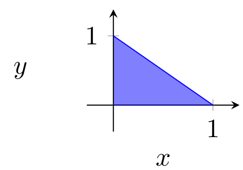
Rukmal Weerawarana (1337197)
CFRM 460
Homework 4 Solutions
2/5/16

Question 1

Part (b)

$$\iint_D e^{y^2}\, dA, \text{ where } D = \{(x, y) : 0 \leq y \leq 1, 0 \leq x \leq y\}$$

The region D is as follows:



$$\Rightarrow \iint_D e^{y^2}\, dA = \int_0^1 \int_0^y e^{y^2}\, dx dy$$

$$= \int_0^1 \left[x e^{y^2} \right]_0^y dy = \int_0^1 y e^{y^2}\, dy = \left[\frac{1}{2} e^{y^2} \right]_0^1 = \underline{\underline{\frac{e}{2} - \frac{1}{2}}}$$

Question 2

Part (a)

$$\iint_D e^{\frac{x+y}{x-y}}\, dA$$

Changing variables using:

$$u = x + y \tag{1}$$

$$v = x - y \tag{2}$$

Expressing x and y in terms of u and v:

$$(1){+}(2) \Rightarrow x = \frac{u+v}{2}$$

$$(1){-}(2) \Rightarrow y = \frac{u-v}{2}$$

Substituting these values of x and y in the initial equation:

$$e^{\frac{x+y}{x-y}} = \exp\left[\frac{x+y}{x-y}\right] = \exp\left[\frac{\frac{u+v}{2} + \frac{u-v}{2}}{\frac{u+v}{2} - \frac{u-v}{2}}\right] = \exp\left[\frac{\frac{u+v+u-v}{2}}{\frac{u+v-u+v}{2}}\right]$$

$$= \exp\left[\frac{\frac{2u}{2}}{\frac{2v}{2}}\right] = \exp\left[\frac{u}{v}\right] = e^{\frac{u}{v}}$$

Calculating the Jacobian for the change of variables:

$$\Rightarrow \mathrm{D}(x(u,v),y(u,v)) = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{bmatrix}$$

Finding the determinant of the Jacobian:

$$\Rightarrow \det\left[D(x(u,v),y(u,v))\right] = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \left(\frac{1}{2}\cdot -\frac{1}{2}\right) - \left(\frac{1}{2}\cdot\frac{1}{2}\right) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Applying the 2-dimensional change of variable formula:

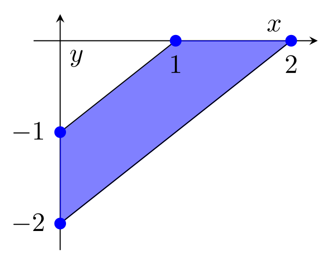
$$\Rightarrow \iint_D e^{\frac{x+y}{x-y}}\,dA = \iint_S e^{\frac{u}{v}} \cdot \left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right] dS = \underline{\underline{\iint_S -\frac{1}{2}e^{\frac{u}{v}}\,dvdu}}$$

Part (b)

Region D is bound by the following vertices on the xy plane:

$$(x_1,y_1) = (1,0)$$
$$(x_2,y_2) = (2,0)$$
$$(x_3,y_3) = (0,-2)$$
$$(x_4,y_4) = (0,-1)$$

Plotting Region D on the xy plane:

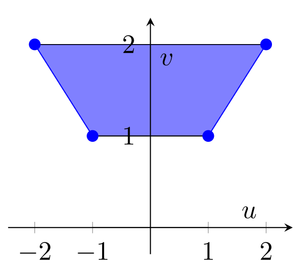


Transforming the vertices of Region D from the xy plane to the uv plane using $u = x+y$ and $v = x-y$:

$$(x_1,y_1) = (1,0) \Rightarrow (u_1,v_1) = (1,1)$$
$$(x_2,y_2) = (2,0) \Rightarrow (u_2,v_2) = (2,2)$$
$$(x_3,y_3) = (0,-2) \Rightarrow (u_3,v_3) = (-2,2)$$
$$(x_4,y_4) = (0,-1) \Rightarrow (u_4,v_4) = (-1,1)$$

$\therefore$ Region S has vertices $(u_1,v_1),(u_2,v_2),(u_3,v_3),(u_4,v_4)$ on the uv plane.

Plotting Region S on the uv plane:



Part (c)

$$\iint_S -\frac{1}{2}e^{\frac{u}{v}}\,dvdu$$

Using the vertices of Region S from Part (b), Region S can be defined as follows:

$$S = \{(u,v) : -v \le u \le v, 1 \le v \le 2\}$$

$$\Rightarrow \iint_S -\frac{1}{2}e^{\frac{u}{v}}\,dvdu = \int_1^2\int_{-v}^{v} -\frac{1}{2}e^{\frac{u}{v}}\,dudv = \int_1^2 \left[-\frac{v}{2}e^{\frac{u}{v}}\right]_{-v}^{v} dv = \int_1^2 \left(-\frac{v}{2}e - \left(-\frac{v}{2}e^{-1}\right)\right) dv$$

$$\int_1^2 \frac{v}{2}\left(\frac{1}{e} - e\right) dv = \left[\frac{v^2}{4}\left(\frac{1}{e}-e\right)\right]_1^2 = \left(\frac{1}{e}-e\right) - \frac{1}{4}\left(\frac{1}{e}-e\right) = \underline{\underline{\frac{3}{4}\left(\frac{1}{e}-e\right)}}$$

Question 3

Recall, the Black-Scholes formula for the price of a European call option is as follows:

$$C(\cdot) = Se^{-q(T-t)}\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

Also, recall the Greeks Δ & Γ, and how they are related to $C(\cdot)$:

$$\text{Delta, } \Delta(S) = \frac{\partial C(S)}{\partial S}$$

$$\text{Gamma, } \Gamma(S) = \frac{\partial^2 C(S)}{\partial S^2}$$

The taylor polynomial for a function $f(x)$ of order n about a point a is:

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) = \sum_{k=0}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a)$$

$\Rightarrow$ By using the information above, and treating $C(\cdot)$ as a function of a single variable S, the second-order taylor polynomial around point S_0 for $C(S)$ is as follows:

$$P_2(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a)$$

$$\Rightarrow P_2(S) = C(S_0) + (S-S_0)\frac{\partial}{\partial S}(C(S_0)) + \frac{(S-S_0)^2}{2}\frac{\partial^2}{\partial S^2}(C(S_0))$$

$$= C(S_0) + (S-S_0)\Delta(S_0) + \left(\frac{S^2}{2} - S_0S + \frac{S_0^2}{2}\right)\Gamma(S_0) = C(S_0) + S\Delta(S_0) - S_0\Delta(S_0) + \frac{\Gamma}{2}S^2 - S_0\Gamma(S_0)S + \frac{\Gamma(S_0)S_0^2}{2}$$

$$\therefore P_2(S) = \underline{\underline{\frac{\Gamma(S_0)}{2}S^2 + (\Delta(S_0) - S_0\Gamma(S_0))\,S + \left(C(S_0) - S_0\Delta(S_0) + \frac{S_0^2\Gamma(S_0)}{2}\right)}}$$

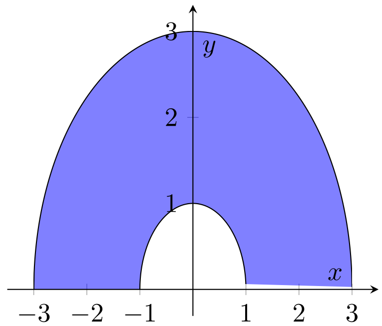
Question 4

Part (a)

$$\iint_D \sqrt{x^2+y^2}\,dxdy$$

Where $D = \{(x,y) : 1 \le x^2+y^2 \le 9, y \ge 0\}$

Plotting Region D on the xy plane:



3
2
1
y
x
−3
−2
−1
1
2
3