1. The Black-Scholes price for a European put option is

$$P(S, t, K, T, r, q, \sigma) = Ke^{-r(T-t)}\Phi(-d_{-}) - Se^{-q(T-t)}\Phi(-d_{+})$$
(1)

where

$$d_{+} = \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^{2}}{2}\right)\left(T - t\right)}{\sigma\sqrt{T - t}}$$

and

$$d_{-} = d_{+} - \sigma\sqrt{T - t} = \frac{\log\left(\frac{S}{K}\right) + \left(r - q - \frac{\sigma^{2}}{2}\right)\left(T - t\right)}{\sigma\sqrt{T - t}}$$

Compute each of

(a) 
$$\Delta(P) = \frac{\partial P}{\partial S}$$

(b) 
$$\Gamma(P) = \frac{\partial^2 P}{\partial S^2}$$

(c) 
$$\theta(P) = \frac{\partial P}{\partial t}$$

(d) 
$$\rho(P) = \frac{\partial P}{\partial r}$$

by taking derivatives of (1). Verify that your answers are correct using put-call parity.

You can find expressions for *The Greeks* on pages 92 and 93 of the Stefanica text. Verify that your answer matches the expression for the put option and that put-call parity gives the expression for the call option. And as always, verify your calculations with Mathematica.

## Example

Compute the vega of a European put option.

$$\operatorname{vega}(P) = \frac{\partial P}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left[ Ke^{-r(T-t)} \Phi(-d_{-}) - Se^{-q(T-t)} \Phi(-d_{+}) \right]$$

$$= Ke^{-r(T-t)} \phi(-d_{-}) \frac{\partial}{\partial \sigma} (-d_{-}) - Se^{-q(T-t)} \phi(-d_{+}) \frac{\partial}{\partial \sigma} (-d_{+})$$

$$= Ke^{-r(T-t)} \phi(d_{-}) \frac{\partial}{\partial \sigma} (-d_{-}) - Se^{-q(T-t)} \phi(d_{+}) \frac{\partial}{\partial \sigma} (-d_{+})$$

Lemma 3.15 states that  $Ke^{-r(T-t)}\phi(d_-) = Se^{-q(T-t)}\phi(d_+)$ , thus

$$\operatorname{vega}(P) = \frac{\partial P}{\partial \sigma} = Se^{-q(T-t)}\phi(d_{+})\frac{\partial}{\partial \sigma}(-d_{-}) - Se^{-q(T-t)}\phi(d_{+})\frac{\partial}{\partial \sigma}(-d_{+})$$

$$= Se^{-q(T-t)}\phi(d_{+})\left[\frac{\partial}{\partial \sigma}(-d_{-}) - \frac{\partial}{\partial \sigma}(-d_{+})\right]$$

$$= Se^{-q(T-t)}\phi(d_{+})\left[\frac{\partial}{\partial \sigma}(d_{+}) - \frac{\partial}{\partial \sigma}(d_{-})\right]$$

$$= Se^{-q(T-t)}\phi(d_{+})\frac{\partial}{\partial \sigma}\left[d_{+} - d_{-}\right]$$

But 
$$d_- = d_+ - \sigma \sqrt{T - t} \implies d_+ - d_- = \sigma \sqrt{T - t}$$
, thus

$$\operatorname{vega}(P) = \frac{\partial P}{\partial \sigma} = Se^{-q(T-t)}\phi(d_{+}) \frac{\partial}{\partial \sigma} \left[\sigma\sqrt{T-t}\right]$$
$$\operatorname{vega}(P) = \frac{\partial P}{\partial \sigma} = Se^{-q(T-t)}\phi(d_{+}) \sqrt{T-t}$$

Check the result using put-call parity:

$$P = C - Se^{-q(T-t)} + Ke^{-r(T-t)}$$
$$vega(P) = \frac{\partial P}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left[ C - Se^{-q(T-t)} + Ke^{-r(T-t)} \right]$$
$$= \frac{\partial C}{\partial \sigma}$$

The vega of a European put option is the same as the vega for a European call option.