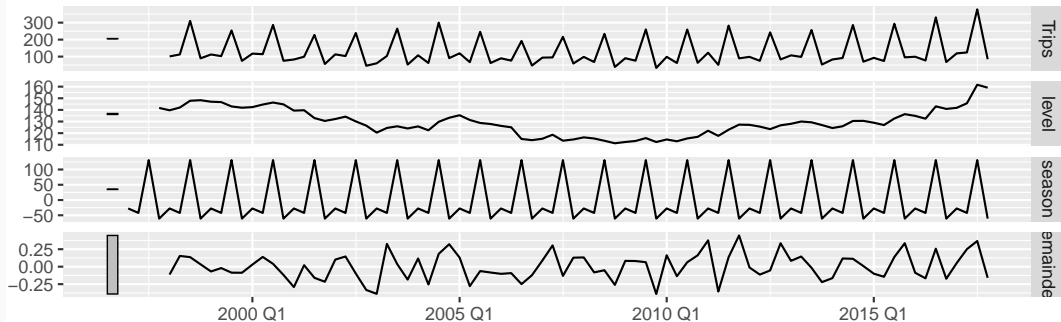


# Example: Australian holiday tourism

```
fit |>  
  filter(Region == "Snowy Mountains") |>  
  components(fit) |>  
  autoplot()
```

ETS(M,N,A) decomposition

$\text{Trips} = (\text{lag}(\text{level}, 1) + \text{lag}(\text{season}, 4)) * (1 + \text{remainder})$



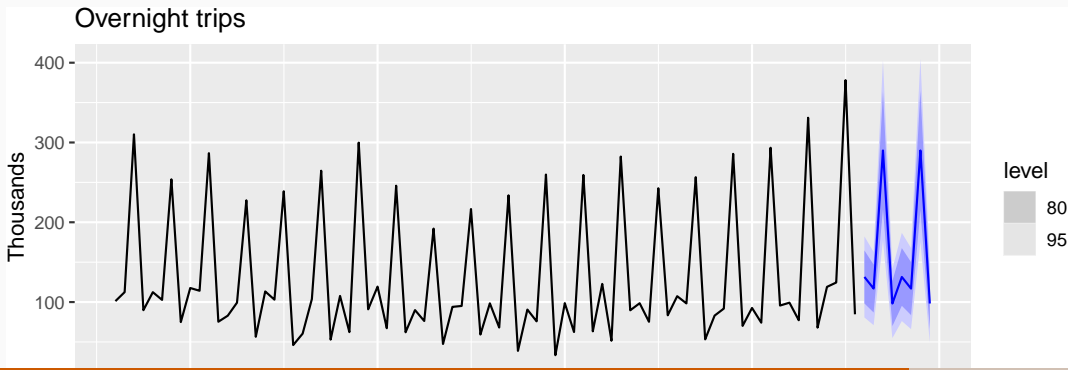
# Example: Australian holiday tourism

```
fit |> forecast()
```

```
## # A fable: 608 x 7 [1Q]
## # Key:      Region, State, Purpose, .model [76]
##   Region      State      Purpose .model Quarter      Trips .mean
##   <chr>        <chr>      <chr>  <chr>    <qtr>      <dist> <dbl>
## 1 Adelaide     South Australia Holiday ets    2018 Q1 N(210, 457) 210.
## 2 Adelaide     South Australia Holiday ets    2018 Q2 N(173, 473) 173.
## 3 Adelaide     South Australia Holiday ets    2018 Q3 N(169, 489) 169.
## 4 Adelaide     South Australia Holiday ets    2018 Q4 N(186, 505) 186.
## 5 Adelaide     South Australia Holiday ets    2019 Q1 N(210, 521) 210.
## 6 Adelaide     South Australia Holiday ets    2019 Q2 N(173, 537) 173.
## 7 Adelaide     South Australia Holiday ets    2019 Q3 N(169, 553) 169.
## 8 Adelaide     South Australia Holiday ets    2019 Q4 N(186, 569) 186.
## 9 Adelaide Hills South Australia Holiday ets    2018 Q1 N(19, 36) 19.4
## 10 Adelaide Hills South Australia Holiday ets    2018 Q2 N(20, 36) 19.6
## # ... with 598 more rows
```

# Example: Australian holiday tourism

```
fit |>  
  forecast() |>  
  filter(Region == "Snowy Mountains") |>  
  autoplot(holidays) +  
  labs(y = "Thousands", title = "Overnight trips")
```



# Residuals

## Response residuals

$$\hat{e}_t = y_t - \hat{y}_{t|t-1}$$

## Innovation residuals

Additive error model:

$$\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$$

Multiplicative error model:

$$\hat{\varepsilon}_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}}$$

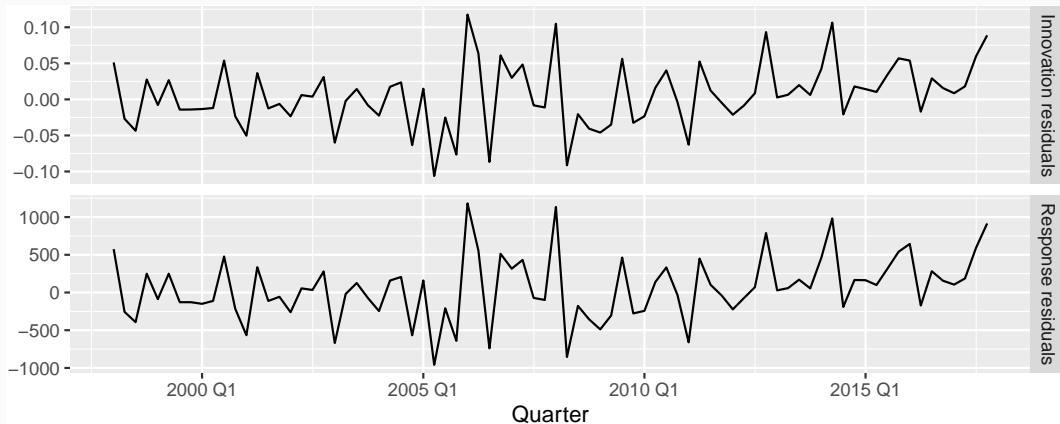
# Example: Australian holiday tourism

```
aus_holidays <- tourism |>
  filter(Purpose == "Holiday") |>
  summarise(Trips = sum(Trips))
fit <- aus_holidays |>
  model(ets = ETS(Trips)) |>
  report()
```

```
## Series: Trips
## Model: ETS(M,N,M)
##   Smoothing parameters:
##     alpha = 0.358
##     gamma = 0.000969
##
##   Initial states:
##   l[0]  s[0] s[-1] s[-2] s[-3]
## 9667 0.943 0.927 0.968 1.16
##
##   sigma^2: 0.0022
```

# Example: Australian holiday tourism

```
residuals(fit)  
residuals(fit, type = "response")
```



# Example: Australian holiday tourism

```
fit |>  
  augment()
```

```
## # A tsibble: 80 x 6 [1Q]  
## # Key:           .model [1]  
##   .model Quarter  Trips .fitted .resid  .innov  
##   <chr>    <qtr>  <dbl>   <dbl>  <dbl>   <dbl>  
## 1 ets      1998 Q1 11806.  11230.  576.    0.0513  
## 2 ets      1998 Q2  9276.   9532. -257.   -0.0269  
## 3 ets      1998 Q3  8642.   9036. -393.   -0.0435  
## 4 ets      1998 Q4  9300.   9050.  249.    0.0275  
## 5 ets      1999 Q1 11172.  11260.  -88.0  -0.00781  
## 6 ets      1999 Q2  9608.   9358.  249.    0.0266  
## 7 ets      1999 Q3  8914.   9042. -129.   -0.0142  
## 8 ets      1999 Q4  9026.   9154. -129.   -0.0140  
## 9 ets      2000 Q1 11071.  11221. -150.   -0.0134  
## 10 ets     2000 Q2  9196.   9308. -111.   -0.0120  
## #           with 70 more rows
```

# Example: Australian holiday tourism

```
fit |>
  augment()
```

Innovation residuals (`.innov`) are given by  $\hat{\varepsilon}_t$  while regular residuals (`.resid`) are  $y_t - \hat{y}_{t-1}$ . They are different when the model has multiplicative errors.

```
## # A tsibble: 80 x 6 [1Q]
## # Key:           .model [1]
##   .model Quarter  Trips .fitted .resid  .innov
##   <chr>    <qtr>   <dbl>   <dbl>  <dbl>   <dbl>
## 1 ets      1998 Q1  11806.  11230.   576.    0.0513
## 2 ets      1998 Q2   9276.   9532.  -257.   -0.0269
## 3 ets      1998 Q3   8642.   9036.  -393.   -0.0435
## 4 ets      1998 Q4   9300.   9050.   249.    0.0275
## 5 ets      1999 Q1  11172.  11260.  -88.0  -0.00781
## 6 ets      1999 Q2   9608.   9358.   249.    0.0266
## 7 ets      1999 Q3   8914.   9042.  -129.   -0.0142
## 8 ets      1999 Q4   9026.   9154.  -129.   -0.0140
## 9 ets      2000 Q1  11071.  11221.  -150.   -0.0134
## 10 ets     2000 Q2   9196.   9308.  -111.   -0.0120
## #           with 70 more rows
```



## Some unstable models

- Some of the combinations of (Error, Trend, Seasonal) can lead to numerical difficulties; see equations with division by a state.
- These are:  $ETS(A,N,M)$ ,  $ETS(A,A,M)$ ,  $ETS(A,A_d,M)$ .
- Models with multiplicative errors are useful for strictly positive data, but are not numerically stable with data containing zeros or negative values. In that case only the six fully additive models will be applied.

# Exponential smoothing models

## Additive Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	A,N,N	A,N,A	<del>A,N,M</del>
A	(Additive)	A,A,N	A,A,A	<del>A,A,M</del>
A <sub>d</sub>	(Additive damped)	A,A <sub>d</sub> ,N	A,A <sub>d</sub> ,A	<del>A,A<sub>d</sub>,M</del>

## Multiplicative Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A <sub>d</sub>	(Additive damped)	M,A <sub>d</sub> ,N	M,A <sub>d</sub> ,A	M,A <sub>d</sub> ,M

# Outline

- 1 Exponential smoothing
- 2 Simple exponential smoothing
- 3 Models with trend
- 4 Models with seasonality
- 5 Innovations state space models
- 6 Forecasting with exponential smoothing

# Forecasting with ETS models

**Traditional point forecasts:** iterate the equations for  $t = T + 1, T + 2, \dots, T + h$  and set all  $\varepsilon_t = 0$  for  $t > T$ .

# Forecasting with ETS models

**Traditional point forecasts:** iterate the equations for  $t = T + 1, T + 2, \dots, T + h$  and set all  $\varepsilon_t = 0$  for  $t > T$ .

- Not the same as  $E(y_{t+h}|\mathbf{x}_t)$  unless seasonality is additive.
- fable uses  $E(y_{t+h}|\mathbf{x}_t)$ .
- Point forecasts for  $\text{ETS}(A, *, *)$  are identical to  $\text{ETS}(M, *, *)$  if the parameters are the same.

## Example: ETS(A,A,N)

$$y_{T+1} = \ell_T + b_T + \varepsilon_{T+1}$$

$$\hat{y}_{T+1|T} = \ell_T + b_T$$

$$y_{T+2} = \ell_{T+1} + b_{T+1} + \varepsilon_{T+2}$$

$$= (\ell_T + b_T + \alpha\varepsilon_{T+1}) + (b_T + \beta\varepsilon_{T+1}) + \varepsilon_{T+2}$$

$$\hat{y}_{T+2|T} = \ell_T + 2b_T$$

etc.

## Example: ETS(M,A,N)

$$y_{T+1} = (\ell_T + b_T)(1 + \varepsilon_{T+1})$$

$$\hat{y}_{T+1|T} = \ell_T + b_T.$$

$$y_{T+2} = (\ell_{T+1} + b_{T+1})(1 + \varepsilon_{T+2})$$

$$= \{(\ell_T + b_T)(1 + \alpha\varepsilon_{T+1}) + [b_T + \beta(\ell_T + b_T)\varepsilon_{T+1}]\} (1 + \varepsilon_{T+2})$$

$$\hat{y}_{T+2|T} = \ell_T + 2b_T$$

etc.

# Forecasting with ETS models

**Prediction intervals:** can only be generated using the models.

- The prediction intervals will differ between models with additive and multiplicative errors.
- Exact formulae for some models.
- More general to simulate future sample paths, conditional on the last estimate of the states, and to obtain prediction intervals from the percentiles of these simulated future paths.



# Prediction intervals

PI for most ETS models:  $\hat{y}_{T+h|T} \pm c\sigma_h$ , where  $c$  depends on coverage probability and  $\sigma_h$  is forecast standard deviation.

$$(A,N,N) \quad \sigma_h = \sigma^2 \left[ 1 + \alpha^2(h-1) \right]$$

$$(A,A,N) \quad \sigma_h = \sigma^2 \left[ 1 + (h-1) \left\{ \alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h-1) \right\} \right]$$

$$(A,A_d,N) \quad \sigma_h = \sigma^2 \left[ 1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2} \{2\alpha(1-\phi) + \beta\phi\} - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)} \{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\} \right]$$

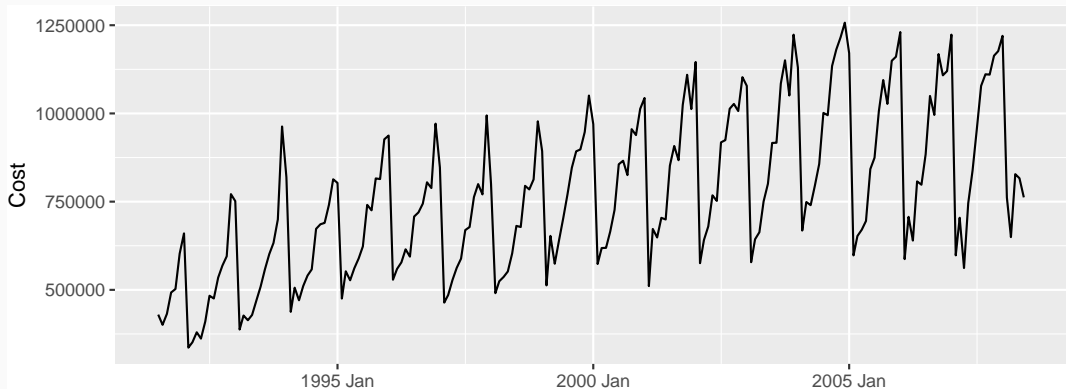
$$(A,N,A) \quad \sigma_h = \sigma^2 \left[ 1 + \alpha^2(h-1) + \gamma k(2\alpha + \gamma) \right]$$

$$(A,A,A) \quad \sigma_h = \sigma^2 \left[ 1 + (h-1) \left\{ \alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h-1) \right\} + \gamma k \{2\alpha + \gamma + \beta m(k+1)\} \right]$$

$$(A,A_d,A) \quad \sigma_h = \sigma^2 \left[ 1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2} \{2\alpha(1-\phi) + \beta\phi\} - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)} \{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\} \right. \\ \left. + \gamma k(2\alpha + \gamma) + \frac{2\beta\gamma\phi}{(1-\phi)(1-\phi^m)} \{k(1-\phi^m) - \phi^m(1-\phi^{mk})\} \right]$$

# Example: Corticosteroid drug sales

```
h02 <- PBS |>  
  filter(ATC2 == "H02") |>  
  summarise(Cost = sum(Cost))  
h02 |> autoplot(Cost)
```



# Example: Corticosteroid drug sales

```
h02 |>  
  model(ETS(Cost)) |>  
  report()
```

```
## Series: Cost  
## Model: ETS(M,Ad,M)  
## Smoothing parameters:  
##   alpha = 0.307  
##   beta  = 0.000101  
##   gamma = 0.000101  
##   phi   = 0.978  
##  
## Initial states:  
##   l[0] b[0] s[0] s[-1] s[-2] s[-3] s[-4] s[-5] s[-6] s[-7] s[-8] s[-9]  
## 417269 8206 0.872 0.826 0.756 0.773 0.687 1.28 1.32 1.18 1.16 1.1  
## s[-10] s[-11]  
## 1.05 0.981  
##  
## sigma^2: 0.0046  
##  
## AIC AICc BIC  
## 5515 5519 5575
```

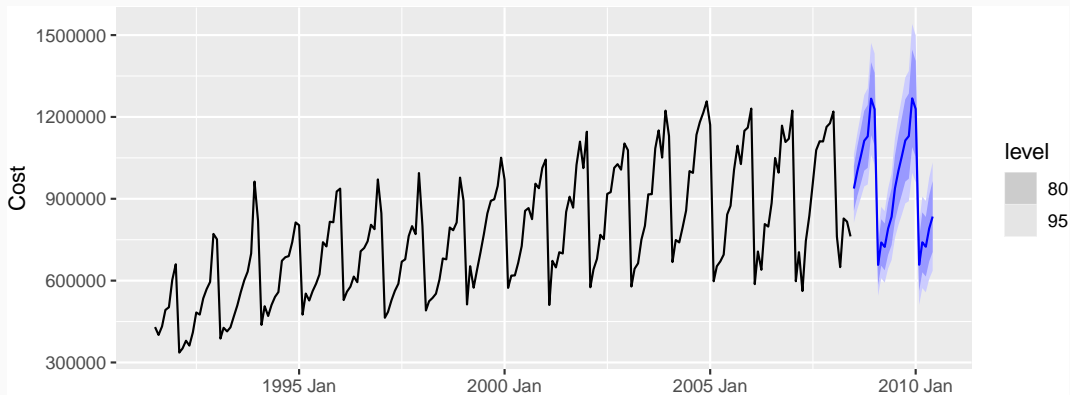
# Example: Corticosteroid drug sales

```
h02 |>
  model(ETS(Cost ~ error("A") + trend("A") + season("A"))) |>
  report()
```

```
## Series: Cost
## Model: ETS(A,A,A)
## Smoothing parameters:
##   alpha = 0.17
##   beta  = 0.00631
##   gamma = 0.455
##
## Initial states:
##   l[0] b[0]   s[0]   s[-1]   s[-2]   s[-3]   s[-4]   s[-5]   s[-6]   s[-7]
## 409706 9097 -99075 -136602 -191496 -174531 -241437 210644 244644 145368
##   s[-8] s[-9] s[-10] s[-11]
## 130570 84458  39132 -11674
##
##   sigma^2: 3.5e+09
##
## AIC AICc BIC
## 5585 5589 5642
```

# Example: Corticosteroid drug sales

```
h02 |>  
  model(ETS(Cost)) |>  
  forecast() |>  
  autoplot(h02)
```



# Example: Corticosteroid drug sales

```
h02 |>
  model(
    auto = ETS(Cost),
    AAA = ETS(Cost ~ error("A") + trend("A") + season("A"))
  ) |>
  accuracy()
```

Model	MAE	RMSE	MAPE	MASE	RMSSE
auto	38649	51102	4.99	0.638	0.689
AAA	43378	56784	6.05	0.716	0.766