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10 Levels of Celestial Mechanics

Astronomy for Beginners

This document is made for personal purposes.

I write this to practice my English writing skills.

I also write this to teach others to prepare OSN in Astronomy.

I apologise if there are any writing errors in this document.

If you have any questions regarding this, please contact me at hilmizaman2002@gmail.com.

Contents

1 Level 1: Newton's Theory of Gravity

Celestial mechanics is a field in astronomy that studies how objects move in the sky as a result of gravity. In this document, all of the calculations and equations stated are based on Newton's theory of gravity. If you want to read more about Einstein's general relativity, you can read other sources instead.

1.1 Newton's Law of Motion

To understand how objects move, including a celestial body, we need to understand all three Newton's laws of motion. This will critically help us understand almost every equation in this document.

1.1.1 Newton's first law of motion

"Every body continues in its state of rest, or of uniform motion in a right [straight] line, unless it is compelled to change that state by forces impressed upon it."

The first law of motion basically states that if the total force acting on the body is equal to zero, the body will continue to maintain its inertia. That's why Newton's first law of motion is sometimes called the "inertia law." Newton's first law of motion can be written in equation 1.

$$\sum_{i=0}^{n} F_i = 0 \tag{1}$$

The implication of zero total force acting on the body is that there will be no change in momentum if the system is closed. One of the techniques to solve an equation about momentum is choosing the system so that the outside force is equal to zero; hence, the momentum will be conserved. One of the examples is choosing 2 bodies that orbit each other. If we include both bodies in the system, the total angular momentum will be conserved because there will be no other forces acting on the system from outside the system.

1.1.2 Newton's second law of motion

"The change of motion is proportional to the motive force impressed and is made in the direction of the right line in which that force is impressed."

The second law of motion states that if a force is acting on a body, the change of velocity (acceleration) is proportional to the force. The second law also states that the acceleration is inversely proportional to the inertia of the object. Newton's second law of motion states that the change of momentum over time is proportional to the force. Newton's second law of motion can be written in equation 2.

$$\sum_{i=0}^{n} F_i = \frac{\mathrm{d}p}{\mathrm{d}t} \tag{2}$$

and if the mass is constant, the equation will be,

$$\sum_{i=0}^{n} F_i = m\frac{\mathrm{d}v}{\mathrm{d}t} \tag{3}$$

Note that the acceleration vector will always be in the same direction as the force acting on it. For example, in celestial mechanics, the shape of an orbit is determined by the gravitational force vector and the object's velocity vector, more on that in the next levels.

1.1.3 Newton's third law of motion

"To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts."

The third law of motion states that for every action force, there will be a reaction force which acts on the other object. This reaction force has the same magnitude as the action, but acts in the opposite direction. Newton's third law of motion can be written in equation 4.

$$\sum F_{act} = -\sum F_{react} \tag{4}$$

One of the examples is the gravitational force between two bodies that always acts on both bodies with the same magnitude but in opposite directions.

1.2 Newton's Theory of Gravity

Newton's theory of gravity started when he observed a falling apple and wondered what force caused it. Newton then wondered if such a force could make an apple fall; could it make the moon fall? To solve this question, he developed calculus and, after that, the theory of gravity.

Newton's theory of gravity states that if there are two objects with mass, both of them will attract each other proportional to their masses and inversely proportional to their distance squared. The simple terms for Newton's theory of gravity can be written in equation 5.

$$F_g = G\frac{m_1 m_2}{r_{12}^2} \tag{5}$$

With G being the gravitational constant, with modern measurement, the value of G is approximately

$$G = 6.6743 \times 10^{-11} m^3 kg^{-1} s^{-2} \tag{6}$$

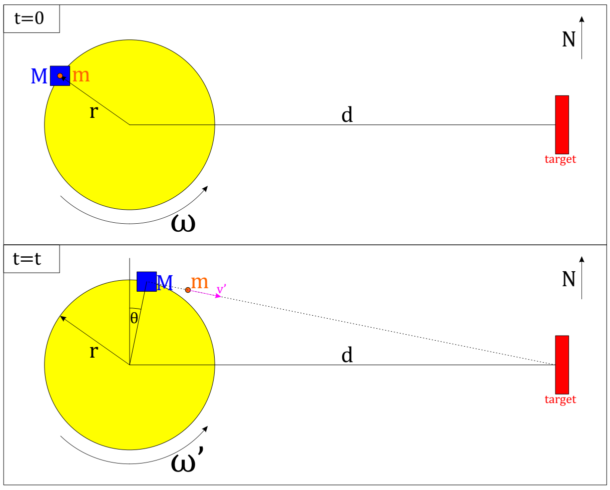
Note that gravity is a form of force, so Newton's second law of motion is applicable. If we assume that both of the masses are constant, we can calculate the acceleration of object 1 caused by the gravitational force from object 2 (g).

$$\begin{aligned} F_g &= m_1 g \\ G\frac{m_1 m_2}{r_{12}^2} &= m_1 g \\ g &= G\frac{m_2}{r_{12}^2} \end{aligned} \tag{7}$$

The equation states that the gravitational acceleration is a function of the mass of the pulling object and the distance between the two objects. For an object on the surface of the Earth, the gravitational acceleration due to Earth's gravity is approximately 9.8 m/s^2. This value can be changed depending on the location of the measurement, but as a rule of thumb, when solving a problem, the approximate value is more than enough to solve the problem.

1.3 Exercise Problem

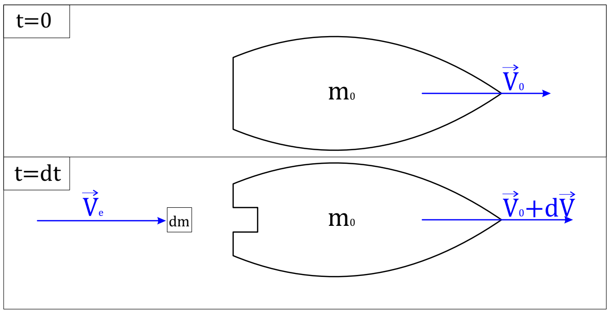
1. A man with mass $M = 80$kg carrying a ball with a mass $m = 0.5$kg while rotating on the edge of a disc with a radius of $r = 2$m with angular velocity of $\omega = 15$rpm. A target is placed on the east side of the disc, with the distance from the disc's centre $d = 10$m. The man wants to throw the ball at the



target perpendicular to the disc radius. The man is also throwing the ball in the opposite direction with the tangential velocity. The man threw the ball with a velocity of $v_b = 15$m/s relative to the man.

   What is the angle between the north line and the man's throwing position (θ) so that the man hits the target? What is the final radial velocity of the disc (ω')?

2. The Tsiolkovsky rocket equation is an equation that describes the motion of an object that propels itself by ejecting some of its mass with high velocity due to the conservation of momentum.



The figure above shows the principle of a rocket. Prove that the difference in total momentum is equal to

$$\mathrm{d}P = m_0 \mathrm{d}m + (\vec{V}_e - \vec{V}_0)\mathrm{d}m$$

We know that if there is no outside force, the total momentum will be conserved over time. If the exhausted mass velocity relative to the rocket $(\vec{v}_e)$ is constant, where

$$\vec{v}_e = \vec{V}_e - \vec{V}_0$$

Prove that the final velocity is equal to

$$V - V_0 = -v_e \ln \frac{m}{m_0}$$

Where V is the final velocity and m is the final mass of the rocket.

3. This is the classical problem in physics. Assume the Earth is perfectly spherical with homogeneous density. If we drill a straight hole through the Earth's core, prove that if we only move in one dimension along r, the equation of motion is,

$$\ddot{r} + \frac{g}{R} r = 0$$

With g being the Earth's surface gravitational acceleration and R being the Earth's radius. The solution for that equation is the simple harmonic oscillation, where,

$$\omega = \sqrt{\frac{g}{R}}$$

Calculate the period of the oscillation!

2 Level 2: Kepler's Law

Johann Kepler analysed the observation data about planetary movements. The data he used to analyse are based on Tycho Brahe's observations. This analysis took place before Newton was born, so there was no theory of gravity to support the observation data and the analysis. After Newton developed the theory of gravity, each of Kepler's laws was proved right, and the theory of gravity further developed each law.

2.1 Kepler's First Law

"The orbit of each planet is an ellipse with the Sun at one focus."

Nicolaus Copernicus stated that planets orbit around the Sun in circular orbits. He also calculated how much each observed planet revolved around the Sun. However, when Kepler analysed the observation data, he found that the orbit of a planet is not perfectly circular; instead, it forms an ellipse with the Sun at one focus.

Newton's theory of gravity further supported Kepler's discovery. Deriving the equation of motion for an object in an orbit, Newton found that the shape of an orbit is a part of a conic section. There are four possible shapes of an orbit that are determined by their orbital elements: circle, ellipse, parabola, and hyperbola.

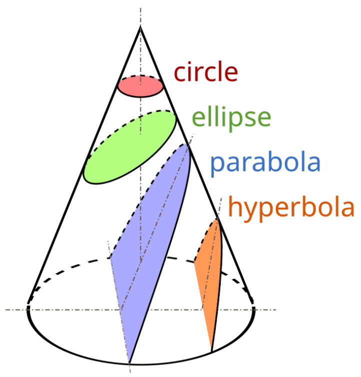


Figure 1: *Conic Section*

2.2 Kepler's Second Law

"The line joining the planet to the Sun sweeps out equal areas in equal times."

During his analysis, Kepler discovered that a planet moves faster when it's closer to the Sun. Kepler found that the time a planet takes to move is proportional to the area that it sweeps out. Figure 2 shows that the planet's movement is faster if it's closer to the Sun.

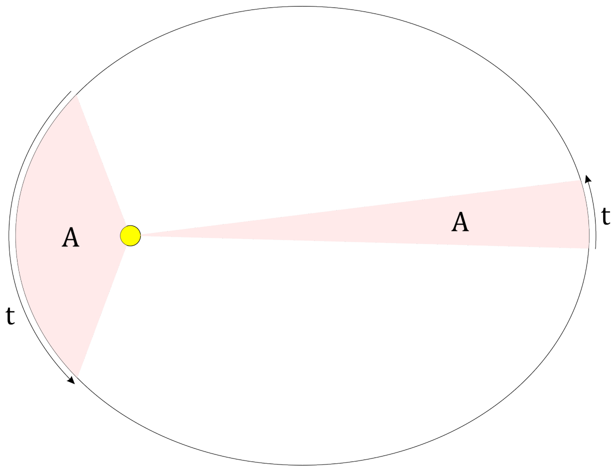


Figure 2: *Kepler's second law*

Newton's law of motion further proved that Kepler's second law can be derived analytically. Kepler's second law states that the change of area a planet sweeps over time is constant. We can write Kepler's second law in equation 8.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \text{constant} \tag{8}$$

To prove Kepler's second law, we can create a small triangle that represents a small area that a planet sweeps as in Figure 3. Because the area is so small, we can assume that the area the planet sweeps is a triangle. The area of the triangle is written in equation 9.

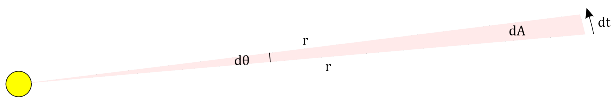


Figure 3: *Small area that planet sweeps (dA) in a small increment of time (dt)*

$$\mathrm{d}A = \frac{1}{2} r^2 \sin \mathrm{d}\theta \tag{9}$$

Because the value of $\mathrm{d}\theta$ is so small, we can assume that $\sin \mathrm{d}\theta \approx \mathrm{d}\theta$. If we derive equation 9 over time, we can get.

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2} r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2} r^2 \dot{\theta} \end{aligned} \tag{10}$$

The value of $r^2\dot{\theta}$ is the specific angular momentum for an orbiting object (h). This value will always be constant if Newton's first law of motion is applied. Because the Sun's gravitational force is much larger than other objects in the solar system, if we choose a system that consists of a planet and the Sun, other forces outside the system are not strong enough to tweak the observational data that Kepler analysed.

2.3 Kepler's Third Law

"The square of the period of a planet is proportional to the cube of its mean distance to the Sun."

Kepler realised that there is a relation between a planet's revolution period and its distance from the Sun. Kepler calculated that the square of a planet's period is always proportional to the cube of its mean distance from the Sun. Because a planet's orbit is discovered to be an ellipse, the planet's mean distance from the sun is its major axis of the ellipse (a). Kepler's third law can be written in equation 11.

$$a^3 \propto T^2 \tag{11}$$

Further discoveries found that Kepler's third law is also applicable to every orbiting object, as long as it's orbiting the same object. For example, all of Jupiter's moons follow Kepler's third law because they all orbit Jupiter. Newton's theory of gravity further proves that Kepler's third law applies to every orbiting object. Assuming that an object has a circular orbit, we can apply Newton's second law of motion; the acceleration caused by the gravitational force will act as a centripetal force to keep the orbit circular. If the object orbiting has (m) mass and the object in the centre has (M) mass, and the distance between the two objects is (r), the equation is written in equation 12.

$$\begin{aligned}
F &= ma \\
G\frac{Mm}{r^2} &= m\omega^2 r \\
G\frac{M}{r^3} &= \left(\frac{2\pi}{T}\right)^2 \\
\frac{r^3}{T^2} &= \frac{GM}{4\pi^2}
\end{aligned} \tag{12}$$

From equation 12, we can see that if the value of (M) is constant, the ratio between the cube of the distance and the square of the period will stay constant; thus, Kepler's third law is proved to be correct.[1]

2.4 Exercise Problem

1. Another method to derive Kepler's third law is by integrating Kepler's second law. From Kepler's second law, prove that.

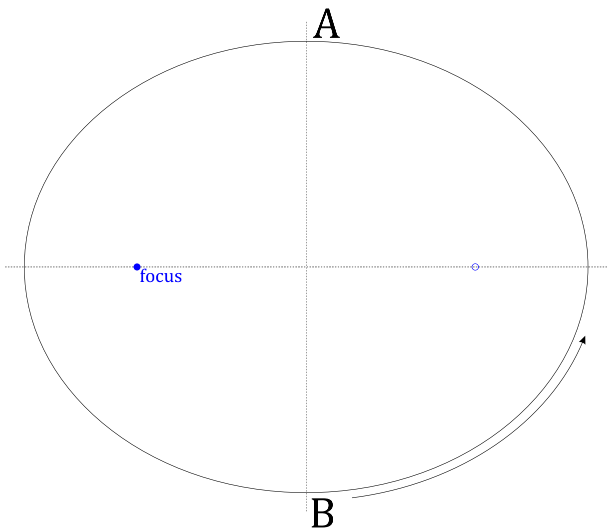
   $$A = \frac{1}{2}hT$$

   Take the case of an ellipse, where the area of an ellipse is equal to $A = \pi ab$. You might want to look at the next couple of levels to get the equation $h^2 = \mu p$. Prove that for an ellipse, Kepler's third law can be written as

   $$\frac{a^3}{T^2} = \frac{\mu}{4\pi^2}$$

[1] This equation will be important in celestial mechanics, especially in OSN and IOAA, so make sure to understand it fully.

2. From the Figure below, what is the ellipse eccentricity so that the ratio between the time it takes for an object to go from point B to A is two times the time it takes for the object to go from point A to B?



3. Orbital resonance is a phenomenon in celestial mechanics where orbiting objects exert a gravitational force on other orbiting objects, causing the orbit to resonate so that a ratio of small integers relates the periods.

   One of the most famous examples of orbital resonance is Jupiter's moons' orbits. Ganymede, Europa, and Io have a 1:2:4 orbital resonance, so every time Ganymede completes one orbit, Europa will complete exactly two orbits. Assuming the gravitational influence of each moon is only affected by Jupiter, calculate the ratio of the orbital semi-major axis between Ganymede, Europa, and Io!

3 Level 3: Conic Section

As we learn from the previous level, there are four possible orbit shapes. All of those shapes are part of the conic section. In this level, we will talk more about the characteristics of each conic section.

All of the conic section follows equation 13; this equation describes the relation between the distances from the focus (r) and the angle sweeps from the closest distance (θ). This equation depends on the semi-latus rectum (p) and eccentricity (e) value, which will vary for each orbit.

$$r = \frac{p}{1 + e\cos\theta} \tag{13}$$

Semi-latus rectum is half of latus rectum, which is the chord passing through the focus and perpendicular to the axis of a conic section. Semi-latus rectum represents the size of the conic section in this equation. On the other hand, eccentricity represents the shape of the conic section in this equation. The relation between semi-latus rectum and eccentricity is shown in equation 14, with a represents the value of semi-major axis.

$$p = a\left(1 - e^2\right) \tag{14}$$

3.1 Circle

The first and simplest conic section is a circle. A circle will form if you cut a cone with the same angle as its base, as seen in Figure 1. A circle has an eccentricity value of 0. Because of that, a circular orbit will always have a constant distance to the focus, which is at the centre of the circle.

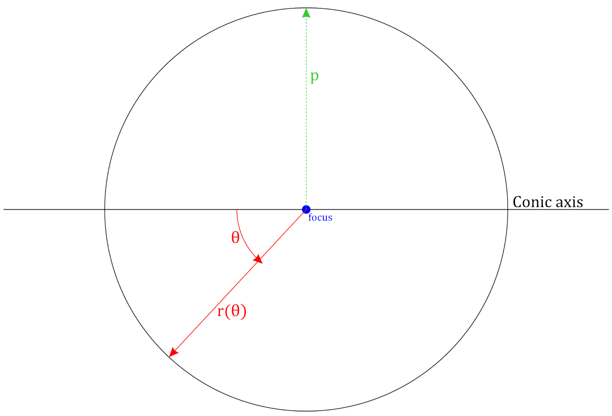


Figure 4: *Circle's property*

Figure 4 shows that for a circle, the value of the radius (r) is the same as the semi-latus rectum (p). You can prove this by inserting the eccentricity value in equation 13 with zero, achieving $r = p$ as the result.

We can form a circle as a Cartesian function with equation 5. The m and n values are the coordinates of the circle's centre in x and y directions, respectively. Meanwhile, the r value is the circle's radius.

$$(x-m)^2 + (y-n)^2 = r^2 \tag{15}$$

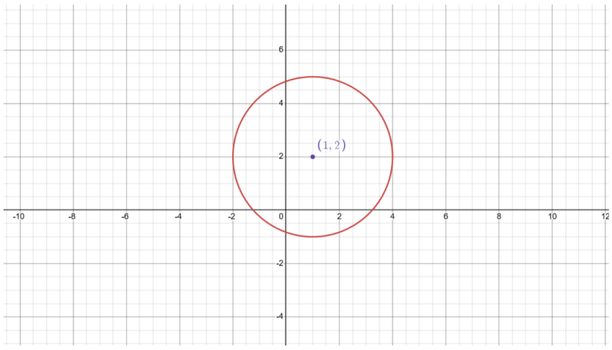


Figure 5: *An example for a circle with a radius of 3 and centre coordinates at (1,2)*

3.2 Ellipse

The second conic section that we will cover is an ellipse. An ellipse will form if you cut a cone with the angle between the cutting plane and the cone axis that is more than the cone angle itself, as seen in Figure 1. An ellipse's eccentricity value ranges between 0 and 1; with a higher eccentricity value, the ellipse will be more elongated and less circular.

If an object has an elliptical or a circular orbit, the orbit can be called a 'closed' orbit. This is because for an ellipse or a circle, the object will return to the initial position in the orbit over time.

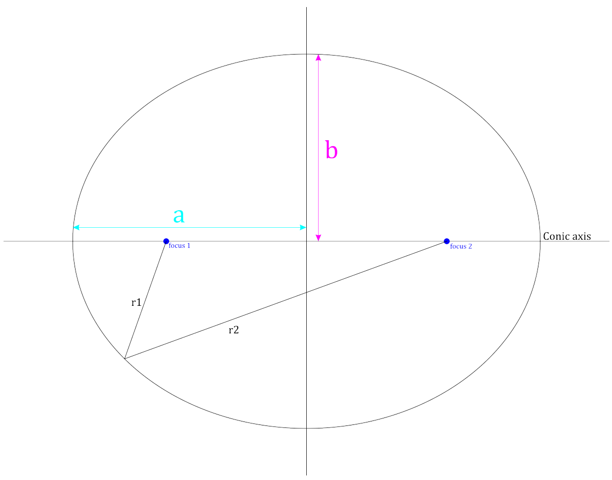


Figure 6: *The total distance* $(r1 + r2)$ *will always be constant throughout the ellipse*

An ellipse also has a special feature, which is that the total distance between the position and both focal points $(r1 + r2)$ is always constant. This total distance will always equal the length of the major axis $(2a)$ in the ellipse. To prove this, you can take the case when the object is on the conic axis.[2]

Kepler's first law stated that the primary object in orbit is at one of the ellipse's focal points. In an elliptical orbit, one of the focal points is empty, while the other is the place where the primary object is. Because of this, an orbiting object with an elliptical trajectory will sometimes be closer to the primary object throughout the period of the orbit. When an orbiting object is at the minimum distance from the primary object, we call it periapsis. Meanwhile, if the orbiting object is at the furthest distance from the primary object, we call it the apoapsis.[3]

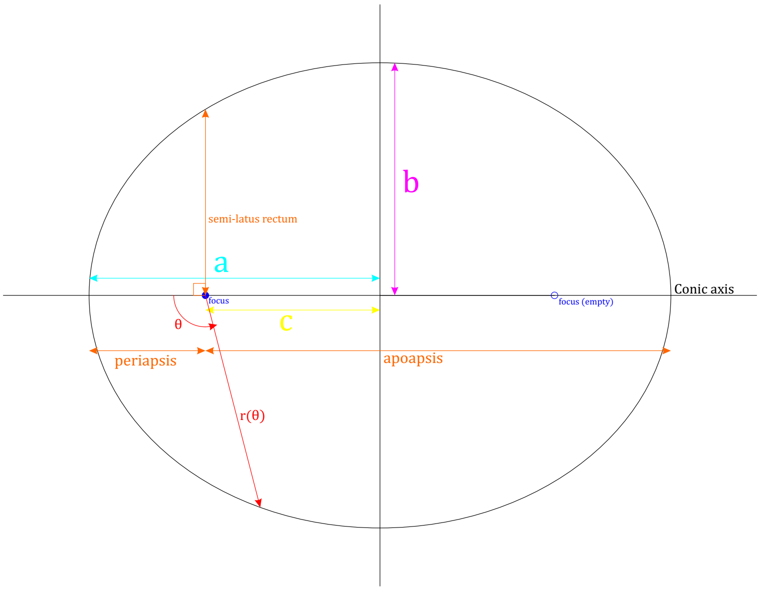


Figure 7: *Ellipse parameter visualisation*

To calculate the periapsis and apoapsis distance, we must define a parameter representing the distance between the ellipse centre and the focal point (c) as seen in Figure 7. Ellipse eccentricity can also be calculated as a ratio between c and a as in equation 16.

$$e = \frac{c}{a} \tag{16}$$

To calculate c, we can use the fact in Figure 6 and put the orbiting object at the minor axis as in Figure 8. Because an ellipse is a symmetric shape, the distance from each focal point will be the same, and because the total distance is always $2a$, the distance to one of the focal points will be a. From the figure, we can clearly see that a, b, and c are forming a right triangle. With that, we can calculate the relation between a, b, and c with the Pythagorean theorem as in equation 17.

[2] Try solving this :)

[3] Sometimes the term 'apsis' is replaced with the primary object's name, such as 'helion' for the Sun, 'gee' for Earth, or 'astron' for a star

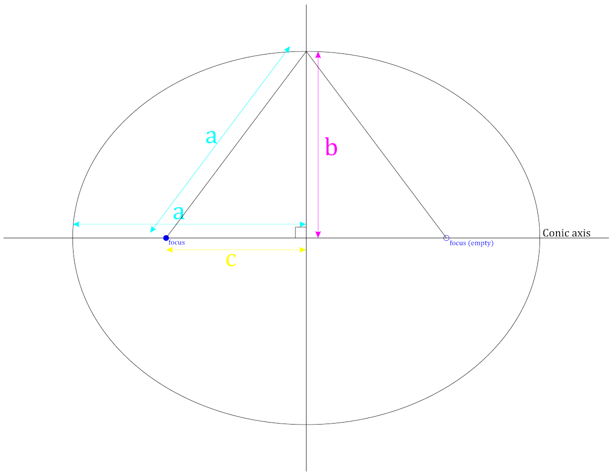


Figure 8: *Relation between a, b, and c in ellipse*

$$a^2 = b^2 + c^2 \tag{17}$$

Most of the time, the problem that we face will have the eccentricity and the semi-major axis data, and we need to know how to connect each elliptical parameter to those data. To calculate b, we can use equation 18.

$$\begin{aligned} a^2 &= b^2 + (ae)^2 \\ b^2 &= a^2 - a^2e^2 \\ b &= a\sqrt{1-e^2} \end{aligned} \tag{18}$$

To calculate the periapsis distance, we can use equation 19.

$$\begin{aligned} r_{peri} &= a - c \\ r_{peri} &= a - ae \\ r_{peri} &= a(1-e) \end{aligned} \tag{19}$$

To calculate the apoapsis distance, we can use equation 20.

$$\begin{aligned} r_{apo} &= a + c \\ r_{apo} &= a + ae \\ r_{apo} &= a(1+e) \end{aligned} \tag{20}$$

We can form an ellipse as a Cartesian function with equation 9. The m and n values are the coordinates of the ellipse's centre (not focus) in x and y directions, respectively. Meanwhile, a and b represent the values of both of the ellipse axes.

$$\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{b^2} = 1 \tag{21}$$

Figure 9 gives an example of an ellipse with major axis of 4 and minor axis of 3, and the centre is located at $(1, -1)$. The equation of this ellipse is written below.

$$\frac{(x-1)^2}{16} + \frac{(y+1)^2}{9} = 1 \tag{22}$$

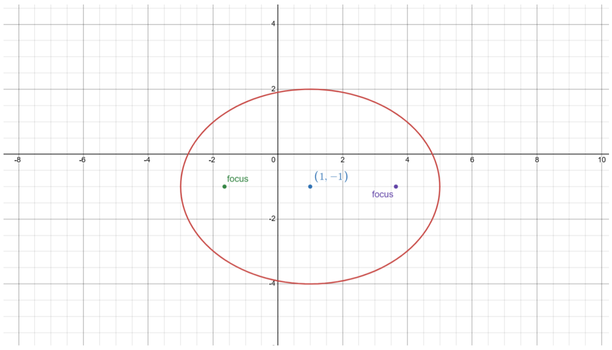


Figure 9: *An example of an ellipse in Cartesian coordinates*

3.3 Parabola

The third conic section is a parabola. A parabola will form if you cut a cone with the angle between the cutting plane and the cone axis equal to the cone angle, as in Figure 1. A parabola has an eccentricity value of 1.

A parabola orbit in celestial mechanics acts like a 'gate' between an ellipse and a hyperbola orbit. If an ellipse orbit is considered a 'closed' orbit and a hyperbola orbit is considered an 'open' orbit, in a parabola orbit, you can't really call it a 'closed' or an 'open' orbit. But in practice, if an object has a parabolic orbit, it will never return to the initial position in the orbit.

A parabola orbit has one focus. The most important parameter in a parabolic orbit is the closest distance from the focus, or the periapsis. Based on my experience, sometimes the periapsis in a parabolic orbit is also symbolised with a like the semi-major axis in an ellipse. In this document, I symbolised the periapsis as a' to distinguish it from the semi-major axis, because it is fundamentally different.

If we calculate the semi-latus rectum for a parabola with equation 14, we will find that the p value there will be zero, since the parabola has an eccentricity value of 1. But as we can see in Figure 10, the semi-latus rectum for a parabola is not zero. This is because a parabola can be treated like a really elongated ellipse, so that the semi-major axis is infinite, and we can't multiply infinity and zero. A parabola has a special feature where the semi-latus rectum value is always double the periapsis value.

$$p = 2a' \tag{23}$$

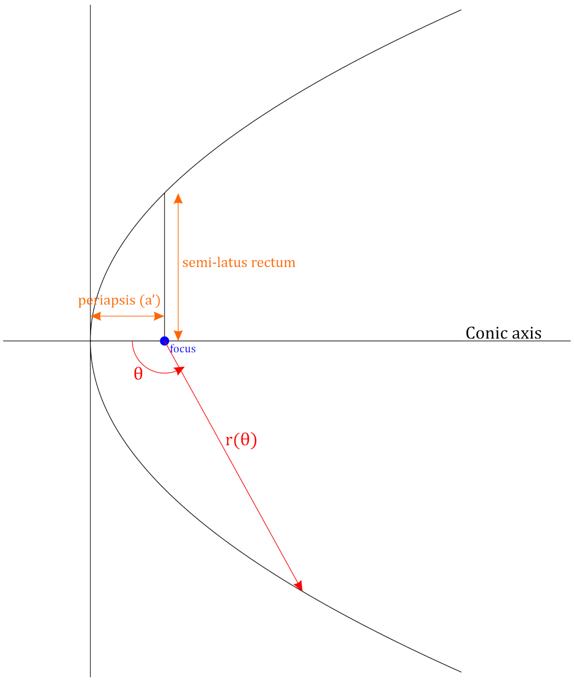


Figure 10: *Parabola parameter visualisation*

We can form a parabola as a Cartesian function with equation 11. The m and n values are the coordinates of the parabola's peak (or the closest point to the focus, not the focus itself) in x and y directions, respectively. We can use either of these equations to form the parabola.

$$
\begin{aligned}
(y-n)^2 &= 2p(x-m) \\
(y-n)^2 &= 4a'(x-m)
\end{aligned}
\tag{24}
$$

Figure 11 gives an example of a parabola with the equation below.

$$(y-1)^2 = 4(x+2) \tag{25}$$

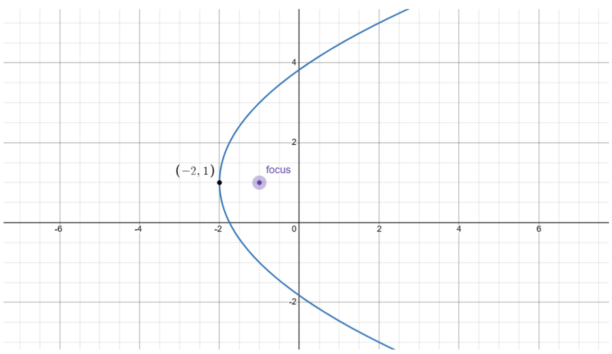


Figure 11: *An example of a parabola with p of 2 and peak at (-2,1)*

3.4 Hyperbola

The fourth and final conic section is a hyperbola. A hyperbola will form if you cut a cone with the angle between the cutting plane and the cone axis less than the cone angle, as in Figure 1. A hyperbola has an eccentricity value that is more than 1; with a higher eccentricity value, the hyperbola will be flatter.

An object with a hyperbolic orbit will not return to its initial position in the orbit. This is why a hyperbolic orbit is considered an 'open' orbit. Because of this, an orbit with a hyperbolic shape doesn't have a period.

Similar to the ellipse, a hyperbola also has a special feature, which is that the difference between the distance from the object and both focal points $|r1 - r2|$ is always constant, as in Figure 12. This difference is equal to double the absolute value of the semi-major axis in the hyperbola ($|2a|$). The proof is also similar to the ellipse, where you can take the case when the object is on the conic axis.[4]

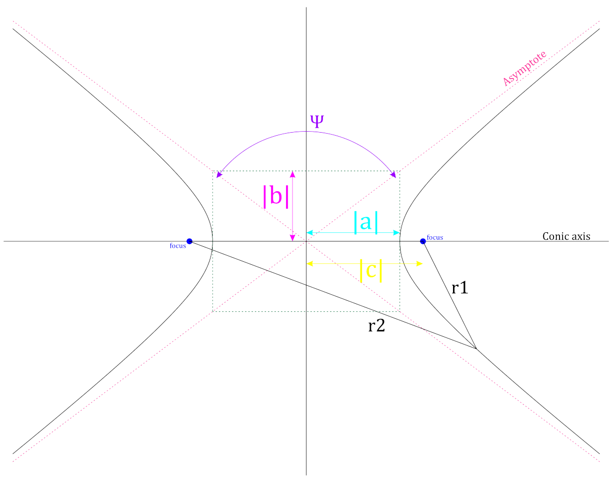


Figure 12: *The difference between $r1$ and $r2$ will always be $|2a|$ throughout the hyperbola*

If you notice that in Figure 12, I put an absolute bracket for every parameter in the hyperbola. I purposely did that because I want equation 13 and 14 to be consistent throughout this level. A hyperbola will form if we put a negative a value and an eccentricity value that is more than one for those equations. This is why throughout this document, if you find a negative a, b, or c value, that is because the conic section is a hyperbola.

Similar to the ellipse, if an orbiting object has a hyperbolic orbit, the primary object is located at one of the focal points. The orbiting object's trajectory will only follow the side where the focal point is located.[5] Similar to the parabola, because a hyperbolic orbit is an 'open' orbit, it only has a periapsis.

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[4] Also try to solve this :)

[5] If the primary object is located on the right side focal point, then the orbiting object will only follow the right side curve as the trajectory, where the left side of the curve is basically neglected.

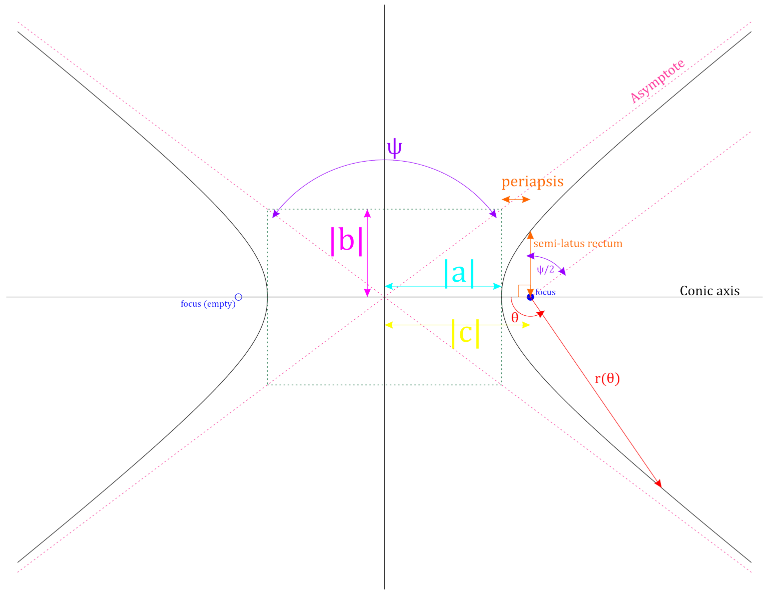


Figure 13: *Hyperbola parameter visualisation*

If we take a look at Figure 13, we can see that for a hyperbola, the θ value is actually limited because of the asymptote. The maximum θ value for a hyperbolic orbit is actually $90°$ plus half the angle between the two asymptotes (ψ).

$$\theta_{max} = 90° + \frac{\psi}{2} \tag{26}$$

We can calculate ψ with some geometry, where,

$$\tan\frac{\psi}{2} = \frac{|a|}{|b|} \tag{27}$$

Same with ellipse, a hyperbola's eccentricity value is defined as the ratio between c and a as in equation 16. To calculate c, we can use the fact in Figure 12 and put the orbiting object at the furthest distance from both focal points so that the lines $r1$ and $r2$ will be parallel as in Figure 14. We can construct a line that is perpendicular to $r1$ and $r2$ lines through one of the focal points (line gf). Because of that, the difference between $r1$ and $r2$ is visualised in line gf', where the value is equal to $|2a|$. We can also create a new variable that is the angle between an asymptote and the conic axis (α).

We know that the value of line ff' is the same as $|2c|$, and we know that line gf' is the same as $|2a|$. With that, we can rewrite α in equation 28.

$$\cos\alpha = \frac{a}{c} \tag{28}$$

We also know that from the triangle ofq, the value α can be written as in equation 29.

$$\tan\alpha = \frac{b}{a} \tag{29}$$

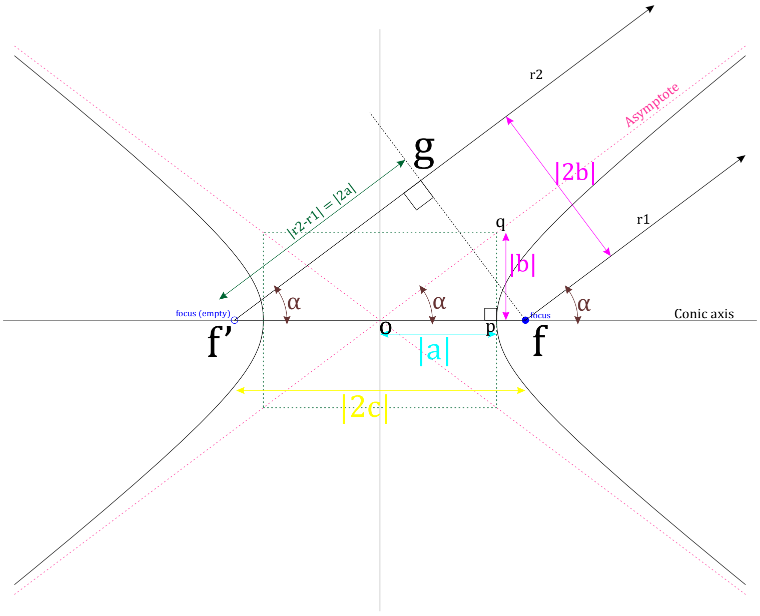


Figure 14: *Relation between a, b, and c in hyperbola*

By using trigonometric identities, we can solve the relation between a, b, and c from equations 28 and 29. The relation for those parameters in a hyperbola can be written in equation 30.

$$a^2 + b^2 = c^2 \tag{30}$$

Just like an ellipse, most of the time the problem that we face will have the eccentricity and the semi-major axis data, and we need to know how to connect each hyperbolic parameter to those data. To calculate b, we can use equation 31.

$$\begin{aligned} a^2 + b^2 &= (ae)^2 \\ b^2 &= a^2e^2 - a^2 \\ b &= a\sqrt{e^2 - 1} \end{aligned} \tag{31}$$

And to calculate the periapsis distance, we can use the same equation as an ellipse in equation 19. But keep in mind that a hyperbola has a negative value of a and an e value that is larger than 1.

To create a hyperbola as a Cartesian function, we can use equation 32. The m and n in this equation are the crossing point coordinates for both asymptotes in x and y directions, respectively. Meanwhile, a and b represent the value of semi-major and semi-minor axes for the hyperbola.

$$\frac{(x-m)^2}{a^2} - \frac{(y-n)^2}{b^2} = 1 \tag{32}$$

Figure 15 gives an example of a hyperbola with a and b values of 4 and 3, respectively. The crossing point for both asymptotes is located at $(3, 2)$. The equation is written below.

$$\frac{(x-3)^2}{16} - \frac{(y-2)^2}{9} = 1 \tag{33}$$

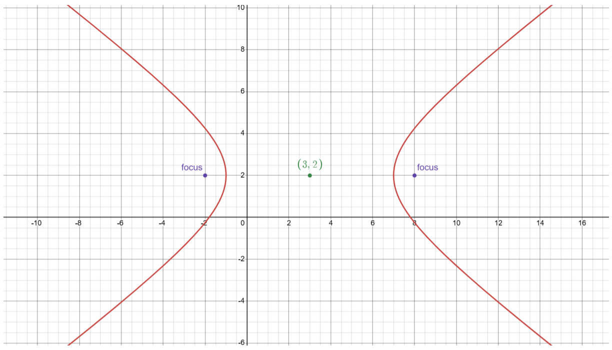


Figure 15: *An example of a hyperbola in Cartesian coordinates*

3.5 Exercise Problem

1. From the conic equation at equation 13, by substituting this equation that connects the polar and cartesian coordinate systems,

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Prove that the general equation of a conic in a Cartesian coordinate system is

$$x^2 + y^2 = p^2 - 2pex + e^2x^2$$

2. Assume Mars and Earth have circular orbits and lie in the same ecliptic plane. A rocket launched from Mars towards Earth with an elliptical orbit. The rocket orbit has an aphelion distance of 1.52 AU and a perihelion distance of 0.88 AU.

If we create a Cartesian coordinate system where the Sun is located at the centre, and the rocket is launched at point $(1.52, 0)$, determine the coordinates where the rocket's orbit will intersect the Earth's orbit!

3. The Rotation matrix is very useful to rotate an object in a coordinate system. We can rotate an object by multiplying its coordinates with the rotation matrix. If an object has initial coordinates at (x, y), and we want to rotate it counter-clockwise by θ, the final coordinates (x', y') can be calculated with this equation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Suppose we have an ellipse with a semi-latus rectum value of 3 and an eccentricity of 0.5. The ellipse's focal point is located at (0,0). If we want to rotate the ellipse by $30°$ counter-clockwise with the rotation centre at (0,0), determine the final equation of the ellipse in the Cartesian coordinate system!

4 Level 4: Vector

In celestial mechanics, we learn about how an object can move through space in an orbit. The keyword here is movement, because movement itself is a vector; we need to learn how to represent a vector in an orbit. On this level, we will discuss more about vectors, especially in an orbital plane.

4.1 Orbital Plane

We know that the orbit has a flat geometry because it's a conic section. The orbital plane is a plane where the orbit is placed. This term is important because an orbit isn't always placed in a reference plane in a coordinate system. Defining an orbital plane can simplify a problem, because we know that every conic element will lie in the orbital plane, so the analysis will only require 2D geometry.

In celestial mechanics, we usually represent a conic element vector in the orbital coordinates. Orbital coordinates are basically a polar coordinate with the coordinate centre as the focus of the orbit, and the line of reference is the periapsis, as in Figure 16.

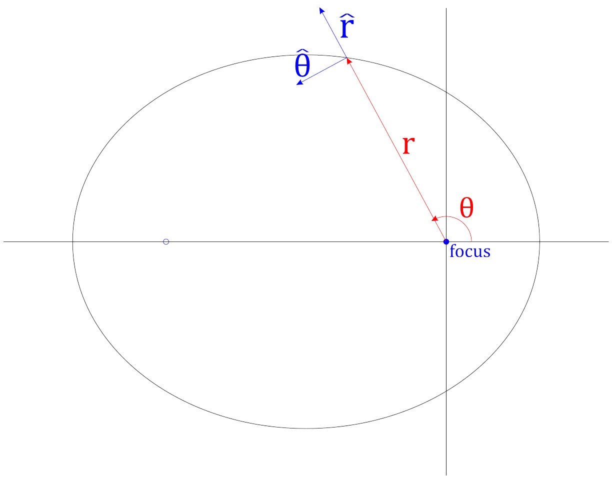


Figure 16: *Orbital coordinates*

Because orbital coordinate is the same as polar coordinate, each point in the orbital coordinate is represented by the distance from the focus (r) and the angle between the position vector and the line of reference (θ). To represent a vector in a polar coordinate, we usually introduce a pair of new unit vectors called $\hat{r}$ and $\hat{\theta}$. The unit $\hat{r}$ always has the same direction as the radius vector, or radial direction. The unit $\hat{\theta}$, on the other hand, is always perpendicular to the radius vector. The direction of unit vector $\hat{\theta}$ is always following the right-hand rule, so it is always directed at $90°$ counterclockwise from the $\hat{r}$ unit, as in Figure 16.[6]

[6] We will always use the right-hand rule throughout this document

4.2 Motion in Polar

In polar coordinates, we represent a point with r and θ values, while in Cartesian coordinates, we represent a point with x and y values. The relation between polar and Cartesian coordinates is visualised in Figure 27.

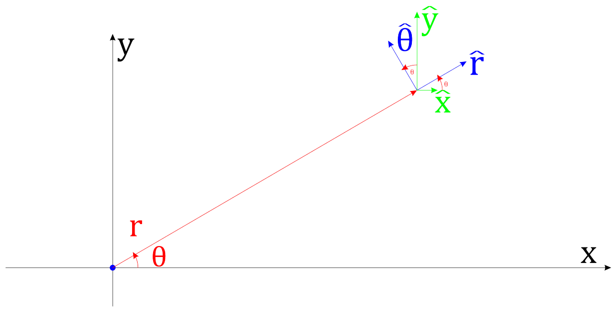


Figure 17: *Relation between polar and Cartesian coordinates*

From the figure, we can represent each of the polar unit vectors in Cartesian as equation 34.

$$\begin{aligned} \hat{r} &= (\cos\theta, \sin\theta) \\ \hat{\theta} &= (-\sin\theta, \cos\theta) \end{aligned} \tag{34}$$

Because of that relation, if the polar unit vectors are derived to θ, the result will be as in equations 35 and 36.

$$\begin{aligned} \frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta} &= (-\sin\theta, \cos\theta) \\ \frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta} &= \hat{\theta} \end{aligned} \tag{35}$$

$$\begin{aligned} \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\theta} &= (-\cos\theta, -\sin\theta) \\ \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\theta} &= -\hat{r} \end{aligned} \tag{36}$$

4.2.1 Position vector

To represent a position vector in polar coordinates, we can write it as a scalar value that represents the distance to the centre (r) times the unit vector $(\hat{r})$.

$$\vec{r} = r\hat{r} \tag{37}$$

4.2.2 Velocity vector

As we know, the velocity vector is the derivative of time from a position vector. Velocity vector in polar coordinates can be achieved by deriving the position vector from equation 37. Using the product rule, we can achieve this.

$$\begin{aligned}\dot{\vec{r}} &= \frac{\mathrm{d}r}{\mathrm{d}t}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} \\ \dot{\vec{r}} &= \frac{\mathrm{d}r}{\mathrm{d}t}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ \dot{\vec{r}} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\end{aligned} \tag{38}$$

As we can see from equation 38, there are two components of velocity in polar coordinates. The term $\dot{r}\hat{r}$ is called a radial velocity, basically stating how the r value changes over time. Meanwhile, the term $r\dot{\theta}\hat{\theta}$ is called a tangential velocity, stating the movement in θ direction.

4.2.3 Acceleration vector

We can continue to derive equation 38 over time to get the acceleration vector in polar coordinates.

$$\begin{aligned}\ddot{\vec{r}} &= \frac{\mathrm{d}\dot{r}}{\mathrm{d}t}\hat{r} + \dot{r}\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} + \frac{\mathrm{d}r}{\mathrm{d}t}\dot{\theta}\hat{\theta} + r\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t}\hat{\theta} + r\dot{\theta}\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} \\ \ddot{\vec{r}} &= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} \\ \ddot{\vec{r}} &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}\end{aligned} \tag{39}$$

From equation 39, we can see that there are four components of acceleration in polar coordinates. The first component $(\ddot{r}\hat{r})$ is called radial acceleration, along with the second component $(-r\dot{\theta}^2\hat{r})$ that is called centripetal acceleration; both of them act in the radial direction. The third component $(2\dot{r}\dot{\theta}\hat{\theta})$ is called coriolis acceleration, while the last component $(r\ddot{\theta}\hat{\theta})$ is called tangential acceleration; both of them act in the tangential direction.

4.3 Gravitational Force Vector

Gravity is the main force in celestial mechanics. Newton stated that the gravitational force vector is always in line with the position vector between the two objects, but the direction is inverted. With that in mind, we can rewrite the equation 5 so that we can represent the force vector $(\vec{F}_g)$ in position vector $(\vec{r})$, as in equation 41.

$$\begin{aligned}\vec{F}_g &= -G\frac{Mm}{|\vec{r}|^2}\frac{\vec{r}}{|\vec{r}|} \\ \vec{F}_g &= -G\frac{Mm}{|\vec{r}|^3}\vec{r} \\ \vec{F}_g &= -G\frac{Mm}{r^2}\hat{r}\end{aligned} \tag{40}$$

With Newton's second law of motion, we can further derive the acceleration vector caused by the gravitational force. Substituting the force vector in equation 41 with Newton's second law in equation 3, we can get

$$\begin{aligned} m\ddot{\vec{r}} &= -G\frac{Mm}{r^2}\hat{r} \\ \ddot{\vec{r}} &= \left(-\frac{GM}{r^2}\right)\hat{r} \end{aligned} \tag{41}$$

An orbiting object will only be influenced by the gravitational force from the primary object. Because of that, for an orbiting object, it only has an acceleration in the radial direction $(\hat{r})$, because gravity doesn't influence acceleration in the tangential $(\hat{\theta})$ direction. As a result, for an orbiting body, the acceleration equation is written in equation 42.

$$\begin{aligned} \left(-\frac{GM}{r^2}\right)\hat{r} &= (\ddot{r} - r\dot{\theta}^2)\hat{r} \\ \frac{GM}{r^2} &= r\dot{\theta}^2 - \ddot{r} \end{aligned} \tag{42}$$

4.4 Angular Momentum Vector

One of the most important parameters in celestial mechanics is angular momentum. As we know, angular momentum (L) is the cross product between a radius vector and a linear momentum vector as in equation 43.

$$\vec{L} = \vec{r} \times m\vec{v} \tag{43}$$

In celestial mechanics, we usually talk about angular momentum as a specific angular momentum (h). A specific angular momentum is defined as angular momentum divided by the mass of the moving object, as in equation 44. We can calculate a specific angular momentum vector by doing a cross product between the position vector and the velocity vector.

$$\begin{aligned} \vec{h} &= \frac{\vec{L}}{m} \\ \vec{h} &= \vec{r} \times \dot{\vec{r}} \end{aligned} \tag{44}$$

If we substitute the $\dot{\vec{r}}$ value with equation 38, we get.

$$\begin{aligned} \vec{h} &= (r\hat{r}) \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \\ \vec{h} &= (r\dot{r})(\hat{r} \times \hat{r}) + (r^2\dot{\theta})(\hat{r} \times \hat{\theta}) \\ \vec{h} &= (r^2\dot{\theta})\hat{h} \end{aligned} \tag{45}$$

The $\hat{h}$ unit vector represents the direction of the specific angular momentum, which is perpendicular to the orbital plane.[7] With h, we can also solve the equation 42 to find a relation between gravity and

[7] This is because of the cross rule

orbit parameter. We can calculate the $\ddot{r}$ value by deriving the conic equation in equation 13 over time. The result of the first position derivative is written in equation 46.

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\left(\frac{p}{1+e\cos\theta}\right)}{\mathrm{d}(1+e\cos\theta)}\frac{\mathrm{d}(1+e\cos\theta)}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\
\dot{r} &= \frac{p}{(1+e\cos\theta)^2}e\sin\theta\dot{\theta} \\
\dot{r} &= \left(\frac{p}{1+e\cos\theta}\right)^2\frac{e\sin\theta\dot{\theta}}{p} \\
\dot{r} &= r^2\dot{\theta}\frac{e\sin\theta}{p} \\
\dot{r} &= \frac{h}{p}e\sin\theta
\end{aligned} \tag{46}$$

The value of h, p, and e will be constant over time, so the second position derivative is written in equation 47.

$$\begin{aligned}
\frac{\mathrm{d}\dot{r}}{\mathrm{d}t} &= \frac{he}{p}\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\
\ddot{r} &= \frac{h}{p}e\cos\theta\dot{\theta}
\end{aligned} \tag{47}$$

Substituting equation 47 to equation 42, and adding a μ constant where $\mu = GM$ gives us

$$\begin{aligned}
\frac{\mu}{r^2} &= r\dot{\theta}^2 - \frac{h}{p}e\cos\theta\dot{\theta} \\
\frac{\mu}{r^2} &= \frac{r^4\dot{\theta}^2}{r^3} - \frac{hr^2\dot{\theta}}{r^2}\frac{e\cos\theta}{p} \\
\frac{\mu}{r^2} &= \frac{h^2}{r^3} - \frac{h^2}{r^2}\frac{e\cos\theta}{p} \\
\mu &= h^2\left(\frac{1}{r} - \frac{e\cos\theta}{p}\right) \\
\mu &= h^2\left(\frac{1+e\cos\theta}{p} - \frac{e\cos\theta}{p}\right) \\
\mu &= \frac{h^2}{p} \\
h^2 &= \mu p
\end{aligned} \tag{48}$$

Equation 48 shows the relation between angular momentum, gravitational force, and orbit parameter. This equation is fundamental in celestial mechanics, and we will review it frequently in the next level.

4.5 Exercise Problem

1. Using matrix multiplication, prove that $\vec{h}$ is constant over time!

   (Hint: We know that $\vec{h} = \vec{r} \times \dot{\vec{r}}$, and we know that $\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}$)

2. An asteroid that orbits the Sun is observed from the Earth. The asteroid's position relative to the Earth is

$$r_{a,\oplus} = \begin{bmatrix} -2.21 \\ 9.02 \\ 0.12 \end{bmatrix} AU$$

If the Sun's position relative to the Earth is

$$r_{\odot,\oplus} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} AU$$

Assume that the Earth is moving in a circular orbit with a velocity of 30km/s along the x-y plane. The velocity vector is always perpendicular to the position vector in a circular orbit. Assuming the Earth isn't rotating, if we observe the asteroid's velocity vector from the Earth as

$$v_{a,\oplus} = \begin{bmatrix} 10.31 \\ -14.87 \\ -0.09 \end{bmatrix} \text{km/s}$$

Calculate the asteroid's semi-latus rectum!

3. Assuming the Earth and Mars have circular orbits and lie in the same plane, if the Earth and Mars' distances from the sun are $a_\oplus$ and a_m, and the velocities of the Earth and Mars are $v_\oplus$ and v_m, prove that during the western quadrature, the relative velocity of mars will be

$$v_{m,\oplus} = \begin{bmatrix} v_m \frac{a_m}{a_\oplus} - v_\oplus \\ v_m \sqrt{1 - \frac{a_m^2}{a_\oplus^2}} \end{bmatrix}$$

5 Level 5: Orbital Energy

An orbiting object's trajectory is determined by its total energy in the orbit. In an orbit, the total energy is represented as the total mechanical energy of the object. We know that total mechanical energy (E) is the total of the kinetic energy (T) and the potential energy (V), as in equation 49.

$$E = T + V \tag{49}$$

Kinetic energy in an object will increase following the object's velocity. The formula for kinetic energy can be seen in equation 50. Potential energy, on the other hand, is a form of energy that can potentially be converted into kinetic energy. In general, potential energy has many forms, such as gravitational, elastic, heat, etc.

$$T = \frac{1}{2}mv^2 \tag{50}$$

5.1 Gravitational Potential Energy

As we discussed before, potential energy has many forms, but in the case of orbital mechanics, we will only consider gravitational potential energy as the potential energy. Gravitational potential energy is defined as the work done by gravity to move an object from a reference point (usually infinity) to a point at a distance r. Work done is defined as the integral of force over distance. The result can be seen in equation 51.

$$\begin{aligned}
V &= \int_{\infty}^{r} F_g(r')\mathrm{d}r' \\
V &= \int_{\infty}^{r} \frac{GMm}{r'^2}\mathrm{d}r' \\
V &= GMm \int_{\infty}^{r} \frac{1}{r'^2}\mathrm{d}r' \\
V &= GMm \left[-\frac{1}{r'}\right]_{\infty}^{r} \\
V &= GMm \left(-\frac{1}{r} - 0\right) \\
V &= -\frac{GMm}{r}
\end{aligned} \tag{51}$$

The result in equation 51 is a gravitational potential energy between two objects with masses of M and m, respectively. If we have more than two objects in our system, the total gravitational potential energy is equal to the sum of each unique pair of masses. The formula for total gravitational potential energy is written in equation 52.

$$V = -G \sum_{1 \le i < j \le n} \frac{m_i m_j}{r_{ij}} \tag{52}$$

5.1.1 Gravitational potential energy for a homogeneous spherical object

One of the most common problems in celestial mechanics is calculating an object's total gravitational potential energy by itself.[8] For example, calculating the total gravitational potential energy in a star cluster, or calculating the total gravitational potential energy in the Sun.

One of the common approaches to solving this problem is to assume that the object is a homogeneous spherical object. This approach works decently for most astronomical objects because most of them are spherical, and the density variation inside the object is not that significant. The approach starts with dividing the object into a thin layer, like an onion, with a thickness of dr as in Figure 18.

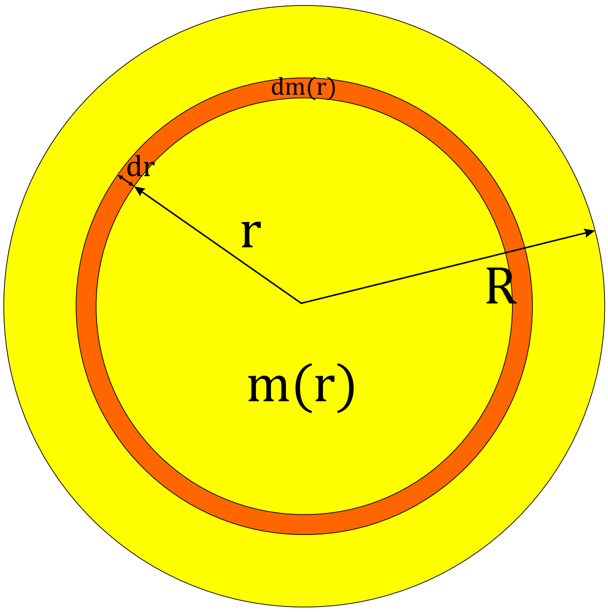


Figure 18: *Homogeneous spherical object approach*

For each layer, the potential energy can be calculated as the potential energy between that layer and all of the layers below it combined. Each layer will have a mass of dm, and all the layers below it will have a combined mass of m. Because of that, the potential energy $(\mathrm{d}V)$ for each layer can be calculated using equation 53.

$$\mathrm{d}V = -G\frac{m(r)\mathrm{d}m(r)}{r} \tag{53}$$

We can calculate the $m(r)$ value for a homogeneous sphere using equation 54.

$$m(r) = \frac{4}{3}\pi r^3 \rho \tag{54}$$

[8] gravitational potential energy by itself means that an object has its own value of potential energy without the influence from other objects' gravitational field.

Because dr is so small, we can approximate the volume for each layer as a tube with a base of the surface area at r and height of dr. Because of this, we can calculate the d$m(r)$ value for a homogeneous sphere using equation 55.

$$\mathrm{d}m(r) = 4\pi r^2 \mathrm{d}r\rho \tag{55}$$

Substitute equations 54 and 55 into equation 53, and integrate the equation to give us the total potential energy for a homogeneous spherical object.

$$\begin{aligned}
\mathrm{d}V &= -G\frac{\left(\frac{4}{3}\pi r^3\rho\right)\left(4\pi r^2\rho\mathrm{d}r\right)}{r} \\
\mathrm{d}V &= -\frac{16}{3}G\pi^2\rho^2 r^4\mathrm{d}r \\
V &= -\frac{16}{3}G\pi^2\rho^2\int_0^R r^4\mathrm{d}r \\
V &= -\frac{16}{3}G\pi^2\rho^2\left[\frac{1}{5}r^5\right]_0^R \\
V &= -\frac{16}{15}G\pi^2\rho^2 R^5
\end{aligned} \tag{56}$$

Substituting the total mass of the object $(M = m(R))$ with equation 54, we get the total potential energy in equation 57.

$$\begin{aligned}
V &= -\frac{16}{15}G\pi^2\rho^2 R^5 \\
V &= -\frac{3G}{5R}\left(\frac{4}{3}\pi\rho R^3\right)^2 \\
V &= -\frac{3}{5}\frac{GM^2}{R}
\end{aligned} \tag{57}$$

Equation 57 represents the total gravitational potential energy of a homogeneous spherical object. Whenever we solve a problem in the competition, we need to be able to distinguish when to use equation 57 or equation 51. We only use equation 57 when the problem mentions something about the potential energy for the object itself.[9]

5.2 Energy and Conic Section (Kepler's First Law)

In this section, we will prove Kepler's first law, which states that an orbit is the shape of a conic section. The total energy of an orbiting object is formulated in equation 49. In this section, we will use the term specific energy (ε), which is the total energy divided by the orbiting object's mass, as stated in equation 58.

$$\varepsilon = \frac{E}{m} \tag{58}$$

By expanding equation 49, we can calculate the total specific energy of an orbiting object.

$$\varepsilon = \frac{1}{2}\dot{\vec{r}}^2 - \frac{GM}{r} \tag{59}$$

[9] Remember this, because it will help you during the competition.

Substitute equation 38 into equation 59, and adding constant μ will give us,

$$\begin{aligned}
\varepsilon &= \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{r} \\
\frac{1}{2}\dot{r}^2 &= \varepsilon - \frac{1}{2}r^2\dot{\theta}^2 + \frac{\mu}{r} \\
\frac{\dot{r}^2}{r^2\dot{\theta}^2} &= \frac{2\varepsilon}{r^2\dot{\theta}^2} + \frac{2\mu}{r^3\dot{\theta}^2} - 1 \\
\frac{\dot{r}^2}{r^4\dot{\theta}^2} &= \frac{2\varepsilon}{r^4\dot{\theta}^2} + \frac{2\mu}{r^5\dot{\theta}^2} - \frac{1}{r^2}
\end{aligned} \tag{60}$$

We can add a constant value for angular momentum (h), remember that $h = r^2\dot{\theta}$. We can also simplify the term $\frac{\dot{r}}{\dot{\theta}}$ into $\frac{\mathrm{d}r}{\mathrm{d}\theta}$

$$\left(\frac{\mathrm{d}r}{r^2\mathrm{d}\theta}\right)^2 = \frac{2\varepsilon}{h^2} + \frac{2\mu}{rh^2} - \frac{1}{r^2} \tag{61}$$

To solve the equation, we can introduce a new variable u. This will help us solve the derivative.

$$\begin{aligned}
u &= \frac{1}{r} \\
\mathrm{d}u &= -\frac{1}{r^2}\mathrm{d}r
\end{aligned} \tag{62}$$

Substituting equation 62 into 61, we get

$$\begin{aligned}
\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 &= \frac{2\varepsilon}{h^2} + \frac{2\mu u}{h^2} - u^2 \\
\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 &= \left(-u^2 + \frac{2\mu u}{h^2} - \frac{\mu^2}{h^4}\right) + \left(\frac{2\varepsilon}{h^2} + \frac{\mu^2}{h^4}\right) \\
\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 &= -\left(u - \frac{\mu}{h^2}\right)^2 + \left(\frac{\mu^2}{h^4}\right)\left(1 + \frac{2\varepsilon h^2}{\mu^2}\right)
\end{aligned} \tag{63}$$

We then introduce new variables Z and B to simplify the equation further, the value of variables Z and B are

$$\begin{aligned}
Z &= u - \frac{\mu}{h^2} \\
B &= \frac{\mu}{h^2}\sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}} \\
\mathrm{d}Z &= \mathrm{d}u \\
\mathrm{d}B &= 0
\end{aligned} \tag{64}$$

Substituting equation 64 into 63 gives us

$$\begin{aligned}
\left(\frac{\mathrm{d}Z}{\mathrm{d}\theta}\right)^2 &= -Z^2 + B^2 \\
\frac{\mathrm{d}Z}{\mathrm{d}\theta} &= \sqrt{-Z^2 + B^2} \\
-\frac{\mathrm{d}Z}{\sqrt{-Z^2 + B^2}} &= -\mathrm{d}\theta
\end{aligned} \tag{65}$$

Equation 65 can be solved because it's one of the standard integrals in calculus.

$$\begin{aligned}\int_{Z(\theta_0)}^{Z(\theta)} -\frac{\mathrm{d}Z}{\sqrt{-Z^2+B^2}} &= \int_{\theta_0}^{\theta} -\mathrm{d}\theta \\ \arccos\frac{Z}{B} &= -\theta + C \\ Z &= B\cos{(\theta - C)}\end{aligned} \tag{66}$$

We can substitute the value of Z and B in equation 64 into equation 66.[10]

$$\begin{aligned}\frac{1}{r} - \frac{\mu}{h^2} &= \frac{\mu}{h^2}\sqrt{1+\frac{2\varepsilon h^2}{\mu^2}}\cos{(\theta - C)} \\ \frac{1}{r} &= \frac{\mu}{h^2}\left(1+\sqrt{1+\frac{2\varepsilon h^2}{\mu^2}}\cos{(\theta - C)}\right) \\ r &= \frac{\frac{h^2}{\mu}}{1+\sqrt{1+\frac{2\varepsilon h^2}{\mu^2}}\cos{(\theta - C)}}\end{aligned} \tag{67}$$

We can clearly see that equation 67 has a similar form to equation 13. From there, we can represent a conic parameter if we know the total energy of the orbit, which is calculated in equation 68.

$$\begin{aligned}p &= \frac{h^2}{\mu} \\ e &= \sqrt{1+\frac{2\varepsilon h^2}{\mu^2}}\end{aligned} \tag{68}$$

We still have a variable that we don't know (C). To calculate C, we need to introduce our initial condition. As we learn from the previous level, we know that the value θ in the conic equation is always measured from the periapsis. Because of that, we know that if the value of θ equals to 0, the value of r must be the same as the periapsis value.

$$r(\theta = 0) = a(1-e) \tag{69}$$

We can put the value of θ that is equal to zero in equation 67.

$$\begin{aligned}a(1-e) &= \frac{p}{1+e\cos{(C)}} \\ 1+e\cos{(C)} &= \frac{a(1-e^2)}{a(1-e)} \\ 1+e\cos{(C)} &= 1+e \\ \cos{(C)} &= 1 \\ C &= 0\end{aligned} \tag{70}$$

[10] If you're wondering why $-\theta + C$ becomes $\theta - C$ in the equation 66, it's because of the cosine rule where $\cos x = \cos{(-x)}$.

We can put the result in equation 70 into equation 67 to get the final result.

$$\begin{aligned} r &= \frac{\frac{h^2}{\mu}}{1 + \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}} \cos\theta} \\ r &= \frac{p}{1 + e\cos\theta} \end{aligned} \tag{71}$$

With this, we can conclude that an orbit will always have a trajectory of a conic section, and thus Kepler's first law is proven to be correct.

5.3 Total Orbital Energy

As indicated in equation 68, we know that the total orbital energy has a direct influence on the orbit's eccentricity. In this section, we will derive the equation to calculate the total orbital energy. From equation 68 we get the relation between the total energy and semi-major axis.

$$\begin{aligned} e &= \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}} \\ e^2 - 1 &= \frac{2\varepsilon p}{\mu} \\ e^2 - 1 &= \frac{2\varepsilon a(1 - e^2)}{\mu} \\ \varepsilon &= -\frac{\mu}{2a} \end{aligned} \tag{72}$$

This total orbital energy equation also shows more about the difference between a closed and an open orbit. A closed orbit (circle and ellipse) will have a total orbital energy that is negative. For an open orbit (hyperbola), because the value of a will be negative, the total orbital energy will be positive. On the other hand, a parabola will have a total orbital energy of zero.

Table 1: *Total energy for each orbital shape*

Orbit shape	Total orbital energy
Circle and Ellipse	$\varepsilon < 0$
Parabola	$\varepsilon = 0$
Hyperbola	$\varepsilon > 0$

Negative total orbital energy means that the value of potential energy is greater than the kinetic energy, which results in the orbiting object unable to escape from the primary object's gravitational field. The same concept goes for the positive total orbital energy; the kinetic energy value is greater, and because of that, the orbiting object will escape from the primary object's gravitational field.

5.3.1 Orbital velocity formula

Orbital velocity is arguably one of the most important topics in celestial mechanics. The problem of orbital velocity is frequently asked about in competitions. In this section, we will derive the orbital velocity equation from equation 72.

We know that, other than in equation 72, total orbital energy is also defined as the total value of kinetic and potential energy, as in equation 49. The specific total orbital energy is formulated in equation 73.

$$\begin{aligned} \varepsilon &= \frac{1}{2}v^2 - \frac{\mu}{r} \\ -\frac{\mu}{2a} &= \frac{1}{2}v^2 - \frac{\mu}{r} \\ \frac{1}{2}v^2 &= \frac{\mu}{r} - \frac{\mu}{2a} \\ v^2 &= \mu\left(\frac{2}{r} - \frac{1}{a}\right) \end{aligned} \tag{73}$$

We learn the difference between closed and open orbits. In a closed orbit, the orbiting object can't escape the gravitational field of the primary object, thus, it will return to the initial position over time. To escape the gravitational field of the primary object, we need to have an open orbit so that the total orbital energy is not negative. An object must form a parabolic orbit in order to escape the primary object's gravitational field. From this concept, we can calculate the escape velocity (v_{esc}) in equation 74.

$$\begin{aligned} 0 &= \frac{1}{2}v_{esc}^2 - \frac{\mu}{r} \\ v_{esc}^2 &= \frac{2\mu}{r} \end{aligned} \tag{74}$$

As we learn from the previous level, there are two main velocity components in orbital coordinates: radial and tangential. If you realise, we already derive the radial velocity component in equation 46.

$$v_r = \dot{r} = \frac{h}{p}e\sin\theta \tag{75}$$

The tangential velocity component can also be easily derived from the angular momentum concept because the tangential component will always be perpendicular to the radius.

$$\begin{aligned} h &= r \times v = rv_t \\ v_t &= r^2\dot{\theta} = \frac{h}{r} \\ v_t &= \frac{h}{p}(1 + e\cos\theta) \end{aligned} \tag{76}$$

The relation between the velocity components is visualised in Figure 19. Sometimes, we introduce a new variable that connects both velocity components, called the flight path angle (ϕ), that represents the direction of the orbiting object. Flight path angle can be calculated using equation 77.

$$\sin\phi = \frac{v_r}{v}$$
$$\cos\phi = \frac{v_t}{v} \tag{77}$$

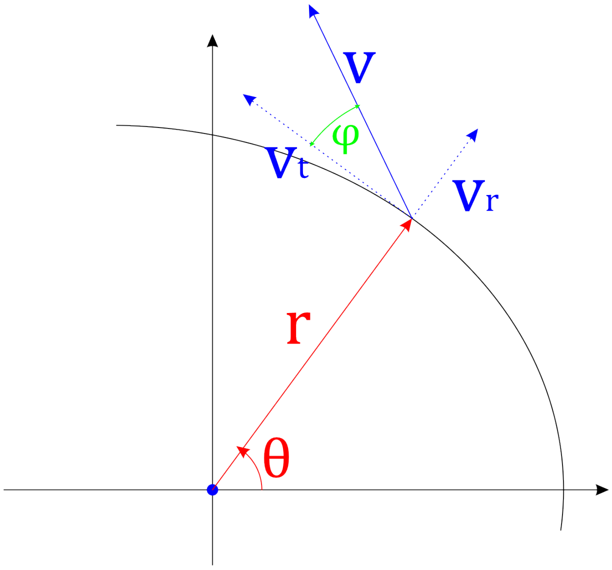


Figure 19: *Velocity vector and flight path angle*

5.4 Exercise Problem

1. Suppose there is a large spherical cloud in space with a mass of M and a radius of R. This cloud consists of a bunch of small particles with the mass of m_p. The particles inside the cloud behave like an ideal gas. If the number of particles inside the cloud is N and the density of the cloud is ρ, the total internal kinetic energy in this cloud can be calculated with the ideal gas kinetic energy, where

   $$K = \frac{3}{2}NkT$$

   Where k is the Boltzmann constant.

   If the cloud is stable, it will follow the virial theorem, where

   $$2K + V = 0$$

To become stable, this cloud must have a certain amount of mass, which is called the Jeans mass. Derive the equation to calculate the Jeans mass, and prove the equation is equal to this!

$$M = \left(\frac{2kT}{Gm_p}\right)^{\frac{3}{2}} \left(\frac{3}{4\pi\rho}\right)^{\frac{1}{2}}$$

2. If there is no potential energy $(V = 0)$, prove that an object will have a straight line as its trajectory! (Hint: if you want to use polar, the general equation for a straight line is $r = \frac{d}{\cos(\theta - \alpha)}$)
3. An alien ship is coming towards the solar system with a hyperbolic orbit. The ship is having a velocity of 300km/s before entering the sun's gravitational influence. The ship trajectory is designed so that they have an eccentricity of 5. The trajectory is also designed so that the ship will hit Earth exactly at its perihelion. After hitting the Earth, the ship will combine with the Earth, forming a new orbital trajectory. What is the minimum mass of the ship so that after colliding with the Earth, the trajectory will be enough to escape the solar system? (Ignore the relativistic effect and both Earth and the ship's gravitational influence, and assume the Earth isn't rotating)

6 Level 6: 2-Body Problem Solution

We learn that the gravitational force follows Newton's third law of motion. This implies that both objects in an orbit are affected by each other's gravity. Until now, we have assumed that the primary object in an orbit is always stationary, which is not. Because both objects are orbiting each other, we need to find a new stationary point so that the equation of motion can be solved.

6.1 Centre of Mass

In this section, we will introduce a concept called the centre of mass. This concept is important to solve the orbital equation. A centre of mass is the unique point at any given time where the weighted relative position of the distributed mass sums to zero. In an orbit, the centre of mass position vector $(\vec{R})$will always be stationary; this is why it is important for solving the equation of motion.

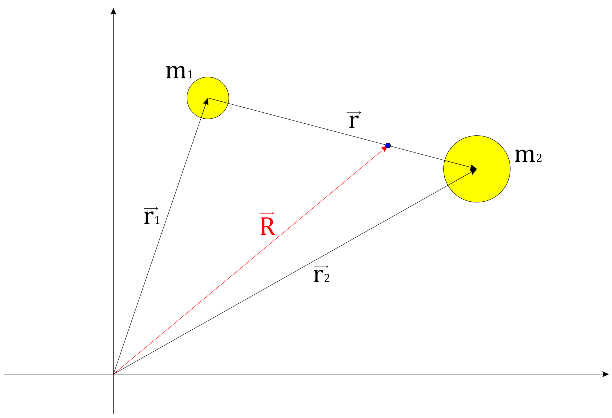


Figure 20: *Centre of mass vector*

To find the centre of mass position vector $(\vec{R})$ in Figure 20, we can calculate using this equation 78.

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2} \tag{78}$$

Position vector between both objects is defined by $\vec{r}$, which can be calculated with,

$$\vec{r} = \vec{r_2} - \vec{r_1} \tag{79}$$

Because of this, and adding the total mass of both objects $M = m_1 + m_2$, we can invert equation 78 to this,

$$\begin{aligned} \vec{r_1} &= \vec{R} - \frac{m_2}{M}\vec{r} \\ \vec{r_2} &= \vec{R} + \frac{m_1}{M}\vec{r} \end{aligned} \tag{80}$$

Because the centre of mass is a fixed point in space, sometimes we put the origin right at the centre of mass. This results in $\vec{R}$ value being equal to zero. Equation 80 is modified to,

$$\begin{aligned} \vec{r_1} &= -\frac{m_2}{M}\vec{r} \\ \vec{r_2} &= \frac{m_1}{M}\vec{r} \end{aligned} \tag{81}$$

Equation 81 is also beneficial for searching the distance value between the centre of mass to one of the objects.

In a 2-body problem, both objects and the centre of mass will always form a line. The distance between the two objects might change over time, but both position vector for the objects will always follows equation 80.

6.2 Reduced Mass

Now we know that both objects in an orbit are moving, the total orbital energy in a system needs to be revisited. Because the primary object is also moving, it will have its kinetic energy. The total orbital energy can be calculated with equation 82.

$$E = \frac{1}{2}m_1\dot{\vec{r_1}}^2 + \frac{1}{2}m_2\dot{\vec{r_2}}^2 - \frac{Gm_1m_2}{r} \tag{82}$$

Substituting equation 80 into equation 82, we get.

$$\begin{aligned} E = \frac{1}{2}m_1\left(\dot{\vec{R}} - \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} + \frac{m_1}{M}\dot{\vec{r}}\right)^2 - \frac{Gm_1m_2}{r} \\ E = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)\dot{\vec{r}}^2 - \frac{Gm_1m_2}{r} \\ E = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}x\dot{\vec{r}}^2 - \frac{Gm_1m_2}{r} \end{aligned} \tag{83}$$

We know that the centre of mass position is constant in space. This means that the value $\dot{\vec{R}}$ in equation 83 is equal to zero.

$$E = \frac{1}{2}x\dot{\vec{r}}^2 - \frac{Gm_1m_2}{r} \tag{84}$$

If we observe equation 83, we can see that to solve the 2-body problem, we can analyse it by reducing it to a 1-body problem where an object with mass x is orbiting a stationary object with mass M. The term x in this equation is the reduced mass; it is just a tool to solve the 2-body problem.

$$\frac{1}{x} = \frac{1}{m_1} + \frac{1}{m_2} \tag{85}$$

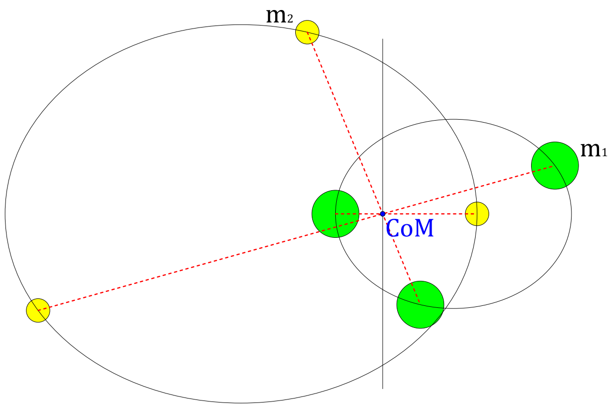


Figure 21: *Kepler's first law for the 2-body problem*

Because this is another 1-body problem, we can rederive equation 58 until equation 71 to calculate each conic parameter in the orbit. The specific energy in equation 58 can be rewritten as

$$\begin{aligned}\varepsilon &= \frac{E}{x} \\ \varepsilon &= \frac{1}{2}\dot{\vec{r}}^2 - \frac{Gm_1m_2}{xr} \\ \varepsilon &= \frac{1}{2}\dot{\vec{r}}^2 - \frac{G(m_1+m_2)}{r} \\ \varepsilon &= \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{r}\end{aligned} \tag{86}$$

From now on, we will redefine the variable μ as

$$\mu = G(m_1 + m_2) \tag{87}$$

Equation 45 is the same as equation 60. We can continue to derive the conic parameter using the same method. The result in the equation 68 will be the same, but remember that we redefine the μ value in equation 87.

6.3 Kepler's Law

The 2-body problem solution forces us to redefine Kepler's law. This is because all of our analyses before are based on the 1-body problem.

Kepler's first law is redefined so that the centre of mass for the system is always located on the conic focus. The orbit system is visualised in Figure 21. Because both objects and the centre of mass have to be in line, the period for both orbiting objects will always be the same.

$$T_1 = T_2 \tag{88}$$

Kepler's second law is also redefined so that the sweep is relative to the centre of mass, not the primary object. The area swept from the centre of mass will be proportional to the time elapsed.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}h \tag{89}$$

To get Kepler's third law, we can integrate equation 89.

$$\begin{aligned} A &= \frac{1}{2}h\int_0^T \mathrm{d}t \\ A &= \frac{1}{2}hT \end{aligned} \tag{90}$$

Substituting the area of an ellipse and the value of h, we get

$$\begin{aligned} \pi ab &= \frac{1}{2}\sqrt{\mu p}T \\ 2\pi a^2\sqrt{1-e^2} &= \sqrt{\mu a(1-e^2)}T \\ 4\pi^2 a^3 &= \mu T^2 \\ \frac{a^3}{T^2} &= \frac{\mu}{4\pi^2} \\ \frac{a^3}{T^2} &= \frac{G(m_1+m_2)}{4\pi^2} \end{aligned} \tag{91}$$

As we see in the equation 91, the value $\frac{a^3}{T^2}$ is constant; thus, Kepler's third law is proved.

6.4 3-Body Problem Cases

There is no analytical solution for a 3-body problem. If we want to analyse an orbit that consists of 3 objects, we have to solve it numerically. However, there are some restricted 3-body problems that we can look into. In this section, we will cover 3 different problems that sometimes appear in the competition. Each of the problems' solutions has a restricted 3-body problem.

6.4.1 Lagrange points

Lagrange points are points of equilibrium for small-mass objects under the gravitational influence of two massive orbiting bodies. At Lagrange points, the gravitational pull of two large masses precisely equals the centripetal force required for a small object to move with them.

There are a total of five Lagrange points, with three of them being unstable and the rest being stable. The unstable Lagrange points, namely L1, L2, and L3, lie along the line connecting two larger masses. The stable Lagrange points, L4 and L5, form the apex of two equilateral triangles that have the large masses at their vertices.

In the Earth-Sun system, L1 offers uninterrupted sun observation, this is why some solar observing satellites are located there. L2, on the other hand, offers a clear view of deep space, which is why some

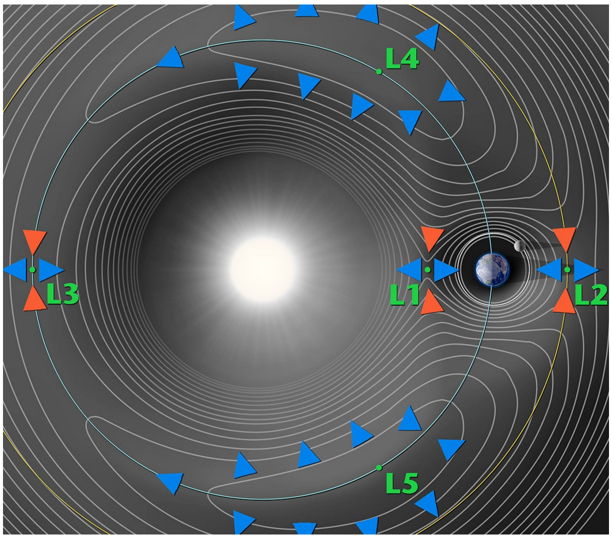


Figure 22: *Lagrange points*

astronomical observation satellites are located there. L4 and L5 are home to a cluster of asteroids called Trojans.

We will derive the location of Lagrange point 1, we can also follow this equation to solve L2 and L3. If the distance between L1 and object 2 is d as in Figure 23, we can calculate the forces using the definition of Lagrange point in equation 92, assuming object 2 is orbiting object 1.

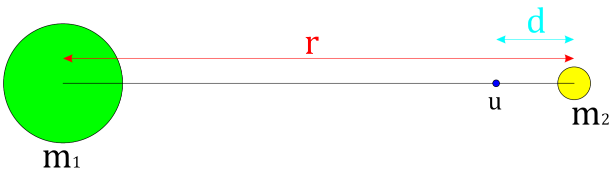


Figure 23: *L1 point*

$$\begin{aligned} \sum F_g &= u\omega_L^2(r-d) \\ \sum F_g &= u\omega_2^2(r-d) \\ \sum F_g &= u\frac{Gm_1}{r^3}(r-d) \end{aligned} \tag{92}$$

The total force acting in the Lagrange point can be described as the total gravitational force from the two massive objects.

$$\begin{aligned}
\frac{Gm_1u}{(r-d)^2} - \frac{Gm_2u}{d^2} &= \frac{Gm_1u}{r^3}(r-d) \\
\frac{m_1}{(r-d)^3} - \frac{m_1}{r^3} &= \frac{m_2}{(r-d)d^2} \\
\frac{m_1}{r^3}\left(\frac{1}{(1-\frac{d}{r})^3} - 1\right) &= \frac{m_2}{(r-d)d^2} \\
\frac{m_1}{r^3}\left(\left(1-\frac{d}{r}\right)^{-3} - 1\right) &= \frac{m_2}{(r-d)d^2}
\end{aligned} \tag{93}$$

We can use Newton's binomial equation, where if $x << 1$, then $(1+x)^n \approx 1+nx$. Because the value of d is much smaller than r, this applies,

$$\begin{aligned}
\frac{m_1}{r^3}\left(1+\frac{3d}{r} - 1\right) &= \frac{m_2}{(r-d)d^2} \\
\frac{(r-d)d^3}{r^4} &= \frac{m_2}{3m_1} \\
\frac{d^3}{r^3} - \frac{d^4}{r^4} &= \frac{m_2}{3m_1}
\end{aligned} \tag{94}$$

The value $\frac{d^4}{r^4}$ in equation 94 is negligible, so the distance to L1 can be calculated using equation 95.

$$d^3 = \frac{m_2}{3m_1}r^3 \tag{95}$$

The distance between object 2 and L1 is also called a Hill sphere. The Hill sphere is a sphere where the gravitational force from object 2 is more influential than that of object 1. If an object is located inside object 2's Hill sphere, it will orbit object 2 rather than object 1.

6.4.2 Tidal force

Tidal force is defined as the difference in gravitational force between two points in an object. This force is causing the object to stretch unevenly towards the attraction. On Earth, one of the factors that causes ocean waves is the tidal force from the Moon and the Sun.

We can calculate the tidal force on an object's surface caused by another object's gravitational force. The tidal force will be the difference between the gravitational force on the surface and in the centre of the object. The tidal force on the surface of object 1 in Figure 24 is calculated by

$$\begin{aligned}
\vec{F_T} &= \vec{F_S} - \vec{F_C} \\
\vec{F_T} &= F_S \cos\alpha\hat{x} - F_S \sin\alpha\hat{y} - F_C\hat{x}
\end{aligned} \tag{96}$$

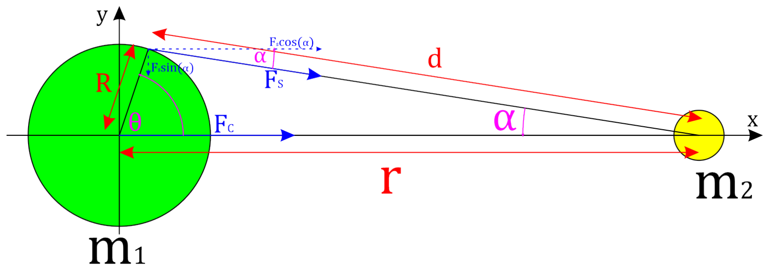


Figure 24: *Tidal force*

Using trigonometry, we can calculate the relation between θ and α in Figure 24.

$$
\begin{aligned}
\vec{F_T} &= F_S \frac{r - R\cos\theta}{d}\hat{x} - F_C\hat{x} - F_S\frac{R\sin\theta}{d}\hat{y} \\
\vec{F_T} &= \frac{Gm_1m_2}{d^2}\frac{r - R\cos\theta}{d}\hat{x} - \frac{Gm_1m_2}{r^2}\hat{x} - \frac{Gm_1m_2}{d^2}\frac{R\sin\theta}{d}\hat{y} \\
\vec{F_T} &= Gm_1m_2\left(\frac{r - R\cos\theta}{d^3} - \frac{1}{r^2}\right)\hat{x} - Gm_1m_2\frac{R\sin\theta}{d^3}\hat{y}
\end{aligned}
\tag{97}
$$

Using the cosine rule, we can get a relation between d and θ.

$$
\begin{aligned}
\vec{F_T} &= Gm_1m_2\left(\frac{r - R\cos\theta}{(R^2 + r^2 - 2Rr\cos\theta)^{3/2}} - \frac{1}{r^2}\right)\hat{x} - Gm_1m_2\frac{R\sin\theta}{(R^2 + r^2 - 2Rr\cos\theta)^{3/2}}\hat{y} \\
\vec{F_T} &= Gm_1m_2\left(\frac{r - R\cos\theta}{r^3}\left(1 + \frac{R^2}{r^2} - 2\frac{R}{r}\cos\theta\right)^{-3/2} - \frac{1}{r^2}\right)\hat{x} - Gm_1m_2\frac{R\sin\theta}{r^3}\left(1 + \frac{R^2}{r^2} - 2\frac{R}{r}\cos\theta\right)^{-3/2}\hat{y}
\end{aligned}
\tag{98}
$$

Assume that the distance between two objects is much larger than the object radius, $R << r$. We can ignore the R^2/r^2 in equation 98. We can also use Newton's binomial equation to solve this

$$
\begin{aligned}
\vec{F_T} &= Gm_1m_2\left(\frac{r - R\cos\theta}{r^3}\left(1 - 2\frac{R}{r}\cos\theta\right)^{-3/2} - \frac{1}{r^2}\right)\hat{x} - Gm_1m_2\frac{R\sin\theta}{r^3}\left(1 - 2\frac{R}{r}\cos\theta\right)^{-3/2}\hat{y} \\
\vec{F_T} &= \frac{Gm_1m_2}{r^2}\left(\left(1 - \frac{R\cos\theta}{r}\right)\left(1 + \frac{3R}{r}\cos\theta\right) - 1\right)\hat{x} - Gm_1m_2\frac{R\sin\theta}{r^3}\left(1 + \frac{3R}{r}\cos\theta\right)\hat{y} \\
\vec{F_T} &= \frac{Gm_1m_2}{r^2}\left(1 + \frac{3R}{r}\cos\theta - \frac{R}{r}\cos\theta - \frac{3R^2}{r^2}\cos^2\theta - 1\right)\hat{x} - Gm_1m_2\frac{R\sin\theta}{r^3}\left(1 + \frac{3R}{r}\cos\theta\right)\hat{y}
\end{aligned}
\tag{99}
$$

Again, because $R << r$, the value of $\frac{3R^2}{r^2}\cos^2\theta$ is negligible. The equation for tidal force is written in equation 100.

$$
\vec{F_T} = \frac{2Gm_1m_2R\cos\theta}{r^3}\hat{x} - \frac{Gm_1m_2R\sin\theta}{r^3}\left(1 + \frac{3R}{r}\cos\theta\right)\hat{y}
\tag{100}
$$

As we can see, the maximum tidal force will occur at $\theta = 0$. And at $\theta = 90°$, the tidal force will only have a y component, and the direction of the tidal force is going into the object. The distribution of tidal force on the surface of an object is visualised in Figure 25

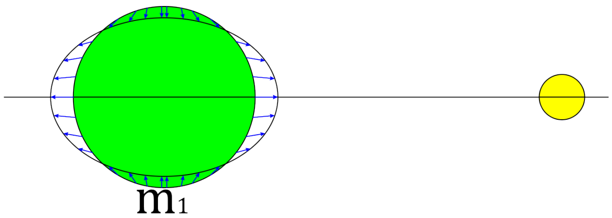


Figure 25: *Tidal force distribution*

6.4.3 Roche limit

The Roche limit is defined as the distance from a celestial body at which, if there is another object held only by its gravitational force, that object will disintegrate because the tidal force acting is greater than its gravitational force, as we see from Figure 26.

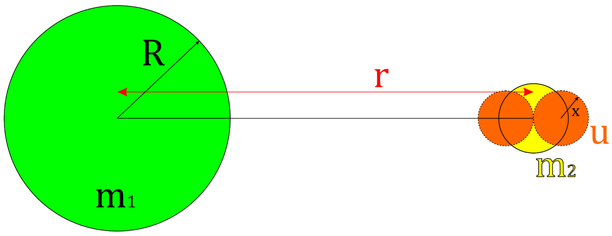


Figure 26: *Roche limit approximation*

To calculate the Roche limit, we can approximate it by dividing the closing object into two separate objects with the same total mass, but keeping the density of the object. The divided object will have a radius of x as visualised in Figure 26. Because Roche's limit is the distance where the tidal force is equal to an object's gravitational force, we can calculate it using equation 101.

$$\begin{aligned} F_T &= F_g \\ \frac{Gm_1u}{(r-x)^2} - \frac{Gm_1u}{(r+x)^2} &= \frac{Gu^2}{(2x)^2} \\ \frac{m_1}{r^2}\left(\left(1-\frac{x}{r}\right)^{-2} - \left(1+\frac{x}{r}\right)^{-2}\right) &= \frac{u}{4x^2} \end{aligned} \tag{101}$$

Using Newton's binomial, we get

$$\begin{aligned} \frac{m_1}{r^2}\left(1+\frac{2x}{r}-1+\frac{2x}{r}\right) &= \frac{u}{4x^2} \\ \frac{4m_1}{r^3} &= \frac{u}{4x^3} \end{aligned} \tag{102}$$

Substituting the mass value with a sphere volume times the density, we will get

$$\begin{aligned} \frac{\frac{4}{3}\pi R^3 \rho_1}{r^3} &= \frac{\frac{4}{3}\pi x^3 \rho_2}{16x^3} \\ r^3 &= \frac{16\rho_1}{\rho_2} R^3 \end{aligned} \tag{103}$$

The result in the equation 103 is just a rough approximation for a Roche limit formula. You can explore it further by using another method. But as of now, equation 103 is satisfactory for the competition.

6.5 Exercise Problem

1. A double star detected with a telescope. The double star has a period of 3.23 days. From the spectroscopy data, it's concluded that the first star has an orbital velocity of 80km/s and the second star has an orbital velocity of 120km/s. Assume both stars have a circular orbit. Calculate each star's masses!
2. Two stars with the masses of $1.23M_\odot$ and $4.12M_\odot$ orbiting each others. The stars are separated by $5.83AU$ and the velocity of the first star relative to the second star is 24.23km/s. The velocity vector is perpendicular to the distance vector. Calculate the value of the semi-major axis and eccentricity of the orbit!
3. You're stranded on an island. To escape from the island, you need to calculate how much the total tidal force on your island. From the GPS device that you bring, you know that your geographic coordinate is ($\lambda = 104.29$°E, $\phi = 5.78$°N. You anticipate a lunar eclipse that will occur on September 7, 2025. During the lunar eclipse, the Moon's distance from the Earth is approximately $370,000$km. You need to calculate how much the tidal force is at the maximum lunar eclipse, which happens at $6:58$GMT.

7 Level 7: 3D Orbital Elements

We live in three-dimensional space, so we need to understand how it works in three-dimensional space to analyse celestial mechanics. In this level, we will learn to understand what is needed to solve the orbit in three dimensions.

7.1 3D Coordinate System

As we know, there is no analytical solution for the 3-body problem or more. To solve the problem, we can use a numerical method by integrating the acceleration vector over time. With this method, we can add as many acceleration sources as possible, such as gravity or other perturbations, to mimic the real environment.

To solve the integration, we need to add an initial condition. If we integrate the acceleration vector, the result will be a velocity vector, and we need to set the initial condition for it. To achieve the position vector over time, we can integrate the velocity vector once more, and we also need to set the initial position vector.

This is why, to solve an orbit numerically, we need to have both the position and the velocity vector at a given time. The acceleration vector can be achieved if we assume gravity as the sole force source because gravity is only affected by both objects' masses and their distance to each other.

$$\begin{aligned}\ddot{\vec{r}} &= -\frac{\mu}{|r|^2}\frac{\vec{r}}{|r|} \\ \dot{\vec{r}} &= \int \ddot{\vec{r}}\mathrm{d}t \\ \vec{r} &= \int \dot{\vec{r}}\mathrm{d}t\end{aligned} \tag{104}$$

This document uses two main coordinate systems: polar and Cartesian. The Cartesian coordinate system is commonly used to analyse the orbit in a computer. The polar coordinate system (spherical), however, is used to intuitively imagine an object's position in the coordinate system. If you have already learned about coordinate systems in astronomy, such as equatorial, elliptical, horizon, etc., you should be familiar with the polar coordinate system in three-dimensional space because all of those are types of polar coordinate systems.

In a three-dimensional polar coordinate system, we have three variables to represent a position in the coordinate system. Two types of angles are used in the polar coordinate system, the first one lies on the reference plane of the coordinate system. In contrast, the other angle is calculated perpendicular to the reference plane. The reference plane in the polar coordinate system is usually the same as the x-y plane in the Cartesian coordinate system. The relation between polar and Cartesian coordinate systems is visualised in Figure 27.

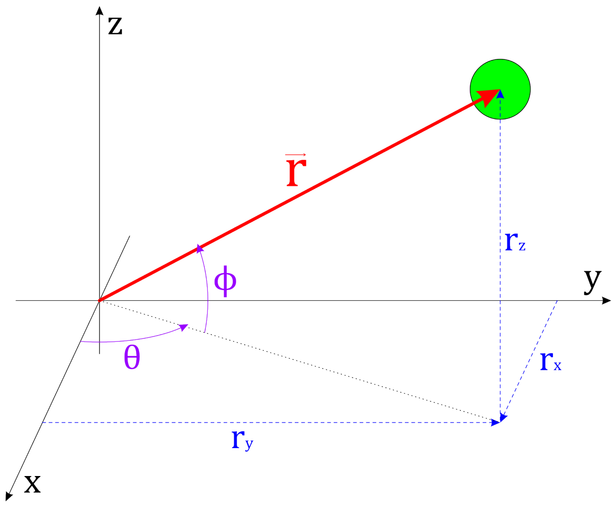


Figure 27: *Three-dimensional polar and Cartesian coordinate systems*

The relation between two coordinate systems can be calculated using equation 105.

$$\vec{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} r\cos\phi\cos\theta \\ r\cos\phi\sin\theta \\ r\sin\phi \end{bmatrix} \tag{105}$$

7.2 Keplerian Orbital Elements

Keplerian orbital elements consist of six variables that determine an object's position in a three-dimensional space orbit. Keplerian orbital elements are used because it's easier for us to imagine an orbit with these elements compared to having position and velocity vectors.

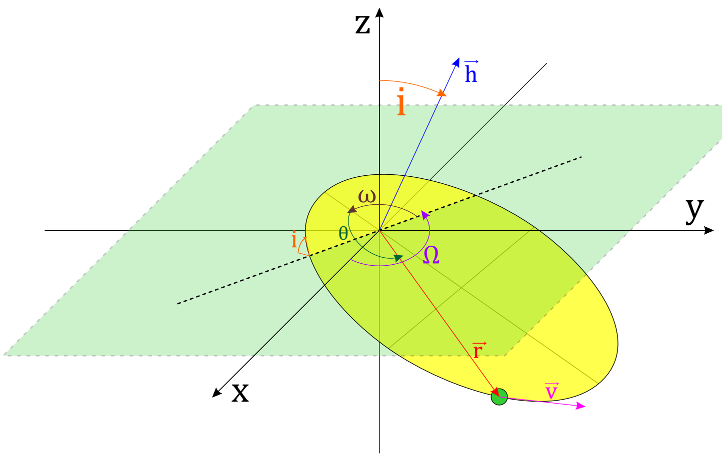


Figure 28: *Keplerian orbital elements visualisation*

In Keplerian orbital elements, the semi-major axis (a) and the eccentricity (e) represent the orbit's shape. The inclination (i), longitude of ascending node (Ω), and argument of periapsis (ω) show the orientation of the orbital plane in three-dimensional space. The true anomaly (θ) shows the object's position in the orbit. The Keplerian orbital elements are visualised in Figure 28.

We have learn that the specific angular momentum vector $(\vec{h})$ is the cross product of the position vector $(\vec{r})$ and the velocity vector $(\vec{v})$. The position and velocity vectors will always lie in the orbital plane. This means that $\vec{h}$ is always perpendicular to the orbital plane. To calculate the Keplerian orbital elements from the position and velocity vectors, we will often use this $\vec{h}$ vector.

$$\vec{h} = \vec{r} \times \vec{v} \tag{106}$$

7.2.1 Semi-major axis

Semi-major axis (a) represents the orbit size. This element has the same dimension as length. In practical use, sometimes we represent it in kilometres if we're calculating a satellite orbit, or in astronomical units if we're calculating a celestial object. We can use the specific energy equation to calculate the semi-major axis from the position and velocity vectors in three-dimensional space, as in equation 107.

$$\begin{aligned} \varepsilon &= \frac{1}{2}|\vec{v}|^2 - \frac{\mu}{|\vec{r}|} \\ a &= -\frac{\mu}{2\varepsilon} \end{aligned} \tag{107}$$

In this equation, we can also predict the type of orbit. As I stated in the third level, if the a value is negative, the orbit is a hyperbola, while if the a value is undefined because the ε value is equal to zero, the orbit is a parabola.

7.2.2 Eccentricity

As we know from the previous levels, eccentricity (e) determines an orbit's shape, whether it's a circle, ellipse, parabola, or hyperbola. This element is dimensionless, and the value ranges from zero to infinity.

One of the ways to get the eccentricity value is by using the specific angular momentum as in equation 48. The other way is by defining the eccentricity vector $(\vec{e})$. The eccentricity vector is a vector that has the same direction as the periapsis of the conic. This vector will be useful later on because some of the orbital elements are dependent on the direction of the periapsis. We know that from the conic equation, the relation between r and e is described in equation 108.

$$\begin{aligned} r &= \frac{p}{1 + e\cos\theta} \\ r + re\cos\theta &= \frac{h^2}{\mu} \\ re\cos\theta &= \frac{h^2}{\mu} - r \end{aligned} \tag{108}$$

We know that θ is an angle between the position vector $(\vec{r})$ and the periapsis, and the periapsis has the same direction as the eccentricity vector $(\vec{e})$ as in Figure 29. We can also calculate the angle between two vectors using a dot product, as in equation 109.

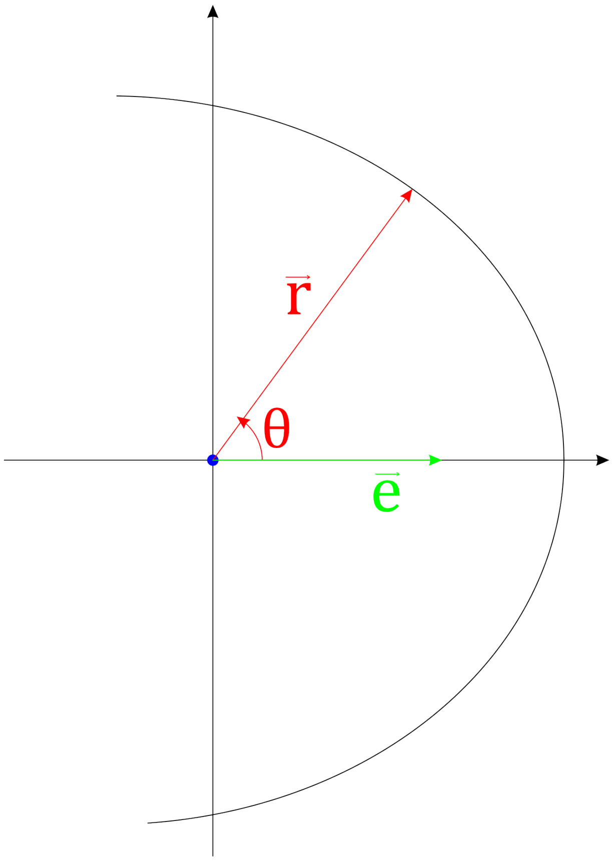


Figure 29: *Relation between θ, $\vec{r}$, and $\vec{e}$*

$$\vec{r} \cdot \vec{e} = re\cos\theta \tag{109}$$

Substituting the equation, we can get

$$\begin{aligned}
\vec{r} \cdot \vec{e} &= \frac{h^2}{\mu} - r \\
\vec{r} \cdot \vec{e} &= \frac{\vec{h} \cdot \vec{h}}{\mu} - \frac{\vec{r} \cdot \vec{r}}{r} \\
\vec{r} \cdot \vec{e} &= \frac{\vec{h} \cdot (\vec{r} \times \vec{v})}{\mu} - \frac{\vec{r} \cdot \vec{r}}{r}
\end{aligned} \tag{110}$$

By using the scalar triple product rule, we get

$$
\begin{aligned}
\vec{r}\cdot\vec{e} &= \frac{\vec{r}\cdot(\vec{v}\times\vec{h})}{\mu} - \frac{\vec{r}\cdot\vec{r}}{r} \\
\vec{r}\cdot\vec{e} &= \vec{r}\cdot\left(\frac{\vec{v}\times\vec{h}}{\mu} - \frac{\vec{r}}{r}\right) \\
\vec{e} &= \frac{\vec{v}\times\vec{h}}{\mu} - \frac{\vec{r}}{r} \\
e &= |\vec{e}|
\end{aligned}
\tag{111}
$$

Equation 111 shows the calculation of the eccentricity vector.

7.2.3 Inclination

Inclination (i) is defined as the angle between the orbital plane and the reference plane. Together with the longitude of ascending node and the argument of periapsis, these element represents the orientation of the orbital plane.

We know that $\vec{h}$ is always perpendicular to the orbital plane. Thus, we can represent the orbital plane with the $\vec{h}$ vector. To calculate the angle between the orbital and reference plane, we can calculate the angle formed by the vector that is perpendicular to the orbital plane and the vector that is perpendicular to the reference plane. If the reference plane is the same as the xy plane, the vector that is perpendicular to the reference plane will have a direction in the z axis, which we will use $\hat{z}$ as the unit vector that has the magnitude of one. The inclination angle can be calculated using the dot product rule.

If the h vector is

$$
\vec{h} = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}
\tag{112}
$$

Then we can calculate the i angle with

$$
\begin{aligned}
\vec{h}\cdot\hat{z} &= |\vec{h}|\cos i \\
\cos i &= \frac{h_z}{|\vec{h}|}
\end{aligned}
\tag{113}
$$

As we see from equation 113, the inclination can be easily calculated if we know the h vector.

7.2.4 Longitude of ascending node

The intersection between two planes in three-dimensional space will be a line. This goes for the intersection between the orbital and reference planes. Because of the orbit's shape, there are two points where the orbiting object lies on the reference plane. The two points are called nodes; the first node is the ascending node, which is defined as the point where the orbiting object goes from under to over the reference plane.[11] The other node is called the descending node, which is the opposite of the ascending node.

---

[11] Or from negative z value to positive z value.

The longitude of ascending node (Ω) is defined as the angle between the reference point (usually in the direction of the x axis) to the ascending node. Because we know that the node must lie on the orbiting plane and the reference plane, we can calculate the node vector ($\vec{n}$) by crossing two vectors, one that are perpendicular to the reference plane ($\hat{z}$), and other that are perpendicular to the orbital plane ($\vec{h}$). The direction of $\vec{n}$ will always point to the ascending node if we calculate it using equation 114.

$$\begin{aligned}
\vec{n} &= \hat{z} \times \vec{h} \\
\vec{n} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \\
\vec{n} &= \begin{bmatrix} -h_y \\ h_x \\ 0 \end{bmatrix}
\end{aligned} \tag{114}$$

To calculate Ω, we can use the same method as we used to calculate i. If the reference point is at the x axis direction, we can calculate it using the dot product rule.

$$\begin{aligned}
\vec{n} \cdot \hat{x} &= |\vec{n}| \cos \Omega \\
\cos \Omega &= \frac{n_x}{|\vec{n}|}
\end{aligned} \tag{115}$$

Remember that the arccos function has two values. If the value of n_y is negative, Ω must be above $180°$ because the Ω is in the 3$^{\text{rd}}$ or 4$^{\text{th}}$ quadrants.

7.2.5 Argument of periapsis

The argument of periapsis (ω) represents the orbit's rotation in the orbital plane, which states the direction of the periapsis from the node. This can be calculated using the angle between the ascending node and the periapsis. We know that the vector $\vec{n}$ has the same direction as the ascending node, and the vector $\vec{e}$ has the same direction as the periapsis. We can calculate ω using equation 116.

$$\begin{aligned}
\vec{n} \cdot \vec{e} &= |\vec{n}||\vec{e}| \cos \omega \\
\cos \omega &= \frac{\vec{n} \cdot \vec{e}}{|\vec{n}||\vec{e}|}
\end{aligned} \tag{116}$$

If the periapsis is below the reference plane, or the e_z value is below zero, the ω must be above $180°$ because ω is in the 3$^{\text{rd}}$ or 4$^{\text{th}}$ quadrants.

7.2.6 True anomaly

The true anomaly (θ) is defined as the angle between the object's position vector and the periapsis. We already know how to get this using the conic equation as in equation 13. In this section, we will derive it using the angle between the eccentricity vector ($\vec{e}$) (which has the same direction as the periapsis) and the position vector ($\vec{r}$).

$$\begin{aligned} \vec{r} \cdot \vec{e} &= |\vec{r}||\vec{e}| \cos\theta \\ \cos\theta &= \frac{\vec{r} \cdot \vec{e}}{|\vec{r}||\vec{e}|} \end{aligned} \tag{117}$$

If the angle between the position and velocity vector is above $90°$, or the value of $\vec{r} \cdot \vec{v}$ is less than zero, the θ must be above $180°$ because θ is in the 3rd or 4th quadrants.

Remember that the true anomaly and argument of periapsis lie on the same plane. This creates a relation. If we don't want to calculate the eccentricity vector, we can calculate the angle between the node angle and the position vector, and then the result can be subtracted from the true anomaly in equation 13 to get the argument of periapsis.

$$\vec{r} \cdot \vec{n} = |\vec{r}||\vec{n}| \cos(\omega + \theta) \tag{118}$$

7.3 Exercise Problem

1. You're boarding a spaceship with a position and velocity vectors relative to the sun equal to the values below. The vectors have the ecliptic plane as the reference plane (xy-plane) and the vernal equinox as the reference point (x-axis).

$$\vec{r} = \begin{bmatrix} -2.12 \\ 0.45 \\ 0.86 \end{bmatrix} AU; \vec{v} = \begin{bmatrix} 1.78 \\ -18.73 \\ 5.23 \end{bmatrix} \text{km/s}$$

   You know you will pass through the orbit's perihelion on June 21. Calculate the Earth's ecliptic coordinate relative to the spaceship at the perihelion! And which constellation is the Earth in? (Assume the Earth has a circular orbit with zero inclination)

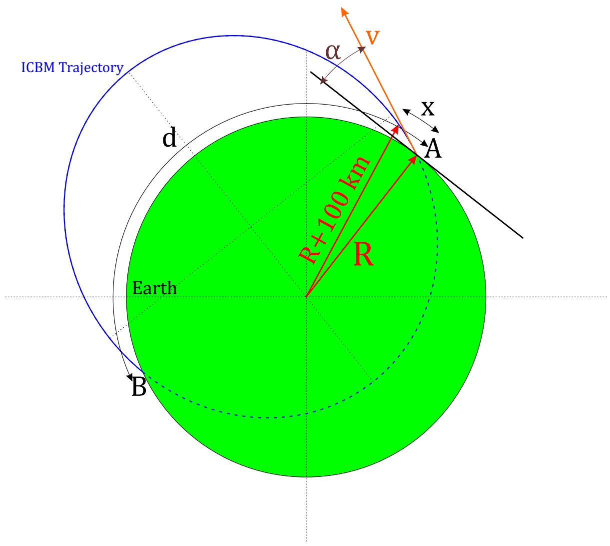
2. A space junk of significant size is identified orbiting the Earth. The position and velocity vector of the space junk are

$$\vec{r} = \begin{bmatrix} -4102.12 \\ 5398.58 \\ 898.12 \end{bmatrix} \text{km}; \vec{v} = \begin{bmatrix} 6.22 \\ 4.59 \\ 0.99 \end{bmatrix} \text{km/s}$$

   The ISS orbit has a semi-major axis value of 6794.47km, eccentricity of 0.0002, inclination of $51.64°$, longitude of ascending node of $290.32°$, and argument of periapsis of $257.44°$. To analyse whether this space junk will impact the ISS, calculate the distance between the space junk orbit and the ISS orbit at the intersection node! (the line that is located in both orbital planes).

3. Countries A and B are in a war. Country A wants to nuke down Country B with an intercontinental ballistic missile (ICBM). The distance between the two countries is $d = 12000$km. To avoid interfering with another country's aerospace defence, the ICBM trajectory must have an altitude of

over 100 km when the missile is above another country. An ICBM trajectory can be approximated as an elliptical orbit, as we see in the figure below.



If the distance between the launch site and the closest border within the ICBM trajectory is x = 300km, calculate the velocity needed (v) and the launch angle (α) to launch the ICBM! (Assume the Earth is perfectly spherical and stationary and ignore other forces other than gravity)

8 Level 8: Coordinate Transformation

We will incorporate your positional astronomy skills with celestial mechanics in this level. The coordinate transformation is important in celestial mechanics because we want to know where the orbiting object will go in an orbit and predict its position in a coordinate frame.

8.1 Orbital Elements in Spherical Coordinates

In positional astronomy, we are already familiar with spherical coordinates. In this section, we will discuss how to convert the orbital elements of an orbit into spherical coordinates.

In spherical coordinates, we need to define the reference plane and reference point to represent an object's position. The orbital elements are also tied to the reference point and plane; by changing the reference point or plane, the orbital elements will also be altered.

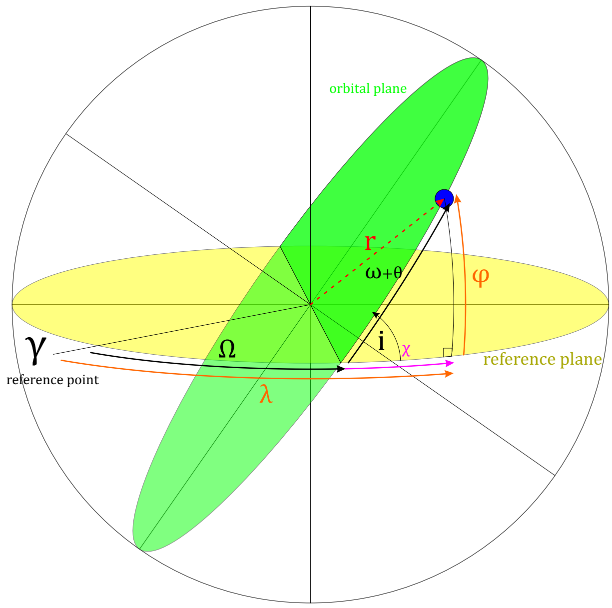


Figure 30: *Orbital coordinate transformation*

Figure 30 shows the visualisation of the orbital elements in a spherical coordinate system. As we know, to represent an object's position in the spherical coordinate system, we can represent it by its longitude (λ) and latitude (φ).

Using the spherical triangle formed in Figure 30, we can use the sine rule to get the latitude value as in the equation below.

$$\begin{aligned} \frac{\sin\varphi}{\sin i} &= \frac{\sin(\omega+\theta)}{\sin 90^\circ} \\ \sin\varphi &= \sin(\omega+\theta)\sin i \end{aligned} \tag{119}$$

And to calculate the longitude, we can use the four-part formula of the spherical triangle.

$$\begin{aligned} \cos\chi\cos i &= \sin\chi\cot(\omega+\theta) - \sin i\cot 90^\circ \\ \tan\chi &= \frac{\cos i}{\cot(\omega+\theta)} \\ \lambda &= \Omega + \chi \end{aligned} \tag{120}$$

Both of those equations above are the backbone of this level. For the next sections, we will use those equations to apply to a variety of cases.

8.2 Earth's Satellite Orbit and Geocentric Coordinates

In this section, we will learn how to calculate an orbiting object's geocentric coordinate. There are two coordinate systems that we will discuss in this section: equatorial and geographic.

Earth's satellite orbital elements are usually determined with the equator as the reference plane and the vernal equinox point as the reference point. If you are familiar with it, it's an equatorial coordinate system that we have already familiar with.

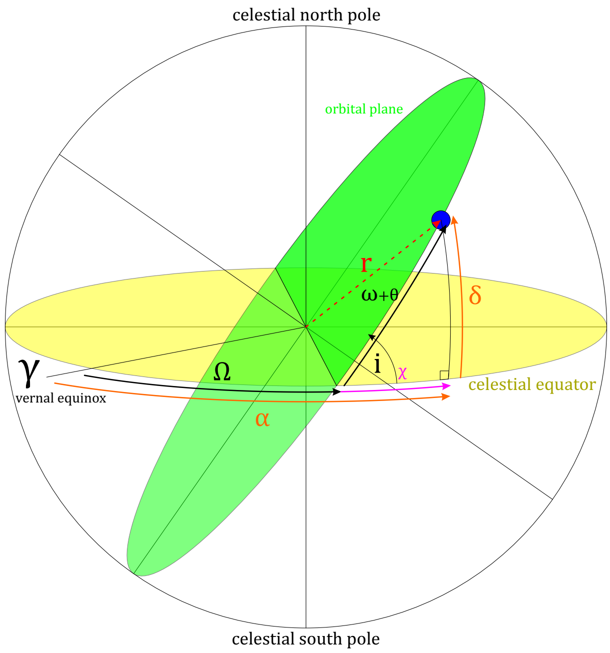


Figure 31: *Orbital elements in equatorial coordinate system*

Utilising the spherical triangle, we can get the value of α and δ in Figure 31 with this equation.

$$\begin{aligned} \sin\delta &= \sin(\omega+\theta)\sin i \\ \alpha &= \Omega + \arctan\frac{\cos i}{\cot(\omega+\theta)} \end{aligned} \tag{121}$$

Sometimes, a satellite can also be represented by its geographic coordinates, especially for the geostationary satellite, it's much more common to represent its position with the satellite's geographic coordinates. But as we know, the reference point of geographic coordinates (null island)[12] is always rotating following the rotation of the Earth. Meanwhile, the orbit of the satellite isn't rotating with the Earth.[13]

One of the methods to calculate the satellite's geographic coordinates is to find the right ascension of the null island (α_{GMT}). The reference plane for geographic and equatorial coordinate systems is the same because of that, we only need the right ascension angle of the reference point for the geographic coordinate system.

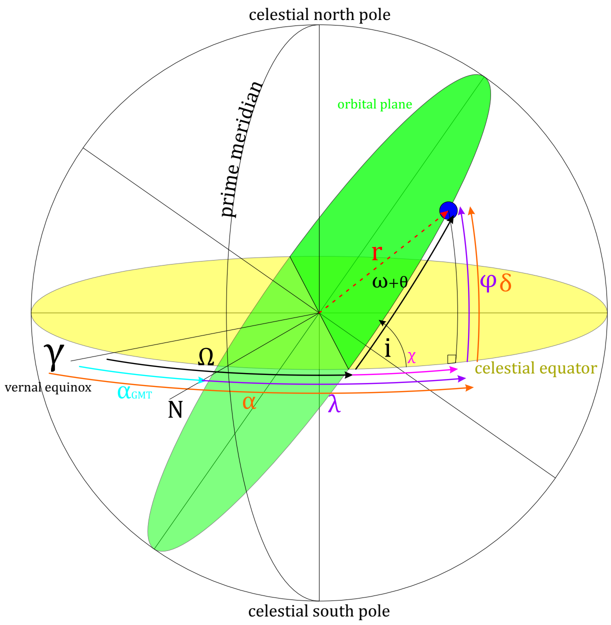


Figure 32: *Orbital elements in equatorial and geographic coordinate systems*

As we can see from Figure 32, the relation between geographic longitude (λ) and right ascension (α) can be represented with this equation

$$\begin{aligned} \lambda &= \alpha - \alpha_{\text{GMT}} \\ \varphi &= \delta \end{aligned} \tag{122}$$

To calculate the null point's right ascension, we can use the Earth's rotation angle. If the time when the null point is facing the vernal equinox at t_0, the value of α_{GMT} after t time can be calculated using equation 123.

$$\begin{aligned} \dot{\alpha}_{\text{GMT}} &= \frac{2\pi}{23^h 56^m} \\ \alpha_{\text{GMT}} &= \dot{\alpha}_{\text{GMT}}(t - t_0) \end{aligned} \tag{123}$$

[12] the longitude is the same as GMT, but the latitude is zero, that's why it's called null island

[13] what I mean is the satellite orbital elements (apart from true anomaly) don't change as the Earth is rotating

8.3 Planet Orbit and Ecliptical Coordinates

For a planet orbit, the orbital elements are usually stated relative to the ecliptical coordinate system. To calculate the ecliptical coordinates, we can use a similar equation as before.

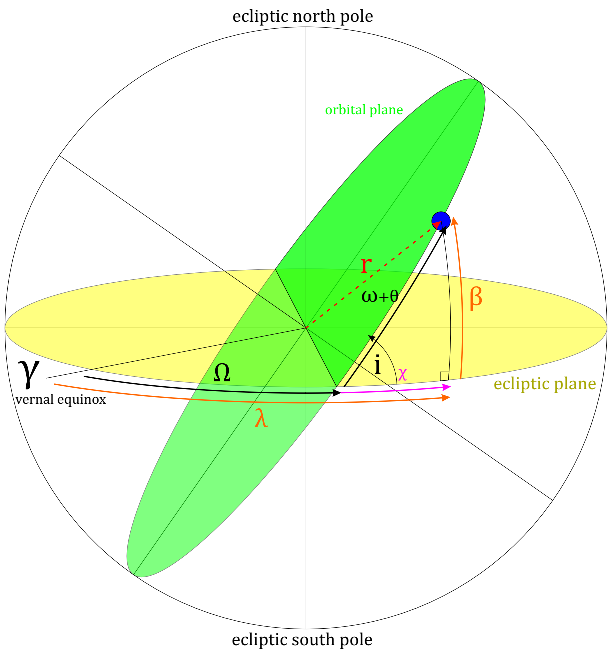


Figure 33: *Orbital elements in ecliptic coordinate system*

$$\begin{aligned} \sin\beta &= \sin(\omega+\theta)\sin i \\ \lambda &= \Omega + \arctan\frac{\cos i}{\cot(\omega+\theta)} \end{aligned} \tag{124}$$

We can convert the orbital elements between the references by converting the object's coordinates.

8.4 Exoplanet and Viewing Plane

If we observe a distant object, such as an exoplanet or a binary star, we can use any frame of reference to determine the orbit. However, defining the orbit will be difficult because the reference point or plane is hard to calculate.

One of the solutions is to make the viewing plane the reference frame. The viewing plane is a plane that is perpendicular to the viewing line. To detect an exoplanet, the method will be different with different orbit relative to the viewing plane.

As we can see from Figure 34, the z direction in the viewing plane reference frame is pointed to the observer. A radial velocity is one of the easiest data points we can get from an exoplanet. The radial velocity is tied to the inclination between the orbital and viewing planes. The higher the inclination,

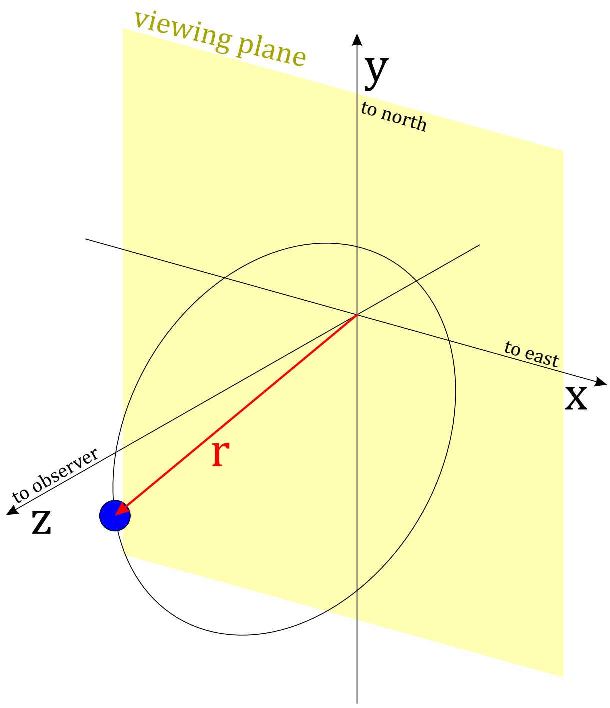


Figure 34: *Orbit in a viewing plane*

the higher the maximum radial velocity that we can detect, with an orbit with $90°$ inclination (edge on) having the highest radial velocity compared to others.

8.5 Exercise Problem

1. A satellite is observed in Medan $(03°35'N, 98°40'E)$ and Purwakarta $(6°33'S, 107°26'E)$. From the observation data, we can get the distance from the observer to the satellite and the satellite's horizontal coordinates from each observer. The satellite orbit is a perfect circle $(e = 0)$ and has a

Observer	Distance (km)	horizontal coordinates (deg)
Medan	1292.028	Az = 137.19 ; alt = 30.886
Purwakarta	915.163	Az = 323.02 ; alt = 52.747

positive inclination $(i > 0)$. It is known that during observation, the satellite will pass through its ascending node after 3 minutes. Determine the satellite geographic coordinates during observation and the period, semi-major axis, inclination, and longitude of the ascending node!

2. A sun-synchronous orbit is a type of orbit where the orbit is purposely perturbed so that the satellite will be at the same location during the specific local time. This orbit is especially useful for an Earth observation satellite because the image will have consistent lighting from the Sun. The angular precession per orbit $\Delta\Omega$ for an Earth-orbiting satellite is approximately

$$\Delta\Omega = -3\pi \frac{J_2 R_\oplus^2}{p^2} \cos i$$

Where $R_\oplus$ is the Earth's mean radius, p is the orbit's semi-latus rectum, i is the orbital inclination, and J_2 is a coefficient for perturbation, where the value is

$$J_2 = 1.08263 \times 10^{-3}$$

Suppose we want to design a circular orbit for an Earth-observing satellite with a period of 96 minutes. We want to consistently capture Tokyo (35°41′23″N 139°41′32″E) with the satellite at 10.00 AM GMT+8. Calculate the satellite's inclination, semi-major axis, and the longitude of the ascending node during the vernal equinox!

3. An exoplanet is detected to have position and velocity vectors in the viewing plane coordinate system.

$$\vec{r} = \begin{bmatrix} 0.234 \\ 0.128 \\ -0.034 \end{bmatrix} AU, \quad \vec{v} = \begin{bmatrix} -15.33 \\ -24.01 \\ 57.86 \end{bmatrix} \text{km/s}$$

The exoplanet's equatorial coordinates are

$$\alpha = 20^h 45^m 56^s, \delta = 65°57'01''$$

Calculate the exoplanet's orbital elements if the reference plane is the equatorial plane!

9 Level 9: Transfer Orbit

We have learn that an orbit can be solved by having the position and velocity vector of the orbiting object. Sometimes, we should control these elements to put the orbiting object into the intended orbit.

We can change an orbit by manipulating its position or velocity vector. In an orbit, we couldn't change an object's position directly, but we can change the velocity vector by giving the object more velocity (Δv) with a thruster. The direction of the thrust will determine which velocity vector components have changed and further change the orbital elements.

$$\vec{v}_f = \vec{v}_i + \Delta\vec{v} \tag{125}$$

9.1 Coplanar Manoeuvre

A coplanar manoeuvre will result in the change of orbital elements in the orbital plane. The semi-major axis, eccentricity, argument of periapsis, and true anomaly lie on the orbital plane. Note that in an orbital plane, the direction of $\vec{h}$ is always perpendicular to the $\vec{r}$ and $\vec{v}$. This means that for a coplanar manoeuvre, the change of velocity $(\Delta\vec{v})$ should also be perpendicular to the $\vec{h}$, otherwise the direction of $\vec{h}$ will change and thus the orbital plane will change.

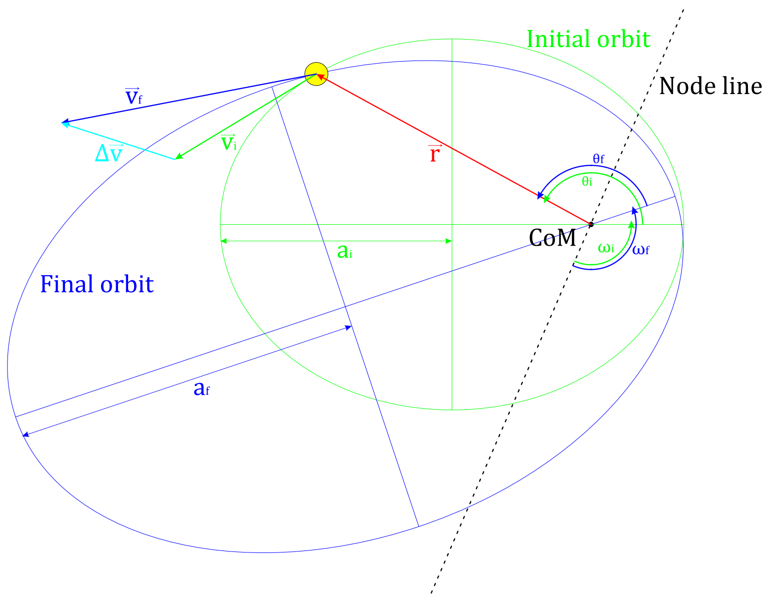


Figure 35: *Coplanar manoeuvre visualisation*

By adding $\Delta\vec{v}$ to the orbiting object, the value of orbital kinetic energy will also change. Because there is no change in the position vector, the potential energy will stay the same; thus, the total orbital energy for the orbiting object will change.

$$\varepsilon_f = \frac{1}{2}(\vec{v}_i + \Delta\vec{v})^2 - \frac{\mu}{r} \tag{126}$$

The final orbit's semi-major axis and eccentricity values can be calculated using equation 127.

$$
\begin{aligned}
a_f &= -\frac{\mu}{2\varepsilon_f} \\
h_f^2 &= \mu a_f(1-e_f^2) \\
e_f &= \sqrt{1-\frac{h_f^2}{\mu a_f}}
\end{aligned}
\tag{127}
$$

If we look at the Figure 35, we can see that the position vector and node line are constant. We know that the angle between the position vector and the node line is the sum of the argument of periapsis and true anomaly. We can calculate the true anomaly value by using the conic equation. The final true anomaly and argument of periapsis values can be calculated using equation 128.

$$
\begin{aligned}
r &= \frac{p_f}{1+e_f\cos\theta_f} \\
\cos\theta_f &= \frac{a_f(1-e_f^2)-r}{e_f} \\
\omega_f+\theta_f &= \omega_i+\theta_i
\end{aligned}
\tag{128}
$$

We already covered the basis of the coplanar manoeuvre, but we are limited to a certain orbit with one manoeuvre. The fact that we can't change the position vector directly means that the final conic we create with a coplanar manoeuvre will always have two intersecting points with the initial orbit.

To cover the problem, we can combine two or more manoeuvres to achieve the intended orbit. The first manoeuvre changes the initial orbit into the transfer orbit, while the last manoeuvre changes the transfer orbit into the final intended orbit. The core of changing an orbit is to choose suitable transfer orbit(s) to maximise or minimise certain variables, usually the most important is the total change in velocity (Δv_{tot}). This Δv_{tot} is defined as the sum of the absolute value of each velocity change. This variable is important because the main source to add Δv is fuel that is carried by the orbiting object, and we can't bring an infinite amount of fuel to space.

$$
\Delta v_{\text{tot}} = \sum_{i=1}^{n} |\Delta\vec{v}_i| \tag{129}
$$

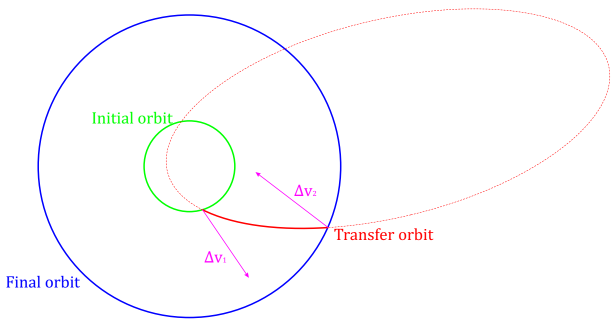


Figure 36: *Transfer orbit design*

9.1.1 Hohmann Transfer Orbit

Hohmann transfer orbit is one of the common ways to design a transfer orbit. As we see in Figure 37, the transfer orbit is an ellipse with the periapsis value that is the same as the initial orbit, and the apoapsis value that is the same as the final orbit. Two manoeuvres are needed to complete a Hohmann transfer orbit, the first is at the orbit periapsis and the second is at the orbit apoapsis. During transfer, the orbiting object will complete half an ellipse of the transfer orbit.

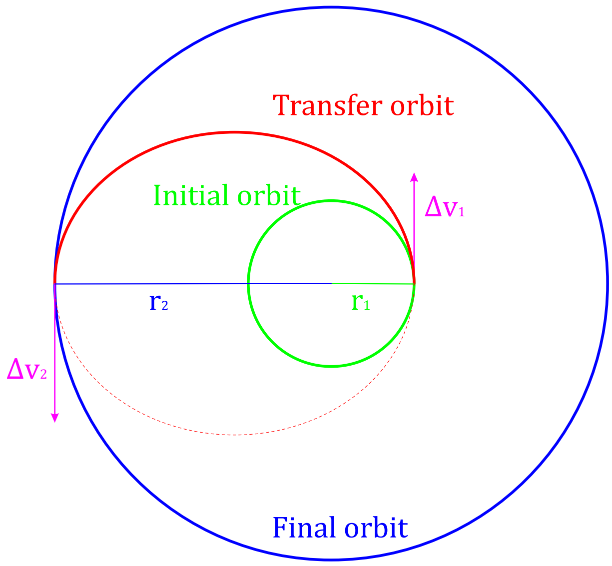


Figure 37: *Hohmann transfer orbit visualised*

From Figure 37, we know that the Hohmann transfer orbit will have a semi-major axis value of,

$$\begin{aligned} a_{\text{trn}} &= \frac{r_1 + r_2}{2} \\ e_{\text{trn}} &= 1 - \frac{r_1}{a_{\text{trn}}} = \frac{r_2 - r_1}{r_1 + r_2} \end{aligned} \tag{130}$$

The total transfer time for this case can be calculated using half the period of the transfer orbit's ellipse.

$$\begin{aligned} \Delta t_{\text{trn}} &= \frac{1}{2} T \\ \Delta t_{\text{trn}} &= \pi \sqrt{\frac{a_{\text{trn}}^3}{\mu}} \end{aligned} \tag{131}$$

We can calculate the $\Delta \vec{v}_1$ and $\Delta \vec{v}_2$ values using equation 125. Remember that in a Hohmann orbit, point 1 is always located at the periapsis and point 2 is always at the apoapsis; thus, the velocity vector will only have a tangential direction at both points. The velocity in this case can be calculated using equation 75 and 76.

$$
\begin{aligned}
\Delta\vec{v}_1 &= \vec{v}_p - \vec{v}_i \\
\Delta\vec{v}_1 &= \sqrt{\frac{\mu(1+e_{\text{trn}})}{a_{\text{trn}}(1-e_{\text{trn}})}}\hat{\theta}_1 - \sqrt{\frac{\mu}{a_i(1-e_i^2)}}(1+e_i\cos\theta_i)\hat{\theta}_1 - \sqrt{\frac{\mu}{a_i(1-e_i^2)}}e_i\sin\theta_i\hat{r}_1 \\
\\
\Delta\vec{v}_2 &= \vec{v}_f - \vec{v}_a \\
\Delta\vec{v}_2 &= \sqrt{\frac{\mu}{a_f(1-e_f^2)}}(1+e_f\cos\theta_f)\hat{\theta}_2 + \sqrt{\frac{\mu}{a_f(1-e_f^2)}}e_f\sin\theta_f\hat{r}_2 - \sqrt{\frac{\mu(1-e_{\text{trn}})}{a_{\text{trn}}(1+e_{\text{trn}})}}\hat{\theta}_2
\end{aligned}
\tag{132}
$$

If both initial and final orbits are circles, which is the common problem for this application, we can calculate the $\Delta\vec{v}_1$ and $\Delta\vec{v}_2$ values,

$$
\begin{aligned}
\Delta\vec{v}_1 &= \left(\sqrt{\frac{\mu(1+e_{\text{trn}})}{a_{\text{trn}}(1-e_{\text{trn}})}} - \sqrt{\frac{\mu}{r_1}}\right)\hat{\theta}_1 \\
\Delta\vec{v}_1 &= \left(\sqrt{\frac{\mu}{r_1}\frac{2r_2}{r_1+r_2}} - \sqrt{\frac{\mu}{r_1}}\right)\hat{\theta}_1 \\
\Delta\vec{v}_1 &= \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right)\hat{\theta}_1 \\
\\
\Delta\vec{v}_2 &= \left(\sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{\mu(1-e_{\text{trn}})}{a_{\text{trn}}(1+e_{\text{trn}})}}\right)\hat{\theta}_2 \\
\Delta\vec{v}_2 &= \left(\sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{\mu}{r_2}\frac{2r_2}{r_1+r_2}}\right)\hat{\theta}_2 \\
\Delta\vec{v}_2 &= \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_2}{r_1+r_2}}\right)\hat{\theta}_2
\end{aligned}
\tag{133}
$$

If both initial and final orbits are circles, the total change in velocity for a Hohmann transfer can be calculated with equation 134.

$$
\begin{aligned}
\Delta v_{\text{tot}} &= |\Delta\vec{v}_1| + |\Delta\vec{v}_2| \\
\Delta v_{\text{tot}} &= \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right) + \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_2}{r_1+r_2}}\right)
\end{aligned}
\tag{134}
$$

9.1.2 Bi-Elliptic Transfer Orbit

Another method to transfer orbit is using a bi-elliptic transfer orbit. This transfer orbit consumes less Δv_{trn} compared to the Hohmann transfer orbit in some cases.

The bi-elliptic method utilises the fact that we can change orbital elements with less velocity change at the apoapsis. This method launches the orbiting object into a very elliptical and high orbit with the apoapsis value of r_b. At the apoapsis, a velocity change is made to change the transfer orbit's periapsis into the same value as the final orbit. The final Δv changes the transfer orbit into the final orbit. This method takes much longer time compared to the Hohmann transfer orbit, but potentially reduces the Δv_{tot} needed.

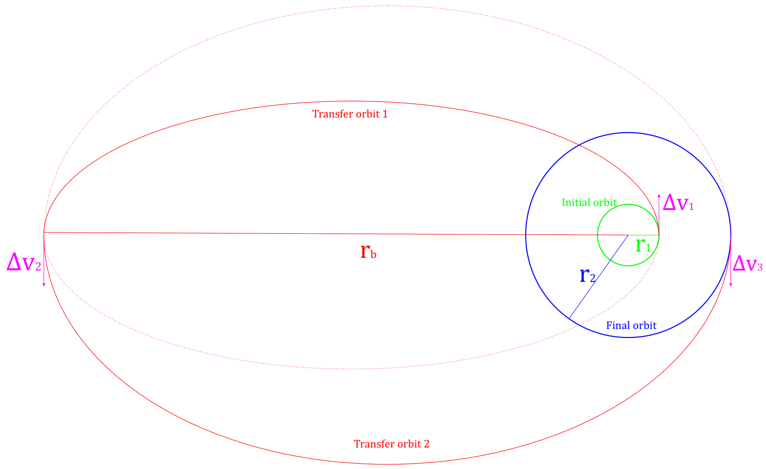


Figure 38: *Bi-elliptic transfer orbit visualised*

From Figure 38, we can calculate the orbital parameter for the bi-elliptic transfer orbit.

$$
\begin{aligned}
a_{\text{trn}_1} &= \frac{r_1 + r_b}{2} \\
e_{\text{trn}_1} &= 1 - \frac{r_1}{a_{\text{trn}_1}} = \frac{r_b - r_1}{r_1 + r_b} \\
\\
a_{\text{trn}_2} &= \frac{r_2 + r_b}{2} \\
e_{\text{trn}_2} &= 1 - \frac{r_2}{a_{\text{trn}_2}} = \frac{r_b - r_2}{r_2 + r_b}
\end{aligned}
\tag{135}
$$

The total transfer time for this case can be calculated similarly to Hohmann's transfer orbit.

$$
\begin{aligned}
\Delta t_{\text{trn}} &= \frac{1}{2} T_{\text{trn}_1} + \frac{1}{2} T_{\text{trn}_2} \\
\Delta t_{\text{trn}} &= \pi \sqrt{\frac{a^3_{\text{trn}_1}}{\mu}} + \pi \sqrt{\frac{a^3_{\text{trn}_2}}{\mu}}
\end{aligned}
\tag{136}
$$

If both initial and final orbits are circles, we can easily calculate the Δv for the bi-elliptic method.

$$
\begin{aligned}
\Delta v_1 &= \sqrt{\frac{\mu}{r_1}} \left(\sqrt{\frac{2r_b}{r_1 + r_b}} - 1 \right) \\
\Delta v_2 &= \sqrt{\frac{\mu}{r_b}} \left(\sqrt{\frac{2r_2}{r_2 + r_b}} - \sqrt{\frac{2r_1}{r_1 + r_b}} \right) \\
\Delta v_3 &= \sqrt{\frac{\mu}{r_2}} \left(\sqrt{\frac{2r_b}{r_2 + r_b}} - 1 \right)
\end{aligned}
\tag{137}
$$

9.2 Phasing

One of the common reasons why the transfer orbit is needed is that we want an orbiting object to arrive at another orbiting object, for example, launching a spacecraft into the Moon. Because the target is also orbiting the primary object and not stationary, we need to calculate when the manoeuvre is needed.

We want the object that is launched to meet the target at the same point. This means that we need to time the manoeuvre so that the target object will arrive at the same place as the launched object after the transfer is finished. This implies that before manoeuvring, the target object needs to be placed in a way so that after the transfer time has elapsed, the target is positioned at the final point of the transfer orbit.

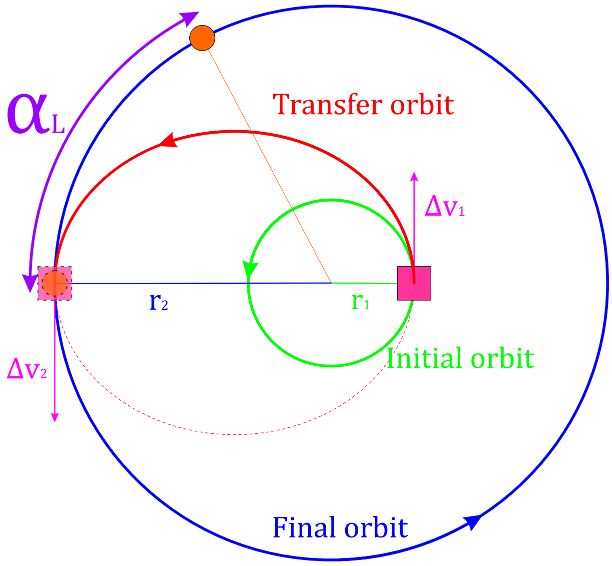


Figure 39: *Target position before launch*

As visualised in Figure 39, we need to calculate the leading angle before launching the object into the target(α_L). In this level, we will only cover a circular orbit; other shapes will be covered later. If the target follows a circular path, the leading angle can be calculated using equation 138.

$$\begin{aligned} \alpha_L &= \frac{2\pi}{T_2} t_{\text{trn}} \\ \alpha_L &= \sqrt{\frac{\mu}{r_2^3}} t_{\text{trn}} \end{aligned} \tag{138}$$

If we want to change an object's position in a circular orbit, we can utilise a technique called circular rendezvous. This method utilises the difference in orbital period between the transfer orbit and the initial orbit to reach the target.

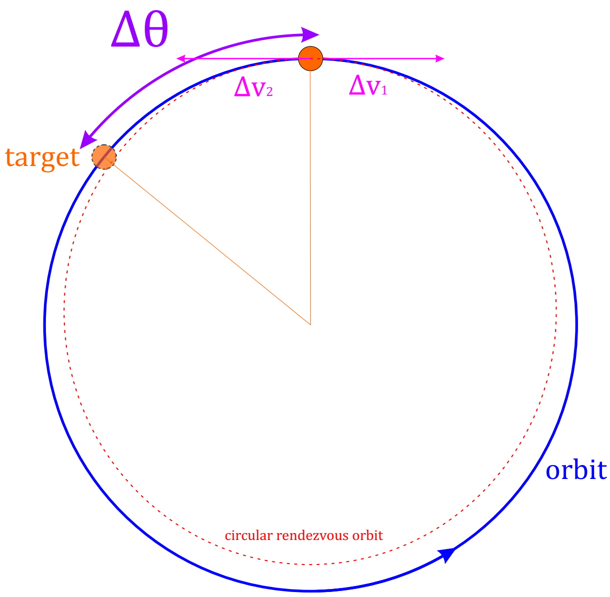


Figure 40: *Circular rendezvous visualisation*

From Figure 40, we can see that to meet the target, the angle that the target can sweep needs to follow this equation

$$\begin{aligned} \omega_1 t_{\text{trn}} &= 2\pi n - \Delta\theta \\ t_{\text{trn}} &= \frac{2\pi n - \Delta\theta}{\omega_1} \end{aligned} \tag{139}$$

n is a natural number representing how many revolutions the target object makes before meeting the object. This corresponds with how long the circular rendezvous time is. This n also corresponds with the change in velocity; the higher n value, the less Δv needed to do a circular rendezvous. The circular rendezvous orbit has a period of T_{cr}, the circular rendezvous orbit parameter can be calculated using equation 140.

$$\begin{aligned} a_{\text{cr}}^3 &= \frac{\mu}{4\pi^2} T_{\text{cr}}^2 \\ a_{\text{cr}}^3 &= \frac{\mu}{4\pi^2} \left(\frac{t_{\text{trn}}}{n}\right)^2 \end{aligned} \tag{140}$$

The total change in velocity can be calculated using this equation

$$\begin{aligned} \Delta v &= 2|v_{\text{cr}} - v_1| \\ \Delta v &= 2\left|\sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_{\text{cr}}}\right)} - \sqrt{\frac{\mu}{r_1}}\right| \end{aligned} \tag{141}$$

9.3 Non-Coplanar Manoeuvre

We have already discussed the method of performing manoeuvres in the same orbital plane. However, sometimes, we need to alter the orbital plane to fulfil the mission requirements. To change the orbital plane, we need to do a non-coplanar manoeuvre.

We already know that the intersection of two planes will form a line. This is the main concept of the non-coplanar manoeuvre. For a non-coplanar manoeuvre, the velocity change must happen within the intersection of the initial and the final orbit.

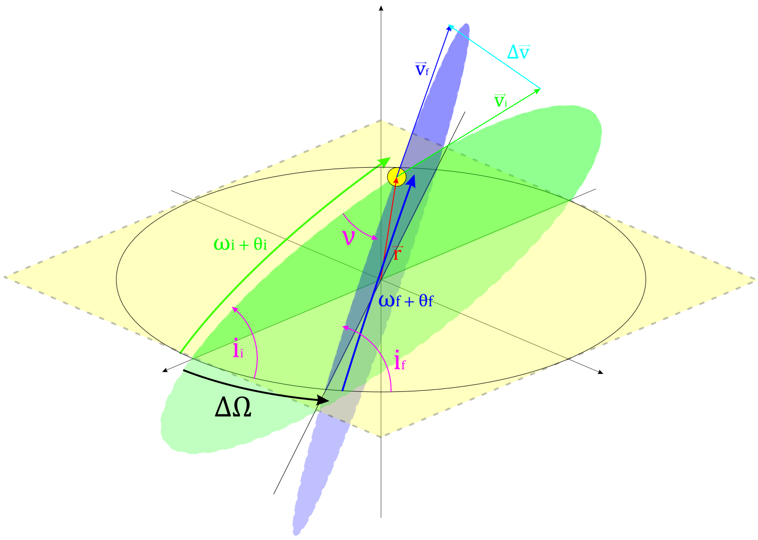


Figure 41: *Non-coplanar manoeuvre visualisation*

To only change an object's orbital plane without other elements, we need only to change its velocity vector's direction without changing its value. As we can see from Figure 41, if the value of v_i and v_f is the same, we can get the angle between the two velocity vectors (ν) to calculate the change in Ω and i.

$$
\begin{aligned}
\Delta v^2 &= v_i^2 + v_f^2 - 2v_i v_f \cos\nu \\
\Delta v^2 &= 2v_i^2(1 - \cos\nu) \\
\Delta v &= 2v_i \sin\frac{1}{2}\nu
\end{aligned}
\tag{142}
$$

Using the spherical triangle formula, we can get the relation between ν, i, and Ω as seen in Figure 41. If we know the initial orbital elements, we can calculate the change in Ω value by using the four-part formula of a spherical triangle.

$$
\begin{aligned}
\cos(\omega_i + \theta_i)\cos i_i &= \sin(\omega_i + \theta_i)\cot\Delta\Omega - \sin i_i \cot\nu \\
\cot\Delta\Omega &= \frac{\cos(\omega_i + \theta_i)\cos i_i + \sin i_i \cot\nu}{\cos(\omega_i + \theta_i)}
\end{aligned}
\tag{143}
$$

To calculate the final inclination, we can do it using the sine equation of a spherical triangle

$$
\begin{aligned}
\frac{\sin(\pi - i_f)}{\sin(\omega_i + \theta_i)} &= \frac{\sin\nu}{\sin\Delta\Omega} \\
\sin i_f &= \frac{\sin\nu \sin(\omega_i + \theta_i)}{\sin\Delta\Omega}
\end{aligned}
\tag{144}
$$

9.4 Interplanetary Transfer

Interplanetary transfer is a complex field because transferring an object between two or more primary bodies will involve the three-body problem, which doesn't have an analytical solution. However, if the orbiting object has far less mass compared to both its primary objects, the problem can be simplified into two separate two-body problems. In this section, we will discuss the technique that can be used to calculate an interplanetary transfer orbit.

To tackle the problem, we need to separate the 'interplanetary transfer' into two separate problems. The first problem is to transfer the object to 'escape' the first primary body. The second problem is to transfer the object from the initial primary body to the target by using the second body's gravity.

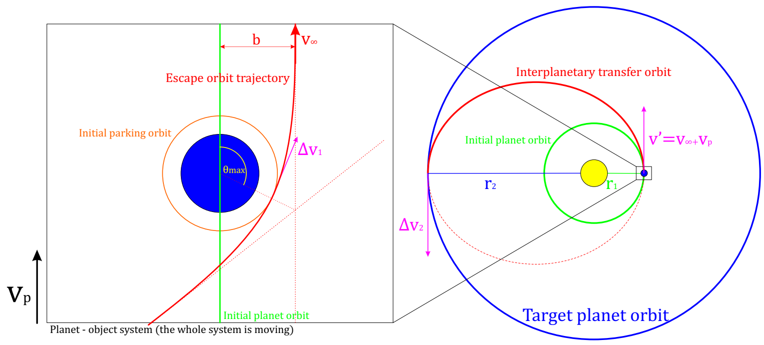


Figure 42: *Interplanetary transfer orbit approximation*

As we learn from the previous levels, the orbiting object should form a parabolic or hyperbolic trajectory to escape the primary object. As we can see from Figure 42, the first step should be escape from the 'planet'. This requires the $\Delta v_1 + v_1$ to be at least the escape velocity from the initial parking orbit.

However, we must calculate the required velocity to transfer between planets (v') to design the escape orbit. To calculate v', we can use the Hohmann transfer orbit equation as in equation 133. After we get the v' value, we can calculate the excess velocity that is required for the transfer. As we see in Figure 42, if we design such an orbit so that the excess velocity is in the same direction as the planet's movement, the relation between v', the planet's velocity (v_p), and the excess velocity (v_∞) can be calculated using this

$$v' = v_p + v_\infty \tag{145}$$

After we get the v_∞ value, we can design the escape orbit for the object. One of the most common ways is to design the escape orbit so that the escape orbit's periapsis is located at the initial parking orbit. To calculate the Δv_1 needed in this case, we can utilise the h vector once again. If the parking orbit is in the same plane as the planet orbit, and has the radius of r_1, and the periapsis velocity of the escape orbit is v_{peri}, we can calculate the Δv_1 with this equation.

$$\begin{aligned} h &= r_1 v_{peri} = b v_\infty \\ v_{peri} &= \frac{b}{r_1} v_\infty \\ v_{peri} &= \sqrt{\frac{e+1}{e-1}} v_\infty \\ \Delta v_1 &= v_{peri} - v_{park} \end{aligned} \tag{146}$$

To make sure the excess velocity vector is in the same direction as the planet, we need to launch the object at an angle θ_{max} to the planet's velocity vector. The value of θ_{max} can be calculated using equation 26 and 27.

9.4.1 Slingshot effect

For an interplanetary transfer, the requirement for Δv is much larger. So, rather than bringing more fuel to do a manoeuvre, we can utilise another planet's gravitational field. By entering one planet's gravitational field, we can control the orbiting object's velocity relative to another primary object (The Sun). One of the most famous examples is the Voyager 1 and Voyager 2 missions that enabled them to escape the solar system with gravitational assistance from outer planets.

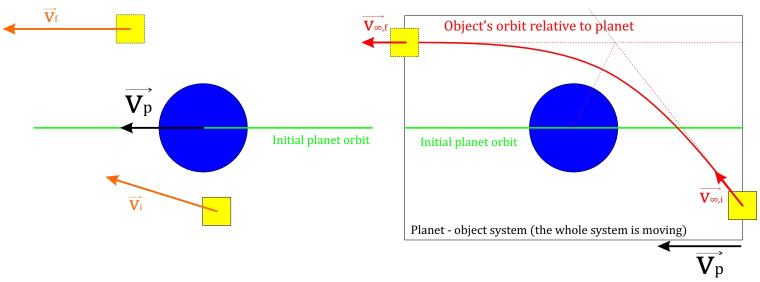


Figure 43: *Slingshot effect visualised*

We have already learn that we can approach a three-body problem by isolating the orbiting object with one primary body. The main concept of the slingshot effect is relative movement. As we can see from Figure 43, if an object is approaching a primary object (in this case, a planet), relative to the planet itself, the object is approaching the planet with a hyperbolic orbit, with an excess velocity of v_∞. The excess velocity vector can be calculated using this equation.

$$\vec{v_{\infty,i}} = \vec{v_i} - \vec{v_p} \tag{147}$$

Because of the symmetrical identity of a hyperbola, we know that the value of the final excess velocity should be the same as the initial excess velocity. The only difference is the direction of it. Because of this, the final velocity of the object is

$$\vec{v_f} = \vec{v_{\infty,f}} + \vec{v_p} \tag{148}$$

The initial velocity of an object can heavily influence the slingshot effect. Not only that, but the slingshot effect is also affected by the timing and the object's position relative to the planet. If the object is leading the planet, the object's velocity will be different compared to if the planet is leading, as we see in Figure 44.

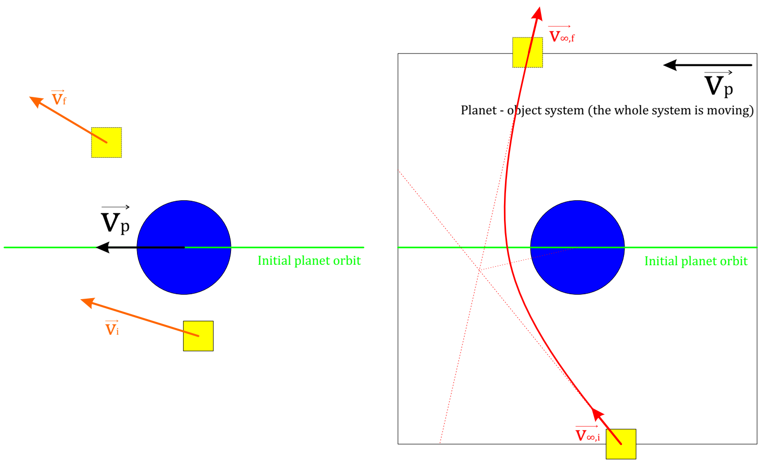


Figure 44: *Leading slingshot effect visualised*

9.5 Exercise Problem

1. Prove that the minimum inclination of an orbit during the launch is the launch place geographic latitude! Or

$$i_{\text{launch}} \geq \phi_{\text{launch}}$$

2. We have a satellite with the initial inclination and ascending node of,

$$i_i = 70.1°, \quad \Omega_i = 267.4°$$

We want to change the inclination and the ascending node to

$$i_f = 97.7°, \quad \Omega_f = 154.2°$$

Where are the two points to transfer an orbit from the initial orbit to the final orbit?

3. An alien ship is approaching Earth in a hyperbolic orbit. The position and velocity vector at one time relative to the ecliptic plane are

$$\vec{r} = \begin{bmatrix} 689,456 \\ -245,111 \\ 367,576 \end{bmatrix} \text{km}, \quad \vec{v} = \begin{bmatrix} -0.92 \\ 0.85 \\ -0.56 \end{bmatrix} \text{km/s}$$

If the alien ship escapes Earth on January 31, calculate its orbital elements after it escapes Earth and orbits the Sun!

10 Level 10: Position Over Time

In this level, we will derive the equation for an object's position over time for every type of conic. The relation between position and time is difficult to derive because it includes some integrals that are difficult to solve.

From the specific angular momentum vector, we can get the relation between true anomaly (θ) and time elapsed (t) in this equation.

$$\begin{aligned} r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \sqrt{\mu p} \\ \mathrm{d}t &= \sqrt{\frac{p^3}{\mu}}\frac{\mathrm{d}\theta}{(1+e\cos\theta)^2} \\ \int_\tau^t \mathrm{d}t = t-\tau &= \int_0^\theta \sqrt{\frac{p^3}{\mu}}\frac{\mathrm{d}\theta}{(1+e\cos\theta)^2} \end{aligned} \tag{149}$$

The solution for the integral in equation 149 varies with the value of eccentricity (e). We will cover the solution for each type of orbit.

10.1 Circle

The solution for a circle is easy because the e value in equation 149 is equal to zero. The integral can be rewritten as in equation 150.

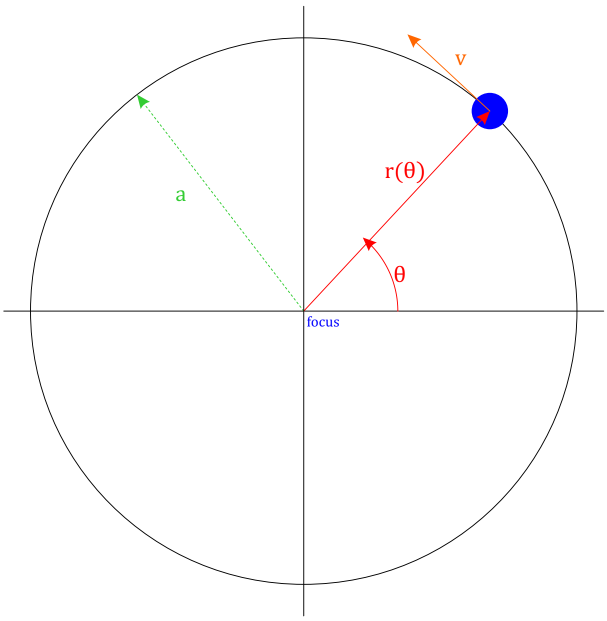


Figure 45: *Object position over time in a circle*

$$
\begin{aligned}
\int_{\tau}^{t} \mathrm{d}t = t - \tau &= \int_{0}^{\theta} \sqrt{\frac{p^3}{\mu}} \mathrm{d}\theta \\
\theta(t) &= \frac{v}{a}(t - \tau)
\end{aligned} \tag{150}
$$

We know that the value of $\frac{v}{a}$ in a circle is equal to the angular velocity of the orbit.

10.2 Ellipse

To solve the ellipse equation, we need to introduce another parameter called eccentric anomaly (E) to simplify the position equation.

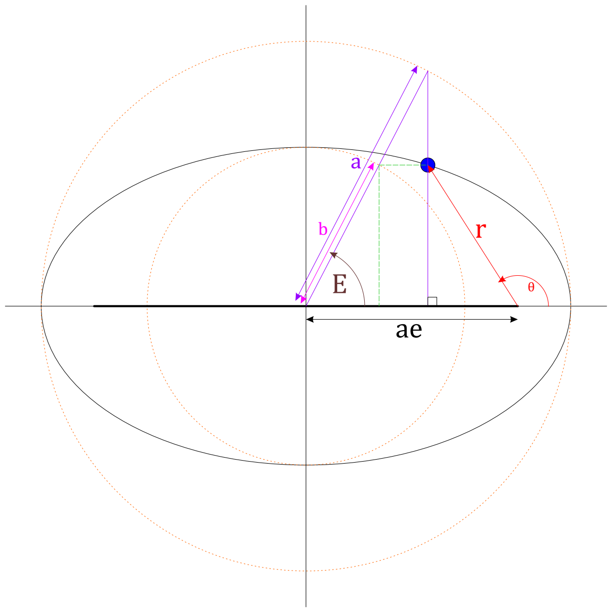


Figure 46: *Eccentric anomaly visualisation*

From Figure 46, we can calculate r with this equation.

$$
\begin{aligned}
\vec{r} &= (a\cos E - ae)\hat{x} + b\sin E\hat{y} \\
r^2 &= a^2\cos^2 E + a^2e^2 - a^2e\cos E + a^2(1-e^2)\sin^2 E \\
r^2 &= a^2(1 - e\cos E + e^2\cos^2 E) \\
r &= a(1 - e\cos E)
\end{aligned} \tag{151}
$$

We need to find the relation between eccentric anomaly E and true anomaly θ. We can do this by substituting equation 151 for equation 13.

$$
\begin{aligned}
a(1 - e\cos E) &= \frac{a(1-e^2)}{1 + e\cos\theta} \\
(1 + e\cos\theta)(1 - e\cos E) &= 1 - e^2
\end{aligned} \tag{152}
$$

The result will be

$$
\begin{aligned}
\cos\theta &= \frac{\cos E - e}{1 - e\cos E}\\
\sin\theta &= \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}
\end{aligned}
$$

$$
\begin{aligned}
\cos E &= \frac{\cos\theta + e}{1 + e\cos\theta}\\
\sin E &= \frac{\sqrt{1-e^2}\sin\theta}{1 + e\cos\theta}
\end{aligned}
\tag{153}
$$

The relation between θ and E can be simplified further using the tan half-angle identity.

$$
\begin{aligned}
\tan\frac{1}{2}\theta &= \frac{1-\cos\theta}{\sin\theta}\\
\tan\frac{1}{2}\theta &= \frac{1+e-e\cos E-\cos E}{\sqrt{1-e^2}\sin E}\\
\tan\frac{1}{2}\theta &= \frac{(1+e)(1-\cos E)}{\sqrt{1-e^2}\sin E}\\
\tan\frac{1}{2}\theta &= \sqrt{\frac{1+e}{1-e}}\frac{(1-\cos E)}{\sin E}\\
\tan\frac{1}{2}\theta &= \sqrt{\frac{1+e}{1-e}}\tan\frac{1}{2}E
\end{aligned}
\tag{154}
$$

We need to find the relation between the derivative of θ and E to solve equation 149.

$$
\begin{aligned}
\left(\frac{1}{1+\cos\theta}\right)\mathrm{d}\theta &= \sqrt{\frac{1+e}{1-e}}\left(\frac{1}{1+\cos E}\right)\mathrm{d}E\\
\mathrm{d}\theta &= \sqrt{\frac{1+e}{1-e}}\left(\frac{e+e\cos\theta}{e+e\cos E}\right)\mathrm{d}E\\
\mathrm{d}\theta &= \sqrt{\frac{1+e}{1-e}}\left(\frac{(e-1)+(1+e\cos\theta)}{(e+1)-(1-e\cos E)}\right)\mathrm{d}E\\
\mathrm{d}\theta &= \sqrt{\frac{1+e}{1-e}}\left(\frac{(e-1)+\frac{a(1-e^2)}{r}}{(e+1)-\frac{r}{a}}\right)\mathrm{d}E\\
\mathrm{d}\theta &= \sqrt{\frac{1+e}{1-e}}\left(\frac{a}{r}\frac{r(e-1)+a(1-e^2)}{a(e+1)-r}\right)\mathrm{d}E\\
\mathrm{d}\theta &= \sqrt{\frac{1+e}{1-e}}\left(\frac{a}{r}\frac{(e-1)(a(e+1)-r)}{a(e+1)-r}\right)\mathrm{d}E\\
\mathrm{d}\theta &= \frac{\sqrt{1-e^2}a}{r}\mathrm{d}E
\end{aligned}
\tag{155}
$$

Putting the result into equation 149, we get

$$
\begin{aligned}
\mathrm{d}t &= \sqrt{\frac{p^3}{\mu}} \frac{a\sqrt{1-e^2}\mathrm{d}E}{r(1+e\cos\theta)^2} \\
\mathrm{d}t &= \frac{a\sqrt{1-e^2}r}{\sqrt{\mu p}}\mathrm{d}E \\
\mathrm{d}t &= \frac{a^2\sqrt{1-e^2}(1-e\cos E)}{\sqrt{\mu a(1-e^2)}}\mathrm{d}E \\
\sqrt{\frac{\mu}{a^3}}\mathrm{d}t &= (1-e\cos E)\mathrm{d}E
\end{aligned}
\tag{156}
$$

From Kepler's third law, we know that the value of $\sqrt{\frac{\mu}{a^3}}$ is equal to $\frac{2\pi}{T}$. With this information, we can introduce a new variable called mean motion (n). And because the value of n is constant in an orbit, we can complete the equation

$$
\begin{aligned}
n\int_{\tau}^{t} \mathrm{d}t &= \int_{0}^{E} (1-e\cos E)\mathrm{d}E \\
n(t-\tau) &= E + e\sin E \\
M &= E + e\sin E
\end{aligned}
\tag{157}
$$

The variable M in equation 157 is called mean anomaly, which tells us where the object's position is in a circular orbit with the radius of a. We can easily calculate the value of M if we know the period of the ellipse. We must do it numerically to solve for E; there is no analytical solution for E.[14]

10.3 Parabola

For a parabola, because the value of e is equal to 1, we can rewrite the equation 149 into

$$
\int_{\tau}^{t} \mathrm{d}t = \sqrt{\frac{p^3}{\mu}} \int_{0}^{\theta} \frac{\mathrm{d}\theta}{(1+\cos\theta)^2} \tag{158}
$$

This integral form has its solution that is available in a lot of mathematical handbooks.

$$
t - \tau = \sqrt{\frac{p^3}{\mu}} \left(\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}\right) \tag{159}
$$

To solve for θ, we can consider a mean anomaly for a parabolic object, M_p. The solution is written in equation 160.

$$
\begin{aligned}
M_p &= \sqrt{\frac{\mu}{p^3}}(t-\tau) \\
\tan\frac{\theta}{2} &= \left(3M_p + \sqrt{(3M_p)^2+1}\right)^{\frac{1}{3}} - \left(3M_p + \sqrt{(3M_p)^2+1}\right)^{-\frac{1}{3}}
\end{aligned}
\tag{160}
$$

[14] If you want to use this equation, make sure each angle unit is in radians

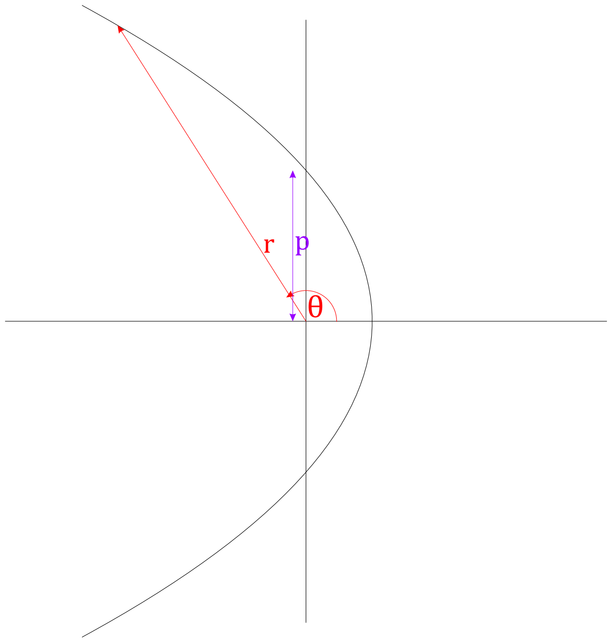


Figure 47: *Object position over time in a parabola*

10.4 Hyperbola

Similar to an ellipse, to solve the hyperbola equation, we need to define another variable named Gudermannian (ζ). This variable will help us to solve the equation.

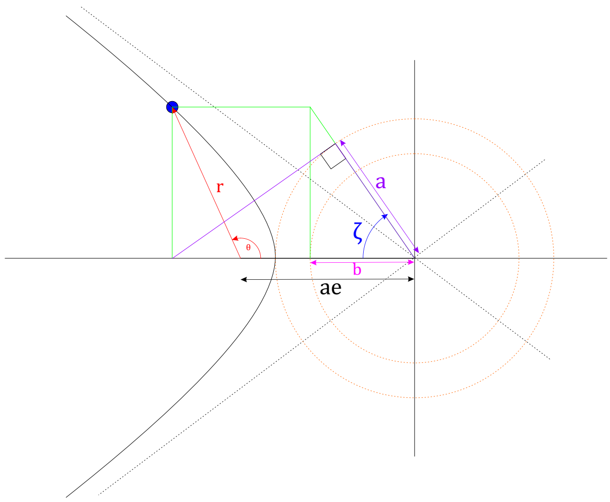


Figure 48: *Gudermannian visualisation*

From figure 48, we can calculate r with this equation.

$$\begin{aligned}\vec{r} &= (a\sec\zeta - ae)\hat{x} + b\tan\zeta\hat{y}\\ r^2 &= a^2(\sec\zeta - e)^2 + a^2(1-e^2)\tan^2\zeta \\ r &= a(1 - e\sec\zeta)\end{aligned} \tag{161}$$

We can introduce the eccentric anomaly for a hyperbola (H). This variable will help us to solve the integral needed. We know that from an identity function,

$$\sec^2 x - \tan^2 x = 1$$

and

$$\cosh^2 x - \sinh^2 x = 1$$

We will define the eccentric anomaly for a hyperbola as,

$$\begin{aligned} \tan\zeta &= \sinh H \\ \sec\zeta &= \cosh H \end{aligned} \tag{162}$$

So the equation 161 can be written as

$$r = a(1 - e\cosh H) \tag{163}$$

The relation between θ and H can be derived with a similar method to an ellipse as in equation 152. The result will be,

$$\begin{aligned} \cos\theta &= \frac{e - \cosh H}{e\cosh H - 1} \\ \sin\theta &= \frac{\sqrt{e^2-1}\sinh H}{e\cosh H - 1} \\ \\ \cosh H &= \frac{e + \cos\theta}{1 + e\cos\theta} \\ \sinh H &= \frac{\sqrt{e^2-1}\sin\theta}{1 + e\cos\theta} \end{aligned} \tag{164}$$

With the tan half-angle identity, we can get,

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{H}{2} \tag{165}$$

The relation between the derivative of θ and H can be derived as in equation 155.

$$\begin{aligned} \left(\frac{1}{1+\cos\theta}\right)\mathrm{d}\theta &= \sqrt{\frac{e+1}{e-1}}\left(\frac{1}{1+\cosh H}\right)\mathrm{d}H \\ \mathrm{d}\theta &= \frac{-a\sqrt{e^2-1}}{r}\mathrm{d}H \end{aligned} \tag{166}$$

Putting the result in equation 149, we will get

$$\begin{aligned} \sqrt{\frac{\mu}{(-a)^3}}\int_\tau^t \mathrm{d}t &= \int_0^H (e\cosh H - 1)\mathrm{d}H \\ N &= e\sinh H - H \end{aligned} \tag{167}$$

The variable N in equation 167 refers to the mean anomaly in hyperbolic orbit. Same with an ellipse, we need to solve H numerically to find the value of θ.

10.5 Exercise Problem

1. An ICBM has a position and velocity vector relative to the geographic plane of

$$r = \begin{bmatrix} -2325.12 \\ 5476.88 \\ 4516.28 \end{bmatrix} \text{km}, \quad v = \begin{bmatrix} -5.04 \\ 1.82 \\ -0.86 \end{bmatrix} \text{km/s}$$

Calculate when and where (geographic coordinates) the ICBM will land!

2. A satellite has a parabolic orbit orbiting the Earth. The satellite's equatorial coordinates at the perigee relative to the centre of the Earth are

$$\alpha = 159.24°, \quad \delta = 54.87°$$

It's known that the satellite will pass through Altair ($\alpha = 19^h 51^m, \delta = 8°52'$). The satellite is at perigee on March 20, 2025, at 00:52 UTC. Calculate when the satellite will pass Altair if the satellite's velocity at the perigee is equal to $v_p = 10.24$km/s! (Assume the Earth is stationary)

3. The Earth has orbital elements relative to the J2000 ecliptic plane that are listed below.

$$a = 1.00000011AU$$
$$e = 0.01671022$$
$$i = 0.00005°$$
$$\Omega = 102.94719°$$
$$\omega = -11.26064°$$

The Earth is in perihelion on January 4, 2025, at 13:28 UTC. We would like to observe Pallas on May 5, 2025, at 00:00 UTC. The orbital elements and the mean anomaly of Pallas relative to the J2000 ecliptic plane are

$$a = 2.7701937AU$$
$$e = 0.23054039$$
$$i = 34.92402°$$
$$\Omega = 172.89531°$$
$$\omega = 310.91036°$$
$$M = 168.7987110°$$

Calculate the ecliptic coordinates of Pallas during the observation relative to the Earth!

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