1. Let A be an $m \times n$ matrix.

   (a) The full $A = QR$ factorization contains more information than necessary to reconstruct A. What are the smallest matrices $\tilde{Q}$ and $\tilde{R}$ such that $\tilde{Q}\tilde{R} = A$?

   (b) Let $\tilde{A}$ be an $m \times n$ matrix ($m > n$) whose columns each sum to zero, and let $\tilde{A} = \tilde{Q}\tilde{R}$ be the reduced QR factorization of $\tilde{A}$. The squared *Mahalanobis* distance to the point $\tilde{x}_i^{\mathrm{T}}$ (the i^{th} row of $\tilde{A}$) is

   $$d_i^2 = \tilde{x}_i^{\mathrm{T}} \hat{S}^{-1} \tilde{x}_i$$

   where $\hat{S} = \frac{1}{m-1} \tilde{A}^{\mathrm{T}} \tilde{A}$ is a covariance matrix. Compute d_i^2 without inverting a matrix.

2. Consider the constrained optimization problem

   $$\begin{aligned} \text{minimize:} \quad & x^2 + 4xy + 3y^2 \\ \text{subject to:} \quad & 2x + 8y = 12 \end{aligned}$$

   (a) Write down the Lagrangian for the problem and compute its gradient.

   (b) The critical point of the Lagrangian can be found by solving a linear system. Write this system in matrix notation.

   (c) Use elimination followed by back substitution to compute the critical point. What is the value of the objective at the critical point?

   (d) Use the constraint to reduce the objective to a function of 1 variable. Find the critical point of this new 1 variable objective and verify that it is a minimum.

3. Consider using Lagrange's method to solve the following optimization problem.

   $$\begin{aligned} \text{minimize:} \quad & 3x_1 - 4x_2 + x_3 - 2x_4 \\ \text{subject to:} \quad & -x_2^2 + x_3^2 + x_4^2 = 1 \\ & 3x_1^2 + x_3^2 + 2x_4^2 = 6 \end{aligned}$$

   Write down the Lagrangian and its gradient and explain why it is difficult to find a critical point.

4. Compute the gradient of the Lagrangian for the maximum expected return portfolio subject to risk $= (20\%)^2$.