CH5440- Assignment 1 MM17B113

1) a) $u_1, u_2, u_3 \ldots u_N$, $y_1, y_2, y_3 \ldots y_N$ } N measurements

u & $y \rightarrow$ linearly related

$\rho = \dfrac{\sigma_\epsilon^2}{\sigma_s^2}$

WTLS

Optimization Prob

$$\min_{\alpha, \beta, u_i^*} \sum (y_i - a\hat{u}_i - b)^2/\sigma_\epsilon^2 + (u_i - \hat{u}_i)^2/\sigma_s^2$$

multiply with σ_ϵ^2

$$\min_{\alpha, \beta, u_i^*} \sum (y_i - a\hat{u}_i - b)^2 + (u_i - \hat{u}_i)^2 \rho \rightarrow \text{Let this be } f$$

derivation with b

$$\frac{\partial f}{\partial b} = \sum_{i=1}^{N} 2(y_i - a\hat{u}_i - b)(-1) = 0$$

$$\Rightarrow \sum y_i - \alpha \sum \hat{u}_i - \sum b = 0$$

$$\Rightarrow N\bar{y} - \alpha N \bar{u} - Nb = 0$$

$$\Rightarrow \boxed{b = \bar{y} - \alpha \bar{u}} \rightarrow \text{assuming } \sum \text{errors} = 0 \text{ (mean=0)}$$

derivation with a

$$\frac{\partial f}{\partial a} = \sum_{i=1}^{N} 2(y_i - a\hat{u}_i - b)(-\hat{u}_i) = 0$$

$$\Rightarrow \sum \hat{u}_i y_i - a\hat{u}_i^2 - b\hat{u}_i = 0$$

$$\Rightarrow \sum \hat{u}_i y_i - a\sum \hat{u}_i^2 - b\sum \hat{u}_i = 0$$

$$\Rightarrow \sum \hat{u}_i y_i - a\sum \hat{u}_i^2 - \bar{y}\sum \hat{u}_i + \alpha \bar{u} \hat{u}_i = 0$$

$$\Rightarrow \boxed{\alpha = \frac{\sum \hat{u}_i y_i - \bar{y}\sum \hat{u}_i}{\sum \hat{u}_i^2 - \bar{u}\sum \hat{u}_i}}$$

derivation with u_i^*

$\boxed{u_i^* = \hat{u}_i}$

$$\frac{\partial f}{\partial u_i^*} = \sum 2(y_i - a u_i^* - b)(-a) + 2(-\rho)(u_i - u_i^*) = 0$$

$$\Rightarrow \sum a y_i - \sum a^2 u_i^* - \sum ab + \rho \sum (u_i - u_i^*) = 0$$

$$\Rightarrow a\sum y_i - a^2 \sum u_i^* - a(N\bar{y} - aN\bar{u}) + \rho \sum (u_i - u_i^*) = 0$$

$$\Rightarrow \sum u_i^*(-a^2 - \rho) = a(N\bar{y} + aN\bar{u}) + \rho \sum u_i - a\sum y_i = 0$$

$$\Rightarrow \sum u_i^* = \frac{a\sum y_i + \rho \sum u_i - a(N\bar{y} - aN\bar{u})}{a^2 + \rho}$$

$$\Rightarrow \sum u_i^* = \frac{a(\sum(y_i - \bar{y} + a\bar{u})) + \rho \sum u_i}{a^2 + \rho}$$

$$\Rightarrow \boxed{\sum u_i^* = \frac{a(\sum(y_i - \bar{y})) + a^2\bar{u} + \rho \sum u_i}{a^2 + \rho}}$$

Substitute $\sum u_i^*$ in a

$$a = \frac{\sum \hat{u}_i y_i - \bar{y}\sum \hat{u}_i}{\sum \hat{u}_i^2 - \bar{u}\sum \hat{u}_i}$$

From $\sum_{i=1}^{N} (x_i - \bar{x})^2 = \sum_{i=1}^{N} x_i^2 - \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2$

$$\sum \hat{u}_i^2 = \sum (\hat{u}_i - \bar{u})^2 + \frac{1}{N}\left(\sum_{i=1}^{N} \hat{u}_i\right)^2$$

$$= \sum (\hat{u}_i - \bar{u})^2 + \frac{1}{N}(N\bar{u}_i)^2$$

$$\boxed{\sum \hat{u}_i^2 = \sum (\hat{u}_i - \bar{u})^2 + N\bar{u}^2}$$

$$a = \frac{\sum \hat{u}_i y_i - \bar{y}\sum \hat{u}_i}{\sum \hat{u}_i^2 - \bar{u}\sum \hat{u}_i}$$

$$a\left(\sum(\hat{u}_i - \bar{u})^2 + \cancel{N\bar{u}^2} - \underbrace{\bar{u}\sum \hat{u}_i}_{-\cancel{N\bar{u}^2}}\right) = \sum \hat{u}_i y_i - \bar{y}\sum \hat{u}_i$$

$$a\left(\sum(\hat{u}_i - \bar{u})^2\right) = \sum \hat{u}_i y_i - \underbrace{\bar{y}\sum \hat{u}_i}_{\sum \bar{y}\hat{u}_i}$$

$$= \sum \hat{u}_i (y_i - \bar{y})$$

substitute $\hat{u}_i$

$$\hat{u}_i = \frac{a(\Sigma(y_i-\bar{y})) + a^2\bar{u} + \rho\,\Sigma_i u_i}{a^2+\rho}$$

$$a\left(\Sigma\left(\frac{a(y_i-y)+\rho(u_i-\bar{u})}{a^2+\rho}\right)^2\right) = \Sigma\left(\frac{a(y_i-\bar{y})^2 + a^2\bar{u}(y_i-\bar{y}) + \rho u_i(y_i-\bar{y})}{a^2+\rho}\right)$$

$$\Sigma\frac{a}{(a^2+\rho)^2}\left(a^2(y_i-y)^2+\rho^2(u_i-\bar{u})^2+2a\rho(y_i-y)(u_i-\bar{u})\right) = \frac{\Sigma(a(y_i-\bar{y})^2+a^2\bar{u}(y_i-\bar{y})+\rho u_i(y_i-\bar{y}))}{a^2+\rho}$$

$$\Rightarrow \Sigma\left[a^3(y_i-y)^2+a\rho^2(u_i-\bar{u})^2+2a^2\rho(y_i-\bar{y})(u_i-\bar{u})\right] = \Sigma(a^3(y_i-\bar{y})^2+a^4\bar{u}(y_i-\bar{y})+a^2\rho u_i(y_i-\bar{y})+a\rho(y_i-\bar{y})^2+a^2\rho\bar{u}(y_i-\bar{y})+\rho^2u_i(y_i-\bar{y}))$$

RHS solving

$$\Rightarrow \Sigma a^3(y_i-\bar{y})^2+a^4\bar{u}(y_i-\bar{y})+a^2\rho u_i(y_i-\bar{y})+a\rho(y_i-\bar{y})^2+a^2\rho\bar{u}(y_i-\bar{y})+\rho^2u_i(y_i-\bar{y})$$

$\Rightarrow$ 1) add $\rho^2\bar{u}(y_i-\bar{y})$ and subtract $\rho^2\bar{u}(y_i-\bar{y})$

2) add and subtract $\alpha^2\rho\bar{u}(y_i-\bar{y}) - \alpha^2\rho\bar{u}(y_i-\bar{y})$

$$\Rightarrow \Sigma a^3(y_i-\bar{y})^2+a^4\bar{u}(y_i-\bar{y})+a^2\rho u_i(y_i-\bar{y})+a^2\rho\bar{u}(y_i-\bar{y})-a^2\rho\bar{u}(y_i-\bar{y}) + a\rho(y_i-\bar{y})^2+a^2\rho\bar{u}(y_i-\bar{y})+\rho^2u_i(y_i-\bar{y})+\rho^2\bar{u}(y_i-\bar{y})-\rho^2\bar{u}(y_i-\bar{y})$$

$$\Rightarrow \Sigma a^3(y_i-\bar{y})^2+a^4\bar{u}(y_i-\bar{y})+a^2\rho(u_i-\bar{u})(y_i-\bar{y})+a^2\rho\bar{u}(y_i-\bar{y}) + \rho^2(u_i-\bar{u})(y_i-\bar{y})+\rho^2\bar{u}(y_i-\bar{y}) + a\rho(y_i-\bar{y})^2+a^2\rho\bar{u}(y_i-\bar{y})$$

$$\Rightarrow \Sigma a^3(y_i-\bar{y})^2+\rho^2(u_i-\bar{u})(y_i-\bar{y})+a^2\rho(u_i-\bar{u})(y_i-\bar{y})+\bar{u}(y_i-\bar{y})\left[a^4+\rho^2+2a^2\rho\right]+a\rho(y_i-\bar{y})^2$$

$$\Rightarrow \Sigma a^3(y_i-\bar{y})^2+\rho^2(u_i-\bar{u})(y_i-\bar{y})+a^2\rho(u_i-\bar{u})(y_i-\bar{y})+\bar{u}(y_i-\bar{y})(a^2+\rho)^2+a\rho(y_i-\bar{y})^2$$

divide by N

$$\Rightarrow a^3\frac{\Sigma(y_i-\bar{y})^2}{N}+\rho^2\frac{1}{N}\Sigma(u_i-\bar{u})(y_i-\bar{y})+a^2\rho\frac{1}{N}\Sigma(u_i-\bar{u})(y_i-\bar{y})+\Sigma\bar{u}(y_i-\bar{y})(a^2+\rho)^2+a\rho\frac{1}{N}\Sigma(y_i-\bar{y})^2$$

we know that

$$S_{yy} = \frac{1}{N}\Sigma(y_i-\bar{y})^2 \qquad S_{uu} = \frac{1}{N}\Sigma(u_i-\bar{u})^2$$

$$S_{yu} = \frac{1}{N}\Sigma(y_i-\bar{y})(u_i-\bar{u})$$

$$\Rightarrow a^3S_{yy}+\rho^2S_{yu}+a^2\rho S_{yu}+\Sigma\bar{u}\cancelto{0}{(y_i-\bar{y})}(a^2+\rho)^2+a\rho S_{yy}$$

$$\hookrightarrow \Sigma\bar{u}y_i - \Sigma\bar{u}\bar{y} = N\bar{u}\bar{y} - N\bar{u}\bar{y} = 0$$

• LHS solving

$$\sum a^3 (y_i-\bar{y})^2 + a\rho^2 \sum (u_i-\bar{u})^2 + 2a^2\rho \sum (y_i-\bar{y})(u_i-\bar{u})$$

$\Rightarrow$ divide with N

$$\Rightarrow a^3 \frac{\sum (y_i-\bar{y})^2}{N} + a\rho^2 \frac{1}{N} \sum (u_i-\bar{u})^2 + 2a^2\rho \frac{1}{N} \sum (y_i-\bar{y})(u_i-\bar{u})$$

$$\Rightarrow a^3 S_{yy} + a\rho^2 S_{uu} + 2a^2\rho S_{yu}$$

Now LHS = RHS

$$\Rightarrow a^3 \cancel{S_{yy}} + a\rho^2 S_{uu} + 2a^2\rho S_{yu} = a^3 \cancel{S_{yy}} + \rho^2 S_{yu} + a^2\rho S_{yu} + a\rho S_{yy}$$

$$\Rightarrow a^2 \rho S_{yu} + a^2\rho S_{yu} = a\rho S_{yy} + \rho^2 S_{yu} + \cancel{a^2\rho S}$$

$$\Rightarrow a\rho S_{uu} + a^2 S_{yu} = a S_{yy} + \rho S_{yu}$$

$$\Rightarrow a^2 S_{yu} + a(\rho S_{uu} - S_{yy}) - \rho S_{yu} = 0$$

Solving quadratic equation

$$a = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = \frac{(S_{yy} - \rho S_{uu}) + \sqrt{(S_{yy} - \rho S_{uu})^2 - 4(S_{yu})(-\rho S_{yu})}}{2S_{yu}}$$

$$a = \frac{(S_{yy} - \rho S_{uu}) + \sqrt{(S_{yy} - \rho S_{uu})^2 + 4\rho S_{yu}^2}}{2S_{yu}}$$

Therefore

$$\boxed{\begin{aligned} b &= \bar{y} - a\bar{u} \\ a &= \frac{S_{yy} - \rho S_{uu} + \sqrt{(S_{yy} - \rho S_{uu})^2 + 4\rho S_{yu}^2}}{2S_{yu}} \end{aligned}}$$

) given $b=0$

initial prob

$$\min_{\alpha,\beta,u_i^*} \sum (y_i - a\hat{u}_i - b)^2/\sigma_\epsilon^2 + (u_i - \hat{u}_i)^2/\sigma_s^2$$

$$\Rightarrow \min_{\alpha,\beta,u_i^*} \sum (y_i - a\hat{u}_i)^2 + (u_i - \hat{u}_i)^2 \rho \rightarrow f$$

derivation w.r.t α

$$\frac{\partial f}{\partial a} = \sum 2(y_i - a\hat{u}_i)(-\hat{u}_i) = 0$$

$$\Rightarrow \sum y_i\hat{u}_i - a\sum \hat{u}_i^2 = 0$$

$$\Rightarrow \boxed{a = \frac{\sum y_i\hat{u}_i}{\sum \hat{u}_i^2}}$$

changed from $a = \dfrac{\sum \hat{u}_i y_i - \bar{y}\sum \hat{u}_i}{\sum \hat{u}_i^2 - \bar{u}\sum \hat{u}_i}$

i.e, $a = \dfrac{\sum \hat{u}_i (y_i - \bar{y})}{\sum \hat{u}_i(\hat{u}_i - \bar{u})}$

derivation w.r.t $\hat{u}_i$

$$\frac{\partial f}{\partial \hat{u}_i} = \sum 2(y_i - a\hat{u}_i)(-a) + \sum 2\rho(u_i - \hat{u}_i)(-1) = 0$$

$$\Rightarrow \sum (ay_i - a^2\hat{u}_i + \rho u_i - \rho\hat{u}_i) = 0$$

$$\Rightarrow a\sum y_i - a^2\sum \hat{u}_i + \rho\sum u_i - \rho\sum \hat{u}_i = 0$$

$$\boxed{\sum \hat{u}_i = \frac{a\sum y_i + \rho\sum u_i}{a^2+\rho}}$$

from previous part $\sum \hat{u}_i^2 = \sum(\hat{u}_i - \bar{u})^2 + N\bar{u}^2$

substitute $\hat{u}_i$ in a

$$a\sum \hat{u}_i^2 = \sum y_i\hat{u}_i$$

$$a\sum(\hat{u}_i - \bar{u})^2 + aN\bar{u}^2 = \sum y_i\hat{u}_i$$

from previous part $a\sum(\hat{u}_i - \bar{u})^2$, we can't substitute bcz diff $\hat{u}_i$

From a equation if $b=0$; we can observe that $(y_i - \bar{y})$ is replaced by y_i and $\hat{u}_i - \bar{u}$ is replaced by $\hat{u}_i$. Similarly in the solution, we can write it as

$$a = \frac{\left[\sum y_i^2 - \rho\sum x_i^2\right] \pm \sqrt{\left(\sum y_i^2 - \rho\sum x_i^2\right)^2 + 4\rho\left(\sum x_i y_i\right)^2}}{2\sum(x_i y_i)}$$

From the above observation, we can also say that if $b=0$, there is no need of mean shifting the data.

b) parameters of IOLS and OLS

$$\alpha = \frac{S_{yy} - \rho S_{uu} + \sqrt{(S_{yy} - \rho S_{uu})^2 + 4\rho S_{yu}^2}}{2 S_{yu}} \qquad \beta = \bar{y} - \alpha \bar{u}$$

for IOLS $\rho \to 0$ $\rightarrow$ y - accurate u - not accurate

$$\alpha = \frac{S_{yy} + \sqrt{S_{yy}^2}}{2 S_{yu}}$$

$$\alpha = \frac{S_{yy}}{S_{yu}}$$

Therefore for IOLS; $\alpha = \frac{S_{yy}}{S_{yu}}$ and $\beta = \bar{y} - \alpha\bar{u}$

for OLS $\rho \to \infty$ $\rightarrow$ u - accurate y - not accurate

$$\alpha = \frac{\left(\frac{1}{\rho}S_{yy} - S_{uu} + \sqrt{\left(\frac{1}{\rho}S_{yy} - S_{uu}\right)^2 + \frac{4\rho S_{yu}^2}{\rho^2}}\right)\rho}{2 S_{yu}}$$

$$= \frac{\left(-S_{uu} + \sqrt{S_{uu}^2 + \frac{4\rho S_{yu}^2}{\rho^2}}\right)\rho}{2 S_{yu}}$$

$$= \frac{\left(-S_{uu} + S_{uu}\sqrt{1 + \frac{4 S_{yu}^2 \rho}{S_{uu}^2 \rho^2}}\right)\rho}{2 S_{yu}}$$

$$= \frac{-S_{uu} + S_{uu}\left(1 + \frac{1}{2}\left(\frac{4 S_{yu}^2 \rho}{S_{uu}^2 \rho^2}\right)\right)\rho}{2 S_{yu}}$$

$$= \frac{S_{uu}\left(\cancel{-1} \cancel{+1} + \frac{1}{\cancel{2}}\frac{\overset{2}{\cancel{4}} S_{yu}^2 \rho}{S_{uu}^2 \rho^2}\right)\rho}{2 S_{yu}}$$

$$= \frac{S_{uu}}{\cancel{2} S_{yu}} \times \frac{\cancel{2} S_{yu}^2 \cancel{\rho^2}}{S_{uu} \cancel{\rho^2}}$$

$$\boxed{\alpha = \frac{S_{yu}}{S_{uu}}} \quad \text{and} \quad \beta = \bar{y} - \alpha\bar{u}$$

OLS

$$y_i^* = \alpha u_i^* + \beta$$

$$y_i^* = \frac{S_{yu}}{S_{uu}} u_i^* + \bar{y} - \frac{S_{yu}}{S_{uu}} \bar{u}$$

$$y_i^* = \frac{S_{yu}}{S_{uu}} (u_i^* - \bar{u}) + \bar{y} \quad \& \quad u_i^* = \frac{y_i^* - \left(\bar{y} - \frac{S_{yu}}{S_{uu}} \bar{u}\right)}{\frac{S_{yu}}{S_{uu}}}$$

IOLS

$$y_i = \hat{y}_i$$

$$\hat{y}_i = \hat{\alpha} \hat{x}_i + \hat{\beta}$$

in iols u_i is y_i y_i is u_i in ols

$$u_i^* = \alpha y_i^* + \beta$$

$$u_i^* = \frac{S_{yu}}{S_{yy}} y_i^* + \bar{y} - \alpha \bar{u}$$

$$\frac{S_{yy}}{S_{yu}} u_i^* + \bar{u} = y_i^*$$

$$y_i^* = \frac{S_{yy}}{S_{yu}} u_i^* + \bar{u} \quad \& \quad u_i^* = \frac{S_{yu}}{S_{yy}} y_i^* + \bar{y} - \alpha \bar{u}$$

WTLS

$$y_i^* = \alpha u_i^* + \beta$$

$$\alpha = \frac{S_{yy} - \rho S_{uu} + \sqrt{(S_{yy} - \rho S_{uu})^2 + 4\rho S_{yu}^2}}{2 S_{yu}}$$

$$u_i^* = \frac{\alpha (y_i - \bar{y} + \alpha \bar{u}) + \rho \Sigma u_i}{\alpha^2 + \rho}$$

$$\beta = \bar{y} - \alpha \bar{u}$$

Substitute α, u_i^*, β in y_i^*