EE1206 Linear Systems and Signal Processing
Exam 1, Oct 15 2023

Time: 150 min
41 points

Instructions

- This is a closed book closed notes exam. No phones, calculators, or laptops can be used during the exam. The notation used is the same as from class.
- There are no negative marks. Use a blue or a black pen to answer the questions.

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1. *(6 pts)* Let $\underline{x} = (p, q, r, s, t, u)$ be a vector of length 6 and assume that $\underline{x}$ has the 6-point DFT $\mathcal{F}\underline{x} = (P, Q, R, S, T, U)$, where $p, \cdots, u, P, \cdots, U$ are complex scalars. For each signal (i)-(iii), indicate its corresponding discrete Fourier transform from parts (a)-(j). Provide a brief explanation for each answer.

   (i) $(p, -q, r, -s, t, -u)$

   (ii) (s, t, u, p, q, r)

   (iii) $(p, q, r, s, t, u) * (1, \cos(2\pi/3), \cos(4\pi/3), \cos(6\pi/3), \cos(8\pi/3), \cos(10\pi/3))$

   (a) $\frac{1}{3}(P + R + T, Q + S + U)$

   (b) (P, S)

   (c) $\frac{1}{2}(P + S, Q + T, R + U)$

   (d) (S, T, U, P, Q, R)

   (e) (P, R, T)

   (f) (T, U, P, Q, R, S)

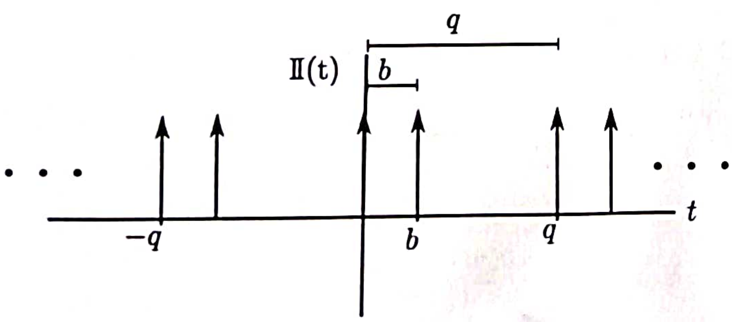
   (g) $(P, -Q, R, -S, T, -U)$

   (h) $3(0, 0, R, 0, T, 0)$

   (j) $(P + S, 0, Q + T, 0, R + U, 0)$

2. *(8 pts) Sampling with half a shah*

   We define a new sampling function $\mathrm{II}(t)$, which is train of evenly spaced delta function *pairs*, as shown in the figure. This sampling function is parameterized by two parameters: q, the spacing between the delta function pairs, and b, the spacing within the pairs of delta functions.



   We will apply this sampling scheme to a function $f(t)$, which has a Fourier transform $\mathcal{F}f(s)$ and is bandlimited with bandwidth p (i.e. $\mathcal{F}f(s) = 0$ for $|s| \geq p/2$).

(a) Write $\mathrm{II}(t)$ as the sum of two Shah functions (in terms of q and b).

(b) Define $g(t)$ to be the sampled version of $f(t)$ using our new sampling scheme, so that $g(t) = \mathrm{II}(t)f(t)$. Write the Fourier transform of $g(t)$ in terms of $\mathcal{F}f(s)$, q, and b.

(c) Consider the case where $b = q/2$. What is the maximum bandwidth p for which we can guarantee reconstruction of $f(t)$ from its sampled version $g(t)$? What is a possible reconstruction scheme?

(d) Now consider the case when $b = q/4$. We further assume that $f(t)$ is real and even. What is the maximum bandwidth p for which we can guarantee reconstruction of $f(t)$ from its sampled version $g(t)$? For a signal of maximum bandwidth, what is a possible reconstruction scheme?
Hint: consider reconstruction based only on the real or the imaginary part of the Fourier transform of $g(t)$.

3. *(6 pts) Oversampling*

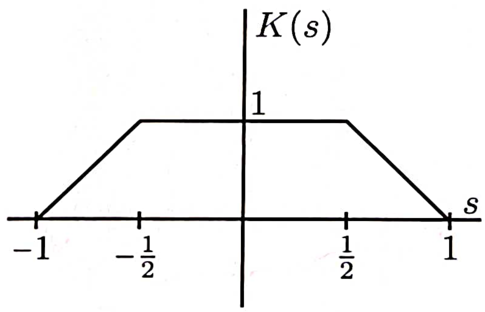
Let $f(t)$ be a bandlimited signal with spectrum contained in the interval $-1/2 < s < 1/2$. Suppose you sample $f(t)$ at intervals of $1/2$ (that is, at *twice* the Nyquist rate), to obtain

$$f_{\text{sampled}}(t) = \frac{1}{2}\text{III}_{1/2}(t)\, f(t).$$

1. Qualitatively explain why the following equation is correct:

$$f(t) = \mathcal{F}^{-1}\{K(s)\ \mathcal{F}f_{\text{sampled}}(s)\}$$

where $K(s)$ is defined by



2. Show that you can reconstruct $f(t)$ by

$$f(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} f\left(\frac{n}{2}\right)k\left(t - \frac{n}{2}\right) \qquad \text{where } k(y) = \frac{\cos(\pi y) - \cos(2\pi y)}{\pi^2 y^2}.$$

You may use the fact that

$$K(s) = \mathcal{F}\left\{\frac{\cos(\pi t) - \cos(2\pi t)}{\pi^2 t^2}\right\}.$$

3. Describe an advantage the reconstruction formula in part (b) has over the usual sinc interpolation formula, say in terms of the accuracy of the series if one only uses a finite number of terms.

4. *(3 pts)* Let $X(s)$ be the Fourier transform of the signal $x(t)$. The real part of $X(s)$ is $\left[4/(4+4\pi^2 s^2)\right]$, and the imaginary part of $X(s)$ is $[2/(\pi s)]$. Find $x(t)$.

5. *(6 pts)* Let $f(x)$ be a signal, and for $h > 0$ let $A_h f(x)$ be the averaging operator,

$$A_h f(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy\,.$$

   1. Show that we have the alternate expressions for $A_h f(x)$:

$$A_h f(x) = \frac{1}{2h} \int_{-h}^{h} f(x+y)\, dy = \frac{1}{2h} \int_{-h}^{h} f(x-y)\, dy\,.$$

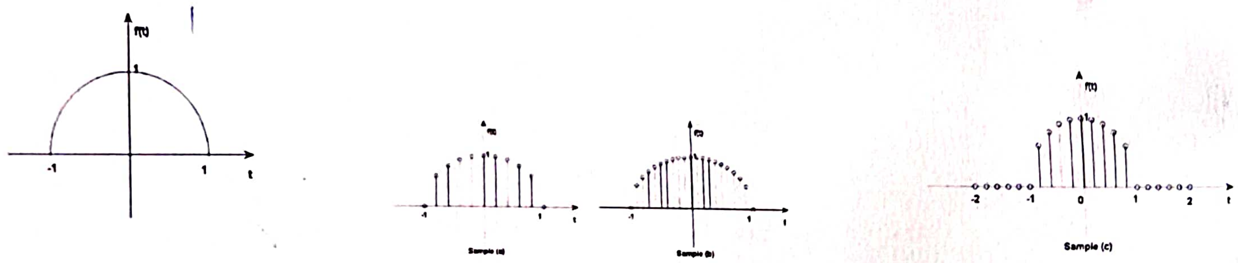
   2. Show that we should define $A_h T$ for a distribution T by

$$\langle A_h T\,,\, \varphi \rangle = \langle T\,,\, A_h \varphi \rangle\,.$$

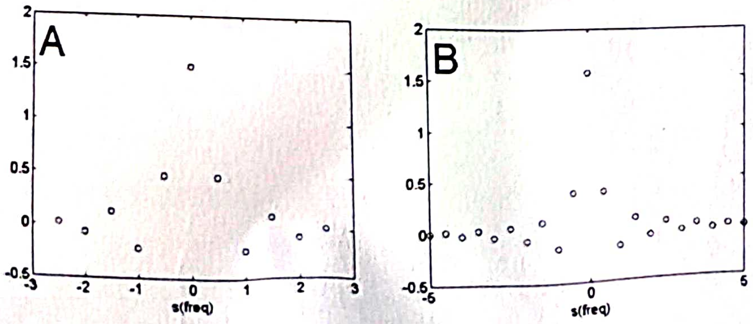
   3. What is $A_h \delta$?

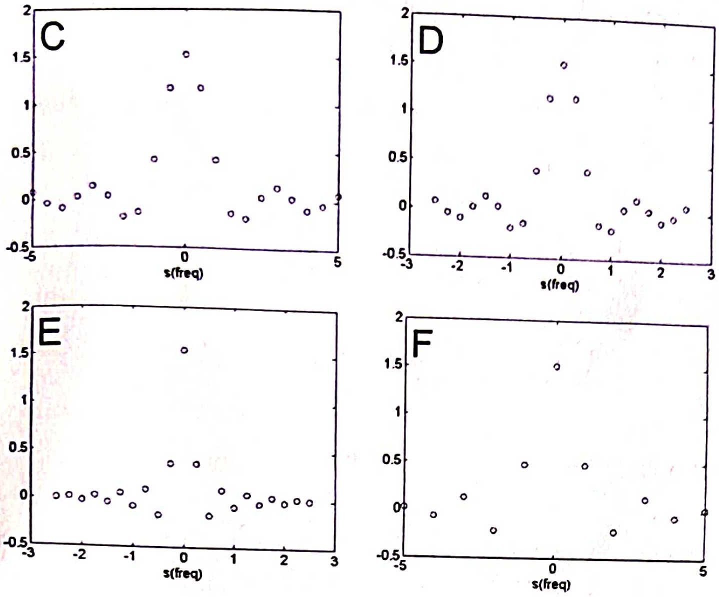
6. *(6 pts) Approximation of Continuous Fourier Transform*

   We can view the DFT as a discrete approximation to the Fourier transform of a continuous-time signal, and we know how to take the DFT in numpy (fft, fftshift, ifftshift, ...) by sampling the continuous-time signal. In this problem, we want to approximate the Fourier transform of the time-limited signal $f(t)$ shown below.



   Possible results, obtained using MATLAB, are shown below. What you need to do is match (a), (b) and (c) to the approximation of the Fourier transform in (A) through (F). Give a clear explanation for your answers.





7. *(6 pts) Solving a cubic, the hard way*

In this problem we will find the complex roots of the cubic

$$f(x) = x^3 - 3\alpha\beta x - (\alpha^3 + \beta^3) = 0.$$

It can be shown (not required for this problem) that any cubic can be written in this form.

1. Find a 3×3 circulant matrix A such that $\det(xI - A) = f(x)$:

$$A = \begin{pmatrix} a_1 & a_3 & a_2 \\ a_2 & a_1 & a_3 \\ a_3 & a_2 & a_1 \end{pmatrix}, \quad \det(xI - A) = f(x).$$

   Hint: Pick $a_1 = 0$ and compute $\det(xI - A)$. You should be able to read off a_2, a_3 by inspection.

2. Use (a) to find all the roots of $f(x)$. Recall that the roots of $\det(xI - A)$ are the eigenvalues of A.