Module-5: Linear Algebra
Introduction of linear algebra related to EC & EE engineering applications. Elementary row transformation of a matrix, Rank of a matrix. Consistency and Solution of system of linear equations - Gauss-elimination method, Gauss-Jordan method and approximate solution by Gauss-Seidel method. Eigenvalues and Eigenvectors, Rayleigh's power method to find the dominant Eigenvalue and Eigenvector. **Self-Study:** Solution of system of equations by Gauss-Jacobi iterative method. Inverse of a square matrix by Cayley- Hamilton theorem. **Applications:** Network Analysis, Markov Analysis, Critical point of a network system. Optimum solution. **(RBT Levels: L1, L2 and L3)**

L1-Elementary row transformation of a matrix, Rank of a matrix

Recall:

1. What is a Matrix and their types.
2. What are the properties of a Matrix.
3. What is Elementary row transformation of a matrix.
4. What is a Minor and order of a Matrix.

Elementary transformation of a matrix:

1. The interchange of any two rows (columns)
2. The multiplication of any row (column) by a non-zero number.
3. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column)

Two matrices A and B are said to be **equivalent** if one can be obtained from the other by a sequence of Elementary transformation. Equivalent matrices are denoted by $A\sim B$.

A matrix is obtained from the unit matrix by any one of the elementary transformations is called **Elementary matrix.**

Rank: A matrix is said to be of rank $\boldsymbol{r}$**,** if it has at least one nonzero minor of order r and every minor of order higher then r vanishes. Rank of A is denoted by $\rho(A)$.

Note: 1. If a matrix has nonzero minor of order r , then its rank is $\geq r$.
2. If all the minors of order $r+1$ are zero, then its rank is $\leq r$.
3. Elementary transformations do not change the rank of a matrix.

Echelon Form: A rectangular matrix is in echelon form if,
1. All nonzero rows are above any zero rows.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

Row reduced Echelon Form: An echelon form is said to be row reduced if, the leading entry in each nonzero row is 1 and each leading 1 is the only nonzero entry in its column.

If a matrix $\boldsymbol{A}$ is equivalent to an echelon matrix $\boldsymbol{E}$, then $\boldsymbol{\rho(A)=}$ Number of nonzero rows in $\boldsymbol{E}$.

Examples: Find the rank of the following matrix:

1. $\begin{bmatrix}1 & 2 & 3\\1 & 4 & 2\\2 & 6 & 5\end{bmatrix}$ 2. $\begin{bmatrix}2 & 3 & -1 & -1\\1 & -1 & -2 & -4\\3 & 1 & 3 & -2\\6 & 3 & 0 & -7\end{bmatrix}$ 3. $\begin{bmatrix}90 & 91 & 92 & 93 & 94\\91 & 92 & 93 & 94 & 95\\92 & 93 & 94 & 95 & 96\\93 & 94 & 95 & 96 & 97\\94 & 95 & 96 & 97 & 98\end{bmatrix}$ 4. $\begin{bmatrix}0 & 1 & -3 & -1\\1 & 0 & 1 & 1\\3 & 1 & 0 & 2\\1 & 1 & -2 & 0\end{bmatrix}$

Solutions: 1. $\begin{bmatrix}1 & 2 & 3\\1 & 4 & 2\\2 & 6 & 5\end{bmatrix} \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - R_1} \begin{bmatrix}1 & 2 & 3\\0 & 2 & -1\\0 & 2 & -1\end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix}1 & 2 & 3\\0 & 2 & -1\\0 & 0 & 0\end{bmatrix}$

Clearly reduced matrix is in echelon form with 2 nonzero rows. $\therefore \boldsymbol{\rho(A) = 2}$ **.**

2. $\begin{bmatrix}2 & 3 & -1 & -1\\1 & -1 & -2 & -4\\3 & 1 & 3 & -2\\6 & 3 & 0 & -7\end{bmatrix} \xrightarrow{R_4 = R_4 - (R_1 + R_2 + R_3)} \begin{bmatrix}2 & 3 & -1 & -1\\1 & -1 & -2 & -4\\3 & 1 & 3 & -2\\0 & 0 & 0 & 0\end{bmatrix}$

$\xrightarrow[R_3 = 2R_3 - 3R_1]{R_2 = -(2R_2 - R_1)} \begin{bmatrix}2 & 3 & -1 & -1\\0 & 5 & 3 & 7\\0 & -7 & 9 & -1\\0 & 0 & 0 & 0\end{bmatrix} \xrightarrow{R_3 = 5R_3 + 7R_2} \begin{bmatrix}2 & 3 & -1 & -1\\0 & 5 & 3 & 7\\0 & 0 & 66 & 44\\0 & 0 & 0 & 0\end{bmatrix}$

Clearly reduced matrix is in echelon form with 3 nonzero rows. $\therefore \boldsymbol{\rho(A) = 3}$ **.**

3. $\begin{bmatrix}90 & 91 & 92 & 93 & 94\\91 & 92 & 93 & 94 & 95\\92 & 93 & 94 & 95 & 96\\93 & 94 & 95 & 96 & 97\\94 & 95 & 96 & 97 & 98\end{bmatrix} \xrightarrow[R_3 = R_3 - R_2,\ R_2 = R_2 - R_1]{R_5 = R_5 - R_4,\ R_4 = R_4 - R_3} \begin{bmatrix}90 & 91 & 92 & 93 & 94\\1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\\1 & 1 & 1 & 1 & 1\end{bmatrix}$

$\xrightarrow[R_3 = R_3 - R_2]{R_5 = R_5 - R_2,\ R_4 = R_4 - R_2} \begin{bmatrix}90 & 91 & 92 & 93 & 94\\1 & 1 & 1 & 1 & 1\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix} \ R_2 = -(90R_2 - R_1) \begin{bmatrix}90 & 91 & 92 & 93 & 94\\0 & 1 & 2 & 3 & 4\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$

Clearly reduced matrix is in echelon form with 2 nonzero rows. $\therefore \boldsymbol{\rho(A) = 2}$ **.**

4. $\begin{bmatrix}0 & 1 & -3 & -1\\1 & 0 & 1 & 1\\3 & 1 & 0 & 2\\1 & 1 & -2 & 0\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix}1 & 1 & -2 & 0\\1 & 0 & 1 & 1\\3 & 1 & 0 & 2\\0 & 1 & -3 & -1\end{bmatrix}$

$\xrightarrow[R_3 = -\frac{1}{2}(R_3 - 3R_1)]{R_2 = R_2 - R_1} \begin{bmatrix}1 & 1 & -2 & 0\\0 & -1 & 3 & 1\\0 & 1 & -3 & -1\\0 & 1 & -3 & -1\end{bmatrix} \xrightarrow[R_3 = R_3 + R_2]{R_4 = R_4 + R_2} \begin{bmatrix}1 & 1 & -2 & 0\\0 & -1 & 3 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$

Clearly reduced matrix is in echelon form with 2 nonzero rows. $\therefore \boldsymbol{\rho(A) = 2}$ **.**

Review:

1. What is an Elementary matrix.
2. What are the Elementary transformation of a matrix.
3. What is a Rank of a matrix.
4. What is Echelon and row reduced echelon form.

L2- Consistency and Solution of system of linear equations

Recall:

1. What are the three types of elementary row operations? Provide an example of each.
2. How does applying an elementary row transformation affect the determinant of a matrix?
3. Can the rank of a matrix change when applying row transformations? Justify your answer.
4. When adding a multiple of one row to another, does it alter the rank of the matrix?

Consistency of Homogeneous linear equations, $AX = 0$:

$X = 0$ is the trivial solution. Thus the homogeneous system is always consistent.

Note: 1. If $\rho(A) =$ number of unknowns, then the system has only trivial solution.

2. If $\rho(A) <$ number of unknowns, then the system has an infinite number of solutions.

Consistency of non-homogeneous linear equations, $AX = B$:

1. If $\rho(A) = \rho(A|B) =$ number of unknowns, then the system has unique solution.

2. If $\rho(A) = \rho(A|B) <$ number of unknowns, then the system has an infinite number of solutions.

3. If $(A) \neq \rho(A|B)$, then system has no solution.

Examples:

1. Test for consistency and solve the system $x + 4 + 3z = 0, \quad x - y + z = 0, \; 2x - y + 3z = 0$.

Solution: Augmented matrix $[A|B]$ is

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 = -(R_3 - 2R_1)]{R_2 = -(R_2 - R_1)} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 9 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = 5R_3 - 9R_2} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

Clearly $\rho(A) = \rho(A|B) = 3 =$ number of unknowns, the system has unique solution that is trivial.

$x = y = z = 0$.

2. For what values of λ and μ do the system of equations: $x + y + z = 6, \quad x + 2y + 3z = 10,$

$x + 2y + \lambda z = \mu$ have (i) no solution (ii) unique solution (iii) infinite solutions.

Solution: Augmented matrix $[A|B]$ is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

(i) If $(A) \neq \rho(A|B)$, then the system has no solution.

If $\lambda - 3 = 0$ and $\mu - 10 \neq 0$ then $\rho(A) = 2 \neq \rho(A|B) = 3$.
Therefore, if $\lambda = 3$ and $\mu \neq 10$ then the system has no solution

(ii) If $\rho(A) = \rho(A|B) =$ number of unknowns, then the system has unique solution.
If $\lambda - 3 \neq 0$ and for any value of μ, $\rho(A) = \rho(A|B) = 3 =$ number of unknowns.
Hence for $\lambda \neq 3$, the system has unique solution.

(iii) If $\rho(A) = \rho(A|B) <$ number of unknowns, then the system has an infinite number of solutions.
If $\lambda - 3 = 0$ and $\mu - 10 = 0$ then $\rho(A) = 2 = \rho(A|B) < 3$.
Therefore if $\lambda = 3$ and $\mu = 10$ then the system has an infinite number of solutions.

3. Show that if $\lambda \neq -5$, the system $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ have a unique solution. Find the solution if $\lambda = -5$.

Solution: Augmented matrix $[A|B]$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & 4 & 3 \\ 6 & 5 & \lambda & -3 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 6R_1]{R_2 = R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & -7 & \lambda + 18 & 9 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & \lambda + 5 & 0 \end{array}\right]$$

Clearly if $\lambda + 5 \neq 0$, $\rho(A) = \rho(A|B) = 3 =$ number of unknowns.
Therefore if $\lambda \neq -5$ then the system has unique solution.
if $\lambda = -5$ then $\rho(A) = 2 = \rho(A|B) < 3$, the system has an infinite number of solutions.

$x + 2y - 3z = -2$ and $-7y + 13z = 9 \Longrightarrow y = \frac{13z-9}{7}$, $x = -2 - 2\left(\frac{13z-9}{7}\right) + 3z = \frac{4-5z}{7}$

Therefore solutions are $\begin{pmatrix} \frac{4-5z}{7} \\ \frac{13z-9}{7} \\ z \end{pmatrix}$ for any value of z.

Review:

1. What is the trivial solution of the homogeneous system AX=0.
2. If the Rank of a matrix is equal to the number of unknowns, what type of solution does the homogeneous system have.
3. If the Rank of a matrix is less then the number of unknowns, what type of solution does the homogeneous system have.
4. Give the condition for nonhomogeneous system to be consistent.

L3- Gauss-elimination method

Recall:

1. When we say that the system of equation $AX = B$ has a unique solution?
2. What is the condition for system of equation $AX = B$ to have infinitely many solutions?

3. If the Rank of a matrix is less then the number of unknowns, what type of solution does the homogeneous system have.
4. What is the maximum rank a m×n matrix can have?
5. What is the condition for the system of equation to have no solution?

Solution of linear simultaneous equations:

1. **Gauss elimination method:**

Consider the equations $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$.
Reduce augmented matrix into an upper triangular matrix as below

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$$\xrightarrow[R_3 = a_1R_3 - a_3R_1]{R_2 = a_1R_2 - a_2R_1} \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & b'_3 & c'_3 & d'_3 \end{array}\right]$$

$$\xrightarrow{R_3 = b'_2R_3 - b'_3R_2} \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & 0 & c''_3 & d''_3 \end{array}\right]$$

Then $z = \frac{d''_3}{c''_3}$, $y = \frac{d'_2 - zc'_2}{b'_2}$, $x = \frac{d_1 - yb_1 - zc_1}{a_1}$

Example: 1. Solve by Gauss elimination method,
$2x - 3y + z = -1$, $x + 4y + 5z = 25$, $3x - 4y + z = 2$.
Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 1 & 4 & 5 & 25 \\ 3 & -4 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 = 2R_3 - 3R_1]{R_2 = 2R_2 - R_1} \left[\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 0 & 11 & 9 & 51 \\ 0 & 1 & -1 & 7 \end{array}\right]$$

$$\xrightarrow{R_3 = 11R_3 - R_2} \left[\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 0 & 11 & 9 & 51 \\ 0 & 0 & -20 & 26 \end{array}\right]$$

$\therefore$ $z = -\frac{26}{20} = -1.3$, $y = \frac{51-9\times(-1.3)}{11} = 5.7$ and $z = \frac{-1+3\times5.7+1.3}{2} = 8.7$.

2. Solve by Gauss elimination method, $2x + 3y + z = -1$, $x - y + z = 6$, $3x + 2y - z = -4$.

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 2 & 3 & 1 & -1 \\ 3 & 2 & -1 & -4 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 5 & -1 & -13 \\ 0 & 5 & -4 & -22 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 5 & -1 & -13 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

$\therefore \quad -3z = -9, \quad 5y - z = -13, \quad x - y + z = 6 \implies \quad z = 3, \quad y = -2, \; and \quad x = 1$.

Review:

1. What is the primary goal of the Gauss Elimination method.
2. In the Gauss Elimination method, what is the first step typically performed on the augmented matrix.
3. Discuss how Gauss- elimination can fail if the pivot element is zero. How is this issue resolved?
4. In Gauss- elimination, does the rank of a matrix change during row operations?

L4- Gauss-Jordan method

Recall:

1. Define the role of pivot elements in Gaussian elimination.
2. In the Gauss Elimination method, what is the first step typically performed on the augmented matrix.
3. State the condition under which Gauss-elimination can be applied to solve a linear system.
4. What is the main objective of the Gauss-elimination method?

2. **Gauss Jordan method:**

Reduce augmented matrix into a diagonal matrix as below

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$$\xrightarrow[R_3 = a_1R_3 - a_3R_1]{R_2 = a_1R_2 - a_2R_1} \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & b'_3 & c'_3 & d'_3 \end{array}\right]$$

$$\xrightarrow[R_3 = b'_2R_3 - b'_3R_2]{R_1 = b'_2R_1 - b_1R_2} \left[\begin{array}{ccc|c} a'_1 & 0 & c'_1 & d'_1 \\ 0 & b'_2 & c'_2 & d'_2 \\ 0 & 0 & c''_3 & d''_3 \end{array}\right]$$

$$\xrightarrow[R_2 = c''_3R_2 - c'_2R_3]{R_1 = c''_3R_1 - c'_1R_3} \left[\begin{array}{ccc|c} a''_1 & 0 & 0 & d''_1 \\ 0 & b''_2 & 0 & d''_2 \\ 0 & 0 & c''_3 & d''_3 \end{array}\right]$$

Then $x = \frac{d''_1}{a''_1}$, $y = \frac{d''_2}{b''_2}$ and $z = \frac{d''_3}{c''_3}$.

Example:

1. Solve by Gauss Jordan method, $2x - y + 3z = 1, \quad -3x + 4y - 5z = 0, \quad x + 3y - 6z = 0.$

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ -3 & 4 & -5 & 0 \\ 1 & 3 & -6 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 = 2R_3 - R_1]{R_2 = 2R_2 + 3R_1} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 5 & -1 & 3 \\ 0 & 7 & -15 & -1 \end{array}\right]$$

$$\xrightarrow[R_3 = 5R_3 - 7R_2]{R_1 = 5R_1 + R_2} \left[\begin{array}{ccc|c} 10 & 0 & 14 & 8 \\ 0 & 5 & -1 & 3 \\ 0 & 0 & -68 & -26 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 5 & 0 & 7 & 4 \\ 0 & 5 & -1 & 3 \\ 0 & 0 & 34 & 13 \end{array}\right]$$

$$\xrightarrow[R_2 = 34R_2 + R_3]{R_1 = 34R_1 - 7R_3} \left[\begin{array}{ccc|c} 170 & 0 & 0 & 45 \\ 0 & 170 & 0 & 115 \\ 0 & 0 & 34 & 13 \end{array}\right]$$

$\therefore \quad x = \frac{45}{170} = \frac{9}{34} = 0.2647$, $\quad y = \frac{115}{170} = \frac{23}{34} = 0.6765$ and $\quad z = \frac{13}{34} = 0.3824.$

2. Solve by Gauss Jordan method, $2x + y + z = 10, \quad 3x + 2y + 3z = 18, \quad x + 4y + 9z = 16.$

Solution: Augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$$\xrightarrow[R_3 = 2R_3 - R_1]{R_2 = 2R_2 - 3R_1} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{array}\right]$$

$$\xrightarrow[R_3 = -\frac{1}{4}(R_3 - 7R_2)]{R_1 = \frac{1}{2}(R_1 - R_2)} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\xrightarrow[R_2 = R_2 - 3R_3]{R_1 = R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$\therefore \quad x = 7$, $\quad y = -9$ and $\quad z = 5.$

Review:

1. How does the Gauss-Jordan method differ from Gauss- elimination?
2. What is the primary advantage of converting a matrix to reduced row echelon form using the Gauss-Jordan method?
3. What are the additional steps in Gauss-Jordan compared to Gauss-elimination?
4. State one real-world application of the Gauss-Jordan method.

T1- Problems on system of linear equations

1. Test for consistency and solve the system $x + y + z = 3, \quad 2x - y + 3z = 10, \; 4x + y + 5z = 16$.
2. For what values of λ and μ do the system of equations: $2x + 3y + 5z = 9, \; 7x + 3y - 2z = 8,$ $2x + 3y + \lambda z = \mu$ have (i) no solution (ii) unique solution (iii) infinite solutions.
3. Test for consistency of the system $x + y + z = 3, \quad 2x + y + 3z = 5, \; x + 2y = 3$.
4. Applying Gauss elimination method solve

1. $2x + 3y - z = 5, \quad 4x + 4y - 3z = 3, \; 2x - 3y + 2z = 2.$
2. $x + y + z = 6, \quad x - 2y + 3z = 8, \; 2x + y - z = 3$.

5. Applying Gauss Jordan method solve
1. $2x + 3y - z = 5, \quad 4x + 4y - 3z = 3, \; 2x - 3y + 2z = 2.$
2. $x + y + z = 6, \quad x - 2y + 3z = 8, \; 2x + y - z = 3$

L5- Approximate solution by Gauss-Seidel method.

Recall:

1. Compare the efficiency of the Gauss-Jordan method and Gauss-elimination for large systems.
2. Does the Gauss-Jordan method require back substitution? Why or why not?
3. Why we use Gauss-Jordan method and Gauss- elimination method?
4. What is the basic requirement about the rank of a matrix for solving a system of linear equations using either the Gauss-Jordan method or Gauss- elimination method?

3. Gauss-Seidel iteration method: Consider the equations $a_1x + b_1y + c_1z = d_1, \; a_2x + b_2y + c_2z = d_2,$ $a_3x + b_3y + c_3z = d_3$ If $a_1, \; b_2, \; c_3$ are large as compared to other coefficients in their respective equations. Then iterative formula for $x, y \; and \; z$ are given by

$$x_{n+1} = \frac{1}{a_1}(d_1 - c_1z_n - b_1y_n), \quad y_{n+1} = \frac{1}{b_2}(d_2 - a_2x_{n+1} - c_2z_n), \quad z_{n+1} = \frac{1}{c_3}(d_3 - b_3y_{n+1} - a_3x_{n+1})$$

Start with initial approximations $x_0, \; y_0, \; z_0$ (each = 0) for x, y, z respectively .

Note: **Gauss-Seidel method converges if in each equation, the absolute value of the largest coefficient is Greater than the sum of the absolute values of the remaining coefficients.**

Example:

1. Solve $54x + y + z = 110$, $2x + 15y + 6z = 72$, $-x + 6y + 21z = 85$ by Gauss-Seidel iteration method.

$x_{n+1} = \frac{1}{54}(110 - z_n - y_n)$	$y_{n+1} = \frac{1}{15}(72 - 2x_{n+1} - 6z_n)$	$z_{n+1} = \frac{1}{21}(85 - 6y_{n+1} + x_{n+1})$
Let $x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = \frac{1}{54}(110 - z_0 - y_0)$ $= 2.037$	$y_1 = \frac{1}{15}(72 - 2x_1 - 6z_0)$ $= 4.528$	$z_1 = \frac{1}{21}(85 - 6y_1 + x_1)$ $= 2.851$
$x_2 = \frac{1}{54}(110 - z_1 - y_1)$ $= 1.900$	$y_2 = \frac{1}{15}(72 - 2x_2 - 6z_1)$ $= 3.406$	$z_2 = \frac{1}{21}(85 - 6y_2 + x_2)$ $= 3.165$
$x_3 = \frac{1}{54}(110 - z_2 - y_2)$ =1.915	$y_3 = \frac{1}{15}(72 - 2x_3 - 6z_2)$ =3.279	$z_3 = \frac{1}{21}(85 - 6y_3 + x_3)$ =3.202

$x_4 = \frac{1}{54}(110 - z_3 - y_3)$ =1.917	$y_4 = \frac{1}{15}(72 - 2x_4 - 6z_3)$ =3.264	$z_4 = \frac{1}{21}(85 - 6y_4 + x_4)$ =3.206
$x_5 = \frac{1}{54}(110 - z_4 - y_4)$ =1.917	$y_5 = \frac{1}{15}(72 - 2x_5 - 6z_4)$ =3.262	$z_5 = \frac{1}{21}(85 - 6y_5 + x_5)$ =3.207

$\therefore \quad x = 1.917 \,, \quad y = 3.262 \quad and \quad z = 3.207 \,.$

2. Use Gauss-Seidel method to solve $20x + y - 2z = 17$, $3x + 20y - z = 18$, $2x - 3y + 20z = 25$.

Carry out 2 iterations with $x_0 = 0,\ y_0 = 0,\ z_0 = 0$.

$x_{n+1} = \frac{1}{20}(17 + 2z_n - y_n)$	$y_{n+1} = \frac{1}{20}(18 - 3x_{n+1} + z_n)$	$z_{n+1} = \frac{1}{20}(25 + 3y_{n+1} - 2x_{n+1})$
Let $x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = \frac{1}{20}(17 + 2z_0 - y_0)$ $= 0.85$	$y_1 = \frac{1}{20}(18 - 3x_1 + z_0)$ $= 0.7725$	$z_1 = \frac{1}{20}(25 + 3y_1 - 2x_1)$ $= 1.2809$
$x_2 = \frac{1}{20}(17 + 2z_1 - y_1)$ $= 0.9395$	$y_2 = \frac{1}{20}(18 - 3x_2 + z_1)$ $= 0.8231$	$z_2 = \frac{1}{20}(25 + 3y_2 - 2x_2)$ $= 1.2795$

$\therefore \quad x = 0.9395 \,, \quad y = 0.8231 \quad and \quad z = 1.2795 \,.$

Exercise:

Solve by Gauss-Seidel method.

1. $2x + y + 6z = 9$, $8x + 3y + 2z = 13$, $x + 5y + z = 7$.
2. $83x + 11y - 4z = 95$, $7x + 52y + 13z = 104$, $3x + 8y + 29z = 71$.

Review:

1. For the Gauss-Seidel method to converge, what condition must be satisfied?
2. In the Gauss-Seidel method, how is the updated value of a variable used in the iteration process?
3. Discuss the conditions under which the Gauss-Seidel method converges to a solution
4. What are the advantages of using the Gauss-Seidel method over direct methods like Gaussian elimination?

L6- Eigenvalues and Eigen Vectors

Recall:

1. What type of system is suitable for solving with the Gauss-Seidel method?
2. Define the diagonal dominance condition and its importance for the Gauss-Seidel method.
3. Explain how the initial guess affects the convergence of the Gauss-Seidel method.
4. Compare the Gauss-Seidel method with the Jacobi method for solving linear systems.

Characteristic equation: $|A-\lambda I|=0$ is the characteristic equation of the square matrix A. Roots are called **Characteristic roots** or **Eigen values** or **latent roots** of A.

Any vector X satisfying $[A-\lambda I]X=0$ is called **Eigen vector** corresponding to the Eigen value .

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then Characteristic equation is $\lambda^2-(a+d)\lambda+(ad-cb)=0$.

If $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, then Characteristic equation is

$$\lambda^3-(a_1+b_2+c_3)\lambda^2+(sum\ of\ the\ minors\ of\ a_1, b_2 \& c_3)\lambda-|A|=0 .$$

Properties of Eigen values:

1) The sum of the Eigen values of a matrix is the sum of the principal diagonal elements.
2) The product of the Eigen values of a matrix is equal to its determinant.
3) If λ is the Eigen value of A , then $1/\lambda$ is Eigen value of A^{-1}.
4) If λ is the Eigen value of an orthogonal matrix, then $1/\lambda$ is also its Eigen value .
5) If λ is the Eigen value of A, then λ^n is the Eigen value of A^n . But Eigen vectors are same.

Cayley-Hamilton theorem: Every square matrix satisfies its characteristic equation.

1) Find the Eigen values and Eigen vectors of the following matrices.

i) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ iii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution:

i) Let $A=\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Characteristic equation is $|A-\lambda I|=0$.

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}=0$$

$\sum D=1+5+1=7.$

$\sum M\ D=4-8+4=0$

$|A|=-36$

$\Rightarrow\ \lambda^3-(\ 7\ )\lambda^2+(\ \ 0\ )\lambda-(\ -36)=0$

$\Rightarrow\ \lambda^3-7\lambda^2+0\lambda+36=0$

Roots are $-2,\ 3,\ 6$

$\lambda_1=-2$	$\lambda_2=3$	$\lambda_3=6$
$3x+y+3z=0$	$-2x+y+3z=0$	$-5x+y+3z=0$
$x+7y+z=0$	$x+2y+z=0$	$x-y+z=0$
$\Rightarrow 20y=0$, and $z=-x$	$\Rightarrow\ y=-z$	$\Rightarrow\ z=x$
$\therefore X_1=[1,\ \ 0,\ \ -1]'$	$X_2=[1,\ \ -1,\ \ 1]'$	$X_3=[1,\ \ 2,\ \ 1]'$

Eigen values are $-2,\ 3$ and 6 , the corresponding Eigen vectors are $[1,\ \ 0,\ \ -1]'$, $[1,\ \ -1,\ \ 1]'$ and $[1,\ \ 2,\ \ 1]'$ respectively.

ii) Let $A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$\sum D=8+7+3=18.$

Characteristic equation is $|A-\lambda I|=0$.

$\sum M\ D = 5+20+20=45$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$|A|=0.$

$\Rightarrow\ \lambda^3-(\ 18\ )\lambda^2+(\ 45\ )\lambda-(\ 0)=0$

$\Rightarrow\ \lambda^3-18\lambda^2+45\lambda+0=0$

Roots are 0, 3, 15

$\lambda_1=0$	$\lambda_2=3$	$\lambda_3=15$
$8x-6y+2z=0$	$5x-6y+2z=0$	$-7x-6y+2z=0$
$-6x+7y-4z=0$	$-6x+4y-4z=0$	$-6x-8y-4z=0$
$\Rightarrow 10x-5y=0$,	$\Rightarrow\ 4x-8y=0$	$\Rightarrow\ -20x-20y=0$
$\Rightarrow\ y=2x$	$\Rightarrow\ x=2y$	$\Rightarrow\ y=-x$
$\therefore\ X_1=[1,\ \ 2,\ \ 2]'$	$X_2=[2,\ \ 1,\ \ -2]'$	$X_3=\left[1,\ \ -1,\ \ \frac{1}{2}\right]'$

Eigen values are 0, 3 and 15 , the corresponding Eigen vectors are

$[1,\ \ 2,\ \ 2]'$, $[2,\ \ 1,\ \ -2]'$ and $[2,\ \ -2,\ \ 1]'$ respectively.

iii) Let $A=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$\sum D = 6+3+3=12.$

Characteristic equation is $|A-\lambda I|=0$.

$\sum M\ D = 8+14+14=36$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$|A|=32$

$\Rightarrow\ \lambda^3-(\ 12\ )\lambda^2+(\ 36)\lambda-(\ 32)=0$

$\Rightarrow\ \lambda^3-12\lambda^2+36\lambda-32=0$

Roots are 2, 2, 8

(The sum of the Eigen values of a matrix is the sum of the principal diagonal elements.

$\lambda_1=2$	$\lambda_2=2$	$\lambda_3=8$
$4x-2y+2z=0$	$4x-2y+2z=0$	$-2x-2y+2z=0$
Or $2x-y+z=0$	Or $2x-y+z=0$	$-2x-5y-z=0$
Let $y=0$, and $z=-2x$	Let $z=0$, and $y=2x$	$\Rightarrow 3y+3z=0$
$\therefore\ X_1=[1,\ \ 0,\ \ -2]'$	$X_2=[1,\ \ 2,\ \ 0]'$	$X_3=[2,\ \ -1,\ \ 1]'$

Eigen values are 2, 2 and 8 , the corresponding Eigen vectors are

$[1,\ \ 0\ \ -2]'$, $[1,\ \ 2,\ \ 0]'$ and $[2\ \ -1\ \ 1]'$ respectively.

Review:

1. What is the characteristic equation of a square matrix A.
2. What does the term λ represent in the characteristic equation.
3. What are the characteristic roots of a matrix also known as.
4. What is an Eigen vector of a matrix A.
5. If λ_1, λ_2,…….. λ_n are the Eigen values of a matrix A, what is the product of the eigen values equal to.
6. What does the Cayley-Hamilton theorem state.

L7- Rayleigh's power method to find the dominant Eigenvalue and Eigenvector

Recall:

1. State the condition for a square matrix A to have eigenvalues.
2. What does it mean if an eigenvalue of a matrix is zero?
3. State the eigenvalues of an identity matrix of order n.
4. How many eigenvalues does an n×n matrix have?
5. If λ is an eigenvalue of A, what is an eigenvalue of kA, where k is a scalar?
6. How does the determinant of a matrix relate to its eigenvalues?
7. .

Determination of largest Eigen value by Rayleigh's power method:

Let A be the given square matrix and a column vector X_0 be the initial Eigen vector. Evaluate $AX_0 = \lambda_1 X_1$ where λ_1 is the first approximation of the Eigen value and X_1 is the corresponding Eigen vector. $AX_1 = \lambda_2 X_2$. Where λ_2 is the 2^{nd} approximation of the Eigen value and X_2 is the corresponding Eigen vector. $AX_2 = \lambda_3 X_3$. Where λ_3 is the 3^{rd} approximation of the Eigen value and X_3 is the corresponding Eigen vector. Repeat this process till $X_n - X_{n-1}$ becomes negligible.

Example:1. Find the largest Eigen value and corresponding Eigen vector of the matrix by power method.

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ taking initial eigen vector } X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution:

$$AX_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda_1 X_1, \qquad AX_1 = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.43 \\ 0 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 3.57 \\ 1.86 \\ 0 \end{bmatrix} = 3.57 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \lambda_3 X_3 , \qquad AX_3 = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda_5 X_5$$

Since X_4 and X_5 are same, the largest Eigen value is 4 and the corresponding Eigen vector is $[1,\ 0.5,\ 0]'$

2. Find the largest Eigen value and corresponding Eigen vector of the matrix by power method.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \text{ taking initial Eigen vector } X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \text{ Carry out 4 iterations.}$$

Solution:

$$AX_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda_1 X_1, \qquad AX_1 = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda_2 X_2,$$

$$AX_2 = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \lambda_3 X_3, \qquad AX_3 = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = 2.93 \begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \lambda_4 X_4.$$

The largest Eigen value is 2.93 and the corresponding Eigen vector is $[1, \ 0, \ 0.98]'$.

3. Find the largest Eigen value and corresponding Eigen vector of the matrix by power method $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Carry out 4 iterations.

Solution:

$$AX_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda_1 X_1, \qquad AX_1 = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \lambda_2 X_2,$$

$$AX_2 = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = 4 \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \lambda_3 X_3, \qquad AX_3 = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = 3.5 \begin{bmatrix} 0.71 \\ -1 \\ 0.71 \end{bmatrix} = \lambda_4 X_4.$$

The largest Eigen value is 3.5 and the corresponding Eigen vector is $[0.71, -1, \ 0.71]'$.

Review:

1. What is the primary purpose of Rayleigh's method.
2. In Rayleigh's power method, what is the initial vector X_0 used for.
3. What condition indicates that the iterations in Rayleigh's power method should stop.
4. What is the role of normalization in Rayleigh's power method? Why is it necessary?
5. How can you determine the corresponding eigenvector after approximating the dominant eigenvalue?

T2-Problems on Eigenvalues, Eigen Vectors and Rayleigh's power method

1. Find the Eigen values and Eigen vectors of the following matrices.

i) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

2. Find the largest Eigen value and corresponding Eigen vector of the following matrix by power method.

1. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ 2. $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$, $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ 3. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$, $X_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $X_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

T3-Self Study: Solution of system of equations by Gauss-Jacobi iterative method and Inverse of a square matrix by Cayley- Hamilton theorem

Self-study:

Gauss-Jacobi's iteration method:

Consider the equations $a_1x+b_1y+c_1z=d_1$; $a_2x+b_2y+c_2z=d_2$; $a_3x+b_3y+c_3z=d_3$,

If a_1, b_2, c_3 are numerically large as compared to other coefficients in their respective equations.

Then iterative formula for $x, y\ and\ z$ are given by

$$x=\frac{1}{a_1}(d_1-b_1y-c_1z), \quad y=\frac{1}{b_2}(d_2-a_2x-c_2x) \text{ and } z=\frac{1}{c_3}(d_3-a_3x-b_3y)\cdots\cdots(1)$$

If not given assume that initial value of $(x,\ y,\ z)\equiv(0,\ 0,\ 0)$. Substitute these values in (1) and find

$$x_1=\frac{1}{a_1}(d_1), \quad y_1=\frac{1}{b_2}(d_2) \text{ and } z_1=\frac{1}{c_3}(d_3)$$

Then find, $x_2=\frac{1}{a_1}(d_1-b_1y_1-c_1z_1)$, $y_2=\frac{1}{b_2}(d_2-a_2x_1-c_2z_1)$ and $z_2=\frac{1}{c_3}(d_3-a_3x_1-b_3y_1)$

Continuing like this using ,

$$x_{n+1}=\frac{1}{a_1}(d_1-b_1y_n-c_1z_n), \quad y_{n+1}=\frac{1}{b_2}(d_2-a_2x_n-c_2z_n) \text{ and } z_{n+1}=\frac{1}{c_3}(d_3-a_3x_n-b_3y_n)$$

Until two set of values coincides.

<u>Example</u>:

1. Solve $54x+y+z=110$, $2x+15y+6z=72$, $-x+6y+21z=85$

   by **Gauss-Jacobi's** iteration method by taking initial values $(x,\ y,\ z)\equiv(2,\ 3,\ 4)$.

$x_{n+1}=\frac{1}{54}(110-z_n-y_n)$	$y_{n+1}=\frac{1}{15}(72-2x_n-6z_n)$	$z_{n+1}=\frac{1}{21}(85-6y_n+x_n)$
Let $x_0=2$	$y_0=3$	$z_0=4$
$x_1=\frac{1}{54}(110-z_0-y_0)$ $=1.907$	$y_1=\frac{1}{15}(72-2x_0-6z_0)$ $=2.933$	$z_1=\frac{1}{21}(85-6y_0+x_0)$ $=3.286$
$x_2=\frac{1}{54}(110-z_1-y_1)$ $=1.922$	$y_2=\frac{1}{15}(72-2x_1-6z_1)$ $=3.231$	$z_2=\frac{1}{21}(85-6y_1+x_1)$ $=3.300$
$x_3=\frac{1}{54}(110-z_2-y_2)$ $=1.916$	$y_3=\frac{1}{15}(72-2x_2-6z_2)$ $=3.224$	$z_3=\frac{1}{21}(85-6y_2+x_2)$ $=3.216$
$x_4=\frac{1}{54}(110-z_3-y_3)$ $=1.918$	$y_4=\frac{1}{15}(72-2x_3-6z_3)$ $=3.258$	$z_4=\frac{1}{21}(85-6y_3+x_3)$ $=3.218$
$x_5=\frac{1}{54}(110-z_4-y_4)$ $=1.917$	$y_5=\frac{1}{15}(72-2x_n-6z_4)$ $=3.257$	$z_5=\frac{1}{21}(85-6y_4+x_4)$ $=3.208$
$x_6=\frac{1}{54}(110-z_5-y_5)$	$y_6=\frac{1}{15}(72-2x_5-6z_5)$	$z_6=\frac{1}{21}(85-6y_5+x_5)$

=1.917	=3.261	=3.208

$\therefore \quad x = 1.917 \ , \quad y = 3.261 \quad and \quad z = 3.208 \ .$

2. Use **Gauss-Jacobi's** iteration to solve $20x + y - 2z = 17 \ , \ 3x + 20y - z = 18 \ , \ 2x - 3y + 20z = 25 \ .$

with $x_0 = 0, \ \ y_0 = 0, \ z_0 = 1$.

$x_{n+1} = \frac{1}{20}(17 + 2z_n - y_n)$	$y_{n+1} = \frac{1}{20}(18 - 3x_n + z_n)$	$z_{n+1} = \frac{1}{20}(25 + 3y_n - 2x_{n+1})$
Let $x_0 = 0$	$y_0 = 0$	$z_0 = 1$
$x_1 = \frac{1}{20}(17 + 2z_0 - y_0)$ $= 0.95$	$y_1 = \frac{1}{20}(18 - 3x_0 + z_0)$ $= 0.95$	$z_1 = \frac{1}{20}(25 + 3y_0 - 2x_0)$ $= 1.25$
$x_2 = \frac{1}{20}(17 + 2z_1 - y_1)$ $= 0.928$	$y_2 = \frac{1}{20}(18 - 3x_1 + z_1)$ $= 0.820$	$z_2 = \frac{1}{20}(25 + 3y_1 - 2x_1)$ $= 1.298$
$x_3 = \frac{1}{20}(17 + 2z_2 - y_2)$ $= 0.939$	$y_3 = \frac{1}{20}(18 - 3x_2 + z_2)$ $= 0.826$	$z_3 = \frac{1}{20}(25 + 3y_2 - 2x_2)$ $= 1.28$
$x_4 = \frac{1}{20}(17 + 2z_3 - y_3)$ $= 0.937$	$y_4 = \frac{1}{20}(18 - 3x_3 + z_3)$ $= 0.823$	$z_4 = \frac{1}{20}(25 + 3y_3 - 2x_3)$ $= 1.28$

$\therefore \quad x = 0.937 \ , \quad y = 0.823 \quad and \quad z = 1.28 \ .$

<u>Cayley-Hamilton theorem</u>: Every square matrix satisfies its characteristic equation.

Example: 1. Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, by Cayley-Hamilton theorem.

Solution: Characteristic equation is $|A - \lambda I| = 0$.

$$\Longrightarrow \quad \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Longrightarrow \lambda^2 - 7\lambda + 10 = 0 \ .$$

By Cayley-Hamilton theorem $A^2 - 7A + 10I = 0$

$$\Longrightarrow \ 10A^{-1} = 7I - A$$

$$\therefore A^{-1} = \frac{1}{10}\left\{\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}\right\} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} .$$

2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, by Cayley-Hamilton theorem.

Solution:

i) Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\sum D = 1 + 5 + 1 = 7.$

$\sum M \;\; D = 4 - 8 + 4 = 0$

$|A| = -36$

$$\Longrightarrow \lambda^3 - (\ 7\ )\lambda^2 + (\ 0\ )\lambda - (\ -36) = 0$$
$$\Longrightarrow \lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$$

By Cayley-Hamilton theorem $A^3 - 7A^2 + 36I = 0$

$$\Longrightarrow 36A^{-1} = 7A - A^2$$

$$\therefore A^{-1} = \frac{1}{36}\left\{\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} - \begin{bmatrix} 11 & 9 & 7 \\ 9 & 27 & 9 \\ 7 & 9 & 11 \end{bmatrix}\right\} = \frac{1}{36}\begin{bmatrix} -4 & -9 & -7 \\ -9 & -20 & -9 \\ -7 & -9 & -4 \end{bmatrix}$$

Course outcome

- Solve the system of linear equations using matrix theory and compute eigenvalues and eigenvectors and demonstrate using python.

PRACTICE QUESTION BANK

MODULE 5: Introduction of linear algebra related to Computer Science & Engineering.

1. Find the **Rank** of the following Matrices:

a. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$. b. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.

c. $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 12 & 13 & 14 & 15 \\ 13 & 14 & 15 & 16 \\ 14 & 15 & 16 & 17 \end{bmatrix}$. d. $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 3 & 2 & 5 & 2 \\ 2 & -1 & 1 & -3 \end{bmatrix}$.

2. Solve the following system of equations by **Gauss elimination** method:

a. $2x + 3y + z = -1, \quad x - y + z = 6, \quad 3x + 2y - z = -4.$

b. $3x + y + 2z = 3, \quad 2x - 3y - z = -3, \quad x + 2y + z = 4$.

c. $2x + y + 4z = 12, \quad 4x + 11y - z = 33, \quad 8x - 3y + 2z = 20.$

d. $2x - 3y + z = -1, \quad x + 4y + 5z = 25, \quad 3x - 4y + z = 2$.

3. Solve the following system of equations by **Gauss Jordan** method:

a. $2x + y + z = 10, \quad 3x + 2y + 3z = 18, \quad x + 4y + 9z = 16.$

b. $x + y + z = 11, \quad 3x - y + 2z = 12, \quad 2x + y - z = 3$.

c. $2x + 5y + 7z = 52, \quad 2x + y - z = 0, \quad x + y + z = 9.$

d. $2x - y + 3z = 1, \quad -3x + 4y - 5z = 0, \quad x + 3y - 6z = 0.$

4. Solve the following system of equations by **Gauss Seidel** iterative method:

a. $54x + y + z = 110$, $2x + 15y + 6z = 72$, $-x + 6y + 21z = 85.$

b. $2x - 3y + 20z = 25$, $20x + y - 2z = 17$, $3x + 20y - z = -18$.

c. $83x + 11y - 4z = 95$, $3x + 8y + 29z = 71$, $7x + 52y + 13z = 104.$

d. $2x + y + 6z = 9$, $8x + 3y + 2z = 13$, $x + 5y + z = 7.$

5. Test for **consistency** and solve the system if consistent:

a. $5x + 3y + 7z = 4, \quad 3x + 26y + 2z = 9, \quad 7x + 2y + 10z = 5$.

b. $x + y + z = 3, \quad 2x - y + 3z = 10, \quad 4x + y + 5z = 16.$

c. $x + y + z = 3, \quad 2x + y + 3z = 5, \quad x + 2y = 3.$

6. Find the **Eigen values and Eigen vectors** of the following matrices.

i) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

7. Find the largest Eigen value and corresponding Eigen vector of the matrix by **power method**:

a. $\begin{bmatrix}1 & 6 & 1\\1 & 2 & 0\\0 & 0 & 3\end{bmatrix}$.

b. $A=\begin{bmatrix}2 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 2\end{bmatrix}$, $X_0=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

c. $A=\begin{bmatrix}6 & -2 & 2\\-2 & 3 & -1\\2 & -1 & 3\end{bmatrix}$, $X_0=\begin{bmatrix}1\\0\\1\end{bmatrix}$.

d. $\begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}$.

8. For what values of λ and μ do the system of equations: $x+y+z=6,\quad x+2y+3z=10,$ $x+2y+\lambda z=\mu$ have (i) no solution (ii) unique solution (iii) infinite solutions.

9. For what values of λ and μ do the system of equations: $2x+3y+5z=9,\ 7x+3y-2z=8,$ $2x+3y+\lambda z=\mu$ have (i) no solution (ii) unique solution (iii) infinite solutions.