Module-1: Calculus(8 hours)
Introduction to polar coordinates and curvature relating to Computer Science and Engineering. Polar coordinates, Polar curves, angle between the radius vector and the tangent, angle between two curves. Pedal equations. Curvature and Radius of curvature - Cartesian, Parametric, Polar and Pedal forms. Problems. **Self-study:** Center and circle of curvature, evolutes and involutes. **Applications:** Computer graphics, Image processing. **(RBT Levels: L1, L2 and L3)**

L1- Introduction to Polar coordinates. Polar curves

Recall:

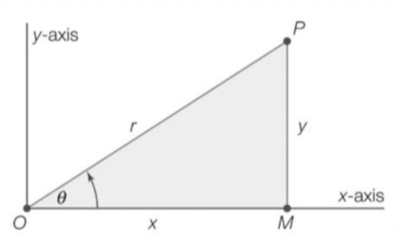
1. What is a Cartesian Co-ordinate System?
2. How are Cartesian Co-ordinates used in real life?
3. How is the point represented in the Polar Co-ordinates?
4. What is the relationship between Cartesian and Polar Co-ordinates.

Note:
- $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
- $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

- $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$.
- $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$, $\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$.

Polar coordinates:

Initial reference is chosen by spotting a point O in the plane called a pole. A line OM drawn through O is called the initial line. If P is any given point in the plane, join the points O and P with the results an angle is formed at O. The length of OP denoted by r is called the radius vector of the point P and the angle MOP denoted by θ measured in the anticlockwise direction is called the vectorial angle. The pair r and θ represented by P=(r, θ) or P(r, θ) are called as the polar co-ordinates of the point P.



Relationship between the Cartesian Co-ordinates (x, y) and the Polar Co-ordinates (r, θ):

Let (x, y) and (r, θ) respectively represent the Cartesian and Polar Co-ordinates of any point P in the plane where the origin O is taken as the pole and the x- axis is taken as the initial line. From the figure we have OM=x, PM=y. Also from the right angled triangle OMP we have

$\cos\theta = \frac{OM}{OP} = \frac{x}{r} \Rightarrow x = rcos\theta$ ……………(1)

$\sin\theta = \frac{PM}{OP} = \frac{y}{r} \Rightarrow y = rsin\theta$ …………….(2)

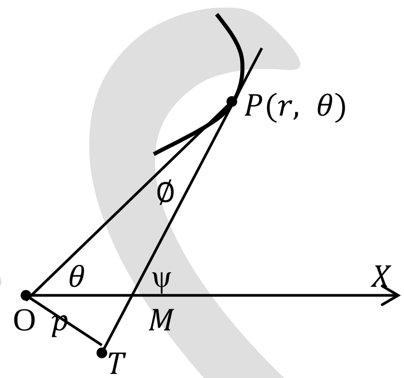
Further squaring and adding (1) and (2) we get $x^2 + y^2 = r^2(cos^2\theta + sin^2\theta)$

$\therefore r = \sqrt{x^2 + y^2}$ $\quad \because (cos^2\theta + sin^2\theta = 1)$ …………….(3)

Also dividing (2) by (1) we get $\frac{rsin\theta}{rcos\theta} = \frac{y}{x} \Rightarrow \tan\theta = \frac{y}{x}$ $\quad \therefore \theta = \tan^{-1}(\frac{y}{x})$ …………….(4)

The relations (1) and (2) determine the Cartesian co-ordinates in terms of polar co-ordinates whereas relations (3) and (4) determine the Polar co-ordinates in terms of Cartesian Co-ordinates.

Polar curves:



O is the pole, OX is the initial line, OP the radius vector , PT is the tangent to the curve at P .

And $OT = p$. In ΔOPM , $\psi = \theta + \phi$.

Review:

1. What is the relationship between the angle ψ and the angle θ and ϕ in polar coordinates.
2. If the radius vector OP makes an angle θ with the positive x-axis, what is the angle ϕ between the radius vector and the tangent PM.
3. Express the cartesian coordinate $(1,1)$ interms of polar coordinate (r, ϑ).
4. Express the cartesian coordinate $(1, -1)$ interms of polar coordinate (r, ϑ).
5. Express the cartesian coordinate $(-1,1)$ interms of polar coordinate (r, ϑ).
6. Express the cartesian coordinate $(-1, -1)$ interms of polar coordinate (r, ϑ).

L2- Angle between the radius vector and tangent

Recall:

1. What is Polar Co-ordinates?
2. What is the relationship between Cartesian and Polar Co-ordinates?
3. What is a Polar Curve?
4. How to convert from cartesian coordinate to polar coordinates?
5. What is radius vector and polar angle?
6. Express slope of the tangent in terms of angle ϕ along the initial line.

1. Prove that $\boldsymbol{\tan\phi = r\frac{d\theta}{dr}}$ **.**

Proof: Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$. $\therefore \widehat{XOP} = \theta$ and $OP = r$.

Let PM be the tangent to the curve at P subtending an angle ψ with the positive direction of initial line (x-axis) and ϕ be the angle between the radius vector OP and the tangent PM i.e $\widehat{OPM} = \phi$.

From the figure we have $\psi = \theta + \phi$ ($\because$ exterior angle = sum of the interior opposite angles).

$$tan\,\psi = \tan(\theta + \phi) \Rightarrow tan\,\psi = \frac{tan\,\theta + tan\,\phi}{1 - tan\,\theta tan\,\phi} \quad ....1)$$

Let $P(x, y)$ be the Cartesian co-ordinates of P so that $x = r\cos\theta$, $y = r\sin\theta$.

Also wkt, Slope of the tangent $= \tan\psi = \frac{dy}{dx} = \frac{dy/dr}{dx/dr}$

$$= \frac{\sin\theta + r\cos\theta\frac{d\theta}{dr}}{\cos\theta - r\sin\theta\frac{d\theta}{dr}} = \frac{\tan\theta + r\frac{d\theta}{dr}}{1 - \tan\theta\, r\frac{d\theta}{dr}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

By (1) and (2) $\quad \boldsymbol{\tan\phi = r\frac{d\theta}{dr}}$ **or** $\boldsymbol{\cot\phi = \frac{1}{r}\frac{dr}{d\theta}}$.

Problems:

1. Find the angle between the radius vector and the tangent and also find the slope of the tangent r=a(1+$cos\theta$) at θ=π/3

Solution: $r = a(1 + \cos\theta)$

Diff. w.r.to θ we get, $r_1 = a(-\sin\theta)$

$$\therefore \tan\phi = \frac{r}{r_1} = -\frac{(1+\cos\theta)}{\sin\theta} = -\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = -\cot\left(\frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$\Longrightarrow \phi = \frac{\pi}{2} + \frac{\theta}{2}$ is the angle between the radius vector and the tangent.

Now, $\psi = \theta + \phi = \theta + \frac{\pi}{2} + \frac{\theta}{2} = \frac{\pi}{2} + \frac{3\theta}{2}$

At $\theta = \frac{\pi}{3}$, $\psi = \pi$ $\therefore$ The slope of the tangent = $\tan\psi = \tan\pi = 0$.

2. Find the angle between the radius vector and the tangent to the curve $r = a(\sin\theta + \cos\theta)$

Solution: $r = a(\sin\theta + \cos\theta)$

Diff. w.r.to θ we get,

$$r_1 = a(\cos\theta - \sin\theta)$$

$$\therefore \tan\phi = \frac{r}{r_1} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} = \frac{1 + \tan\theta}{1 - \tan\theta}$$

$$= \tan\left(\theta + \frac{\pi}{4}\right)$$

$$\Longrightarrow \quad \phi = \theta + \frac{\pi}{4}$$

Review:

1. What is the formula for the tangent of the angle $\emptyset$ between the radius vector and the tangent?.
2. If the angle between the radius vector and the tangent is $\emptyset$, what is the relationship between $\emptyset$ and the slope of the tangent?
3. What is the angle ϕ between the radius vector and the tangent to the curve $\mathrm{r} = asin\theta$ at $\theta = \frac{\pi}{3}$?
4. What is the angle ϕ between the radius vector and the tangent to the curve $\mathrm{r} = a$?
5. What is the slope of the tangent to the curve $\mathrm{r} = a\,\theta$ at $\theta = \frac{\pi}{6}$?
6. Is tangent to the curve $\mathrm{r} = a$ is perpendicular? Explain.

L3- Angle between two polar curves

Recall:

1. What is polar curves?
2. In an polar curve r=f (θ) what is the relation between θ & the coordinates (x,y)?
3. What is the formula for the tangent of the angle ∅ between the radius vector and the tangent?
4. If the angle between the radius vector and the tangent is ∅, what is the relationship between ∅ and the slope of the tangent?
5. For any curve what is the slope of the tangent at $\theta = 0$?
6. Find the angle between the radius vector and the tangent for the curve $r = ae^{\theta cot\alpha}$.

Note: Angle between the two polar curves is $|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2|$

Find $\tan\phi_1 = \frac{r}{r_1}$ for the first curve and $\tan\phi_2 = \frac{r}{r_1}$ for the second curve

And if $\tan\phi_1 . \tan\phi_2 = -1$, then angle of intersection is $\frac{\pi}{2}$.

Problems:

Find the angle between the following two curves:

a) $r = a(1-\sin\theta)$, $r = b(1+\sin\theta)$

Solution $r = a(1-\sin\theta)$

Diff. w.r.to θ we get, $r_1 = a(-\cos\theta)$

$\therefore \tan\phi_1 = \frac{r}{r_1} = -\frac{(1-\sin\theta)}{\cos\theta}$

$r = b(1+\sin\theta)$

Diff. w.r.to θ we get, $r_1 = b(\cos\theta)$

$\therefore \tan\phi_2 = \frac{r}{r_1} = \frac{(1+\sin\theta)}{\cos\theta}$

$$\Longrightarrow \tan\phi_1 . \tan\phi_2 = -\frac{(1-\sin^2\theta)}{\cos^2\theta} = -1$$

Hence angle between them is $\frac{\pi}{2}$.

b) $r^n = a^n \cos n\theta$, $r^n = b^n \sin n\theta$.

Solution: $r^n = a^n \cos n\theta$

Diff. w.r.to θ we get,

$nr^{n-1}r_1 = -na^n \sin n\theta$

Or $r^n \frac{\boldsymbol{r_1}}{\boldsymbol{r}} = -a^n \sin n\theta$

$\therefore \tan\phi_1 = \frac{r}{r_1} = -\cot n\theta$

$r^n = b^n \sin n\theta$

Diff. w.r.to θ we get,

$nr^{n-1}r_1 = nb^n \cos n\theta$

Or $r^n \frac{\boldsymbol{r_1}}{\boldsymbol{r}} = b^n \cos n\theta$

$\therefore \tan\phi_2 = \frac{r}{r_1} = \tan n\theta$

$$\Longrightarrow \tan\phi_1 . \tan\phi_2 = -\cot n\theta \tan n\theta = -1$$

Hence the angle of intersection is $\frac{\pi}{2}$.

c) $r = \frac{2a}{(1-\cos\theta)}$, $r = \frac{2b}{(1+\cos\theta)}$

Solution: $r(1-\cos\theta) = 2a$

Diff. w.r.to θ we get,

$r_1(1-\cos\theta) + r\sin\theta = 0$

Or $r\sin\theta = -r_1(1-\cos\theta)$

$r(1+\cos\theta) = 2b$

Diff. w.r.to θ we get,

$r_1(1+\cos\theta) - r\sin\theta = 0$

Or $r\sin\theta = r_1(1+\cos\theta)$

$\therefore \tan\phi_1 = \frac{r}{r_1} = -\frac{(1-\cos\theta)}{\sin\theta}$ $\qquad$ $\therefore \tan\phi_2 = \frac{r}{r_1} = \frac{(1+\cos\theta)}{\sin\theta}$

$\Rightarrow \tan\phi_1 . \tan\phi_2 = -\frac{(1-\cos^2\theta)}{\sin^2\theta} = -1.$ $\qquad$ Hence the angle of intersection is $\frac{\pi}{2}$.

d) $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$

Solution: $r = \sin\theta + \cos\theta$

Diff. w.r.to θ we get,

$r_1 = \cos\theta - \sin\theta$

$\therefore \tan\phi_1 = \frac{r}{r_1} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} = \frac{1+\tan\theta}{1-\tan\theta}$

$= \tan\left(\theta + \frac{\pi}{4}\right)$

$\Rightarrow \quad \phi_1 = \theta + \frac{\pi}{4}$

$r = 2\sin\theta$

Diff. w.r.to θ we get,

$r_1 = 2\cos\theta$

$\therefore \tan\phi_2 = \frac{r}{r_1} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$\Rightarrow \quad \phi_2 = \theta$

Hence the angle of intersection $= |\phi_1 - \phi_2| = \frac{\pi}{4}$.

Review:

1. What is the formula to calculate angle between the polar curves?
2. If $\tan\phi_1 . \tan\phi_2 = -1$, then the angle of intersection is?
3. What is the angle of intersection between $r = e^{-\theta} and\ r = e^{\theta}$?
4. Is $r = \sin\theta\ and\ r = \cos\theta$ orthogonal?
5. What is the condition for two curves to be orthogonal?
6. At what angle the curves $r = \theta$ and $r = \frac{1}{\theta}$ intersect?

L4- Pedal equations

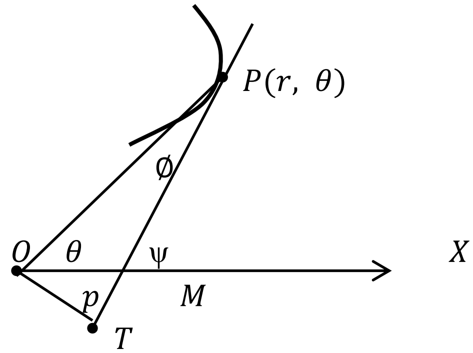
Recall:

1. What is the formula to calculate angle between the polar curves?.
2. What you can conclude when $\tan\phi_1 . \tan\phi_2 = -1$?
3. If two curves are orthogonal with $\tan\phi_1 = 1$ then $\tan\phi_2$?
4. For any two curves if $\tan\phi_1 . \tan\phi_2 = -1$,then angle of intersection between them is?

Length of the Perpendicular from the pole to the tangent:

2. Prove that $\quad \boldsymbol{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$.

Proof: Let O be the pole and OX be the initial line. Let $P(r,\theta)$ be any point on the curve and hence we have $X\hat{O}P = \theta$ and $OP = r$. Draw OT=p perpendicular from the pole on the tangent at P and let ϕ be the angle made by the radius vector with the tangent.



From the figure ,$O\hat{T}P = 90^0$.

In ΔOPT, $\frac{OT}{OP} = \sin\phi \quad \Longrightarrow \quad \frac{p}{r} = \sin\phi \quad or \quad p = r\sin\phi$ …………(1)

Squaring (1) and taking reciprocal we get, $\frac{1}{p^2} = \frac{1}{r^2}\ \text{cosec}^2\,\phi$

$$= \frac{1}{r^2}\ (1 + \cot^2\phi) \qquad (\tan\phi = r\frac{d\theta}{dr} \Longrightarrow \cot\phi = \frac{1}{r}\frac{dr}{d\theta}\ )$$

$$= \frac{1}{r^2}\,[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2] \ = \boldsymbol{\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}.$$

Equation involving only $\boldsymbol{p}$ *and* $\boldsymbol{r}$ is called **pedal equation.**

To find the pedal equation, find $\boldsymbol{\frac{r_1}{r}}$ and use it in $\boldsymbol{\frac{1}{p^2} = \frac{1}{r^2}\ [1 + \left(\frac{r_1}{r}\right)^2]}$ and then eliminate θ .

Find the pedal equation of the following curves:

a) $r = a(1 - \sin\theta)$

Solution: Diff. w.r.to θ we get, $r_1 = a(-\cos\theta)$

$$\therefore\ \boldsymbol{\frac{r_1}{r}} = -\frac{\cos\theta}{(1-\sin\theta)}$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{\cos\theta}{1-\sin\theta}\right)^2\right] = \frac{1}{r^2}\left[\frac{2(1-\sin\theta)}{(1-\sin\theta)^2}\right] = \boldsymbol{\frac{2a}{r^3}}$$

Hence Pedal equation is $\boxed{r^3 = 2ap^2}$.

b) $r^n = a^n\cos n\theta$

Solution: Diff. w.r.to θ we get, $nr^{n-1}r_1 = -na^n\sin n\theta$

Or $r^n\,\boldsymbol{\frac{r_1}{r}} = -a^n\sin n\theta$

$\therefore \frac{r_1}{r} = -\tan n\theta \quad \Longrightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}\ [1 + \tan^2 n\theta\ ] = \frac{1}{r^2\cos^2 n\theta}$

$\Longrightarrow\ p = r\cos n\theta \quad \Longrightarrow$ Pedal equation is $\boxed{pa^n = r^{n+1}}$

c) $r(1 - \cos\theta) = 2a$

Solution: Diff. w.r.to θ we get, $r_1(1 - \cos\theta) + r\sin\theta = 0$

Or $r\sin\theta = -\,r_1(1 - \cos\theta) \quad \Longrightarrow \frac{r_1}{r} = -\frac{\sin\theta}{(1-\cos\theta)}$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{\sin\theta}{1-\cos\theta}\right)^2\right] = \frac{1}{r^2}\left[\frac{2(1-\cos\theta)}{(1-\cos\theta)^2}\right] = \frac{\mathbf{1}}{ar}$$

Hence Pedal equation is $\boxed{p^2 = ar}$.

d) $r = m\theta$

Solution: Diff. w.r.to θ we get, $r_1 = m$ $\Longrightarrow$ $\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{m^2}{r^2}\right]$

Hence Pedal equation is $\boxed{r^4 = [r^2 + m^2]p^2}$

Review:

1. What is Pedal equation?
2. Find the Pedal equation for the circle centered at origin with radius a.
3. What is the expression for pedal in terms of angle between radius vector and tangent?
4. Find the pedal equation of the curve $r = e^{\theta}$.
5. Give the pedal equation expression for the curve $r = sec\theta$.
6. What is the length of the perpendicular from the pole to the tangent for the curve $r = sin\theta$ at $\theta = \frac{\pi}{4}$.

T1- Problems on Angle between two polar curves and Pedal equations

1. Find the angle between the following pair of curves:
   1. $r^2 = a^2 \operatorname{cosc} 2\theta$ and $r^2 = b^2 \sec 2\theta$, 2. $r = ae^{\theta}$ and $re^{\theta} = b$,
   3. $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$, 4. $r = a\theta$ and $r = \frac{a}{\theta}$,
   5. $r = alog\theta$ and $r = \frac{a}{log\theta}$.
2. Find the pedal equation of the following curves:
   1. $r^n = a^n \sec n\theta$, 2. $r = ae^{\theta \cot\propto}$, 3. $r^m = a^m(\cos m\theta + \sin m\theta)$,
   4. $r^n = a^n \sin n\,\theta + b^n \cos n\,\theta$, 5. $r^2 = a^2 \sec 2\theta$.

L5- Curvature and Radius of curvature – Cartesian form

Recall:

1. What we call for length of perpendicular from the pole to the tangent drawn to any curve?
2. What is the expression for pedal.
3. What is the expression for pedal in terms of angle between radius vector and tangent?
4. Find the pedal equation of the curve $re^{\theta} = 1$.

Derivative of arc length: $\frac{ds}{dx} = \sqrt{1 + y_1^2} = \sec\psi$, $\frac{ds}{dy} = \sqrt{1 + \frac{1}{y_1^2}} = \operatorname{cosec}\psi$.

Parametric: $\frac{ds}{dt} = \sqrt{\dot{\boldsymbol{x}}^2 + \dot{\boldsymbol{y}}^2}$ where $\dot{\boldsymbol{x}} = \frac{dx}{dt}$, $\dot{\boldsymbol{y}} = \frac{dy}{dt}$

Polar form: $\frac{ds}{d\theta} = \sqrt{r^2 + r_1^2} = r \operatorname{cosec}\emptyset$, $\frac{ds}{dr} = \sqrt{\left(\frac{r}{r_1}\right)^2 + 1} = \sec\emptyset$,

Therefore $\sin\emptyset = r\frac{d\theta}{ds}$ and $\cos\emptyset = \frac{dr}{ds}$.

Curvature $\boldsymbol{K} = \frac{d\psi}{ds}$, **Radius of curvature** $\boldsymbol{\rho} = \frac{ds}{d\psi}$.

<u>Radius of curvature in Cartesian form</u>: $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$

Proof: We know that $\tan\psi = y_1$ or $\psi = \tan^{-1} y_1$.

Differentiating both sides w.r.t. x, $\frac{d\psi}{dx} = \frac{1}{1+y_1^2} \cdot \frac{dy_1}{dx} = \frac{y_2}{1+y_1^2}$.

$$\rho = \frac{ds}{d\psi} = \frac{ds}{dx} \cdot \frac{dx}{d\psi} = \sqrt{1+y_1^2} \cdot \frac{1+y_1^2}{y_2} = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} .$$

Problems: 1. Find the radius of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the Folium $x^3 + y^3 = 3axy$.

Solution: Differentiating with respect to x, we get

$$3x^2 + 3y^2 y_1 = 3a(xy_1 + y) \implies y_1 = -\frac{x^2-ay}{y^2-ax} \quad \text{........}(i)$$

$$\therefore y_1 \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right) = -1 .$$

Differentiating (i) , $y_2 = -\frac{(y^2-ax)(2x-ay_1)-(x^2-ay)(2yy_1-a)}{(y^2-ax)^2}$

$$\therefore y_2 \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right) = -\frac{32}{3a} .$$

Hence ρ at $\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{3a}{8\sqrt{2}}$.

2. Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point $(1, 1)$.

Solution: $x^4 + y^4 = 2$

Differentiating with respect to x, we get

$$4x^3 + 4y^3 y_1 = 0 \implies y_1 = -\frac{x^3}{y^3} \quad \text{........ }(i) \quad \therefore y_1 \text{ at } (1, 1) = -1 .$$

Differentiating (i) with respect to x, we get,

$$y_2 = -\frac{y^3 3x^2 - x^3 3y^2 y_1}{y^6} \qquad \therefore y_2 \text{ at } (1, 1) = -6 .$$

Hence ρ at $(1, 1) = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$.

3. Find the radius of curvature of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at any point (x, y).

Solution: $y_1 = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \implies y_1^3 = -\frac{y}{x} \implies xy_1^3 = -y \implies 3xy_1^2 y_2 + y_1^3 = -y_1$

$$\implies 3xy_1 y_2 + y_1^2 = -1 \implies y_2 = -\left(\frac{1+y_1^2}{3xy_1}\right)$$

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+y_1^2)^{\frac{3}{2}}}{-\left(\frac{1+y_1^2}{3xy_1}\right)} = -3xy_1(1+y_1^2)^{\frac{1}{2}}$$

$$= 3x\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\left(\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{1}{2}} = 3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}} = 3\sqrt[3]{axy} .$$

Review:

1. What is the formula for the radius of curvature in Cartesian form?
2. What does a larger radius of curvature indicate about a curve?
3. What is the curvature for straight line?
4. What is the curvature of circle with radius r?
5. Find R.C for the curve $y = \cosh(x)$.
6. Find the radius the curve $y = \log sin\,(x)$.

L6- Curvature and Radius of curvature – Parametric form

Recall:

1. What is the formula for the radius of curvature in Cartesian form.
2. The curvature of a function f(x) is zero. Which function could be f(x)?
3. Find R.C for the curve $y = e^x$ at $x = 0$.
4. Find the Curvature of the curve $y = e^x$ at $x = 0$.
5. Find the RC of the curve $y = \mathrm{x}^2$ at the point where it crosses y axis.

Radius of curvature in Parametric form:

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} \qquad \text{Where } \dot{\boldsymbol{x}} = \frac{dx}{dt}\,,\ \dot{\boldsymbol{y}} = \frac{dy}{dt}\,, \ddot{x} = \frac{d^2x}{dt^2}\,, \ddot{y} = \frac{d^2y}{dt^2}$$

Problems:

1. Find the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta),\ \ y = a(1 - \cos\theta)$.

Solution: $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$.

$$x = a(\theta + \sin\theta)\ , \qquad y = a(1 - \cos\theta)$$

$$\Longrightarrow \quad \dot{x} = a(1 + \cos\theta)\ , \qquad \dot{y} = a\sin\theta$$

$$\text{And} \quad \ddot{x} = -a\sin\theta \quad , \qquad \ddot{y} = a\cos\theta$$

$$\therefore \quad \rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{\left(a^2(1+\cos\theta)^2+a^2\sin^2\theta\right)^{\frac{3}{2}}}{a^2(1+\cos\theta)\cos\theta+a^2\sin^2\theta}$$

$$= a2^{\frac{3}{2}}\sqrt{(1+\cos\theta)} = 4a\cos\frac{\theta}{2}.$$

2.Find the radius of curvature of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at any point $(x,\ y)$.

Solution: Parametric equation is $x = a\cos^3 t \quad and \quad y = a\sin^3 t$

$$\Longrightarrow \quad \dot{x} = -3a\cos^2 t\ \sin t\ , \qquad \dot{y} = 3a\sin^2 t\cos t$$

$$\ddot{x} = -3a(-2\cos t\sin^2 t + \cos^3 t) \quad , \quad \ddot{y} = 3a(2\sin t\cos^2 t - \sin^3 t)$$

$$\therefore \quad \rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{(3a)^3\cos^3 t\ \sin^3 t}{-9a^2\cos^2 t\ \sin t(2\sin t\cos^2 t-\sin^3 t)+9a^2\sin^2 t\ \cos t(2\sin t\cos^2 t-\sin^3 t)}$$

$$= \frac{3a\cos^3 t\ \sin^3 t}{-\cos^2 t\ \sin^2 t} = 3a\cos t\sin t\ = 3a\left(\frac{x}{a}\right)^{\frac{1}{3}}\left(\frac{y}{a}\right)^{\frac{1}{3}} = 3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}} = 3\sqrt[3]{axy}.$$

3. Find the radius of curvature of the curve $x = a\log(\sec t + \tan t)$, $y = a\sec t$.

Solution: Given that, $x = a\log(\sec t + \tan t)$, $y = a\sec t$.

$$\Longrightarrow \quad \dot{x} = a\frac{\sec t\ \tan t + \sec^2 t}{\sec t + \tan t} = a\sec t\ , \qquad \dot{y} = a\sec t\ \tan t$$

$$\ddot{x} = a\sec t\ \tan t\ , \qquad \ddot{y} = a(\sec^3 t + \sec t\tan^2 t)$$

$$\therefore \quad \rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{a^3(\sec^2 t+\sec^2 t\tan^2 t)^{\frac{3}{2}}}{a^2\sec^2 t\,(\sec^2 t+\tan^2 t)-a^2\sec^2 t\tan^2 t}$$

$$= \frac{a\sec^6 t}{\sec^4 t} = a\ \sec^2 t.$$

Review:

1. What is the formula for the radius of curvature in parametric form.
2. Find ρ ,for the curve $x = a\cos t$, $y = a\sin t$ at any point t.
3. Find R.C at $t = 0$, for the curve $x = e^t$, $y = e^{-t}$.
4. For $x = x(t), y = y(t)$ what is $\frac{dy}{dt}$?
5. Can we obtain the radius of curvature of a curve in parametric form by using ρ expression in Cartesian form?
6. Give the parametric form of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

L7- Curvature and Radius of curvature –Polar & Pedal form

Recall:

1. What is the formula for the radius of curvature in parametric form?
2. What is the RC for the circle with radius r?
3. What does the reciprocal of the curvature of the curve at any point 'P' is called?
4. Find R.C of the circle $x^2 + y^2 = 9$.

Radius of curvature in Polar form: $\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$.

Proof: We know that $\tan\phi = \frac{r}{r_1}$, diff. w. r. t. θ we get $\sec^2\emptyset\,\frac{d\phi}{d\theta} = \frac{r_1^2-rr_2}{r_1^2}$

$$\frac{d\phi}{d\theta} = \frac{r_1^2-rr_2}{r_1^2\left[\left(\frac{r}{r_1}\right)^2+1\right]} = \frac{r_1^2-rr_2}{r^2+r_1^2} \ .$$

But $\psi = \theta + \phi$, therefore $\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta} = 1 + \frac{r_1^2-rr_2}{r^2+r_1^2} = \frac{r^2+2r_1^2-rr_2}{r^2+r_1^2}$.

Finally $\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta}\cdot\frac{d\theta}{d\psi} = \sqrt{r^2+r_1^2}\ .\frac{r^2+r_1^2}{r^2+2r_1^2-rr_2} = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$.

Radius of curvature in Pedal form: $\rho = r\frac{dr}{dp}$.

Problems:

1. For the cardioid $r = a(1+\cos\theta)$, show that ρ^2 is proportional to r .

Solution: $r = a(1+\cos\theta) \implies r_1 = -a\sin\theta$ and $r_2 = -a\cos\theta$

$$\therefore\ \rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{\left(a^2(1+\cos\theta)^2+a^2\sin^2\theta\right)^{\frac{3}{2}}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2(1+\cos\theta)\cos\theta}$$

$$= \frac{a[2(1+\cos\theta)]^{\frac{3}{2}}}{3(1+\cos\theta)}$$

$$\Longrightarrow \quad \rho^2 = \frac{8a^2(1+\cos\theta)^3}{9(1+\cos\theta)^2} = \frac{8a}{9}r, \qquad \text{and hence} \qquad \rho^2 \propto r\ .$$

2. Find the radius of curvature for $p^2 = ar$.

Solution: $p^2 = ar \Longrightarrow r = \frac{p^2}{a}$

$$\therefore \frac{dr}{dp} = \frac{2p}{a} = 2\sqrt{\frac{r}{a}} \qquad \Longrightarrow \rho = r\frac{dr}{dp} = \frac{2r\sqrt{r}}{\sqrt{a}}\ .$$

3. Find the radius of curvature of the curve $r^n = a^n \cos n\theta$.

Solution: Given curve is $r^n = a^n \cos n\theta$

Diff. w.r.to θ we get,

$$nr^{n-1}r_1 = -na^n \sin n\theta$$

Or $\quad r^n\ \frac{r_1}{r} = -a^n \sin n\theta$

$$\therefore \frac{r_1}{r} = -\tan n\theta \qquad \Longrightarrow \qquad \frac{1}{p^2} = \frac{1}{r^2}\ [1+\tan^2 n\theta\ ] = \frac{1}{r^2\cos^2 n\theta}$$

$\Longrightarrow\ p = r\cos n\theta \qquad \Longrightarrow$ Pedal equation is $\boxed{pa^n = r^{n+1}}$

Diff. w.r.to r we get, $\frac{dp}{dr}a^n = (n+1)r^n$

Therefore $\rho = r\frac{dr}{dp} = \frac{a^n}{(n+1)r^{n-1}}$.

Or

Given curve is $r^n = a^n \cos n\theta$

Diff. w.r.to θ we get,

$$nr^{n-1}r_1 = -na^n \sin n\theta$$

$$\therefore \quad r_1 = -r\tan n\theta$$

$$r_2 = -r_1\tan n\theta - nr\sec^2 n\theta$$

$$= r\tan^2 n\theta - nr\sec^2 n\theta$$

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{r^3(1+\tan^2 n\theta)^{\frac{3}{2}}}{r^2+2r^2\tan^2 n\theta - r(r\tan^2 n\theta - nr\sec^2 n\theta)}$$

$$= \frac{r\sec^3 n\theta}{1+\tan^2 n\theta + n\sec^2 n\theta} = \frac{r\sec n\theta}{n+1}$$

$$= \frac{ra^n}{(n+1)r^n} = \frac{a^n}{(n+1)r^{n-1}}\ .$$

4. Find the radius of curvature of the curve $r^n = a^n \sin n\theta$.

Solution: $\quad r^n = a^n \sin n\theta$

Diff. w.r.to θ we get, $nr^{n-1}r_1 = na^n \cos n\theta$

Or $r^n \frac{r_1}{r} = a^n \cos n\theta$

$\therefore \frac{r_1}{r} = \cot n\theta \quad \Longrightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}[1 + \cot^2 n\theta] = \frac{1}{r^2 \sin^2 n\theta}$ (3)

$\Longrightarrow p = r \sin n\theta \quad \Longrightarrow$ Pedal equation is $\boxed{pa^n = r^{n+1}} \Longrightarrow \frac{dp}{dr} = \frac{(n+1)r^n}{a^n}$

$\therefore \rho = r\frac{dr}{dp} = \frac{a^n}{(n+1)r^{n-1}}$.

5. Find the radius of curvature of the curve $y^2 = \frac{a^2(a-x)}{x}$ at the point $(a, 0)$.

Solution: $y^2 = \frac{a^2(a-x)}{x}$

$y^2 x = a^3 - a^2 x$

Differentiating w.r.t x

$2xyy_1 + y^2 = -a^2 \quad \Longrightarrow y_1 = -\frac{a^2+y^2}{2xy}$.

Since $y_1 = \infty$ at $(a, 0)$, therefore $\rho = \frac{1}{x_2}$.

$x_1 = -\frac{2xy}{y^2+a^2}$ then $x_1 = 0\ at\ (a, 0)$

$(y^2 + a^2)x_1 = -2xy$.

Again differentiating w.r.t y

$(y^2 + a^2)x_2 + 2yx_1 = -2x - 2yx_1$

$\Longrightarrow (y^2 + a^2)x_2 = -2x - 2yx_1 - 2yx_1$

Then $x_2 = -\frac{2}{a}$ at $(a, 0)$

$\therefore \rho = \frac{1}{x_2} = -\frac{a}{2}$

$\therefore$ The radius of curvature of the given curve is $\frac{a}{2}$.

Review:

1. What is the formula for the radius of curvature in polar form.
2. What is the formula for the radius of curvature in pedal form.
3. Find the radius of curvature at any point (r, θ) for the curve $r = a\cos\theta$.
4. Find the radius of curvature of the curve $x^2 + y^2 + 4x - 2y - 4 = 0$.
5. Find ρ for the curve $pr = a^2$.
6. What happens to curvature when radius of curvature increases?

T2- Problems on Curvature and Radius of curvature

Recall:

1. Find the radius of the curvature of the curve $r = a\sin\theta$ at the pole.
2. What is the RC for the circle with radius 3?

3. Which of the parameters do you need to determine the radius of curvature in polar coordinates?
4. Find R.C for the curve $x = t^2, y = t$ at $t = 1$.

Problems

1. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = 4$ at the point where it cuts the line $y = x$.
2. Find the radius of curvature of the following curves

   1. $xy^3 = a^4$ at the point $(a,\ a)$
   2. $y = 4\sin x - \sin 2x$ at $(\frac{\pi}{2},\ 4)$
   3. $r = a(1 + \cos\theta)$.

3. If ρ_1 and ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $a(1 + \cos\theta)$ which passes through the pole, show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.

Review:

1. If the pedal equation of a curve is $p = cs\sin\theta$, where c is a constant, Find the radius of curvature.
2. What is the formula for the radius of curvature in pedal form.
3. Find the radius of curvature at any point (r, θ) for the curve $r = a\sin\theta$.
4. Find the radius of curvature of the curve $x^2 + y^2 + 4x = 0$.
5. Find the curvature for the curve $y = x$.
6. What happens to curvature when radius of curvature increases?

T3- Self Study: Center and circle of curvature, Evolutes and Involutes

Centre of curvature: A point $\boldsymbol{C}$ on the normal to any point $\boldsymbol{P}$ of a curve at a distance $\boldsymbol{\rho}$ from it, is called center of curvature.

Circle of curvature: A circle with center $\boldsymbol{C}$ (center of curvature at $\boldsymbol{P}$) and radius $\boldsymbol{\rho}$ is called circle of curvature Or osculating circle at $\boldsymbol{P}$.

Centre of curvature at any point $P(x,\ y)$ on the curve $y = f(x)$ is given by

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} \quad , \qquad \bar{y} = y + \frac{1+y_1^2}{y_2} \quad .$$

Evolute: The locus of the center of the curvature for a curve is called its **evolute** and the curve is called an **Involute** of its evolute.

Problems:

1. If the center of curvature of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at one end of the minor axis lies at the other end, then show that the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$.

Solution: Given that the center of curvature at $(0,\ b)$ is $(0, -b)$. Therefore $\bar{y} = -b$.

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $y_1 = -\frac{b^2x}{a^2y}$, and $y_2 = -\frac{b^2}{a^2}\left(\frac{y - xy_1}{y^2}\right)$.

At the point $(0,\ b)$, $y_1 = 0$, and $y_2 = -\frac{b}{a^2}$.

$$\bar{y} = y + \frac{1+y_1^2}{y_2} \Longrightarrow \quad -b = b - \frac{a^2}{b} \quad \Longrightarrow 2b^2 = a^2 \quad \text{or} \quad \frac{b^2}{a^2} = \frac{1}{2} \ .$$

Hence eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$.

2. Find the coordinates of the center of curvature at any point of the parabola $y^2 = 4ax$. Hence find its evolute.

Solution: Differentiating with respect to x, we get

$$2yy_1 = 4a \quad \Longrightarrow \quad y_1 = \frac{2a}{y} \quad \text{........} (i)$$

Differentiating (i), $y_2 = -\frac{2ay_1}{y^2} \quad \Longrightarrow y_2 = -\frac{4a^2}{y^3}$.

Centre of curvature at any point is $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x + \frac{\frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)}{\frac{4a^2}{y^3}} = x + \frac{y^2+4a^2}{2a} = 3x + 2a$,

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = y - \frac{\left(1+\frac{4a^2}{y^2}\right)}{\frac{4a^2}{y^3}} = y - \frac{y(y^2+4a^2)}{4a^2} = -\frac{y^3}{4a^2} \ .$$

To find the evolute, $(\bar{y})^2 = \frac{y^6}{16a^4} = \frac{(4ax)^3}{16a^4} = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3$ or $(\bar{y})^2 = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3$.

Therefore the locus of $(\bar{x},\ \bar{y})$ i.e., evolute, is $27ay^2 = 4(x-2a)^3$.

Assignment questions:

1. Find the coordinates of the center of curvature at $(2,\ 1)$ on the parabola $x^2 = 4y$.
2. Find the coordinates of the center of curvature at any point of the parabola $x^2 = 4ay$. Hence find its evolute.
3. Show that the equation of the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

Course outcome

- Apply the knowledge of calculus to solve problems related to polar curves and its applications in determining the bentness of a curve and demonstrate using python.

PRACTICE QUESTION BANK

Polar Curves – angle between radius vector and tangent, angle between two curves, pedal equations:

1. Prove that $\tan\phi = r\frac{d\theta}{dr}$.
2. Prove that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.
3. Find the angle between the radius vector and the tangent and also find the slope of the tangent r=a(1+$cos\theta$) at θ=π/3.
4. Find the angle between the following two curves.
   a. $r = a(1-\sin\theta)$, $r = b(1+\sin\theta)$
   b. $r^n = a^n\cos n\theta$, $r^n = b^n \sin n\theta$.
   c. $r = \frac{2a}{(1-\cos\theta)}$, $r = \frac{2b}{(1+\cos\theta)}$
   d. $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$.
   e. $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$
   f. $r^2 = a^2 \operatorname{cosc} 2\theta$ and $r^2 = b^2\sec 2\theta$
   g. $r = ae^{\theta}$ and $re^{\theta} = b$
   h. $r^n = a^n\sec(n\theta + \propto)$ and $r^n = b^n\sec(n\theta+\beta)$
   i. $r = a\log\theta$, $r = \frac{a}{\log\theta}$

5. Show that the curves r=a(1+$sin\theta$) and r=a(1−$sin\theta$) cuts each other orthogonally
6. Find the pedal equation of the following curves.
   a. $r = a(1-\sin\theta)$
   b. $r^n = a^n\cos n\theta$
   c. $r(1-\cos\theta) = 2a$
   d. $r^2 = a^2\sec 2\theta$
   e. $r^m = a^m(\cos m\theta + \sin m\theta)$
   f. $r^n = a^n\sin n\theta + b^n\cos n\theta$
   g. $r^n = a^n \operatorname{sech} n\theta$
   h. $r = ae^{\theta\cot\propto}$
   i. $r = a\log\tan\theta$
   j. $r = a + b\sin\theta$

Curvature – radius of curvature:

7. Derive the radius of curvature in Cartesian form.
8. Derive the radius of curvature in polar form.
9. Find the radius of curvature for the following curves.
   a. $x^3 + y^3 = 3axy$ at $(\frac{3a}{2}, \frac{3a}{2})$
   b. $xy^3 = a^4$ at the point (a, a)
   c. $r^n = a^n\sin n\theta$ at any point
   d. For the cardioid $r = a(1+\cos\theta)$
   e. $y = 4\sin x - \sin 2x$ at $(\frac{\pi}{2}, 4)$
   f. $x = a\log\sec\theta$, $y = a(\tan\theta - \theta)$
   g. $r^2 = a^2\sec 2\theta$
   h. $p^2 = ar$
10. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = 4$ at the point where it cuts the line $y = x$.
11. Find the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$.
12. If ρ_1 and ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a(1+\cos\theta)$ which passes through the pole, show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$
13. Find the radius of curvature of the curve $x = a\log(\sec t + \tan t)$, $y = a\sec t$.
14. Find the radius of curvature of the curve $pa^n = r^{n+1}$
15 Find the radius of curvature of the curve $r^3 = 2ap^2$
16. Find the radius of curvature for the curve $y^2 = \frac{4a^2(2a-x)}{x}$ where the curve meets the x-axis.
17. Find the radius of curvature of the curve $\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$ at any point on it.
18. Prove that $\rho = p + \frac{d^2p}{d\psi^2}$ with usual notations.
19. Find the radius of curvature of the curve $y = x^3(x-a)$ at the point (a,0).
20. Find the radius of curvature of the curve $x^4 + y^4 = 2$ $at\ the\ point$ (1,1).
21. Find that the radius of curvature of $a^2y = x^3 - a^3$ at the point where the curves cut the X-axis.
22. Find the radius of curvature for the curve $y = ax^2 + bx + c$ at $x = \frac{1}{2a}(\sqrt{a^2-1} - b)$.
23. Show that the radius of curvature for the curve $r^n = a^n\cos n\theta$ varies inversely as r^{n-1}.