# The Cross-Section of Average Delta-Hedge Option Returns Under Stochastic Volatility*

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**Abstract**

Existing evidence indicates that average returns of purchased market-hedge S&P 500 index calls, puts, and straddles are non-zero but large and negative, which implies that options are expensive. This result is intuitively explained by means of volatility risk and a negative volatility risk premium, but there is a recent surge of empirical and analytical studies which also attempt to find the sources of this premium. An important question in the line of a priced volatility explanation is if a standard stochastic volatility model can also explain the cross-sectional findings of these empirical studies. The answer is fairly positive. The volatility elasticity of calls and puts is several times the level of market volatility, depending on moneyness and maturity, and implies a rich cross-section of negative average option returns – even if volatility risk is not priced heavily, albeit negative. We introduce and calibrate a new measure of option overprice to explain these results. This measure is robust to jump risk if jumps are not priced.

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# 1 Introduction

Understanding how risk is priced is a main topic in economics. Option markets provide a significant example because of their large volume and rich data available and because of their economic importance, since options can be used as hedging and/or leveraged securities. The non-redundancy of options and sources of the observed unexplained option premium is a topic of interest: The empirical results of Aït-Sahalia et al. (2001), Buraschi and Jackwerth (2001), Coval and Shumway (2001), and Bakshi and Kapadia (2003a), among others, show that options are non-redundant assets and their prices are "too expensive." For example, Coval and Shumway (2001) report that purchased zero-beta straddles produce average losses of approximately three percent per week for the S&P 500 index. This indicates that, besides market risk, other risk factors are also priced. Evidence from these works suggests that among all candidates of additional risk factors, stochastic volatility seems to be the most important extra risk that is factored into asset prices, and, more importantly, that the premium associated with purchasing into volatility is negative.

Bakshi and Kapadia (2003a), Coval and Shumway (2001), and Jones (2006) indeed provide a comprehensive empirical study of the S&P500 index option risk premium for call, put, and straddle options using option returns.[1] Therefore, an important extension is whether a standard stochastic volatility model can, or where it fails to, explain the rich findings of these empirical studies – providing further rationale for volatility risk and a negative volatility risk-premium. In addition, since "stochastic volatility" and "jumps" are essential components of option-pricing models in explaining volatility smiles/smirks (e.g., Bates (1996), Pan (2001)), it is worth studying the implications of volatility risk for option returns in depth. Affine jump-diffusion models are routinely used for pricing purposes by practitioners and researchers alike, but option returns (and the connection between option prices and option returns) in jump-diffusion models have received less attention.

This paper introduces a new measure of option overprice which quantifies the impact on option prices of the volatility risk premium. This measure is given by the difference of two option prices. The basic idea is that an option price can be factorized into its risk components (Ibáñez (2008)). When applied to a stochastic volatility model, one can factorize out the volatility risk-premium. That is, the difference between the true option price and a fictitious price, which is associated to a pricing measure which assumes that volatility risk is not priced, is equal to the expectation of the stream of volatility risk-premiums under the previous probability measure. This price difference is denoted as the option overprice.

[1] Coval and Shumway (2001) and Bakshi and Kapadia (2003a) show their empirical findings are robust to downfall risk and jumps. Jones (2006) finds that, unlike market and volatility risks, jump risk is comparatively not priced.

On the other hand, the expected return of a dynamically delta-hedged option provides the same stream of volatility risk-premiums (Bakshi and Kapadia (2003a)). Since these two terms, expected option returns under the objective measure and option overprice, depend on the same volatility risk premiums and assume the same dynamics for volatility, the difference between them is less sensitive to a non zero equity premium. Accordingly, average delta-hedged option returns and option overprice are closely related – allowing us to focus on the overprice measure which is analytical.

The option overprice measure is derived in a standard stochastic volatility model, but it is robust to jump risk,[2] if jumps are not priced. This is as follows. Since the option overprice is given by the difference of two option prices, this difference offsets the absence of jump risk in the two option prices if jumps are not priced, which is a form of robustness against model risk. This is formally developed in the paper – and implies that we can compute the option overprice in a stochastic volatility model instead of a (more realistic) model which also includes jumps. This is convenient for us, since the option overprice is related to a tracking delta-hedging portfolio, and thus, we do not have to deal with jumps. As it can be too strong or unrealistic to assume that jumps are not priced (e.g., Bates (1996)), we provide a numerical example of the impact on option prices of priced jump risk.

Our results are a novel application of option-pricing in incomplete markets and are based on standard arguments of dynamic spanning (Ibáñez (2008)). For other examples of option-pricing in incomplete markets, see Naik and Lee (1990) and Merton (1998). The large negative option premium has led to the surge of empirical and analytical studies which also attempt to find the sources of this premium. See Benzoni et al. (2005), Bondarenko (2003), Branger and Schlag (2008), Broadie et al. (2008), Cao and Huang (2008), Constantinides et al. (2008), Garleanu et al. (2007), Low and Zhang (2005), Santa-Clara and Saretto (2007), and Wu (2007), among others.[3] For instance, Driessen et al. (2008) relate the option risk-premium to correlation risk. And Carr and Wu (2008) also find a robust negative variance risk-premia by using variance swaps. We also show how the option overprice measure relates to variance risk-premia.

Then, we compute the overprice or price impact of a negative market volatility risk premium by using Heston (1993) stochastic volatility model. The parameter values of Heston's model are based on Pan (2002) empirical results for the S&P 500 index and option markets. Let $C(0)$ denote the true option price, and $\widetilde{C}(0)$ a fictitious price associated with a zero market volatility price of risk. We conduct an extensive numerical exercise from which we derive the following robust findings: The

[2] Eraker et al. (2003) report that jumps in equity indices typically generates about 10% to 15% of returns variance. Carr and Wu (2003) show that jumps are important to generate the smiles that we observe in short-maturity options.

[3] For single equities, the empirical evidence indicates a much less significant option risk-premium compared to the S&P500 index; see Bakshi and Kapadia (2003b), Carr and Wu (2008), and Driessen et al. (2008).

(total and percent) option overprice is optimized across strike prices as follows,

$$
\begin{cases}
C(0) - \widetilde{C}(0) : \begin{cases} \text{Maximized: at-the-money calls/puts/straddles} \\ \text{Minimized: away-from-the-money calls/puts/straddles} \end{cases} \\
\frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)} : \begin{cases} \text{Maximized:} \begin{cases} \text{out-of-the-money calls/puts} \\ \text{away-from-the-money straddles} \end{cases} \\ \text{Minimized:} \begin{cases} \text{in-the-money calls/puts} \\ \text{near at-the-money straddles} \end{cases} \end{cases}
\end{cases}
$$

The premium "$C(0) - \widetilde{C}(0)$" is maximized for at-the-money options, since they have the largest volatility risk. The premium over the price "$(C(0) - \widetilde{C}(0))/\widetilde{C}(0)$" is maximized (minimized) for out-of-the-money (in-the-money) call/puts and the relationship is convex. This implies large differences across strike prices and is primarily driven by the low prices of out-of-the-money calls/puts. The percent impact is maximized for away-from-the-money straddles, since they include out-of-the-money calls or puts, though the difference with at-the-money straddles is smaller. We also find a term effect, the total and percent price impact go up with maturity for calls/puts/straddles, except for the percent impact of short-term out-of-the-money calls which lowers with maturity. If jumps imply more leptokurtic returns, the percent impact on out-the-money puts (calls) can lower (rise).

These results are consistent with the cross-section findings Bakshi and Kapadia (2003a) for calls/puts and Coval and Shumway (2001) for straddles. For example, a conservative average volatility risk premium of $-8.45\%$ ($-12\%$) per year implies that the percent price impact on a ten percent out-of-the-money call/put with three months to maturity is 25% (36%); i.e., approximately, 2% (3%) per week! This large option premium is fully arbitrage-free and is explained by a large volatility risk. That is, the volatility elasticity (option Vega over option price) of at- and especially out-of-the-money calls/puts is up to ten times the level of market volatility implying large option risk premiums. The main discrepancy is that the percent impact on out-of-the-money puts rises with maturity, indicating that only a volatility risk premium cannot totally explain expensive short-term out-of-the-money puts (e.g., Bakshi and Kapadia (2003a), Jones (2006)).

We also provide a simulation study of dynamically delta-hedged option returns to prove the robustness of the previous findings. 1) Expected option returns and the option overprice are very close if the equity risk premium is not large, which is a sensible assumption and confirms our intuition. 2) We compute the standard deviation of delta-hedged option returns and find it is large, which is consistent with the large volatility risk of at- and out-of-the-money calls/puts. 3) Consider a call option with a longer maturity, e.g., three months. The expected daily/weekly delta-hedged option return, over one day/week, can be twice the expected return, over three months, when the option is dynamically delta-hedged until maturity. Hence, the option maturity and the investment horizon matter when computing expected option returns.

In sum, this paper has two main contributions: First, we introduce a new measure to address the option risk premium under stochastic volatility (i.e., under market and volatility risks). This measure is analytical, relates option prices to option returns, and is robust to jump risk if jumps are not priced. Second, in a simple Heston's (1993) setting, an extensive numerical exercise shows that volatility risk can explain an important piece of the cross-section of this premium for the S&P 500 index. For the well studied out-of-the-money puts, we do not find that they are necessarily mispriced, but that their large delta-hedged average returns are compensation for a volatility risk which is several times the level of market volatility. This risk is exacerbated if jumps, model risk, or other frictions (such as margin requirements, Santa-Clara and Saretto (2007)) are also considered.

The rest of the paper is as follows. Section 2 derives the measure of option overprice under stochastic volatility. Then, we expand these ideas using Heston's model. Section 3 studies the price impact of a negative volatility risk premium from a calibrated model. In Section 4 we divide the volatility process into two components, one perfectly correlated and one orthogonal to the market, and provide the same analysis for the orthogonal component of volatility. Section 5 computes delta-hedged option returns by simulation. Section 6 concludes.

# 2 A measure of option overprice under stochastic volatility

In this section, we introduce a measure of option overprice which quantifies the impact on option prices of the volatility risk-premium. We show that this measure is closely related to the expected return of a dynamically delta-hedged option and derive several of its properties.

## 2.1 Expected Option Returns and the Pricing of Options: Bridging the Gap

A negative volatility risk premium can be explained following Bakshi and Kapadia (2003a). The instantaneous expected gain of a purchased delta–hedged call option is equal to "$\lambda(v) \times \frac{\partial C}{\partial v} \times dt$," where $\frac{\partial C}{\partial v}$ is the call Vega and $v$ is the (instantaneous) variance. Since $\frac{\partial C}{\partial v} > 0$, the volatility risk premium is negative if the market volatility price of risk is negative; i.e., "$\lambda(v) < 0$." The same result follows if the call option is dynamically delta-hedged until maturity, $T$; i.e.,

$$E_0 \left[ \int_0^T e^{r(T-t)} \lambda(v) \frac{\partial C}{\partial v} dt \right] < 0 \quad \text{if} \quad \lambda(v) < 0, \tag{1}$$

where $E_0$ denotes the expectation under the objective $\mathcal{P}-$measure, and the risk-premiums are invested/financed at the riskless rate, $r$.

Hence, the cross-section of average option returns can be studied from equation (1). For example, let $\frac{\partial P}{\partial v}$ denote the put Vega; then $\frac{\partial C}{\partial v} = \frac{\partial P}{\partial v}$ from put-call parity and implies that average gains of purchased delta-hedged calls and puts and zero-delta straddles with the same strike price and maturity are the same. However, because the expectation in equation (1) is not analytically known, we introduce a second question. A negative volatility risk premium implies that options are expensive; e.g., Black-Scholes implied volatility goes up. Let $C(0)$ be the call price which is associated with $\lambda$. The question is to quantify how expensive options are if $\lambda < 0$.

We introduce a fictitious call price, denoted by $\widetilde{C}(0)$, associated with a fictitious zero market volatility price of risk, $\widetilde{\lambda} = 0$. The price $\widetilde{C}(0)$ is equal to the cost of a tracking portfolio which delta-hedges the market risk of the call. This implies the following: Since $C(0)$ is the market price of the call, the difference $C(0) - \widetilde{C}(0)$ can be interpreted as a compensation for volatility risk. Hence, $C(0) - \widetilde{C}(0)$ is a measure of option overprice. More importantly, it holds that

$$C(0) - \widetilde{C}(0) = -E_0^{\widetilde{Q}} \left[ \int_0^T e^{-r(T-t)} \lambda(v) \frac{\partial C}{\partial v} dt \right] > 0 \text{ if } \lambda(v) < 0, \tag{2}$$

where $\widetilde{Q}$ is a (risk–neutral) probability measure associated with the fictitious $\widetilde{\lambda} = 0$. These results are an application of option-pricing theory in incomplete markets and Feynman-Kac Theorem (Ibáñez (2008) and Duffie (2001)). We will write out this result in Heston's (1993) model below.

Equations (1) and (2) allow us to bridge the gap between option returns and the pricing of options with regard to stochastic volatility. The average gain of purchased delta–hedged options in equation (1) is negative, if and only if, the option overprice in equation (2) is positive. Let us compare the two expectations in equations (1) and (2). Both terms depend on the same stream of volatility risk premiums, differing only in the probability measures, $\mathcal{P}$ and $\widetilde{Q}$, respectively. $\mathcal{P}$ and $\widetilde{Q}$ have the same zero price of volatility risk, $\widetilde{\lambda} = 0$. Since both terms depend essentially on volatility paths, the difference between them is less sensitive to a non-zero equity premium. Accordingly, average delta-hedged option returns and option overprice are economically and quantitatively closely related, allowing us to directly focus on the option overprice, $C(0) - \widetilde{C}(0)$, which is analytical.

We now explain several additional properties of the measure of option overprice.

**The overprice as an expected delta-hedged option return** The price difference $C(0) - \widetilde{C}(0)$ is also equal to the expected gain of an option writing strategy which does not have market risk either. That is, consider the delta–hedging tracking portfolio that is based on the fictitious price. Its initial cost is $\widetilde{C}(0)$. The difference $C(0) - \widetilde{C}(0)$ can be invested in a bank account. As the volatility risk of this new delta-hedging portfolio has zero mean (since we assume that $\widetilde{\lambda} = 0$), then $(C(0) - \widetilde{C}(0)) \times e^{rT}$ is the expected gain.

Therefore, for a selling price $C(0)$, the option writer can decide between receiving either an up-front premium $C(0) - \widetilde{C}(0)$ or, alternatively, a stochastic stream of volatility risk-premiums. This is a novel implication of our results and can be studied in depth in future research.

**The overprice can be computed without the market volatility price of risk** Assume that the stochastic volatility model is correctly specified. Then, we can substitute $C(0)$ by the market price of the option, which is observable. Since $\widetilde{C}(0)$ assumes that volatility risk is not priced and that the equity risk-premium is zero, we only need the dynamics of volatility under the objective measure. This simplifies the implementation of this measure in practice.

**The overprice is robust to jump risk if jumps are not priced** An interesting property of the option overprice in equation (2) is the robustness to model risk and, in particular, to jump risk if jumps are not priced. Since the option overprice is given by the difference of two option prices, this difference offsets the absence of jump risk in the two option prices if jumps are not priced. In the same way, the measure of option overprice is robust to the mispecification of the volatility process. Another way to see this result, from the right hand side of equations (1) or (2), is if the option vegas are similar with or without jumps; i.e., $C_v \approx C_v^{jd}$, where $C_v$ ($C_v^{jd}$) is the option Vega in a model without (in the correctly specified model, with) jumps. We delay the details of this result until the stochastic volatility model is specified. The robustness of equation (1) to jump risk is also addressed in Bakshi and Kapadia (2003a).

**Variance Swaps** Equations (1) and (2) hold, in particular, for those securities which do not depend on the underlying price $S$; i.e., the associated delta is equal to zero, $\frac{\partial C}{\partial S} = 0$. One example are variance swaps. We will show how these two equations are related for variance swaps below.

In sum, the new measure has several properties. Namely, (i), it is analytical if option prices are analytical. (ii) It relates option prices to option returns. (iii) It does not depend on the equity risk-premium, which is difficult to estimate in general. (iv) If the option-pricing model is properly specified, one can substitute the option price derived from the model by the market price, and in this way, we do not need the market price of volatility risk which is not observable. (v) Since the overprice is given by the difference of two option prices, it is robust to model risk given that this difference can offset the mispecification. And (vi), equations (1) and (2) can be generalized to multi-factor stochastic volatility models (e.g., Christoffersen et al. (2007), Duffie et al. (2000)). Next, we consider a specific volatility model.

## 2.2 The Stochastic Volatility Model

We consider Heston (1993) model, which is tractable and has good empirical properties and is most suitable to our analysis. Under the actual probability measure, $\mathcal{P}$,

$$dS = \mu S dt + \sqrt{v} S dw_{1,t}, \text{ and} \tag{3}$$

$$dv = \kappa(\theta - v)dt + \sigma\sqrt{v}\left(\rho dw_{1,t} + \sqrt{1-\rho^2}dw_{2,t}\right), \tag{4}$$

where $S$ is the Index or risky asset with drift $\mu$, and where $v$ is the stochastic variance which is mean reverting with mean reversion rate $\kappa$, long-term level $\theta$, and volatility $\sigma$.

The shocks $dw_1$ and $dw_2$ are two standard orthogonal Wiener processes, $E_t[dw_1 dw_2] = 0$. The parameter $\rho$ measures the correlation between returns and volatility, $E_t[dSdv] = \rho\sigma v S dt$. The shock $dw_2$ affects only volatility $v$. We define a new process, $dw_{v,t} = \rho dw_{1,t} + \sqrt{1-\rho^2}dw_{2,t}$.

Let $r$ be the constant riskless rate and $C(S, v, t)$ be the price of a derivative security (e.g., a call or put option). Following non-arbitrage arguments, the option price satisfies the following partial differential equation (PDE, see Heston's equation (6)),

$$C_t + rSC_S + \frac{1}{2}vS^2C_{SS} + \left[\kappa(\theta - v) - \lambda(S, v, t)\right]C_v + \frac{1}{2}\sigma^2 v C_{vv} + \rho\sigma v S C_{vS} = rC. \tag{5}$$

For simplicity in the notation, we suppress the dependence of $C$ on $S, v$, and $t$. The parameter $\lambda(S, v, t)$ is the risk premium associated with stochastic volatility. Because volatility $v$ is not a tradable asset, one must make assumptions about this risk premium. Using equilibrium arguments, Heston shows that an appropriate risk premium is proportional to $v$, i.e., $\lambda(S, v, t) = \lambda v$.

It is appropriate to specify the market prices of risk of the orthogonal shocks

$$\text{“}\lambda_1\sqrt{v}\text{”} \quad \text{and} \quad \text{“}\lambda_2\sqrt{v}\text{”} \tag{6}$$

associated with $dw_1$ and $dw_2$, respectively. Therefore, the equity risk premium and the market volatility risk premium are given, respectively, by

$$\mu - r = \lambda_1 v \quad \text{and} \quad \lambda v = \left(\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2\right)\sigma v. \tag{7}$$

The value of $\lambda_2\sigma v$ is similar to the market volatility risk premium in Heston and the equity risk premium $\lambda_1 v$ is like Pan (2002). If $|\rho| = 1$, we define $\lambda_2 = 0$. Note that $\lambda = 0$ if $\lambda_2 = \frac{-\rho}{\sqrt{1-\rho^2}}\lambda_1$.

The previous PDE pricing equation (5) can be rewritten now as

$$C_t + \mu SC_S + \frac{1}{2}vS^2C_{SS} + \kappa(\theta - v)C_v + \frac{1}{2}\sigma^2 v C_{vv} + \rho\sigma v S C_{vS} = rC + \lambda_1 v S C_S + \frac{\lambda}{\sigma}\sigma v C_v. \tag{8}$$

The option provides two risk premiums, a market-related premium and a volatility risk premium.

### 2.2.1 Average Delta-Hedged Option Returns

We follow Bakshi and Kapadia (2003a). Equation (3) can be rewritten as

$$dS = (r + \lambda_1 v)\, Sdt + \sqrt{v} S dw_{1,t}, \tag{9}$$

and by Ito's Lemma, and from the PDE pricing equation (8), the option price $C$ satisfies

$$dC = \left( rC + \lambda_1 vSC_S + \frac{\lambda}{\sigma}\sigma vC_v \right) dt + \sqrt{v}SC_S dw_{1,t} + \sigma\sqrt{v}C_v dw_{v,t}. \tag{10}$$

Consider a delta-hedging strategy on the risky asset $S$,

$$h_1 = C_S. \tag{11}$$

Let $X_t^h$ be the price of a portfolio that consists of a short position in the option, a long position $h_1$ of stock, plus a net cash position that earns the riskless rate. The net cash position implies this portfolio is self-financing, any additional cash that it is needed to maintain the hedged portfolio over time will be borrowed at the riskless rate and become part of the net cash position. From equations (9) to (11), the portfolio gain is given by

$$\begin{aligned} dX_t^h &= h_1 dS - dC - (h_1 S - C)\, rdt \\ &= \left( (r + \lambda_1 v)\, Sh_1 - \left( rC + \lambda_1 vSC_S + \frac{\lambda}{\sigma}\sigma vC_v \right) - (h_1 S - C)\, r \right) dt - \sigma\sqrt{v}C_v dw_{v,t} \\ &= -\left( \frac{\lambda}{\sigma}\sigma vC_v dt + \sigma\sqrt{v}C_v dw_{v,t} \right). \end{aligned} \tag{12}$$

Now, integrating $dX_t^h$ until the option maturity $T$, and investing these gains to the riskless rate,

$$\int_0^T e^{r(T-t)} dX_t^h = -\lambda \int_0^T e^{r(T-t)} vC_v dt - \sigma \int_0^T e^{r(T-t)} \sqrt{v}C_v dw_{v,t}. \tag{13}$$

Taking expectations under the actual probability measure, which is denoted by $E_t[.]$,

$$E_t\left[ dX_t^h \right] = -\lambda vC_v dt, \tag{14}$$

and, by the law of the iterated expectation,

$$E_0\left[ \int_0^T e^{r(T-t)} dX_t^h \right] = -\lambda E_0\left[ \int_0^T e^{r(T-t)} vC_v dt \right]. \tag{15}$$

For a short call/put option, $-E_0\left[\int_0^T e^{r(T-t)} vC_v dt\right] < 0$ since $C_v > 0$, and

$$E_0\left[\pi_{0,T}\right] = E_0\left[ \int_0^T e^{r(T-t)} dX_t^h \right] > 0, \text{ if and only if, } \lambda < 0\ . \tag{16}$$

Note the option return is the option gain divided by the option price; i.e., $\frac{1}{C(0)}\pi_{0,T}$.

### 2.2.2 The Option Overprice

We quantify the impact on option prices of the volatility risk premium. Let $Q$ be the unique risk-neutral probability measure given the prices of risk $\lambda_1$ and $\lambda_2$. Let $C(0) = E_0^Q \left[e^{-rT}C(T)\right]$. We define $Z(0) = C(0)$ and

$$dZ(t) - Z(t)rdt = h_1(t)\left(dS_t - rS_t dt\right) = dC(t) - rC(t)dt + dX_t^h, \tag{17}$$

where the tracking or hedging error, $dX_t^h$, is given by equation (12) above. $Z(t)$ is the value process of a self-financing tracking portfolio, which is designed to dynamically delta-hedge the option payoff $C(T)$. Integrating over time, and since $d\left[e^{-rt}C(t)\right] = e^{-rt}\left(dC(t) - rC(t)dt\right)$, it follows that

$$\begin{aligned} Z(T) &= e^{rT}Z(0) + C(T) - e^{rT}C(0) + \int_0^T e^{r(T-t)}dX_t^h \\ &= C(T) - \int_0^T e^{r(T-t)}\lambda v C_v dt - \int_0^T e^{r(T-t)}\sigma\sqrt{v}C_v dw_{v,t}. \end{aligned} \tag{18}$$

The tracking portfolio has the same price as the option, since $Z(0) = C(0)$. However, the payoffs at maturity are not identical because of the residual volatility risk. Now, we provide an application of option-pricing theory to incomplete markets. We show that it is possible to delta-hedge the market risk of the option, but assuming that the volatility risk premium is zero. Then, we show that the cost of a new delta-hedging tracking portfolio is lower than the option price if these premiums are negative.

Let $\widetilde{Q}$ be an equivalent probability measure with the same equity price of risk $\widetilde{\lambda}_1\sqrt{v} = \lambda_1\sqrt{v}$ for $dw_1$, but a fictitious market volatility price of risk $\widetilde{\lambda} = 0$ (which implies that $\widetilde{\lambda}_2 = \frac{-\rho}{\sqrt{1-\rho^2}}\lambda_1$ for $dw_2$). Consider the corresponding fictitious option price process, $\widetilde{C}(t)$,

$$d\widetilde{C} = \left(r\widetilde{C} + \lambda_1 v S\widetilde{C}_S\right)dt + \sqrt{v}S\widetilde{C}_S dw_{1,t} + \sigma\sqrt{v}\widetilde{C}_v dw_{v,t}, \tag{19}$$

where $\widetilde{C}(T) = C(T)$ at maturity, since the option price would be independent of the risk premia. Define $\widetilde{C}(0) = E_0^{\widetilde{Q}}\left[e^{-rT}\widetilde{C}(T)\right]$.

Now, define the hedging strategy

$$\widetilde{h}_1 = \widetilde{C}_S, \tag{20}$$

and define the processes $d\widetilde{X}$ and $d\widetilde{Z}$ as $dX$ and $dZ$, respectively, but based on $\widetilde{h}_1$ instead of $h_1$. Let $\widetilde{Z}(0) = \widetilde{C}(0)$. Then,

$$\begin{aligned} \widetilde{Z}(T) &= e^{rT}\widetilde{Z}(0) + \widetilde{C}(T) - e^{rT}\widetilde{C}(0) + \int_0^T e^{r(T-t)}d\widetilde{X}_t^h \\ &= \widetilde{C}(T) - \int_0^T e^{r(T-t)}\sigma\sqrt{v}\widetilde{C}_v dw_{v,t}, \end{aligned} \tag{21}$$

since we assume that $\widetilde{\lambda} = 0$. Thus, the hedging strategy $\widetilde{h}_1$ associated with $\widetilde{Z}$ tracks the option payoff (since $\widetilde{C}(T) = C(T)$) except for the volatility risk, and has a price of $\widetilde{Z}(0) = \widetilde{C}(0)$.

The two tracking portfolios, $Z(T)$ and $\widetilde{Z}(T)$, produce a residual volatility risk, which only differs in the option vegas, $C_v$ and $\widetilde{C}_v$, respectively. We numerically confirm that these two vegas are similar. Next, we compare the initial prices of the two portfolios, $Z(0)$ and $\widetilde{Z}(0)$, which are equal to the option prices, $C(0)$ and $\widetilde{C}(0)$, respectively. The process $e^{-rt}Z(t)$, $t \leq T$, is a martingale under both $Q$ and $\widetilde{Q}$.[4] We take expectations under $\widetilde{Q}$.

**Proposition 1**.

$$\begin{aligned} C(0) &= Z(0) = E_0^{\widetilde{Q}}\left[e^{-rT}Z(T)\right] \\ &= E_0^{\widetilde{Q}}\left[e^{-rT}C(T)\right] - \frac{\lambda}{\sigma}E_0^{\widetilde{Q}}\left[\int_0^T e^{-rt}\sigma v C_v dt\right] \\ &= \widetilde{C}(0) - \frac{\lambda}{\sigma}E_0^{\widetilde{Q}}\left[\int_0^T e^{-rt}\sigma v C_v dt\right]. \end{aligned} \tag{22}$$

Moreover, since Vega is positive for put and call options (i.e., $C_v > 0$), then

$$C(0) - \widetilde{C}(0) = -\lambda E_0^{\widetilde{Q}}\left[\int_0^T e^{-rt} v C_v dt\right] > 0 \text{ if and only if } \lambda < 0. \quad \blacksquare \tag{23}$$

We have decomposed the option price $C(0)$ into the price of a tracking delta-hedged portfolio, $\widetilde{C}(0)$, plus a second term which depends on minus the market volatility risk premiums. (For more on the factorization of option prices, see Ibáñez (2008)). Average delta-hedged option gains/returns and the option overprice, equations (15) and (23), respectively, depend on the same stream of market volatility risk premiums. The expected gain/return of purchased delta-hedged options is negative, if and only if, the option overprice is positive. These two terms are not equal since the two expectations depend on different probability measures, the actual measure $\mathcal{P}$ and a fictitious pricing measure $\widetilde{Q}$, respectively.

**The probability measure $\widetilde{Q}$** Let us write the dynamics of market returns and stochastic volatility under the new measure $\widetilde{Q}$. From $\widetilde{\lambda}_1 = \lambda_1$ and $\widetilde{\lambda} = 0$, it follows

$$dS = rSdt + \sqrt{v}S dw_{1,t}^{\widetilde{Q}}, \text{ and} \tag{24}$$

$$dv = \kappa(\theta - v)dt + \sigma\sqrt{v}\left(\rho dw_{1,t}^{\widetilde{Q}} + \sqrt{1-\rho^2}dw_{2,t}^{\widetilde{Q}}\right). \tag{25}$$

Since $\widetilde{\lambda} = 0$, the dynamics of volatility is the same under $\mathcal{P}$ and $\widetilde{Q}$. (i) Since the market volatility risk premiums (i.e., the terms $\lambda v C_v$) depend mostly on volatility paths, the difference between the r.h.s.

[4] The process $dZ(t)$ depends only on $dS_t$ (i.e., $dw_1$), and $Q$ and $\widetilde{Q}$ assume the same price of risk for $dw_1$, $\widetilde{\lambda}_1\sqrt{v} = \lambda_1\sqrt{v}$.

of equations (15) and (23) is less sensitive to a positive equity premium. (ii) For out-of-the-money calls, a positive $\lambda_1$ implies that near at-the-money trajectories, which are associated with a larger option Vega, are more plausible; hence, expectations under $\mathcal{P}$ rise for out-of-the-money calls (relative to the expectation under the $\widetilde{Q}$ measure). The opposite holds for in-the-money calls. A reciprocal analysis follows for puts. These two results are confirmed in Section 5, where average delta-hedged option returns are computed by simulation.

The measures $\mathcal{P}$ and $\widetilde{Q}$ are the same, and hence, expected option gains and option overprice are equal too, if the equity risk premium is zero ($\lambda_1 = 0$ and $\mu = r$). The left-hand-side of equation (23) can be computed as the difference of two option prices from Heston's formula.

**The probability measures $Q$ and $\widetilde{Q}$** We compare the probability measures $Q$ and $\widetilde{Q}$. $Q$ is the pricing measure but $\widetilde{Q}$ is a fictitious measure. If the market is complete, the stock and a second option can be traded, $Q$ is the unique risk-neutral probability measure. If the market is incomplete, only the stock can be traded, $Q$ and $\widetilde{Q}$ are two of the multiple risk-neutral probability measures. $Q$ and $\widetilde{Q}$ imply the same dynamics for the risky asset (a drift equal to $r$). $Q$ and $\widetilde{Q}$ differ in the dynamics of stochastic volatility.

The dynamics of volatility under the risk-neutral probability measure, $Q$, is given by

$$dv = (\kappa + \lambda)\left(\frac{\kappa}{\kappa + \lambda}\theta - v\right)dt + \sigma\sqrt{v}\left(\rho dw_{1,t}^{Q} + \sqrt{1-\rho^2}dw_{2,t}^{Q}\right). \tag{26}$$

Let us assume that $\kappa > 0$ and $\kappa + \lambda > 0$ (i.e., volatility is stationary under $\widetilde{Q}$ and $Q$). If $\lambda < 0$, volatility is more volatile under under $Q$ than under $\widetilde{Q}$. That is, if $\lambda < 0$, the long-term volatility is larger and the mean reversion rate is lower under $Q$; $\frac{\kappa}{\kappa+\lambda}\theta > \theta$ and $\kappa + \lambda < \kappa$, respectively.

## 2.3 A Shorter Holding Period of Time

The average option gain and the option overprice can be generalized if the option is not held until maturity, but a shorter period of time $\tau$, $0 < \tau < T$. In this case, equation (15) is just given by

$$E_0\left[\int_0^\tau e^{r(\tau-t)}dX_t^h\right] = -\lambda E_0\left[\int_0^\tau e^{r(\tau-t)}vC_v dt\right] \tag{27}$$

and equation (23) by

$$C(0) - E_0^{\widetilde{Q}}\left[e^{-r\tau}C(\tau)\right] = -\lambda E_0^{\widetilde{Q}}\left[\int_0^\tau e^{-rt}vC_v dt\right] \tag{28}$$

which can be proved like Proposition 1 but defining the tracking portfolio $\widetilde{Z}(\tau) = C(\tau)$ at time $\tau$ (instead of $\widetilde{Z}(T) = C(T)$ at maturity $T$).

We do not know a closed-form solution for

$$E_0^{\widetilde{Q}}\left[e^{-r\tau}C(\tau)\right] = E_0^{\widetilde{Q}}\left[e^{-r\tau}E_\tau^{Q}\left[e^{-r(T-\tau)}C(T)\right]\right] = e^{-rT}E_0^{\widetilde{Q}}\left[E_\tau^{Q}\left[C(T)\right]\right], \tag{29}$$

since we cannot directly apply the law of the iterated expectation, if $Q \neq \widetilde{Q}$. However, the most left-hand-side expectation can be computed by simulation since $C(\tau)$ is analytical (and the price $\widetilde{C}(0)$ can be used as a control variable because $\widetilde{C}(0) = E_0^{\widetilde{Q}}[e^{-r\tau}\widetilde{C}(\tau)]$).

On the other hand, for a very short $\tau$, it follows (for both measures $\mathcal{P}$ and $\widetilde{Q}$)

$$E_0\left[\int_0^\tau e^{r(\tau-t)}dX_t^h\right] \approx -\lambda v C_v \tau \text{ and } E_0\left[\left(\int_0^\tau e^{r(\tau-t)}dX_t^h\right)^2\right] \approx vC_v^2\tau \text{ , if } \tau \to 0,$$

which can be used to approximate a brief holding period (e.g., a day or a week). One observation is worth to made here. If we consider the option overprice divided by the holding period; i.e.,

$$\frac{C(0) - E_0^{\widetilde{Q}}\left[e^{-r\tau}C(\tau)\right]}{\tau} = -\lambda E_0^{\widetilde{Q}}\left[\int_0^\tau e^{-rt}v\frac{C_v}{\tau}dt\right],$$

we expect this quantity to be bound by the instantaneous gain, $-\lambda v C_v$ (assuming $\lambda < 0$), if $\frac{d}{dt}C_v(t) < 0$, which is confirmed in the simulation exercise. For example, $C_v^{BS}(t) = S\sqrt{T-t}N'(d_1) < S\sqrt{T}N'(d_1) = C_v^{BS}(0)$, for $S$ and $d_1$ constants, in the Black and Scholes model.

## 2.4 The Impact on Prices of Calls, Puts, and Straddles

**Calls/Puts** Let $P(0)$ denote the price of put options. For the same strike price, from put-call parity, the option Vega is the same for calls and puts; i.e., $C(0) + e^{-rT}K = S_0 + P(0)$, then $C_v = P_v$.

This implies that average option gains and option overprice, associated with stochastic volatility, are the same for calls and puts with the same strike price, $P(0) - \widetilde{P}(0) = C(0) - \widetilde{C}(0)$. That is, the option overprice is the same for in-the-money calls and out-of-the-money puts. The price impact is maximized for at-the-money calls and puts because they have the largest Vega.

We can focus on the percent price impact $\frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)}$, which in addition depends on the option moneyness, $e^{-rT}K/S_0$.[5] Then,

$$\frac{P(0) - \widetilde{P}(0)}{\widetilde{P}(0)} = \frac{C(0) - \widetilde{C}(0)}{\widetilde{C}(0) + e^{-rT}K - S_0}. \tag{30}$$

Consequently, if $\lambda < 0$,

$$\frac{P(0) - \widetilde{P}(0)}{\widetilde{P}(0)} \gtrless \frac{C(0) - \widetilde{C}(0)}{\widetilde{C}(0)} \quad \text{if } S_0 \gtrless e^{-rT}K, \tag{31}$$

[5] We can divide all option prices by $S_0$; i.e., $\frac{\widetilde{C}(0)}{S_0} = E_0^{\widetilde{Q}}\left[\frac{e^{-rT}S_T}{S_0}1_{\{S_T>K\}}\right] - \frac{e^{-rT}K}{S_0}E_0^{\widetilde{Q}}\left[1_{\{S_T>K\}}\right]$.

and, if $\lambda > 0$,

$$\frac{P(0) - \widetilde{P}(0)}{\widetilde{P}(0)} \lesseqgtr \frac{C(0) - \widetilde{C}(0)}{\widetilde{C}(0)} \quad \text{if } S_0 \gtreqless e^{-rT}K. \tag{32}$$

For example, if $\lambda < 0$ ($\lambda > 0$), in percent terms, out-of-the-money put prices go up (down) more than the corresponding in-the-money call prices since $S_0 > e^{-rT}K$. Therefore, although puts and calls have (i) the same implied volatility and (ii) the same price impact "$C(0) - \widetilde{C}(0)$," this result implies that out-of-the-money puts are more expensive than in-the-money calls in percent terms if $\lambda < 0$. For at-the-money options ($S_0 = e^{-rT}K$), the price impact is the same for calls and puts.

From our numerical exercise, we also find that the measure $\frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)}$ is maximized for out-of-the-money calls/puts, which is driven by a low price $\widetilde{C}(0)$ in the denominator. Therefore, we have the following result: "If $\lambda < 0$, out-of-the-money (at-the-money) calls/puts maximize the premium "$C(0) - \widetilde{C}(0)$" subject to a constraint in the option price (in the number of options) to sell."

**Straddles** A zero-delta straddle (denoted by $D$) is a convex combination of calls and puts with the same strike price, and therefore, has the same Vega as the corresponding call/put ($D_v = C_v = P_v$). Let $D(0)$ be the price of a zero-delta straddle, which contains $\alpha$ calls and $1-\alpha$ puts, such that its delta sensitivity is zero (i.e., $D_S(0) = \alpha C_S(0) + (1-\alpha)P_S(0) = 0$). From put-call parity, $C_S - P_S = 1$. Then, $\alpha = 1 - C_S(0)$ and $0 < \alpha < 1$. This implies that zero-delta straddle average gains, and overprice, are the same as for calls/puts. The price impact is maximized for at-the-money (minimized for away-from-the-money) straddles as well.

In percent terms, since

$$\frac{D(0) - \widetilde{D}(0)}{\widetilde{D}(0)} = \frac{C(0) - \widetilde{C}(0)}{\widetilde{C}(0) + \widetilde{C}_S(0)\left(e^{-rT}K - S_0\right)},$$

it follows, if $\lambda < 0$,

$$\frac{P(0) - \widetilde{P}(0)}{\widetilde{P}(0)} \gtreqless \frac{D(0) - \widetilde{D}(0)}{\widetilde{D}(0)} \gtreqless \frac{C(0) - \widetilde{C}(0)}{\widetilde{C}(0)} \quad \text{if } S_0 \gtreqless e^{-rT}K. \tag{33}$$

Then, (i) the percent price impact is larger for out-of-the-money calls/puts than for away-from-the-money straddles with the same strike price. (ii) For straddles, the measure is maximized (minimized) for away-from-the-money (near at-the-money) straddles.[6] However, from our numerical exercise, the moneyness effect is small for straddles (e.g., if they are less than ten percent away-from-the-money).

[6] If $\widetilde{C}_S(0) \to 1$ (a deep in-the-money call), $\frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)+\widetilde{C}_S(0)\left(e^{-rT}K-S_0\right)} \approx \frac{P(0)-\widetilde{P}(0)}{\widetilde{P}(0)}$, and if $\widetilde{C}_S(0) \to 0$ (a deep out-of-the-money call), $\frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)+\widetilde{C}_S(0)\left(e^{-rT}K-S_0\right)} \approx \frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)}$.

### 2.5 Robustness with regard to jump risk and model risk

An important question is whether the new measure is robust to jump risk. We assume that jump risk is not priced. Consider a stochastic volatility jump-diffusion model (Bates (1996)). Under $\mathcal{P}$,

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{v} dw_{1,t} + J_t dN_t - \eta_t \gamma_t dt, \tag{34}$$

where $J_t \geq -1$ is the jump size, and $N$ is a Poisson process with intensity $\gamma_t$ and compensator $\eta_t \gamma_t dt$. Since jump risk is not priced, jumps are the same under $\mathcal{P}$ and $Q$. Using similar notation as above, where "$jd$" highlights the jump-diffusion model, let $Q^{jd}$ denote the risk-neutral measure; i.e., $C^{jd}(0) = E_0^Q\left[e^{-rT}C(T)\right]$. And $\widetilde{Q}^{jd}$ denotes a fictitious measure, which assumes that $\lambda = 0$; i.e., $\widetilde{C}^{jd}(0) = E_0^{\widetilde{Q}^{jd}}\left[e^{-rT}C(T)\right]$.

A Feynman-Kac theorem for jump-diffusions (Duffie (2001)) implies that

$$\begin{aligned}
E_0^{Q^{jd}}\left[e^{-rT}C(T)\right] &= E_0^{\widetilde{Q}^{jd}}\left[e^{-rT}C(T)\right] + \lambda E_0^{\widetilde{Q}^{jd}}\left[\int_0^T e^{-rt} v C_v^{jd} dt\right], \text{ and} \\
C^{jd}(0) - \widetilde{C}^{jd}(0) &= \lambda E_0^{\widetilde{Q}^{jd}}\left[\int_0^T e^{-rt} v C_v^{jd} dt\right].
\end{aligned} \tag{35}$$

Intuitively, if the option vegas are similar with or without jumps (i.e., $C_v \approx C_v^{jd}$), the r.h.s. terms of equation (23) in Proposition 1 and of equation (35) are too. This implies that, even though option prices are different in both models (i.e., $C \neq C^{jd}$ and $\widetilde{C} \neq \widetilde{C}^{jd}$), we have $C(0) - \widetilde{C}(0) \approx C^{jd}(0) - \widetilde{C}^{jd}(0)$. If $C_v$ is not sensitive to the volatility process, we have the same robustness with regard to the volatility model.

Let us formally show this result. Let denote by $\Pi^{jd}(x, \lambda) = C^{jd}(0) - \widetilde{C}^{jd}(0)$ the true option overprice. The variable $x$, if $x \neq 0$, denotes a measure of model risk; e.g., absence of jumps and/or mispecification of the volatility process. $\lambda$ is the price of volatility risk and we assume that jumps are not priced. Let $\Pi(\lambda) = C(0) - \widetilde{C}(0)$ be the option overprice under the no jumps model. We assume that the function $\Pi^{jd}$ is smooth. First, $\Pi^{jd}(0, \lambda) = \Pi(\lambda)$ if there is not model risk. Second, $\Pi^{jd}(x, 0) = 0$ if volatility risk is not priced, which implies $\Pi_x^{jd}(0,0) = \Pi_{xx}^{jd}(0,0) = 0$. A second order Taylor expansion at the point $(x, \lambda) = (0, 0)$ implies

$$\begin{aligned}
\Pi^{jd}(x, \lambda) &\approx \Pi(0,0) + \Pi_x^{jd}(0,0)x + \Pi_\lambda^{jd}(0,0)\lambda + \frac{1}{2}\left(\Pi_{xx}^{jd}(0,0)x^2 + \Pi_{\lambda\lambda}^{jd}(0,0)\lambda^2 + 2\Pi_{x\lambda}^{jd}(0,0)\lambda x\right) \\
&= \Pi_\lambda^{jd}(0,0)\lambda + \frac{1}{2}\Pi_{\lambda\lambda}^{jd}(0,0)\lambda^2 + \Pi_{x\lambda}^{jd}(0,0)\lambda x \approx \Pi(\lambda) + \Pi_{x\lambda}^{jd}(0,0)\lambda x,
\end{aligned} \tag{36}$$

where the error is order three. Consequently, the approximation $\Pi^{jd}(x, \lambda) \approx \Pi(\lambda)$ depends on the cross-derivative $\Pi_{x\lambda}^{jd}(0,0)$ and the approximation error is second order.

Let us provide a numerical example since intuition indicates that $\Pi_{x\lambda}^{jd}(0,0)$ can be small and insignificant. We assume the following parameters for the stochastic volatility process: $\kappa = 5.0$,

$\sqrt{\theta} = 0.13$, $\sigma = 0.25$, $\rho = -0.4$ and $r = 0.05$, and volatility is equal to its long term level, $\sqrt{v_0} = 0.13$. For the jump component, $\gamma = 0.2$ and $\ln(1 + J_t) \sim \mathcal{N}(-0.10, 0.10^2)$ is a Gaussian random variable. And the price of volatility risk, $\lambda = -1.775$. Model risk is due to the fact that we wrongly assume that the probability of jumps is zero; i.e., $x = \gamma$. For an at-the-money call with one month to maturity ($S_0 = 100$, $K = 100$, and $T = 1/12$), we obtain by finite-differences that $\Pi^{jd}_{x\lambda}(0, 0) = 0.0025$, and

$$\text{“}\Pi^{jd}(\gamma = 0.2, \lambda = -1.775) - \Pi(\lambda = -1.775) = -0.0009\text{” and “}\Pi^{jd}_{x\lambda}(0, 0) \times 0.2 \times (-1.775) = -0.00089,\text{”}$$

which shows that the second order approximation is very accurate.

When we compare $-0.0009$ with $\Pi(-1.775) = 0.0488$, the difference is less than 2% if we leave out jumps. For a three months to maturity call ($K = 100$ and $T = 0.25$), $\Pi^{jd}_{x\lambda}(0, 0) \approx 0.02$ and the difference is less than 4%. This shows the robustness of the volatility risk premium with regard to the absence/mispecification of jump risk if jumps are not priced. In percent terms, where the option price appears in the denominator, if jump risk implies a steeper volatility smile, the percent impact on puts (calls) can be smaller (larger).

**Jumps are priced** On the other hand, if jump risk is priced, the “volatility risk premium” is not a true volatility risk premium, but includes a “jump risk premium” component. We have followed the literature on the option risk premium, which explains this premium as a compensation for volatility risk and shows the robustness against downfall risk and jumps. However, estimates of option pricing models find that jump risk is priced (e.g., Bates (1996)). The measure of option overprice can be extended to take into account that both volatility risk and jump risk are priced. We need good estimators of these two risk parameters, since the overprice measure is sensitive to them. Therefore, an extension of this paper is to explain the option risk premium from both volatility risk and jump risk.

For completeness, we provide a numerical illustration of the option overprice when, in addition to volatility risk, jump risk is also priced. We assume that the probability of jumps increases under the risk-neutral measures (Bates(1996)); i.e., the ratio $\frac{\gamma^Q}{\gamma}$ measures how jump risk is priced. Let now denote by $\widetilde{C}^{jd}(0)$ a fictitious price associated with $\lambda = 0$ and $\gamma^Q = \gamma$, and $C^{jd}(0)$ the true option price. We redefine the overprice measure as $\Pi^{jd}(\gamma, \lambda, \frac{\gamma^Q}{\gamma}) = C^{jd}(0) - \widetilde{C}^{jd}(0)$, and $\Pi^{jd}(0, \lambda, \frac{\gamma^Q}{\gamma}) = \Pi(\lambda)$ if there are not jumps. We follow the example above, then,

| $\mathbf{\Pi}^{jd}\left(\gamma, \lambda = -1.775, \frac{\gamma^Q}{\gamma}\right)$ | $\gamma = 0$ | $\gamma = 0.2, \frac{\gamma^Q}{\gamma} = 1$ | $\gamma = 0.2, \frac{\gamma^Q}{\gamma} = 1.5$ | $\gamma = 0.2, \frac{\gamma^Q}{\gamma} = 2$ |
|---|---|---|---|---|
| $T = 1/12$ : | 0.0488 | 0.0479 | 0.0862 | 0.1245 |
| $T = 6/12$ : | 0.4383 | 0.4134 | 0.5608 | 0.7056 |

The first and second columns confirm that the overprice measure is robust to jump risk if jumps are not priced. Columns two to four illustrates that if jumps are priced, the volatility risk premium contains a jump component, which is more important for short-term options.

## 2.6 Variance Swaps

Equations (1) and (2) also hold for those securities whose payoff does not depend on the price $S$; i.e., $\frac{\partial C}{\partial S} = 0$. One example are variance swaps, which depend not only on the variance $v$, but on the quadratic variation of $S$. In this section, we relate the option overprice in equation (2) to the variance risk-premia of variance swaps, which is also given by equation (1). That is, we derive the variance risk-premia first from variance swaps and second from equation (2). For more on variance swaps, see Carr and Wu (2006) and (2008), Jiang and Tian (2006), and Wu (2007).

We follow Carr and Wu (2008). Assume that $N$ follows an independent Poisson process. First, the expected Realized Variance of the process $S$ under the $\mathcal{P}$–probability is defined as

$$\overline{RV}^{\mathcal{P}}_{0,T} = E^{\mathcal{P}}_0\left[\int_0^T \left(\frac{dS_t}{S_t}\right)^2\right] = E^{\mathcal{P}}_0\left[\int_0^T \left(v_t dt + J_t^2 dN_t\right)\right], \tag{37}$$

and, since the dynamics of $v$ is the same under $\mathcal{P}$ and $\widetilde{Q}$,

$$\overline{RV}^{\mathcal{P}}_{0,T} = E^{\widetilde{Q}}_0\left[\int_0^T v_t dt\right] + E^{\mathcal{P}}_0\left[\int_0^T J_t^2 dN_t\right], \tag{38}$$

The variance swap rate is defined as

$$SW^{Q}_{0,T} = E^{Q}_0\left[\int_0^T \left(\frac{dS_t}{S_t}\right)^2\right] = E^{Q}_0\left[\int_0^T \left(v_t dt + J_t^2 dN_t\right)\right]. \tag{39}$$

A variance swap, which has zero cost, pays the difference between realized variance and the variance swap rate (over time to maturity). Therefore, the expected gain of a variance swap is given by

$$\overline{RV}^{\mathcal{P}}_{0,T} - SW^{Q}_{0,T} = E^{\widetilde{Q}}_0\left[\int_0^T v_t dt\right] - E^{Q}_0\left[\int_0^T v_t dt\right] + E^{\mathcal{P}}_0\left[\int_0^T J_t^2 dN_t\right] - E^{Q}_0\left[\int_0^T J_t^2 dN_t\right], \tag{40}$$

which is the variance risk-premia. If we assume that jumps are not priced,

$$\overline{RV}^{\mathcal{P}}_{0,T} - SW^{Q}_{0,T} = E^{\widetilde{Q}}_0\left[\int_0^T v_t dt\right] - E^{Q}_0\left[\int_0^T v_t dt\right]. \tag{41}$$

In Heston's (1993) model (since $E^{\mathcal{P}}_0[v_t] = \theta + e^{-\kappa t}(v_0 - \theta)$; and by exchanging the integral and the expectation), this simplifies to

$$\begin{aligned} \frac{1}{T}E^{\widetilde{Q}}_0\left[\int_0^T v_t dt\right] - \frac{1}{T}E^{Q}_0\left[\int_0^T v_t dt\right] &= \frac{1}{T}\int_0^T \left(\theta + e^{-\kappa t}(v_0 - \theta)\right) dt - \frac{1}{T}\int_0^T \left(\theta^* + e^{-\kappa^* t}(v_0 - \theta^*)\right) dt \\ &= \theta + \frac{1 - e^{-\kappa T}}{\kappa T}(v_0 - \theta) - \left(\theta^* + \frac{1 - e^{-\kappa^* T}}{\kappa^* T}(v_0 - \theta^*)\right), \end{aligned} \tag{42}$$

where $\theta^* = \frac{\kappa}{\kappa+\lambda}\theta$ and $\kappa^* = \kappa + \lambda$.

Second, the idea is to define $C(0) = SW^Q_{0,T}$ and $\widetilde{C}(0) = \overline{RV}^{\widetilde{Q}}_{0,T}$ (like equations (39) and (38) but under $\widetilde{Q}$, respectively). For simplicity, let us assume that $r = 0$. If jumps are not priced, it follows from equation (41) that

$$C(0) - \widetilde{C}(0) = -\left(\overline{RV}^{\mathcal{P}}_{0,T} - SW^Q_{0,T}\right), \tag{43}$$

which relates the variance risk-premia and the overprice of variance swaps. In Heston's model, let us assume that $C(s, v_s)$ denotes the price of a contract that pays $v_t$ at time $t$; i.e., for $0 \leq s \leq t$,

$$C(s, v_s) = E^Q_s[v_t] = \theta^* + e^{-\kappa^*(t-s)}(v_s - \theta^*).$$

Then, the associated Vega is given by $C_v(s, v_s) = e^{-\kappa^*(t-s)}$, which does not depend on $v$. From the variance risk-premia definition, or equation (1), the expected gain of this contract is given by

$$\begin{aligned} E^{\mathcal{P}}_0[v_t] - E^Q_0[v_t] &= \theta + e^{-\kappa t}(v_0 - \theta) - \left(\theta^* + e^{-\kappa^* t}(v_0 - \theta^*)\right) \\ &= -\left(1 - e^{-\kappa^* t}\right)(\theta^* - \theta) - \left(e^{-\kappa^* t} - e^{-\kappa t}\right)(v_0 - \theta) \\ &= \frac{1 - e^{-\kappa^* t}}{\kappa^*}\theta\lambda - \left(e^{-\kappa^* t} - e^{-\kappa t}\right)(v_0 - \theta). \end{aligned} \tag{44}$$

From equation (2), by exchanging integral and expectations,

$$\begin{aligned} C(0, v_0) - \widetilde{C}(0, v_0) &= -\lambda E^{\widetilde{Q}}_0\left[\int_0^t v C_v ds\right] = -\lambda \int_0^t \left(\theta + e^{-\kappa s}(v_0 - \theta)\right) e^{-\kappa^*(t-s)} ds \\ &= -\lambda\left(\frac{1 - e^{-\kappa^* t}}{\kappa^*}\theta + \frac{e^{-\kappa t} - e^{-\kappa^* t}}{\kappa^* - \kappa}(v_0 - \theta)\right) \\ &= -\frac{1 - e^{-\kappa^* t}}{\kappa^*}\theta\lambda + \left(e^{-\kappa^* t} - e^{-\kappa t}\right)(v_0 - \theta). \end{aligned} \tag{45}$$

If we consider that $C$ pays the average $\frac{1}{T}\int_0^T v_t dt$, instead of $v_t$, we have the desired result.

# 3 The Impact on Option Prices: Numerical Evidence

We now provide numerical evidence of the impact on option prices of a (negative) market volatility risk premium. We consider straddle, call, and put options. We first explain the results of a calibrated stochastic volatility model, then compare them with the rich empirical work of Bakshi and Kapadia (2003a), Coval and Shumway (2001), and Jones (2006). The parameter values of this numerical exercise are similar to those of Liu and Pan (2003, 414), which are based on Pan (2002) empirical results for the S&P 500 index and option markets.

For the stochastic volatility process, $\kappa = 5.0$, $\sqrt{\theta} = 0.13$, $\sigma = 0.25$, and $\rho = -0.4$. The riskless rate is $r = 0.05$. The pricing parameters are as follows. For the market component, $\lambda_1 = 4$ which implies an average equity price of risk of $\lambda_1\sqrt{\theta} = 0.52$ (and an average equity premium of $\lambda_1\theta = 0.0676$) per year. For the second component, the base case parameter is a conservative $\lambda_2 = -6$, and $\lambda_2\sqrt{\theta} = -0.78$. This implies that $\lambda/\sigma = \rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2 = -7.0991$ and the average market volatility price of risk is equal to $\lambda/\sigma\sqrt{\theta} = -0.9229$ (an average volatility risk premium of $\lambda/\sigma\theta = -0.12$). Note that $\kappa + \lambda > 0$, volatility is stationary under $Q$.

We assume the initial value of volatility is its long term level, $\sqrt{v_0} = 0.13$, and $S_0 = 100$.

We consider call options, $C(T) = \{S_T - K\}^+$, where $K$ is the strike price and $T = 0.25$ is the maturity. We compute call prices using Heston's formula, where $\lambda = 0$ for the price $\widetilde{C}$. The prices of puts and straddles are obtained from put-call parity. We provide several figures and focus on the price differences $C(0) - \widetilde{C}(0)$ and $\frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)}$.

In Figure 1, we compute the price impact as a function of the market volatility price of risk, $\lambda/\sigma$. A negative (positive) $\lambda$ rises (lowers) the price of calls/puts/straddles. The curves are decreasing and convex and the pricing impact of a negative $\lambda$ can be notable. The upper-left picture shows $C(0) - \widetilde{C}(0)$ for calls, puts, and straddles with the same strike price. One can see the price impact is maximized for at-the-money (minimized for away-from-the-money) calls/puts/straddles, since Vega is maximized for near at-the-money options. For example, for the near at-the-money option with $K = 100$, $\lambda/\sigma = -7$ (i.e., $\lambda/\sigma\sqrt{\theta} = -0.91$) implies a price increase of 0.20 dollars (6.20 per cent of the call price). For away-from-the-money calls ($K \neq S_0$), the impact of out-of-the-money calls is larger than of in-the-money calls. If we consider the percent impact, $\frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)}$, these results change.

The lower-left and lower-right pictures show that this measure is maximized for out-of-the-money calls and puts, respectively. This is driven by the small price of out-of-the-money calls/puts, implying that the price difference divided by the option price goes up quickly. As a straddle is a convex combination of a put and a call with the same strike price, the price impact is maximized for away-from-the-money straddles, but the effect is small across strike prices. For a strike price of $K = 110$, a conservative $\lambda/\sigma = -7$ implies a percent price increase of 36.40, 8.41, and 0.98 for the call, straddle, and put, respectively, for a holding period of three months (as $T = 0.25$).

In Figure 2, we compute the price impact across strike prices and across maturities, where $\lambda/\sigma = -7.0991$. As Figure 1, $C(0) - \widetilde{C}(0)$ is maximized for at-the-money options. $\frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)}$ is maximized for out-of-the-money call/puts and the impact is convex. The percent impact is minimized for near at-the-money straddles, but the moneyness effect is small. The term effect is clear. As the flow of volatility risk premiums and the option Vega rise with maturity, $C(0) - \widetilde{C}(0)$ is maximized for long term options. However, for deep out-of-the-money calls, $\frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)}$ is maximized for short term calls.

In Figure 3 we show the effect of the initial value of volatility, $\sqrt{v_0}$, which is mean-reverting.

The difference $C(0) - \widetilde{C}(0)$ is maximized for at-the-money options and is increasing and convex with regard to volatility. In percent terms, the effect of a larger volatility is more difficult to analyze, since it also rises the option price $\widetilde{C}(0)$. We find that $\frac{C(0)-\widetilde{C}(0)}{\widetilde{C}(0)}$ is more constant for near at-the-money calls/puts, but it is decreasing in volatility for out-of-the-money calls/puts. For straddles, the moneyness and volatility effects are not as large.

Figure 4 provides additional insights on two key volatility parameters. The two lower pictures consider $\sigma$, which is the volatility of volatility. The price impact of $\sigma$ is positive and almost linear (which is consistent with $\lambda$ is linear in $\sigma$, see equation (7)). The slopes are equal to 0.25 and 1.50 for $K = 100$ and $K = 110$, respectively, in the lower-right picture. Hence, an increase of 1% in $\sigma$ implies an increase in the percent price impact of 0.25% and 1.50%, respectively, for a fixed $\lambda/\sigma = -7.0991$. The two upper pictures show the effect of the speed of mean reversion of volatility, $\kappa$. As expected, increasing $\kappa$, volatility is less volatile and the price impact lowers.

### 3.1 Relationship with the Empirical Evidence

We summarize the existing empirical evidence for the S&P 500 index option market. We remark that the previous numerical results, where only volatility risk is priced, are generally consistent with the cross-section of the empirical evidence for negative plausible values of $\lambda$.

**Calls and Puts** Bakshi and Kapadia (2003a, 540-546) report losses and losses over the option price ($\pi$ and $\pi/C$, respectively). In their Table 1 they report losses for calls across strike prices and for two different maturities. These results are consistent with the upper- and lower-left pictures of our Figures 1 and 2. That is, $\pi$ is negative and is maximized for at-the-money calls, the impact is larger for near out-of-the-money than for near in-the-money calls. $\pi/C$ is maximized for out-of-the-money calls and the impact is convex across strike prices. There is also a term effect, $\pi$ rises with maturity, but $\pi/C$ is larger for short term out-of-the-money calls. The only discrepancy is that $\pi$ is positive for deep in-the-money calls; although they suffer from illiquidity, and therefore, a frictionless option-pricing model cannot be applied straightaway.

In their Table 2 they report losses for puts across strike prices and for two different maturities, which are consistent with the upper-left and lower-right pictures of our Figures 1 and 2. The larger discrepancy is that $\pi/P$ decreases with maturity ($\pi/P$ is larger for the put with shorter maturity), but our numerical exercise indicates the opposite. In their Table 3, they show that the losses of calls $\pi$ and $\pi/C$ rise with volatility (as our Figure 3, upper-left picture, and lower-left picture for near at-the-money options), and that $\pi/C$ of slightly out-of-the-money calls are larger than $\pi/C$ of slightly in-the-money calls (as our Figure 3 lower-left picture for $K = 105$ and $K = 100$, respectively).

Bakshi and Kapadia show that negative average gains are related to the option Vega (their Table 4). For example, in Black-Scholes, the option Vega is $C_v^{BS} = S\sqrt{T}N'(d_1)$, which is larger for near at-the-money and long-term calls/puts. The upper-left pictures of our Figures 2 and 3 clearly show that the price impact is maximized for near at-the-money and long-term options and high volatility regimens. Note, however, these results change if we consider the price impact over the option price.

In a recent study Jones (2006) finds that a flexible class of nonlinear models fails to explain the large negative returns of short-term out-of-the-money S&P 500 index puts. Only volatility risk fails too, since the overprice of puts in percent terms is maximized for out-of-the-money ones, but it rises with maturity.[7]

**Straddles** Coval and Shumway (2001, 995-998) find large negative returns for zero-beta straddle positions in their Table III (Panel A for SPX and Panel B for OEX). The most negative return corresponds with the farthest away-from-the-money straddle (and with the lowest strike price for the S&P index) and some convexity appears across strike prices. They report weekly and daily losses over the zero-beta straddle price, $\pi/D$. Their results are consistent with the upper-right picture of our Figure 2, where the percent price impact is convex across strike prices. For the four maturities, this relationship is minimized for near at-the-money straddles and is maximized for straddles with the lowest strike price (as the empirical evidence for the S&P index). We can increase the magnitude of the premium by increasing the market volatility price of risk (see the upper-right picture of our Figure 1) and by shortening the three months holding period (if we consider the premium per holding period, see our Tables 1 to 3 below).

# 4 The Orthogonal Component of Volatility

The volatility process can be divided into two orthogonal components. A first component, which is perfectly negatively correlated with market returns,[8] plus a second one, which is orthogonal to the market. The literature has focused on the overall market volatility risk premium. However, one can analyze the market volatility risk premium by analyzing separately each component of volatility.

The first component, because it is perfectly negatively correlated with market returns, includes a negative risk premium that is equal to the negative correlation multiplied by the equity risk premium.

[7] The price impact and the price of short-term out-of-the-money puts, i.e., $P(0) - \widetilde{P}(0)$ divided by $\widetilde{P}(0)$, are two very small numbers, and it is necessary to compute them with great numerical accuracy.

[8] This is referred to as the leverage effect. See, e.g., Nandi (1998).

That is, delta-hedged returns of purchased options are still negatively correlated with market returns, yielding an extra negative equity risk premium. In addition, the orthogonal component of volatility may also be priced. Since it is orthogonal to market returns, it cannot be used as a market hedge and hence is not necessarily demanded in equilibrium. If it can be diversified, economic theory dictates that its risk premium should be zero (see Heston (1993) and Merton (1998)). The empirical evidence, however, suggests that the orthogonal volatility risk premium is negative. For example, Pan (2002) directly estimates the price of risk of this component and finds that is negative.[9]

We provide (i) the hedging strategy, (ii) the average option gains, and (iii) the option overprice, which are associated with the orthogonal component of volatility. Then, we show that the average option gains and the option overprice are also closely related for this orthogonal component. We start by relating the orthogonal volatility risk premium to the overall volatility risk premium.

## 4.1 The Orthogonal Volatility Risk Premium

From equation (7), the market volatility risk premium parameter, $\lambda$, is equal to

$$\lambda = \left(\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2\right)\sigma. \tag{46}$$

If $\sigma = 0$, the variance is deterministic ($v_t = \theta + e^{-kt}(v_0 - \theta)$) and the volatility risk premium is zero, $\lambda = 0$. If we assume that the parameters $\rho, \sigma$, and $\lambda_1$ can be estimated, $\lambda$ depends only on $\lambda_2$.

It is quite common in the literature to motivate $\lambda v dt = \gamma \mathrm{cov}\left(\frac{d\widehat{c}_t}{\widehat{c}_t}, dv_t\right)$.[10] Hence, if $\mathrm{cov}\left(\frac{d\widehat{c}_t}{\widehat{c}_t}, dw_{2,t}\right) \approx 0$, $\lambda_2 \approx 0$ and equation (46) simplifies to $\lambda \approx \rho\sigma\lambda_1$. Moreover, if $\lambda_1 > 0$ and $\rho < 0$, then $\lambda < 0$. Indeed, $\lambda \leq 0$ if $\lambda_2 \leq \frac{-\rho}{\sqrt{1-\rho^2}}\lambda_1$. This boundary is positive $\frac{-\rho}{\sqrt{1-\rho^2}}\lambda_1 > 0$, if $\lambda_1 > 0$ and $\rho < 0$.

A few interesting intuitions follow: 1) the total volatility risk premium can be decomposed into market-related and orthogonal ones, 2) with a positive equity risk premium, the overall volatility risk premium can be negative as long as the orthogonal risk premium does not exceed an upper boundary which is solely determined by the correlation structure between volatility and returns, and lastly, 3) under rare conditions (e.g., if $\rho$ is very negative), there exists a set of positive risk premia for both market related and orthogonal components that makes the overall volatility risk premium negative.

[9] There is also a recent literature on dynamic derivative strategies, which theoretical framework is motivated partly by the existence of a non-zero (orthogonal) volatility risk premium, see Liu and Pan (2003).

[10] See Heston (1993, 329)) where $\widehat{c}$ is the equilibrium consumption and $\gamma$ is the relative-risk aversion of a representative investor. We can make a similar assumption to determine the risk premium associated with $dw_2$, i.e., $\lambda_2\sigma v dt = \gamma \mathrm{cov}\left(\frac{d\widehat{c}_t}{\widehat{c}_t}, \sigma\sqrt{v}dw_{2,t}\right)$. Note that we are not assuming that the consumption $\widehat{c}$ is equal to the the market portfolio, $S$. For example, $\widehat{c}$ could depend on $dw_2$ as well. However, if $\widehat{c} = S$, $\mathrm{cov}\left(\frac{d\widehat{c}_t}{\widehat{c}_t}, \sigma\sqrt{v}dw_{2,t}\right) = 0$ since $E_t[dw_{1,t}dw_{2,t}] = 0$.

## 4.2 The Hedging Strategy and the Associated Option Return

Consider the following hedging portfolio for a call option,

$$g_1 = C_S + \rho \frac{\sigma C_v}{S}. \tag{47}$$

The portfolio $g_1$ allows us to hedge the whole market risk of the option. In addition to delta-hedging, $g_1$ hedges the correlated stochastic volatility, which depends on the option Vega. If $\rho < 0$, $g_1 < C_S$, and therefore, $C_S$ overhedges the market risk of calls and puts which have positive Vega. From this hedging strategy, we do the same analysis as in Section 3.

We define the hedging error $X_t^g$ similar to $X_t$, but based on $g_1$ instead of $h_1$. Then,

$$\begin{aligned} dX_t^g &= g_1 dS - dC - (g_1 S - C)\, r dt \\ &= \Big( (r + \lambda_1 v)\, S g_1 - \Big( rC + \lambda_1 v S C_S + \Big( \rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2 \Big)\, \sigma v C_v \Big) - (g_1 S - C)\, r \Big)\, dt \\ &\quad -\sigma\sqrt{v} C_v \sqrt{1-\rho^2} dw_{2,t} \\ &= -\Big( \lambda_2 \sqrt{1-\rho^2} \sigma v C_v dt + \sqrt{1-\rho^2} \sigma \sqrt{v} C_v dw_{2,t} \Big) \end{aligned} \tag{48}$$

and

$$\int_0^T e^{r(T-t)} dX_t^g = -\lambda_2 \sqrt{1-\rho^2} \sigma \int_0^T e^{r(T-t)} v C_v dt - \sqrt{1-\rho^2} \sigma \int_0^T e^{r(T-t)} \sqrt{v} C_v dw_{2,t}. \tag{49}$$

**Proposition 2**. Taking expectations under the actual probability measure,

$$E_t \left[ dX_t^g \right] = -\lambda_2 \sqrt{1-\rho^2} \sigma v C_v dt, \tag{50}$$

and, by the law of the iterated expectation,

$$E_0 \left[ \int_0^T e^{r(T-t)} dX_t^g \right] = -\lambda_2 \sqrt{1-\rho^2} \sigma E_0 \left[ \int_0^T e^{r(T-t)} v C_v dt \right]. \tag{51}$$

Therefore, given a short call/put position ($C_v > 0$), $-E_0 \left[ \int_0^T e^{r(T-t)} v C_v dt \right] < 0$ and

$$E_0 \left[ \pi_{0,T}^{ort} \right] = E_0 \left[ \int_0^T e^{r(T-t)} dX_t^g \right] > 0, \text{ if and only if, } \lambda_2 < 0 \ . \quad \blacksquare \tag{52}$$

Proposition 2 extends equations (14) to (16). The expected gain of a selling option strategy, which is fully orthogonal to market returns, is positive if $\lambda_2$ is negative. The expected gain is the same for puts and calls with the same strike price, since they have the same Vega. Note that Proposition 2 allows a means of estimating the separate contributions of the two risk premia without having to estimate the equity risk premium (which is difficult to do).

### 4.3 The Option Overprice

Let $\widehat{Q}$ be an equivalent probability measure with the same market price of risk $\widehat{\lambda}_1\sqrt{v} = \lambda_1\sqrt{v}$ for $dw_1$, but a fictitious $\widehat{\lambda}_2 = 0$ for $dw_2$. Consider the corresponding fictitious option price process $\widehat{C}(t)$,

$$d\widehat{C} = \left(r\widehat{C} + \lambda_1 v\left(S\widehat{C}_S + \rho\sigma\widehat{C}_v\right)\right)dt + \sqrt{v}S\widehat{C}_S dw_{1,t} + \sigma\sqrt{v}\widehat{C}_v\left(\rho dw_{1,t} + \sqrt{1-\rho^2}dw_{2,t}\right), \tag{53}$$

with the boundary condition $\widehat{C}(T) = C(T)$. Define $\widehat{C}(0) = E_0^{\widehat{Q}}\left[e^{-rT}\widehat{C}(T)\right]$.

Now, we define the hedging strategy

$$\widehat{g}_1 = \widehat{C}_S + \rho\frac{\sigma\widehat{C}_v}{S}, \tag{54}$$

and define $d\widehat{X}$ and $d\widehat{Z}$ (similar to $d\widetilde{X}$ and $d\widetilde{Z}$, respectively) but based on $\widehat{g}_1$. Let $\widehat{Z}(0) = \widehat{C}(0)$. Then,

$$\begin{aligned}\widehat{Z}(T) &= e^{rT}\widehat{Z}(0) + \widehat{C}(T) - e^{rT}\widehat{C}(0) + \int_0^T e^{r(T-t)}d\widehat{X}_t^g \\ &= \widehat{C}(T) - \int_0^T e^{r(T-t)}\sqrt{1-\rho^2}\sigma\sqrt{v}\widehat{C}_v dw_{2,t},\end{aligned} \tag{55}$$

since we assume that $\widehat{\lambda}_2 = 0$.

As we assume that the residual risk is not priced, this application is consistent with Merton (1998) in the sense that the portfolio represented by $\widehat{g}_1$ is the instantaneous "minimum variance portfolio" that tracks the option payoff except for a residual error term associated with the orthogonal component of volatility. Next, we compare the price of this hedging portfolio and the price of the option, $\widehat{Z}(0)$ and $C(0)$, respectively. Similar to equation (18), we define (where $Z^g(0) = C(0)$)

$$\begin{aligned}Z^g(T) &= e^{rT}Z^g(0) + C(T) - e^{rT}C(0) + \int_0^T e^{r(T-t)}dX_t^g \\ &= C(T) - \int_0^T e^{r(T-t)}\lambda_2\sqrt{1-\rho^2}\sigma v C_v dt - \int_0^T e^{r(T-t)}\sqrt{1-\rho^2}\sigma\sqrt{v}C_v dw_{2,t}.\end{aligned} \tag{56}$$

Note that $e^{-rt}Z^g(t)$, $t \in [0,T]$, is a martingale under both $Q$ and $\widehat{Q}$; since $\widehat{g}_1$ invests only in $dS_t$ which is conditionally orthogonal to $dw_{2,t}$. We take expectations under $\widehat{Q}$.

**Proposition 3**.

$$\begin{aligned}C(0) &= Z^g(0) = E_0^{\widetilde{Q}}\left[e^{-rT}Z^g(T)\right] \\ &= E_0^{\widetilde{Q}}\left[e^{-rT}C(T)\right] - \lambda_2 E_0^{\widehat{Q}}\left[\int_0^T e^{-rt}\sqrt{1-\rho^2}\sigma\sqrt{v}C_v dt\right] \\ &= \widehat{C}(0) - \lambda_2 E_0^{\widehat{Q}}\left[\int_0^T e^{-rt}\sqrt{1-\rho^2}\sigma v C_v dt\right].\end{aligned} \tag{57}$$

In addition, since Vega is positive for put and call options (i.e., $C_v > 0$), then

$$C(0) - \widehat{C}(0) = -\lambda_2 E_0^{\widehat{Q}}\left[\int_0^T e^{-rt}\sqrt{1-\rho^2}\sigma v C_v dt\right] > 0 \text{ if and only if } \lambda_2 < 0. \quad \blacksquare \tag{58}$$

Proposition 3 quantifies the impact on the option price of the orthogonal volatility risk. The impact is the same for puts and calls with the same strike price. The left-hand-side of the equality of equation (58) can be computed as the difference of two option prices.

Like the overall volatility risk premium, average market-hedged option gains and option overprice in Propositions 2 and 3, respectively, are related. If $\lambda_1 = 0$, the expected gains/losses of buying options and hedging all market risk is equal to the price impact of the orthogonal risk premium. Also, if $\lambda_1 = 0$, equations (51) and (58) are valid if we only delta-hedge the option regardless of the correlation structure between returns and volatility, $\rho$.

## 4.4 Additional Properties

**The probability measure $\widehat{Q}$** Let us write the dynamics of market returns and stochastic volatility under the new measure $\widehat{Q}$. Since $\widehat{\lambda}_1 = \lambda_1$ and $\widehat{\lambda}_2 = 0$ (implying $\widehat{\lambda} = \rho\widehat{\lambda}_1\sigma$),

$$\begin{aligned} dS &= rSdt + \sqrt{v}Sdw_{1,t}^{\widehat{Q}}, \text{ and} &(59)\\ dv &= (\kappa + \rho\lambda_1\sigma)\left(\frac{\kappa\theta}{\kappa + \rho\lambda_1\sigma} - v\right)dt + \sigma\sqrt{v}\left(\rho dw_{1,t}^{\widehat{Q}} + \sqrt{1-\rho^2}dw_{2,t}^{\widehat{Q}}\right). &(60)\end{aligned}$$

The dynamics of volatility is not the same under $\mathcal{P}$ and $\widehat{Q}$ (different to previous $\widetilde{Q}$). Again, if $\lambda_1 = 0$, $\mathcal{P}$ and $\widehat{Q}$ are the same. If $\lambda < \rho\lambda_1\sigma < 0$, $\frac{\kappa}{\kappa+\lambda}\theta > \frac{\kappa}{\kappa+\rho\lambda_1\sigma}\theta > \theta$, and hence, the long-term level of volatility is larger under $Q$ than under $\widehat{Q}$ and $\widetilde{Q}$, in this order.

Therefore, if $\lambda_1 > 0$, the drift and the long-term volatility differ under $\widehat{Q}$ and $\mathcal{P}$. We expect the option overprice to be larger than the average option gain, which is confirmed in the simulation examples below, since $\frac{\kappa}{\kappa+\rho\lambda_1\sigma}\theta > \theta$ . In addition, for out-of-the-money call options, a large $\lambda_1$ implies that near at-the-money trajectories, which are associated with a larger option Vega, are more plausible; therefore, expectations under $\mathcal{P}$ relative to $\widehat{Q}$ should rise in this case.

**The percent price impact** One straddle option, which, in addition to be delta-hedged, is only exposed to the orthogonal component of volatility, contains $\beta$ calls and $1-\beta$ puts, where $\beta = 1 - \left(C_S + \rho\frac{\sigma C_v}{S}\right)$. Then, this straddle, with price $D$, satisfies

$$\frac{D(0) - \widehat{D}(0)}{\widehat{D}(0)} = \frac{C(0) - \widehat{C}(0)}{\widehat{C}(0) + \left(\widehat{C}_S(0) + \rho\frac{\sigma\widehat{C}_v}{S}\right)(e^{-rT}K - S_0)}.$$

Let us assume that $0 < \widehat{C}_S(0) + \rho\frac{\sigma\widehat{C}_v}{S} < 1$. It follows, if $\lambda_2 < 0$,

$$\frac{P(0) - \widehat{P}(0)}{\widehat{P}(0)} \gtrless \frac{D(0) - \widehat{D}(0)}{\widehat{D}(0)} \gtrless \frac{C(0) - \widehat{C}(0)}{\widehat{C}(0)} \quad \text{if } S_0 \gtrless e^{-rT}K, \tag{61}$$

and therefore, the percent impact is largest for out-of-the-money calls and puts.

**The price impact of the overall volatility and of the orthogonal component** We assume $\rho\lambda_1 < 0$ and $\lambda_2 < 0$. We copy both option overprice expressions

$$C(0) - \widetilde{C}(0) = -\left(\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2\right)\sigma E_0^{\widetilde{Q}}\left[\int_0^T e^{-rt} v C_v dt\right], \tag{62}$$

$$C(0) - \widehat{C}(0) = -\lambda_2\sqrt{1-\rho^2}\sigma E_0^{\widehat{Q}}\left[\int_0^T e^{-rt} v C_v dt\right]. \tag{63}$$

We expect the right-hand-side of equation (62) to be larger than the r.h.s. of equation (63). If $\widehat{Q}$ and $\widetilde{Q}$ are the same, this is right. However, if $\lambda_1 > 0$, $\widehat{Q}$ and $\widetilde{Q}$ are different and it is analytically difficult to compare these two terms.

From the left-hand-sides of these two equations, we must prove that $\widetilde{C}(0) < \widehat{C}(0)$. This seems to be true since the long-term volatility is larger under $\widehat{Q}$ than under $\widetilde{Q}$; i.e., $\frac{\kappa}{\kappa+\rho\lambda_1\sigma}\theta > \theta$. This can be formally proved extending Proposition 1; i.e.,

$$\widehat{C}(0) - \widetilde{C}(0) = -\rho\lambda_1\sigma E_0^{\widetilde{Q}}\left[\int_0^T e^{-rt} v \widehat{C}_v dt\right] > 0 \text{ if and only if } \rho\lambda_1 < 0. \tag{64}$$

Therefore, it follows, if $\rho\lambda_1 < 0$,

$$C(0) - \widetilde{C}(0) > C(0) - \widehat{C}(0) \text{ and } \frac{C(0) - \widetilde{C}(0)}{\widetilde{C}(0)} > \frac{C(0) - \widehat{C}(0)}{\widehat{C}(0)}. \tag{65}$$

For completeness, the overall volatility risk premium also satisfies

$$C(0) - \widetilde{C}(0) = -\lambda_1\rho\sigma E_0^{\widetilde{Q}}\left[\int_0^T e^{-rt} v \widehat{C}_v dt\right] - \lambda_2\sqrt{1-\rho^2}\sigma E_0^{\widehat{Q}}\left[\int_0^T e^{-rt} v C_v dt\right]. \tag{66}$$

## 4.5 Numerical Evidence from a Calibrated Model

In Figures 5 and 6 we compute the price impact of the orthogonal component of volatility. The impact on the option price of the orthogonal volatility risk premium is similar to the impact of the overall volatility risk premium across strike prices, maturity, and volatility regimens, and for calls, puts, and straddles. This is intuitive since $\lambda$ is linear in $\lambda_2$; see equation (46). The price impact of orthogonal volatility is lower than the impact of the overall volatility risk, as all the Figures show, but the impact of a negative $\lambda_2$ is notable. Therefore, the previous arguments in Figures 1 to 3 above extend to Figures 5 and 6 as well.

We find a difference for straddles. Its price is given by $\widehat{C}(0) + \left(\widehat{C}_S(0) + \rho\frac{\sigma\widehat{C}_v}{S}\right)\left(e^{-rT}K - S_0\right)$, and the term $\rho\frac{\sigma\widehat{C}_v}{S}\left(e^{-rT}K - S_0\right)$ is negative if $e^{-rT}K > S_0$ for $\rho < 0$. Consequently, when the straddle price appears is the denominator, the percent impact is maximized for away-from-the-money straddles and for strike prices such that $e^{-rT}K > S_0$, as the upper-right pictures of Figures 5 and 6 show. Although not reported, this result is more evident for low levels of volatility (e.g., if $\sqrt{v} \leq 0.10$).

The effect on option prices of the orthogonal volatility risk premium can be notable, and, in addition, is unrelated to the market since it has offsetting covariance with the market. The evidence indicates that this component is large and negative. Therefore, an important component of the overall volatility risk premium is its orthogonal part. An interesting extension is to empirically disentangle this risk premium from option returns, which is left as avenue of future research. Some evidence of a simulation exercise is provided next.

## 5 A Simulation Exercise: Market-Hedged Option Returns

We complete the paper by computing expected market-hedged option gains/returns by simulation. See Figlewski and Green (1999) for a pioneer simulation exercise for options.

We focus on two problems. First, we relate the average option gain to the new measure of option overprice, for the overall and for the orthogonal volatility risk premiums. We have three main hypothesis. (i) If $\lambda_1 \approx 0$, both measures are the same (which is exact if $\lambda_1 = 0$), except for the discount term $e^{rT}$. (ii) For the overall volatility risk premium, if $\lambda_1 > 0$, we expect delta-hedged option gains go up (down) relative to the option overprice for out-of-the-money (in-the-money) calls. (iii) For the orthogonal component, in addition to (ii), we expect another rise of the option overprice for puts and calls because a long-term level of volatility (under $\widehat{Q}$).

In Table 1 and 2, we provide an analysis in detail of option returns. The value of the parameters of this exercise are described in detail in Section 3. We consider two scenarios for the orthogonal risk premium, a conservative $\lambda_2 = -6$ and a larger $\lambda_2 = -12$. We provide the option price, the option Vega, and also the instantaneous volatility risk premium multiplied by maturity (i.e., $-\lambda v_0 C_v T$). We provide the two first sample moments of market returns and its Sharpe ratio ($\frac{\overline{S}_T - S_0}{S_0} \frac{1}{\sqrt{\theta T}}$) and compare with option returns. We consider five strike prices. The hedging period is twice per week.[11]

Tables 1 and 2 show the two option overprices are closely related to the two corresponding option gains and confirm the predicted moneyness effect. The option overprice is a close substitute of average delta-hedged option returns and we can explain the small bias across strike prices. For example, for the near at-the-money call, $K = 100$, the difference between the sample average of $\pi_{0,T}$ and $\pi_{0,T}^{ort}$ and the two corresponding overprices is not statistically significant (though they must be different since $\lambda_1 = 4$, $\mathcal{P} \neq \widetilde{Q}$ and $\mathcal{P} \neq \widehat{Q}$).

Second, we take advantage of the simulations and compute the second moment of option gains.

[11] We discretize $S$ and $v$ processes (equations (3) and (4), respectively) using a simple Euler scheme, where $dt$ is equal to one hour. We use $16,000$ paths for all simulations, which implies a sample standard error less than 0.005 in parentheses. For the five strike prices, we use the same random numbers.

Heston's model produces large average losses of purchased delta-hedged options. These losses should be a compensation for volatility risk, even if skewness and kurtosis are not taking into account. The market prices of risk parameters, $\lambda_1$ and $\lambda_2$, relate the instantaneous expected return to the standard deviation of returns. Therefore, if $\lambda_1$ and $\lambda_2$ are economically meaningful, large average gains/returns must be associated with large volatility risk.

Tables 1 clearly shows that large average delta-hedged option gains are associated with high risk. For example, take $K = 100$ in Table 1, $\overline{\pi}_{0,T} = 0.2063$, but $\left(\overline{\text{Variance}\left(\pi^o_{0,T}\right)}\right)^{0.5} = 0.6704$ for a three months to maturity call.[12] Indeed, the Sharpe ratios across the five strike prices go from 0.21 to 0.31, which are not far away from the Sharpe ratio of equity risk, 0.2615. In brief, these large average returns are a compensation for volatility risk.

For $K = 100$ the elasticities of the call option with regard to market risk and to volatility risk are equal to $\frac{C_S}{C}\sqrt{v} = 17.83\sqrt{v}$ and $\frac{\sigma C_v}{C}\sqrt{v} = 3.45\sqrt{v}$, respectively, which are substantially larger than market risk $\sqrt{v} = 0.13$. For $K = 105$, $\frac{C_S}{C}\sqrt{v} = 25.71\sqrt{v}$ and $\frac{\sigma C_v}{C}\sqrt{v} = 8.64\sqrt{v}$. Note also the terms $-\lambda v_0 C_v T$ are approximately twice larger the terms $\pi_{0,T}$ (i.e., $-\lambda v_0 C_v T \approx 2\pi_{0,T}$).

This implies that the average gain/return per unit of time of a shorter investment period (e.g., one day/week), which can be approximated by $-\lambda v_0 C_v$, is twice the average gain/return of holding the option over the three months (i.e., $-\lambda v_0 C_v \approx 2\frac{\pi_{0,T}}{T}$). On the other hand, the standard deviation of the instantaneous gain (multiplied by the root-square of maturity) and of the gains until maturity are given by $\sigma\sqrt{v_0}C_v\sqrt{T} = 0.77$ and $\left(\overline{\text{Variance}\left(\pi_{0,T}\right)}\right)^{0.5} = 0.6704$, respectively.

**The average gain of the orthogonal component** Similar results hold for the orthogonal component of volatility in Table 1. This component is important because, for $\rho = -0.4$, $\sqrt{1-\rho^2} = 0.9165$, which is close to one. In Table 2, we repeat the same exercise where average gains rise notably from the orthogonal part of volatility, since $\lambda_2 = -12$. In Table 3, we consider a higher correlation between market returns and volatility, $\rho = -0.80$, which is associated with a stronger leverage effect. In this case, the orthogonal component, and its associated risk premiums, are less important.

The orthogonal risk premium is lower in the three tables, but its associated risk is not reduced as much, which is important if this strategy is practiced. This risk can be reduced by hedging more. For example, by hedging eight times per week instead of twice, for $K = 95$, the standard deviation

[12] For example, in the Black-Scholes model with a volatility of 0.13, the standard deviation of the error/gain of the same three months to maturity call is given by 0.40 for delta-hedging twice per week. By delta-hedging four and eight times per week, the standard deviation of the error is reduced to 0.29 and 0.21, respectively. By dividing by the 3.2452 Black-Scholes call price and multiplying by $\sqrt{\frac{1}{0.25}}$, the standard deviation of annually delta-hedged returns is given by 0.25, 0.18, and 0.13, respectively, for the three different lengths of the hedging period.

is reduced from 0.4680 to 0.3561, and from 0.4174 to a lower 0.2922 for the orthogonal component.

**A last picture** Figure 7 shows the value of a tracking delta-hedging portfolio at maturity ($Z(T)$, see equation (18)), as a function of the final stock price, for a near at-the-money call, $K = 100$ and $T = 0.25$. We show 5,000 simulated paths. The difference between the tracking portfolio and the call option are the gains/losses (financed to the riskless rate), associated with delta-hedging twice per week. In average, the gain is positive for a short call position, $\overline{\pi}_{0,T} = 0.2035$, and has a standard deviation of 0.6730 (similar to Table 1). We have separated those paths where volatility ends above the long-term level (i.e., $v_T > \theta$) from those less volatile paths ($v_T \leq \theta$). As expected, most of the gains (losses) of this short call position correspond with low (high) volatile scenarios. Note also how the hedging errors are larger for those paths finishing near at-the-money.

In Figure 8, we directly show the gains/losses of Figure 7 tracking portfolio. Using put-call parity, these gains/losses associated with stochastic volatility are the same for a delta-hedged call, a delta-hedged put, or a delta-hedged straddle. That is, for a simple one-period hedging,

$$\begin{aligned}\frac{\partial C}{\partial S}\left(S_{t+\Delta t} - S_t e^{r\Delta t}\right) - \left(C_{t+\Delta t} - C_t e^{r\Delta t}\right) &= \frac{\partial P}{\partial S}\left(S_{t+\Delta t} - S_t e^{r\Delta t}\right) - \left(P_{t+\Delta t} - P_t e^{r\Delta t}\right) \\ &= \frac{\partial P}{\partial S}\left(C_{t+\Delta t} - C_t e^{r\Delta t}\right) + \frac{\partial C}{\partial S}\left(P_{t+\Delta t} - P_t e^{r\Delta t}\right)\end{aligned}$$

Now, one can clearly see that larger hedging errors and losses correspond with those trajectories finishing near at-the-money and where volatility is above its long-term level.

## 6 Concluding Remarks

That relatively less attention has been paid to option returns, compared with the pricing of options, is one of the original motivations in Bakshi and Kapadia (2003a) and Coval and Shumway (2001). These studies find that volatility is a prevalent source of risk and that the premium associated with purchasing into volatility is negative. Therefore, a natural extension is whether a standard stochastic volatility model can explain the rich cross-sectional findings of these empirical studies, providing further rationale for volatility risk and an associated negative risk premium.

This paper has two main contributions. First, we introduce a new measure of option overprice which quantifies the impact of the volatility risk premium on option prices. This measure is closely related to the expected delta-hedged return of the same option, but with the advantage that it is analytical. Moreover, it is robust to the absence of jump risk in the stochastic volatility model if jumps are not priced. This allows us to address the option risk-premium under rich return generator processes such as jump-diffusion models. Second, an extensive calibration exercise shows the option overprice is consistent with the cross-section of average delta-hedged option returns reported by the literature for straddles, calls, and puts for the S&P 500 index market. The volatility elasticity of calls and puts is several times the level of market volatility, depending on moneyness and maturity, and implies a rich cross-section of negative average option returns – even if volatility risk is not priced heavily, albeit negative. The main discrepancy is that the percent price impact on out-of-the-money puts rises with maturity.

The measure of option overprice is sufficiently general to allow it to be applied to many other problems. First, if we estimate an option-pricing model, we can directly compute the volatility risk-premium from the option overprice. Second, we can consider multi-factor stochastic volatility models such as Christoffersen et al. (2007). Third, another interesting area is fixed-income securities (e.g., Duffee (2002)). Fixed income markets are truly multi-factor, state variables are generally modelled as diffusions, issues such as spanned/unspanned stochastic volatility are prevalent, and bonds, caps and swaptions have received a lot of attention. Fourth, the option overprice can be considered as an up-front premium, and we can study its optimality compared with a stream of volatility risk-premiums. And Fifth, to consider that both volatility risk and jump risk are priced is an important extension too. These questions are left as avenues of future research.

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| **Table 1**: Simulated average market-hedged call gains and call overprice ($\lambda_2 = -6$) | | | | | |
|---|---|---|---|---|---|
| $S_0 = 100, r = 0.05$, and $T = 0.25$. Volatility parameters: $\kappa = 5.0, \sqrt{\theta} = 0.13, \sigma = 0.25, \rho = -0.4$ | | | | | |
| Pricing parameters: $\lambda_1 = +4, \lambda_1\theta = 0.0676, \lambda_2 = -6, \frac{\lambda}{\sigma} = \rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2 = -7.0991$ | | | | | |
| $\left(e^{\lambda_1\theta T} - 1\right)e^{rT} = 0.0173, \sqrt{\theta T} = 0.065; \frac{\overline{S}_T - S_0 e^{rT}}{S_0} = 0.0170, \left(\overline{\text{Variance}\left(\frac{S_T - S_0 e^{rT}}{S_0}\right)}\right)^{0.5} = 0.0649$ | | | | | |
| | Strike Price | | | | |
| | $K = 90$ | $K = 95$ | $K = 100$ | $K = 105$ | $K = 110$ |
| $C(0)$ , $C_v$ | 11.33 , 13.22 | 6.95 , 30.76 | 3.43 , 47.36 | 1.26 , 43.43 | 0.33 , 22.31 |
| $-\lambda v_0 C_v T$ | 0.0991 | 0.2307 | 0.3552 | 0.3257 | 0.1673 |
| $C(0) - \widetilde{C}(0)$ | 0.0592 | 0.1362 | 0.2035 | 0.1791 | 0.0882 |
| $\overline{\pi}_{0,T}$ | 0.0522 | 0.1292 | 0.2063 | 0.1913 | 0.1035 |
| | (0.0019) | (0.0036) | (0.0053) | (0.0053) | (0.0038) |
| $\left(\overline{\text{Variance}\left(\pi_{0,T}\right)}\right)^{0.5}$ | 0.2403 | 0.4554 | 0.6704 | 0.6704 | 0.4807 |
| $\overline{\pi}_{0,T} \times \left(\overline{\text{Variance}\left(\pi_{0,T}\right)}\right)^{-0.5}$ | 0.2172 | 0.2837 | 0.3077 | 0.2854 | 0.2153 |
| $-\lambda_2\sqrt{1-\rho^2}\sigma v_0 C_v T$ | 0.0768 | 0.1787 | 0.2751 | 0.2523 | 0.1296 |
| $C(0) - \widehat{C}(0)$ | 0.0475 | 0.1080 | 0.1606 | 0.1417 | 0.0707 |
| $\overline{\pi}_{0,T}^{ort}$ | 0.0402 | 0.1002 | 0.1595 | 0.1457 | 0.0797 |
| | (0.0019) | (0.0036) | (0.0051) | (0.0050) | (0.0036) |
| $\left(\overline{\text{Variance}\left(\pi_{0,T}^{ort}\right)}\right)^{0.5}$ | 0.2403 | 0.4554 | 0.6451 | 0.6325 | 0.4554 |
| $\overline{\pi}_{0,T}^{o} \times \left(\overline{\text{Variance}\left(\pi_{0,T}^{ort}\right)}\right)^{-0.5}$ | 0.1673 | 0.2200 | 0.2472 | 0.2304 | 0.1750 |

We use the following notation for a call option: $C(0)$ is the call price, $C_v$ is the call Vega, $-\lambda v_0 C_v T$ is the instantaneous risk premium multiplied by maturity, and $C(0) - \widetilde{C}(0)$ is the call overprice. $\overline{\pi}_{0,T}$ denotes sample average delta-hedged gains (with the standard error in parentheses), $\overline{\text{Variance}\left(\pi_{0,T}\right)}$ is the corresponding variance, and the Sharpe ratio is given by $\overline{\pi}_{0,T} \times \left(\overline{\text{Variance}\left(\pi_{0,T}\right)}\right)^{-0.5}$. Similar notation is used for the orthogonal component of volatility, where $\widehat{C}(0)$ is the fictitious price and $\pi_{0,T}^{ort}$ is the gain. We use 16,000 paths, with the same random numbers for the five strike prices.

The terms $\left(e^{\lambda_1\theta T} - 1\right)e^{rT}$ and $\sqrt{\theta T}$ are the first two average moments of market returns, where $\frac{\overline{S}_T - S_0 e^{rT}}{S_0}$ and $\overline{\text{Variance}\left(\frac{S_T - S_0 e^{rT}}{S_0}\right)}$ are the sample values corresponding to the 16,000 simulated paths. These terms are also used to check the accuracy of the simulation.

| **Table 2**: Simulated average market-hedged option returns and option overprice ($\lambda_2 = -12$) | | | | | |
|---|---|---|---|---|---|
| $S_0 = 100$ and $T = 0.25$. Volatility parameters: $\kappa = 5.0, \sqrt{\theta} = 0.13, \sigma = 0.25, \rho = -0.4$. | | | | | |
| Pricing parameters: $\lambda_1 = +4, \lambda_1\theta = 0.0676, \lambda_2 = -12, \frac{\lambda}{\sigma} = \rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2 = -12.5982$; $r = 0.05$ | | | | | |
| $\left(e^{\lambda_1\theta T} - 1\right)e^{rT} = 0.0173, \sqrt{\theta T} = 0.065; \frac{\overline{S}_T - S_0 e^{rT}}{S_0} = 0.0170, \left(\overline{\text{Variance}\left(\frac{S_T - S_0 e^{rT}}{S_0}\right)}\right)^{0.5} = 0.0649$ | | | | | |
| | Strike Price | | | | |
| | $K = 90$ | $K = 95$ | $K = 100$ | $K = 105$ | $K = 110$ |
| $C(0)$ , $C_v$ | 11.39 , 16.46 | 7.07 , 35.17 | 3.62 , 51.81 | 1.42 , 48.30 | 0.42 , 27.10 |
| $-\lambda v_0 C_v T$ | 0.2190 | 0.4680 | 0.6895 | 0.6427 | 0.3607 |
| $C(0) - \widetilde{C}(0)$ | 0.1202 | 0.2642 | 0.3869 | 0.3444 | 0.1784 |
| $\overline{\pi}_{0,T}$ | 0.1077 | 0.2507 | 0.3898 | 0.3679 | 0.2065 |
| | (0.0019) | (0.0036) | (0.0053) | (0.0054) | (0.0039) |
| $\left(\overline{\text{Variance}\left(\pi_{0,T}\right)}\right)^{0.5}$ | 0.2403 | 0.4554 | 0.6704 | 0.6831 | 0.4933 |
| $\overline{\pi}_{0,T} \times \left(\overline{\text{Variance}\left(\pi_{0,T}\right)}\right)^{-0.5}$ | 0.4481 | 0.5505 | 0.5814 | 0.5386 | 0.4186 |
| $-\lambda_2\sqrt{1-\rho^2}\sigma v_0 C_v T$ | 0.1912 | 0.4085 | 0.6019 | 0.5611 | 0.3149 |
| $C(0) - \widehat{C}(0)$ | 0.1085 | 0.2361 | 0.3439 | 0.3070 | 0.1609 |
| $\overline{\pi}_{0,T}^{ort}$ | 0.0938 | 0.2187 | 0.3394 | 0.3187 | 0.1797 |
| | (0.0020) | (0.0037) | (0.0054) | (0.0050) | (0.0036) |
| $\left(\overline{\text{Variance}\left(\pi_{0,T}^{ort}\right)}\right)^{0.5}$ | 0.2530 | 0.4680 | 0.6831 | 0.6325 | 0.4554 |
| $\overline{\pi}_{0,T}^{o} \times \left(\overline{\text{Variance}\left(\pi_{0,T}^{ort}\right)}\right)^{-0.5}$ | 0.3708 | 0.4673 | 0.5160 | 0.5039 | 0.3946 |

We use the following notation for a call option: $C(0)$ is the call price, $C_v$ is the call Vega, $-\lambda v_0 C_v T$ is the instantaneous risk premium multiplied by maturity, and $C(0) - \widetilde{C}(0)$ is the call overprice. $\overline{\pi}_{0,T}$ denotes sample average delta-hedged gains (with the standard error in parentheses), $\overline{\text{Variance}\left(\pi_{0,T}\right)}$ is the corresponding variance, and the Sharpe ratio is given by $\overline{\pi}_{0,T} \times \left(\overline{\text{Variance}\left(\pi_{0,T}\right)}\right)^{-0.5}$. Similar notation is used for the orthogonal component of volatility, where $\widehat{C}(0)$ is the fictitious price and $\pi_{0,T}^{ort}$ is the gain. We use 16,000 paths, with the same random numbers for the five strike prices.

The terms $\left(e^{\lambda_1\theta T} - 1\right)e^{rT}$ and $\sqrt{\theta T}$ are the first two average moments of market returns, where $\frac{\overline{S}_T - S_0 e^{rT}}{S_0}$ and $\overline{\text{Variance}\left(\frac{S_T - S_0 e^{rT}}{S_0}\right)}$ are the sample values corresponding to the 16,000 simulated paths. These terms are also used to check the accuracy of the simulation.

| **Table 3**: Simulated average market-hedged option returns and option overprice ($\rho = -0.8$) | | | | | |
|---|---|---|---|---|---|
| $S_0 = 100$ and $T = 0.25$. Volatility parameters: $\kappa = 5.0, \sqrt{\theta} = 0.13, \sigma = 0.25, \rho = -0.8$. | | | | | |
| Pricing parameters: $\lambda_1 = +4, \lambda_1\theta = 0.0676, \lambda_2 = -6, \frac{\lambda}{\sigma} = \rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2 = -6.80$; $r = 0.05$ | | | | | |
| $\left(e^{\lambda_1\theta T} - 1\right)e^{rT} = 0.0173, \sqrt{\theta T} = 0.065; \frac{\overline{S}_T - S_0e^{rT}}{S_0} = 0.0168, \left(\overline{\text{Variance}\left(\frac{S_T - S_0e^{rT}}{S_0}\right)}\right)^{0.5} = 0.0620$ | | | | | |
| | Strike Price | | | | |
| | $K = 90$ | $K = 95$ | $K = 100$ | $K = 105$ | $K = 110$ |
| $C(0)$ , $C_v$ | 11.39 , 14.83 | 7.02 , 31.04 | 3.45 , 46.82 | 1.14 , 44.06 | 0.19 , 19.21 |
| $-\lambda v_0 C_v T$ | 0.1065 | 0.2229 | 0.3363 | 0.3164 | 0.1380 |
| $C(0) - \widetilde{C}(0)$ | 0.0663 | 0.1359 | 0.1954 | 0.1678 | 0.0631 |
| $\overline{\pi}_{0,T}$ | 0.0595 | 0.1308 | 0.2009 | 0.1889 | 0.0795 |
| | (0.0021) | (0.0037) | (0.0053) | (0.0053) | (0.0034) |
| $\left(\overline{\text{Variance}\left(\pi_{0,T}\right)}\right)^{0.5}$ | 0.2656 | 0.4680 | 0.6704 | 0.6704 | 0.4301 |
| $\overline{\pi}_{0,T} \times \left(\overline{\text{Variance}\left(\pi_{0,T}\right)}\right)^{-0.5}$ | 0.2240 | 0.2795 | 0.2997 | 0.2818 | 0.1849 |
| $-\lambda_2\sqrt{1-\rho^2}\sigma v_0 C_v T$ | 0.0564 | 0.1180 | 0.1780 | 0.1675 | 0.0731 |
| $C(0) - \widehat{C}(0)$ | 0.0374 | 0.0753 | 0.1073 | 0.0930 | 0.0365 |
| $\overline{\pi}^{ort}_{0,T}$ | 0.0311 | 0.0697 | 0.1061 | 0.0979 | 0.0418 |
| | (0.0020) | (0.0033) | (0.0044) | (0.0041) | (0.0027) |
| $\left(\overline{\text{Variance}\left(\pi^{ort}_{0,T}\right)}\right)^{0.5}$ | 0.2530 | 0.4174 | 0.5566 | 0.5186 | 0.3415 |
| $\overline{\pi}^{o}_{0,T} \times \left(\overline{\text{Variance}\left(\pi^{ort}_{0,T}\right)}\right)^{-0.5}$ | 0.1229 | 0.1670 | 0.1906 | 0.1888 | 0.1224 |

We use the following notation for a call option: $C(0)$ is the call price, $C_v$ is the call Vega, $-\lambda v_0 C_v T$ is the instantaneous risk premium multiplied by maturity, and $C(0) - \widetilde{C}(0)$ is the call overprice. $\overline{\pi}_{0,T}$ denotes sample average delta-hedged gains (with the standard error in parentheses), $\overline{\text{Variance}\left(\pi_{0,T}\right)}$ is the corresponding variance, and the Sharpe ratio is given by $\overline{\pi}_{0,T} \times \left(\overline{\text{Variance}\left(\pi_{0,T}\right)}\right)^{-0.5}$. Similar notation is used for the orthogonal component of volatility, where $\widehat{C}(0)$ is the fictitious price and $\pi^{ort}_{0,T}$ is the gain. We use 16,000 paths, with the same random numbers for the five strike prices.

The terms $\left(e^{\lambda_1\theta T} - 1\right)e^{rT}$ and $\sqrt{\theta T}$ are the first two average moments of market returns, where $\frac{\overline{S}_T - S_0e^{rT}}{S_0}$ and $\overline{\text{Variance}\left(\frac{S_T - S_0e^{rT}}{S_0}\right)}$ are the sample values corresponding to the 16,000 simulated paths. These terms are also used to check the accuracy of the simulation.

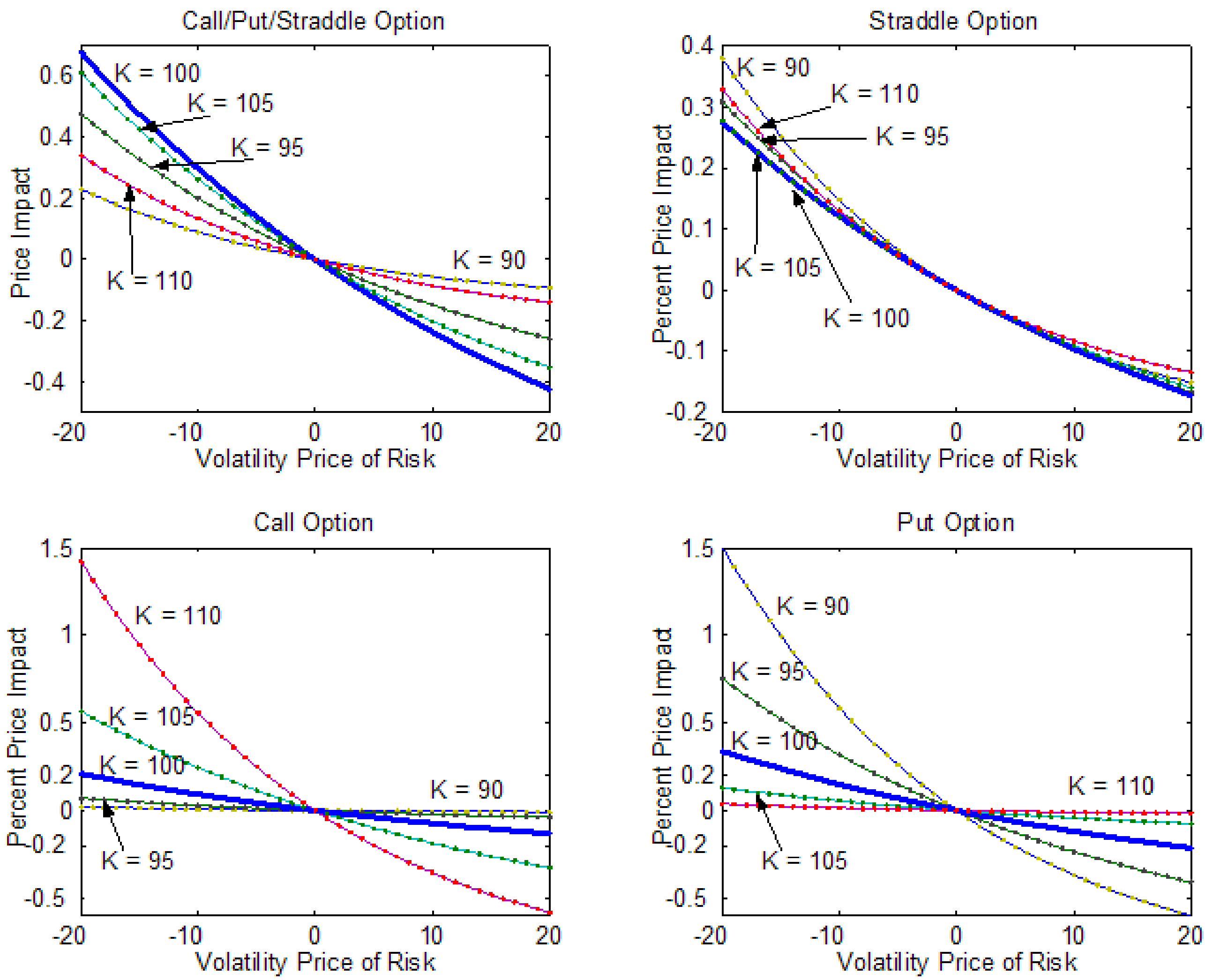


Figure 1: The impact of a nonzero market volatility risk premium on option prices ($C(0) - \widetilde{C}(0)$ and percent price impact) as a function of the market volatility price of risk, $\lambda/\sigma$. The parameters are as follows. For stochastic volatility, $\kappa = 5.0$, $\sqrt{\theta} = 0.13$, $\sigma = 0.25$, and $\rho = -0.4$. The risk-free rate is $r = 0.05$. Note for, $\frac{\lambda}{\sigma} < 20$, $\kappa + \lambda > 0$ and volatility is stationary under $Q$. We assume that the initial value of volatility is its long term level, $\sqrt{v_0} = 0.13$, and $S_0 = 100$. The option maturity is $T = 0.25$ years and $K$ is the strike price.

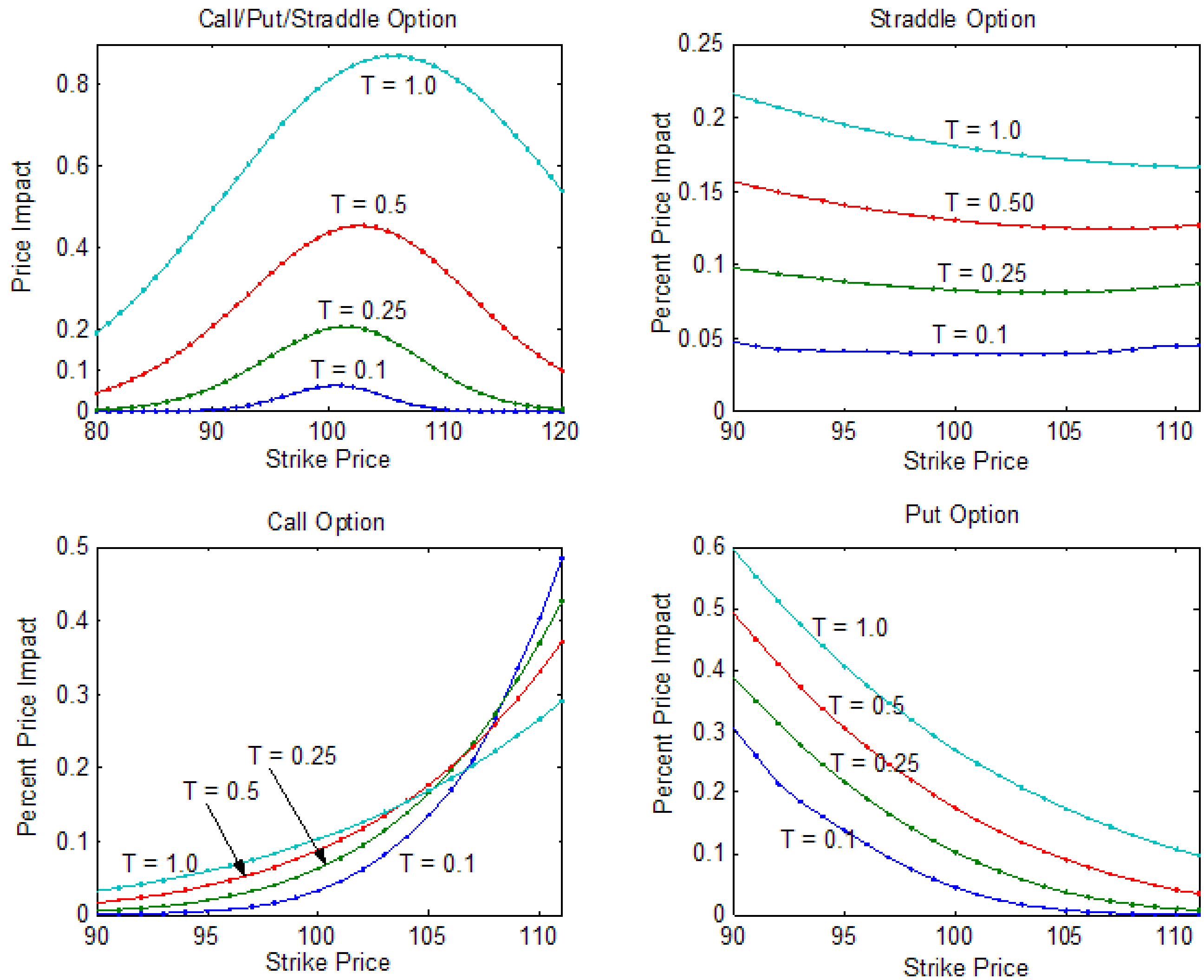


Figure 2: The impact of a negative market volatility risk premium on option prices ($C(0) - \widetilde{C}(0)$ and percent price impact) across strike prices, $K$. The parameters are as follows. For stochastic volatility, $\kappa = 5.0$, $\sqrt{\theta} = 0.13$, $\sigma = 0.25$, and $\rho = -0.4$. The pricing parameters are $r = 0.05$, $\lambda_1 = 4$, and $\lambda_2 = -6$, which implies $\frac{\lambda}{\sigma} = \rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2 = -7.0991$ and the average market volatility price of risk is equal to $\frac{\lambda}{\sigma}\sqrt{\theta} = -0.9229$. We assume that the initial value of volatility is its long term level, $\sqrt{v_0} = 0.13$, and $S_0 = 100$. The option maturity is $T$ years.

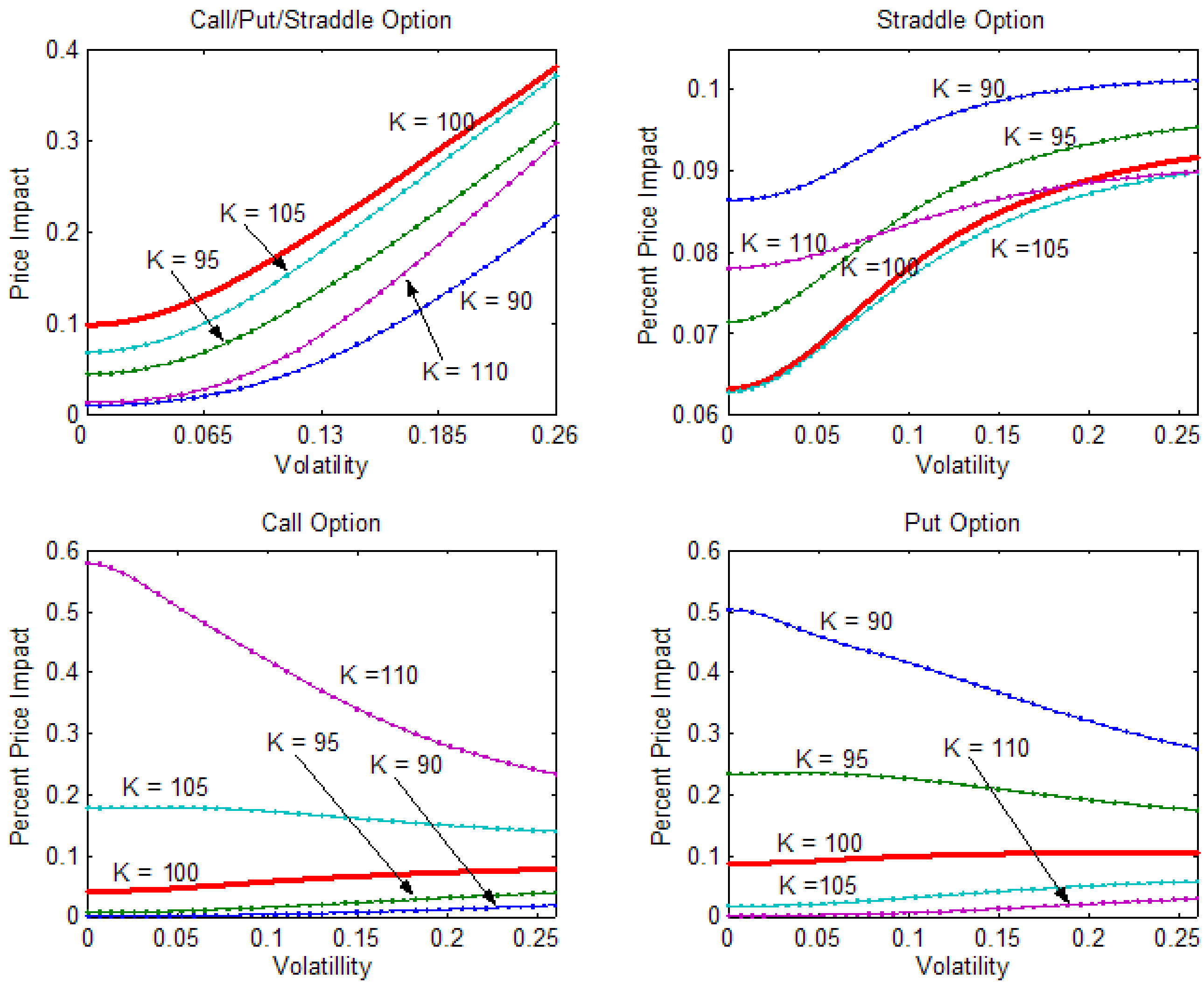


Figure 3: The impact of a negative market volatility risk premium on option prices ($C(0) - \widetilde{C}(0)$ and percent price impact) as a function of the initial volatility, $\sqrt{v_0}$. The parameters are as follows. For stochastic volatility, $\kappa = 5.0$, $\sqrt{\theta} = 0.13$, $\sigma = 0.25$, and $\rho = -0.4$. The pricing parameters are $r = 0.05$, $\lambda_1 = 4$, and $\lambda_2 = -6$, which implies $\frac{\lambda}{\sigma} = \rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2 = -7.0991$ and the average market volatility price of risk is equal to $\frac{\lambda}{\sigma}\sqrt{\theta} = -0.9229$. We assume $S_0 = 100$. The option maturity is $T = 0.25$ years and $K$ is the strike price.

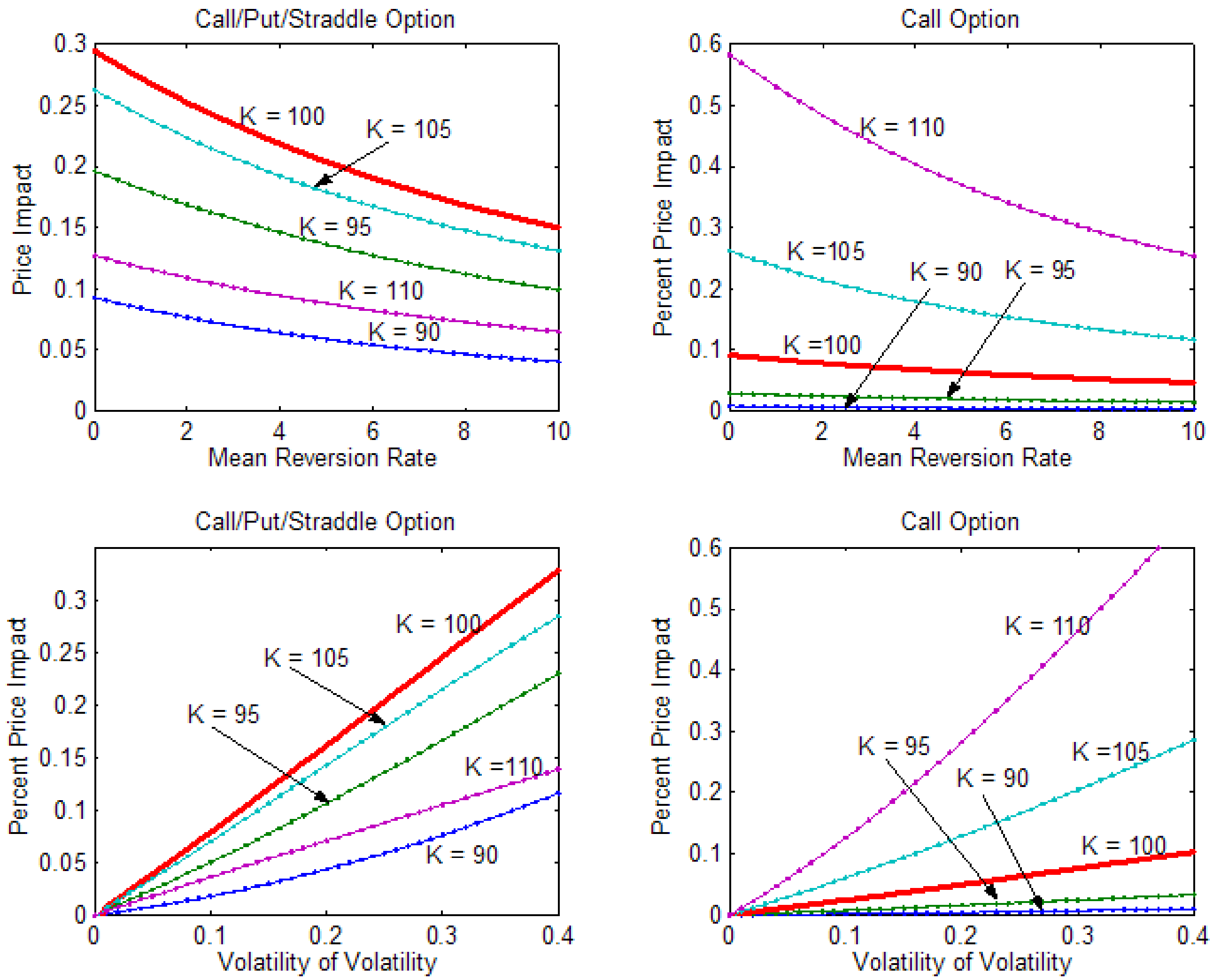


Figure 4: The impact of a negative market volatility risk premium on option prices ($C(0) - \widetilde{C}(0)$ and percent price impact) as a function of two volatility parameters, $\kappa$ and $\sigma$. The other parameters are as follows. For stochastic volatility, $\sqrt{\theta} = 0.13$ and $\rho = -0.4$ (where $\kappa = 5.0$ or $\sigma = 0.25$). The pricing parameters are $r = 0.05$, $\lambda_1 = 4$, and $\lambda_2 = -6$, which implies $\frac{\lambda}{\sigma} = \rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2 = -7.0991$ and the average market volatility price of risk is equal to $\frac{\lambda}{\sigma}\sqrt{\theta} = -0.9229$. We assume $S_0 = 100$. The option maturity is $T = 0.25$ years and $K$ is the strike price.

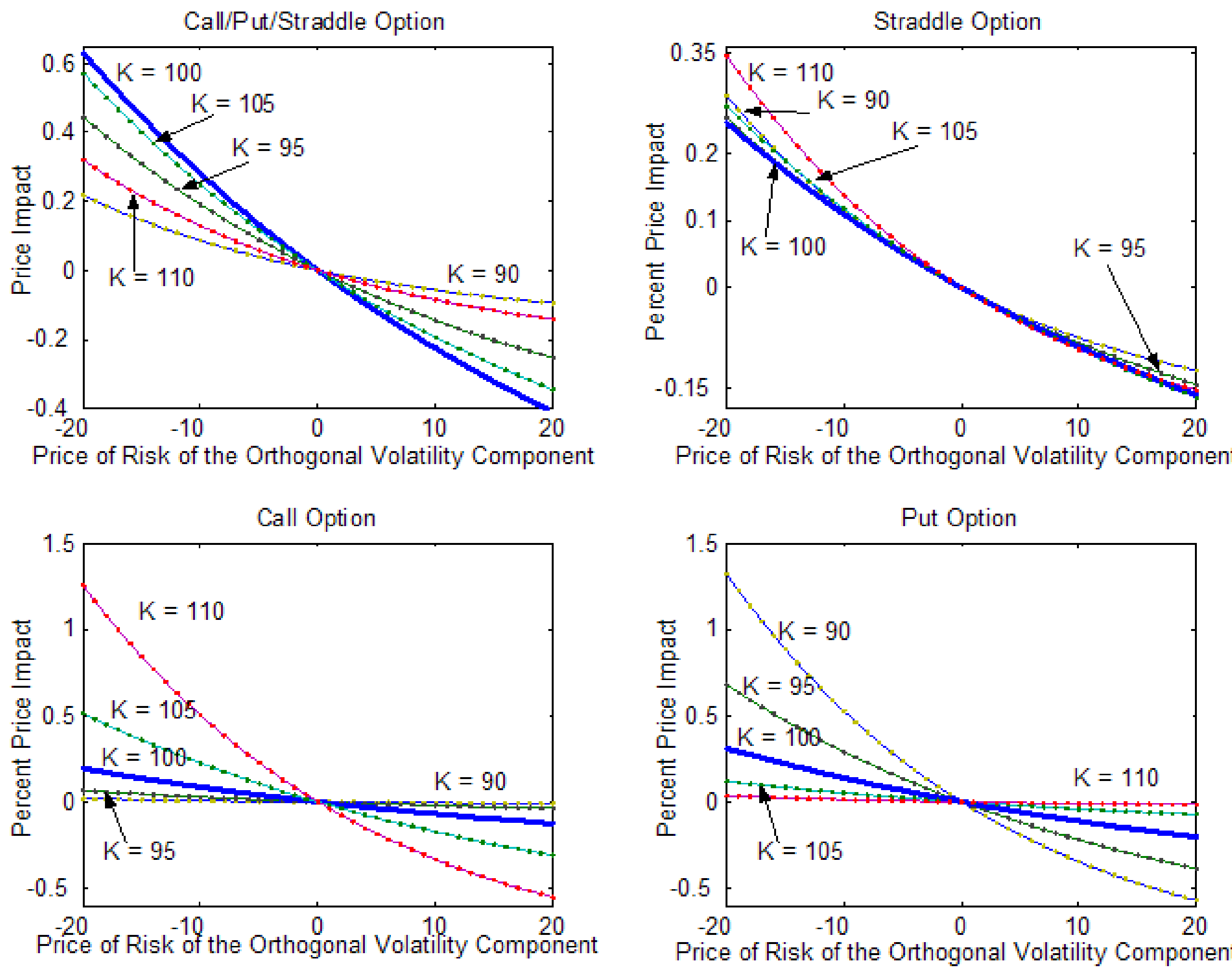


Figure 5: The impact of a nonzero orthogonal volatility risk premium on option prices ($C(0) - \widehat{C}(0)$ and percent price impact) as a funtion of the orthogonal volatility market price of risk, $\lambda_2$. The parameters are as follows. For stochastic volatility, $\kappa = 5.0$, $\sqrt{\theta} = 0.13$, $\sigma = 0.25$, and $\rho = -0.4$. The risk-free rate is $r = 0.05$ and $\lambda_1 = 4$. We assume that the initial value of volatility is its long term level, $\sqrt{v_0} = 0.13$, and $S_0 = 100$. The option maturity is $T = 0.25$ years and $K$ is the strike price.

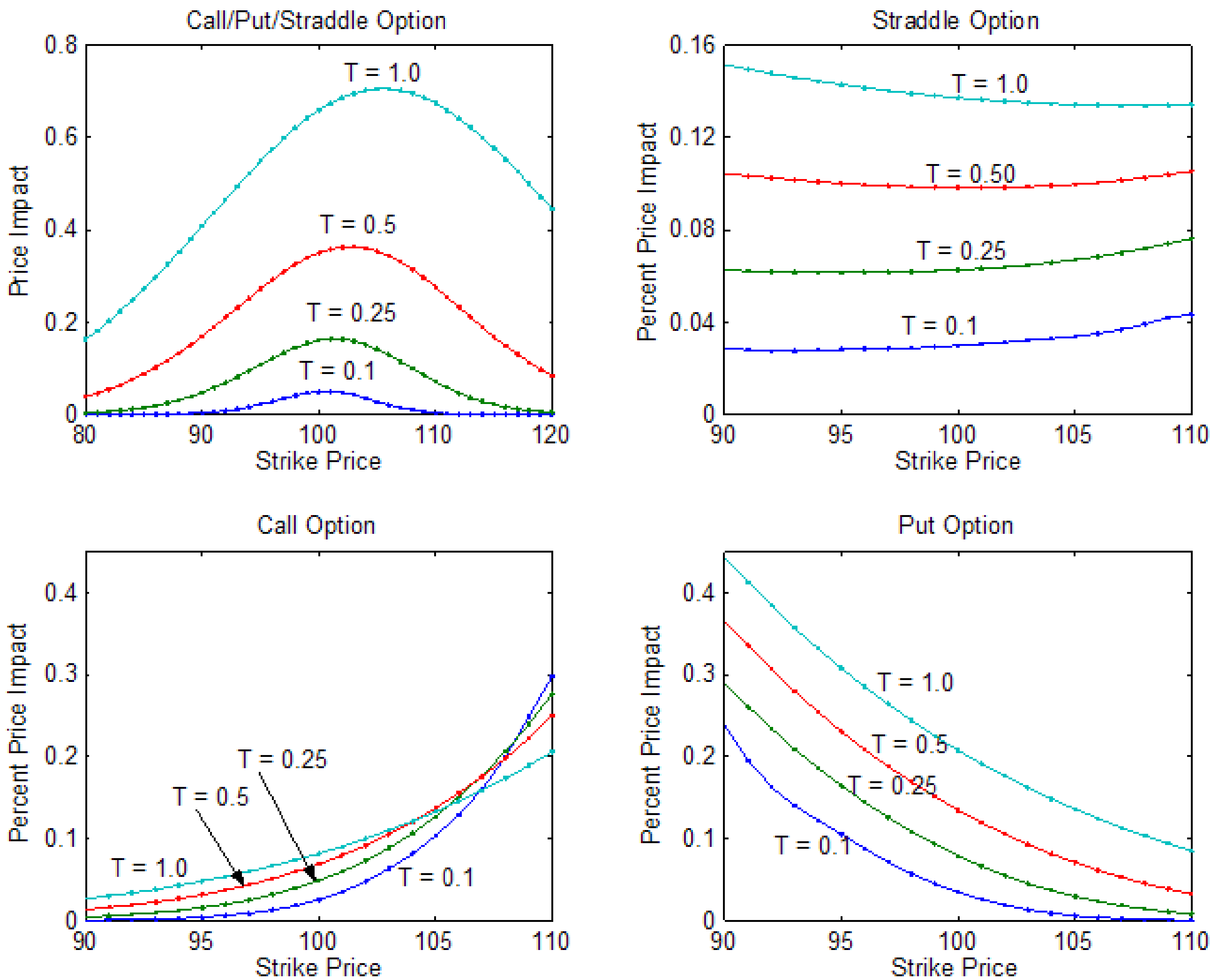


Figure 6: The impact of a negative orthogonal volatility risk premium on option prices ($C(0) - \widehat{C}(0)$ and percent price impact) across strike prices, $K$. The parameters are as follows. For stochastic volatility, $\kappa = 5.0$, $\sqrt{\theta} = 0.13$, $\sigma = 0.25$, and $\rho = -0.4$. The pricing parameters are$r = 0.05$, $\lambda_1 = 4$, and $\lambda_2 = -6$. We assume that the initial value of volatility is its long term level, $\sqrt{v_0} = 0.13$, and $S_0 = 100$. The option maturity is $T$ years.

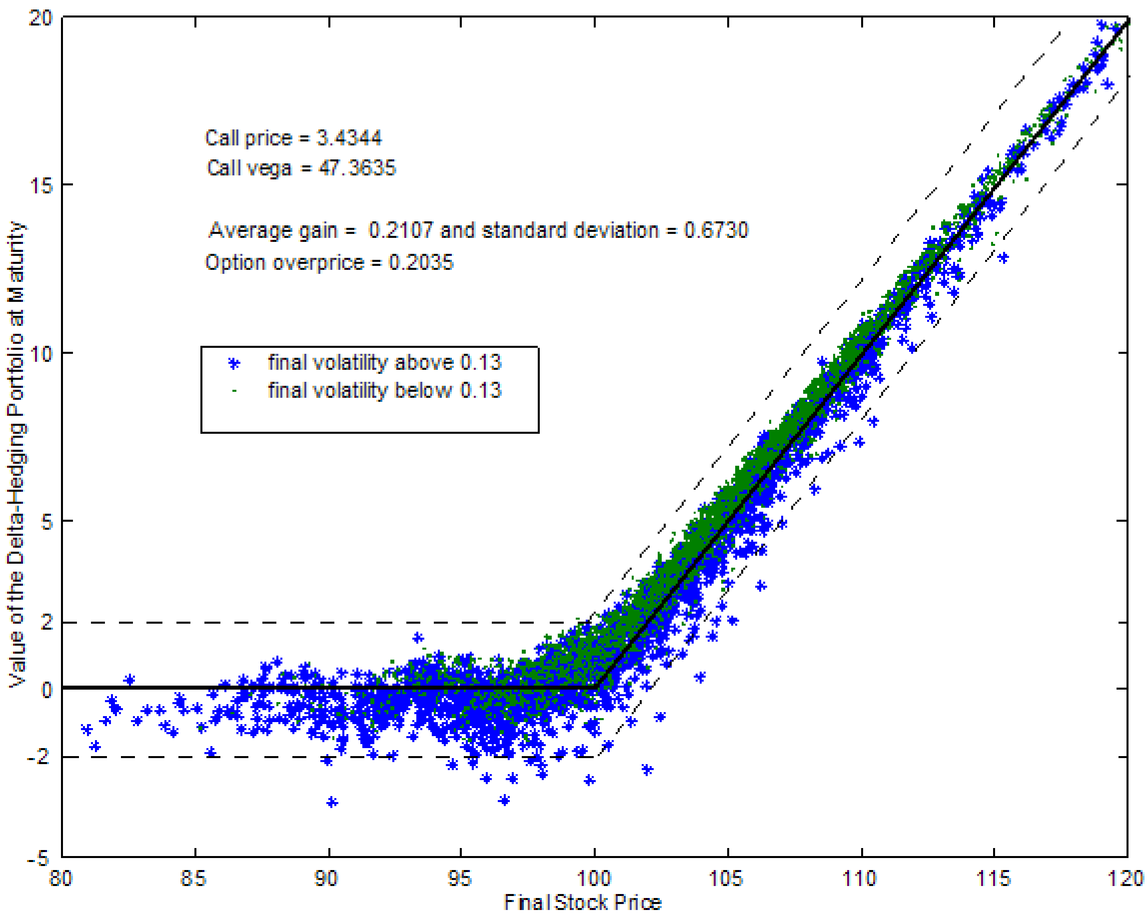


Figure 7: Value at maturity of a tracking delta-hedging portfolio of a short call option. The hedging errors, if the tracking portfolio value is above (below) the intrinsic value, are the gains (losses) of the short call position. The average gain is positive, but most gains (losses) correspond with low (high) volatility paths. The parameters are as follows. For stochastic volatility, $\kappa = 5.0$, $\sqrt{\theta} = 0.13$, $\sigma = 0.25$, and $\rho = -0.4$. The risk-free rate is $r = 0.05$ and $\lambda_1 = 4$ and $\lambda_2 - 6$. We assume that the initial value of volatility is its long term level, $\sqrt{v_0} = 0.13$, and $S_0 = 100$. The option maturity is $T = 0.25$ years and $K = 100$ is the strike price. We delta-hedge twice per week and show 5,000 simulations.

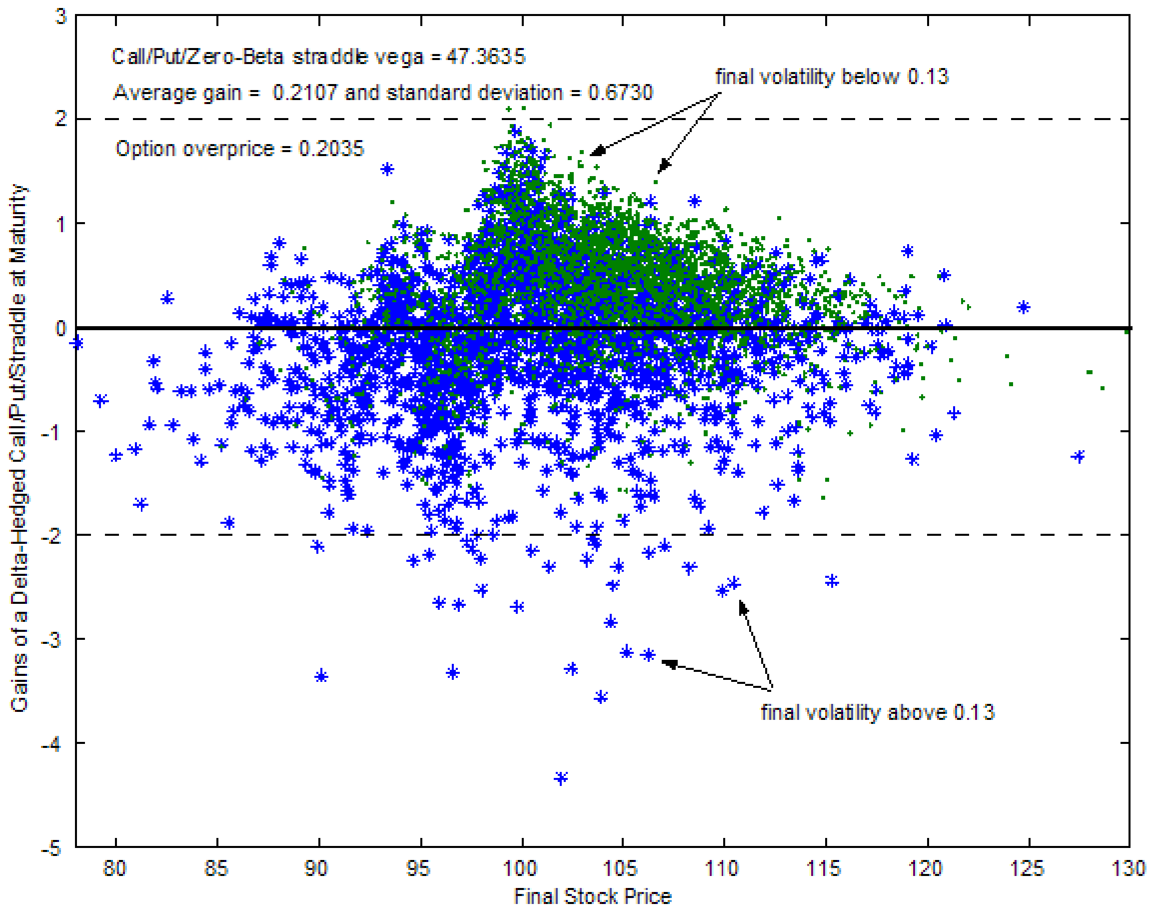


Figure 8: Gains/losses at maturity of a delta-hedged short call/put/straddle with the same strike price. The average gain is positive, but most gains (losses) correspond with low (high) volatility paths. The gains in this figure are from the tracking portfolio in Figure 7. The parameters are as follows. For stochastic volatility, $\kappa = 5.0$, $\sqrt{\theta} = 0.13$, $\sigma = 0.25$, and $\rho = -0.4$. The risk-free rate is $r = 0.05$ and $\lambda_1 = 4$ and $\lambda_2 - 6$. We assume that the initial value of volatility is its long term level, $\sqrt{v_0} = 0.13$, and $S_0 = 100$. The option maturity is $T = 0.25$ years and $K = 100$ is the strike price. We delta-hedge twice per week and show 5,000 simulations.