MATHEMATICS 2022-2

February 9, 2025

1. let α and β be real numbers such that $-\frac{\pi}{4} < \beta < 0 < \alpha < \frac{\pi}{4}$.If $\sin(\alpha + \beta) = \frac{1}{3} and \cos(\alpha - \beta) = \frac{2}{3}$ then the greatest integer less than or equal to $(\frac{\sin\alpha}{\cos\beta} + \frac{\cos\beta}{\sin\alpha} + \frac{\cos\alpha}{\sin\beta} + \frac{\sin\beta}{\cos\alpha})^2$ is _.

2. If $y(x)$ is the solution of the differential equation

   xdy - $(y^2 - 4y) = 0$, for x ¿ 0, y(1) = 2,

   and the slope of the curve $y = y(x)$ is never zero, then the value of $10y(\sqrt{2})$ is _.

3. The greatest integer less than or equal to

   $\int_1^2 \log_2(x^3 + 1)$ dx + $\int_1^{\log_2 9} (2^x - 1)^{1/3}$ dx

   is _.

4. The product of all positive real values of x satisfying the equation

   $x^{16(\log_5 x)^3 - 68\log_5 x}$ $=5^{-16}$

   is _.

5. If

   $\beta = \lim_{x\to 0} \frac{e^{x^3} - (1-x^3)^{1/3} + \left(((1-x^2)^{1/2} - 1)\right)\sin x}{x\sin^2 x}$,

   then the value of 6β is _.

6. Let β be a real number. Consider the matrix

$$A = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix} \tag{1}$$

   If $A^7 - (\beta - 1)A^6 - \beta A^5$ is a singular matrix, then the value of 9β is _.

7. Consider the hyperbola

   $\frac{x^2}{100} - \frac{y^2}{64} =$

   with foci at S and S_1, where S lies on the positive x-axis. Let P be a point on the hyperbola in the first quadrant. Let $\angle SPS_1 = \alpha$, with $\alpha < \frac{\pi}{2}$. The straight line passing through the point S and having the same slope as that of the tangent at P to the hyperbola intersects the straight line S_1P at P_1.

Let δ be the distance of P from the straight line SP_1, and let $\beta = S_1P$. Then the greatest integer less than or equal to

$$\frac{\beta\delta}{9\sin\frac{\alpha}{2}} \tag{2}$$

is _.

8. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by
   $f(x) = \frac{x^2+5}{12}$
   and g(x) =

$$2\left((1-\frac{4|x|}{3})\right), |x| \leq \frac{3}{4}, \tag{3}$$

$$0, |x| > \frac{3}{4}. \tag{4}$$

   If α is the area of the region

$$\{(x,y) \in \mathbb{R}\times\mathbb{R} \mid |x| \leq \frac{3}{4}, 0 \leq y \leq \min\{f(x), g(x)\}\}, \tag{5}$$

   then the value of 9α is _.

9. Let PQRS be a quadrilateral in a plane, where $QR = 1$, $\angle PQR = \angle QRS = 70°$, $\angle PQS = 15°$, and $\angle PRS = 40°$. If $\angle RPS = \theta°$, $PQ = \alpha$, and $PS = \beta$, then the interval(s) that contain(s) the value of $4\alpha\beta\sin\theta°$ is/are:

   (a) $(0, \sqrt{2})$
   (b) $(1, 2)$
   (c) $(\sqrt{2}, 3)$
   (d) $(2\sqrt{2}, 3\sqrt{2})$

10. Let
    $\alpha = \sum_{k=1}^{\infty} \sin^{2k}\left((\frac{\pi}{6})\right)$.
    Let $g : [0,1] \to \mathbb{R}$ be the function defined by
    g(x) = $2^{\alpha x}$ $+2^{\alpha(1-x)}$.
    Then, which of the following statements is/are TRUE?

    (a) The minimum value of $g(x)$ is $2^{7/6}$.
    (b) The maximum value of $g(x)$ is $1 + 2^{1/3}$.

(c) The function $g(x)$ attains its maximum at more than one point.

(d) The function $g(x)$ attains its minimum at more than one point.

11. Let $\bar{z}$ denote the complex conjugate of a complex number z. If z is a non-zero complex number for which both real and imaginary parts of $(\bar{z})^2 + \frac{1}{z^2}$ are integers, then which of the following is/are possible value(s) of $|z|$?

(a) $\left((\frac{43+3\sqrt{205}}{2})\right)^{1/4}$

(b) $\left((\frac{7+\sqrt{33}}{4})\right)^{1/4}$

(c) $\left((\frac{9+\sqrt{65}}{4})\right)^{1/4}$

(d) $\left((\frac{7+\sqrt{13}}{6})\right)^{1/4}$

12. Let G be a circle of radius $R > 0$. Let $G_1, G_2, \ldots, G_n$ be n circles of equal radius $r > 0$. Suppose each of the n circles $G_1, G_2, \ldots, G_n$ touches the circle G externally. Also, for $i = 1, 2, \ldots, n-1$, the circle G_i touches G_{i+1} externally, and G_n touches G_1 externally. Then, which of the following statements is/are TRUE?

(a) If $n = 4$, then $(\sqrt{2} - 1)r < R$.

(b) If $n = 5$, then $r < R$.

(c) If $n = 8$, then $(\sqrt{2} - 1)r < R$.

(d) If $n = 12$, then $\sqrt{2}(\sqrt{3} + 1)r > R$.

13. Let $\hat{\imath}, \hat{\jmath}, \hat{k}$ be the unit vectors along the three positive coordinate axes. Let

$$\overrightarrow{a} = 3\hat{\imath} + \hat{\jmath} - \hat{k}, \tag{6}$$

$$\overrightarrow{b} = \hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}, \quad b_2, b_3 \in \mathbb{R}, \tag{7}$$

$$\overrightarrow{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}, \quad c_1, c_2, c_3 \in \mathbb{R} \tag{8}$$

be three vectors such that $b_2 b_3 > 0$, $\overrightarrow{a} \cdot \overrightarrow{b} = 0$, and

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix} =$$

$\begin{pmatrix} 3 - c_1 \\ 1 - c_2 \\ -1 - c_3 \end{pmatrix}$ Then, which of the following statements is/are TRUE?

(a) $\overrightarrow{a} \cdot \mathbf{c} = 0$

(b) $\overrightarrow{b} \cdot \overrightarrow{c} = 0$

(c) $|\overrightarrow{b}| > \sqrt{10}$

(d) $|\overrightarrow{c}| \leq \sqrt{11}$

14. For $x \in \mathbb{R}$, let the function $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + 12y = \cos\left((\frac{\pi}{12}x)\right), \quad y(0) = 0. \tag{9}$$

Then, which of the following statements is/are TRUE?

(a) $y(x)$ is an increasing function.

(b) $y(x)$ is a decreasing function.

(c) There exists a real number β such that the line $y = \beta$ intersects the curve $y = y(x)$ at infinitely many points.

(d) $y(x)$ is a periodic function.

15. Consider 4 boxes, where each box contains 3 red balls and 2 blue balls. Assume that all 20 balls are distinct. In how many different ways can 10 balls be chosen from these 4 boxes so that from each box at least one red ball and one blue ball are chosen?

(a) 21816

(b) 85536

(c) 12096

(d) 156816

16. If $M = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$ then which of the following matrices is equal to M^{2022}?

(a) $(A) \begin{pmatrix} 3034 & 3033 \\ -3033 & -3032 \end{pmatrix}$

(b) $(B) \begin{pmatrix} 3034 & -3033 \\ 3033 & -3032 \end{pmatrix}$

(c) $(C) \begin{pmatrix} 3033 & 3032 \\ -3032 & -3031 \end{pmatrix}$

(d) $(D) \begin{pmatrix} 3032 & 3031 \\ -3031 & -3030 \end{pmatrix}$

17. Suppose that:
Box-I contains 8 red, 3 blue, and 5 green balls.
Box-II contains 24 red, 9 blue, and 15 green balls.
Box-III contains 1 blue, 12 green, and 3 yellow balls.

Box-IV contains 10 green, 16 orange, and 6 white balls.
A ball is chosen randomly from Box-I; call this ball b. If b is red, then a ball is chosen randomly from Box-II. If b is blue, then a ball is chosen randomly from Box-III. If b is green, then a ball is chosen randomly from Box-IV.The conditional probability of the event "one of the chosen balls is white" given that the event "at least one of the chosen balls is green" has happened, is equal to:

(a) $(A)\frac{15}{256}$
(b) $(B)\frac{3}{16}$
(c) $(C)\frac{5}{52}$
(d) $(D)\frac{1}{8}$

18. For a positive integer n, define: f(n) = n + $\frac{16+5n-3n^2}{4n+3n^2}+\frac{32+n-3n^2}{8n+3n^2}+\frac{48-3n-3n^2}{12n+3n^2}+\cdots+\frac{25n-7n^2}{7n^2}$ Then, the value of $\lim_{n\to\infty} f(n)$ isequal to:

(a) $(A)3+\frac{4}{3}\log_e 7$
(b) $(B)4-\frac{3}{4}\log_e \frac{7}{3}$
(c) $(C)4-\frac{4}{3}\log_e \frac{7}{3}$
(d) $(D)3+\frac{3}{4}\log_e 7$