MONASH University

MONASH BUSINESS SCHOOL

# ETC3550/ETC5550 Applied forecasting

## Ch9. ARIMA models

OTexts.org/fpp3/

# Seasonal ARIMA models

| ARIMA | $\underbrace{(p,d,q)}$ | $\underbrace{(P,D,Q)_m}$ |
|---|---|---|
| | $\uparrow$ | $\uparrow$ |
| | Non-seasonal part of the model | Seasonal part of of the model |

where $m$ = number of observations per year.

# Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

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$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t \;=\; (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$

# Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

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(Non-seasonal AR(1))

(Seasonal AR(1))

(Non-seasonal difference)

(Seasonal difference)

(Non-seasonal MA(1))

(Seasonal MA(1))

# Seasonal ARIMA models

E.g., ARIMA$(1, 1, 1)(1, 1, 1)_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t \;=\; (1+\theta_1 B)(1+\Theta_1 B^4)\varepsilon_t.$$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t &= (1+\phi_1)y_{t-1} - \phi_1 y_{t-2} + (1+\Phi_1)y_{t-4} \\ &\quad - (1+\phi_1+\Phi_1+\phi_1\Phi_1)y_{t-5} + (\phi_1+\phi_1\Phi_1)y_{t-6} \\ &\quad - \Phi_1 y_{t-8} + (\Phi_1+\phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} \end{aligned}$$

# Common ARIMA models

The US Census Bureau uses the following models most often:

| | |
|---|---|
| ARIMA(0,1,1)(0,1,1)$_m$ | with log transformation |
| ARIMA(0,1,2)(0,1,1)$_m$ | with log transformation |
| ARIMA(2,1,0)(0,1,1)$_m$ | with log transformation |
| ARIMA(0,2,2)(0,1,1)$_m$ | with log transformation |
| ARIMA(2,1,2)(0,1,1)$_m$ | with no transformation |

# Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)$_{12}$ will show:**

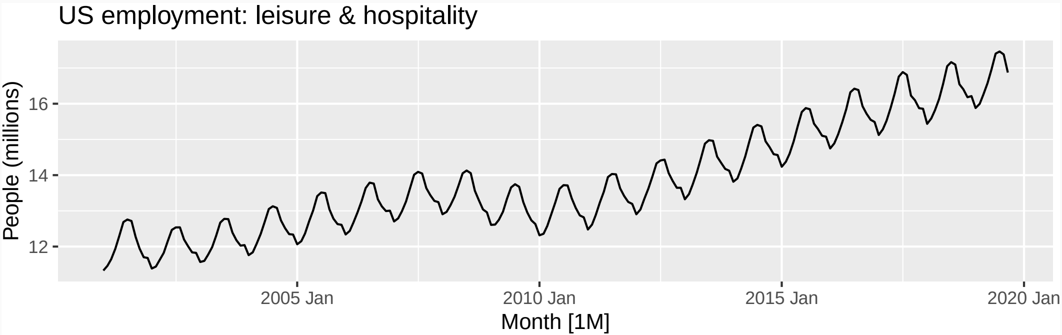
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, . . . .

**ARIMA(0,0,0)(1,0,0)$_{12}$ will show:**

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

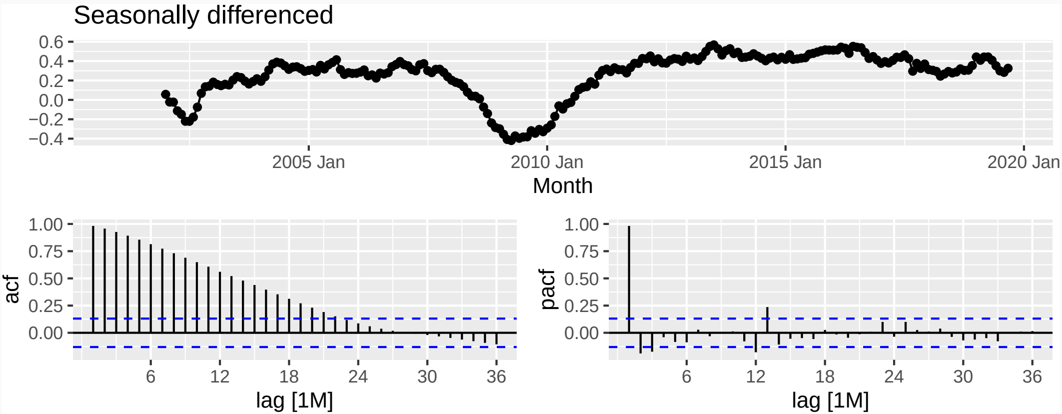
# US leisure employment

```
leisure <- us_employment |>
  filter(Title == "Leisure and Hospitality", year(Month) > 2000) |>
  mutate(Employed = Employed / 1000) |>
  select(Month, Employed)
autoplot(leisure, Employed) +
  labs(title = "US employment: leisure & hospitality", y = "People (millions)")
```



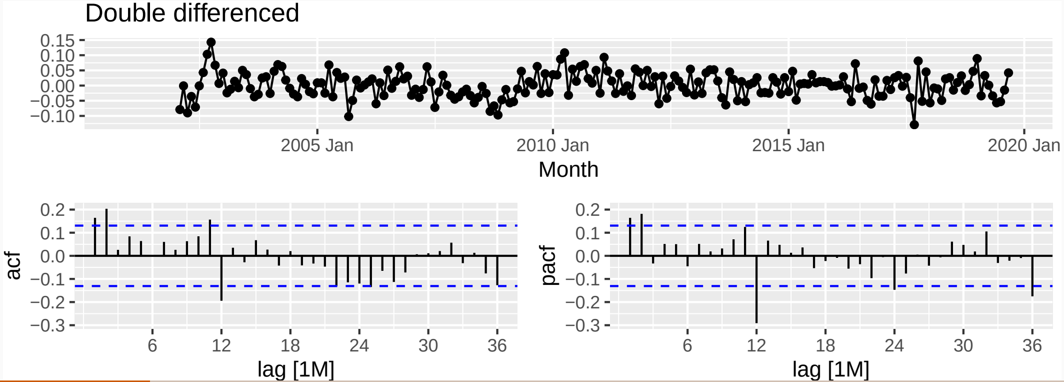
# US leisure employment

```
leisure |>
  gg_tsdisplay(difference(Employed, 12), plot_type = "partial", lag = 36) +
  labs(title = "Seasonally differenced", y = "")
```



# US leisure employment

```r
leisure |>
  gg_tsdisplay(difference(Employed, 12) |> difference(),
    plot_type = "partial", lag = 36
  ) +
  labs(title = "Double differenced", y = "")
```



# US leisure employment

```r
fit <- leisure |>
  model(
    arima012011 = ARIMA(Employed ~ pdq(0, 1, 2) + PDQ(0, 1, 1)),
    arima210011 = ARIMA(Employed ~ pdq(2, 1, 0) + PDQ(0, 1, 1)),
    auto = ARIMA(Employed, stepwise = FALSE, approx = FALSE)
  )
fit |> pivot_longer(everything(),
  names_to = "Model name",
  values_to = "Orders"
)
```

```
# A mable: 3 x 2
# Key:     Model name [3]
  `Model name`                          Orders
  <chr>                                <model>
1 arima012011  <ARIMA(0,1,2)(0,1,1)[12]>
2 arima210011  <ARIMA(2,1,0)(0,1,1)[12]>
3 auto         <ARIMA(2,1,0)(1,1,1)[12]>
```

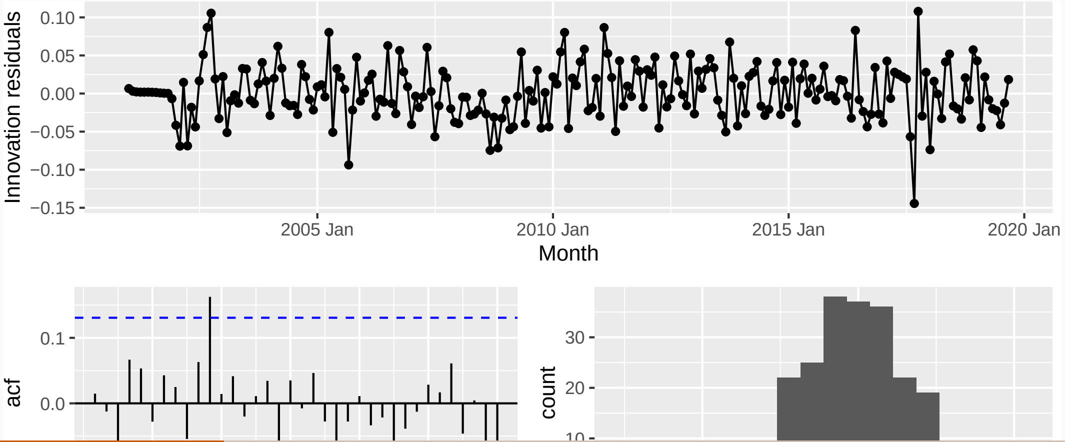
# US leisure employment

```r
glance(fit) |>
  arrange(AICc) |>
  select(.model:BIC)
```

```
# A tibble: 3 x 6
  .model       sigma2 log_lik   AIC  AICc   BIC
  <chr>         <dbl>   <dbl> <dbl> <dbl> <dbl>
1 auto        0.00142    395. -780. -780. -763.
2 arima210011 0.00145    392. -776. -776. -763.
3 arima012011 0.00146    391. -775. -775. -761.
```

# US leisure employment

```
fit |>
  select(auto) |>
  gg_tsresiduals(lag = 36)
```



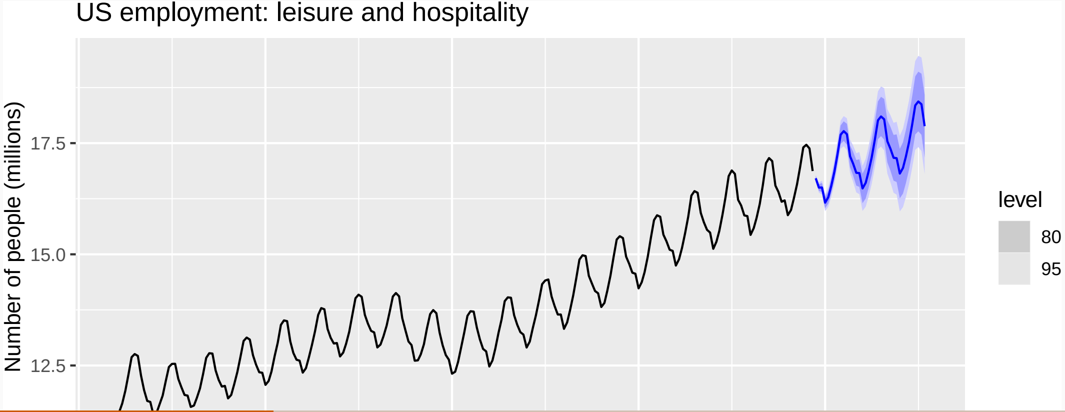
# US leisure employment

```r
augment(fit) |> features(.innov, ljung_box, lag = 24, dof = 4)
```

```
# A tibble: 3 x 3
  .model      lb_stat lb_pvalue
  <chr>         <dbl>     <dbl>
1 arima012011    22.4     0.320
2 arima210011    18.9     0.527
3 auto           16.6     0.680
```

# US leisure employment

```
forecast(fit, h = 36) |>
  filter(.model == "auto") |>
  autoplot(leisure) +
  labs(title = "US employment: leisure and hospitality", y = "Number of people (mill
```

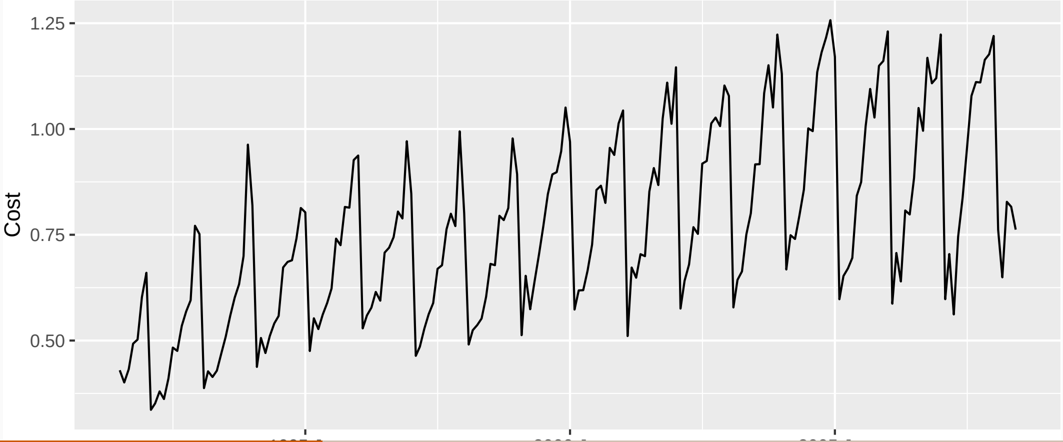


# Cortecosteroid drug sales

```r
h02 <- PBS |>
  filter(ATC2 == "H02") |>
  summarise(Cost = sum(Cost) / 1e6)
```

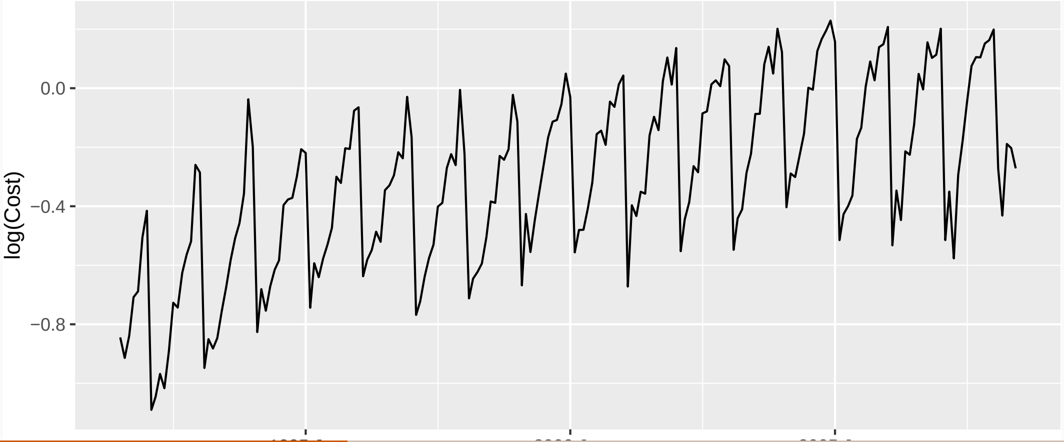
# Cortecosteroid drug sales

```r
h02 |> autoplot(
  Cost
)
```



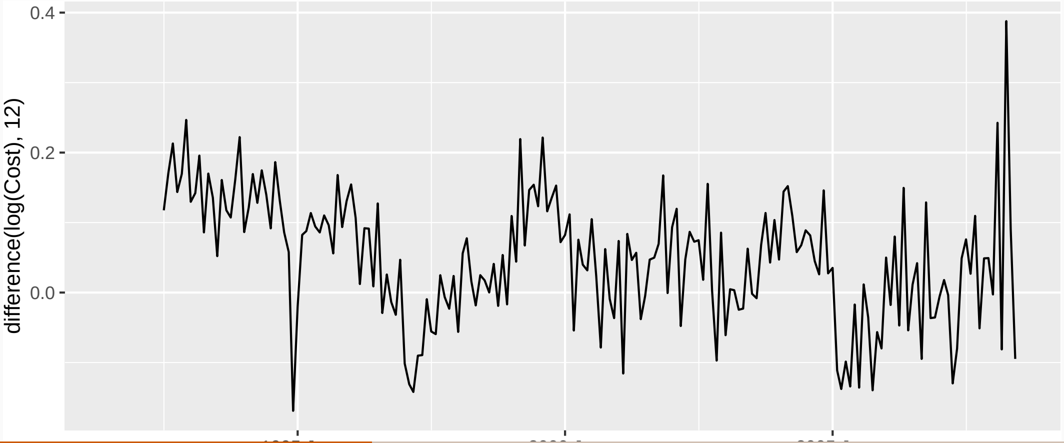
# Cortecosteroid drug sales

```r
h02 |> autoplot(
  log(Cost)
)
```



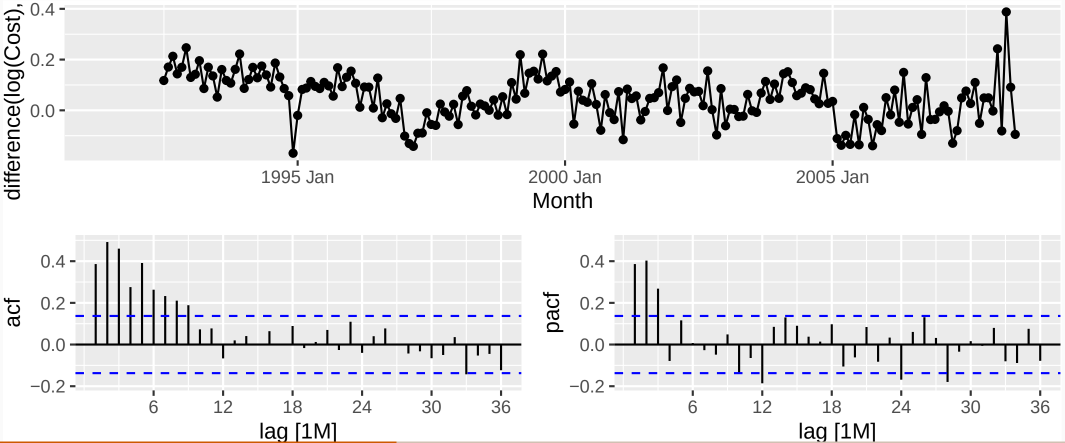
# Cortecosteroid drug sales

```r
h02 |> autoplot(
  log(Cost) |> difference(12)
)
```



# Cortecosteroid drug sales

```r
h02 |> gg_tsdisplay(difference(log(Cost), 12),
  lag_max = 36, plot_type = "partial"
)
```



# Cortecosteroid drug sales

- Choose $D = 1$ and $d = 0$.
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggests possible non-seasonal AR(3) term.
- Initial candidate model: ARIMA$(3,0,0)(2,1,0)_{12}$.

# Cortecosteroid drug sales

| .model | AICc |
|---|---|
| ARIMA(3,0,1)(0,1,2)[12] | -485 |
| ARIMA(3,0,1)(1,1,1)[12] | -484 |
| ARIMA(3,0,1)(0,1,1)[12] | -484 |
| ARIMA(3,0,1)(2,1,0)[12] | -476 |
| ARIMA(3,0,0)(2,1,0)[12] | -475 |
| ARIMA(3,0,2)(2,1,0)[12] | -475 |
| ARIMA(3,0,1)(1,1,0)[12] | -463 |

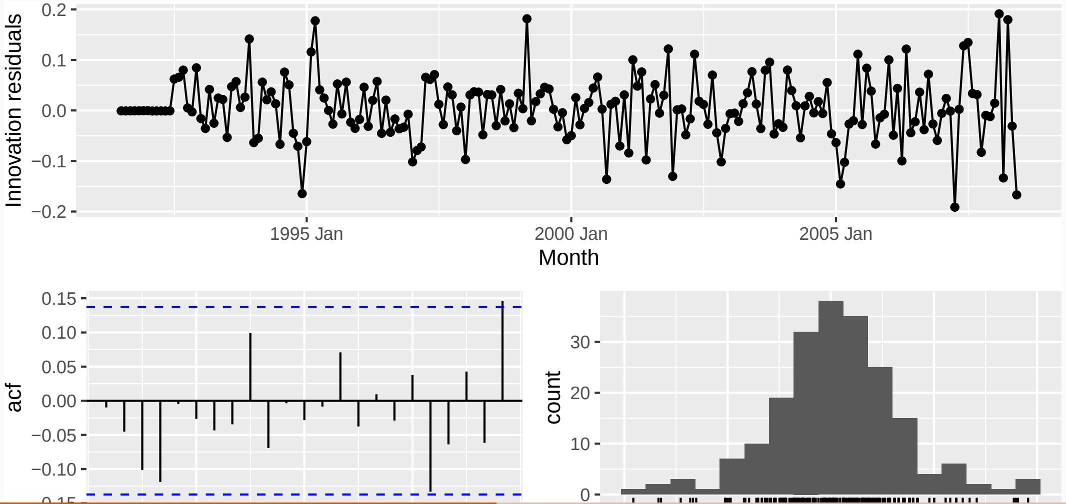
# Cortecosteroid drug sales

```
fit <- h02 |>
  model(best = ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(0, 1, 2)))
report(fit)
```

```
Series: Cost
Model: ARIMA(3,0,1)(0,1,2)[12]
Transformation: log(Cost)

Coefficients:
         ar1     ar2     ar3    ma1     sma1     sma2
      -0.160  0.5481  0.5678  0.383  -0.5222  -0.1768
s.e.   0.164  0.0878  0.0942  0.190   0.0861   0.0872

sigma^2 estimated as 0.004278:  log likelihood=250
AIC=-486   AICc=-485   BIC=-463
```

# Cortecosteroid drug sales

```
gg_tsresiduals(fit)
```



# Cortecosteroid drug sales

```r
augment(fit) |>
  features(.innov, ljung_box, lag = 36, dof = 6)
```

```
# A tibble: 1 x 3
  .model lb_stat lb_pvalue
  <chr>    <dbl>     <dbl>
1 best      50.7    0.0104
```

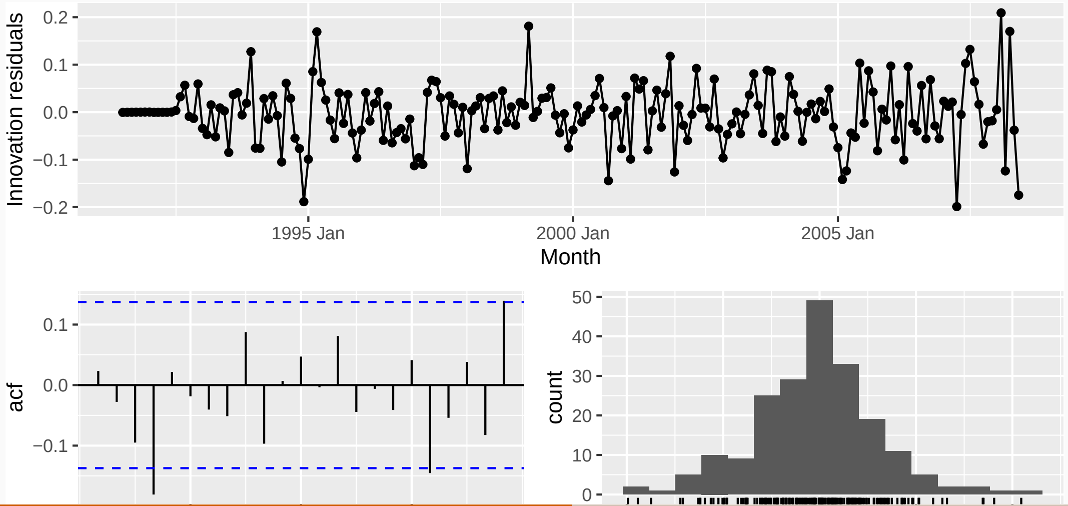
# Cortecosteroid drug sales

```r
fit <- h02 |> model(auto = ARIMA(log(Cost)))
report(fit)
```

```
Series: Cost
Model: ARIMA(2,1,0)(0,1,1)[12]
Transformation: log(Cost)

Coefficients:
          ar1      ar2     sma1
      -0.8491  -0.4207  -0.6401
s.e.   0.0712   0.0714   0.0694

sigma^2 estimated as 0.004387:  log likelihood=245
AIC=-483   AICc=-483   BIC=-470
```

# Cortecosteroid drug sales

```r
gg_tsresiduals(fit)
```



# Cortecosteroid drug sales

```r
augment(fit) |>
  features(.innov, ljung_box, lag = 36, dof = 3)
```

```
# A tibble: 1 x 3
  .model lb_stat lb_pvalue
  <chr>    <dbl>     <dbl>
1 auto      59.3   0.00332
```

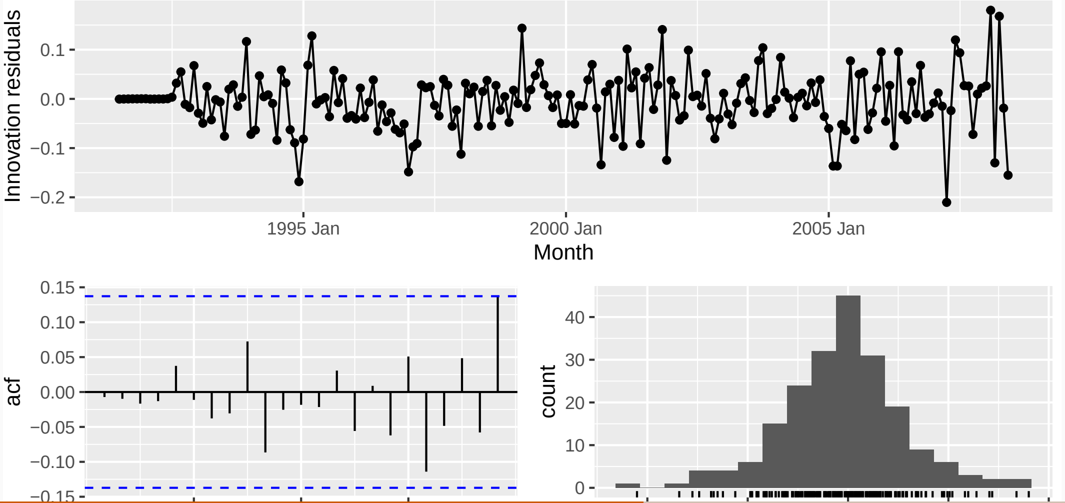
# Cortecosteroid drug sales

```r
fit <- h02 |>
  model(best = ARIMA(log(Cost),
    stepwise = FALSE,
    approximation = FALSE,
    order_constraint = p + q + P + Q <= 9
  ))
report(fit)
```

```
Series: Cost
Model: ARIMA(4,1,1)(2,1,2)[12]
Transformation: log(Cost)

Coefficients:
          ar1    ar2    ar3     ar4     ma1   sar1    sar2    sma1   sma2
      -0.0425  0.210  0.202  -0.227  -0.742  0.621  -0.383  -1.202  0.496
s.e.   0.2167  0.181  0.114   0.081   0.207  0.242   0.118   0.249  0.213

sigma^2 estimated as 0.004049:  log likelihood=254
```

# Cortecosteroid drug sales

```
gg_tsresiduals(fit)
```



# Cortecosteroid drug sales

```r
augment(fit) |>
  features(.innov, ljung_box, lag = 36, dof = 9)
```

```
# A tibble: 1 x 3
  .model lb_stat lb_pvalue
  <chr>    <dbl>     <dbl>
1 best      36.5     0.106
```

# Cortecosteroid drug sales

Training data: July 1991 to June 2006

Test data: July 2006–June 2008

```
fit <- h02 |>
  filter_index(~ "2006 Jun") |>
  model(
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 0) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 2) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(1, 1, 0))
    # ... #
  )

fit |>
  forecast(h = "2 years") |>
  accuracy(h02)
```

# Cortecosteroid drug sales

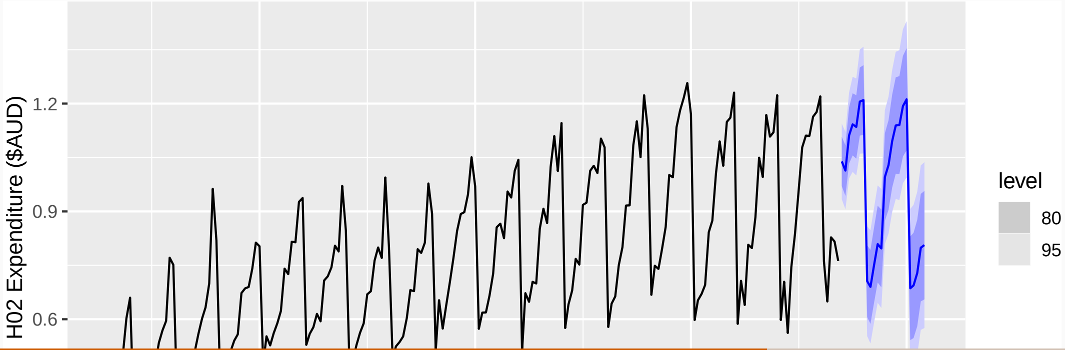
| .model | RMSE |
|---|---|
| ARIMA(3,0,1)(1,1,1)[12] | 0.0619 |
| ARIMA(3,0,1)(0,1,2)[12] | 0.0621 |
| ARIMA(3,0,1)(0,1,1)[12] | 0.0630 |
| ARIMA(2,1,0)(0,1,1)[12] | 0.0630 |
| ARIMA(4,1,1)(2,1,2)[12] | 0.0631 |
| ARIMA(3,0,2)(2,1,0)[12] | 0.0651 |
| ARIMA(3,0,1)(2,1,0)[12] | 0.0653 |
| ARIMA(3,0,1)(1,1,0)[12] | 0.0666 |
| ARIMA(3,0,0)(2,1,0)[12] | 0.0668 |

# Cortecosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- Use the best model available, even if it does not pass all tests.

# Cortecosteroid drug sales

```
fit <- h02 |>
  model(ARIMA(Cost ~ 0 + pdq(3, 0, 1) + PDQ(0, 1, 2)))
fit |>
  forecast() |>
  autoplot(h02) +
  labs(y = "H02 Expenditure ($AUD)")
```

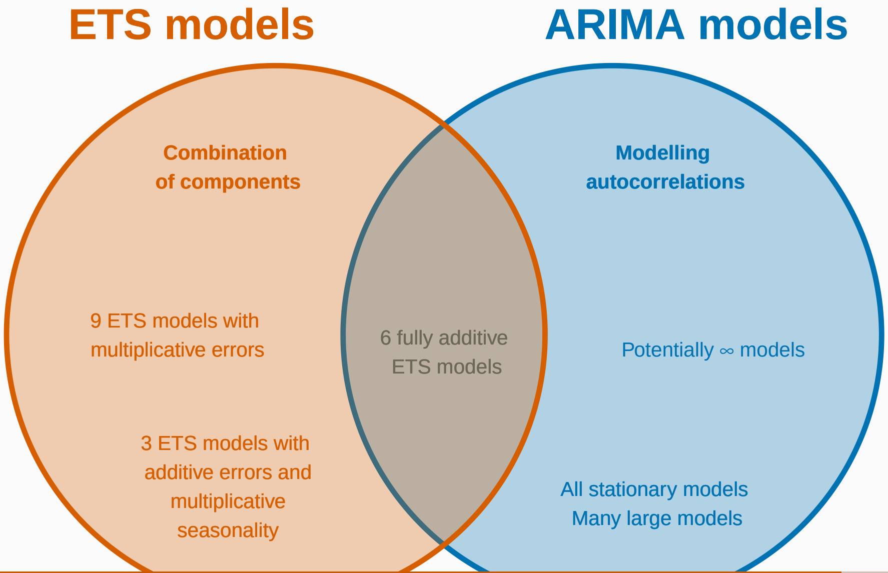


# Outline

# ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

# ARIMA vs ETS


ETS models
ARIMA models
Combination of components
Modelling autocorrelations
9 ETS models with multiplicative errors
6 fully additive ETS models
Potentially ∞ models
3 ETS models with additive errors and multiplicative seasonality
All stationary models
Many large models

# Equivalences

| ETS model | ARIMA model | Parameters |
|---|---|---|
| ETS(A,N,N) | ARIMA(0,1,1) | $\theta_1 = \alpha - 1$ |
| ETS(A,A,N) | ARIMA(0,2,2) | $\theta_1 = \alpha + \beta - 2$ |
| | | $\theta_2 = 1 - \alpha$ |
| ETS(A,$\text{A}_\text{d}$,N) | ARIMA(1,1,2) | $\phi_1 = \phi$ |
| | | $\theta_1 = \alpha + \phi\beta - 1 - \phi$ |
| | | $\theta_2 = (1 - \alpha)\phi$ |
| ETS(A,N,A) | ARIMA$(0,0,m)(0,1,0)_m$ | |
| ETS(A,A,A) | ARIMA$(0,1,m+1)(0,1,0)_m$ | |
| ETS(A,$\text{A}_\text{d}$,A) | ARIMA$(1,0,m+1)(0,1,0)_m$ | |

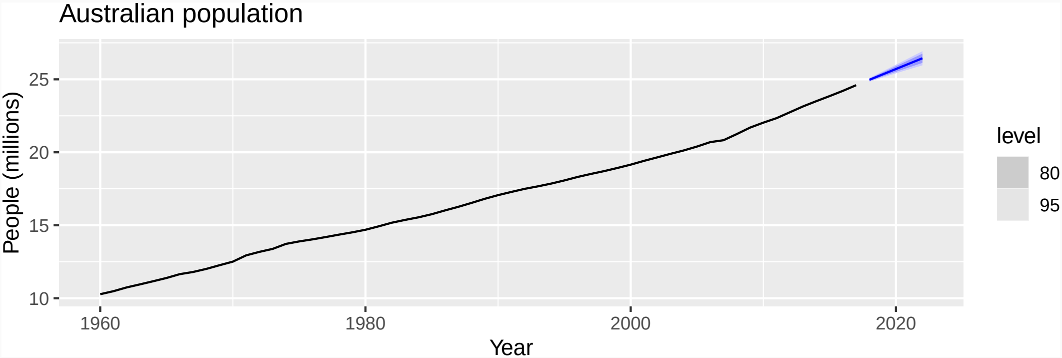
# Example: Australian population

```
aus_economy <- global_economy |>
  filter(Code == "AUS") |>
  mutate(Population = Population / 1e6)
aus_economy |>
  slice(-n()) |>
  stretch_tsibble(.init = 10) |>
  model(
    ets = ETS(Population),
    arima = ARIMA(Population)
  ) |>
  forecast(h = 1) |>
  accuracy(aus_economy) |>
  select(.model, ME:RMSSE)
```

```
# A tibble: 2 x 8
  .model      ME   RMSE     MAE    MPE   MAPE   MASE  RMSSE
  <chr>    <dbl>  <dbl>   <dbl>  <dbl>  <dbl>  <dbl>  <dbl>
1 arima   0.0420 0.194   0.0789 0.277  0.509  0.317  0.746
```

# Example: Australian population

```
aus_economy |>
  model(ETS(Population)) |>
  forecast(h = "5 years") |>
  autoplot(aus_economy) +
  labs(title = "Australian population", y = "People (millions)")
```



# Example: Cement production

```r
cement <- aus_production |>
  select(Cement) |>
  filter_index("1988 Q1" ~ .)
train <- cement |> filter_index(. ~ "2007 Q4")
fit <- train |>
  model(
    arima = ARIMA(Cement),
    ets = ETS(Cement)
  )
```

# Example: Cement production

```r
fit |>
  select(arima) |>
  report()
```

```
Series: Cement
Model: ARIMA(1,0,1)(2,1,1)[4] w/ drift

Coefficients:
         ar1     ma1   sar1    sar2    sma1  constant
      0.8886  -0.237  0.081  -0.234  -0.898      5.39
s.e.  0.0842   0.133  0.157   0.139   0.178      1.48

sigma^2 estimated as 11456:  log likelihood=-464
AIC=941   AICc=943   BIC=957
```

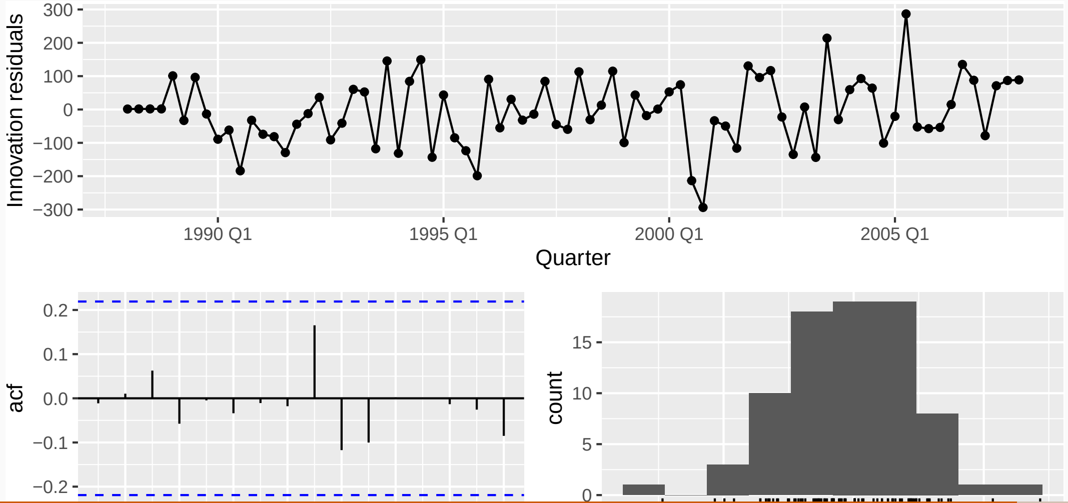
# Example: Cement production

```r
fit |>
  select(ets) |>
  report()
```

```
Series: Cement
Model: ETS(M,N,M)
  Smoothing parameters:
    alpha = 0.753
    gamma = 1e-04

  Initial states:
 l[0] s[0] s[-1] s[-2] s[-3]
 1695 1.03  1.05  1.01 0.912

  sigma^2:  0.0034

 AIC AICc  BIC
1104 1106 1121
```

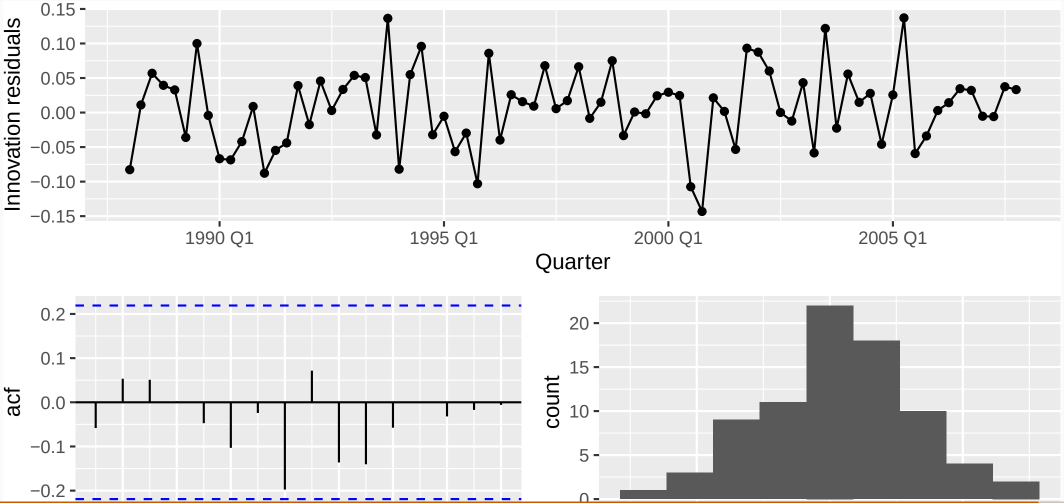
# Example: Cement production

```
gg_tsresiduals(fit |> select(arima), lag_max = 16)
```



# Example: Cement production

```r
gg_tsresiduals(fit |> select(ets), lag_max = 16)
```



# Example: Cement production

```
fit |>
  select(arima) |>
  augment() |>
  features(.innov, ljung_box, lag = 16, dof = 6)
```

```
# A tibble: 1 x 3
  .model lb_stat lb_pvalue
  <chr>    <dbl>     <dbl>
1 arima     6.37     0.783
```

# Example: Cement production

```r
fit |>
  select(ets) |>
  augment() |>
  features(.innov, ljung_box, lag = 16, dof = 6)
```

```
# A tibble: 1 x 3
  .model lb_stat lb_pvalue
  <chr>    <dbl>     <dbl>
1 ets       10.0     0.438
```

# Example: Cement production

```r
fit |>
  forecast(h = "2 years 6 months") |>
  accuracy(cement) |>
  select(-ME, -MPE, -ACF1)
```

```
# A tibble: 2 x 7
  .model .type  RMSE   MAE  MAPE  MASE RMSSE
  <chr>  <chr> <dbl> <dbl> <dbl> <dbl> <dbl>
1 arima  Test   216.  186.  8.68  1.27  1.26
2 ets    Test   222.  191.  8.85  1.30  1.29
```

# Example: Cement production

```
fit |>
  select(arima) |>
  forecast(h = "3 years") |>
  autoplot(cement) +
  labs(title = "Cement production in Australia", y = "Tonnes ('000)")
```