Rukmal Weerawarana (1337197)
CFRM 460
Homework 1 Solutions
1/15/16

Question 1

Part (a)

$$\lim_{h\to 0}\frac{4(x+h-3)^2-4(x-3)^2}{h}=\lim_{h\to 0}\frac{4((x+h)^2-6(x+h)+9)-4(x^2-6x+9)}{h}$$
$$=\lim_{h\to 0}\frac{4(x^2+2xh+h^2-6x-6h+9)-4x^2+24x-36}{h}$$
$$=\lim_{h\to 0}\frac{4x^2+8xh+4h^2-24x-24h+36-4x^2+24h-36}{h}=\lim_{h\to 0}\frac{4h^2+8xh-24h}{h}$$
$$=\lim_{h\to 0}(\frac{4h^2}{h}+\frac{8xh}{h}-\frac{24h}{h})=8x-24-(\lim_{h\to 0}4h)$$
$$=\underline{\underline{8x-24}}$$

Mathematica: `Limit[(4 (x + h - 3)^2 - 4 (x - 3)^2)/h, h -> 0]`

Part (b)

$$\lim_{x\to\infty}\frac{1}{\sqrt{4x^2-2x-10}+2x}=\lim_{x\to\infty}\frac{1}{\sqrt{(2x-\frac{1}{2})^2-\frac{41}{4}}+2x}$$

Thus, it is clear that the denominator of the function will approach infinity as $x\to\infty$

$$\Rightarrow\lim_{x\to\infty}\frac{1}{\sqrt{4x^2-2x-10}+2x}=\underline{\underline{0}}$$

Mathematica: `Limit[1/(Sqrt[4 x^2 - 2 x - 10] + 2 x), x -> Infinity]`

Question 2

Part (a)

$$\Rightarrow f'(x)=\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{d}{dx}\left((1-x)^{-1}\right)$$

Using the chain rule:

$$=-(1-x)^{-2}\cdot -1=\underline{\underline{\frac{1}{(1-x)^2}}}$$

Mathematica: `D[1/(1 - x), x]`

Part (b)

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\sum_{k=1}^{7} k e^{-a_k x^3}\right) = \sum_{k=1}^{7} k \frac{d}{dx}\left(e^{-a_k x^3}\right)$$

Using the chain rule:

$$= \sum_{k=1}^{7} k(e^{-a_k x^3} \cdot -3a_k x^2) = \underline{\underline{-3\sum_{k=1}^{7} k a_k x^2 e^{-a_k x^3}}}$$

Mathematica: `D[Sum[k * Exp[-a[k] x^3], {k, 1, 7}], x]`

Part (c)

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{\log\left(\frac{x}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) = \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{d}{dx}\left(\log\left(\frac{x}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)\right)$$

Using the chain rule, and eliminating constants:

$$= \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{1}{\frac{x}{K}} \cdot \frac{1}{K} = \frac{1}{\sigma\sqrt{T-t}} \cdot \frac{K}{x} \cdot \frac{1}{K} = \underline{\underline{\frac{1}{(\sigma\sqrt{T-t})x}}}$$

Mathematica: `D[(Log[x/K] + (r - q + (\[Sigma]^2)/2) (T - t))/(\[Sigma] Sqrt[T - t]), x]`

Part (d)

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{x^2}{2}\right)(T-t)}{x\sqrt{T-t}}\right) = \frac{1}{\sqrt{T-t}} \cdot \frac{d}{dx}\left(\frac{\log\left(\frac{S}{K}\right)}{x} + \frac{\left(r - q + \frac{x^2}{2}\right)(T-t)}{x}\right)$$

$$= \frac{1}{\sqrt{T-t}}\left(-\frac{\log\left(\frac{S}{K}\right)}{x^2} + \frac{d}{dx}\left(\frac{(r-q)(T-t)}{x} + \frac{\frac{x^2}{2}(T-t)}{x}\right)\right)$$

$$= \frac{1}{\sqrt{T-t}}\left(-\frac{\log\left(\frac{S}{K}\right)}{x^2} + \frac{d}{dx}\left(\frac{(r-q)(T-t)}{x} + \frac{x(T-t)}{2}\right)\right) = \frac{1}{\sqrt{T-t}}\left(-\frac{\log\left(\frac{S}{K}\right)}{x^2} - \frac{(r-q)(T-t)}{x^2} + \frac{(T-t)}{2}\right)$$

$$= \underline{\underline{\frac{\sqrt{T-t}}{2} - \frac{\log\left(\frac{S}{K}\right) + (r-q)(T-t)}{x^2\sqrt{T-t}}}}$$

Mathematica: `D[(Log[S/K] + (r - q + (x^2/2)) (T - t))/(x Sqrt[T - t]), x]`

Part (e)

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{\log\left(\frac{S}{K}\right) + (x - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) = \frac{d}{dx}\left(\frac{\log\left(\frac{S}{K}\right)}{\sigma\sqrt{T-t}} + \frac{(\frac{\sigma^2}{2} - q)(T-t)}{\sigma\sqrt{T-t}} + \frac{x(T-t)}{\sigma\sqrt{T-t}}\right)$$

Eliminating constants:

$$= \frac{d}{dx}\left(\frac{x(T-t)}{\sigma\sqrt{T-t}}\right) = \frac{d}{dx}\left(\frac{x\sqrt{T-t}}{\sigma}\right) = \underline{\underline{\frac{\sqrt{T-t}}{\sigma}}}$$

Mathematica: `D[(Log[S/K] + (x - q + (\[Sigma]^2/2)) (T - t))/(\[Sigma] Sqrt[T - t]), x]`

Question 3

Part (a)

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

Thus, as both the numerator and denominator $\to 0$ as $x \to 0$, l'Hôpital's rule can be used:

$$\Rightarrow \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \lim_{x \to 0} \cos(x) = \cos(0) = \underline{\underline{1}}$$

Mathematica: `Limit[Sin[x]/x, x -> 0]`

Part (b)

$$\lim_{x \to \infty} \frac{\log(x^3)}{x} = 3 \cdot \lim_{x \to \infty} \frac{\log(x)}{x}$$

Thus, as both the numerator and denominator $\to \infty$ as $x \to \infty$, l'Hôpital's rule can be used:

$$\Rightarrow 3 \cdot \lim_{x \to \infty} \frac{\log(x)}{x} = 3 \cdot \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 3 \cdot \lim_{x \to \infty} \frac{1}{x} = 3 \cdot 0 = \underline{\underline{0}}$$

Mathematica: `Limit[Log[x^3]/x, x -> Infinity]`