Precession Pattern of Spinning Top Using SIMULINK Simulation

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***Abstract*—Rotational dynamics has been a subject of interest in classical mechanics for a long time. One of the topics that regularly intrigued physicist is the movement of the spinning top. In rotational dynamics, the movement of the spinning top is divided into spin, nutation, and precession. This study simulates the movements of two spinning tops using SIMULINK to observe the precession pattern. The results show that the precession pattern changes with the initial precession angular velocity**

***Index Terms*—Spinning top, precession, SIMULINK**

I. Introduction

The study of rotational dynamics and gyroscopic motions has long been a subject of classical mechanics, which has provided insight into the fundamental principles governing the behaviour of rotating bodies. One items that have intrigued many physicists and has been discussed for a long time is the spinning top. The spinning top is one of the oldest recognizable toys found in archaeological sites, and has appeared independently many times in different cultures from around the world [1]. It often comes in very simple shape and form, yet the physics behind the spinning top is far from simple.

The movement of the spinning top depends on gyroscopic effect for its motion. The movements of the spinning top can be analyzed by dividing it into three, namely spin, nutation, and precession. Each movement can be described with its own equation. A simulation was carried out to determine the precession pattern exhibited by the spinning top using SIMULINK as a tool for the basis of the simulation.

II. Theoretical Framework

A. Rotation About Fixed Point

Consider a top rotating about a fixed point O on a flat plane with presence of gravity. To represent attitude of the top in the inertial frame, we introduce three Euler angle variables as stated in Fig. 1. Three Euler angles to represent attitude are ψ (spin), ϕ (precession), and θ (nutation) [2].

The first derivative of the Euler angles, $\dot{\psi}$, $\dot{\phi}$, and $\dot{\theta}$ is shown in Fig. 1. The relationships between angular velocities in body axes and the first derivative of the Euler angles are described in (1), (2), and (3).

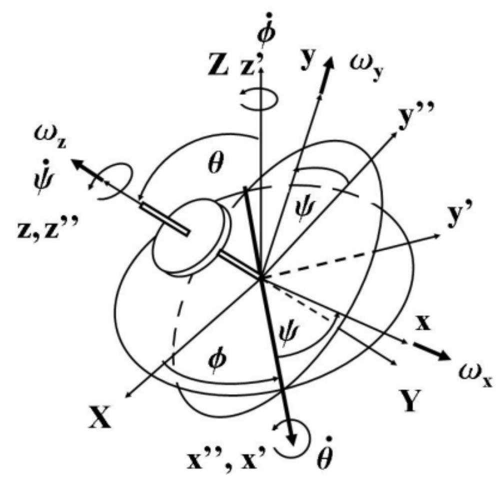


Fig. 1: Top attitude with respect to inertial frame [2].

$$\omega_x = \dot{\phi} \sin\theta \sin\phi + \dot{\theta} \cos\psi, \tag{1}$$
$$\omega_y = \dot{\phi} \sin\theta \cos\phi - \dot{\theta} \sin\psi, \tag{2}$$
$$\omega_z = \dot{\phi} \cos\theta + \dot{\psi}. \tag{3}$$

B. Euler Equation of Symmetric Top

Consider a symmetric top with principal moment of inertia $I_{xx} = I_{yy} = I_0$ and $I_{zz} = I$. The angular momentum of the top are described in (4), (5), and (6).

$$H_x = I_0 \omega_x, \tag{4}$$
$$H_y = I_0 \omega_y, \tag{5}$$
$$H_z = I \omega_z. \tag{6}$$

Euler equation of rotating body in body frame is described in (7). Because the rotation is in rotating frame, the inertia of top is constant over time ($\frac{d}{dt} I = 0$).

$$\dot{\boldsymbol{H}} = \frac{d}{dt}\boldsymbol{H} + \Omega \times \boldsymbol{H}. \tag{7}$$

Term $\dot{\boldsymbol{H}}$ is equal to external moment applied to the top. For symmetric top rotating about fixed point, we assume that

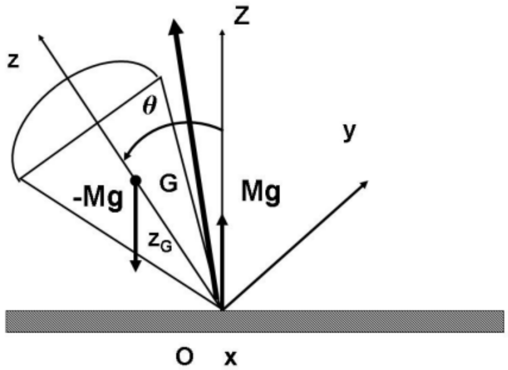


Fig. 2: Free body diagram of a top [2].

moment applied into the top is caused by gravity. Free body diagram of the top can be seen in Fig 2. From the free body diagram, we can conclude that the moment applied to the top is equal to

$$M = \begin{bmatrix} Mgz_g \sin\theta & 0 & 0 \end{bmatrix}^T \tag{8}$$

Assuming that the moment of inertia is equal to principal axes, solution of the Euler equation in (7) is only consist of moment of inertia and the Euler angles. The solution of Euler equation is described in (9), (10), and (11) [2].

$$M_x = I_0(\ddot{\theta} - \dot{\phi}^2 \sin\theta\cos\theta) + I\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi}), \tag{9}$$

$$M_y = I_0(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) - I\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}), \tag{10}$$

$$M_z = I(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta). \tag{11}$$

C. Precession Pattern

Assumed that a top is rotating with initial nutation angle of θ_0. The top is given initial spin velocity of $\dot{\psi}_0$ and initial precession velocity of $\dot{\phi}_0$. Result of the Euler equation is the top will continue to do precession and the value of nutation angle will oscillate between two value, θ_1 and θ_2 as seen in Fig 3.

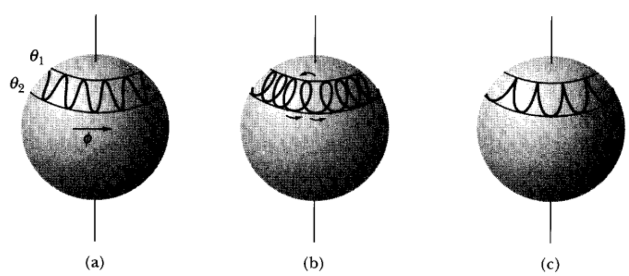


Fig. 3: Precession pattern of a spinning top. (a). Unidirectional, (b). Looping, and (c). Cuspidial [3].

The pattern in Fig 3 is caused by different value of initial condition. For a constant body frame angular velocity ω_z, there is two possible point for precession. If the top has initial precession velocity that around $\dot{\phi}_{0(-)}$ then the precession is called slow precession. If the top has initial precession velocity around $\dot{\phi}_{0(+)}$ then the precession is called fast precession. The approximate value of $\dot{\phi}_{0(-)}$ and $\dot{\phi}_{0(+)}$ can be seen in (12) and (13) [3].

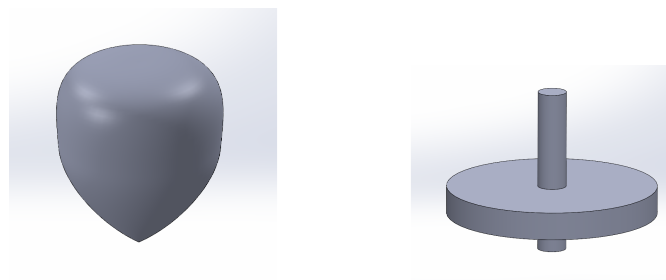
$$\dot{\phi}_{0(+)} \approx \frac{I\omega_z}{I_0 \cos\theta_0}, \tag{12}$$

$$\dot{\phi}_{0(-)} \approx \frac{Mgz_g \cos\theta_0}{I\omega_z}. \tag{13}$$

III. Simulation Setup

A. Spinning Top Properties

For this simulation, we use two spinning tops with the same mass but different shapes. The difference in shape resulting in difference in moment of inertia and also center of gravity. First top is shaped like nut and have high center of gravity, and the second top is shaped like nail and have low center of gravity as seen in Fig. 9. The top properties is tabulated in Table I.



(a) (b)

Fig. 4: Spinning top used in simulation (a). Nut-like top and (b). Nail-like top.

TABLE I: Spinning top properties

Top Properties	Value	
	Nut-like	***Nail-like***
Mass	9.750×10^{-2} kg	9.750×10^{-2} kg
z_g	4.814×10^{-2} m	2.656×10^{-2} m
I_0	2.789×10^{-4} kgm^2	1.386×10^{-4} kgm^2
I	0.456×10^{-4} kgm^2	1.228×10^{-4} kgm^2

B. Initial Condition

In order to show the precession pattern, we must define the initial condition for the top. For this simulation we define the initial nutation angle (θ_0) and the angular velocity at body z-axis (ω_{z0}) is constant. The precession angular velocity ($\dot{\phi}_0$) will be the variable that we manipulate to see the precession pattern of the top. Spin angular velocity ($\dot{\psi}_0$) will change following equation (3). The value of initial condition is tabulated in Table II [4].

C. SIMULINK Setup

To program the Euler equation in SIMULINK, we need to rearrange the equation (9), (10), and (11) so that the left side of the equation is second derivative of Euler angles as in (14),

TABLE II: Spinning top initial condition

Variable	Value
θ_0	0.1745 rad
ω_{z0}	418.9 rad/s

(15), and (16). The purpose of rearranging the equation is to input the second derivative of Euler angles into integrator block so that the output is the first derivative of Euler angles. After that we can add another integrator block so that the output will be Euler angles.

$$\ddot{\theta} = \frac{M_x - I\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})}{I_0} + \dot{\phi}^2 \sin\theta\cos\theta, \quad (14)$$

$$\ddot{\phi} = \frac{I\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}) - 2I_0\dot{\phi}\dot{\theta}\cos\theta}{I_0 \sin\theta}, \quad (15)$$

$$\ddot{\psi} = \dot{\phi}\dot{\theta}\sin\theta - \ddot{\phi}\cos\theta. \quad (16)$$

SIMULINK model is run with ode8 (Dormand-Prince) solver with fixed time step. Because the top is rotating very fast, time step size is defined at 10^{-6} s. This allows the output of simulation to avoid aliasing because the frequency of signal from simulation is about 67 Hz, far below the time step size.

To observe the precession pattern of a spinning top, the properties at Table I and initial condition at Table II is submitted in the SIMULINK model. Precession pattern can be observed by plotting precession angle (ϕ) in the x axis and nutation angle (θ) in the y axis.

IV. Data Result and Analysis

A. Precession Pattern of Nut-Like Spinning Top

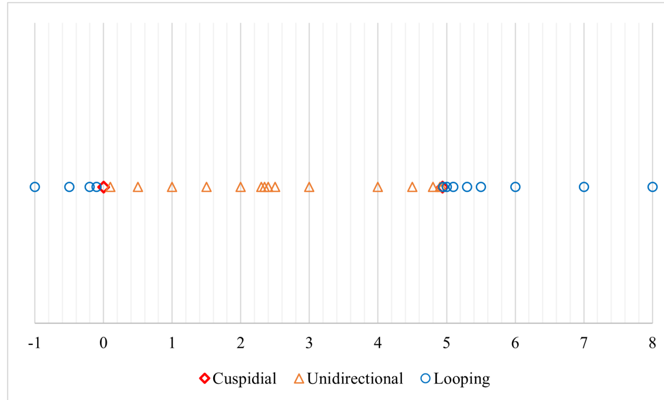
Based on equation (12) and (13), the nut-like spinning top will have $\dot{\phi}_{0(+)}$ and $\dot{\phi}_{0(-)}$ value as described in Table III.

TABLE III: Nut-like spinning top precession point

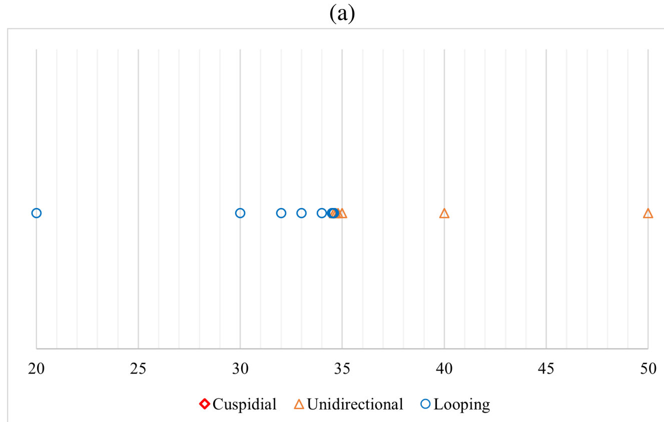
Variable	Value
$\dot{\phi}_{0(-)}$	69.74 rad/s
$\dot{\phi}_{0(+)}$	2.373 rad/s

The simulation result were observed and then grouped based on the precession pattern. Each group were then plotted into the same graph which can be seen in Fig 5a and Fig 5b accordingly.

We can observe the precession pattern around the slow precession point. If the initial precession angular velocity is equals to zero, the precession pattern will be cuspidial as shown in Fig 6. The unidirectional precession pattern will be shown if the initial precession angular velocity is around the value of $\dot{\phi}_{0(+)}$ or $\dot{\phi}_{0(-)}$, as shown in Fig 7. Looping precession pattern is shown if the initial precession angular velocity is not around the value of $\dot{\phi}_{0(+)}$ or $\dot{\phi}_{0(-)}$ as shown in Fig 8. The maximum and minimum value of nutation angle for every sample here is tabulated in Table IV.



(a)



(b)

Fig. 5: Precession pattern change of the Nut-like spinning top with: (a). Slow precession and (b). Fast precession.

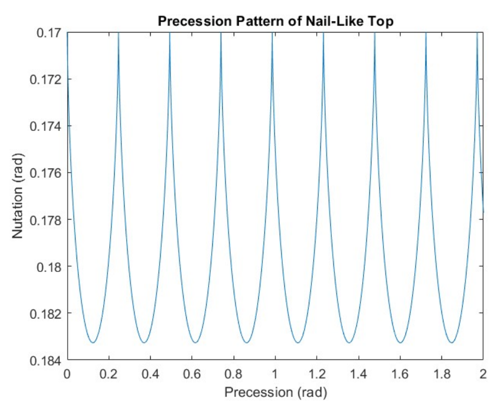


Fig. 6: Precession pattern when $\dot{\phi}_0 = 0$ rad/s

TABLE IV: Maximum and minimum value of nutation angle, Nut-like spinning top

$\dot{\phi}_0$ (rad/s)	θ_2 (rad)	θ_1 (rad)
0	0.1833	0.1700
1.5	0.1753	0.1700
10	0.1700	0.1302

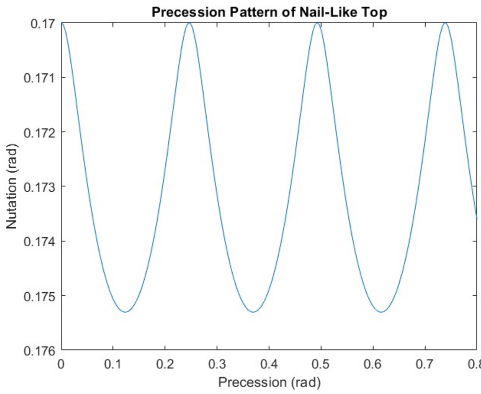


Fig. 7: Precession pattern when $\dot{\phi}_0 = 0.5$ rad/s

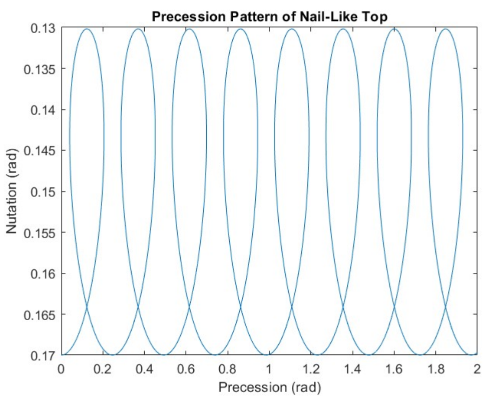


Fig. 8: Precession pattern when $\dot{\phi}_0 = 10$ rad/s

B. *Precession Pattern of Nail-Like Spinning Top*

Similarly, equation (12) and (13) gives the nail-like spinning top the $\dot{\phi}_{0(+)}$ and $\dot{\phi}_{0(-)}$ value as described in Table V.

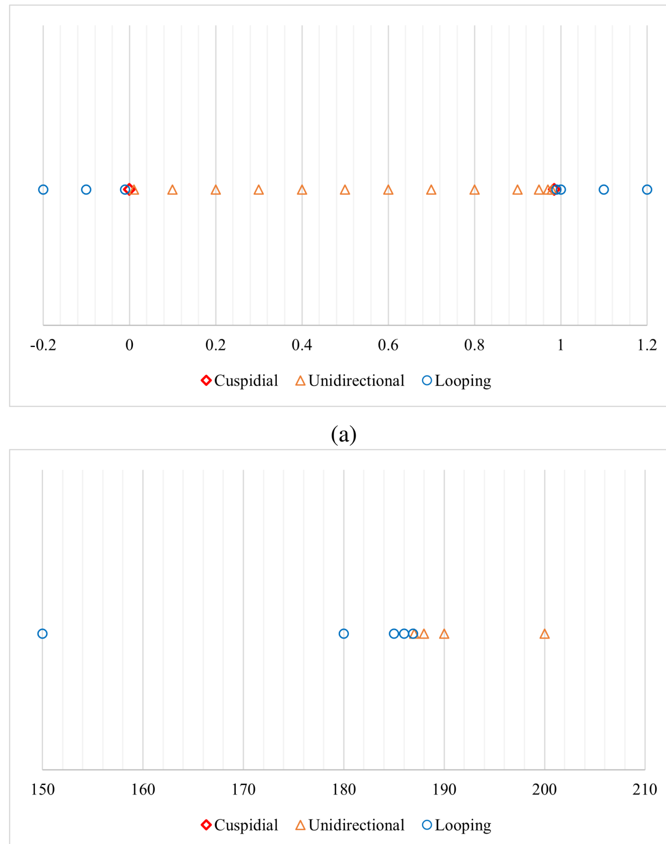
TABLE V: Nail-like spinning top precession point

Variable	Value
$\dot{\phi}_{0(-)}$	376.8 rad/s
$\dot{\phi}_{0(+)}$	0.4863 rad/s

Similar to the previous section, the precession pattern change can be seen in Fig 9a and Fig 9b.

We can observe the precession pattern around the slow precession point. If the initial precession angular velocity is equals to zero, the precession pattern shown will be cuspidial as shown in Fig 10. If the initial precession angular velocity is around the value of $\dot{\phi}_{0(+)}$ or $\dot{\phi}_{0(-)}$, the precession pattern will be unidirectional as shown in Fig 11. We can see the loop precession pattern if the initial precession angular velocity is not aroud the value of $\dot{\phi}_{0(+)}$ or $\dot{\phi}_{0(-)}$ as shown in Fig 12.

The maximum and minimum value of nutation angle for every sample here is tabulated in Table VI. We can see that if the initial precession angular velocity is lower than $\dot{\phi}_{0(-)}$ then the initial nutation angle will be minimum nutation angle.



(a)

(b)

Fig. 9: Precession pattern change of the Nail-like spinning top with: (a). Slow precession and (b). Fast precession.

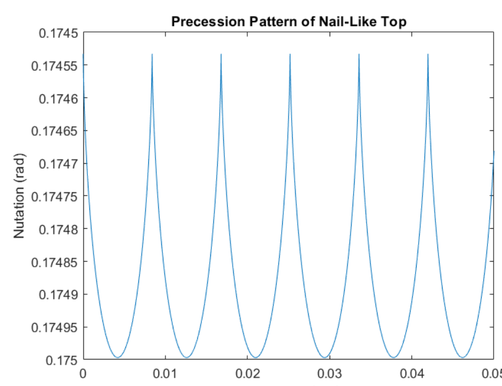


Fig. 10: Precession pattern when $\dot{\phi}_0 = 0$ rad/s

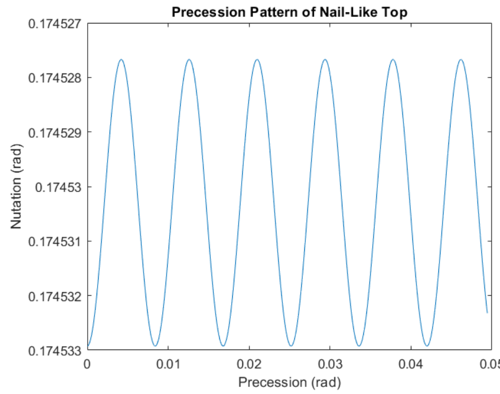


Fig. 11: Precession pattern when $\dot{\phi}_0 = 0.5$ rad/s

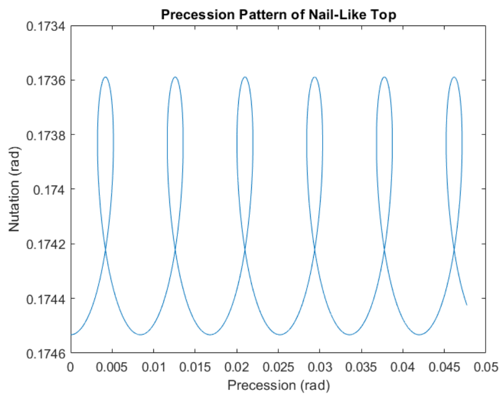


Fig. 12: Precession pattern when $\dot{\phi}_0 = 1.5$ rad/s

Otherwise, the initial nutation angle will be the maximum nutation angle.

TABLE VI: Maximum and minimum value of nutation angle, nail-line spinning top

$\dot{\phi}_0$ (rad/s)	θ_2 (rad)	θ_1 (rad)
0	0.174996902	0.174532925
0.5	0.174532925	0.174527672
1.5	0.174532925	0.173589211

C. Result Analysis

Based from section IV-A and IV-B we can conclude the precession pattern of spinning top is affected by the initial condition. We can tabulate the initial condition value and precession pattern in Table VII.

We can clearly observe that for slow precession, the precession pattern is changing with the initial precession angular velocity. If the initial precession angular velocity is significantly faster or slower than the value of slow precession point ($\dot{\phi}_{0(-)}$) the precession pattern will shown loop precession. If the initial precession angular velocity is around the value of $\dot{\phi}_{0(-)}$ then the precession pattern will be unidirectional precession. Cuspidial pattern can be observed in the transition point of unidirectional precession and loop precession.

TABLE VII: Precession pattern relation with initial condition

Precession Pattern	**Initial condition (rad/s)**	
	Nut-like	***Nail-like***
Loop	<0	<0
Cuspidial	=0	=0
Unidirectional	$0 < \dot{\phi} < 4.94$	$0 < \dot{\phi} < 0.985$
Cuspidial	=4.94	=0.985
Loop	$4.94 < \dot{\phi} \leq 34.59$	$0.985 < \dot{\phi} < 187$
Unidirectional	>34.59	≥ 187

For fast precession, if the initial precession angular velocity is significantly slower than the value of fast precession point ($\dot{\phi}_{0(+)}$) then the precession pattern will shown loop precession. Otherwise the precession pattern is unidirectional. Cuspidial pattern is not observed within fast precession, this probably due to the transition between loop precession and unidirectional precession is not occur in the θ_2 point. Transition in the fast precession is occur in the $\theta_1 = 0$ point, the consequences for this is there will be singularity in the $\ddot{\phi}$ as in (15).

This phenomenon is called *gimbal lock*, the top is losing one degree of freedom and the solution to the equation is going towards singularity. Gimbal lock is occur because we use Euler angles to calculate rotation dynamics of the top. To prevent this from happening, instead of calculate the dynamics with Euler angles, we can calculate the dynamics with angular velocities in body frame, then convert it to Euler angles.

V. Conclusion

Precession pattern of a spinning top depends on the top properties and initial condition. For this simulation, two spinning tops with different properties are used to compare how will it affect the precession pattern. Both tops have initial nutation angle of 10°and rotation about z-axis in body frame 4000 rpm.

Nail-like top have smaller inertia at x-axis compared to nut-like top, but higher inertia at z-axis compared to nut-like top. This properties make nail-like top experience slow precession at the lower precession angular velocity compared to nut-like top. This properties also make nail-like top experiences fast precession at the faster precession angular velocity compared to nut-like top. This means nail-like top is more likely experience loop precession compared to nut-like top.

In the slow precession, cuspidial precession is occur at the boundary between loop and unidirectional precession. In the fast precession there is no cuspidial precession occur at the boundary between loop and unidirectional.

Acknowledgment

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Appendix

The SIMULINK simulation file (.slx) can be downloaded using this link.

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