Module-2: Series Expansion and Multivariable Calculus
Taylor's and Maclaurin's theorems for function of one variable (statement only) – problems on Maclaurin's Series. Evaluation of Indeterminate forms - L'Hospital's rule –Problems. Partial differentiation, total derivative - differentiation of composite functions. Jacobian and problems. Maxima and minima for a function of two variables. Problems. **Self-study:** Euler's Theorem and problems. Method of Lagrange's undetermined multipliers with single constraint. **(RBT Levels: L1, L2 and L3)**

L1- Taylor's and Maclaurin's series expansion for one variable – problems

Recall:

1. What is a sequence.
2. What is a series.
3. What are the different types of progressions.
4. When we say a function is continuous at any given point.

Taylor's theorem: If i) $f(x)$ and its first $(n-1)$ derivatives be continuous in the interval $[a,\ a+h]$, and

ii) n^{th} derivative of $f(x)$ exists for every values of x in $(a,\ a+h)$, then there is at least one number θ in $(0,\ 1)$ such that,

$$f(a+h) = f(a) + hf\,'(a) + \frac{h^2}{2!}f\,''(a) + \cdots\cdots\cdots + \frac{h^n}{n!}f^n\,(a+\theta h)$$

If $a = 0$ then the Taylor's theorem is called Maclaurin's theorem.

Taylor's series: Expansion of $f(x)$ about $x = a$ (or in powers of $(x-a)$) is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f'^{v}(a) + \cdots$$

Or $$y = y(a) + y_1(a)(x-a) + \frac{y_2(a)}{2!}(x-a)^2 + \frac{y_3(a)}{3!}(x-a)^3 + \frac{y_4(a)}{4!}(x-a)^4 + \cdots$$

If $a = 0$ then series is called **Maclaurin's series** i.e.

Expansion of $f(x)$ about $x = 0$ (or in powers of x) is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f'^{v}(0) + \cdots$$

Or $$y = y(0) + y_1(0)x + \frac{y_2(0)}{2!}x^2 + \frac{y_3(0)}{3!}x^3 + \frac{y_4(0)}{4!}x^4 + \cdots$$

Examples:

1. Expand $y = \sin x$ in powers of $(x - \frac{\pi}{2})$.

Clearly $a = \frac{\pi}{2}$ and $y = \sin x$, $y_1 = \cos x$, $y_2 = -\sin x$, $y_3 = -\cos x$, $y_4 = \sin x$ and so on.

$$\therefore\ y\left(\frac{\pi}{2}\right) = 1,\quad y_1\left(\frac{\pi}{2}\right) = 0,\quad y_2\left(\frac{\pi}{2}\right) = -1,\quad y_3\left(\frac{\pi}{2}\right) = 0,\quad y_4\left(\frac{\pi}{2}\right) = 1 \ \ \ldots\ldots..$$

Substituting in the Taylor's formula

$$y = y(a) + y_1(a)(x-a) + \frac{y_2(a)}{2!}(x-a)^2 + \frac{y_3(a)}{3!}(x-a)^3 + \frac{y_4(a)}{4!}(x-a)^4 + \cdots$$

$$\sin x = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} - \cdots$$

2. Find Maclaurin's series of a) e^x b) $\cos x$ c) $\sin x$ d) $\tan x$ e) $\cosh x$ f) $\sinh x$ g) $\log(1+x)$ h) $\log \sec x$ i) $e^{\sin x}$ j) $\tan^{-1} x$ k) $\sqrt{(1+\sin 2x)}$.

Solutions:

a) $y = e^x = y_1 = y_2 = y_3 = \cdots$ And hence $y(0) = 1 = y_1(0) = y_2(0) = y_3(0) = y_4(0) =$ ……..

Maclaurin's series is $y = y(0) + y_1(0)x + \frac{y_2(0)}{2!}x^2 + \frac{y_3(0)}{3!}x^3 + \frac{y_4(0)}{4!}x^4 + \cdots$

$$\Longrightarrow \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

b) $y = \cos x$, $y_1 = -\sin x$, $y_2 = -\cos x$, $y_3 = \sin x$, $y_4 = \cos x$ ………

$y(0) = 1$, $y_1(0) = 0$, $y_2(0) = -1$, $y_3(0) = 0$, $y_4(0) = 1$, ……..

$$\Longrightarrow \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

c) $y = \sin x \Longrightarrow y_1 = \cos x$, $y_2 = -\sin x$, $y_3 = -\cos x$, $y_4 = \sin x$ ………

$y(0) = 0$, $y_1(0) = 1$, $y_2(0) = 0$, $y_3(0) = -1$, $y_4(0) = 0$, ……..

$$\Longrightarrow \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

d) $y = \tan x \Longrightarrow y_1 = \sec^2 x = 1 + y^2$, $\Longrightarrow$ $y(0) = 0$, $y_1(0) = 1$

$y_2 = 2yy_1$, $\Longrightarrow$ $y_2(0) = 0$

$y_3 = 2yy_2 + 2y_1^2$, $\Longrightarrow$ $y_3(0) = 2$

$y_4 = 2yy_3 + 6y_1y_2$, $\Longrightarrow$ $y_4(0) = 0$

$y_5 = 2yy_4 + 8y_1y_3 + 6y_2^2$, $\Longrightarrow$ $y_5(0) = 16.$ ……

$$\Longrightarrow \quad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

e) $y = \cosh x$ f) $\sinh x$

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$$\Longrightarrow \quad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$ and $\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

g) $y = \log(1+x) \Longrightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$\therefore$ $y(0) = 0$, $y_1(0) = 1$, $y_2(0) = -1$, $y_3(0) = 2$, $y_4(0) = -6$, ……

$$\Longrightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots\cdots$$

h) $y = \log \sec x \Longrightarrow y_1 = \tan x$ $\Longrightarrow$ $y(0) = 0$, $y_1(0) = 0$

$y_2 = \sec^2 x = 1 + y_1^2$, $\Longrightarrow$ $y_2(0) = 1$

$$y_3 = 2y_1y_2\,, \qquad \Rightarrow \quad y_3(0) = 0$$
$$y_4 = 2y_1y_3 + 2y_2^2\,, \qquad \Rightarrow \quad y_4(0) = 2$$
$$y_5 = 2y_1y_4 + 6y_2y_3, \qquad \Rightarrow \quad y_5(0) = 0$$
$$y_6 = 2y_1y_5 + 8y_2y_4 + 6y_3^2, \qquad \Rightarrow \quad y_6(0) = 16. \qquad \cdots\cdots$$
$$\therefore \log\sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}\cdots$$

i) $y = e^{\sin x} \;\Rightarrow y_1 = y\cos x \qquad \Rightarrow \quad y(0) = 1, \qquad y_1(0) = 1$

$$y_2 = y_1\cos x - y\sin x, \qquad \Rightarrow \quad y_2(0) = 1$$
$$y_3 = y_2\cos x - 2y_1\sin x - y_1 \qquad \Rightarrow \quad y_3(0) = 0$$
$$y_4 = y_3\cos x - 3y_2\sin x - 2y_1\cos x - y_2\,, \qquad \Rightarrow \quad y_4(0) = -3$$
$$\Rightarrow \; e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}\cdots\cdots$$

j) $y = \tan^{-1}x \;\Rightarrow y_1 = \frac{1}{1+x^2}$ or $(1+x^2)y_1 = 1 \qquad \Rightarrow \quad y(0) = 0, \qquad y_1(0) = 1$

$$(1+x^2)y_2 + 2xy_1 = 0 \qquad \Rightarrow y_2(0) = 0$$
$$(1+x^2)y_3 + 4xy_2 + 2y_1 = 0 \qquad \Rightarrow y_3(0) = -2$$
$$(1+x^2)y_4 + 6xy_3 + 6y_2 = 0 \qquad \Rightarrow y_4(0) = 0$$
$$(1+x^2)y_5 + 8xy_4 + 12y_3 = 0 \qquad \Rightarrow y_5(0) = 24$$
$$\Rightarrow \; \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5}\cdots\cdots$$

k) $y = \sqrt{(1+\sin 2x)} = \sqrt{(\sin x + \cos x)^2} = \sin x + \cos x$

$y_1 = \cos x - \sin x$, $y_2 = -\sin x - \cos x$, $y_3 = -\cos x + \sin x,$ $y_4 = \sin x + \cos x.$

$$\Rightarrow y(0) = 1,\;\; y_1(0) = 1,\;\; y_2(0) = -1,\;\; y_3(0) = -1,\;\; y_2(0) = -1$$
$$\therefore \sqrt{(1+\sin 2x)} = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots$$

Review:

1. What is the general form of Taylor's series expansion of a function $f(x)$ about $x = a$.
2. When is Taylor's series referred to as a Maclaurin's series.
3. What is the general form of Maclaurin's series expansion of a function $f(x)$ about $x = 0$.
4. Give the series expansion of e^x.
5. Express $\sin x$ and $\cos x$ in ascending powers of x.

L2- Indeterminate forms - L'Hospital's rule . Problems

Recall:

1. What is a limit.
2. What is limits of Trigonometric functions.
3. Evaluate: $\lim_{x\to 0}\frac{\sin x}{x}$
4. What is $\lim_{x\to a}\frac{x^n - a^n}{x-a}$
5. Find $\lim_{x\to 0}\frac{a^x - 1}{x}$

Indeterminate forms:

$\left(\frac{0}{0}\right)$ form: If $f(a) = 0 = g(a)$ then $\lim_{x\to a}\frac{f(x)}{g(x)}$ is called $\frac{0}{0}$ form.

L'Hospital's rule: If $f(a) = 0 = g(a)$ then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided $f'(x), g'(x)$ exists in the neighborhood of $x = a$ and $g'(x) \neq 0$.

Note: Forms 1^∞, $\infty^0, 0^0$ can be reducible to form $\left(\frac{0}{0}\right)$ or form $\frac{\infty}{\infty}$ by taking log.

Examples: Evaluate the following limits.

1. $\underset{x\to\frac{\pi}{2}}{Lt}\ (\sin x)^{\tan x} \quad \cdots\cdots\cdots (1^\infty\ form)$

Solution: Let $k = \underset{x\to\frac{\pi}{2}}{Lt}\ (\sin x)^{\tan x}$

Taking log on both sides,

$$\log k = \underset{x\to\frac{\pi}{2}}{Lt}\ \frac{\log\sin x}{\cot x} \quad \cdots\cdots\cdots \left(\frac{0}{0}\ form\right)$$

$$= \underset{x\to\frac{\pi}{2}}{Lt}\ \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x} = 0$$

And hence $k = e^0 = 1$.

2. $\underset{x\to 0}{Lt}\ \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} \quad \cdots\cdots\cdots (1^\infty\ form)$

Solution: Let $k = \underset{x\to 0}{Lt}\ \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Taking log on both sides,

$$\log k = \underset{x\to 0}{Lt}\ \frac{\log\tan x - \log x}{x^2} \quad \cdots\cdots\cdots \left(\frac{0}{0}\ form\right)$$

$$= \underset{x\to 0}{Lt}\ \frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{2x} = \underset{x\to 0}{Lt}\ \frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{2x}$$

$$= \underset{x\to 0}{Lt}\ \frac{1}{2x}\left[\frac{1}{\sin x\cos x} - \frac{1}{x}\right] = \underset{x\to 0}{Lt}\ \frac{1}{2x}\left[\frac{2}{\sin 2x} - \frac{1}{x}\right]$$

$$= \underset{x\to 0}{Lt}\ \frac{2x - \sin 2x}{2x^2\sin 2x} = \underset{x\to 0}{Lt}\ \frac{2x - \sin 2x}{2x^2 2x\frac{\sin 2x}{2x}} = \underset{x\to 0}{Lt}\ \frac{2x - \sin 2x}{4x^3} \quad \cdots\cdots\cdots \left(\frac{0}{0}\ form\right)$$

$$= \underset{x\to 0}{Lt}\ \frac{2 - 2\cos 2x}{12x^2} = \underset{x\to 0}{Lt}\ \frac{1 - \cos 2x}{6x^2}$$

$$= \underset{x\to 0}{Lt}\ \frac{2\sin^2 x}{6x^2} = \frac{1}{3}$$

And hence $k = e^{\frac{1}{3}}$.

Or $\underset{x\to 0}{Lt}\ \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = \underset{x\to 0}{Lt}\ \left(\frac{x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\cdots}{x}\right)^{\frac{1}{x^2}}$

$$= \underset{x\to 0}{Lt}\ (1 + tx^2)^{\frac{1}{x^2}}, \quad \text{where } t = \frac{1}{3} + \frac{2x^2}{15} + \cdots\cdots$$

$$= \underset{x \to 0}{Lt}\; e^{t} = e^{\frac{1}{3}}\;. \qquad \because \underset{z \to 0}{Lt}\;(1+tz)^{\frac{1}{z}} = e^{z}$$

3. $\underset{x \to 0}{Lt}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ ……… $(1^{\infty}\; form)$

Solution: Let $k = \underset{x \to 0}{Lt}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt}\;\frac{\log \sin x - \log x}{x^2} \quad \cdots\cdots\cdots \left(\frac{0}{0}\; form\right)$$

$$= \underset{x \to 0}{Lt}\;\frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x}$$

$$= \underset{x \to 0}{Lt}\;\frac{1}{2x}\left[\frac{x\cos x - \sin x}{x \sin x}\right] = \underset{x \to 0}{Lt}\;\frac{x\cos x - \sin x}{2x^3} \quad \cdots\cdots\cdots \left(\frac{0}{0}\; form\right)$$

$$= \underset{x \to 0}{Lt}\;\frac{\cos x - x\sin x - \cos x}{6x^2} = \underset{x \to 0}{Lt}\;\frac{-\sin x}{6x} = -\frac{1}{6}\;.$$

And hence $k = e^{-\frac{1}{6}}$.

Or $\underset{x \to 0}{Lt}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = \underset{x \to 0}{Lt}\left(\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots}{x}\right)^{\frac{1}{x^2}}$

$$= \underset{x \to 0}{Lt}\;(1+tx^2)^{\frac{1}{x^2}}\;, \quad \text{where } t = -\frac{1}{3!} + \frac{x^2}{5!} - \cdots\cdots$$

$$= \underset{x \to 0}{Lt}\; e^{t} = e^{-\frac{1}{6}}\;. \qquad \because \underset{z \to 0}{Lt}\;(1+tz)^{\frac{1}{z}} = e^{z}$$

4. $\underset{x \to a}{Lt}\left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ ……… $(1^{\infty}\; form)$

Solution: Let $k = \underset{x \to a}{Lt}\left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Taking log on both sides,

$$\log k = \underset{x \to a}{Lt}\;\frac{\log\left(2 - \frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \quad \cdots\cdots\cdots \left(\frac{0}{0}\; form\right)$$

$$= \underset{x \to a}{Lt}\;\frac{\frac{1}{2 - \frac{x}{a}} \times \left(-\frac{1}{a}\right)}{-\left(\frac{\pi}{2a}\right)\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)} = \frac{2}{\pi}\;.$$

And hence $k = e^{\frac{2}{\pi}}$.

5. $\underset{x \to 0}{Lt}\left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}}$ ……… $(1^{\infty}\; form)$

Solution: Let $k = \underset{x \to 0}{Lt}\left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt}\;\frac{\log(a^x + b^x + c^x) - \log 3}{x} \quad \cdots\cdots\cdots \left(\frac{0}{0}\; form\right)$$

$$= \underset{x \to 0}{Lt} \frac{\frac{1}{a^x+b^x+c^x}(a^x \log a + b^x \log b + c^x \log c)}{1} = \frac{1}{3}\log(abc) = \log\left(\sqrt[3]{abc}\right).$$

And hence $k = \sqrt[3]{abc}$.

6. $\underset{x \to 0}{Lt} (a^x + x)^{\frac{1}{x}}$ ……… $(1^\infty \ form)$

Solution: Let $k = \underset{x \to 0}{Lt} (a^x + x)^{\frac{1}{x}}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt} \frac{\log(a^x+x)}{x} \quad \cdots\cdots\cdots \left(\frac{0}{0} \ form\right)$$

$$= \underset{x \to 0}{Lt} \frac{\frac{1}{a^x+x}(a^x \log a + 1)}{1} = \log a + 1 = \log a + \log e = \log(ea).$$

Hence $k = ea$.

7. $\underset{x \to 0}{Lt} (1 + \sin x)^{\cot x}$ ……… $(1^\infty \ form)$

Solution: Let $k = \underset{x \to 0}{Lt} (1 + \sin x)^{\cot x}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt} \frac{\log(1+\sin x)}{\tan x} \quad \cdots\cdots\cdots \left(\frac{0}{0} \ form\right)$$

$$= \underset{x \to 0}{Lt} \frac{\frac{\cos x}{1+\sin x}}{\sec^2 x} = 1.$$

And hence $k = e^1 = e$.

Review:

1. Which are the indeterminate forms.
2. What is L'Hospital's rule.
3. Find the value of $\underset{x \to \frac{\pi}{2}}{Lt} (\sin x)^{\cot x}$.
4. The value of $\underset{x \to \frac{\pi}{2}}{Lt} (\sec x)^{\cot x}$.
5. What is the value of $\lim_{x \to 0} x^x$.

L3- Problems on Limits

Recall:

1. How to evaluate $\lim_{x \to a} f(x)^{g(x)}$ when $f(a) = g(a) = 0$.
2. For which indeterminate form L'Hospital's rule can be applied?
3. The limit $\underset{x \to 0}{Lt} (1 + x)^{\frac{1}{x}}$ is an example of which indeterminate form?
4. The value of $\underset{x \to 0}{Lt} \left(1 + \frac{2}{x}\right)^x$.
5. What is the value of $\lim_{x \to 0} x^{\frac{1}{x}}$.

8. $\underset{x \to \frac{\pi}{2}}{Lt} (\sec x)^{\cot x} \quad \cdots\cdots\cdots \; (\infty^0 \; form)$

Solution: Let $k = \underset{x \to \frac{\pi}{2}}{Lt} (\sec x)^{\cot x}$

Taking log on both sides,

$$\log k = \underset{x \to \frac{\pi}{2}}{Lt} \frac{\log(\sec x)}{\tan x} \quad \cdots\cdots\cdots \left(\frac{\infty}{\infty} \; form\right)$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{\frac{\sec x \tan x}{\sec x}}{\sec^2 x} = \underset{x \to \frac{\pi}{2}}{Lt} \frac{\tan x}{\sec^2 x}.$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \sin x \cos x = 0.$$

And hence $k = e^0 = 1.$

9. $\underset{x \to 0}{Lt} (\cot x)^{\frac{1}{\log x}} \quad \cdots\cdots\cdots \; (\infty^0 \; form)$

Solution:Let $k = \underset{x \to 0}{Lt} (\cot x)^{\frac{1}{\log x}}$

Taking log on both sides,

$$\log k = \underset{x \to 0}{Lt} \frac{\log(\cot x)}{\log x} \quad \cdots\cdots\cdots \left(\frac{\infty}{\infty} \; form\right)$$

$$= \underset{x \to 0}{Lt} \frac{-\frac{\operatorname{cosec}^2 x}{\cot x}}{\frac{1}{x}} = \underset{x \to 0}{Lt} - \frac{x}{\sin x \cos x}.$$

$$= \underset{x \to 0}{Lt} - \frac{1}{\frac{\sin x}{x} \cos x} = -1.$$

And hence $k = e^{-1} = \frac{1}{e}.$

10. $\underset{x \to \frac{\pi}{2}}{Lt} (\tan x)^{\tan 2x} \quad \cdots\cdots\cdots \; (\infty^0 \; form)$

Solution: Let $k = \underset{x \to \frac{\pi}{2}}{Lt} (\tan x)^{\tan 2x}$

Taking log on both sides,

$$\log k = \underset{x \to \frac{\pi}{2}}{Lt} \frac{\log(\tan x)}{\cot 2x} \quad \cdots\cdots\cdots \left(\frac{\infty}{\infty} \; form\right)$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{\frac{\sec^2 x}{\tan x}}{-2 \operatorname{cosec}^2 2x} = \underset{x \to \frac{\pi}{2}}{Lt} - \frac{\sin^2 2x}{2 \sin x \cos x}.$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} - \frac{\sin^2 2x}{\sin 2x} = \underset{x \to \frac{\pi}{2}}{Lt} - \sin 2x = 0.$$

Hence $k = e^0 = 1.$

11. $\underset{x \to \frac{\pi}{2}}{Lt} (\cos x)^{\frac{\pi}{2} - x} \quad \cdots\cdots\cdots \; (0^0 \; form)$

Solution: Let $k = \underset{x \to \frac{\pi}{2}}{Lt} (\cos x)^{\frac{\pi}{2}-x}$

Taking log on both sides,

$$\log k = \underset{x \to \frac{\pi}{2}}{Lt} \frac{\log(\cos x)}{\frac{1}{\frac{\pi}{2}-x}} \quad \cdots\cdots\cdots \left(\frac{\infty}{\infty} \ form\right)$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} \frac{-\tan x}{\frac{1}{\left(\frac{\pi}{2}-x\right)^2}} = \underset{x \to \frac{\pi}{2}}{Lt} -\frac{\left(\frac{\pi}{2}-x\right)^2}{\cot x}.$$

$$= \underset{x \to \frac{\pi}{2}}{Lt} -\frac{\left(\frac{\pi}{2}-x\right)^2}{\tan\left(\frac{\pi}{2}-x\right)} = \underset{x \to \frac{\pi}{2}}{Lt} -\left(\frac{\pi}{2}-x\right) = 0.$$

Hence $k = e^0 = 1.$

12. $\underset{x \to 1}{Lt} (1-x^2)^{\frac{1}{\log(1-x)}} \quad \cdots\cdots\cdots (0^0 \ form)$

Solution: Let $k = \underset{x \to 1}{Lt} (1-x^2)^{\frac{1}{\log(1-x)}}$

Taking log on both sides,

$$\log k = \underset{x \to 1}{Lt} \frac{\log(1-x^2)}{\log(1-x)} \quad \cdots\cdots\cdots \left(\frac{\infty}{\infty} \ form\right)$$

$$= \underset{x \to 1}{Lt} \frac{\frac{2x}{1-x^2}}{\frac{1}{(1-x)}} = \underset{x \to 1}{Lt} \frac{2x}{1+x} = 1$$

Hence $k = e^1 = e.$

Review:

1. In which indeterminate form is $\lim_{x\to 0} x \log x$?
2. For which indeterminate form L'Hospital's rule can be applied?
3. Determine the value of $\underset{x \to 0}{Lt} \left(1-\frac{1}{x}\right)^x$.
4. What is the value of $\lim_{x\to 0} x^{\frac{1}{x}}$.

L4- Partial differentiation

Recall:

1. Definition of a derivative.
2. What is Algebra of derivative of functions.
3. What is Leibnitz rule.
4. Which are the standard derivatives.

Partial derivatives :

Let $z = f(x, y)$ be a function of two variables in x and y.

The first order partial derivative of z w.r.t. x, denoted by $\frac{\partial z}{\partial x}$ or z_x (i.e. Derivative of z w.r.to x keeping 'y' fixed).

Similarly $\frac{\partial z}{\partial y}$ or z_y is the derivative of $\boldsymbol{z}$ w.r.to y keeping 'x' fixed.

$$\frac{\partial z}{\partial x} = \underset{h \to 0}{Lt} \; \frac{f(x+h,\; y)-f(x,y)}{h} \quad . \quad \text{And } \frac{\partial z}{\partial y} = \underset{h \to 0}{Lt} \; \frac{f(x,\; y+h)-f(x,y)}{h}$$

Higher order partial derivatives also obtained in the same way.

In all ordinary cases, it can be verified that $z_{xy} = z_{yx}$.

Examples:

1. Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.

Solution: Differentiating z partially w.r.t. x we get,

$$\frac{\partial z}{\partial x} = 3x^2 - 3ay$$

Again differentiating z partially w.r.t. x we get,

$$\frac{\partial^2 z}{\partial^2 x} = 6x.$$

2. If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$.

Solution: Given $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

Differentiating u partially w.r.t. y we get,

$$\frac{\partial u}{\partial y} = x^2 . \frac{1}{1+\left(\frac{y}{x}\right)^2} . \frac{1}{x} - \left[y^2 . \frac{1}{1+\left(\frac{x}{y}\right)^2} . \left(\frac{-x}{y^2}\right) + 2y \tan^{-1}\left(\frac{x}{y}\right) \right]$$

$$= x - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

Again differentiating w.r.t x we get,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 1 - 2y . \frac{1}{1+\left(\frac{x}{y}\right)^2} . \frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

3. If $z = \sin^{-1}\left(\frac{y}{x}\right)$. Verify that $z_{xy} = z_{yx}$.

Solution: Given $z = \sin^{-1}\left(\frac{y}{x}\right)$

Differentiating z partially w.r.t. x we get,

$$z_x = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} . \left(\frac{-y}{x^2}\right) = \frac{-y}{x\sqrt{x^2-y^2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating z partially w.r.t. y we get,

$$z_y = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} . \left(\frac{1}{x}\right) = \frac{1}{\sqrt{x^2-y^2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Differentiating (2) partially w.r.t. x we get,

$$z_{xy} = \frac{-1}{2}(x^2-y^2)^{\frac{-3}{2}} . 2x = \frac{-x}{(x^2-y^2)^{\frac{3}{2}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Differentiating (1) partially w.r.t. y we get,

$$z_{yx} = \frac{-1}{x}\left(\frac{\sqrt{x^2-y^2}.1 - y.\frac{-2y}{2\sqrt{x^2-y^2}}}{x^2-y^2}\right) = \frac{-x}{(x^2-y^2)^{\frac{3}{2}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

From (3) and (4) we have $z_{xy} = z_{yx}$.

4. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -9(x + y + z)^{-2}$.

Solution: Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$.

Consider $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$ …………(1)

$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} . 3x^2 - 3yz$ ………..(2)

$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} . 3y^2 - 3xz$ …………(3)

$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} . 3z^2 - 3xy$ ..………(4)

Adding (2), (3) and (4) we get,

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz} = \frac{3}{x+y+z}$ ……………(5)

(5) in (1) , $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) . \frac{3}{x+y+z} = \frac{-9}{(x+y+z)^2}$.

5. If $u = f(r)$, where $r^2 = x^2 + y^2 + z^2$. Then show that $u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r)$.

Solution: $u = f(r)$ where r is a function of x, y, z.

$\therefore u_x = f^1(r)\frac{\partial r}{\partial x} = f^1(r)\frac{x}{r}$ $\left(\because r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}\right)$

$u_{xx} = f^1(r)\left(\frac{r.1 - x\frac{\partial r}{\partial x}}{r^2}\right) + f^{11}(r)\frac{\partial r}{\partial x}\frac{x}{r}$ (Using Chain rule)

$= \frac{f^1(r)}{r^3}(r^2 - x^2) + f^{11}(r)\frac{x^2}{r^2}$

Similarly $u_{yy} = \frac{f^1(r)}{r^3}(r^2 - y^2) + f^{11}(r)\frac{y^2}{r^2}$

$u_{zz} = \frac{f^1(r)}{r^3}(r^2 - z^2) + f^{11}(r)\frac{z^2}{r^2}$

Then $u_{xx} + u_{yy} + u_{zz} = \frac{f^1(r)}{r^3}[(r^2 - x^2) + (r^2 - y^2) + (r^2 - z^2)] + \frac{f^{11}(r)}{r^2}(x^2 + y^2 + z^2)$

$= \frac{f^1(r)}{r^3}(3r^2 - r^2) + f^{11}(r)$

$= \frac{2}{r} f'(r) + f''(r)$

6. If $u = e^{ax+by} f(ax - by)$, show that $bu_x + au_y = 2abu$.

Solution: $u = e^{ax+by} f(ax - by)$

$\Longrightarrow u_x = ae^{ax+by} f'(ax - by) + ae^{ax+by} f(ax - by)$

$= ae^{ax+by} f'(ax - by) + au$

$\Longrightarrow bu_x = abe^{ax+by} f'(ax - by) + abu$

And $u_y = -be^{ax+by} f'(ax - by) + be^{ax+by} f(ax - by)$

$= -be^{ax+by} f'(ax - by) + bu$

$\Longrightarrow au_y = -abe^{ax+by} f'(ax - by) + abu$

$\therefore bu_x + au_y = 2abu$

Review:

1. For $f(x,y) = e^{xy}$ compute the partial derivative of f with respect to y.
2. Find u_x, u_y when $u = x^y$.
3. Given $f(x,y) = x^2 + y^2$,evaluate f_{xx}, f_{yy} and f_{xy} .
4. Let $z = r\cos\theta$ verify $z_{r\theta} = z_{\theta r}$.
5. If $u = e^{x+y+z}$ then find $u_x + u_y + u_z$

T1-Problems on Maclaurin's series and Partial differentiation

1. Using Maclaurin's series expand the following functions:

   1. $\log\sqrt{\frac{1+x}{1-x}}$ 2. $\frac{x}{\sin x}$ 3. $\sec x$ 4. $\log(1+\sin x)$ 5. $\log(1+e^x)$ 6. $e^x\cos x$ 7. $e^{x\sin x}$

   8. $\frac{e^x}{e^x+1}$ 9. $\sin x\cosh x$

2. If $u = x^y$, show that $\frac{\partial^3 u}{\partial x^2\partial y} = \frac{\partial^3 u}{\partial x\partial y\partial x}$.
3. If $z(x+y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.
4. If $u = f(r)$, where $r^2 = x^2 + y^2$. Then show that $u_{xx} + u_{yy} = f''(r) + \frac{1}{r}f'(r)$.
5. If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.
6. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x\partial y} = -(x\log\, ex)^{-1}$.

L5- Total derivative - differentiation of composite functions

Recall:

1. What is a composite function.
2. How you describe partial derivative?
3. The temperature T(x,y) in degrees Celsius at a point (x,y) on a metal plate is given by $\mathrm{T(x,y)} = 3\mathrm{x}^2 - 2\mathrm{xy} + \mathrm{y}^2$. Find the rate of change of temperature with respect to y at the point $(2,1)$.
4. A fluid flows through a pipe with velocity function $\mathrm{v(x,y)} = \mathrm{x}^2 - \mathrm{xy} + \mathrm{y}^2$, where x and y are spatial coordinates in the pipe. Calculate the rate of change of the velocity with respect to x at the point $(1,1)$.

Total derivatives:

1. If $u = f(x,y)$ $and\ x = g(t),\ \ y = h(t)\ \ then$ $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.
2. If $f(x,y) = constant$, then $\frac{dy}{dx} = -\frac{f_x}{f_y}$.
3. If $u = f(x,y)$ subject to $\varphi(x,y) = c$. Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$, where $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$.
4. If $u = f(r,s,t)$ where $r, s\ and\ t\ are\ functions\ of\ (x,\ y,\ z)$, then by Chain rule

   $u_x = u_r r_x + u_s s_x + u_t t_x$, $u_y = u_r r_y + u_s s_y + u_t t_y$ and $u_z = u_r r_z + u_s s_z + u_t t_z$.

Problems:

1. If $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$ and $\mathrm{y} = \mathrm{t}^2$ find $\frac{du}{dt}$ $as\ a function\ of\ t.$

Solution: $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)e^t - \frac{x}{y^2}\cos\left(\frac{x}{y}\right)2t$

$$= \frac{1}{\mathrm{t}^2}\cos\left(\frac{e^t}{\mathrm{t}^2}\right)e^t - \frac{e^t}{t^4}\cos\left(\frac{e^t}{\mathrm{t}^2}\right)2t$$

$$= e^t\cos\left(\frac{e^t}{\mathrm{t}^2}\right)\left[\frac{1}{\mathrm{t}^2} - \frac{2}{\mathrm{t}^3}\right]$$

2. If x increases at the rate of 2 cm/sec at the instant when $x = 3\ cm.$ and $y = 1\ cm.$, at what rate must y changing in order that the function $2xy - 3x^2y$ shell be neither increasing nor decreasing?

Solution: Let $u = 2xy - 3x^2y$, given that $\frac{dx}{dt} = 2$, $\frac{du}{dt} = 0$, $x = 3$ and $y = 1$.

So that $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt}$

$\Longrightarrow \quad 0 = 2(2 - 18) + (6 - 27)\frac{dy}{dt} \quad \Longrightarrow \frac{dy}{dt} = -\frac{32}{21}$ cm/sec.

Thus y is decreasing at the rate of $\frac{32}{21}$ cm/sec.

3. If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$ find $\frac{du}{dx}$.

Solution: If $u = f(x,y)$ subject to $\varphi(x,y) = c$. Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$, where $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$.

Given that $u = x \log xy$, $\varphi(x,y) = x^3 + y^3 + 3xy$

Clearly $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} = -\frac{3x^2+3y}{3y^2+3x} = -\frac{x^2+y}{y^2+x}$, $\quad \frac{\partial u}{\partial x} = \log xy + 1$, $\quad \frac{\partial u}{\partial y} = \frac{x}{y}$.

Hence $\frac{du}{dx} = \log xy + 1 - \frac{x(x^2+y)}{y(y^2+x)}$

4. If $u = \sqrt{x^2 + y^2}\ and\ x^3 + y^3 + 3axy = 5a^2, find\ \frac{du}{dx}$ when $x = y = a$.

Solution: If $u = f(x,y)$ subject to $\varphi(x,y) = c$. Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$, where $\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}$.

Hear $u = \sqrt{x^2 + y^2}\ and\ \varphi = x^3 + y^3 + 3axy = 5a^2$

$\Longrightarrow \frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} = -\frac{3x^2+3ay}{3y^2+3ax} = -\frac{x^2+ay}{y^2+ax} = -1$ at $x = y = a$

Then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\left(-\frac{x^2+ay}{y^2+ax}\right)$

$= \frac{a}{\sqrt{2a^2}} - \frac{a}{\sqrt{2a^2}} = 0$, at $x = y = a$.

5. If $u = f(y - z, z - x, x - y), then\ prove\ that\ u_x + u_y + u_z = 0.$

Solution: Let $r = y - z$, $\quad s = z - x$, $\quad t = x - y$

Then $r_x = 0$, $r_y = 1$, $r_z = -1$, $s_x = -1$, $s_y = 0$, $s_z = 1$, $t_x = 1$, $t_y = -1$, $t_z = 0$.

If $u = f(r,s,t)$ where $r, s\ and\ t\ are\ functions\ of\ (x,\ y,\ z)$, then by Chain rule

$u_x = u_r r_x + u_s s_x + u_t t_x$, $\quad u_y = u_r r_y + u_s s_y + u_t t_y$ and $\quad u_z = u_r r_z + u_s s_z + u_t t_z$.

$\Longrightarrow u_x = 0 - u_s + u_t$, $\quad u_y = u_r + 0 - u_t$ and $\quad u_z = -u_r + u_s + 0$.

$\therefore u_x + u_y + u_z = -u_s + u_t + u_r - u_t - u_r + u_s = 0.$

6. If $u = f(2x - 3y, 3y - 4z, 4z - 2x), then\ find\ the\ value\ of\ \frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z$.

Solution: Let $r = 2x - 3y$, $\quad s = 3y - 4z$, $\quad t = 4z - 2x$

Then $r_x = 2$, $r_y = -3$, $r_z = 0$, $s_x = 0$, $s_y = 4$, $s_z = -4$, $t_x = -2$, $t_y = 0$, $t_z = 4$.

If $u = f(r, s, t)$ where $r, s\ and\ t\ are\ functions\ of\ (x,\ y,\ z)$, then by Chain rule

$u_x = u_r r_x + u_s s_x + u_t t_x$, $u_y = u_r r_y + u_s s_y + u_t t_y$ and $u_z = u_r r_z + u_s s_z + u_t t_z$.

$\Rightarrow u_x = 2u_r + 0 - 2u_t$, $u_y = -3u_r + 3u_s + 0$ and $u_z = 0 - 4u_s + 4u_t$.

$$\therefore \frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = u_r - u_t - u_r + u_s - u_s + u_t = 0.$$

7. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

Solution: Let $r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$, $s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$.

Then $r_x = -\frac{1}{x^2}$, $r_y = \frac{1}{y^2}$, $r_z = 0$, $s_x = -\frac{1}{x^2}$, $s_y = 0$, $s_z = \frac{1}{z^2}$,

If $u = f(r, s)$ where $r,\ and\ s\ are\ functions\ of\ (x,\ y,\ z)$, then by Chain rule

$u_x = u_r r_x + u_s s_x$, $u_y = u_r r_y + u_s s_y$ and $u_z = u_r r_z + u_s s_z$

$\Rightarrow u_x = -\frac{1}{x^2}u_r - \frac{1}{x^2}u_s$, $u_y = \frac{1}{y^2}u_r + 0$ and $u_z = 0 + \frac{1}{z^2}u_s$

$\Rightarrow x^2\frac{\partial u}{\partial x} = -u_r - u_s$, $y^2\frac{\partial u}{\partial y} = u_r$ and $z^2\frac{\partial u}{\partial z} = u_s$.

Therefore $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

Review:

1. What is Composite function?
2. When to use chain rule and total derivative rule?
3. If $u(x, y) = c$ is an implicit function then express $\frac{dy}{dx}$ in terms of its partial derivatives.
4. The height of a particle is given by $z = x^2 + y^2$, where $x = \cos t$ and $y = \sin t$. Find the rate of change of height with respect to time t.

L6- Jacobian and problems

Recall:

1. If $u(x, y) = x^2y^2$ where $x = e^t$ and $y = e^{-t}$ find the total derivative $\frac{du}{dt}$.
2. If $z = x^2 + y^2$ and $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial z}{\partial r}$.
3. If $u = f(r, s, t)$ where $r, s\ and\ t\ are\ functions\ of\ (x,\ y,\ z)$ then give u_x, u_y and u_z using chain rule.
4. Find $\frac{dy}{dx}$,given $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

Jacobian:

$$\boldsymbol{J} = \frac{\partial(\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w})}{\partial(\boldsymbol{x},\ \boldsymbol{y},\ \boldsymbol{z})} = \begin{vmatrix} \boldsymbol{u_x} & \boldsymbol{u_y} & \boldsymbol{u_z} \\ \boldsymbol{v_x} & \boldsymbol{v_y} & \boldsymbol{v_z} \\ \boldsymbol{w_x} & \boldsymbol{w_y} & \boldsymbol{w_z} \end{vmatrix}.$$

Problems:

1. If $x = r\cos\theta$, $y = r\sin\theta$, then verify that $JJ' = 1$.

Solution: $x = r\cos\theta$, $y = r\sin\theta$

$$J=\frac{\partial(x,\ y)}{\partial(r,\ \theta)}=\begin{vmatrix}x_r & x_\theta\\ y_r & y_\theta\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r(\cos^2\theta+\sin^2\theta)=r\,.$$

$x=r\cos\theta\,,\quad y=r\sin\theta$

$\Longrightarrow\quad r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$

And $$J'=\frac{\partial(r,\ \theta)}{\partial(x,\ y)}=\begin{vmatrix}r_x & r_y\\ \theta_x & \theta_y\end{vmatrix}=\begin{vmatrix}\frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}}\\ \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) & \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)\end{vmatrix}$$

$$=\begin{vmatrix}\frac{x}{r} & \frac{y}{r}\\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2}\end{vmatrix}=\frac{x^2}{r(x^2+y^2)}+\frac{y^2}{r(x^2+y^2)}=\frac{1}{r}$$

$\therefore\ JJ'=1.$

2. If $x=r\cos\varphi,\ \ y=r\sin\varphi\,,\ \ z=z$, then find $\boldsymbol{J}=\frac{\boldsymbol{\partial(x\ \ y,\ \ z)}}{\boldsymbol{\partial(r,\ \ \varphi,\ \ z)}}$

Solution: $$J=\frac{\partial(x\ \ y,\ z)}{\partial(r,\ \varphi,\ z)}=\begin{vmatrix}x_r & x_\varphi & x_z\\ y_r & y_\varphi & y_z\\ z_r & z_\varphi & z_z\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi & 0\\ \sin\varphi & r\cos\varphi & 0\\ 0 & 0 & 1\end{vmatrix}=r\cos^2\varphi+r\sin^2\varphi=r\ .$$

3. If $x=u(1+v),\ \ y=v(1+u), show\ that\ \frac{\boldsymbol{\partial(x,\ y)}}{\boldsymbol{\partial(u,\ v)}}=1+\mathrm{u}+\mathrm{v}\,.$

Solution: Given that $x=u(1+v), y=v(1+u)$

$$\frac{\partial(x,\ y)}{\partial(u,\ v)}=\begin{vmatrix}x_u & x_v\\ y_u & y_v\end{vmatrix}=\begin{vmatrix}(1+v) & u\\ v & (1+u)\end{vmatrix}=(1+v)(1+u)-uv=1+u+v+uv-uv=1+u+v.$$

4. If $u=x+y+z,\ v=x^2+y^2+z^2\ and\ w=xy+yz+zx, then\ find\ \ \frac{\boldsymbol{\partial(u,\ v,\ w)}}{\boldsymbol{\partial(x,\ y,\ z)}}$.

Solution: Given that, $u=x+y+z,\ v=x^2+y^2+z^2\ and\ w=xy+yz+zx$

$$\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}=\begin{vmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1 & 1\\ 2x & 2y & 2z\\ y+z & x+z & x+y\end{vmatrix}=2\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ y+z & x+z & x+y\end{vmatrix}$$

$$=2\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ x+y+z & x+y+z & x+y+z\end{vmatrix}$$

$$=2(x+y+z)\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ 1 & 1 & 1\end{vmatrix}=0.$$

5. If $u=x^2-2y^2$, $v=2x^2-y^2$ and $=r\cos\theta\,,\ \ y=r\sin\theta$, then show that $\frac{\partial(u,\ v)}{\partial(r,\ \theta)}=6r^3\sin 2\theta$.

Solution: Since $$\frac{\partial(u,\ v)}{\partial(r,\ \theta)}=\frac{\partial(u,\ v)}{\partial(x,\ y)}\times\frac{\partial(x,\ y)}{\partial(r,\ \theta)}=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}\times\begin{vmatrix}x_r & x_\theta\\ y_r & y_\theta\end{vmatrix}$$

$$=\begin{vmatrix}2x & -4y\\ 4x & -2y\end{vmatrix}\times\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=12xyr=12r^3\sin\theta\cos\theta=6r^3\sin 2\theta\,.$$

6. Prove that $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ for $x = r\cos\theta$, $y = r\sin\theta$.

Solution: Clearly $\frac{\partial x}{\partial r} = \cos\theta$ and

Since, $r = \sqrt{x^2 + y^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\therefore \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$$

7. If $z = f(x,y)$ and $x = r\cos\theta$, $y = r\sin\theta$ then show that, $(z_x)^2 + (z_y)^2 = (z_r)^2 + \frac{1}{r^2}(z_\theta)^2$.

Solution: $z_r = z_x x_r + z_y y_r = z_x\cos\theta + z_y\sin\theta \qquad \cdots\cdots\cdots(1)$

And $z_\theta = z_x x_\theta + z_y y_\theta = -rz_x\sin\theta + \mathrm{r}z_y\cos\theta$

$$\Longrightarrow \frac{1}{r} z_\theta = -z_x\sin\theta + z_y\cos\theta \qquad \cdots\cdots\cdots(2)$$

$$(1)^2 + (2)^2 \quad \Longrightarrow (z_r)^2 + \frac{1}{r^2}(z_\theta)^2 = (z_x)^2\cos^2\theta + (z_y)^2\sin^2\theta + 2z_x z_y\cos\theta\sin\theta$$

$$+(z_x)^2\sin^2\theta + (z_y)^2\cos^2\theta - 2z_x z_y\cos\theta\sin\theta$$

$$= (z_x)^2 + (z_y)^2.$$

Review:

1. If $u = u(x,y)$, $v = v(x,y)$ then what is $J\left(\frac{u,v}{x,y}\right)$? ?
2. What is null Jacobian ?
3. Suppose $\frac{\partial(u,\ v)}{\partial(x,\ y)} = 0$ where u, v are functions of x and y. What inference about u and v can we get?
4. If $J\left(\frac{u,v}{x,y}\right) = \frac{1}{x+y}$ then what is $J'\left(\frac{x,y}{u,v}\right)$?
5. Given a transformation $x = r\cos\theta$, $y = r\sin\theta$. Find the Jacobian $\frac{\partial(x,\ y)}{\partial(r,\theta)}$.s

L7- Maxima and minima for a function of two variables.

Recall:

1. If $u = x + y$ and $v = x - y$, determine the value of the Jacobian $\frac{\partial(x,\ y)}{\partial(u,v)}$.
2. Given $u = e^{x+y}$, $v = e^{-x-y}$ compute Jacobian J of u, v with respect to x, y.
3. If the given transformations are associated to each other, then what will be the Jacobian determinant value?
4. What does the Jacobian matrix represent in a multivariable transformation?
5. Determine Jacobian of the transformation $x = u + v$ and $y = \frac{1}{u+v}$.

Maxima and minima of functions of two variables:

1. $f(x,\ y)$ is stationary at $(a,\ b)$ i.e. $f(a,\ b)$ is the stationary value of f if $f_x = 0 = f_y$ at $(a,\ b)$.
2. $f(x,\ y)$ is maximum at $(a,\ b)$ i.e. $f(a,\ b)$ is the maximum value of f

   If at $(a,\ b)$ i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 > 0$ iii) $f_{xx} < 0$.
3. $f(x,\ y)$ is minimum at $(a,\ b)$ i.e. $f(a,\ b)$ is the minimum value of f

   If at $(a,\ b)$ i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 > 0$ iii) $f_{xx} > 0$.

4. $(a,\ b)$ is said to be saddle point of $f(x,\ y)$ if i) $f_x = 0 = f_y$ ii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a,\ b)$.
5. If $f_x = 0 = f_y$ and $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a,\ b)$, then by discussion find maxima and minima.

Examples:

1. Examine the function $f(x,\ y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ for extreme values.

Solution: $f(x,\ y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Differentiating partially we get, $f_x = 4x^3 - 4x + 4y$, $f_y = 4y^3 + 4x - 4y$,

$f_{xx} = 12x^2 - 4$, $f_{yy} = 12y^2 - 4$ and $f_{xy} = 4$.

Now for extreme values $f_x = 0$, $f_y = 0$

$\Longrightarrow 4x^3 - 4x + 4y = 0$ and $4y^3 + 4x - 4y = 0$.

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.

Put $y = -x$ in $x^3 - x + y = 0$, we get $x^3 - 2x = 0$

$\Longrightarrow$ $x = 0, \sqrt{2}, -\sqrt{2}$ and corresponding values of y are $0, -\sqrt{2}, \sqrt{2}$.

Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$(\sqrt{2}, -\sqrt{2})$	$20 > 0$	20	4	$384 > 0$	$f(\sqrt{2}, -\sqrt{2}) = -8$ is minimum
$(-\sqrt{2}, \sqrt{2})$	$20 > 0$	20	4	$384 > 0$	$f(-\sqrt{2},\ \sqrt{2}) = -8$ is minimum
$(0, 0)$	$-4 < 0$	-4	4	0	Since $f_{xx}f_{yy} - f_{xy}^2 = 0$ Further investigation is needed.

Clearly $f(0,\ 0) = 0$, $f(0.1,\ 0) = -0.0199$, $f(0.1,\ 0.1) = 0.0002$.

Thus in the neighborhood of $(0,\ 0)$, $f > f(0,\ 0)$ at some points and $f < f(0,\ 0)$ at some points.

Hence $f(0,\ 0)$ is not an extreme value. The point $(0,\ 0)$ is saddle point.

2. Discuss the maxima and minima of $f(x,\ y) = x^3y^2(1 - x - y)$.

Solution: $f(x,\ y) = x^3y^2 - x^4y^2 - x^3y^3$.

Differentiating partially we get, $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$, $f_y = 2x^3y - 2x^4y - 3x^3y^2$,

$f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$, $f_{yy} = 2x^3 - 2x^4 - 6x^3y$ and $f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$.

Now for extreme values $f_x = 0$, $f_y = 0$

$\Longrightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ and $2x^3y - 2x^4y - 3x^3y^2 = 0$.

$\Longrightarrow 3 - 4x - 3y = 0$ and $2 - 2x - 3y = 0$.

Therefore stationary points are $\left(\frac{1}{2},\ \frac{1}{3}\right)$ and $(0,\ 0)$.

Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$\left(\frac{1}{2},\ \frac{1}{3}\right)$	$-\frac{1}{9} < 0$	$-\frac{1}{8}$	$-\frac{1}{12}$	$\frac{1}{144} > 0$	$f\left(\frac{1}{2},\ \frac{1}{3}\right) = \frac{1}{432}$ is maximum
$(0, 0)$	0	0	0	0	Since $f_{xx}f_{yy} - f_{xy}^2 = 0$ Further investigation is needed.

Clearly $f(0,\ 0) = 0$, $f(0.1,\ 0.1) > 0$, $f(-0.1, -0.1) < 0$.

Thus in the neighborhood of $(0,\ 0)$, $f > f(0,\ 0)$ at some points and $f < f(0,\ 0)$ at some points.

Hence $f(0,\ 0)$ is not an extreme value. The point $(0,\ 0)$ is saddle point.

3. Find the maximum and minimum values of the function $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Solution: Given $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Differentiating partially we get, $f_x = 3x^2 + 3y^2 - 30x + 72$, $f_y = 6xy - 30y$,

$f_{xx} = 6x - 30x$, $f_{yy} = 6x - 30$ and $f_{xy} = 6y$.

Now for extreme values $f_x = 0$, $f_y = 0$

$\Rightarrow\ 3x^2 + 3y^2 - 30x + 72 = 0 \ldots\ldots\ldots\ldots.(1)$,$6xy - 30y = 0 \ldots\ldots\ldots\ldots(2)$

Solving (2) we get $y = 0, x = 5$.

Substituting $y = 0, x = 5$ in (1) we get $(4,0), (6,0), (5,1), (5,-1)$.

Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$(4,0)$	-6 $<$ 0	-6	0	$36 > 0$	$f(4,0)$=112 is maximum
$(6,0)$	$6 > 0$	6	0	$36 > 0$	$f(6,0)$= 108 is minimum
$(5,1)$	0	0	6	$-36 < 0$	$f(5,1)$=110
$(5,-1)$	0	0	-6	$-36 < 0$	$f(5,-1)$=110

Therefore, Maximum value is 112 and minimum value is 108.

Review:

1. When we say $(a,\ b)$ is a stationary value for the function$f(x,\ y)$?
2. What is the condition for $f(x,\ y)$ to be maximum at $(a,\ b)$?
3. When $(a,\ b)$ is said to be saddle point of $f(x,\ y)$?
4. If $f(x,\ y) = x^2 + y^2 - x + y$ then find the critical point
5. Given $f(x,\ y) = x^2 + y^2 - 4x + 6y$ then find the minimum value of $f(x,\ y)$.

T2-Problems on Jacobian and Maxima and minima for a function of two variables.

1. If $ux = yz,\ vy = zx,\ \ wz = xy,$ then show that $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = 4$ **.**

2. 19. If $u = x + y + z,\ uv = y + z\ and\ uvw = z, then\ find$ $\frac{\boldsymbol{\partial(x,\ y,\ z)}}{\boldsymbol{\partial(u,\ v,\ w)}}$**.**

3. 20. If $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$ and $z = r\cos\theta$, then find $J = \frac{\partial(x\ \ y,\ z)}{\partial(r,\ \theta,\ \emptyset)}$**.**

4. Find $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$ where $u = x^2 + y^2 + z^2, v = xy + yz + zx, w = x + y + z$.
5. Find the maximum and minimum values of
   a) $x^3 + y^3 - 3x - 12y + 20$
   b) $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$
   c) $xy(a - x - y)$
6. Show that $f(x,y) = x^3 + y^3 - 3xy + 1$ is maximum at (1,1).

T3- Self Study: Euler's Theorem and problems, Method of Lagrange's undetermined multipliers with single constraint.

Homogeneous Function: If a function $f(x,\ y)$ can be expressed in the form of $x^n\varphi\left(\frac{y}{x}\right)$ is called homogeneous of degree n.

Euler's theorem: - If u is a homogenous function of x and y with degree n, then $xu_x + yu_y = nu$.

1. If u is a homogenous function of x and y with degree n,

   then prove that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = n(n-1)u$.

Proof: By Euler's theorem, $xu_x + yu_y = nu \ \cdots\cdots\cdots (1)$

Differentiating 1 w.r.to x partially, $xu_{xx} + u_x + yu_{xy} = nu_x$

$$\Longrightarrow\ xu_{xx} + yu_{xy} = (n-1)u_x.$$

$$\therefore x^2u_{xx} + xyu_{xy} = (n-1)xu_x \cdots\cdots (2).$$

Similarly $y^2u_{yy} + xyu_{xy} = (n-1)yu_y \cdots\cdots (3)$.

Adding 2 and 3 we get, $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = (n-1)\left[xu_x + yu_y\right] = n(n-1)u$.

2. If $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$, then prove that $xu_x + yu_y = 3\tan u$.

Proof: $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right) \Longrightarrow \sin u = \frac{x^2y^2}{x+y} = \frac{x^4(y^2/x^2)}{x(1+y/x)} = x^3\varphi\left(\frac{y}{x}\right)$.

$\therefore \sin u$ is homogenous function of degree 3,

Then by Euler's theorem,

$$x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = 3\sin u$$

$\Longrightarrow \cos u\left(xu_x + yu_y\right) = 3\sin u$. Or $xu_x + yu_y = 3\tan u$.

3. If $u = \frac{x^3y^3}{x^3+y^3}$, then prove that $xu_x + yu_y = 3u$.

Proof: $u = \frac{x^3y^3}{x^3+y^3} = \frac{x^6(y^3/x^3)}{x^3(1+y^3/x^3)} = x^3\varphi\left(\frac{y}{x}\right)$.

$\therefore u$ is homogenous function of degree 3,

Then by Euler's theorem, $xu_x + yu_y = 3u$.

4. If $u = \frac{x^2y^2}{x+y}$, then find the value of $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$.

Solution: If $u = \frac{x^2y^2}{x+y} = \frac{x^4(y^2/x^2)}{x(1+y/x)} = x^3\varphi\left(\frac{y}{x}\right)$.

$\therefore u$ is homogenous function of degree 3,

Then by Euler's theorem,

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = n(n-1)u = 6u\ .$$

Extremum by Lagrange's multiplier method:

To find extremum of $f(x,y,z)$ subject to $\emptyset(x,y,z)=c$, first we write $F=f(x,y,z)+\lambda\emptyset(x,y,z)$

Next we obtain the equations $F_x=0,\ F_y=0,\ F_z=0.$

Then solve the above equations with $\emptyset(x,y,z)=c$.

The valve of x,y,z so obtained will give the stationary value of $f(x,y,z)$.

Example:

1. A rectangular box open at top is to have volume 32 cubic ft. Find the dimension of the box requiring least material for its construction.

Solution: Clearly $f(x,y,z)=xy+2yz+2zx$ (open at top) and $\emptyset(x,y,z)=xyz=32$.

Let $F=xy+2yz+2zx+\lambda xyz$

$$F_x=0,\ F_y=0,\ F_z=0 \Longrightarrow \quad y+2z+\lambda yz=0, \quad \cdots\text{(i)}$$

$$x+2z+\lambda xz=0\ , \quad \cdots\text{(ii)}$$

$$2y+2x+\lambda xy=0\ . \quad \cdots\text{(iii)}$$

$\text{(i)}x-\text{(ii)}y \Longrightarrow 2z(x-y)=0 \quad \Longrightarrow x=y,$

$\text{(ii)}y-\text{(iii)}z \Longrightarrow x(y-2z)=0 \quad \Longrightarrow y=2z.$

$xyz=32 \Longrightarrow 4z^3=32 \Longrightarrow \ z=2,\ x=4,\ y=4.$

Therefore $x=4ft,\ y=4ft,\ z=2ft.$

2. In a plane triangle find the maximum value of $\cos A\ \cos B\ \cos C$.

Solution: Let $x=A,\ y=B,\ z=C.$

Then the question is to find the maximum value of $f=\cos x\ \cos y\ \cos z$ subject to $x+y+z=\pi$

Let $F=\cos x\ \cos y\ \cos z+\lambda(x+y+z)$

$$F_x=0,\ F_y=0,\ F_z=0 \Longrightarrow \quad -\sin x\ \cos y\ \cos z+\lambda=0, \quad \cdots\text{(i)}$$

$$-\cos x\ \sin y\ \cos z+\lambda=0, \quad \cdots\text{(ii)}$$

$$-\cos x\ \cos y\ \sin z+\lambda=0\ . \quad \cdots\text{(iii)}$$

$$\Longrightarrow \lambda=\sin x\ \cos y\ \cos z=\cos x\ \sin y\ \cos z=\cos x\ \cos y\ \sin z$$

$\Longrightarrow \sin x\ \cos y\ =\cos x\ \sin y$ and $\sin y\ \cos z=\ \cos y\ \sin z$

$\Longrightarrow \sin(x-y)=0$ and $\sin(y-z)=0$.

Therefore $x=y=z$ and $x+y+z=\pi \ \Longrightarrow\ x=y=z=\frac{\pi}{3}$.

Hence the maximum value of $\cos A\ \cos B\ \cos C=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$.

3. Find the maximum and minimum distances of the point $(3,4,12)$ from the sphere $x^2+y^2+z^2=4$.

Solution: Let $P(x,y,z)$ be any point on the sphere, and $A\equiv(3,4,12)$.

Then the distance $AP=\sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$

Without loss of generality consider $f=(x-3)^2+(y-4)^2+(z-12)^2$

and $\emptyset = x^2+y^2+z^2=4$.

Let $F=(x-3)^2+(y-4)^2+(z-12)^2+\lambda(x^2+y^2+z^2)$.

$F_x=0,\ F_y=0,\ F_z=0 \Longrightarrow \quad (x-3)+x\lambda=0, \quad \cdots$(i)

$(y-4)+y\lambda=0, \quad \cdots$(ii)

$(z-12)+z\lambda=0$. $\cdots$(iii)

(i)y − (ii)x $\Longrightarrow 4x-3y=0$ and (ii)z − (iii)y $\Longrightarrow 12y-4z=0$

$\Longrightarrow z=3y \quad \&\ x=\frac{3}{4}y$

$x^2+y^2+z^2=4 \quad \Longrightarrow \left(\frac{9}{16}+1+9\right)y^2=4 \quad \Longrightarrow \left(\frac{169}{16}\right)y^2=4$

$\Longrightarrow y^2=\frac{64}{169} \quad \Longrightarrow y=\pm\frac{8}{13}$.

When $y=\frac{8}{13}$, $x=\frac{6}{13}$, $z=\frac{24}{13}$ and when $y=-\frac{8}{13}$, $x=-\frac{6}{13}$, $z=-\frac{24}{13}$.

$$AP=\sqrt{\left(\frac{6}{13}-3\right)^2+\left(\frac{8}{13}-4\right)^2+\left(\frac{24}{13}-12\right)^2}=11.$$

And $$AP=\sqrt{\left(-\frac{6}{13}-3\right)^2+\left(-\frac{8}{13}-4\right)^2+\left(-\frac{24}{13}-12\right)^2}=15.$$

Hence maximum distance is 15 and minimum distance is 11.

4. The temperature T at any point (x,y,z)in space is $T=400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2+y^2+z^2=1$.

Solution: Let $F=xyz^2+\lambda(x^2+y^2+z^2)$.

$F_x=0,\ F_y=0,\ F_z=0 \Longrightarrow \quad yz^2+2x\lambda=0, \quad \cdots$(i)

$xz^2+2y\lambda=0, \quad \cdots$(ii)

$2xyz+2z\lambda=0$. $\cdots$(iii)

(i)y − (ii)x $\Longrightarrow y^2z^2-x^2z^2=0$ and (ii)z − (iii)y $\Longrightarrow xz^3-2xy^2z=0$

$\Longrightarrow y^2=x^2 \quad \&\ z^2=2y^2$

$x^2+y^2+z^2=1 \Longrightarrow \quad y^2+y^2+2y^2=1 \quad \Longrightarrow \quad y^2=\frac{1}{4}$.

$\Longrightarrow \quad x=y=\frac{1}{2}$ and $z^2=\frac{1}{2}$.

Therefore highest temperature is $T=400xyz^2=\frac{400}{8}=50$ units.

Assignment questions:

1. Find the minimum value of $x^2+y^2+z^2$ subject to the condition, i) $xyz=a^3$ ii) $xy+yz+zx=3a^2$.
2. Find the maximum and minimum distances from the origin to the surface $5x^2+6xy+5y^2=8$.
3. Find the dimensions of the rectangular box, open at top, of maximum capacity whose surface is $432\ cm^2$.

Course outcome

- Outline the notion of partial differentiation to calculate rate of change of multivariate functions solve problems related to composite functions, Jacobian and demonstrate using python.

PRACTICE QUESTION BANK

MODULE 2: SERIES EXPANSION AND MULTIVARIABLE CALCULUS

Maclaurin's Series for function of one variable:

1. Using Maclaurin's series expand the following functions:

a. $y = \log \sec x$ b. $\log(1+\sin x)$ c. $\log(1+e^x)$
d. $\tan^{-1} x$ e. $\sqrt{(1+\sin 2x)}$ f. $\cosh x$
g. $e^{x\sin x}$ h. $e^x \cos x$ i. $\frac{e^x}{e^x+1}$
j. $\log\sqrt{\frac{1+x}{1-x}}$ k. $\frac{x}{\sin x}$ l. $\sin x \cosh x$

Evaluation of Indeterminate forms - L'Hospital's rule:

2. Evaluate the following limits:

a. $\underset{x\to\frac{\pi}{2}}{Lt}\ (\sin x)^{\tan x}$ b. $\underset{x\to 0}{Lt}\ \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ c. $\underset{x\to a}{Lt}\ \left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$
d. $\underset{x\to 0}{Lt}\ \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$ e. $\underset{x\to\frac{\pi}{2}}{Lt}\ (\sec x)^{\cot x}$ f. $\underset{x\to 0}{Lt}\ (\cot x)^{\frac{1}{\log x}}$
g. $\underset{x\to\frac{\pi}{2}}{Lt}\ (\tan x)^{\tan 2x}$ h. $\underset{x\to\frac{\pi}{2}}{Lt}\ (\cos x)^{\frac{\pi}{2}-x}$ i. $\underset{x\to 1}{Lt}\ (1-x^2)^{\frac{1}{\log(1-x)}}$
j. $\underset{x\to 0}{Lt}\ (a^x+x)^{\frac{1}{x}}$ k. $\underset{x\to 0}{Lt}\ (1+\sin x)^{\cot x}$ l. $\underset{x\to 0}{Lt}\ \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Partial differentiation, total derivative - differentiation of composite functions. Jacobian:

3. If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$, show that $\frac{\partial^2 u}{\partial x\partial y} = \frac{x^2-y^2}{x^2+y^2}$.
4. If $z = \sin^{-1}\left(\frac{y}{x}\right)$. Verify that $z_{xy} = z_{yx}$.
5. If x increases at the rate of 2 cm/sec at the instant when $x = 3$ cm and $y = 1$ cm. , at what rate must y changing in order that the function $2xy - 3x^2y$ shall neither be increasing nor decreasing?
6. At a given instant the sides of a rectangle are 4ft and 3 ft, and they are increasing at the rate of 1.5ft/sec and 0.5ft/sec respectively. Find the rate at which the area is increasing at that instant.
7. If $u = e^{ax+by}f(ax-by), prove\ that\ bu_x + au_y = 2abu.$
8. If $u = \log(tanx + tany + tanz)$, show that $\sin(2x)u_x + \sin(2y)u_y + \sin(2z)u_z = 2.$
9. If $z(x+y) = x^2+y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$
10. If $u = \log(x^3+y^3+z^3-3xyz)$, show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -9(x+y+z)^{-2}.$
11. If $u = x\log xy$ where $x^3+y^3+3xy = 1$ find $\frac{du}{dx}$.
12. If $u = \sin(x^2+y^2)$ where $a^2x^2+b^2y^2 = c^2$ find $\frac{du}{dx}$.
13. If $u = \sqrt{x^2+y^2}$ and $x^3+y^3+3axy = 5a^2$, find $\frac{du}{dx}$ when $x = y = a$.
14. If $u = f(ax-by, by-cz, cz-ax)$, then show that $\frac{1}{a}u_x + \frac{1}{b}u_y + \frac{1}{c}u_z = 0$.
15. If $u = f(y-z, z-x, x-y)$, then prove that $u_x + u_y + u_z = 0.$

16. If $u = f(2x-3y, 3y-4z, 4z-2x)$, then find the value $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z$.
17. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.
18. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.
19. If $u = \tan^{-1}\left(\frac{y}{x}\right)$, $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$. find $\frac{du}{dt}$ as a function of t.
20. If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(u, v)}{\partial(r, \theta)}$.
21. If $ux = yz$, $vy = zx$, $wz = xy$, then show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$.
22. If $x = r\cos\theta$, $y = r\sin\theta$, then verify that $JJ' = 1$.
23. If $u = x + y + z$, $uv = y + z$ and $uvw = z$, then find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.
24. If $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$ and $z = r\cos\theta$, then find $\frac{\partial(x\ y, z)}{\partial(r, \theta, \emptyset)}$.
25. If $u = x + y + z$, $v = x^2 + y^2 + z^2$ and $w = xy + yz + zx$, $then\ find$ $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.
26. If $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$, then find $J = \frac{\partial(x\ y, z)}{\partial(r, \varphi, z)}$.
27. If $x = u(1+v)$, $y = v(1+u)$, $show\ that$ $\frac{\partial(x, y)}{\partial(u, v)} = 1 + u + v$.
28. If $u = \frac{2yz}{x}$, $v = \frac{3xz}{y}$ and $w = \frac{4xy}{z}$, $then\ find$ $J\left(\frac{u,v,w}{x,y,z}\right)$.
29. If $u = x + 3y^2 - z^3$, $v = 4x^2yz$ and $w = 2z^2 - xy$, $then\ find$ $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ at $(1,-1,0)$.

Maxima and minima for a function of two variables:

30. Find the maximum and minimum values of
    a. $x^3 + y^3 - 2x^2 - 3axy$
    b. $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$
    c. $\sin x \sin y \sin(x + y)$
    d. $x^3 + y^3 - 3axy,\ a \geq 0$
    e. $xy(1 - x - y)$
    f. $x^3 + y^3 - 3x - 12y + 20$
    g. $x^2 + y^2 + 6x - 12$
    h. $x^3 + 3x^2 + 4xy + y^2$
31. Examine the function $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ for extreme values.
32. Discuss the maxima and minima of $f(x, y) = x^3 + y^3 - 3x - 3y + 20$.
33. Find the points on which the function $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ is extreme.
34. Show that the function of $f(x, y) = x^3 + y^3 - 3xy + 1$ is minimum at the point $(1, 1)$.
35. Discuss the maxima and minima of $f(x, y) = x^3y^2(1 - x - y)$.