

ETC3550/ETC5550

Applied forecasting

Ch8. Exponential smoothing

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Historical perspective

- Developed in the 1950s and 1960s as methods (algorithms) to produce point forecasts.
- Combine a “level”, “trend” (slope) and “seasonal” component to describe a time series.
- The rate of change of the components are controlled by “smoothing parameters”: α , β and γ respectively.
- Need to choose best values for the smoothing parameters (and initial states).
- Equivalent ETS state space models developed in the 1990s and 2000s.

Big idea: control the rate of change

α controls the flexibility of the **level**

- If $\alpha = 0$, the level never updates (mean)
- If $\alpha = 1$, the level updates completely (naive)

β controls the flexibility of the **trend**

- If $\beta = 0$, the trend is linear
- If $\beta = 1$, the trend changes suddenly every observation

γ controls the flexibility of the **seasonality**

- If $\gamma = 0$, the seasonality is fixed (seasonal means)
- If $\gamma = 1$, the seasonality updates completely (seasonal naive)

Models and methods

Methods

- Algorithms that return point forecasts.

Models

- Generate same point forecasts but can also generate forecast distributions.
- A stochastic (or random) data generating process that can generate an entire forecast distribution.
- Allow for “proper” model selection.

Simple Exponential Smoothing

Iterative form

$$\hat{y}_{t+1|t} = \alpha y_t + (1 - \alpha) \hat{y}_{t|t-1}$$

Simple Exponential Smoothing

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Weighted average form

$$\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha (1 - \alpha)^j y_{T-j} + (1 - \alpha)^T \ell_0$$

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Component form

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t$$

Smoothing equation

$$\ell_t = \alpha y_t + (1 - \alpha) \ell_{t-1}$$

Simple Exponential Smoothing

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Forecast error: $e_t = y_t - \hat{y}_{t|t-1} = y_t - \ell_{t-1}$.

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Error correction form

$$y_t = \ell_{t-1} + e_t$$

$$\ell_t = \ell_{t-1} + \alpha(y_t - \ell_{t-1})$$

$$= \ell_{t-1} + \alpha e_t$$

Simple Exponential Smoothing

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Specify probability distribution for e_t , we assume $e_t = \varepsilon_t \sim \text{NID}(0, \sigma^2)$. 6

ETS(A,N,N): SES with additive errors

ETS(A,N,N) model

Measurement equation

$$y_t = l_{t-1} + \varepsilon_t$$

State equation

$$l_t = l_{t-1} + \alpha \varepsilon_t$$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- “innovations” or “single source of error” because equations have the same error process, ε_t .
- Measurement equation: relationship between observations and states.
- State equation(s): evolution of the state(s) through time.

ETS(M,N,N): SES with multiplicative errors.

- Specify relative errors $\varepsilon_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \sim \text{NID}(0, \sigma^2)$
- Substituting $\hat{y}_{t|t-1} = \ell_{t-1}$ gives:
 - ▶ $y_t = \ell_{t-1} + \ell_{t-1}\varepsilon_t$
 - ▶ $e_t = y_t - \hat{y}_{t|t-1} = \ell_{t-1}\varepsilon_t$

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ETS(M,N,N) model

Measurement equation

$$y_t = \ell_{t-1}(1 + \varepsilon_t)$$

State equation

$$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$$

ETS(M,N,N): SES with multiplicative errors.

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ETS(M,N,N) model

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$$y_t = \ell_{t-1}(1 + \varepsilon_t)$$

State equation

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- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.

ETS models

General notation E T S : ExponenTial Smoothing



Error Trend Season

Error: Additive ("A") or multiplicative ("M")

ETS models

General notation E T S : ExponenTial Smoothing



Error Trend Season

Error: Additive ("A") or multiplicative ("M")

Trend: None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

ETS models

General notation E T S : ExponenTial Smoothing



Error Trend Season

Error: Additive ("A") or multiplicative ("M")

Trend: None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

Seasonality: None ("N"), additive ("A") or multiplicative ("M")

ETS(A,A,N)

Holt's methods method with additive errors.

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t + hb_t$$

Observation equation

$$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

- Forecast errors: $\varepsilon_t = y_t - \hat{y}_{t|t-1}$

ETS(A,A,A)

Holt-Winters additive method with additive errors.

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$$

Observation equation

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

$$s_t = s_{t-m} + \gamma\varepsilon_t$$

- Forecast errors: $\varepsilon_t = y_t - \hat{y}_{t|t-1}$
- k is integer part of $(h - 1)/m$.

ETS(M,A,M)

Holt-Winters multiplicative method with multiplicative errors.

Forecast equation	$\hat{y}_{t+h t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$
Observation equation	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$
State equations	$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

- Forecast errors: $\varepsilon_t = (y_t - \hat{y}_{t|t-1})/\hat{y}_{t|t-1}$
- k is integer part of $(h - 1)/m$.

ETS model specification

```
ETS(y ~ error("A") + trend("N") + season("N"))
```

By default, optimal values for α , β , γ , and the states at time 0 are used.

The values for α , β and γ can be specified:

```
trend("A", alpha = 0.5, beta = 0.2)
trend("A", alpha_range = c(0.2, 0.8), beta_range = c(0.1, 0.4))
season("M", gamma = 0.04)
season("M", gamma_range = c(0, 0.3))
```

Exponential smoothing methods

		Seasonal Component		
		N	A	M
Trend Component		(None)	(Additive)	(Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A _d	(Additive damped)	(A _d ,N)	(A _d ,A)	(A _d ,M)

(N,N): Simple exponential smoothing

(A,N): Holt's linear method

(A_d,N): Additive damped trend method

(A,A): Additive Holt-Winters' method

(A,M): Multiplicative Holt-Winters' method

(A_d,M): Damped multiplicative Holt-Winters' method

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N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A _d	(Additive damped)	(A _d ,N)	(A _d ,A)	(A _d ,M)

(N,N): Simple exponential smoothing

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(A,A): Additive Holt-Winters' method

(A,M): Multiplicative Holt-Winters' method

(A_d,M): Damped multiplicative Holt-Winters' method

There are also multiplicative trend methods (not recommended).

ETS models

Additive Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A _d	(Additive damped)	A,A _d ,N	A,A _d ,A	A,A _d ,M

Multiplicative Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A _d	(Additive damped)	M,A _d ,N	M,A _d ,A	M,A _d ,M

ETS models

Additive Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A _d	(Additive damped)	A,A _d ,N	A,A _d ,A	A,A_d,M

Multiplicative Error

		Seasonal Component		
Trend Component		N	A	M
		(None)	(Additive)	(Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A _d	(Additive damped)	M,A _d ,N	M,A _d ,A	M,A _d ,M

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

Automatic forecasting

From Hyndman et al. (IJF, 2002):

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best method using AICc:
- Produce forecasts using best method.
- Obtain forecast intervals using underlying state space model.

Method performed very well in M3 competition.

Residuals

Response residuals

$$\hat{e}_t = y_t - \hat{y}_{t|t-1}$$

Innovation residuals

Additive error model:

$$\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$$

Multiplicative error model:

$$\hat{\varepsilon}_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}}$$