EE1206 Linear Systems and Signal Processing
Exam 2

Time: 120 min
Maximum score:
30 points

Instructions

- This is a closed book closed notes exam. No phones, calculators, or laptops can be used during the exam. The notation used is the same as from class.
- There are 6 questions for 6 marks each. The maximum score you can obtain is 30 points (and not 36 marks)
- There are no negative marks. Use a blue or a black pen to answer the questions.

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1. *(6 pts)* (a) Let $X(t)$ be a wide sense stationary random process with autocorrelation $R_{xx}(\tau)$. Let $Y(t) = aX(ct)$ for some constants $a, c \neq 0$. Is Y WSS (and if so, what is its autocorrelation function) ?

   (b) Let $w(t)$ be a (deterministic, non random) signal, and C a random variable. Identify the conditions on C and $w(t)$ such that $X(t) = Cw(t)$ is WSS.

2. *(6 pts)* Let $X(t)$ be a WSS random process with autocorrelation $R_{xx}(\tau)$ and PSD $S_{xx}(s)$.

   (a) Explain why $R_{xx}(\tau) = 2\int_0^{\infty} S_{xx}(s)\cos 2\pi s\tau ds$.

   (b) Show that
   $$R_{xx}(0) - R_{xx}(\tau) \geq \frac{1}{4^n}\left(R_{xx}(0) - R_{xx}(2^n\tau)\right).$$
   You may find the inequality $1 - \cos\theta \geq (1 - \cos 2\theta)/4$ helpful.

   (c) Let $Y(t)$ be the output of an LTI system with input process $X(t)$. Suppose it is known that the PSDs S_{xx} of the input and S_{yy} of the output satisfy $S_{xx}(s)S_{yy}(s) = 0$ for all s. What can you say about the output process $Y(t)$?

3. *(6 pts)* (a) A randomly phased sinusoid $X(t) = \cos(2\pi ft + \Theta)$, with Θ uniformly distributed in $(0, 2\pi)$ is given as input to an LTI system with frequency response $H(s)$ (assume the LTI system has a real impulse response, as usual). What is the autocorrelation function of the output of this filter?

   (b) A discrete-time white noise random process is given as input to an LTI system with impulse response h. We know that $h(0) = 1$, and that all impulse response values $h(m)$ are nonzero. Can the output power of the filter ever be less than the input power? Justify with a proof or counterexample.

4. *(6 pts)* A zero mean discrete time WSS random process $Y(n)$ with PSD $S_{yy}(s) = 2 - 2\cos(2\pi s)$ is corrupted by zero mean white noise $Z(n)$ with power $E(Z(0)^2) = 1$; so that we have measurements of $X(n) = Y(n) + Z(n)$.

   (a) What is the frequency response of the optimal Weiner filter to recover $Y(t)$? Plot this frequency response as a function of the frequency s, for $-1/2 \leq s \leq 1/2$. Is this a low-pass or a high-pass filter ?

   (b) Does the power of the estimate $\hat{Y}$ match the power in Y, i.e do we have $E(Y(0)^2) = E(\hat{Y}(0)^2)$?

5. *(6 pts)* Let $f(t)$ be a bandlimited signal whose Fourier transform satisfies $|\mathcal{F}f(s)| = 0$ for $|s| \geq 1$. According to the sampling theorem, one has to sample $f(t)$ with $\text{III}_{1/2}(t)$ to reproduce the signal without aliasing.

You try to test this out in the lab, but there is something wrong with the ideal sampler — it can *only* sample using $\text{III}_1(t)$. This will inevitably cause aliasing, but you think you can devise a scheme to somehow reconstruct $f(t)$ without aliasing.

This problem explores how this can be done by using this faulty sampler and some filter $h(t)$.

(a) Let $h(t) = -1/(\pi i t)$. If $g(t) = (f * h)(t)$, express $G(s)$ in terms of $F(s)$ *only*. *Hint: Consider the cases* $s = 0$, $s > 0$ *and* $s < 0$.

(b) Suppose you sample $f(t)$ with $\text{III}_1(t)$ to yield $y(t)$. For the case of $0 < s < 1$, express $Y(s)$ in terms of $F(s)$ and $F(s-1)$ *only.*

(c) Suppose you sample $g(t)$ with $\text{III}_1(t)$ to yield $x(t)$. For the case of $0 < s < 1$, express $X(s)$ in terms of $F(s)$ and $F(s-1)$ *only.*

(d) Using the two equations you have from parts (b) and (c), show how you might reconstruct $f(t)$ without aliasing.

6. *(6 pts)* Earlier in the course we discussed the uncertainty principle for continuous time signals, the main idea being that time width and frequency width are inversely related. A similar relationship holds for discrete signals, as we will see in this problem. Let f be an N−length vector and $F = \mathcal{F}f$ be its DFT. Let $\phi(f)$ indicate the number of nonzero coordinates in f, and similarly let $\phi(\mathcal{F}f)$ be the number of non-zero coordinates in $F = \mathcal{F}f$. Then the discrete uncertainty principle states that

$$\phi(f)\phi(\mathcal{F}f) \geq N.$$

In other words, the time width ($\phi(f)$) and frequency width ($\phi(\mathcal{F}f)$) are inversely related.

Below is an outline of the proof of this uncertainty principle. Complete the proof by briefly justifying the statements (a)-(f) in the proof.

Proof : Let the vector s denote the sign of f:

$$s[n] := \begin{cases} 1 \text{ if } f[n] > 0\,, \\ -1 \text{ if } f[n] < 0\,, \\ 0 \text{ if } f[n] = 0\,, \end{cases}$$

and let (f, s) denote the dot product of the vectors f and s. Assume that $f \neq 0$. Defining $s[n]$ gives us the advantage of expressing $|f[n]|$ as $|f[n]| = f[n]s[n]$.

Now

$$\begin{aligned}
\max_m |F[m]| &\overset{(a)}{\leq} \sum_{n=0}^{N-1} |f[n]| \overset{(b)}{=} (f, s) \\
&\overset{(c)}{\leq} (f, f)^{1/2}(s, s)^{1/2} \overset{(d)}{=} \frac{1}{\sqrt{N}}(F, F)^{1/2}(s, s)^{1/2} \\
&\overset{(e)}{=} \frac{1}{\sqrt{N}}(F, F)^{1/2}\phi(f)^{1/2} \overset{(f)}{\leq} \frac{\max_m |F[m]|}{\sqrt{N}}\phi(F)^{1/2}\phi(f)^{1/2}.
\end{aligned}$$

Canceling out $\max_m |F[m]|$ on both sides, we are left with the desired inequality

$$\phi(f)\phi(F) \geq N.$$