1. Give an example of a 2×2 matrix that has no real eigenvectors. Justify your solution with intuition (without solving completely for the eigenvectors and eigenvalues).

2. Consider an $n \times p$ matrix A. Show that the number of linear independent rows is the same as the number of linearly independent columns.
   Hint: Write $A = CR$ where C is a matrix of the linearly independent columns of A. Why can we write A like this? Then consider the CR product in the "row" interpretation of matrix multiplication.

3. Let A be an $m \times n$ matrix (assume $m > n$). The full singular value factorization $A = U\Sigma V^{\mathrm{T}}$ contains more information than necessary to reconstruct A.
   (a) What are the smallest matrices $\tilde{U}$, $\tilde{\Sigma}$ and $\tilde{V}^{\mathrm{T}}$ such that $\tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}} = A$?
   (b) Let $U = \begin{bmatrix} \tilde{U} & \hat{U} \end{bmatrix}$. That is, think about U from the full singular value factorization as a block matrix consisting of the matrix $\tilde{U}$ found in part (a) and the remaining (unneeded) columns $\hat{U}$.

       Find expressions for $\tilde{U}^{\mathrm{T}}\tilde{U}$ and $\tilde{U}\tilde{U}^{\mathrm{T}}$.
   (c) Use the *reduced* singular value factorization obtained in part (a) to find an expression for the matrix $H = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. How many matrices must be inverted (diagonal and orthogonal matrices don't count)?

4. Let x and y be vectors of m elements. The least squares solution for a best-fit line for a plot of y versus x is

$$\hat{\beta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$$

   where

$$X = \begin{bmatrix} | & | \\ 1 & x \\ | & | \end{bmatrix}$$

   (a) Suppose you know the **full** singular value factorization $X = U\Sigma V^{\mathrm{T}}$. Find an expression for $\hat{\beta}$ in terms of U, Σ, and V. Hint: only square matrices are invertible.
   (b) Repeat part (a) using the reduced singular value factorization $X = \tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}}$.

5. Let $\tilde{X}$ be an $m \times n$ matrix $(m > n)$ whose columns have sample mean zero, and let $\tilde{X} = \tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}}$ be a reduced singular value factorization of $\tilde{X}$. The squared *Mahalanobis* distance to the point $\tilde{x}_i^{\mathrm{T}}$ (the i^{th} row of $\tilde{X}$) is

$$d_i^2 = \tilde{x}_i^{\mathrm{T}} \hat{S}^{-1} \tilde{x}_i$$

where $\hat{S} = \frac{1}{m-1}\tilde{X}^{\mathrm{T}}\tilde{X} = \operatorname{cov}(\tilde{X})$. Explain how to compute d_i^2 without inverting a matrix.