University of Washington

CFRM 462 - Computational Finance and Financial Econometrics

# 401(k) Portfolio Optimization

## A Mutual Fund Asset Allocation Project

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# Executive Summary

# Contents

# Chapter 1

# Introduction

## Dataset Description

### VFINX - S&P 500 Index Fund

The Vanguard 500 Index Fund[1] (VFINX) is an exchange traded fund (ETF) that invests in 500 of the largest U.S. companies. These companies span many different industries, and thus provides investors with full exposure to the domestic stock market. The fund focuses on large-capitalization companies that encompass nearly 75% of the U.S. equity market. The fund treats the Standard & Poor's 500[2] as its benchmark, and thus acts as a measurement of overall stock market performance.

### VEURX - European Stock Index Fund

The Vanguard European Stock Index Fund[3] (VEURX) is an ETF that provides investors with exposure to the major stock markets of Europe. The fund holds positions in approximately 1,200 securities across European markets, which represents nearly half of global (non U.S.) equity. In addition to systematic risk, this fund is also exposed to currency risk, and may have significant regional risk as all markets in which the fund invests in are located in Europe. This fund treated the MSCI Europe Index[4] as its benchmark through March 26, 2013, but has used the FTSE Developed Europe Index[5] as its benchmark thereafter.

### VEIEX - Emerging Markets Index Fund

The Vanguard Emerging Markets Stock Index Fund[6] (VEIEX) is an ETF that provides investors with exposure to emerging markets around the world including but not limited to: Brazil, Russia, India and China. As emerging markets tend to be more volatile, this fund has the potential for higher returns, but with considerably higher risk. Similar to the European Stock Index Fund, the returns of this fund too are exposed to significant currency risk. This fund treated the FTSE Emerging Index[7] as its benchmark through November 2, 2015, but has since switched to the FTSE Emerging Markets All Cap China A Transition Index.[8]

[1] The Vanguard Group Inc. (2016a)
[2] S&P Dow Jones Indices LLC (2016)
[3] The Vanguard Group Inc. (2016c)
[4] MSCI Inc. (2016a)
[5] FTSE Russell (2016c)
[6] The Vanguard Group Inc. (2016b)
[7] FTSE Russell (2016e)
[8] FTSE Russell (2016d)

### VBLTX - Long-Term Bond Index Fund

The Vanguard Long-Term Bond Index Fund[9] (VBLTX) is an ETF that provides investors with exposure to long-term bond (i.e. debt obligation) investments. This fund holds positions in both corporate and U.S. Government bonds with a maturity of 10 years or more. However, due to the fact that long-term bonds are highly exposed to price fluctuations caused by changing interest rate, which is attributable to the high duration and convexity of the underlying long-term bonds. This fund used the Barclays U.S. Long Government Float Adjusted Index[10] as its bechmark through December 31, 2009, but has since swithced to the Barclays U.S. Long Government/Credit Float Adjusted Index.[11]

### VBISX - Short-Term Bond Index Fund

The Vanguard Short-Term Bond Index Fund[12] (VBISX) is an ETF that provides investors with exposure to a diversified portoflio of short-term bonds (i.e. debt obligations). This fund holds positions in both corporate and U.S. Government short-term bonds with maturities of 1 to 5 years. Due to the fact that short-term bonds have low duration and convexity, investors can expect minimal price movement with relation to interest rates from this fund, and thus lower yield. This fund uses the Barclays U.S. Government/Credit Float Adjusted 1-5 Year Index[13] as its benchmark.

### VPACX - Pacific Stock Index Fund

The Vanguard Pacific Stock Index Fund[14] (VPACX) is an ETF that provides investors with exposure to a diversified portfolio of securities in markets of developed nations in the Pacific region. The fund holds positions in over 2,000 securities across the Pacific, with the bulk of them being located in Japan. This investment pool represents approximately a quarter of the global (non U.S.) equity market capitalization. The fund initially used the MSCI Pacific Index[15] as its benchmark until March 26, 2013, before switching to the FTSE Developed Asia Pacific Index[16] through September 30, 2015, until finally switching to the FTSE Developed Asia Pacific All Cap Index,[17] which is uses today.

## ETF Historical Prices

Analyzing the price data each of the ETFs in Figure 1, it is clear that VFINX and VBISX have the most stable stream of returns, which is reflected in the (relatively) steady increase in their respective prices over time. Furthermore, it is apparent from the visible trend of each of the ETF prices that while specific, small-scale fluctuations appear to be random, there are many longer time-horizon trends that are common across all of the funds.

In the case of VFINX, the constant positive progression of the price is to be expected as the stock market has displayed an above-average yearly growth rate over the time horizon considered in the graph. This above-average growth rate is explained by renewed investor confidence in the market in the aftermath of the financial crisis of 2009. Considering the VBISX ETF, which tracks the prices of diversified short-term debt obligations, the price increase can be attributed to a constant reduction in interest rates (yield) of short-term debt in the last five years.[18] The reducing interest rates has a positive effect on the net present value (i.e. price) of the debt, which is reflected in the rising price of the VBISX ETF.

In addition to the price trends inherent to each security, the time plots also indicate that there are significant fluctuations in the prices common to all of the equity ETFs (i.e. except VBLTX and VBISX, which are debt ETFs).

[9] The Vanguard Group Inc. (2016d)
[10] Index Portfolio and Risk Solutions Group (IPRS) (2015b)
[11] Index Portfolio and Risk Solutions Group (IPRS) (2015c)
[12] The Vanguard Group Inc. (2016f)
[13] Index Portfolio and Risk Solutions Group (IPRS) (2015a)
[14] The Vanguard Group Inc. (2016e)
[15] MSCI Inc. (2016b)
[16] FTSE Russell (2016b)
[17] FTSE Russell (2016a)
[18] Organisation for Economic Co-operation and Development (2016)

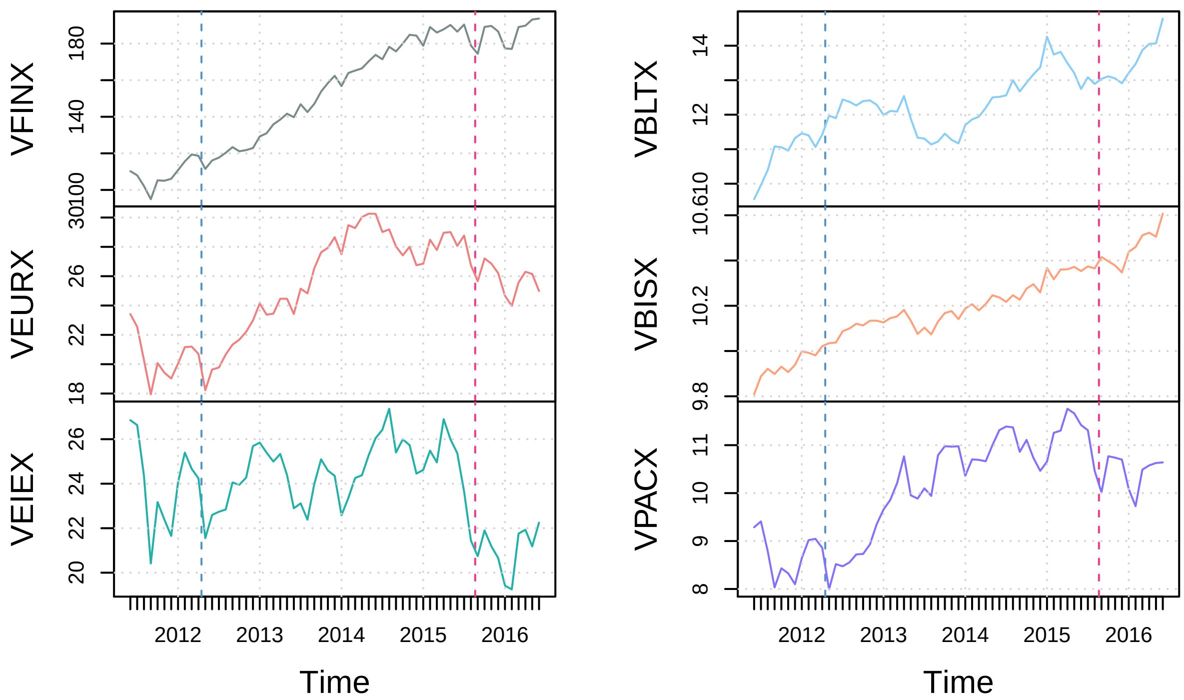



Figure 1: Timeplot of ETF prices

These common fluctuations are triggered by specific macroeconomic events, and illustrate congenial non-diversifiable risk (i.e. market risk) in each of the equity ETFs.

During the middle months of 2012, many European Union (EU) nations (Greece, Spain, Ireland, etc.) were in serious risk of defaulting (or in the case of Greece, already defaulted) on their debt obligations.[19] This caused a significant reduction in investor confidence in capital markets across the globe, which eventually led to massive selloffs in markets across the world, with investors shifting capital from equity investments to perceptively safer and less-risky debt investments.[20] These macroeconomic events are the cause of the simlultaneous significant reduction in equity ETF prices that track multiple regions across the world. This loss in confidence affected markets that have perceptively higher levels of risk compared to others, as is reflected by the sharper drop in emerging markets (VEIEX) when compared to less risky markets such as that of the U.S. (VBLTX). As investors shifted capital to less-risky investments, the surge in invested capital in bond markets caused a drop in bond yields, which was seen in both short-term and long-term bonds. This reduction in bond yields is reflected in the rise in the price of debt ETFs VBLTX and VBISX, as bond yields correlate negatively with the present value (i.e. price) of bonds.

Consider the stock market correction in August 2011, which rattled equity markets worldwide. Investor confidence was damaged when Greece became the first advanced economy to ever default on an International Monetary Fund (IMF) loan payment in July.[21] This was followed by a surprise devaluation of the Chinese Yuan by the Government of China in August,[22] which led to a large drop in the value of companies that use the Chinese Yuan as their primary currency due to their high exposure to foreign exchange risk. Significantly lowered investor confidence, coupled with indicators that the market was inflated led to an eventual global sell-off and inevitable stock market correction in August 2015.[23] This is reflected in the sharp drop in prices of ETFs that track regions across the globe. As with the EU credit crisis of 2012, markets that traditionally display higher levels of risk experienced the largest declines in price, as reflected by steeper drops in Emerging and Pacific Market ETFs (VEIEX and VPACX) compared to American and European Market ETFs (VFINX and VEURX).

[19] Heather Stewart, Larry Elliott, & Giles Tremlett (2012)
[20] Maureen Farrell (2012)
[21] Ian Talley (2015)
[22] Bloomberg News (2015)
[23] Nathaniel Popper & Neil Gough (2015)

## Types of Investment Risk

The presence of market-wide trends is extremely important in the construction of portfolios that comprise ETFs that inherently exhibit these trends. Logically, it can be deduced that trends in the price of a specific ETF that are not common across all other ETFs are attributable to some characteristic of that particular fund, which are fundamentally different from trends that emanate across all of the funds in the market.

Simply put, price trends and fluctuations are illustrations of risk perceived by investors, as a given price reflects the amount of risk an investor is willing to undertake to realize a certain return. This concept can be generalized to the different types of price fluctuations, and would have the following implications:

- Price fluctuations and trends that are specific to a given ETF are an expression of a certain risk factor that is inherent to that particular ETF
- Price fluctuations and trends that are visible across all ETFs are an expression of risk that is perpetuated across the entire market in which the ETFs operate, and is thus not specific to any given ETF

In financial theory, these two types of risk are referred to as diversifiable or unique risk, which is risk that is attributable to some characteristic of a particular ETF, and market or idiosyncratic risk, which is perceived across all ETFs in the market. The main result of this distinction is that combinations of ETFs with particular levels of diversifiable risk can be combined to eliminate their effect, leaving only the idiosyncratic risk of the market, which achieving an enhanced rate of return compared to that of the market as a whole. This is the ideal output of a portfolio, and its realization is the ultimate goal that motivates modern Portfolio Construction Theory.

# Chapter 2

# Analysis

## Continously Compounded Returns

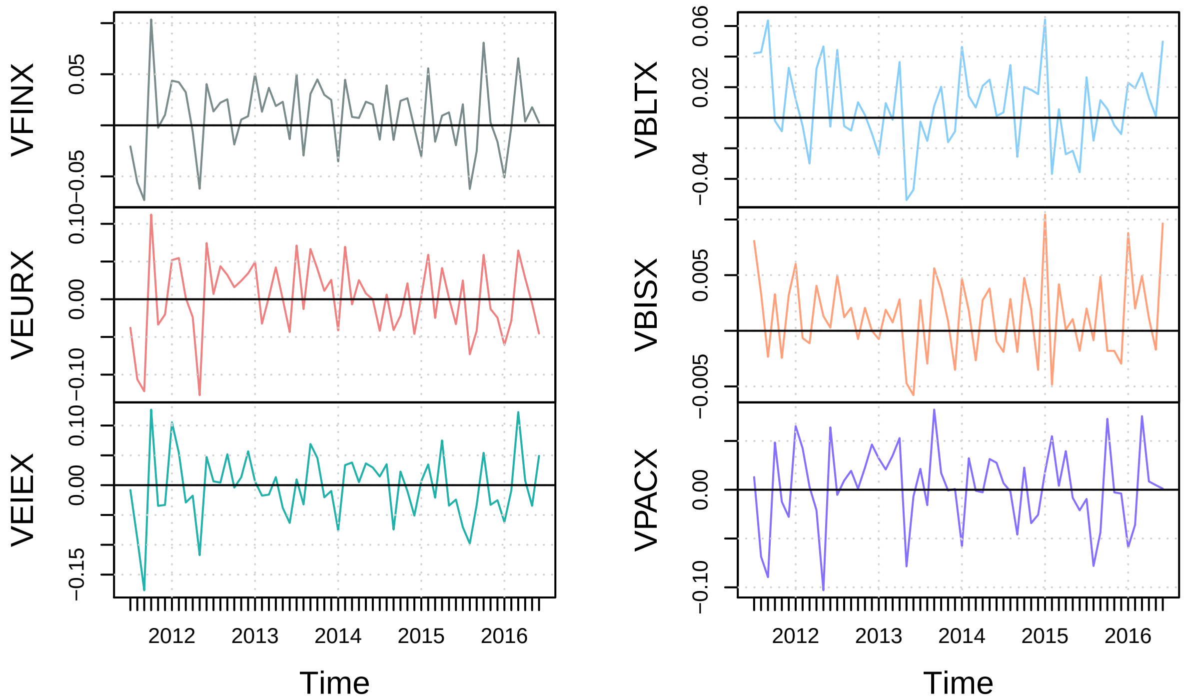



Figure 2: Timeplot of ETF continuously compounded returns

Using the price data from each of the ETFs, we are able to compute a series of *continuously compounded returns* for each ETF. A continuously compounded return (also known as a CC, geometric or log return) is calculated by computing the difference in the log price at each time period for a given asset. That is:

Let $P_t$ and $P_{t-1}$ be the price at time $t$ and $t-1$ respectively

Conventionally, a *simple return* is defined as follows:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

The *continuously compounded return* is defined as:

$$r_t = \log \frac{P_t}{P_{t-1}} = \log P_t - \log P_{t-1}$$

As seen in Figure 2, the graph displays the continuously compounded return at each point in time, for each asset. As seen in the graph, the monthly returns for each asset appear to be extremely jagged, or *volatile*. This is due to the constant price changes in any given asset, which is a consequence of operating in an open market.

Due to the limiting nature of the `log` function, larger returns are often reduced in magnitude when compared to their arithmetic counterparts. This greatly increases the value of the data, as it appears to be better-behaved, and thus a better candidate for statistical evaluation. Furthermore, the additive properties of exponents and the logarithm mean that consecutive returns can be added to determine the return over a contiguous period of time. This is illustrated briefly below:

$$\text{For continuously compounded (geometric) returns, } (1+R_2) = (1+R_0)\times(1+R_1)$$
$$\Rightarrow R_2 = (1+R_0)\times(1+R_1) - 1 = 1 + R_1 + R_0 + R_1R_0 - 1 = R_1 + R_0 + R_1R_0$$
$$\text{However, in the case of simple returns, this would simply be:} R_2 = \frac{R_2 - R_0}{R_0} = \frac{R_2}{R_0} - 1$$
$$\text{As illustrated, this would overlook the effect of the compounded term } R_1R_0,$$
$$\text{which leads to an overstatement of returns in the long run.}$$

$$\text{Furthermore generalizing the geometric returns further, they can be expressed as:}$$
$$\Rightarrow (1+R_k) = (1+R_0)\times(1+R_1)\times(1+R_2)\times\ldots\times(1+R_{k-1}) = \prod_i^{k-1}(1+R_i)$$
$$\text{This result is extremely convenient, as the logarithm can be applied}$$
$$\text{to each side to reduce this to a simple sum:}$$
$$\therefore \log(1+R_k) = \log\left(\prod_i^{k-1}(1+R_i)\right) = \log(1+R_0) + \log(1+R_1) + \ldots + \log(1+R_{k-1})$$
$$\text{This property does not overstate the return of an asset, while also making the process of}$$
$$\text{computing compunded returns mathematically trivial.}$$

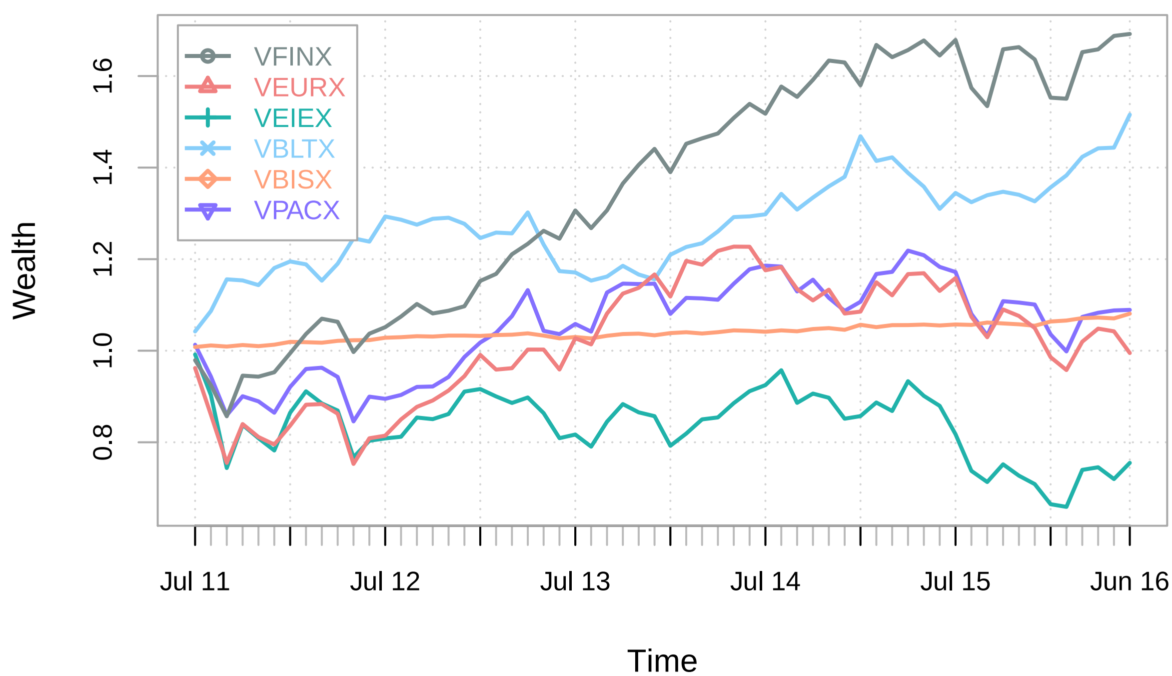



Figure 3: Growth of One Dollar Investment

This concept can be visualized further by considering the equity curve of each of the ETFs (see Figure 3). The equity curve is a generalized method of gauging the performance of a fund, by appying its returns to a benchmark over $1, and viewing its progression over time. The curve is extremely telling, and reiterates the patterns observed in the price graphs of each of the ETFs. It is clear the VFINX has performed the best, yielding a tital return of approximately 70% over the course of the 5-year time horizon of the data. Similarly, it can also be observed that VEIEX performed the worse, with a neagtive return of nearly 25%. The volatility of returns of each of the assets can also be gauged from the equity curve, and it is a fitting precursor to the next section of this report.

## Asset Distribution Analysis

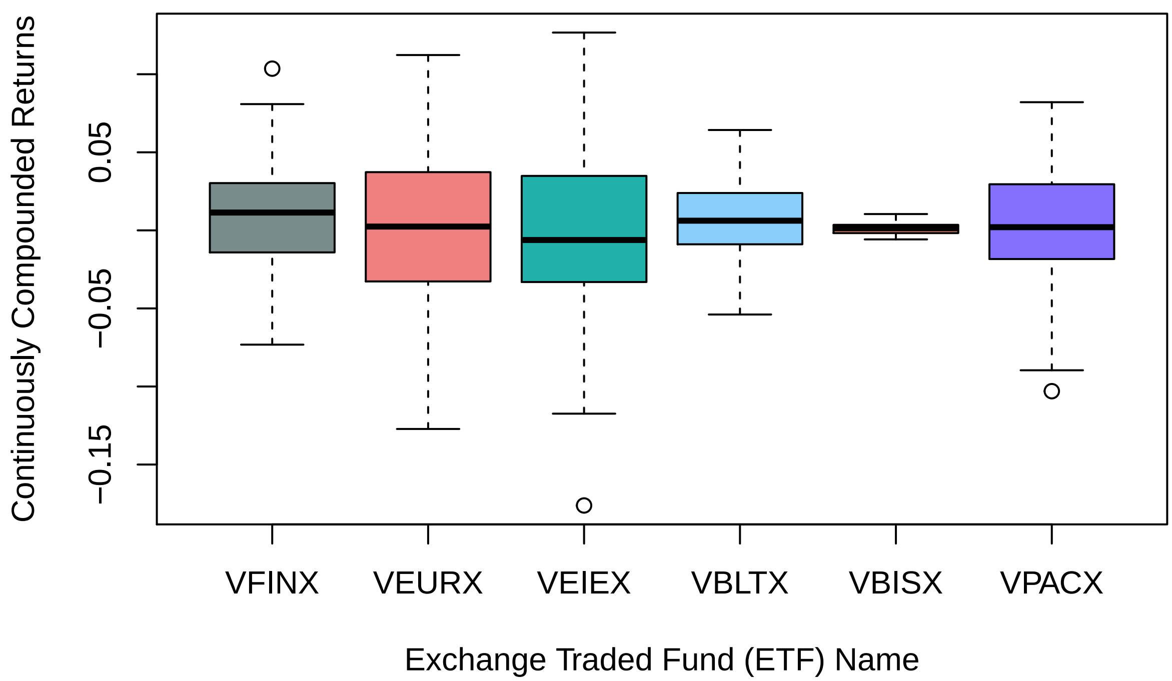



Figure 4: Box Plot of CC Monthly Returns of the ETFs

Modern financial theory perpetuates the notion that all asset returns are random, and that returns are not correlated asynchronously with their past or possible future performance. It is this assumption that has birthed the Constant Expected Return (CER) model, which is rooted in the belief that returns are serially uncorrelated.

Despite this, the CER assumes that the series of past returns produced by a given asset are qunatifiable, and thus regressible. This means that as with any other stationary, ergodic and serially uncorrelated process, assumptions about future behavior may be inferred - but not determined - from past performance. Is is this notion that drives the analysis of financial data, as characteristics of past performance are identified and used to determine possible patterns of future progression.

Referring to the four plot summary charts prepared for each of the return distributions in Appendix A, the returns appear to be mostly normally distributed. This is to be expected, as the law of large numbers states that any repeatedly-sampled dataset will tend towards the normal distribution with time. In all of the datasets, the relationship will the normal distribution appears to be extremely strong towards the middle of the distribution of returns, as evidenced by the mostly-linear behavior of their Normal Q-Q plots near the median of the dataset. However, in the case of VEURX, VEIEX, VPACX and VBISX, this normality breaks down quickly at the tails of the distribution. Although this may seem like a letdown, it is to be expected, as extreme results are often more common and more extreme than predicted by the normal distribution in practice.

It apears that the returns of VBLTX has the most normal-esque distribution, as it displays the most linear shape on the Q-Q plot, as well as having the most symmetric histogram, and no outliers on its box plot. These observations can be scrutinized further by analyzing the descriptive statistics of each of the samples, as displayed below:

Table 1: Univariate Statistics for each of the ETFs

| | VFINX | VEURX | VEIEX | VBLTX | VBISX | VPACX |
|---|---|---|---|---|---|---|
| Mean | 0.009390 | 0.001083 | -0.003133 | 0.007286 | 0.001304 | 0.002265 |
| Variance | 0.001201 | 0.002340 | 0.003079 | 0.000680 | 0.000013 | 0.001689 |
| Std Dev | 0.034656 | 0.048370 | 0.055491 | 0.026083 | 0.003567 | 0.041102 |
| Skewness | -0.121763 | -0.385855 | -0.223684 | 0.015558 | 0.374901 | -0.420863 |
| Excess Kurtosis | 0.330578 | 0.255608 | 0.872732 | -0.269205 | -0.097942 | 0.032793 |

Analyzing the table above, it is clar that the sample statsitics confirm the VEIEX was indeed the worst performer in this category, as evidenced by its negative mean return. Additionally, it can also be confimed that it was also the most volatile, as evidenced by its relatively high standard deviation of returns. The unique shapes of each of the histograms displayed in Appendix A too can be explained by the statistics in Table 1. The negative skewness of VFINX, VEURX, VEIEX and VPACX all indicate that despite an overall positive performance, their returns were battered with large periods of highly negative returns, as seen in the timplots of the price of the assets.

A comparison of the skewness and excess kurtosis displayed by each of the distributions would indicate that VBLTX had the most normally distributed returns, as confirned by its highly linear normal Q-Q plot, and highly symmetric boxplot, as seen in figure. Furthermore, the extreme lack of a centered median in the case of VEURX and VBISX, as they both have skewness values that are far from the Gaussian reference's 0. Considering the excess kurtosis of each of the distributions, it is indicative of VEIEX having an extremely thin distribution, which is highly plausible, as it had the least absolute change in value over the time horizon of the project, which would indicate that the majority of its positive returns would have been negated by the negative.

# Precision of Estimators

Table 2: Standard Errors and Confidence Intervals for ETF Return Means

| | Mean | Mean SE | Mean SE (%) | 95% Confidence Interval |
|---|---|---|---|---|
| VFINX | 0.009390 | 0.004474 | 47.647% | [0.018159, 0.000621] |
| VEURX | 0.001083 | 0.006244 | 576.772% | [0.013322, -0.011156] |
| VEIEX | -0.003133 | 0.007164 | 228.624% | [0.010907, -0.017174] |
| VBLTX | 0.007286 | 0.003367 | 46.214% | [0.013886, 0.000687] |
| VBISX | 0.001304 | 0.000460 | 35.311% | [0.002207, 0.000402] |
| VPACX | 0.002265 | 0.005306 | 234.288% | [0.012665, -0.008135] |

Table 3: Standard Errors and Confidence Intervals for ETF Return Standard Deviations

| | Std Dev | Std Dev SE | Std Dev SE (%) | 95% Confidence Interval |
|---|---|---|---|---|
| VFINX | 0.034656 | 0.003164 | 9.129% | [0.040857, 0.028456] |
| VEURX | 0.048370 | 0.004416 | 9.129% | [0.057024, 0.039715] |
| VEIEX | 0.055491 | 0.005066 | 9.129% | [0.065419, 0.045562] |
| VBLTX | 0.026083 | 0.002381 | 9.129% | [0.030749, 0.021416] |
| VBISX | 0.003567 | 0.000326 | 9.129% | [0.004205, 0.002929] |
| VPACX | 0.041102 | 0.003752 | 9.129% | [0.048456, 0.033748] |

Despite the fact that we may now feel confident in our analysis of the worthiness of a partiuclar asset as a suitable investment vehicle, this is far from the truth. Going back to the core principle that these returns are in fact random, and that we are merely attempting to discern a pattern is a powerful notion, as reflected by the exterme standard errors of estimation in both the means and standard deviations of returns analyzed previously.

In the case of the mean returns in particular, the large standard deviations and small sample size considered have an extremely detrimental effect on the outlook of the inferences made in the initial analysis. In the case of two securities, VEURX and VPACX, the 95% confidence interval as determined by the standard error of the mean sees the average return going into the red. Further, in the case of VFINX - supposedly the most promosing of the ETFs analyze thus far in term of return - too is just 6 basis points from being negative.

As these standard errors were computed using the analytic equations for the standard error of the mean and standard deviation however, they are self-determinant. This is evident from the constant percentage of standard error across all of the standard deviations of returns, which would not be the case if the population of returns were to be sampled repeatedly, and the extreme values observed in Table 2 and Table 3 may be due to a false assumption that the underlying distribution of the means and standard deviations are indeed normal, compounded by the fact only a relatively small smaple size (N = 60) was being considered.

However, despite possible shortcoming that may have caused the large confidence intervals, it is still apparent that the only securities to have displayed a stable - albeit slight - positive rate of return are VFINX, VBLTX and VBISX.

# Risk-Adjusted Return

As evidenced by the amazingly volatility observed above, any exposure to a potential large return is accompanied with a proortional level of risk. This is what is commonly referred to as the *risk-return tradeoff.* Thus, a better measure of gauging an asset's performance would be highly valuable, as a blind evaluation of one aspect (expected return or volatility) without regard to the other is baseless.

The current standard for this type of measurement is *risk-adjusted return.* Risk-adjusted return, which is the notion of measuring the amount of potential returns an asset can provide, relative to the amount of exposure to risk it is undertaking can be measured in multiple forms. However, one of the leading insudtry standards is the *Sharpe Ratio*, which measures the amount of expected return offered by a particular investment, taking into account the prevailing return that is available without any risk (i.e. risk-free asset), per unit of risk, which is measure by standard deviation of returns. The equation for the Sharpe Ratio is:

$$\text{Sharpe Ratio, SR} = \frac{\mathrm{E}[r] - r_f}{\sigma}$$

> Note: For all future use in this project, the risk free rate is assumed to be, $r_f = 0.04167\%$ continuously compounded, per month

By considering both the level of potential returns from an investment along with the risk it presents, the Sharpe Ratio effectively quantifies the risk-return tradeoff, and is used across the industry to rank the value of investments. This relationship is perhaps best illustrated on a plot of an investment's risk versus return, as seen in Figure 5 This is not to say however, that it is the final judge of a measurement, as every investor's level of risk aversion varies, and thus an investment that may seem extremely to one individual may be one that would never be considered by another.

## The Bootstrap

As illustrated by the large confidence intervals in the return of each of the ETFs in Table 2: Standard Errors and Confidence Intervals for ETF Return Means, having a good understanding of the precision of an investment is equally important to having a measurement at all. Thus, the bootstrap technique of gauging the standard error of an estimator will be used to measure the standard error of our calculation of the Sharpe Ratio for each asset.

The Bootstrap method is derived from the notion that any sample, from an existing sample of a population, is reperesentative of the underlying distribution of the population itself. This is similar to the approach taken by us when attempting to approximate characteristics about a population from a sample, except this train of thought extends those notions to samples from the original population as well.

Firstly, advocates of this method justify the risk of bias from the sample it would be gathering data from by arguing that significantly fewer generalizations are made about the distribution of the underlying population when compared to other analytical approaches. A perfect example of this is our assumption that the means of the assets are normally

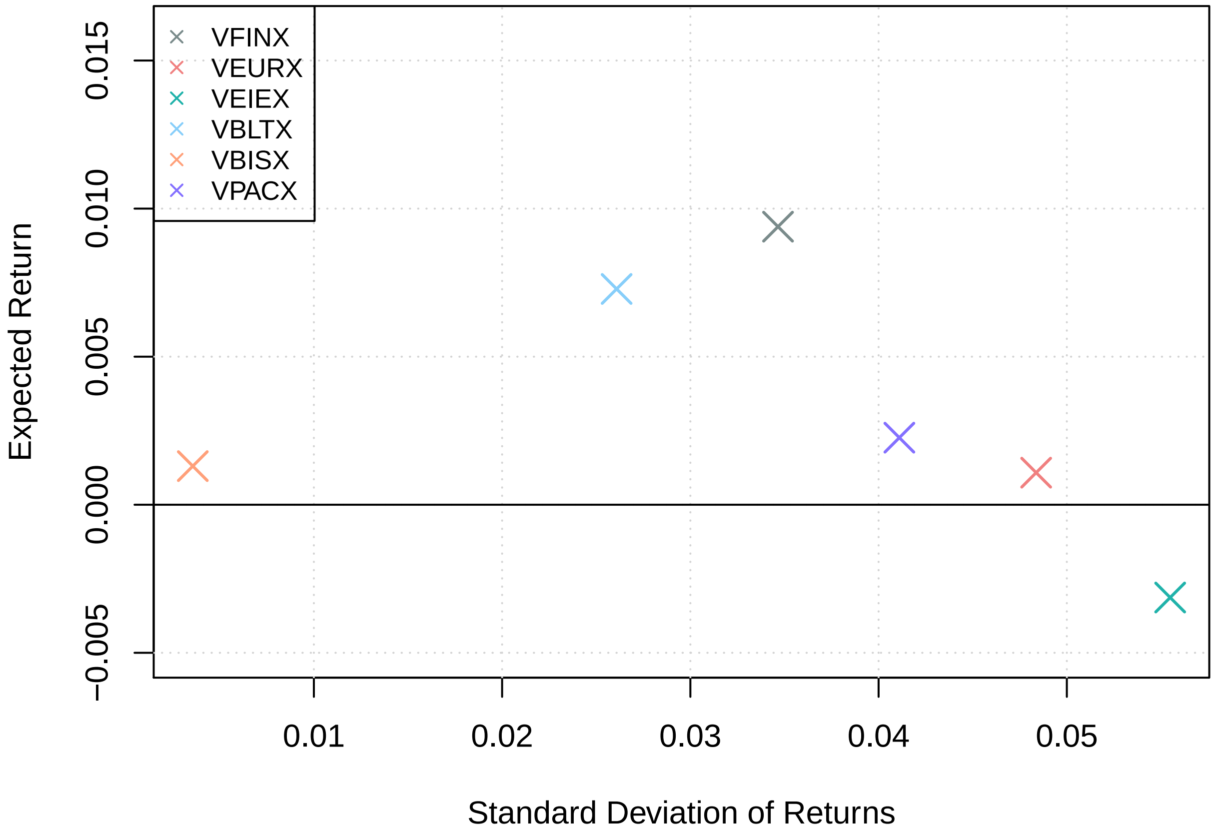



Figure 5: Risk-Return Tradeoff for each of the ETFs

distributed, which was crucial to calculating standard errors and confidence intervals above. Second, this method is also thought to be - in general - more accurate than the analytical assumptions of normality (as per the Central Limit Theorem) that are often made when conducting statistical analyses. Third, this procedure is generally easier to perform, as it is the same regardless of the quantity beign estimated, which is in stark contrast to the wide range of formulas generally used to infer statistics about a population.

The Bootstrap is performed using computer intensive resampling with replacement of a sample data set, calculating the desired estimator for each of these *samples of samples*, and then finally constructing a distribution of the quantity being estimated from the hundreds or thousands of samples taken by the computer. This is then assumed to be an accurate representation of the distribution of the estimator in the underlying population, as per the original assumption that the initial sample accurately reflects that population.

Whiile this method is exposed to the obvious risk of a biased sample from which it will sample data thousands of times, it has many advantages over traditional analytic assumptions of normality about data that are typically made during statistical analysis. One major benefit of quantitatively intensive statistical tenchiques such as the Bootstrap is that the sampled data will undoubtedly reflect the true distribution of the sample; which extends to the population if the underlying assumption about the neutrality of the orignal sample is correct. This, coupled with the simple implementation and high rate of repetition make this method particularly attractive for statistical analysis.

## Bootstrapped Sharpe Ratio

Table 4: Monthly Sharpe Ratios with Bootstrap-estimated Standard Errors (Key: A - Analytical, B - Bootstrap)

| | Sharpe Ratio (A) | Sharpe Ratio (B) | Sharpe Ratio SE (B) | Sharpe Ratio SE % (B) |
|---|---|---|---|---|
| VFINX | 0.258927 | 0.258927 | 0.137560 | 53.127% |
| VEURX | 0.013768 | 0.013768 | 0.131966 | 958.489% |
| VEIEX | -0.063977 | -0.063977 | 0.130389 | 203.805% |

| | Sharpe Ratio (A) | Sharpe Ratio (B) | Sharpe Ratio SE (B) | Sharpe Ratio SE % (B) |
|---|---|---|---|---|
| VBLTX | 0.263376 | 0.263376 | 0.133259 | 50.596% |
| VBISX | 0.248785 | 0.248785 | 0.126489 | 50.843% |
| VPACX | 0.044965 | 0.044965 | 0.132832 | 295.412% |

The bootstrapped values in the table above were calculated by resampling the initial data 9999 times, for each ETF in the dataset. This high rate of sampling, coupled with the small size of the data set (N = 60) would have contributed to the fact that the Bootsrtap-estimated Sharpe Ratio convereged perfectly to the value of the analytical formula.

This high sampling rate provided an extremely good sense of a possible distribution of the Sharpe Ratio of the underlying population, assuming that the sample that we in turn sampled was an accurate reflection of the population it was representing. Despite the bootstrapping process, the Sharpe Ratios have similarly high estimation erros associated with them. This may be attributed to the small sample size of our data, and the face that that Sharpe Ratio is a determinant of two unknow (i.e. risky) quantities, which would result in it inhereting the risk factors of both of its determinants, the mean and standard deviation, which were extremely volatile to begin with.

The bootstrapped standard errors reflect these assumptions well, as the securities that had returns with a lower volatility have a significantly lower standard error of measurement of its Sharpe Ratio compared to the securities that had fatter-tailed distributions. In the case of VEURX, VEIEX and VPACX, the extremely high relative Sharpe Ratio SEs may be due to the shape of the distribution of their distribution in the time horizon of the project. We know from the histograms and the timeplots of the price that the three securities experienced intense fluctionations, and would thus have numerous extreme values at both ends of its distribution, resulting in a large estimation error.

Consirting the rank of the ETFs, it is clear that VBLTX, VFINX and VBISX provide the best risk-adjusted standalone investment opportunies. It is hard to make a distinction betond this, as they all have similar levels of estimation error associated with their respective Sharpe Ratios. This is contrasted with VEIEX, which, with a negative Sharpe Ratio definitely provides the worse opportunity for direct investment. Second, the extremely high estimation error of VEURX's low Sharpe Ratio of 0.013768 should raise concerns, as it is within one standard deviation of measurement to being negative.

## Annualization

Due to the fact that continuously compounded returns were employed in this analysis, annualizing the sample statistics and risk-adjusted return does not add any immediate value to the analysis. This is evidenced by the data in Table 5: , as the measure of risk-adjusted return, the Sharpe Ratio reaffirms the observations made with the monthly Sharpe Ratios. This is attributable to the fact that while not in equal proportions, all of the factors that contribute to the Sharpe Ratio are scaled up by the same, non-parametric amount. The monthly expected returns and risk free rate would be multiplied by 12 to simply account for the 12 months of extra compounding they would be able to undergo, and the standard deviations would be scaled up by $\sqrt{12}$. This finally leads to a scaling factor of $\sqrt{12}$ for the Sharpe Ratio, which would not change the rankings of any of the assets.

Table 5: Annualized Mean, Standard Deviation and Sharpe Ratios for the ETFs

| | Annualized Mean | Annualized Std Dev | Annualized Sharpe Ratio |
|---|---|---|---|
| VFINX | 0.112682 | 0.120053 | 0.896950 |
| VEURX | 0.012992 | 0.167557 | 0.047694 |
| VEIEX | -0.037601 | 0.192226 | -0.221624 |
| VBLTX | 0.087435 | 0.090353 | 0.912361 |
| VBISX | 0.015649 | 0.012356 | 0.861817 |
| VPACX | 0.027178 | 0.142382 | 0.155763 |

This compounding factor is illustrated by the simple additive property of continuously compounded returns. Consdier the following, where the annual and monthly average return on VFINX can be reconciled perfectly to explain the return of a $1 investment in 5 years:

$$\text{Monthly average CC return on VFINX} = 0.00939$$
$$\text{Yearly average CC return on VFINX} = 0.112682$$

$$\Rightarrow \text{Return in 5 years} = 0.00939 \times 60 \text{ months} = \underline{\underline{0.563412}}$$
$$\Rightarrow \text{Return in 5 years} = 0.112682 \times 5 \text{ years} = \underline{\underline{0.563412}}$$

$$\therefore \text{Value of \$1 in 5 years} = \$1 \times \exp 0.112682 = \underline{\underline{1.756656}} = \text{Value of \$1 in 5 years (see Figure 3)}$$

## Covariance and Correlation Analysis

As illustrated in the price trend analysis of each of the ETFs over time, Macroecnomic events affect certain assets in tandem, causing their prices to move in similar patterns. This effect is captured by the covariance between two assets, which is measued by the average deviation of each asset from its mean, similar to the asset's variance:

$$\text{cov}(X, Y) = (\text{E}[X] - \hat{\mu}_x)(\text{E}[Y] - \hat{\mu}_y)$$

Table 6: Covariance matrix showing pairwise covariances between each of the ETFs

| | VFINX | VEURX | VEIEX | VBLTX | VBISX | VPACX |
|---|---|---|---|---|---|---|
| VFINX | 0.001201 | 0.001459 | 0.001474 | -0.000254 | -5.0e-06 | 0.001081 |
| VEURX | 0.001459 | 0.002340 | 0.002162 | -0.000356 | 6.0e-06 | 0.001625 |
| VEIEX | 0.001474 | 0.002162 | 0.003079 | -0.000144 | 5.0e-05 | 0.001941 |
| VBLTX | -0.000254 | -0.000356 | -0.000144 | 0.000680 | 6.4e-05 | -0.000165 |
| VBISX | -0.000005 | 0.000006 | 0.000050 | 0.000064 | 1.3e-05 | 0.000018 |
| VPACX | 0.001081 | 0.001625 | 0.001941 | -0.000165 | 1.8e-05 | 0.001689 |

This relationship is illustrated by the pairwise scatterplots of each of the assets in Figure 6. Analyzing the pairwise scatterplot, a definite insight into the behavior of each asset's prices with relation to another can be gleaned. For example, the positive-linear shape of the VFINX-VEURX scatterplots indicate a clear positive trend, which is reflected by its high covariance (see Table 6: Covariance matrix showing pairwise covariances between each of the ETFs). Similarly, it is also observable that the VEIEX-VPACX and VEURX-VPACX relationships are similarly positive, but not to same extent as that of VFINX-VEURX. In contrast, it can also be observed that there is no significant correlation between VEURX and VBISX, nor between VBLTX and VEIEX. Overall, it appears that the cross-asset correlations between the debt funds with other non-debt funds is extremely low, and is attributable to the negative relationship between the asset's price and its apparent health (i.e. yield).

To better understand and compare the relationships between each of the assets, a correlation matrix is often employed. Derived from the covariance, the correlation between two assets represents a scale-adjusted measure of correlation, which is significantly more useful when comparing the correlations of different pairs of assets. It is calculated by 'correcting' the covariance to the scale of each of the underlying distributions it is tracking, as shown below:

$$\text{corr}(X, Y) = \rho_{xy} = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X) \times \text{var}(Y)}} = \frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x \hat{\sigma}_y}$$

The pairwise correlations between each of the ETFs can be visualized garphically, as seen in Figure 7. The correlation coefficient can range from -1 to 1 for any given comparison, with a correlation of 1 meaning perfect positive corelation, 0 meaning no linear correlation and -1 being perfect negative correlation. As seen in the correlation plot, these values are represented graphically through the use of distorted ellipses, with a larger slant indicating a stronger correlation.

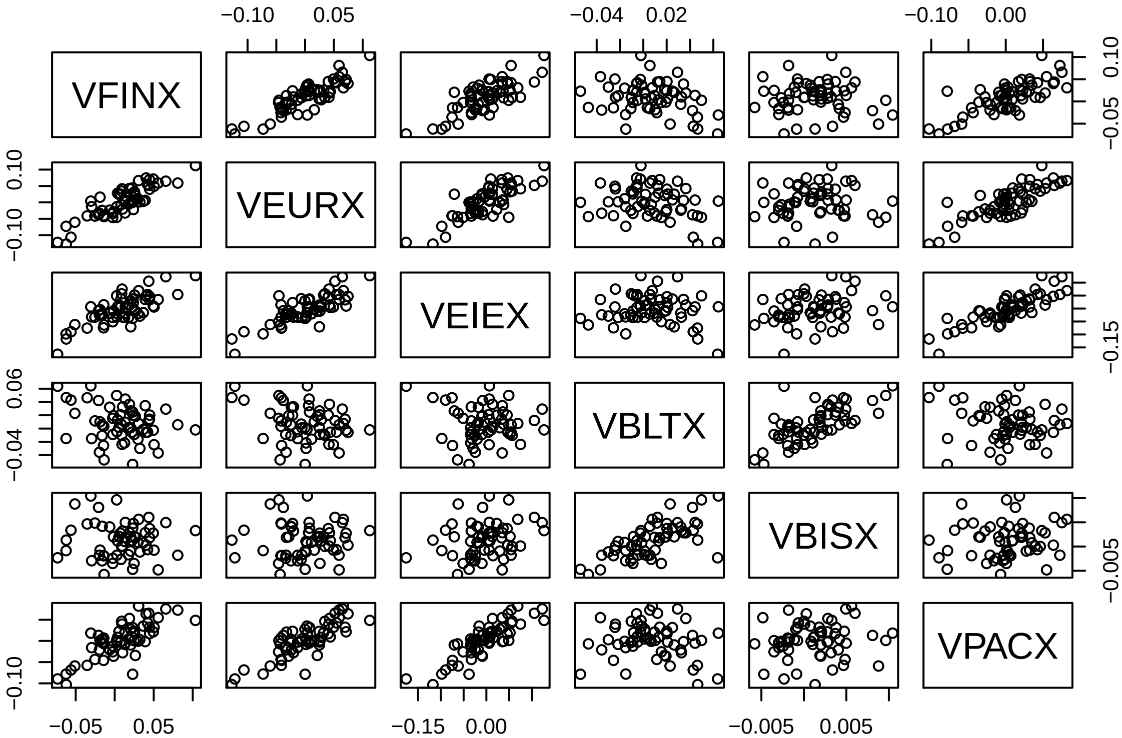


Figure 6: Pairwise Scatterplots of each of the ETF Monthly Returns

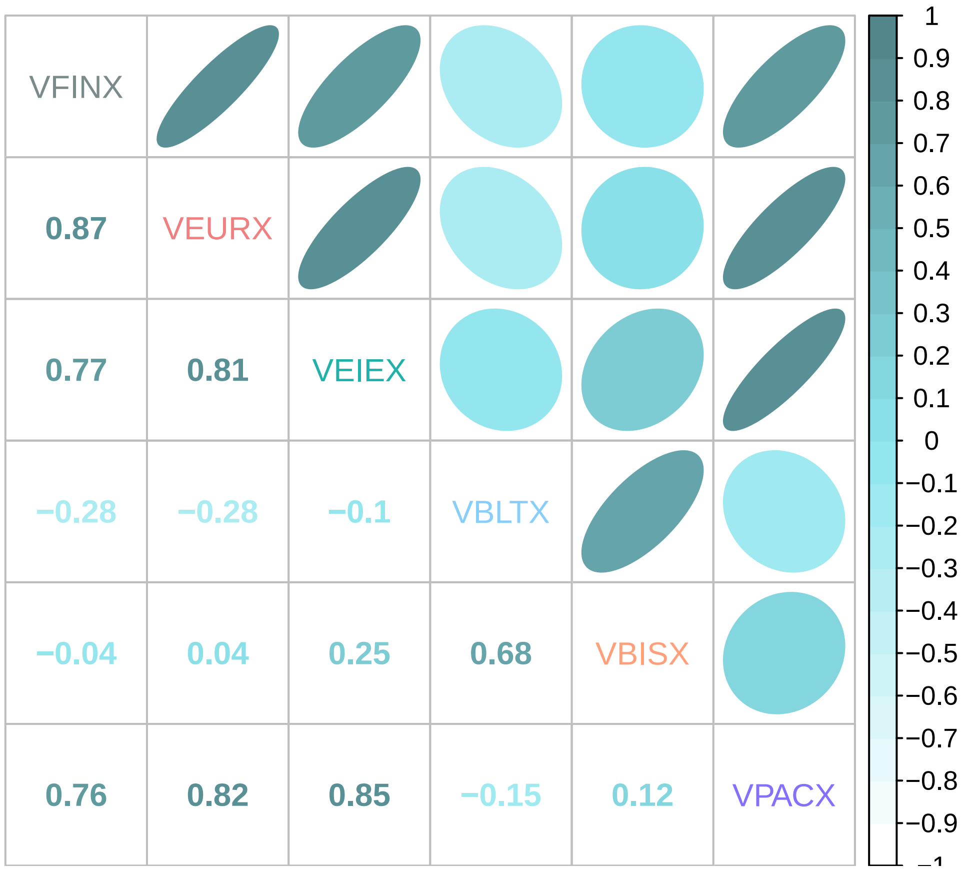


Figure 7: Correlation Plot for the Cross Cross-Correlation between ETFs

By looking at the values for each of the correlations, they reflect the scatterplots of the distributions from Figure 6 extremely well, with the observations were made above being confirmed by the values of the correlations between each of the assets.

These correlations provide a quantifiable basis of diversification benefit. For example, if you have information or insight that a particular asset would be profitable in the future, other assets that are positively correlated with that specific asset too would display this upward trend. Similarly, by buying an assset that is negatively correlated with an asset that you may be considering, it would be possible to reduce the risk of loss in the event that the security being considered does not perform optimally - as its negative returns would be offset by the - anticipated - corresponding positive returns of the assset that was correrlated negatively with it.

By combining different assets, investors are able to reduce their exposure to security-specific, or unique risk. Thus, it is clear from the wide range of possible combinations of the assets, and the different cross-asset correlations that each provides, that the benefit of diversification can indeed be realized through owning combination of the ETFs, or a *portfolio of securities*.

# Chapter 3

# Value at Risk

## Introduction

The concept of *Value at Risk*, or VaR is derived from the reality that in conducting statistical analysis, returns from an asset are treated as a sample from a larger population, which in turn has an observable distribution. Thus, using statistical inference about the positional and scaling parameters for a given security, it is possible to assign probabilities to each return prospect, whithin the bounds of estimation. Similarly, it is also possible to derive the return that is expected to occur at a certain probability.

The VaR quantifies this measure, by attaching a dollar value to the potential losses at a given probability. That is, the 1% VaR would indicate - given the past performance of the sample and value of the initial investment - how much money an investor could expect to lose in the subsequent period, with a probability of 0.01 or 1%. This concept is extremely central to modern finance, as better estimation tools increase the reliability of these measures, and provides a metric by which real-world losses can be gauged, which is sometimes more useful than perhaps citing the standard deviation of a distribution when attempting to illustrate risk.

## ETF Value at Risk

Displayed in Table 7 and Table 8 are the monthly VaRs for each of the ETFs at the 1% and 5% level, calculated using the implied normal distribution quantiles and given an initial investment of $100,000. Monthly VaRs are also determined using the emperical quantiles of each of the samples, which are calculated from the distribution of the sample, rather than from an assumed distribution. Also displayed are the annual VaRs, calculated using the annualized expected returns and standard deviations of each of the ETFs. Additionally, 95% confidence intervals and standard errors of estimation were determined using the bootstrap method, discussed in the previous section. These estimates can be analyzed to develop a better insight into the behavior of the effect of extreme returns on an actual investment in the ETFs.

Consider VEIEX, which has the highest estimated VaR at both the 1% and 5% level, over both monthly and annual time horizons. The relatively high disparity observed between the annualized and emperical VaRs can be attributed to the fact that the distribution of the sample VEIEX returns had smaller tails (i.e. positive excess kurtosis) than those assumed by the normal distribution, which is reflected in the lower estimates for VaR determined with emperical quantiles as opposed to normal quantiles.

On the other hand, VBISX exhibits the lowest VaRs for both levels, across both time horizons. This is to be expected, as it was the ETF that had the lowest variability of returns, and would thus have a higher expected return compared to its mean at the tails of its distribution.

Furthermore, considering the estimation standard error calculated for each of the VaR, it is clear that despite these numbers add a sense of realism to possible returns from the assets, they are in fact estimates, and have high levels of variability. This is reflected in the 95% confidence intervals, where in some cases such as the 95% CI for 5% VaR for VEURX and VEIEX come in close proximity to the correspinding 1% VaR.

Table 7: 1% Value at Risk Analysis for each ETF (Key: A: Analytical Normal, E - Emperical, B - Bootstrap)

| | 1% VaR (A) | 1% Var (E) | 1% VaR (Annual, A) | 1% VaR SE (B, A) | 1% VaR SE 95% CI (B, A) |
|---|---|---|---|---|---|
| VFINX | 6875.49 | 6459.79 | 15346.50 | 874.61 | [5272.00, 8700.40] |
| VEURX | 10545.64 | 11676.16 | 31394.93 | 1237.72 | [8264.83, 13116.61] |
| VEIEX | 12385.55 | 13196.17 | 38417.05 | 1444.59 | [9728.49, 15391.19] |
| VBLTX | 5199.10 | 4866.96 | 11552.23 | 576.39 | [4151.95, 6411.35] |
| VBISX | 696.96 | 520.91 | 1301.06 | 75.55 | [559.78, 855.94] |
| VPACX | 8912.83 | 9072.23 | 26217.80 | 1061.11 | [6944.99, 11104.48] |

Table 8: 5% Value at Risk Analysis for each ETF (Key: A: Analytical Normal, E - Emperical, B - Bootstrap)

| | 5% VaR (A) | 5% Var (E) | 5% VaR (Annual, A) | 5% VaR SE (B, A) | 5% VaR SE 95% CI (B, A) |
|---|---|---|---|---|---|
| VFINX | 4649.88 | 5475.11 | 8129.30 | 710.44 | [3336.95, 6121.82] |
| VEURX | 7547.77 | 7189.17 | 23096.18 | 1020.97 | [5642.26, 9644.40] |
| VEIEX | 9008.82 | 8610.07 | 29797.37 | 1169.97 | [6861.37, 11447.59] |
| VBLTX | 3498.93 | 3507.69 | 5934.87 | 475.77 | [2611.95, 4476.94] |
| VBISX | 455.27 | 356.78 | 466.42 | 59.14 | [346.86, 578.70] |
| VPACX | 6325.33 | 7514.98 | 18699.65 | 872.58 | [4695.75, 8116.22] |

While it may seem that the differences between the annual and monthly VaRs are extremely small, this is to be expected. It is due to the fact that unlike the annualized returns of the ETFs, VaR does not scale linearly, as it is an exponential of the affine combination of the expected return and standard deviation of the assets. Recall, the expected return is scaled by a factor of 12, while the standard deviation is scaled by a factor of $\sqrt{12}$. This illustrates yet again the importance of the precision, and assumed time invariance of these estimates, as small changes may have a large effect on measures such as the VaR. Considerng the annual VaR in particular, an increase in the variance of returns over the course of a year would not be reflected in the monthly statistics, and thus not reflected in the annualized monthly statistics which may lead to a gross misestimation of metrics such as the yearly VaR.

The large confidence intervals and standard errors of estimation can be attributed to the fact that the VaR is calculated using the location and scale parameters of the samples, which too have high levels of variability associated with them, as discussed earlier in the report. Thus, while the implications of VaR should indeed be taken into consideration when making investmnet decisions, the fact that it too is an estimate, derived from an estimate means that VaR should not be the only factor that is analyzed.

# Chapter 4

# Rolling Analysis

## Motivation

A key assumption of the CER model is that the returns being modeled are covariate stationary time series processes. Thus, all of the parameters used in the CER model (mean, standard deviation, etc.) are assumed to be time stationary. However, as evidenced by changing trends over time and the effect of market-wide crises on returns, this cannot always be assumed to be true.

Rolling analysis involves the computation of smaller time horizon estimates for each of the parameters, within the time horizon of the sample itself. These trends can in turn be analyzed to determine the viability of the *stationary* assumption employed when modeling returns.

## Rolling Analysis of Select ETF CER Model Parameters

Figure 8 visualizes 24 month rolling estimated means and standard deviations of returns, along with the assumed constant mean and standard deviation of each return. These graphs provide a necesary insight into the behavior of the mean and standard deviation of each ETF over time, relative to the stationary assumed means and standard deviations, which are also displayed on the graph.

It is clear from the graph that in some instances, rolling assumptions of each of the parameters do not conform to the overall estimates for the mean and standard deivation. In particular, there is clear variation in the rolling standard deviation of VFINX, compared to the stationary estimate. From the graph, it is clear that the standard deviation dropped during the period of 2014-2015, which was coupled with a rise in the rolling mean. Also, analysis of the rolling VEURX mean indicates that it used to be significantly higher than the time-stationary estimation suggests, particularly during the middle of the time horizon.

Overall gowever, there does not appear to be any significant, systematic deviations of the rolling estimates of the standard deviation or mean from their stationary counterparts. This observation suggests that employing the stationary estimations is approapriate for portfolio calculations involving the ETFs.

## Rolling Analysis of Correlation

Perhaps more important to the overall quality of a portfolio calculation would be the time-invariability of correlation. Figure 9 displays a graph of 24 month rolling estimates of the correlation of returns between VFINX and VBLTX, which had a time-invariant approximated correlation of -0.281133.

Analyzing the graph, it is clear that time-invariant assumed estimation of the correlation is heavily skewed by earlier correlations between each of the ETFs. As seen in the graph, the rolling correlation is predominantly over the estimation, and even goes positive towards the end of 2015, and more recently at the beginning of 2016. It appears that the correlation is the lowest at the first observation of the dataset, in mid 2013.

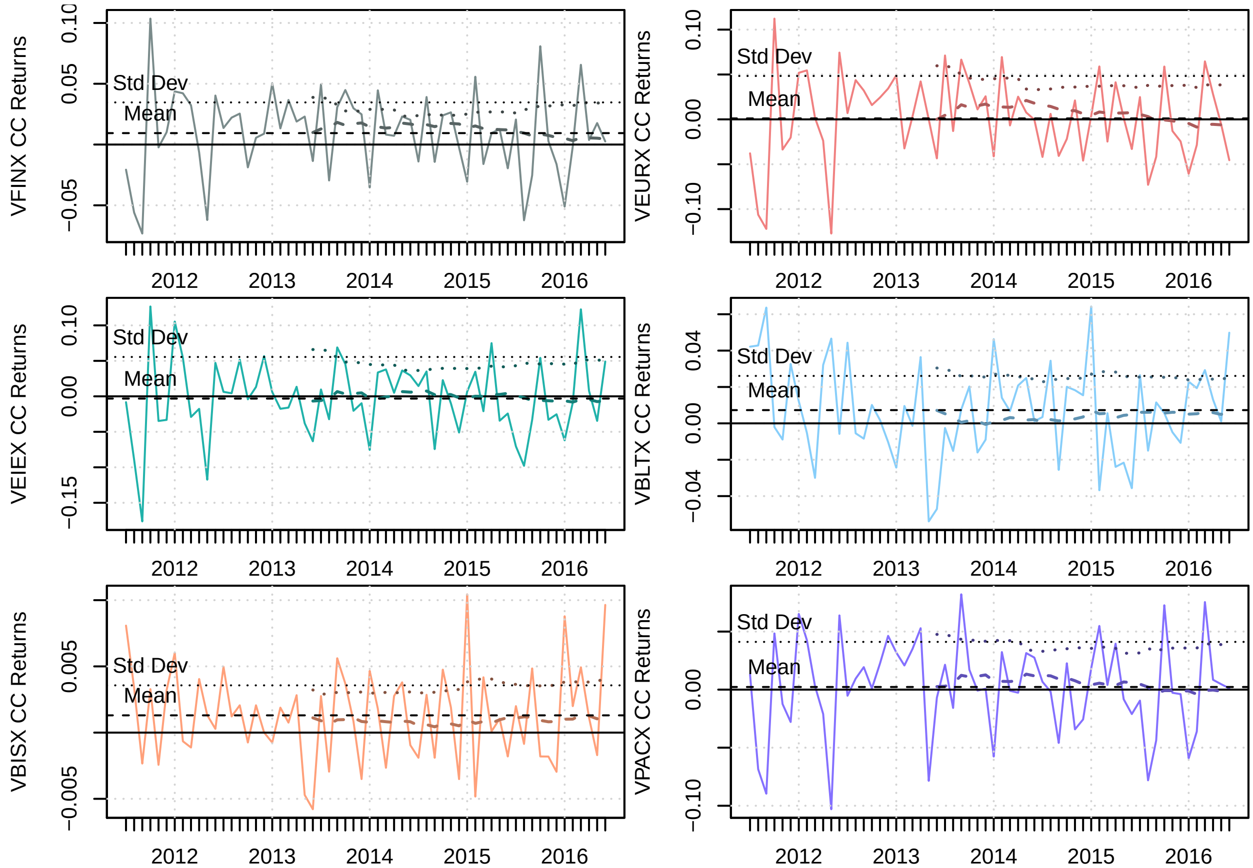



Figure 8: 24 Month Rolling Estimates of ETF Mean and Standard Deviation

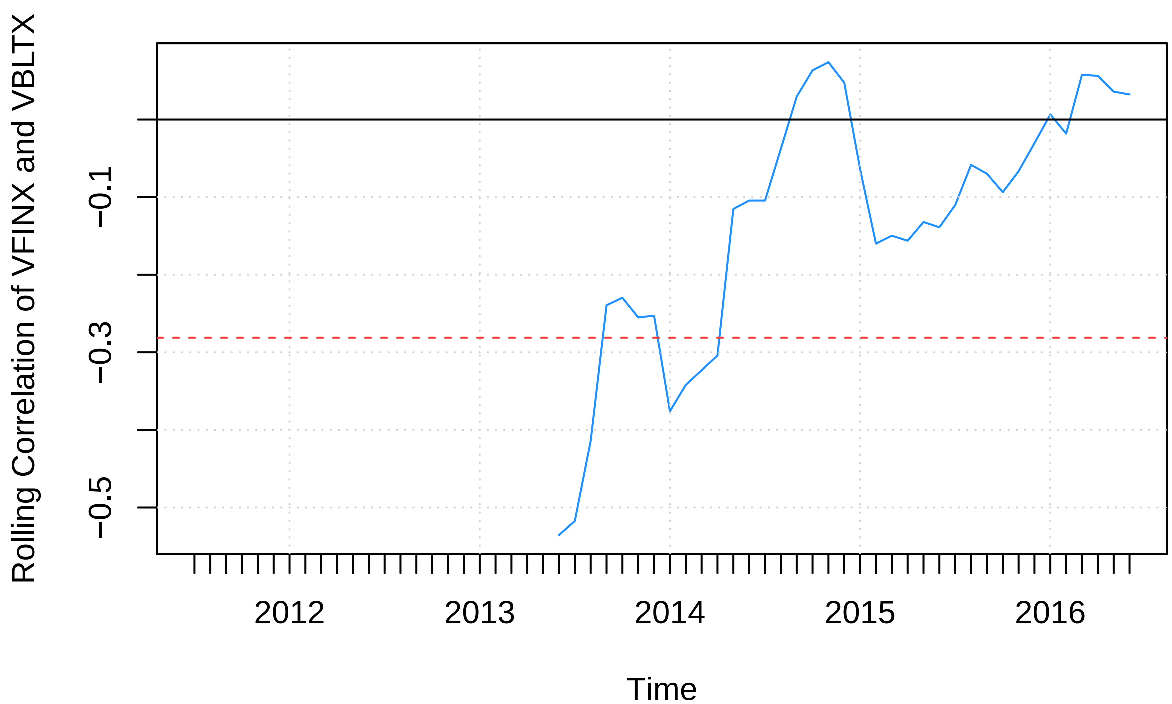



Figure 9: 24 month Rolling Estimates of VFINX-VBLTX correlation

It is the conclusion of observers of financial statistics that correlations between assets tend to increase at time of crisis. This effect is illustrated in the graph, as the correlation increases during the time of the European Debt Crisis, and is lower during the financial market rebound of the early part of this decade in the aftermath of the financial crisis. This analysis reiterates the importance the effect the time-invariant assumption of the CER Model has on implied returns of ETFs portfolios.

# Chapter 5

# Portfolio Theory

## Introduction

Having conducted a through analysis of statistical parameters related to each invdividual ETF in the previous sections, we can now consider combinations, or *portfolios* of these ETFs with confidence. To begin, we must first define certain statistics related to each portfolio, and formulas that can be used to calculate the each statistic.

$$\text{Let } \mathbf{x} \text{ be a vector of the asset weights in the portfolio}$$
$$\text{Let } \boldsymbol{\mu} \text{ be a vector of the expected returns of each of the assets}$$
$$\text{Let } \boldsymbol{\Sigma} \text{ be the covariance matrix of the assets}$$

$$\therefore \text{Portfolio Expected Return, } \mu_p = \boldsymbol{\mu} \cdot \mathbf{x}$$
$$\therefore \text{Portfolio Variance, } \sigma_p^2 = \mathbf{x}^\intercal \times \boldsymbol{\Sigma} \times \mathbf{x}$$

By using the definitions for portfolio characteristics above, portfolios can be constructed that have desirable characteristics with regard to a certain measurement. For example, a portfolio with a target expected return, but minimal variance can be calculated by performing a constrained optimization of the variance of a portfolio, with the constraint that the expected return of the portfolio is equal to the target expected return. Consider the following portfolios, and the notion of efficient portfolios discussed below.

## Portfolios Allowing Short Sales

### Global Minimum Variance Portfolio

The notion of a minimum variance portfolio is derived from the notion that all of the assets have risks associated with them, and thus the possibility of constructing a portfolio that may yield a level of risk lower than that of any single asset is particularly attractive. This type of portfolio can be constructed by determining the asset weights that would minimize the value of the variance of the portfolio, $\sigma_p$. This problem is described mathematically below:

$$\min_{\mathbf{x}} \sigma_p = \mathbf{x}^\intercal \times \boldsymbol{\Sigma} \times \mathbf{x}$$
$$\text{subject to } \mathbf{x}^\intercal \cdot \mathbf{1} = 1$$

This constrained optimization can be performed quickly and efficiently on a computer, and the results are displayed below.

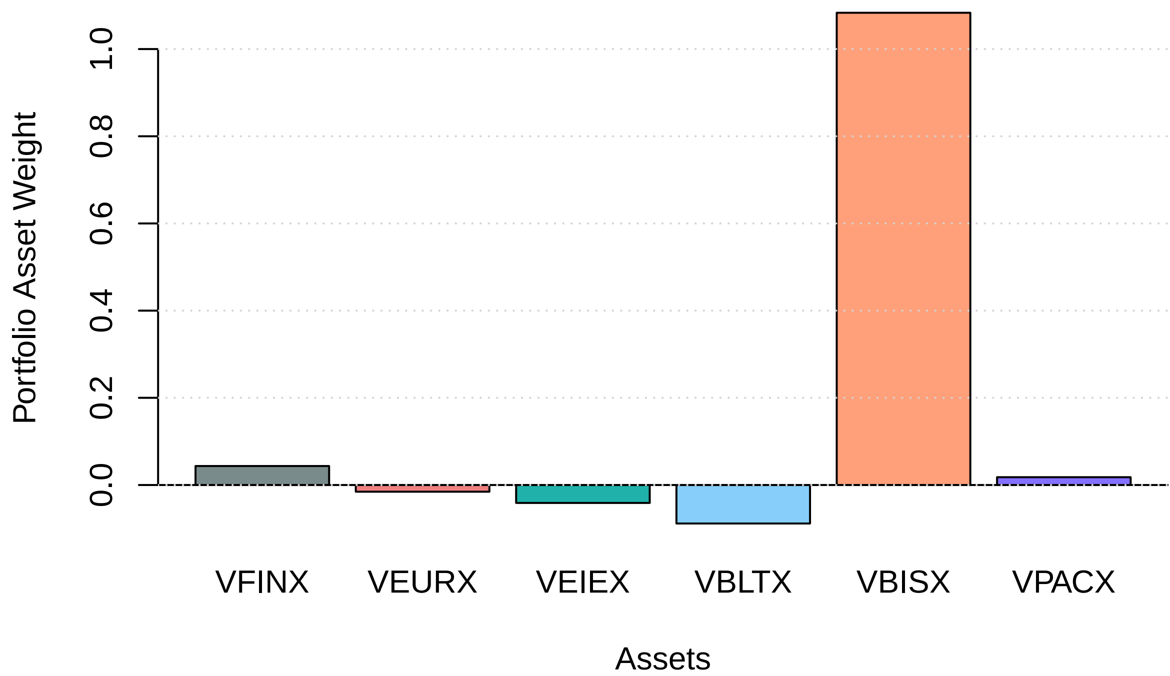



Figure 10: Global Minimum Variance Portfolio Asset Weights allowing Short Sales

Displayed in Figure 10 is a histogram of the asset weights that would contruct a portfolio with minimal variance, given the inital assets, thereby satisfying the constrained optimization described above. While the asset weights alone are not useful, statistical parameters regarding the portfolio can be calculated - in particular, the expected return and standard deviation can be calculated. Using this, we can calculate the Sharpe Ratio of the portfolio, as well as 1% and 5% VaRs that would be offered by this portoflio (assuming an initial investment of $100,000).

Table 9: Descriptive statistics for the Global Minimum Variance Portfolio allowing Short Sales

| | Monthly Returns | Annualzied Returns |
|---|---|---|
| Expected Return | 0.001331 | 0.015977 |
| Standard Deviation | 0.002476 | 0.008577 |
| Sharpe Ratio | 0.369458 | 1.279840 |
| 1% VaR | 441.863582 | 396.748248 |
| 5% VaR | 273.733988 | 187.147023 |

Table 9 shows both the monthly, and implied annual statistics for the global minimum variance portfolio. Looking at the implied standard deviation of the portfolio, it is clearly lower than that of any of the individual assts, immediately illustrating the possible benefits of diversification that are offered in this portoflio. Interestingly, this does not cause the asset to have a significantly rate of return, as it still has a higher expected return that that of the worst-performing ETF, VEIEX.

Furthermore, the VaR of this portfolio clearly quantifies the benefit that can be gained from effective diversification. Both the 1% and 5% VaR for this portfolio reflect the extremely low risk it provides. Considering these VaRs relative to the inidividual asset VaRs, they are significantly lower than the lowest VaRs, offered by VBISX at both time horizons and risk levels.

## Maximum Expected Return Equivalent Efficient Portfolio

An efficient portfolio is defined as a portfolio that can deliver the highest expected return, at a given level of risk. Inverting this statement, it also means that it is any portfolio that provides a target return at the lowest possible risk.

To illustrate this, we will calculate an efficient portoflio that has an expected return equal to highest return out of the ETFs, VFINX. The constrained optimization can thus be set up as follows:

$$\min_{\mathbf{x}} \sigma_p = \mathbf{x}^{\intercal} \times \mathbf{\Sigma} \times \mathbf{x}$$
$$\text{subject to } \mathbf{x}^{\intercal} \cdot \mathbf{1} = 1$$
$$\boldsymbol{\mu} \cdot \mathbf{x} = E[\text{VFINX}] = 0.00939$$

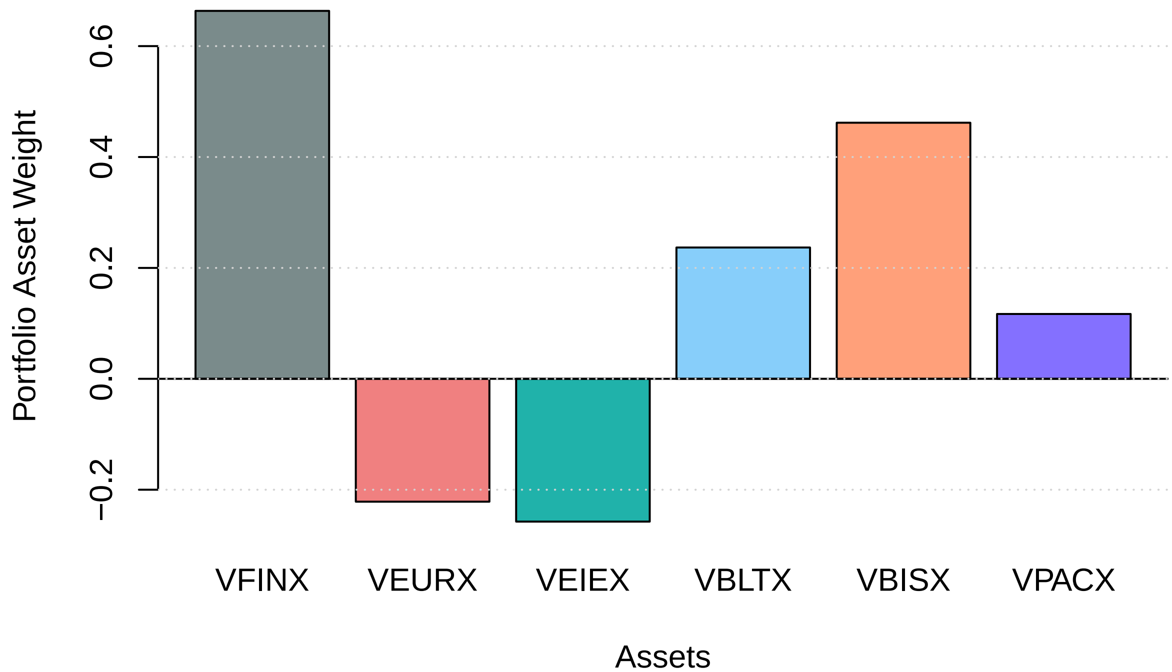



Figure 11: VFINX Return Equivalent Efficient Portfolio Weights allowing Short Sales

Table 10: Descriptive Statistics of VFINX Return Equivalent Efficient Portoflio (EVFINX) and VFINX

| | EVFINX Efficient Portfolio (EVFINX) | VFINX |
|---|---|---|
| Expected Return | 0.009390 | 0.009390 |
| Standard Deviation | 0.012578 | 0.034656 |
| Sharpe Ratio | 0.713409 | 0.258927 |
| 1% VaR | 1967.525923 | 6875.489304 |
| 5% VaR | 1123.573151 | 4649.879691 |

After completing the optimization, the portfolio weights of the VFINX return equivalent efficient portfolio (rerred to as VFINX) are represented visually in Figure 11. Similar to with the Global Minimum Variance Portfolio, descriptive statistics for the portfolio are displayed in Table 10. Also shown are the (repeated) descriptive statistics for VFINX. Consdiering the expected return, they are exactly equal, as required by the conditions of the initial optimization by which the portfolio was calculated. Perhaps more striking however, is the significantly lower standard deviation. By diversifying, an investor could achieve the same expected return as VFINX, but at nearly a quarter of the risk. This effect of this massive reduction in risk is reflected in the extremely high Sharpe ratio of this portfolio compared to that of VFINX, quantifying the significantly higher return per unit of risk. The effect of the lower standard deviation can be quantified to dollars by analyzing the VaRs of both EVFINX and VFINX, as EVFINX has significantly lower VaR at both 1% and 5% compared to that of VFINX.

### Tangency Portfolio

Having displayed the optimization capabilities in terms of both the expected return, and standard deviation, we can now address the Sharpe Ratio. Recall, the Sharpe ratio measures the amount of return that an asset can provide, per unit of risk. Thus, by computing the portofolio that offers the best Sharpe Ratio, we can determine the best possible portfolio to own, given the set of assets. Described below is the constrained optimization that would be performed to compute this portfolio.

$$\min_{\mathbf{x}} SR_p = \frac{\mathbf{x}^\intercal \cdot \boldsymbol{\mu} - r_f}{(\mathbf{x}^\intercal \times \boldsymbol{\Sigma} \times \mathbf{x})^{\frac{1}{2}}}$$
$$\text{subject to } \mathbf{x}^\intercal \cdot \mathbf{1} = 1$$

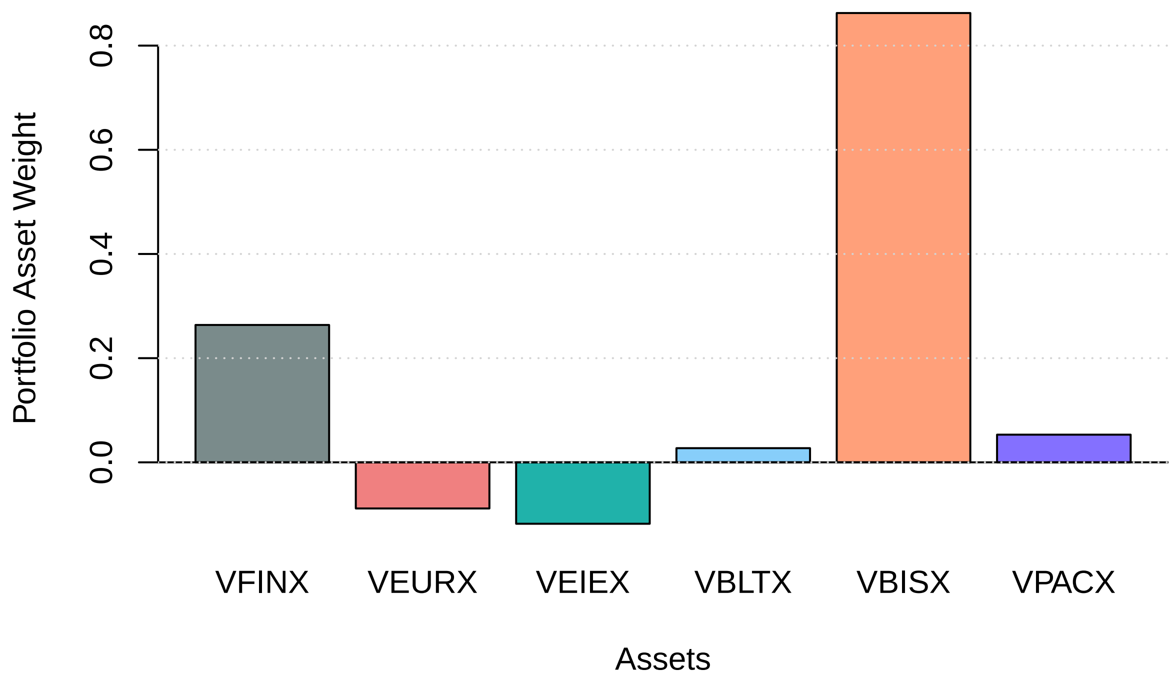



Figure 12: Tangency Portfolio Asset Weights allowing Short Sales

The resulting asset weights from this optimization can be seen in Figure 12. Also, Table 11 displays both monthly and annualized descriptive statistics calculated for the tangency portfolio. Looking at the table, both the monthly and annualized Sharpe ratios of this portfolio are significantly higher than any of the assets, with it being nearly three times higher the highest one displayed by any of the ETFs (VBLTX, SR 0.263376).

Table 11: Monthly and Annualized Descriptive Statistics for the Tangency Portfolio allowing Short Sales (STAN)

| | Monthly Returns | Annualzied Returns |
|---|---|---|
| Expected Return | 0.004193 | 0.050318 |
| Standard Deviation | 0.005031 | 0.017427 |
| Sharpe Ratio | 0.750680 | 2.600433 |

Additionally, by combining the tangency portfolio with the risk free asset, it is possible to construct a new compound portfolio that has a better rate of return compared to any other efficient portfolio at any level of risk. Due to the fact that the risk free asset has a standard deviation of 0, the Sharpe ratio of all of the portfolios that lie on this line all have the same optimal Sharpe ratio as the tangency portfolio. This set of compound mean-variant efficient portfolios are displayed with the Markowitz bullet in Figure 13.

### Markowitz Bullet

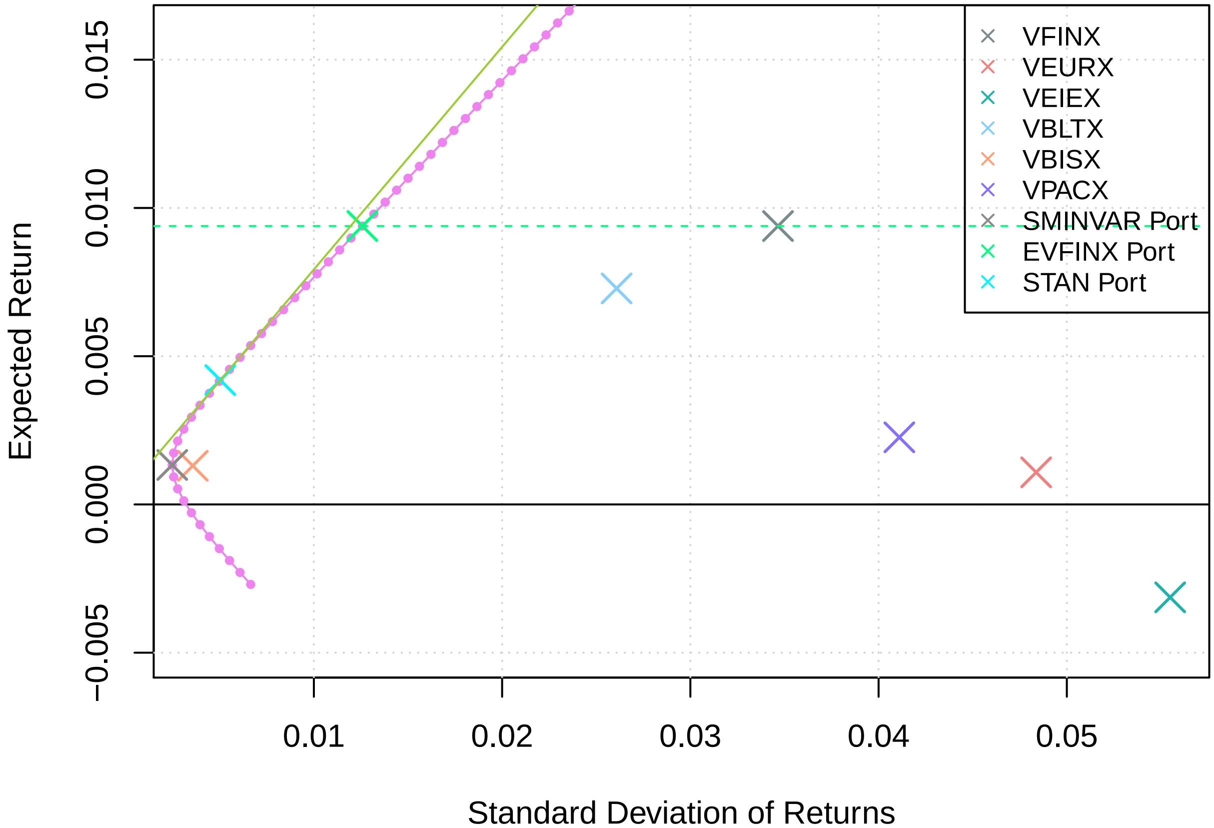



Figure 13: Risk-Return Graph with the Markowitz Bullet and Mean-Variant Efficient Portfolios allowing Short Sales

The set of efficient portfolios, across all levels of risk is called the efficient frontier. Due to the nature of the shape produced by these portfolios on the risk-return graph, it is commonly referred to as th *Markowitz Bullet*. Mathematically the asset weights of any portfolio that lies on the efficient frontier are an affine combination of any other two portfolios that lie on the line, and thus the set of efficient portfolios can be computed very easily.

Figure 13 displayes the efficient frontier on the risk-return graph, along with each of the ETFs themselves, and the other portfolios calculated above. Notice that the set of efficient portfolios provide a higher rate of return at any given level of risk, compared to the individual ETFs.

## Note on Short Sales

The *short sale* of assets refers to the process of borrowing an asset from a broker and selling it on the open market, thereby indebting you to your broker. To close the trade, one would simply purchase the asset again on the open market and return it to the broker. This process allows investors to benefit from declining prices, as an investor would make a profit if they can buy the asset back later for a price lower than what they sold it for.

This is why the portfolios above have negative weights. The negative weight represents a short position in an asset, and the money that is gained from selling that asset is in turn invested in another. However, due to the implications of being indebted to a broker, and due to the regulatory burdens placed on 401(k) investments, the short sale of assets is not allowed. This means that the portfolios computed above are not suitable 401(k) portfolios due to the fact that they all involve the short sale of some assets. To address this, an analysis of portfolios that do not allow the short sale of assets is conducted below.

## Portfolios Not Allowing Short Sales

### Global Minimum Variance Portfolio

Similar to the portoflio computed above, an optimal portfolio that does not allow the short sale of assets can be constructed. The only difference this time would be that an additional constraint will have to be imposed, restricting the weights of the assets to be greater than 0. This constrained optimization is described below:

$$\min_{\mathbf{x}} \sigma_p = \mathbf{x}^\mathsf{T} \times \mathbf{\Sigma} \times \mathbf{x}$$
$$\text{subject to } \mathbf{x}^\mathsf{T} \cdot \mathbf{1} = 1$$
$$\mathbf{x_i} > 0, \ \forall\, i$$

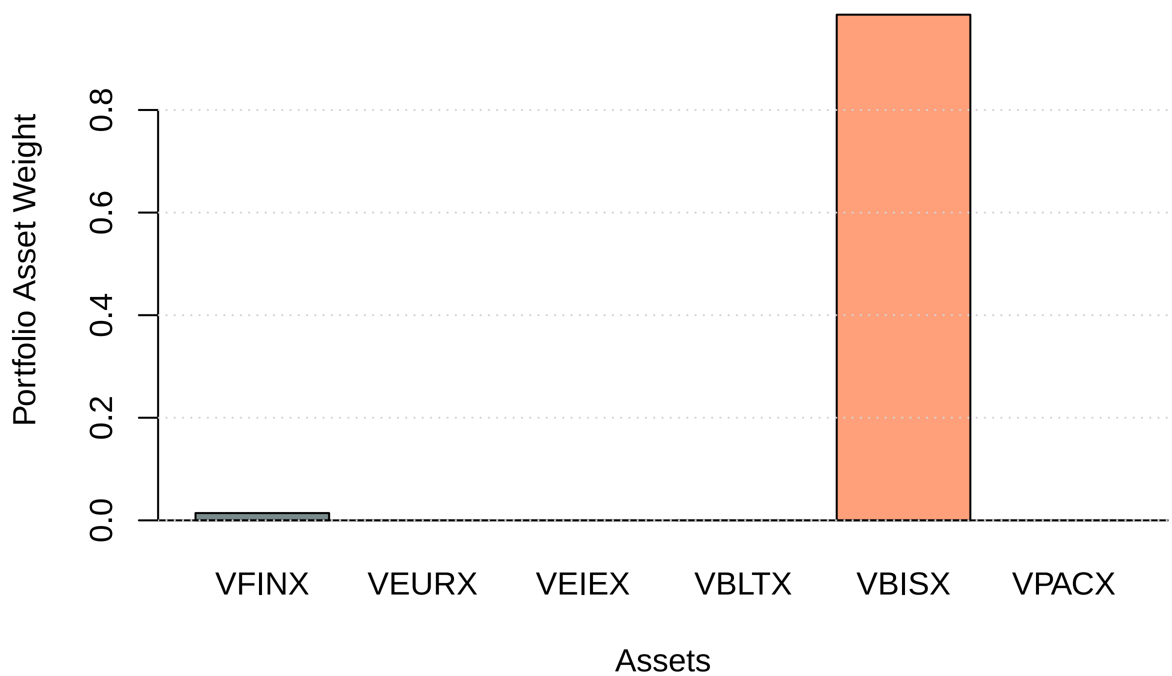



Figure 14: Global Minimum Variance Portfolio Asset Weights not allowing Short Sales

The asset weights of the resulting portfolio are displayed in Figure 14. Additionally, portfolio statistics were computed for this portfolio and are displayed in Table 12. It is immediately apparent that when compared to the minimum variance portfolio that allows the short sale of assets that the restriction of not being able to short-sell severely impacts the performance of the portfolio.

In addition to having significantly higher risk, there is almost no benefit of using this portfolio, as it is nearly almost composed of a single asset, VBLTX and has nearly identical expected return, which would be well within the bounds of estimation error for that asset. It also has a lower Sharpe ratio compared to the portfolio allowing short sales, which confirms that it is not as risk-efficient. The loss of the ability to short sell assets is particularly pronounced in the levels of VaR of each of the portfolios, as the no-shorts minumum variance portfolio has a VaR at risk that is sometimes four times that of its short-selling counterpart.

Table 12: Monthly and Annualized Descriptive Statistics for the Global Minimum Variance Portfolio not allowing Short Sales (NSMINVAR)

| | Monthly Returns | Annualzied Returns |
|---|---|---|
| Expected Return | 0.001419 | 0.017026 |
| Standard Deviation | 0.003532 | 0.012236 |

| | Monthly Returns | Annualzied Returns |
|---|---|---|
| Sharpe Ratio | 0.283717 | 0.982825 |
| 1% VaR | 677.542560 | 1137.409605 |
| 5% VaR | 438.161408 | 309.554544 |

## Tangency Portfolio

As with the short sales scenario, a tangency portfolio that maximizes the the Sharpe ratio can be computed, with the restriction of not being allowed to short sell assets. The optimization that would have to be completed is the same as the scenario that allows short sales, again with the exemption that asset weights cannot be negative.

$$\min_{\mathbf{x}} SR_p = \frac{\mathbf{x}^\intercal \cdot \boldsymbol{\mu} - r_f}{(\mathbf{x}^\intercal \times \boldsymbol{\Sigma} \times \mathbf{x})^{\frac{1}{2}}}$$
$$\text{subject to } \mathbf{x}^\intercal \cdot \mathbf{1} = 1$$
$$\mathbf{x_i} > 0, \; \forall\, i$$

Table 13: Monthly and Annualized Descriptive Statistics for the Global Minimum Variance Portoflio not allowing Short Sales (NSTAN)

| | Monthly Returns | Annualzied Returns |
|---|---|---|
| Expected Return | 0.001419 | 0.017026 |
| Standard Deviation | 0.003532 | 0.012236 |
| Sharpe Ratio | 0.283717 | 0.982825 |

Dispalyed in Figure 15 are the asset weights that would create a mean-variance optimized efficient portfolio. Additionally, descriptive statistics for the computed portfolio were calculated and are displayed in Table 13. Analyzing these statistics, it is immediately clear that the benefit of short selling assets has a large impact on the performance of a mean-variance efficient portfolio. In this case, despite the fact that the standard deviation of the tangency portfolio with no shorts is lower, this is offset by a large gap in the expected returns of the tangency portfolios.

This large disparity is reflected in the extremely large difference in the Sharpe ratios of each of the portfolios. As the tangency portfolio is designed to be optimized to deliver the maximum possible Sharpe ratio, the fact that the lack of short selling causes the Sharpe ratio to be so near that of one of the single assets that comprise the portfolio (VBLTX, SR = 0.026083) is particularly striking.

## Markowitz Bullet

As with the portfolios that allowed short sales, the efficient frontier, which is the set of portfolios that can deliver the maximum possible return at any level of risk can be determined. However, as there is a inequality constraint applied to these portfolios, the same rule of affine combination does not hold. Thus, the efficient frontier must be determined using a *brute force* approach, which entails computing efficient portfolios with the constraint of no short sales applied to a set of expected returns.

Figure 16 illustrates the efficient frontier of portfolios that do not allow short sales, alongside the efficient frontier of portfolios that allow short sales. The position of the efficient frontier allowing short sales completely encompasses the frontier that restricts short sales, indicating that it performs better (i.e. offers higher return per unit of risk) at every level of expected return.

To illustrate this further, consider the green line, which represents the 0.02% level of risk. At this level, an efficient portfolio disallowing short sales would yield an expected return of approximately 0.008, while an efficient portfolio allowing short sales at the same level of risk would yield an expected return of 0.014. At this level of risk, the cost of not being able to short sell assets is approximately 0.006 (0.6%) continously compounded per month, which has a high impact on potential capital gains in the long run.

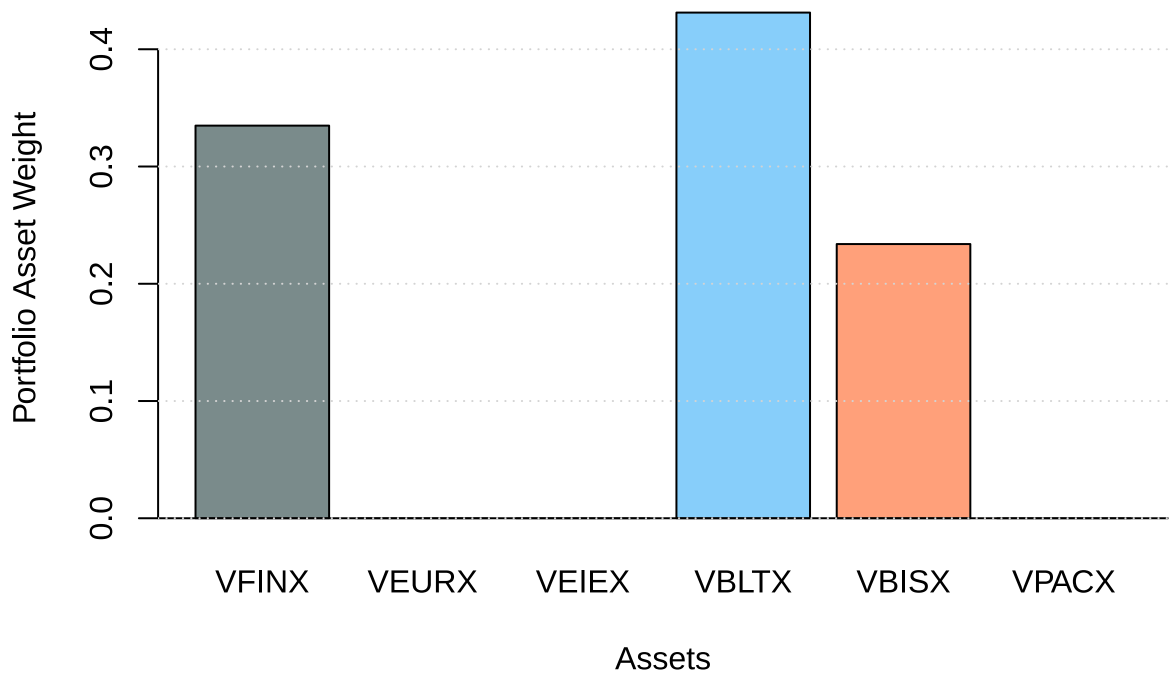



Figure 15: Tangency Portfolio Asset Weights not allowing Short Sales

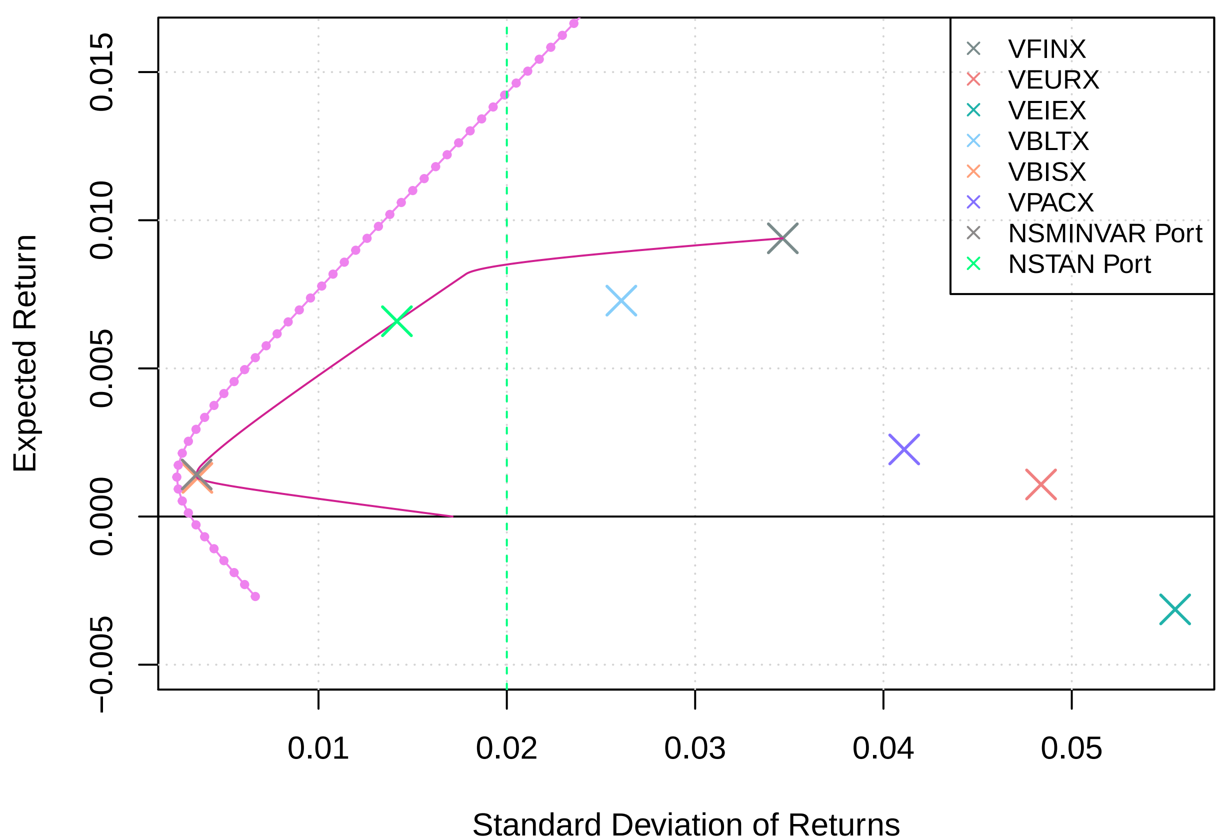
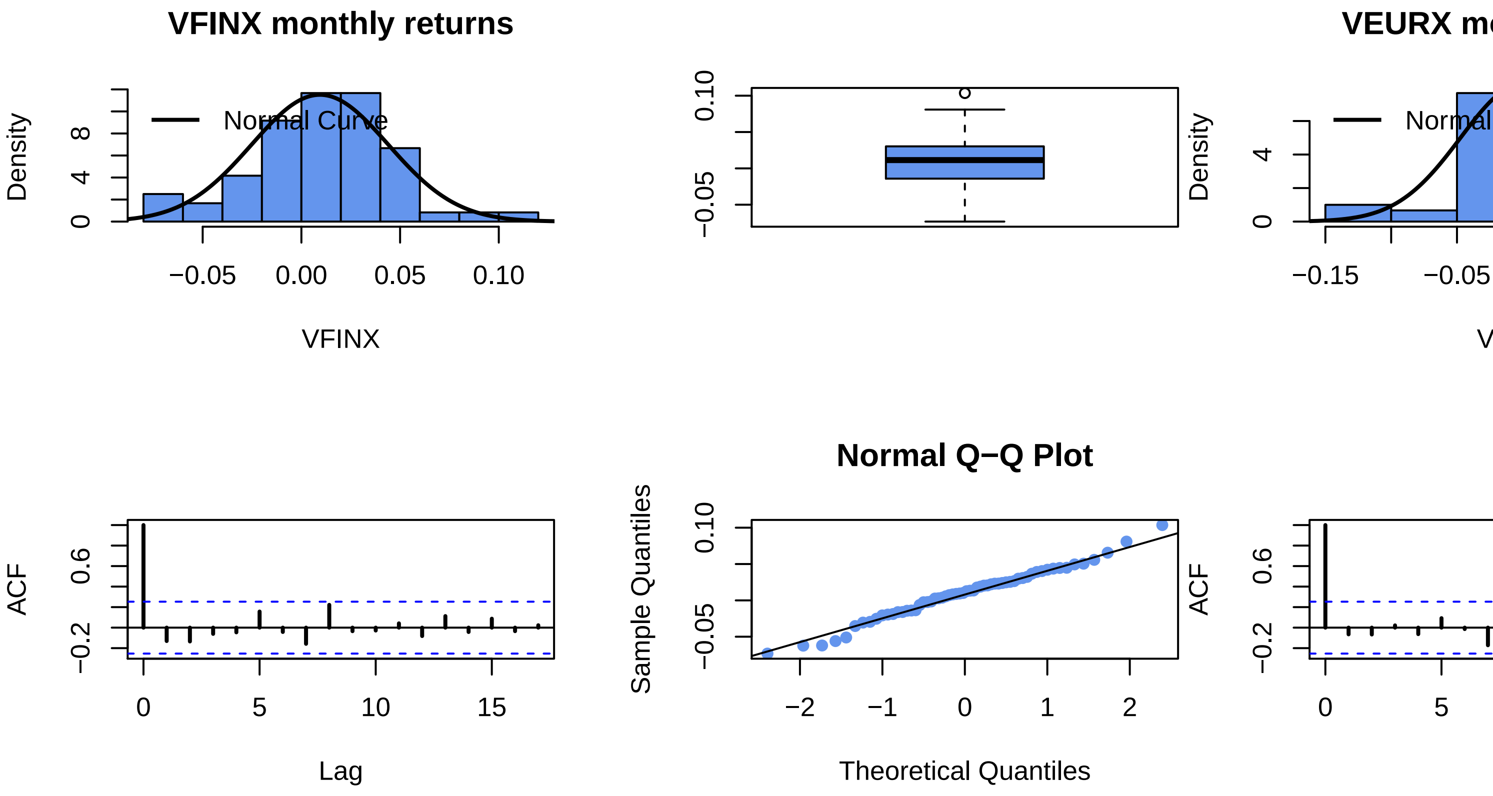
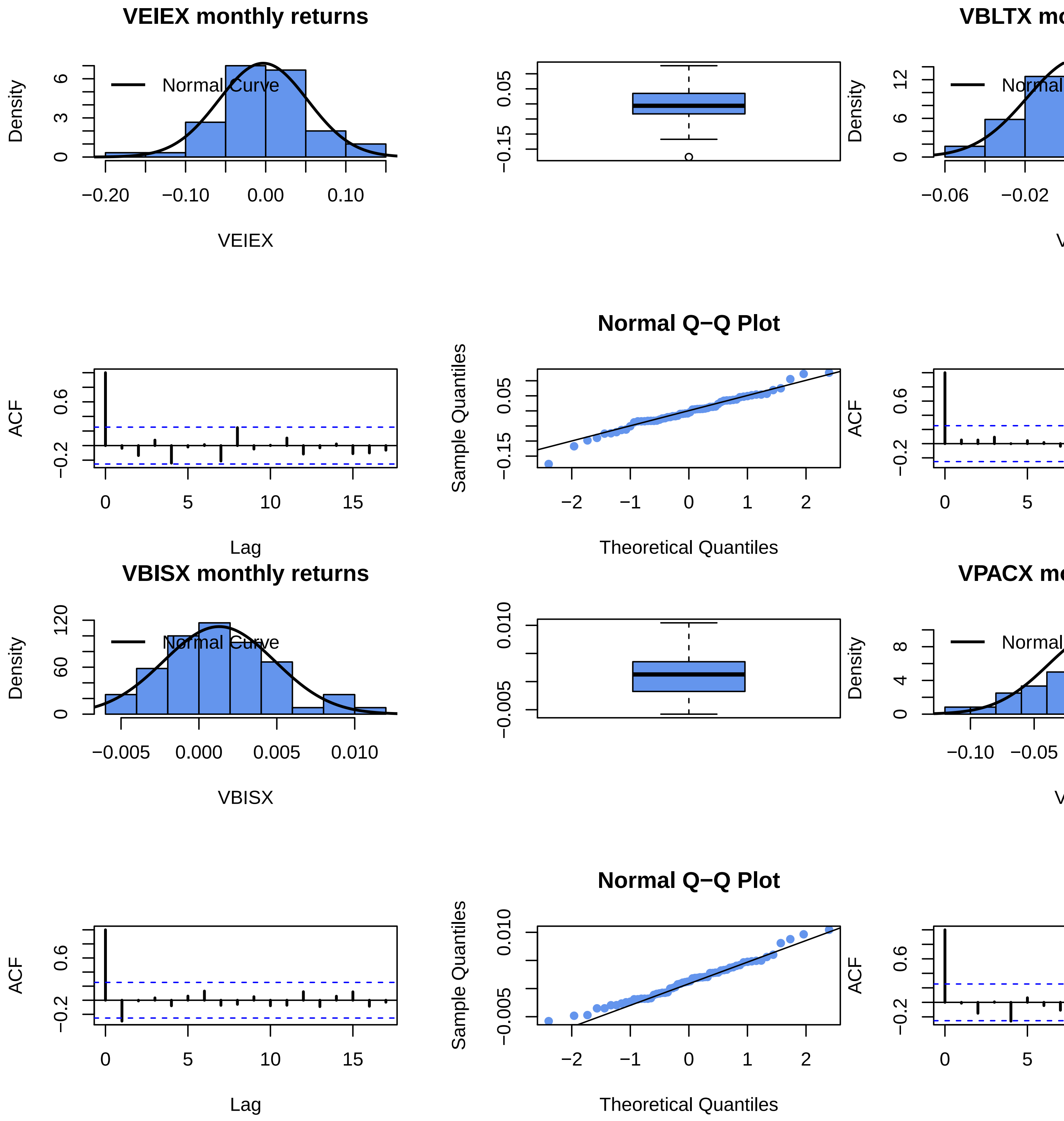



Figure 16: Risk-Return Graph with the Efficient Frontiers of Portfolios allowing Short Sales and those without Short Sales

# Appendix A

# Four Plot Summaries of the ETF CC Returns




VEIEX monthly returns
Density
Normal Curve
VEIEX
VBLTX mo
Normal
ACF
Lag
Normal Q−Q Plot
Sample Quantiles
Theoretical Quantiles
VBISX monthly returns
Density
Normal Curve
VBISX
VPACX mo
Normal
ACF
Lag
Normal Q−Q Plot
Sample Quantiles
Theoretical Quantiles

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