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CFRM 460
Homework 5 Solutions
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Question 1

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

Part (a)

$$\text{Let } L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\Rightarrow L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$
$$\text{Let } L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$
$$\Rightarrow L_2 L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$
$$\text{Let } L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$
$$\Rightarrow L_3 L_2 L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

As the reduced matrix is in Row Echelon form, the number of pivots of A is equivalent to the number of pivots in the reduced Upper-Right triangular form seen above.

$\Rightarrow$ Number of pivots in $A = 2$

Part (b), (c), (d) & (e)

As the square matrix A does not have pivots equal to its dimensions (3), it cannot have an inverse (i.e. matrix A is singular). Thus, the equation $Ax = b$ cannot have any solutions, regardless of the value of b.

Question 2

$$AB = I$$
$$CA = I$$

Part (a)

$$A : n \times n$$
$$B : n \times n$$
$$C : n \times n$$

Part (b)

$$\Rightarrow AB = CA$$
Multiplying both sides by B:
$$ABB = CAB \Rightarrow (AB)B = C(AB)$$
We know $AB = I \Rightarrow IB = CI$
But, $IX = X$ and $XI = X$
$$\therefore B = C$$

Part (c)

We know $AB = I$, $CA = I$ and $B = C$
Using the property that $A^{-1}A = I$ and $AA^{-1} = I$,
we can conclude $B = C = A^{-1}$
$\therefore A$ is invertible.

Question 3

Part (a)

$$(I - A)^2 = I^2 - 2IA + A^2$$
But, we know $A^2 = A$ and $I^n = I$
$$\Rightarrow I^2 - 2IA + A^2 = I - 2A + A = \underline{\underline{I - A}}$$

Part (b)

$$(I - A)^7 = (I - A)\left[(I - A)^2\right]^3$$
Recall, $(I - A)^2 = I - A$
$$\Rightarrow (I - A)[(I - A)^2]^3 = (I - A)(I - A)^3 = (I - A)^4 = [(I - A)^2]^2$$
$$\Rightarrow [(I - A)^2]^2 = (I - A)^2 = \underline{\underline{I - A}}$$

Question 4

Part (a)

$$\Rightarrow a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -5 \end{bmatrix}$$

Multiplying both sides by a constant c, we have:

$$ac \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + bc \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = c \begin{bmatrix} 9 \\ 2 \\ -5 \end{bmatrix} \Rightarrow ac \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + bc \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} - c \begin{bmatrix} 9 \\ 2 \\ -5 \end{bmatrix} = 0$$

Reversing the dot product, we have: $\begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ 3 & 2 & -5 \end{bmatrix} \begin{bmatrix} ac \\ bc \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Using Gaussian elimination to determine solutions:

$$\left[\begin{array}{ccc|c} 1 & 6 & 9 & 0 \\ 2 & 4 & 2 & 0 \\ 3 & 2 & -5 & 0 \end{array}\right]$$

$$R_2 \to \frac{r_2}{2} \left[\begin{array}{ccc|c} 1 & 6 & 9 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 2 & -5 & 0 \end{array}\right]$$

$$R_3 \to r_3 - r_2 \left[\begin{array}{ccc|c} 1 & 6 & 9 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & -6 & 0 \end{array}\right]$$

$$R_1 \to r_1 - 3r_2 \left[\begin{array}{ccc|c} -2 & 0 & 6 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & -6 & 0 \end{array}\right]$$

$$R_3 \to r_3 + r_2 \left[\begin{array}{ccc|c} -2 & 0 & 6 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \to r_2 + \frac{r_1}{2} \left[\begin{array}{ccc|c} -2 & 0 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, as all of the coefficients are 0 as per the solution to the equation, the linear expression of the vector cannot be done.

Part (b)

The Gaussian elimination performed above shows that $\begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ 3 & 2 & -5 \end{bmatrix}$ has 2 pivots.