Module-3: Ordinary Differential Equations (ODEs) of First order
Introduction to first-order ordinary differential equations pertaining to the applications for Computer Science & Engineering. Linear and Bernoulli's differential equations. Exact and reducible to exact differential equations - Integrating factors on $\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]$ *and* $\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]$. Orthogonal trajectories, L-R and C-R circuits. Problems. Non-linear differential equations: Introduction to general and singular solutions, Solvable for p only, Clairaut's equations, reducible to Clairaut's equations. Problems. **Self-Study**: Applications of ODEs, Solvable for x and y. **Applications of ordinary differential equations**: Rate of Growth or Decay, Conduction of heat.(**RBT Levels: L1, L2 and L3**)

Differential Equations:

Solution of first order and first degree differential equations –

Exact equation: A differential equation of the form $M(x,\ y)dx + N(x,\ y)dy = 0$ is said to be **exact** if $M(x,\ y)dx + N(x,\ y)dy = d[u(x,\ y)] = 0$. Then its solution is $u(x,\ y) = c$.

Theorem: The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. (i.e. $M_y = N_x$)

Solution of exact equation is $\underset{(y\ constant)}{\int M\,dx} + \int$ (terms of N not containing x) $dy = c$.

Example: 1. Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$.

Here $M = x^2 - 4xy - 2y^2 \implies \frac{\partial M}{\partial y} = -4x - 4y$.

And $N = y^2 - 4xy - 2x^2 \implies \frac{\partial M}{\partial x} = -4y - 4x$.

Clearly $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and hence the given equation is exact.

The solution is $\underset{(y\ constant)}{\int M\,dx} + \int$ (terms of N not containing x) $dy = c$.

i.e. $\underset{(y\ constant)}{\int (x^2 - 4xy - 2y^2)\,dx} + \int y^2\,dy = c$.

Or $\boxed{\frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c}$.

2. Solve $(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$.

Here $M = y\cos x + \sin y + y \implies \frac{\partial M}{\partial y} = \cos x + \cos y + 1$,

And $N = \sin x + x\cos y + x \implies \frac{\partial N}{\partial x} = \cos x + \cos y + 1$.

Clearly $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and hence the given equation is exact.

The solution is $\underset{(y\ constant)}{\int M\,dx} + \int$ (terms of N not containing x) $dy = c$.

i.e. $\int (y\cos x + \sin y + y)\,dx + \int 0\,dy = c$.

$y\ constant)$

Or $\boxed{y\sin x + x\sin y + xy = c}$.

3. Solve $ye^{xy}dx + (xe^{xy} + 2y)dy = 0$.

Here $M = ye^{xy} \implies \frac{\partial M}{\partial y} = xye^{xy} + e^{xy}$.

And $N = xe^{xy} + 2y \implies \frac{\partial N}{\partial x} = xye^{xy} + e^{xy}$.

Clearly $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and hence the given equation is exact.

The solution is $\underset{(y\ constant)}{\int M\,dx} + \int$ (terms of N not containing x) $dy = c$.

i.e. $\underset{(y\ constant)}{\int (ye^{xy})\,dx} + \int 2y\,dy = c.$

Or $\boxed{e^{xy} + y^2 = c}$.

Reducible to exact:

In the non-exact equation $Mdx + Ndy = 0$,

i) if $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$ i.e. the function of x only, then $\text{I. F.} = e^{\int f(x)dx}$.

ii) if $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$ i.e. the function of y only, then $\text{I. F.} = e^{\int g(y)dy}$.

Examples:

1. Solve $\left(xy^2 - e^{1/x^3}\right)dx - x^2ydy = 0$

$\frac{\partial M}{\partial y} = 2xy,\ \ \frac{\partial N}{\partial x} = -2xy$, but $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{4xy}{-x^2y} = -\frac{4}{x} = f(x)$. $\int f(x)dx = -4\log x = \log\frac{1}{x^4}$

$\therefore\ \text{I. F.} = e^{\int f(x)dx} = \frac{1}{x^4}$

multiplying by I. F. we get $\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}dy = 0$, which is exact.

Hence the solution is $\underset{(y\ constant)}{\int \left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx} + 0 = c.$ or $\boxed{\frac{e^{1/x^3}}{3} - \frac{y^2}{2x^2} = c}$

2. Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0.$

$M = xy^3 + y$, $N = 2x^2y^2 + 2x + 2y^4$

$\frac{\partial M}{\partial y} = 3xy^2 + 1$, $\frac{\partial N}{\partial x} = 4xy^2 + 2$.

$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{xy^2+1}{(xy^2+1)y} = \frac{1}{y} = g(y)$. $\int g(y)dy = \log y$

$\therefore\ \text{I. F.} = e^{\int g(y)dy} = y$

Multiplying by I. F. we get, $(xy^4 + y^2)dx + 2(x^2y^3 + xy + y^5)dy = 0$, which is exact.

Hence the solution is $\underset{(y\ constant)}{\int (xy^4 + y^2)\,dx} + \int 2\,y^5dy = c.$ or $\boxed{\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c}$

3. Solve $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$.

$\frac{\partial M}{\partial y} = 4x + 6y$, $\frac{\partial N}{\partial x} = 2x + 2y$, but $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{2}{x} = f(x)$. $\int f(x)dx = 2\log x = \log x^2$

$\therefore$ I. F. $= e^{\int f(x)dx} = x^2$

Multiplying by I. F. we get $(4x^3y + 3x^2y^2 - x^3)dx + (x^4 + 2x^3y)dy = 0$ which is exact.

Hence the solution is $\underset{(y\ constant)}{\int}(4x^3y + 3x^2y^2 - x^3)\,dx + \ 0 = c.$ or $\boxed{x^4y + x^3y^2 - \frac{x^4}{4} = c}$

4. Solve $(8xy - 9y^2)dx + 2(x^2 - 3xy)dy = 0$

$M = 8xy - 9y^2$, $N = 2x^2 - 6xy$

$\frac{\partial M}{\partial y} = 8x - 18y$, $\frac{\partial N}{\partial x} = 4x - 6y$.

$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{4(x-3y)}{2x(x-3y)} = \frac{2}{x} = f(x)$. $\int f(x)dx = 2\log x = \log x^2$

$\therefore$ I. F. $= e^{\int f(x)dx} = x^2$

Multiplying by I. F. we get $(8x^3y - 9x^2y^2)dx + (2x^4 - 6x^3y)dy = 0$ which is exact.

Hence the solution is $\underset{(y\ constant)}{\int}(8x^3y - 9x^2y^2)\,dx + \ 0 = c.$ or $\boxed{2x^4y - 3x^3y^2 = c}$

Leibnitz's linear equations:

Equation of the type $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x, is called **linear equation.**

And its solution is $y(I.F.) = \int Q(I.F.)dx + c,$ where $I.F. = e^{\int Pdx}$.

Bernoulli's differential equations: Reducible to linear form

Equation of the type $\frac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x, is called **Bernoulli's equation.**

And Dividing by y^n and substituting $z = \frac{1}{y^{n-1}}$ equation reduces to linear form.

1. Solve $x\frac{dy}{dx} + y = x^3y^6$.

Dividing by x , we get, $\frac{dy}{dx} + \frac{1}{x}y = x^2y^6$.

Clearly equation is Bernoulli's equation, dividing by y^6

$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}\left(\frac{1}{y^5}\right) = x^2$, Put $z = \frac{1}{y^5}$ $\Longrightarrow \frac{dz}{dx} = -\frac{5}{y^6}\frac{dy}{dx}$

$\Longrightarrow$ $-\frac{5}{y^6}\frac{dy}{dx} - \frac{5}{x}\left(\frac{1}{y^5}\right) = -5x^2$. Or $\frac{dz}{dx} + \left(-\frac{5}{x}\right)z = -5x^2$

This is linear equation in x on z with $P = -\frac{5}{x}$ and $Q = -5x^2$. $I.F. = e^{\int -\frac{5}{x}dx} = \frac{1}{x^5}$.

Solution is $z(I.F.) = \int Q(I.F.)dx + c$

i.e. $\frac{z}{x^5} = \int \frac{-5}{x^3}dy + c$, or $\boxed{\frac{1}{x^5y^5} = \frac{5}{2x^2} + c}$

2. Solve $xy(1 + xy^2)\frac{dy}{dx} = 1.$

Clearly $\frac{dx}{dy} = xy + x^2y^3$.

Dividing by x^2, we get $\frac{1}{x^2}\frac{dx}{dy} + y\left(-\frac{1}{x}\right) = y^3$ Put $z = -\frac{1}{x}$ Then $\frac{dz}{dy} = \frac{1}{x^2}\frac{dx}{dy}$.

Therefore equation becomes $\frac{dz}{dy} + yz = y^3$, which is linear. $I.F. = e^{\frac{y^2}{2}}$.

Solution is $$ze^{\frac{y^2}{2}} = \int y^3 e^{\frac{y^2}{2}} dy + c$$
$$= \int 2te^t dt + c = 2e^t(t-1) + c$$

Or $$\boxed{\frac{1}{x}e^{\frac{y^2}{2}} = -2e^{\frac{y^2}{2}}\left(\frac{y^2}{2} - 1\right) + k}$$

Applications :

1. **Orthogonal trajectories:** Every member of the one family of curves cuts every member of another family of curves orthogonally then one family is orthogonal trajectory of other.

To find the O.T. of Cartesian curves:

If $f(x,\ y,\ c) = 0$ be the given family of curves with c is the arbitrary constant.

Step1: Find the differential equation of the given family by eliminating c. Let it be $F\left(x,\ y,\ \frac{dy}{dx}\right) = 0$.

Step2: Find the differential equation of the orthogonal trajectory

by replacing $\frac{dy}{dx} = -\frac{dx}{dy}$ i.e. $F\left(x,\ y, -\frac{dx}{dy}\right) = 0$.

Step3: Solve $F\left(x,\ y, -\frac{dx}{dy}\right) = 0$ to get orthogonal trajectory.

To find the O.T. of Polar curves:

If $f(r,\ \theta,\ c) = 0$ be the given family of curves with c is the arbitrary constant.

Step1: Find the differential equation of the given family by eliminating c. Let it be $F\left(r,\ \theta,\ \frac{dr}{d\theta}\right) = 0$.

Step2: Find the differential equation of the orthogonal trajectory

by replacing $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$ i.e. $F\left(r,\ \theta, -r^2\frac{d\theta}{dr}\right) = 0$.

Step3: Solve $F\left(r,\ \theta, -r^2\frac{d\theta}{dr}\right) = 0$ to get orthogonal trajectory.

Examples: Find the O.T. of the following curves:

1. $x^2 - y^2 = c^2$.

Given family is $x^2 - y^2 = c^2$.

Differentiating w.r.t. x, we get $2x - 2y\frac{dy}{dx} = 0$ or $x - y\frac{dy}{dx} = 0$.

Therefore D.E. of O.T. is $+y\frac{dx}{dy} = 0 \implies xdy + ydx = 0$, or $d(xy) = 0$.

Integrating, $\boxed{xy = k}$

2. $x^2+y^2+2gx+c=0$, where g is the parameter.(i.e. system of co-axial circles with center on the x-axis)

Given family is $x^2+y^2+2gx+c=0$.

Differentiating w.r.t. x, we get $2x+2y\frac{dy}{dx}+2g=0 \implies 2x^2+2xy\frac{dy}{dx}+2gx=0$.

Therefore D.E. of given family is $x^2+y^2-2x^2-2xy\frac{dy}{dx}+c=0$ or $y^2-x^2-2xy\frac{dy}{dx}+c=0$

And hence D.E. of O.T. is $y^2-x^2+2xy\frac{dx}{dy}+c=0$.

That is $2x\frac{dx}{dy}-\frac{1}{y}x^2=-\frac{(c+y^2)}{y}$. Put $z=x^2$ then $\frac{dz}{dy}=2x\frac{dx}{dy}$.

$\frac{dz}{dy}+\left(-\frac{1}{y}\right)z=-\frac{(c+y^2)}{y}$. This is linear equation with $P=-\frac{1}{y}$, $Q=-\frac{(c+y^2)}{y}$.

$I.F.=e^{\int -\frac{1}{y}dy}=e^{\log\frac{1}{y}}=\frac{1}{y}$. $\therefore$ Solution is $\frac{z}{y}=\int-\frac{(c+y^2)}{y^2}dy+k$

$\implies \frac{z}{y}=\frac{c}{y}-y+k$. Or $z=c-y^2+ky$

$\therefore$ O.T. is $\boxed{x^2+y^2-ky-c=0}$

3. $r=a(1-\cos\theta)$.

Given family is $r=a(1-\cos\theta)$.

Differentiating w.r.t. θ, we get $\frac{dr}{d\theta}=a\sin\theta \implies \frac{dr}{d\theta}=\frac{r\sin\theta}{(1-\cos\theta)}$.

Therefore D.E. of given family is $\frac{dr}{d\theta}=\frac{r\sin\theta}{(1-\cos\theta)}$. And hence D.E. of O.T. is $-r^2\frac{d\theta}{dr}=\frac{r\sin\theta}{(1-\cos\theta)}$.

Or $-r\frac{d\theta}{dr}=\frac{\sin\theta}{(1-\cos\theta)} \implies \frac{dr}{r}=-\frac{(1-\cos\theta)}{\sin\theta}d\theta$ or $\frac{dr}{r}=(\cot\theta-\operatorname{cosec}\theta)d\theta$

Integrating $\log r=\log\sin\theta-\log(\operatorname{cosec}\theta-\cot\theta)+\log b$

$\implies r=b\frac{\sin\theta}{\operatorname{cosec}\theta-\cot\theta}=b\sin\theta\,(\operatorname{cosec}\theta+\cot\theta)$

$\therefore$ O.T. is $\boxed{r=b(1+\cos\theta)}$

4. $r=2a(\cos\theta+\sin\theta)$

Given family is $r=2a(\cos\theta+\sin\theta)$.

Differentiating w.r.t. θ, we get $\frac{dr}{d\theta}=2a(\cos\theta-\sin\theta) \implies \frac{dr}{d\theta}=\frac{r(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)}$.

Therefore D.E. of given family is $\frac{dr}{d\theta}=\frac{r(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)}$. And hence D.E. of O.T. is $-r^2\frac{d\theta}{dr}=\frac{r(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)}$.

Or $-r\frac{d\theta}{dr}=\frac{(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)} \implies \frac{dr}{r}=\frac{(\cos\theta+\sin\theta)}{(\sin\theta-\cos\theta)}d\theta$

Integrating $\log r=\log(\sin\theta-\cos\theta)+\log b$

$\implies r=b(\sin\theta-\cos\theta)$

$\therefore$ O.T. is $\boxed{r=b(\sin\theta-\cos\theta)}$

5. Show that the system of confocal conics given by $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ is self-orthogonal. (λ is a parameter).

Given family is $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$.

Differentiating w.r.t. x, we get $\frac{2x}{a^2+\lambda}+\frac{2yy_1}{b^2+\lambda}=0 \implies \frac{x}{a^2+\lambda}=\frac{yy_1}{b^2+\lambda}$

$$\Rightarrow \frac{x^2}{a^2+\lambda} = \frac{xyy_1}{b^2+\lambda} \text{ and } \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \Rightarrow \frac{xyy_1}{b^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

$$\Rightarrow xyy_1 + y^2 = b^2 + \lambda$$

Similarly $\frac{xy}{a^2+\lambda} = \frac{y^2y_1}{b^2+\lambda} \Rightarrow \frac{x^2}{a^2+\lambda} + \frac{xy}{y_1(a^2+\lambda)} = 1 \Rightarrow x^2 + \frac{xy}{y_1} = a^2 + \lambda$

$\Rightarrow$ D.E. of given family is $\boldsymbol{x^2 - y^2 + \frac{xy}{y_1} - xyy_1 = a^2 - b^2}$

To get the D.E. of O.T. replace y_1 by $-\frac{1}{y_1}$

Therefore D.E. of O.T. is $x^2 - y^2 - \frac{xy}{\frac{1}{y_1}} + xy\frac{1}{y_1} = a^2 - b^2$.

That is $\boldsymbol{x^2 - y^2 - xyy_1 + \frac{xy}{y_1} = a^2 - b^2}$

Since D.E. of given family and D.E. of its O.T. are same, given family is self-orthogonal.

6. Find O.T. of $r^n \sin n\theta = a^n$.

Given family is $r^n \sin n\theta = a^n$.

Differentiating w.r.t. θ, we get $nr^{n-1}\frac{dr}{d\theta}\sin n\theta + nr^n \cos n\theta = 0 \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\cot n\theta$.

Therefore D.E. of given family is $\frac{1}{r}\frac{dr}{d\theta} = -\cot n\theta$. And hence D.E. of O.T. is $-r\frac{d\theta}{dr} = -\cot n\theta$.

$$\Rightarrow \frac{dr}{r} = \tan n\theta \, d\theta$$

Integrating $\log r = -\frac{1}{n}\log\cos n\theta + \log b$

$$\Rightarrow n\log r + \log\cos n\theta = n\log b \Rightarrow \boxed{r^n \cos n\theta = b^n}$$

2. Flow of electricity:

Note: q: quantity of electricity (in coulomb); i: current (time rate of flow of electricity)(in ampere)
R: Resistance (in ohm); L: Inductance (in Henry); C: Capacitance (farad); V: voltage (in volt).

Basic relation: 1. $i = \frac{dq}{dt}$, $\therefore q = \int i dt$.
2. Voltage drop across resistance $R = Ri$.
3. Voltage drop across inductance $L = L\frac{di}{dt}$.
4. Voltage drop across capacitance $C = \frac{q}{C}$.

Differential equations:

In R, L series circuit with voltage source E, let i be the current at any time t.

Then by Kirchhoff's first law $Ri + L\frac{di}{dt} = E$

or

$$\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$$, clearly this is linear equation with $I.F. = e^{\frac{Rt}{L}}$

And the solution is $ie^{\frac{Rt}{L}} = \int \frac{E}{L}e^{\frac{Rt}{L}}dt + c \Rightarrow ie^{\frac{Rt}{L}} = \frac{E}{L}\frac{L}{R}e^{\frac{Rt}{L}} + c$.

Since initially there is no current in the circuit, $0 = \frac{E}{R} + c \Rightarrow c = -\frac{E}{R}$.

$\therefore\ i = \frac{E}{R}(1 - e^{-\frac{Rt}{L}})$ Which shows that i increases with t and attains the maximum value $\frac{E}{R}$.

Problem :

1. Solve the differential equation $L\frac{di}{dt} + Ri = 200\sin 300t,$ when $L = 0.05,\ R = 100$ and find the current i

At any time t, if initially no current in the circuit. What value does i approach after a long time.

Solution: $L\frac{di}{dt} + Ri = 200\sin 300t, \quad L = 0.05,\ R = 100$

$\Longrightarrow \frac{di}{dt} + 2000i = 4000\sin 300t$ this is linear equation with $I.F. = e^{2000t}$

And the solution is $ie^{2000t} = 4000\int e^{2000t}\sin 300t\,dt + c$

$$= 4000\frac{e^{2000t}}{[2000^2+300^2]}[2000\sin 300t - 300\cos 300t] + c\ .$$

Or, $ie^{2000t} = \frac{2e^{2000t}}{2045}[2000\sin 300t - 300\cos 300t] + c$

since $i = 0$ when $t = 0$, $0 = -\frac{120}{409} + c \Longrightarrow c = \frac{120}{409}$

$\therefore$ Solution is $i = \frac{2}{2045}[2000\sin 300t - 300\cos 300t] + \frac{120}{409}e^{-2000t}$.

After a long time, $i = \frac{2}{2045}[2000\sin 300t - 300\cos 300t]$.

2. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current builds up at a rate given by $L\frac{di}{dt} + Ri = E$. Find i as a function of t. How long will it be, before the current has reached one half its final value if $E = 6$ volts, $R = 100$ ohms and $L = 0.1$ henry?

Solution: $\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$, clearly this is linear equation with $I.F. = e^{\frac{Rt}{L}}$

And the solution is $ie^{\frac{Rt}{L}} = \int\frac{E}{L}e^{\frac{Rt}{L}}dt + c \Longrightarrow ie^{\frac{Rt}{L}} = \frac{E}{L}\frac{L}{R}e^{\frac{Rt}{L}} + c$.

Since initially there is no current in the circuit, $0 = \frac{E}{R} + c \Longrightarrow c = -\frac{E}{R}$.

$\therefore\ i = \frac{E}{R}(1 - e^{-\frac{Rt}{L}})$ Which shows that i increases with t and attains the maximum value $\frac{E}{R}$.

If $E = 6$ volts, $R = 100$ ohms and $L = 0.1$ henry Final value of $i = \frac{E}{R} = \frac{3}{50}$.

If the current reaches half its final value, $\frac{3}{100} = \frac{3}{50}(1 - e^{-1000t}) \Longrightarrow \frac{1}{2} = 1 - e^{-1000t}$

$$\Longrightarrow e^{-1000t} = \frac{1}{2} \quad \Longrightarrow -1000t = \log\left(\frac{1}{2}\right) \Longrightarrow t = \frac{\log\left(\frac{1}{2}\right)}{-1000} \approx 0.0007\text{sec.}$$

Exercise: Solve the following equations.

1. $ye^{xy}dx + (xe^{xy} + 2y)dy = 0$. 2. $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$. 3. $xe^{x^2+y^2}dx + \left(ye^{x^2+y^2} + y\right)dy = 0$.

4. $\frac{2x}{y^3}dx + \frac{y^2-3x^2}{y^4}dy = 0$ 5. $2xydx + 3x^2dy = 0$.

6. $y(2x - y + 1)dx + x(3x - 4y + 3)dy = 0$ 7. $2ydx + (2x\log x - xy)dy = 0$

8. $\frac{dy}{dx} + y = \frac{x}{y}$. 9. $\frac{dy}{dx} + \frac{y}{x} = y^2$. 10. $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

11. $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x}(\log z)^2$ 12. $\tan y\ \frac{dy}{dx} + \tan x = \cos y\cos^2 x$.

13. Find the orthogonal trajectories of the following family of curves.
 i) $r^n\cos n\theta = a^n$. ii) $r = 4a\sec\theta\tan\theta$. iii) $\left(r + \frac{k^2}{r}\right)\cos\theta = a$, with a as parameter.
 iv) $r(1 + \cos\theta) = 2a$. v) $r^2 = a^2\cos 2\theta$ vi) $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$ with λ as parameter.

14. Show that the system of confocal and coaxial parabolas $y^2 = 4a(x + a)$ is self orthogonal.

Equations of first order and higher degree:

Assume that $\frac{dy}{dx} = p$, then the differential equation is of the form $f(x,\ y,\ p) = 0$.

1. **Equations solvable for p** : n^{th}degree, first order equations of the form
 $p^n + P_1p^{n-1} + P_2p^{n-2} + \cdots P_n = 0$, (where $P_1,\ P_2\ ,\cdots P_n$ are functions of $x,\ y$)
 are solvable for p.

 Splitting up the equation in to n linear factors we get , $[p - f_1(x,\ y)][p - f_2(x,\ y)]\cdots[p - f_n(x,\ y)] = 0$.
 Equating each of the factors to zero and solving we get ,
 $$F_1(x,y,c) = 0,\quad F_2(x,y,c) = 0\ ,\quad \cdots\cdots,\quad F_n(x,y,c) = 0.$$
 These n solutions constitute the general solution of the equation.
 Or we can write the general solution as $F_1(x,y,c)F_2(x,y,c)\ \cdots\cdots\ F_n(x,y,c) = 0$.

2. **Clairaut's equations:** An equation of the form $y = px + f(p)$ is called Clairaut's equation.
 General solution is $y = cx + f(c)$.
 To find the singular solution eliminate c from $y = cx + f(c)$ using $x = -f'(c)$.

A. **Equations solvable for p:**

1. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x} \Longrightarrow\ p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$ or $p^2 + \left(\frac{y}{x} - \frac{x}{y}\right)p - 1 = 0$

$$\Longrightarrow\ p^2 + \frac{y}{x}p - \frac{x}{y}p - 1 = 0$$

$$\Longrightarrow\ \left(p + \frac{y}{x}\right)\left(p - \frac{x}{y}\right) = 0$$

And hence $\frac{dy}{dx} = -\frac{y}{x}$, $\frac{dy}{dx} = \frac{x}{y}$.

Or $xdy + ydx = 0,$ $xdx - ydy = 0.$

On integration we get, $xy = c,$ $x^2 - y^2 = c.$

The general solution is $\boxed{(xy - c)\,(x^2 - y^2 - c) = 0}$

2. $p^2 + 2py\cot x = y^2$

Adding $(y\cot x)^2$ on both sides we get, $(p + y\cot x)^2 = (y\,\mathrm{cosec}\,x)^2$

$$\Longrightarrow p + y\cot x = \pm y\,\mathrm{cosec}\,x.$$

$\therefore$ $\frac{dy}{dx} = y(\mathrm{cosec}\,x - \cot x)$ and $\frac{dy}{dx} = -y(\mathrm{cosec}\,x + \cot x)$

Integrating by separating the variables,

$\log y = -\log(\mathrm{cosec}\,x + \cot x) - \log\sin x + \log c$ & $\log y = -\log(\mathrm{cosec}\,x - \cot x) - \log\sin x + \log c.$

Or $y(1 + \cos x) = c$ and $y(1 - \cos x) = c$.

The general solution is $\boxed{y(1 \pm \cos x) = c}$

3. $y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0.$ Or $y\,p^2 + (x - y)p - x = 0$

$$\Longrightarrow (yp + x)(p - 1) = 0$$

$$\Longrightarrow \frac{dy}{dx} = -\frac{x}{y} \quad \text{and} \quad \frac{dy}{dx} = 1$$

$$\Longrightarrow xdx + ydy = 0 \text{ and } dy = dx.$$

Integrating by separating the variables we get, $x^2 + y^2 = c,$ $y - x = c$.

The general solution is $\boxed{(\,x^2 + y^2 - c)(y - x - c) = 0}$

4. $p(p + y) = x(x + y) \Longrightarrow p^2 + (x + y)p - xp - x(x + y) = 0$.

$$\Longrightarrow (p - x)(p + x + y) = 0.$$

$$\Longrightarrow \frac{dy}{dx} = x \quad \text{or} \quad \frac{dy}{dx} = -(x + y)$$

$$\Longrightarrow xdx = dy \quad \text{or} \quad \frac{dy}{dx} + y = -x$$

Solving, we get, $\frac{x^2}{2} = y + k$ or $ye^x = -\int xe^x dx + c$

$\Longrightarrow x^2 = 2y + c$ or $ye^x = -xe^x + e^x + c$

The general solution is $\boxed{(x^2 - 2y - c)(x + y - 1 - ce^{-x}) = 0}$

Note: We can solve $\frac{dy}{dx} = -(x + y)$ by reducing it to variable separable form.

Put $z = x + y,$ we get, $\frac{dz}{dx} = 1 + \frac{dy}{dx}$ or $\frac{dz}{dx} = 1 - z.$

Separating the variables, $\frac{dz}{z-1} = -dx$

On integration we get, $\log(z-1) = -x + k$

$\Longrightarrow \quad z - 1 = e^{-x+k} \quad$ or $\quad x + y - 1 = ce^{-x}$

5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0 \quad \Longrightarrow \; p^2(p+2x) - y^2p(p+2x) = 0.$

Or $p(p-y^2)(p+2x) = 0.$

$\therefore \quad \frac{dy}{dx} = 0, \quad \frac{dy}{dx} = y^2 \quad$ and $\quad \frac{dy}{dx} = -2x$.

$\Longrightarrow dy = 0, \quad \frac{dy}{y^2} = dx \quad$ and $\quad dy = -2xdx$

On integration we get, $y = c, \quad -\frac{1}{y} = x + c \quad$ and $\quad y = -x^2 + c$

The general solution is $\boxed{(y-c)(xy+1+cy)(x^2+y-c) = 0}$

B. Clairaut's equation:

1. $p = \sin(y - xp) \Longrightarrow y = px + \sin^{-1} p$, which is Clairaut's equation.

   Then its general solution is $y = cx + \sin^{-1} c$.

   To find the singular solution, consider $x = -f'(c) = -\frac{1}{\sqrt{1-c^2}} \Longrightarrow 1 - c^2 = \frac{1}{x^2} \quad \Longrightarrow c = \frac{\sqrt{x^2-1}}{x}$

   $\therefore$ Singular solution is $y = \sqrt{x^2-1} + \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right).$

2. $(px - y)(py + x) = a^2p.$

   Put $x^2 = X, \quad y^2 = Y$. Then $2xdx = dX, \quad 2ydy = dY$ and $p = \frac{dy}{dx} = \frac{x}{y}\frac{dY}{dX} = \frac{x}{y}P.$

   The given equation becomes $\left(\frac{x^2}{y}P - y\right)(xP + x) = a^2\frac{x}{y}P \quad \Longrightarrow \quad (XP - Y)(P+1) = a^2P$

   Or $Y = XP - \frac{a^2P}{(P+1)}$ which is Clairaut's equation.

   General solution is $Y = cX - \frac{a^2c}{(c+1)}$.

   $\therefore$ The general solution of the given equation is $y^2 = cx^2 - \frac{a^2c}{(c+1)}$.

3. $y = px - \sqrt{1+p^2}$.

   Since equation is in Clairaut's form. The general solution is $y = cx - \sqrt{1+c^2}$.

   $x = -f'(c) = \frac{c}{\sqrt{1+c^2}} \quad \Longrightarrow \frac{1+c^2}{c^2} = \frac{1}{x^2} \quad \Longrightarrow \frac{1}{c^2} = \frac{1}{x^2} - 1 \quad \Longrightarrow c = \frac{x}{\sqrt{1-x^2}}$.

   $\therefore$ The singular solution is $y = \frac{x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \quad \Longrightarrow y = -\sqrt{1-x^2}.$

   Or $\quad x^2 + y^2 = 1$.

4. $y + 2p^2 = (x+1)p$. $\Longrightarrow y = px + (p - 2p^2)$

The general solution is $y = cx + (c - 2c^2)$.

$$x = -f'(c) = -(1-4c) \quad \Rightarrow \quad c = \frac{x+1}{4}.$$

Substituting in the general solution, $y = \frac{x(x+1)}{4} + \frac{x+1}{4} - \frac{(x+1)^2}{8} = \frac{(x+1)^2}{4} - \frac{(x+1)^2}{8}$

∴ The singular solution is $y = \frac{(x+1)^2}{8}$.

5. $(y - px)(p-1) = p. \Rightarrow y = px + \frac{p}{p-1}$ $\Rightarrow$The general solution is $y = cx + \frac{c}{c-1}$.

$x = -f'(c) = \frac{1}{(c-1)^2}$ $\Rightarrow c = \frac{1+\sqrt{x}}{\sqrt{x}}$ ∴ The singular solution is $y = x + 2\sqrt{x} + 1$.

6. $x^2(y - px) = yp^2$.

Put $x^2 = X, \ \ y^2 = Y$. Then $2xdx = dX, \ \ 2ydy = dY$ and $p = \frac{dy}{dx} = \frac{x}{y}\frac{dY}{dX} = \frac{x}{y}P$.

The given equation becomes $x^2\left(y - \frac{x^2P}{y}\right) = \frac{x^2P^2}{y}$ $\Rightarrow$ $Y = XP + P^2$.

The general solution is $Y = cX + c^2$.

∴ The general solution of the given equation is $y^2 = cx^2 + c^2$.

Exercise: Solve the following equations

A. 1. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$. 2. $p^2 + 2py\cot x = y^2$. 3. $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$.
4. $xyp^2 - (x^2+y^2)p + xy = 0$. 5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$.
6. $p^2 + 2p\cosh x + 1 = 0$.

B. Find general solution and singular solution of the following.

1. $p = \sin(y - xp)$. 2. $y = xp + \frac{a}{p}$. 3. $y = px - \sqrt{1+p^2}$.
4. $y + 2p^2 = (x+1)p$. 5. $xp^2 + px - py + 1 - y = 0$. 6. $p = \log(px - y)$

C. Solve by reducing it into Clairaut's form $(X = x^2, \ Y = y^2)$
1. $(px - y)(py + x) = a^2p$. 2. $x^2(y - px) = yp^2$.
3. Solve by reducing $p^2y + 2px = y$ into Clairaut's form $(X = 2x, \ Y = y^2)$.
4. Solve by reducing $y^2(y - px) = x^4p^2$ into Clairaut's form $(X = \frac{1}{x}, \ Y = \frac{1}{y})$

Self-Study:

1.Equations solvable for x : Equations of the type $x = f(y, \ p)$ are solvable for x .
Differentiating with respect to y we get, $\frac{1}{p} = \emptyset\left(y, \ p, \ \frac{dp}{dy}\right)$ and solve. Let $F(y,p,c) = 0$ be its solution.
Then eliminate p using the given equation to get the solution.
Or write the parametric equation $x = F_1(p, \ c)$, $y = F_2(p, \ c)$.

Solve the following equations

1. $y = 2px + y^2p^3$.

Given equation is $x = \frac{y-y^2p^3}{2p}$

Differentiating w.r.to y, we get,

$$\frac{1}{p} = \frac{p\left(1-3y^2p^2\frac{dp}{dy}-2yp^3\right)-(y-y^2p^3)\frac{dp}{dy}}{2p^2} \implies 2p = p - 2yp^4 - (2y^2p^3+y)\frac{dp}{dy} .$$

$$\implies p(1+2yp^3) + y(1+2yp^3)\frac{dp}{dy} = 0 .$$

Or $p + y\frac{dp}{dy} = 0 \implies p = \frac{c}{y}$.

Substituting in the given equation, $x = \frac{y-y^2\left(\frac{c}{y}\right)^3}{2\left(\frac{c}{y}\right)}$

The general solution is $\boxed{2cx = y^2 - c^3}$.

2. $p^3 - 4xyp + 8y^2 = 0.$

Given equation is $x = \frac{p^3+8y^2}{4yp}$

Differentiating w.r.to y, we get,

$$\frac{1}{p} = \frac{yp\left(3p^2\frac{dp}{dy}+16y\right)-(p^3+8y^2)\left(y\frac{dp}{dy}+p\right)}{4y^2p^2} .$$

$$\implies 4y^2p = (2p^3y - 8y^3)\frac{dp}{dy} + 8y^2p - p^4$$

$$\implies (p^3-4y^2)2y\frac{dp}{dy} - p(p^3-4y^2) = 0 .$$

$\implies 2y\frac{dp}{dy} - p = 0$, Or $2\frac{dp}{p} - \frac{dy}{y} = 0$. $\therefore$ $p = \sqrt{cy}$

Substituting in the given equation, $\sqrt{cy}(cy - 4xy) + 8y^2 = 0$

$$\implies (4x-c)^2 = \frac{64y^2}{cy} .$$

The general solution is $\boxed{y = k(x-k)^2}$.

Or parametric solution is $x = \frac{c^2+8p}{4c}$ *and* $y = \frac{p^2}{c}$.

3. $x - yp = ap^2$.

Given equation is $x = yp + ap^2$

Differentiating w.r.to y, we get, $\frac{1}{p} = y\frac{dp}{dy} + p + 2ap\frac{dp}{dy}$

$$\implies (p^2-1)\frac{dy}{dp} + (yp + 2ap^2) = 0$$

$$\implies \frac{dy}{dp} + \frac{yp}{(p^2-1)} = -\frac{2ap^2}{(p^2-1)} \qquad \text{I.F.} = \sqrt{p^2-1}$$

Solution is $y\sqrt{p^2-1} = -\int \frac{2ap^2}{\sqrt{p^2-1}}dp + c$

Or $y\sqrt{p^2-1} = -a\left(\cosh^{-1}p + p\sqrt{p^2-1}\right) + c$

Parametric solution is $y = \frac{-a\left(\cosh^{-1}p+p\sqrt{p^2-1}\right)+c}{\sqrt{p^2-1}}$ *and* $x = \frac{-a\left(\cosh^{-1}p+p\sqrt{p^2-1}\right)+c}{\sqrt{p^2-1}}p + ap^2$.

4. $p = \tan[x - \frac{p}{1+p^2}].$

Given equation is $x = \tan^{-1}p + \frac{p}{1+p^2}$.

Differentiating w.r.to y, we get, $\frac{1}{p} = \left[\frac{1}{1+p^2} + \frac{(1+p^2)-2p^2}{(1+p^2)^2}\right]\frac{dp}{dy}$

$$\Rightarrow \quad dy = \frac{2p}{(1+p^2)^2}dp$$

On integration we get the general solution is $y = -\frac{1}{1+p^2} + c$ and $x = \tan^{-1}p + \frac{p}{1+p^2}$.

2.Equations solvable for y : Equations of the type $y = f(x,\ p)$ are solvable for y.

Differentiating with respect to x we get, $p = \emptyset\left(x,\ p,\ \frac{dp}{dx}\right)$ and solve. Let $F(x,p,c) = 0$ be its solution.

Then eliminate p using the given equation to get the solution.

Or write the parametric equation $x = F_1(\ p,\ c)$, $y = F_2(p,\ c)$.

Solve the following equations

1. $y - 2px = \tan^{-1}(xp^2)$.

Given equation is $y = 2px + \tan^{-1}(xp^2)$

Differentiating w.r.to x, $p = 2p + 2x\frac{dp}{dx} + \left[\frac{1}{1+x^2p^4}\right]\left(p^2 + 2xp\frac{dp}{dx}\right)$

$$\Rightarrow \left(p + 2x\frac{dp}{dx}\right) + \left(p + 2x\frac{dp}{dx}\right)\left[\frac{p}{1+x^2p^4}\right] = 0$$

$\therefore\ p + 2x\frac{dp}{dx} = 0$ Or $1 + \frac{p}{1+x^2p^4} = 0$

$\Rightarrow \frac{2dp}{p} + \frac{dx}{x} = 0$, on integration , $p^2x = c$ or $p = \sqrt{\frac{c}{x}}$.

Substituting in the given equation, the general solution is $\boxed{y = 2\sqrt{cx} + \tan^{-1}c}$.

Note: $1 + \frac{p}{1+x^2p^4} = 0$ gives singular solution.

2. $y = 2px + p^n$

Differentiating w.r.to x, $p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx}$

$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2}$, which is linear equation.

Solution is $xp^2 = -n\frac{p^{n+1}}{n+1} + c$

$\therefore$ parametric solution of the given equation is

$$x = -n\frac{p^{n-1}}{n+1} + cp^{-2} \quad \text{and} \quad y = -n\frac{2p^n}{n+1} + 2cp^{-1} + p^n = \frac{2c}{p} + \frac{1-n}{1+n}p^n$$

3. $y = x + a\tan^{-1}p$.

Differentiating w.r.to x, $p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$ or $\frac{a}{1+p^2}\frac{dp}{dx} = p - 1$

$$\Rightarrow \quad dx = \frac{a\,dp}{(p-1)(1+p^2)}$$

i.e. $dx = \frac{a}{2}\left[\frac{1}{p-1} - \frac{p}{1+p^2} - \frac{1}{1+p^2}\right]dp$

$\therefore$ Parametric solution is $x = \frac{a}{2}\left[\log\left(\frac{p-1}{\sqrt{1+p^2}}\right) - \tan^{-1}p\right] + c$ and $y = \frac{a}{2}\left[\log\left(\frac{p-1}{\sqrt{1+p^2}}\right) + \tan^{-1}p\right] + c$.

4. $y + px = x^4p^2$.

Given equation is $y = -px + x^4p^2$.

Differentiating w.r.to x, we get, $p = -p - x\frac{dp}{dx} + 2x^4 p\frac{dp}{dx} + 4x^3p^2$

$\Rightarrow \left(2p + x\frac{dp}{dx}\right) - 2x^3p\left(2p + x\frac{dp}{dx}\right) = 0$,

Or, $2p + x\frac{dp}{dx} = 0 \quad \Rightarrow \frac{dp}{p} + 2\frac{dx}{x} = 0. \quad \therefore \quad p = \frac{c}{x^2}$.

Substituting in the given equation,

The general solution is $y + \frac{c}{x} = c^2$. Or $\boxed{xy = xc^2 - c.}$

5. $x^2p^4 + 2px - y = 0.$

Given equation is $y = 2px + x^2p^4$.

Differentiating w.r.to x, we get, $p = 2p + 2x\frac{dp}{dx} + 4x^2p^3\frac{dp}{dx} + 2xp^4$

$\Rightarrow \left(p + 2x\frac{dp}{dx}\right) + 2xp^3\left(p + 2x\frac{dp}{dx}\right) = 0$,

Or, $p + 2x\frac{dp}{dx} = 0 \quad \Rightarrow 2\frac{dp}{p} + \frac{dx}{x} = 0. \quad \therefore \quad p = \sqrt{\frac{c}{x}}$.

Substituting in the given equation,

The general solution is $\boxed{y = 2\sqrt{cx} + c^2}$.

6. $xp^2 + x = 2yp.$

Given equation is $y = \frac{xp^2+x}{2p}.$

Differentiating w.r.to x, we get,

$p = \frac{p\left(2xp\frac{dp}{dx} + p^2 + 1\right) - (xp^2+x)\frac{dp}{dx}}{2p^2} \Rightarrow 2p^3 = x(p^2-1)\frac{dp}{dx} + p^3 + p$

$\Rightarrow x(p^2-1)\frac{dp}{dx} - p(p^2-1) = 0$,

Or, $x\frac{dp}{dx} - p = 0 \quad \Rightarrow \frac{dp}{p} - \frac{dx}{x} = 0. \quad \therefore \quad p = cx$.

Substituting in the given equation,

The general solution is $\boxed{2cy = c^2x^2 + 1}$.

7. $y = xp^2 + p.$

Differentiating w.r.to x, we get,

$p = 2xp\frac{dp}{dx} + p^2 + \frac{dp}{dx}$

$\Rightarrow -p(p-1)\frac{dx}{dp} = 2xp + 1$

Or, $\frac{dx}{dp} + \frac{2x}{p-1} = -\frac{1}{p(p-1)}$.

Solution is $x(p-1)^2 = \log p - p + c$.

The general solution in parametric form is $x = \frac{\log p - p + c}{(p-1)^2}$ and $y = \frac{(\log p - p + c)p^2}{(p-1)^2} + p$.

8. $y = p\sin p + \cos p.$

Differentiating w.r.to x, we get,

$p = p\frac{dp}{dx}\cos p + \sin p - \sin p$ or $\frac{dp}{dx}\cos p = 0$

The general solution in parametric form is $x = \sin p + c$ and $y = p\sin p + \cos p$.

Or $\boxed{y = (x-c)\sin^{-1}(x-c) + \sqrt{1-(x-c)^2}}$.