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CFRM 460
Homework 2 Solutions
1/22/16

Question 1

Part (a)

$$\int x^2 \log(x) dx$$

$$\text{Let } u = \log(x) \quad \text{and} \qquad v = \frac{x^3}{3}$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x} \qquad \Rightarrow \frac{dv}{dx} = x^2$$

$$\text{We know } \int u dv = uv - \int v du$$

$$= \log(x) \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$= \frac{x^3 \log(x)}{3} - \int \frac{x^2}{2} dx$$

$$= \frac{x^3}{9}\left(3\log(x) - 1\right) + C$$

Mathematica: `Integrate[x^2 Log[x], x]`

Part (b)

$$\int x^2 e^x dx$$

$$\text{Let } u = x^2 \quad \text{and} \qquad v = e^x$$

$$\Rightarrow \frac{du}{dx} = 2x \qquad \Rightarrow \frac{dv}{dx} = e^x$$

$$\text{We know } \int u dv = uv - \int v du$$

$$= x^2 \cdot e^x - \int e^x \cdot 2x dx$$

Using integration by parts again with the second part of the equation:

$$\text{Let } m = 2x \quad \text{and} \qquad n = e^x$$

$$\Rightarrow \frac{dm}{dx} = 2 \qquad \Rightarrow \frac{dn}{dx} = e^x$$

$$\text{We know } \int m dn = mn - \int n dm$$

$$= 2x \cdot e^x - \int e^x \cdot 2 dx$$

$$= 2xe^x - 2e^x + C$$

Substituting this back in the original equation:

$$\Rightarrow x^2e^x - \int e^x 2x dx = x^2e^x - (2xe^x - 2e^x) + C$$

$$= \underline{\underline{e^x(x^2 - 2x + 2) + C}}$$

Mathematica: `Integrate[x^2 Exp[x], x]`

Part (c)

$$\int \Big(\log(x)\Big)^2 dx$$

Let $u = \log(x) \Rightarrow x = e^u$

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = xdu$$

$$\Rightarrow \int \Big(\log(x)\Big)^2 dx = \int u^2 x du = \int u^2 e^u du$$

Note, this is the same as Part (b). Thus, using the same solution:

$$\int u^2 e^u du = e^u(u^2 - 2u + 2) + C$$

But, $u = \log(x)$

$$\Rightarrow e^u(u^2 - 2u + 2) + C = e^{\log(x)}\big(\big(\log(x)\big)^2 - 2\log(x) + 2\big) + C$$

$$= \underline{\underline{x\big(\log(x)\big)^2 - 2x\log(x) + 2x + C}}$$

Mathematica: `Integrate[Log[x]^2, x]`

Question 2

Part (a)

$$\int_4^7 x^2 \log(x) dx$$

Note, the integration is the same as Question 1 Part (a)

Thus, the limits can be substituted directly in the solution as follows:

$$\Rightarrow \int_4^7 x^2 \log(x) dx = \left[\frac{x^3}{9}\Big(3\log(x) - 1\Big)\right]_4^7$$

$$= \left(\frac{7^3}{9}\Big(3\log(7) - 1\Big)\right) - \left(\frac{4^3}{9}\Big(3\log(4) - 1\Big)\right) = \frac{3\big(343\log(7) - 64\log(4)\big) - 343 + 64}{9}$$

$$= \underline{\underline{\frac{343\log(7) - 64\log(4)}{3} - 31}}$$

Mathematica: `Integrate[x^2 Log[x], {x, 4, 7}]`

Part (b)

$$\int_0^\infty \frac{1}{(1+x)^2}dx$$
$$\Rightarrow \int_0^\infty \frac{1}{(1+x)^2}dx = \lim_{t\to\infty}\int_0^t \frac{1}{(1+x)^2}dx$$
$$\text{Let } u = 1+x$$
$$\frac{du}{dx} = 1 \Rightarrow dx = du$$
$$\text{Upper limit: } u(t) = t+1$$
$$\text{Lower limit: } u(0) = 1+0 = 1$$
$$\Rightarrow \lim_{t\to\infty}\int_0^t \frac{1}{(1+x)^2}dx = \lim_{t\to\infty}\int_1^{t+1}\frac{1}{u^2}du$$
$$= \lim_{t\to\infty}\left(\left[\frac{-1}{u}\right]_1^{t+1}\right) = \lim_{t\to\infty}\left(\frac{-1}{t+1} - \left(\frac{-1}{1}\right)\right) = 1 + \lim_{t\to\infty}\left(\frac{-1}{t+1}\right) = \underline{\underline{1}}$$

Mathematica: `Integrate[1/(1 + x)^2, {x, 0, Infinity}]`

Question 3

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_0^{b(x)} e^{-\frac{y^2}{2}}dy$$
$$b(x) = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right]$$
$$\text{Such that } \{K, T, \sigma, r\} \in \mathbb{R}_+$$

$$\text{Let } f(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$
$$\text{Also, let there be a function } F(y) \text{ such that } \frac{d}{dx}\Big(F(y)\Big) = f(y)$$

$$\Rightarrow g(x) = \int_0^{b(x)} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy = \int_0^{b(x)} f(y)dy$$

$\therefore$ Based on the above conjectures, and as per the fundamental theorum of calculus (FTC):
$$g(x) = \int_0^{b(x)} f(y)dy = \Big[F(y)\Big]_0^{b(x)} = F\big(b(x)\big) - F(0)$$

As we want to find $\frac{d}{dx}\Big(g(x)\Big)$, the derivative of both sides of the equation can be taken:
$$\Rightarrow g\prime(x) = \frac{d}{dx}\Big(F\big(b(x)\big) - F(0)\Big) = \frac{d}{dx}\Big(F\big(b(x)\big)\Big) - \frac{d}{dx}\Big(F(0)\Big)$$
$$\text{As } F(0) \text{ is a constant, } \frac{d}{dx}\Big(F(0)\Big) = 0$$

$$\therefore g\prime(x) = \frac{d}{dx}\Big(F\big(b(x)\big)\Big)$$

Using the chain rule:

$$\Rightarrow g\prime(x) = F\prime\big(b(x)\big) \cdot b\prime(x)$$

But, $F\prime(y) = f(y)$. Substituting this in the equation, we can show that:

$$g\prime(x) = f(b(x)) \cdot b\prime(x)$$

To solve for $g\prime(x)$, we must first find $b\prime(x)$

$$b\prime(x) = \frac{d}{dx}\left(\frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right]\right) = \frac{d}{dx}\left(\frac{1}{\sigma\sqrt{T}} \cdot \log\left(\frac{x}{K}\right)\right) + \frac{d}{dx}\left(\frac{\sqrt{T}}{\sigma} \cdot \left(r + \frac{\sigma^2}{2}\right)\right)$$

$$= \frac{1}{\sigma\sqrt{T}} \cdot \frac{d}{dx}\left(\log\left(\frac{x}{K}\right)\right) = \frac{1}{\sigma\sqrt{T}} \cdot \frac{1}{\frac{x}{K}} \cdot \frac{1}{K} = \frac{1}{\sigma\sqrt{T}} \cdot \frac{K}{x} \cdot \frac{1}{K}$$

$$\therefore b\prime(x) = \underline{\underline{\frac{1}{\sigma\sqrt{T}x}}}$$

Plugging this into the equation for $g\prime(x)$, we get:

$$g\prime(x) = f(b(x)) \cdot \frac{1}{\sigma\sqrt{T}x} = \frac{1}{\sqrt{2\pi}}e^{-\frac{(b(x))^2}{2}} \cdot \frac{1}{\sigma\sqrt{T}x}$$

$$\therefore g\prime(x) = \underline{\underline{\frac{e^{-\frac{(b(x))^2}{2}}}{\sqrt{2\pi T}\sigma x}}}$$

Mathematica: `SetAttributes[{T, \[Sigma], K, r}, Constant]; b[x_] := (1/(\[Sigma] \[Sqrt](T)))[Log[x/K] + T (r + (\[Sigma]^2)/2)]; f[y_] := 1/(2 Pi)^(1/2) Exp[-y^2/2]; g[x_] := Integrate[f[y], {y, 0, b[x]}]; D[g[x], x]`

Question 4

$$\phi(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}$$

$$\text{Such that } \Phi(x) = \int_{-\infty}^{x} \phi(u)du$$

Part (a)

$$\text{Let } \phi(-x) = \phi(x)$$

$$\text{LHS: } \phi(-x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-(-x)^2}{2}} = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$$

$$\text{RHS: } \phi(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-(x)^2}{2}} = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$$

$$\Rightarrow \text{LHS} = \text{RHS}$$

$$\therefore \phi(-x) = \phi(x)$$

Mathematica: `\[Phi][u_] := (1/Sqrt[2 Pi]) Exp[- u^2 / 2]; \[Phi][-x] == \[Phi][x]`

Part (b)

Mathematica: `\[Phi][u_] := (1/Sqrt[2 Pi]) Exp[-u^2/2]; \[CapitalPhi][x_] := Integrate[\[Phi][u], u, -Infinity, x]; \[CapitalPhi][-x] == 1 - \[CapitalPhi][x]`