Rukmal Weerawarana (1337197)
CFRM 460
Homework 4 Solutions
2/5/16

Question 1

Part (a)

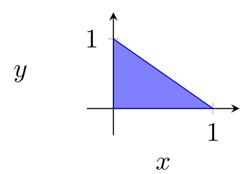
The assumptions of Fubini's theorum are as follows:

$$\iint_D f(x,y)\,dA < \infty$$

Part (b)

$$\iint_D e^{y^2}\,dA, \text{ where } D = \{(x,y) : 0 \le y \le 1, 0 \le x \le y\}$$

The region D is as follows:



$$\Rightarrow \iint_D e^{y^2}\,dA = \int_0^1 \int_0^y e^{y^2}\,dxdy$$

$$= \int_0^1 \left[xe^{y^2}\right]_0^y dy = \int_0^1 ye^{y^2}\,dy = \left[\frac{1}{2}e^{y^2}\right]_0^1 = \underline{\underline{\frac{e}{2} - \frac{1}{2}}}$$

Question 2

Part (a)

$$\iint_D e^{\frac{x+y}{x-y}}\,dA$$

Changing variables using:

$$u = x + y \tag{1}$$

$$v = x - y \tag{2}$$

Expressing x and y in terms of u and v:

$$(1)+(2) \rightarrow x = \frac{u+v}{2}$$

$$(1)-(2) \rightarrow y = \frac{u-v}{2}$$

Substituting these values of x and y in the initial equation:

$$e^{\frac{x+y}{x-y}} = \exp\left[\frac{x+y}{x-y}\right] = \exp\left[\frac{\frac{u+v}{2}+\frac{u-v}{2}}{\frac{u+v}{2}-\frac{u-v}{2}}\right] = \exp\left[\frac{\frac{u+v+u-v}{2}}{\frac{u+v-u+v}{2}}\right]$$
$$= \exp\left[\frac{\frac{2u}{2}}{\frac{2v}{2}}\right] = \exp\left[\frac{u}{v}\right] = e^{\frac{u}{v}}$$

Calculating the Jacobian for the change of variables:

$$\Rightarrow \mathrm{D}(x(u,v),y(u,v)) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Finding the determinant of the Jacobian:

$$\Rightarrow \det\left[D(x(u,v),y(u,v))\right] = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \left(\frac{1}{2}\cdot -\frac{1}{2}\right) - \left(\frac{1}{2}\cdot\frac{1}{2}\right) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Applying the 2-dimensional change of variable formula:

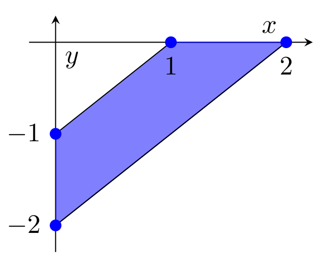
$$\Rightarrow \iint_D e^{\frac{x+y}{x-y}}\, dA = \iint_S e^{\frac{u}{v}} \cdot \left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right] dS = \underline{\underline{\iint_S -\frac{1}{2} e^{\frac{u}{v}}\, dvdu}}$$

Part (b)

Region D is bound by the following vertices on the xy plane:

$$(x_1, y_1) = (1, 0)$$
$$(x_2, y_2) = (2, 0)$$
$$(x_3, y_3) = (0, -2)$$
$$(x_4, y_4) = (0, -1)$$

Plotting Region D on the xy plane:

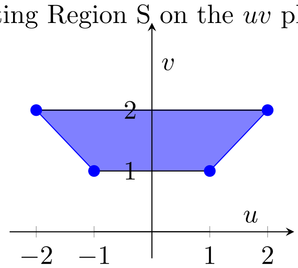


Transforming the vertices of Region D from the xy plane to the uv plane using $u = x + y$ and $v = x - y$:

$$(x_1, y_1) = (1, 0) \Rightarrow (u_1, v_1) = (1, 1)$$
$$(x_2, y_2) = (2, 0) \Rightarrow (u_2, v_2) = (2, 2)$$
$$(x_3, y_3) = (0, -2) \Rightarrow (u_3, v_3) = (-2, 2)$$
$$(x_4, y_4) = (0, -1) \Rightarrow (u_4, v_4) = (-1, 1)$$

$\therefore$ Region S has vertices $(u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4)$ on the uv plane.

Plotting Region S on the uv plane:



Part (c)

$$\iint_S -\frac{1}{2}e^{\frac{u}{v}}\,dvdu$$

Using the vertices of Region S from Part (b), Region S can be defined as follows:

$$S = \{(u,v) : -v \le u \le v, 1 \le v \le 2\}$$

$$\Rightarrow \iint_S -\frac{1}{2}e^{\frac{u}{v}}\,dvdu = \int_1^2 \int_{-v}^{v} -\frac{1}{2}e^{\frac{u}{v}}\,dudv = \int_1^2 \left[-\frac{v}{2}e^{\frac{u}{v}}\right]_{-v}^{v}\,dv = \int_1^2 \left(-\frac{v}{2}e - \left(-\frac{v}{2}e^{-1}\right)\right)dv$$

$$\int_1^2 \frac{v}{2}\left(\frac{1}{e} - e\right)dv = \left[\frac{v^2}{4}\left(\frac{1}{e} - e\right)\right]_1^2 = \left(\frac{1}{e} - e\right) - \frac{1}{4}\left(\frac{1}{e} - e\right) = \underline{\underline{\frac{3}{4}\left(\frac{1}{e} - e\right)}}$$

Question 3

Recall, the Black-Scholes formula for the price of a European call option is as follows:

$$C(\cdot) = Se^{-q(T-t)}\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

Also, recall the Greeks Δ & Γ, and how they are related to $C(\cdot)$:

$$\text{Delta, } \Delta(S) = \frac{\partial C(S)}{\partial S}$$

$$\text{Gamma, } \Gamma(S) = \frac{\partial^2 C(S)}{\partial S^2}$$

The taylor polynomial for a function $f(x)$ of order n about a point a is:

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) = \sum_{k=0}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a)$$

$\Rightarrow$ By using the information above, and treating $C(\cdot)$ as a function of a single variable S, the second-order taylor polynomial around point S_0 for $C(S)$ is as follows:

$$P_2(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a)$$

$$\Rightarrow P_2(S) = C(S_0) + (S - S_0)\frac{\partial}{\partial S}(C(S_0)) + \frac{(S-S_0)^2}{2}\frac{\partial^2}{\partial S^2}(C(S_0))$$

$$= C(S_0) + (S-S_0)\Delta(S_0) + \left(\frac{S^2}{2} - S_0S + \frac{S_0^2}{2}\right)\Gamma(S_0) = C(S_0) + S\Delta(S_0) - S_0\Delta(S_0) + \frac{\Gamma}{2}S^2 - S_0\Gamma(S_0)S + \frac{\Gamma(S_0)S_0^2}{2}$$

$$\therefore P_2(S) = \underline{\underline{\frac{\Gamma(S_0)}{2}S^2 + (\Delta(S_0) - S_0\Gamma(S_0))\,S + \left(C(S_0) - S_0\Delta(S_0) + \frac{S_0^2\Gamma(S_0)}{2}\right)}}$$

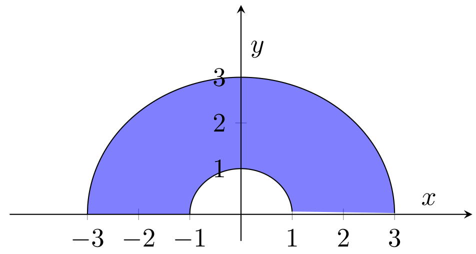
Question 4

Part (a)

$$\iint_D \sqrt{x^2+y^2}\,dxdy$$

Where $D = \{(x,y) : 1 \le x^2 + y^2 \le 9, y \ge 0\}$

Plotting Region D on the xy plane:



Using the change of variable formula to convert the integral to polar coordinates using:

$$x = r\cos(\theta) \tag{1}$$

$$y = r\sin(\theta) \tag{2}$$

Squaring both (1) and (2) results in the following:

$$(1)\cdot(1) \to x^2 = r^2\cos^2(\theta) \tag{3}$$

$$(2)\cdot(2) \to y^2 = r^2\sin^2(\theta) \tag{4}$$

$$(3)+(4) \to x^2+y^2 = r^2\sin^2(\theta) + r^2\cos^2(\theta) = r^2(\cos^2(\theta)+\sin^2(\theta))$$

But, we know $\cos^2(\theta)+\sin^2(\theta) = 1$. Thus:

$$r = \sqrt{x^2+y^2}$$

To find an expression for θ in terms of x and y, consider the following:

$$\frac{(2)}{(1)} \to \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$$

But, we know $\dfrac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$. Thus:

$$\theta = \arctan\left(\frac{y}{x}\right)$$

Using this, the cartesian region can be converted to polar as follows:

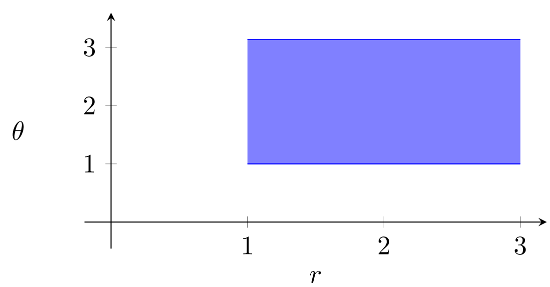
$$1 \le x^2+y^2 \le 9 \Rightarrow 1 \le \sqrt{x^2+y^2} \le 3 \Rightarrow 1 \le \theta \le 3$$

$$y \ge 0 \Rightarrow \theta = \arctan(0) \Rightarrow \theta = 0 \text{ or } \theta = \pi$$

Let the cartesian Region D be equivalent to polar Region S. Thus,

$$D = \{(x,y) : 1 \le x^2+y^2 \le 9, y \ge 0\} \leftrightarrow S = \{(r,\theta) : 1 \le r \le 3, 0 \le \theta \le \pi\}$$

Plotting Region S on the $r\theta$ plane:



Applying the change of variables to polar coordinates formula to the original integral:

$$\iint_D \sqrt{x^2+y^2}\,dxdy = \iint_S r\sqrt{(r\cos(\theta))^2+(r\sin(\theta))^2}\,drd\theta$$

$$= \iint_S r\sqrt{r^2(\cos^2(\theta)+\sin^2(\theta))}\,drd\theta = \iint_S r\sqrt{r^2}\,drd\theta = \iint_S r^2\,drd\theta$$

Recall, $S = \{(r,\theta) : 1 \le r \le 3, 0 \le \theta \le \pi\}$

$$\Rightarrow \iint_S r^2\,drd\theta = \int_0^\pi \int_1^3 r^2\,drd\theta$$

$$= \int_0^\pi \left[\frac{r^3}{3}\right]_1^3 d\theta = \int_0^\pi \left(\frac{27}{3} - \frac{1}{3}\right) d\theta = \int_0^\pi \frac{26}{3}\, d\theta = \left[\frac{26}{3}\theta\right]_0^\pi = \underline{\underline{\frac{26\pi}{3}}}$$

Part (b)

$$\iint_D \sin(\sqrt{x^2+y^2})\, dxdy$$

Where $D = \{(x,y) : \pi^2 \leq x^2 + y^2 \leq 4\pi^2\}$

Converting to polar coordinates using the following from Part (a):

$$r = \sqrt{x^2+y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

Finding the new bounds for r:

$$\pi^2 \leq x^2+y^2 \leq 4\pi^2 \Rightarrow \pi^2 \leq x^2+y^2 \leq (2\pi)^2 \Rightarrow \pi \leq \sqrt{x^2+y^2} \leq 2\pi \Rightarrow \pi \leq r \leq 2\pi$$

As there are no restrictions on the domain of the y variable:

$$0 \leq \theta \leq 2\pi$$

Let cartesian Region D be equivalent to polar Region S. Thus:

$$D = \{(x,y) : \pi^2 \leq x^2+y^2 \leq 4\pi^2\} \leftrightarrow S = \{(r,\theta) : \pi \leq r \leq 2\pi, 0 \leq \theta \leq 2\pi\}$$

Applying the change of variables to polar coordinates formaul to the original integral:

$$\iint_D \sin(\sqrt{x^2+y^2})\, dxdy = \iint_S r\sin(\sqrt{(r\cos(\theta))^2 + (r\sin(\theta))^2}\, drd\theta = \iint_S r\sin(r)\, drd\theta$$

Recall, $S = \{(r,\theta) : \pi \leq r \leq 2\pi, 0 \leq \theta \leq 2\pi\}$

$$\Rightarrow \iint_S r\sin(r)\, drd\theta = \int_0^{2\pi}\int_\pi^{2\pi} r\sin(r)\, drd\theta$$

Consider the inner integral, $\int r\sin(r)\, dr$. Applying integration by parts:

$$\text{Let } u = r \quad \text{and} \quad v = -\cos(r)$$

$$\Rightarrow \frac{du}{dr} = 1 \qquad \Rightarrow \frac{dv}{dr} = \sin(r)$$

$$\text{We know } \int u\, dv = uv - \int v\, du$$

$$= r \cdot -\cos(r) - \int -\cos(r) \cdot 1\, dr$$

$$= -(-\sin(r)) - r\cos(r) = \sin(r) - r\cos(r)$$

Substituting this back into the original integral:

$$\Rightarrow \int_0^{2\pi}\int_\pi^{2\pi} r\sin(r)\, drd\theta = \int_0^{2\pi} [\sin(r) - r\cos(r)]_\pi^{2\pi}\, d\theta = \int_0^{2\pi} ((\sin(2\pi) - 2\pi\cos(2\pi)) - (\sin(\pi) - \pi\cos(\pi)))\, d\theta$$

$$= \int_0^{2\pi} ((0-2\pi) - (0 - \pi(-1)))\, d\theta = \int_0^{2\pi} (-2\pi - \pi)\, d\theta = \int_0^{2\pi} -3\pi\, d\theta$$

$$= [-3\pi\theta]_0^{2\pi} = (-3\pi \cdot 2\pi) - (-3\pi \cdot 0) = \underline{\underline{-6\pi^2}}$$

Question 5

$$f(x) = \frac{1}{1+x}$$

$$\Rightarrow f'(x) = \frac{-1}{(1+x)^2}, f''(x) = \frac{2}{(1+x)^3}, f'''(x) = \frac{-6}{(1+x)^4}, \ldots, f^{(k)}(x) = \frac{(-1)^k \cdot k!}{(1+x)^{(k+1)}}$$

The Taylor series expansion of a function $f(x)$ about a point a is:

$$T(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

Where the power series coefficients, $a_k = \dfrac{f^{(k)}(a)}{k!}$

Thus, the Taylor series expansion of $f(x)$ about the point $a = 0$ is:

$$T(x) = \sum_{k=0}^{\infty} a_k (x-a)^k = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot k! \cdot (x-a)^k}{k! \cdot (1+a)^{(k+1)}} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^k}{1^{(k+1)}} = \sum_{k=0}^{\infty} (-x)^k$$

$$\therefore T(x) = \sum_{k=0}^{\infty} (-x)^k$$

The radius of convergence, R can be determined as follows:

$$R = \frac{1}{\lim_{k\to\infty} |a_k|^{1/k}}$$

$$\Rightarrow \lim_{k\to\infty} |a_k|^{1/k} = \lim_{k\to\infty} \left| \frac{(-1)^k \cdot k!}{(1+a)^{(k+1)} \cdot k!} \right|^{1/k} = \lim_{k\to\infty} \left| \frac{(-1)^k}{1^{(k+1)}} \right|^{1/k} = \lim_{k\to\infty} \left| (-1)^k \right|^{1/k}$$

$$= \lim_{k\to\infty} \left| (-1)^{k\cdot(1/k)} \right| = \lim_{k\to\infty} |(-1)| = |-1|$$

$$\Rightarrow \lim_{k\to\infty} |a_k|^{1/k} = 1$$

$$\therefore \text{ Radius of convergence, } R = \frac{1}{\lim_{k\to\infty} |a_k|^{1/k}} = \frac{1}{1} = 1$$

$T(x) = f(x) = \dfrac{1}{1+x}$ when there is an r such that $0 < r < R$ and

$$\lim_{k\to\infty} \left[\frac{r^k}{k!} \max_{z\in[a-r,a+r]} \left| f^{(k)}(z) \right| \right] = 0 \Rightarrow \lim_{k\to\infty} \left[\frac{r^k}{k!} \max_{z\in[-r,r]} \left| \frac{(-1)^k \cdot k!}{(1+z)^{(k+1)}} \right| \right] = 0$$

Thus, to maximize the function $f^(k)(z)$, the denominator must be minimized (i.e. set to the lower bound of z):

$$\lim_{k\to\infty} \left[\frac{r^k}{k!} \max_{z\in[-r,r]} \left| \frac{(-1)^k \cdot k!}{(1+z)^{(k+1)}} \right| \right] = \lim_{k\to\infty} \left[\frac{r^k}{k!} \cdot \left| \frac{(-1)^k \cdot k!}{(1-r)^{(k+1)}} \right| \right] = \lim_{k\to\infty} \left[r^k \cdot \frac{1}{(1-r)(1-r)^k} \cdot |(-1)|^k \right]$$

$$\Rightarrow \lim_{k\to\infty} \left[\left(\frac{r}{1-r} \right)^k \cdot \frac{1}{1-r} \right] = 0$$

For the above to be true, $\dfrac{r}{1-r} < 1$

$$\Rightarrow \frac{r}{1-r} < 1 \Rightarrow r < 1-r \Rightarrow 2r < 1 \Rightarrow r < \frac{1}{2}$$

$$\therefore T(x) = f(x) = \frac{1}{1+x} \text{ for } |x| < \frac{1}{2}$$

Question 6

$r_t = \log\left(\frac{P_t}{P_{t-1}}\right)$ As we are dealing with small values of R_t, the following can be stated:

$$R_t = 0.01 \Rightarrow \frac{P_t - P_{t-1}}{P_{t-1}} = 0.01$$

$$P_t - P_{t-1} = 0.01P_{t-1} \Rightarrow P_t = 1.01P_{t-1}$$

Finding the first term of the taylor series expansion of r_t:

$$P_1(P_t) = \log\left(\frac{P_t}{P_{t-1}}\right) + \frac{(P_t - a)^k}{k!} f^{(k)}(P_t) = \log\left(\frac{P_t}{P_{t-1}}\right) + (P_t - a)\left(\frac{P_{t-1} - P_t}{P_t P_{t-1}}\right)$$

But a is very small, so $P_t - a \approx P_t$:

$$\Rightarrow r_t \approx \log\left(\frac{P_t}{P_{t-1}}\right) + \frac{P_t - P_{t-1}}{P_{t-1}}$$

Recall, $P_t \approx 0.01P_{t-1} \Rightarrow log\left(\frac{P_t}{P_{t-1}}\right) \approx 0$

$$\therefore r_t \approx \frac{P_t - P_{t-1}}{P_{t-1}} = R_t \text{ for small values of } R_t$$