MM17B113

3) $\bar{x} = \begin{bmatrix} 9 \\ 68 \\ 129 \end{bmatrix}$ $S = \begin{bmatrix} 7 & 21 & 34 \\ 21 & 64 & 102 \\ 34 & 102 & 186 \end{bmatrix}$

a) Largest eigen value = 250.4

$$|S-\lambda I| = \begin{bmatrix} 7-\lambda & 21 & 34 \\ 21 & 64-\lambda & 102 \\ 34 & 102 & 186-\lambda \end{bmatrix}$$

$\Rightarrow -\lambda^3 + 257\lambda^2 - 1653\lambda + 146 = 0$

$\lambda_1 + \lambda_2 + \lambda_3 = 257$

$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = 1653$

$\lambda_1\lambda_2\lambda_3 = 146$

$\Rightarrow \lambda_2 + \lambda_3 = 6.6$ and $\lambda_2\lambda_3 = 0.58$

$\Rightarrow \lambda_2 = 0.089$ and $\lambda_3 = 6.509$

Eigen vector corresponding to 250.4

$$\begin{bmatrix} 7 & 21 & 34 \\ 21 & 64 & 102 \\ 34 & 102 & 186 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 250.4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow 7x_1 + 21x_2 + 34x_3 = 250.4x_1 \rightarrow (1)$

$\Rightarrow 21x_1 + 64x_2 + 102x_3 = 250.4x_2 \rightarrow (2)$

$\Rightarrow 34x_1 + 102x_2 + 186x_3 = 250.4x_3 \rightarrow (3)$

Solving eq (1), eq (2), eq (3)

$x_1 = 0.1887$
$x_2 = 0.5685$
$x_3 = 1$

$V_1 = \begin{bmatrix} 0.1887 \\ 0.5685 \\ 1 \end{bmatrix}$

normalizing V_1

$L = \sqrt{1.887^2 + 0.5685^2 + 1^2} = 1.1656$

$$V_1 = \begin{bmatrix} 0.1887/1.1656 \\ 0.5685/1.1656 \\ 1/1.1656 \end{bmatrix} = \begin{bmatrix} 0.162 \\ 0.487 \\ 0.857 \end{bmatrix}$$

$\leftarrow$ normalized eigen vector corresponding to eigen value: 250.4

Ily for x_2 and x_3

$$\begin{bmatrix} 7 & 21 & 34 \\ 21 & 64 & 102 \\ 34 & 102 & 186 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.089 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$7x_1 + 21x_2 + 34x_3 = 0.089x_1$$
$$21x_1 + 64x_2 + 102x_3 = 0.089x_2$$
$$34x_1 + 102x_2 + 186x_3 = 0.089x_3$$

solving above 3 equations

$$V_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -47.735 \\ 14.0889 \\ 1 \end{bmatrix}$$

<u>normalizing V_2</u>

$$L = \sqrt{-47.735^2 + 14.0889^2 + 1}$$

$$\text{normalized } V_2 = \begin{bmatrix} 0.959 \\ -0.283 \\ -0.020 \end{bmatrix} \leftarrow$$ normalized eigen vector corresponding to eigen value 0.089

$$\begin{bmatrix} 7 & 21 & 34 \\ 21 & 64 & 102 \\ 34 & 102 & 186 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 6.509 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$7x_1 + 21x_2 + 34x_3 = 6.509x_1$$
$$21x_1 + 64x_2 + 102x_3 = 6.509x_2$$
$$34x_1 + 102x_2 + 186x_3 = 6.509x_3$$

solving above 3 equations

$$V_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.454 \\ -1.608 \\ 1 \end{bmatrix}$$

<u>normalizing V_3</u>

$$L = \sqrt{-0.454^2 + (-1.608)^2 + 1^2}$$

$$\text{normalized } V_3 = \begin{bmatrix} -0.233 \\ -0.825 \\ 0.513 \end{bmatrix} \leftarrow$$ normalized eigen vector corresponding to eigen value 6.509

b) $\lambda_1 = 250.4$ $\lambda_2 = 0.089$ $\lambda_3 = 6.509$

in order its $250.4, 6.509, 0.089$

% of variance captured by 1 PC $= \frac{\lambda_1 \times 100}{\lambda_1+\lambda_2+\lambda_3} = \frac{250.4 \times 100}{257}$

$= 0.974 \times 100$

$= 97.4\%$

Therefore, ~~at~~ one principal component should be retained to capture atleast 95% of variance in data.

(c) transformed variables are $z_i = v_i^T(X)$

for two possible linear relationships, consider 2 smallest eigenvalues

$z_2 = v_2^T(X - \bar{X})$ $z_3 = v_3^T(X - \bar{X})$ $\begin{bmatrix} -0.233 \\ -0.825 \\ 0.513 \end{bmatrix}^T$

$z_2 = 0$ and $z_3 = 0$

$$\begin{bmatrix} 0.959 \\ -0.283 \\ -0.020 \end{bmatrix}^T \begin{bmatrix} x_1 - 9 \\ x_2 - 68 \\ x_3 - 129 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} -0.454 \\ -1.608 \\ 1 \end{bmatrix}^T \begin{bmatrix} x_1 - 9 \\ x_2 - 68 \\ x_3 - 129 \end{bmatrix} = 0$$

$\Rightarrow 0.233 z_1 + 0.826 z_2 - 0.513 z_3 + 7.983 = 0$ and

$\Rightarrow 0.958 z_1 \mp 0.283 z_2 - 0.020 z_3 + 13.20 = 0$

are two possible linear relationships

(d) Projecting data onto largest eigen vector, given $X = \begin{bmatrix} 10.1 \\ 73 \\ 135.5 \end{bmatrix}$

$$\text{score} = v_1^T (X - \bar{X})$$

$$= \begin{bmatrix} 0.162 \\ 0.487 \\ 0.857 \end{bmatrix}^T \begin{bmatrix} 10.1 - 9 \\ 73 - 68 \\ 135.5 - 129 \end{bmatrix}$$

$$= \begin{bmatrix} 0.162 & 0.487 & 0.857 \end{bmatrix} \begin{bmatrix} 1.1 \\ 5 \\ 6.5 \end{bmatrix}$$

$\boxed{\text{score} = 8.1837}$ → when considering only one eigen vector

when all three are considered, the scores are

$8.1837, -0.4901, -1.0482$

(e) SVL = 73 mm (z_2)

from two linear relationships from part (c)

$$0.233 z_1 + 0.826 z_2 - 0.513 z_3 + 7.983 = 0$$
$$0.958 z_1 - 0.283 z_2 - 0.02 z_3 + 13.20 = 0$$

replace z_2 by 73 and solve

$$\Rightarrow 0.2330 z_1 + 0.826(73) - 0.513 z_3 + 7.983 = 0$$
$$\Rightarrow 0.2330 z_1 - 0.0513 z_3 = -68.224 \rightarrow (1)$$

$$\Rightarrow 0.958 z_1 - 0.283(73) - 0.02(z_3) + 13.20 = 0$$
$$\Rightarrow 0.958 z_1 - 0.02 z_3 = 7.452 \rightarrow (2)$$

From eq (1) & (2)

$$z_1 = 10.66 \qquad z_2 = 138.01$$

that is, $\boxed{\text{mass} = 10.66 \text{ gms}}$

(f) SVL = 73 mm (z_2)
HLS = 135.5mm (z_3)

find z_1

As both z_2 and z_3 are given, one linear relationship is sufficient to estimate mass (z_1), so lets eliminate z_1 from two equations

$$\Rightarrow 3.681 z_2 - 2.093 z_3 + 19.646 = 0$$

using TLS

$$\min_{\hat{z}} (z - \hat{z})^T (z - \hat{z})$$

s.t $A\hat{z} = b \rightarrow$ this from above equation

$$A = [3.681, -2.093]$$
$$b = -19.646$$

$$\Rightarrow \hat{z} = z - A(AA^T)^{-1}(Az - b)$$ → we get this by substituting linear relationship in objective function

$$\min \tfrac{1}{2}(z - \hat{z})^T (z - \hat{z}) + \lambda (A\hat{z} - b)$$

$$\Rightarrow \hat{z} = [75.97, 140.43]$$

these values are close to given values

From linear relationships;

$$0.233 z_1 + 0.826(75.97) - 0.513(140.43) + 7.983 = 0$$

$$\boxed{z_1 = 11.12 \text{ grams}}$$