Unit III : Fourier Analysis

Joseph Fourier developed a technique to analyse non-sinusoidal waveforms.

Exponential form :- Fourier demonstrated that a periodic function $f(t)$ can be expressed as sum of sinusoidal functions.

As per Fourier representation,

$$f(t) = a_0 + \sum_{n=1}^{\infty} M_n \cos(n\omega_0 t + \theta_n) \quad \text{--- (1)}$$

$\boxed{\text{A function } f(t) \text{ is periodic if } f(t) = f(t+T_0)}$

where $\omega_0 = 2\pi/T_0$,

$T_0 \rightarrow$ time period

when $n=1$, one cycle covers T_0 seconds while $M_1 \cos(\omega_0 t + \theta_1)$ is termed as fundamental. Taking $n=2$, T_0 represents two cycles in T_0 seconds and term $M_2 \cos(2\omega_0 t + \theta_2)$ is called the 2nd harmonic and so on i.e. for $n=K$, K cycles are covered in T_0 seconds and $M_K \cos(K\omega_0 t + \theta_K)$ is called the Kth harmonic term.

Using Euler's identity,

$$f(t) = a_0 + \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$n \neq 0$, $c_n \rightarrow$ complex Fourier coefficients

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{--- Tri}$$

C_n can be evaluated as

$$\because \quad f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

multiplying both sides by $e^{-jk\omega_0 t}$ and integrating over the interval t_1 to $(t_1 + T_0)$ we get

$$\int_{t_1}^{t_1+T_0} f(t)\, e^{-jk\omega_0 t}\, dt = \int_{t_1}^{t_1+T_0} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \cdot e^{-jk\omega_0 t}\, dt$$

$$= C_k T_0$$

$$\because \quad \int_{t_1}^{t_1+T_0} e^{j(n-k)\omega_0 t}\, dt = \begin{cases} 0 & \text{for } n \neq k \\ T_0 & \text{for } n = k \end{cases}$$

$\therefore$ Fourier coefficients are defined by the expression

$$C_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t)\, e^{-jn\omega_0 t}\, dt \quad \text{--- (3)}$$

expression (3) represents the exponential form of Fourier series.

Trigonometric form of Fourier series :-

From eqn (3), we can write

$$2C_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\left[\cos n\omega_0 t - j \sin n\omega_0 t\right] dt$$

or $$2C_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t) \cos n\omega_0 t$$

$$- j\,\frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t) \sin n\omega_0 t\, dt \quad \text{--- (4)}$$

$\because$ C_n is complex coefficient

$\therefore$ $2C_n = a_n - j b_n$ --- (5)

comparing eqns (4) & (5)

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t) \cos n\omega_0 t\, dt \quad \text{--- 6(a)}$$

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t) \sin n\omega_0 t\, dt \quad \text{--- 6(b)}$$

Also from eqn (3) C_0 being written as a_0 and is given by

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t)\, dt \quad \text{--- (7)}$$

$a_0 \rightarrow$ average value and can be don tu-
tly evaluated from the waveform.

Since the periodic function can
represented as sum of sinusoidal fun-
ctions, we can rewrite the eq^n ① as
below

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 \rightarrow$ constant

$a_1, a_2, a_3 \ldots a_n$ and $b_1, b_2, b_3 \ldots b_n$
are amplitudes of different harmonics.
$x =$ variable and $n =$ an integer $1, 2, \ldots$

Fourier series can also be expressed
in terms of either sine or cosine terms.

Let $M = a_n \cos nx + b_n \sin nx$

$$= \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2+b_n^2}} \cos nx + \frac{b_n}{\sqrt{a_n^2+b_n^2}} \sin nx \right]$$

let $\frac{a_n}{\sqrt{a_n^2+b_n^2}} = \sin \phi_n$ and $\frac{b_n}{\sqrt{a_n^2+b_n^2}} = \cos \phi_n$

$$\therefore M = \sqrt{a_n^2+b_n^2} \left[\sin\phi_n \cos nx + \cos\phi_n \sin nx \right]$$

$$M = K_n \sin(nx + \phi_n)$$

$$\boxed{\text{here } \tan\phi_n = \frac{a_n}{b_n} \quad \ldots 8(a)}$$

where $K_n = \sqrt{a_n^2 + b_n^2}$

Thus finally we can write

$$f(t) = a_0 + K_1 \sin(x + \phi_1) + K_2 \sin(2x + \phi_2) + \ldots + K_n \sin(nx + \phi_n) \quad \ldots ⑧$$

on the other hand if we put

$$\frac{a_n}{\sqrt{a_n^2+b_n^2}} = \cos\phi_n \quad \text{and} \quad \frac{b_n}{\sqrt{a_n^2+b_n^2}} = \sin\phi_n$$

then $M = \sqrt{a_n^2+b_n^2}\,[\cos\phi_n \cos n\alpha + \sin\phi_n \sin n\alpha]$

$M = K_n \cos(n\alpha - \phi_n)$

i.e. $f(t) = a_0 + K_1 \cos(\alpha - \phi_1) + K_2 \cos(2\alpha - \phi_n)$

$+ K_3 \cos(3\alpha - \phi_n) + \cdots + K_n \cos(n\alpha - \phi_n)$

Ex: find the period of the function

$f(t) = \cos t/3 + \cos t/4$

Soln:- since $f(t)$ is periodic

$f(t) = f(t+T)$

$\cos t/3 + \cos t/4 = \cos\frac{1}{3}(t+T) + \cos\frac{1}{4}(t+T)$

we know $\cos\theta = \cos(\theta + 2n\pi)$

on equating corresponding terms on both sides

$t/3 + 2m\pi = \frac{1}{3}(t+T)$

$\Rightarrow 2m\pi = \frac{T}{3} \Rightarrow T = 6m\pi$ and

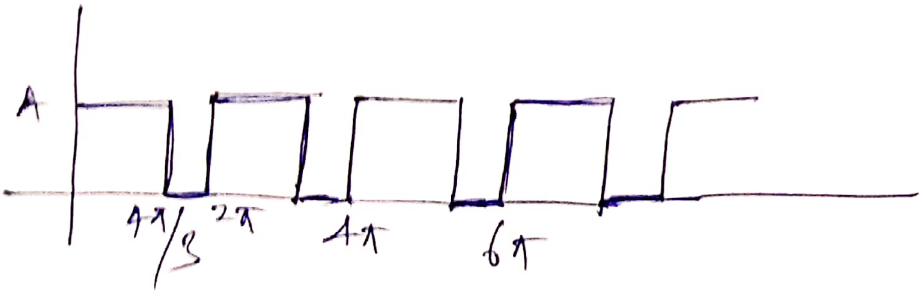
$\Rightarrow T = 6\pi, 12\pi, 18\pi, 24\pi, 30\pi, \ldots$

$\frac{t}{4} + 2n\pi = \frac{1}{4}(t+T) \Rightarrow T = 8n\pi$

$\Rightarrow T = 8\pi, 16\pi, 24\pi, 32\pi \ldots$

therefore, smallest period, $T = 24\pi$

ex 15.1 :- Obtain the coefficient of the exponential Fourier series for the waveform show below:



Soln :-

$$C_n = \frac{1}{T_0}\int_0^{T_0} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{2\pi}\int_0^{4\pi/3} A\, e^{-jn\left(2\pi \times \frac{1}{T_0}\right)t}\, dt$$

$$= \frac{1}{2\pi}\int_0^{4\pi/3} A\, e^{-jnt}\, dt$$

$$= \frac{1}{2\pi}\int_0^{4\pi/3} A\, e^{-jnt}\, dt$$

$$\cancel{\frac{-jA}{2\pi n}\left[e^{-j\left(\frac{4\pi n}{3}\right)} - 1\right]}$$

$$= \frac{A}{2\pi}\int_0^{4\pi/3} e^{-jnt}\, dt$$

$$= \frac{A}{2\pi}\left[\frac{e^{-jnt}}{-jn}\right]_0^{4\pi/3}$$

$$= \frac{-A}{j2\pi n}\left[e^{-jnt}\right]_0^{4\pi/3} = \frac{jA}{2\pi n}\left[e^{\frac{-j4\pi n}{3}} - 1\right]$$

Symmetry in Fourier series :- When a signal exhibit symmetrical properties, simplification procedure can be adopted in Fourier series. There are three types of symmetries :

(a) Even function symmetry
(b) Odd function symmetry
(c) Half wave symmetry

(a) Even function symmetry :- A function is said to be even if

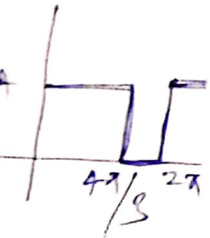
$$f(-t) = f(t)$$

An even function is symmetrical about the vertical axis. In order to determine the coefficients of the Fourier series, the conditions of symmetry being applied, let $t_1 = -T_0/2$

$$\therefore \quad a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t)\,dt = \frac{1}{T_0}\int_{-T_0/2}^{-T_0/2+T_0} f(t)\,dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\,dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{0} f(t)\,dt + \frac{1}{T_0}\int_{0}^{T_0/2} f(t)\,dt$$



Let us now change the variable of the first integral so that $t = -x$ in the first integral.

Then $f(-x) = f(x)$, $dt = d(-x) = -dx$ while the range of integration is from $x = T_0/2$ to 0.

$$\therefore \; a_0 = \frac{1}{T_0}\int_{T_0/2}^{0} f(x)(-dx) + \frac{1}{T_0}\int_0^{T_0/2} f(t)\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0/2} f(x)\,dx + \frac{1}{T_0}\int_0^{T_0/2} f(t)\,dt$$

$$\therefore \; \boxed{a_0 = \frac{2}{T_0}\int_0^{T_0/2} f(t)\,dt}$$

Similarly

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{0} f(t)\cos n\omega_0 t\,dt + \frac{2}{T_0}\int_0^{T_0/2} f(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{T_0}\int_{T_0/2}^{0} f(x)\cos(-n\omega_0 x)(-dx) + \frac{2}{T_0}\int_0^{T_0/2} f(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{T_0}\int_0^{T_0/2} f(x)\cos(n\omega_0 x)\,dx + \frac{2}{T_0}\int_0^{T_0/2} f(t)\cos(n\omega_0 t)\,dt$$

$$\boxed{a_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\cos n\omega_0 t\,dt}$$

x, Also for the other coefficient

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{0} f(t)\sin(n\omega_0 t)\,dt + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_{T_0/2}^{0} f(x)\sin(-n\omega_0 x)(-dx) + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\sin(n\omega_0 t)\,dt$$

$$= -\frac{2}{T_0}\int_{0}^{T_0/2} f(x)\sin(n\omega_0 x)\,dx + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\sin n\omega_0 t\,dt$$

$\therefore b_n = 0$

Thus we see that for even function symmetry

$$a_0 = \frac{2}{T_0}\int_{0}^{T_0/2} ft\,dt$$

$$a_n = \frac{4}{T_0}\int f(t)\cos(n\omega_0 t)\,dt$$

& $b_n = 0$

cosine wave is a even function and the sum or product of two or more even functions is a even function. with addition of a constant, the even nature is still present.

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(5) Odd function symmetry :- A function is said to be odd if

$$f(-t) = -f(t)$$

We have seen that

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{0} f(t)\,dt + \frac{1}{T_0}\int_{0}^{T_0/2} f(t)\,dt$$

for $t = -x$, the first integral becomes $\frac{1}{T_0}\int_{T_0/2}^{0} -f(x)(-dx)$

$$\therefore a_0 = \frac{1}{T_0}\int_{T_0/2}^{0} -f(x)(-dx) + \frac{1}{T_0}\int_{0}^{T_0/2} f(t)\,dt$$

$$= -\frac{1}{T_0}\int_{0}^{T_0/2} f(x)\,dx + \frac{1}{T_0}\int_{0}^{T_0/2} f(t)\,dt$$

$$\therefore a_0 = 0$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{0} f(t)\cos(n\omega_0 t)\,dt + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_{T_0/2}^{0} -f(x)\cos(-n\omega_0 x)(-dx) + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_{T_0/2}^{0} f(x)\cos(n\omega_0 x)\,dx + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\cos(n\omega_0 t)\,dt$$

$$= -\frac{2}{T_0}\int_{0}^{T_0/2} f(x)\cos(n\omega_0 x)\,dx + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\cos(n\omega_0 t)\,dt$$

$$\therefore a_n = 0$$

and $b_n = \frac{2}{T_0}\int_{-T_0/2}^{0} f(t)\sin(n\omega_0 t)\,dt + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\sin(n\omega_0 t)\,dt$

$$= \frac{2}{T_0}\int_{T_0/2}^{0} -f(x)\sin(-n\omega_0 x)(-dx) + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\sin(n\omega_0 t)\,dt$$

$$= -\frac{2}{T_0}\int_{T_0/2}^{0} f(x)\sin(n\omega_0 x)\,dx + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\sin n\omega_0 t\,dt$$

$$= \frac{2}{T_0}\int_{0}^{T_0/2} f(x)\sin(n\omega_0 x)\,dx + \frac{2}{T_0}\int_{0}^{T_0/2} f(t)\sin n\omega_0 t\,dt$$

$$= \frac{4}{T_0}\int_{0}^{T_0/2} f(t)\sin(n\omega_0 t)\,dt$$

Thus for odd symmetry

$a_0 = 0$

$a_n = 0$

& $b_n = \frac{4}{T_0}\int_{0}^{T_0/2} f(t)\sin(n\omega_0 t)\,dt$

Examples of even and odd symmetry are shown below

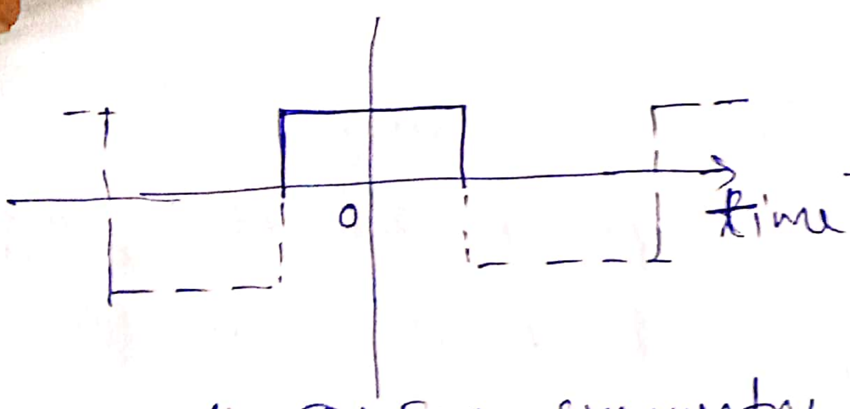


fig (a): Even symmetry

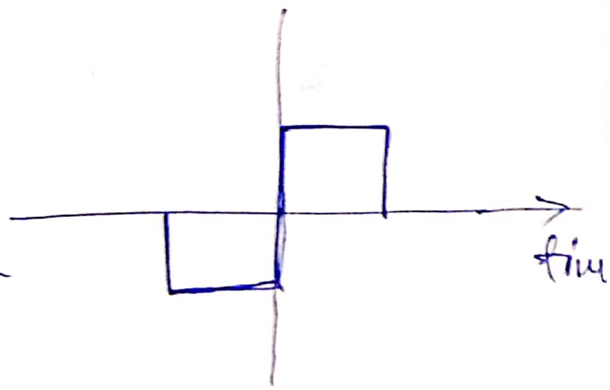
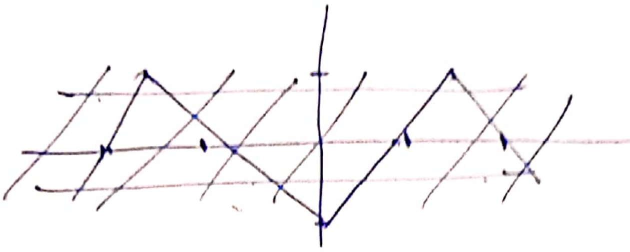


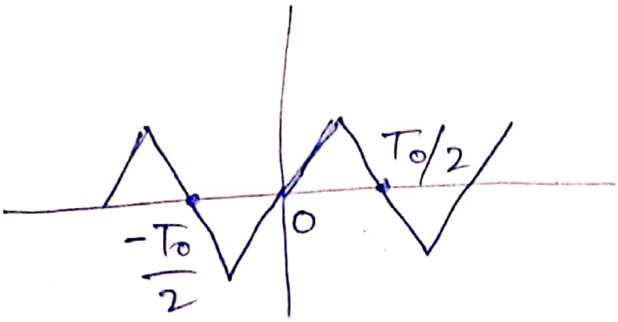
fig (b): odd symmetry

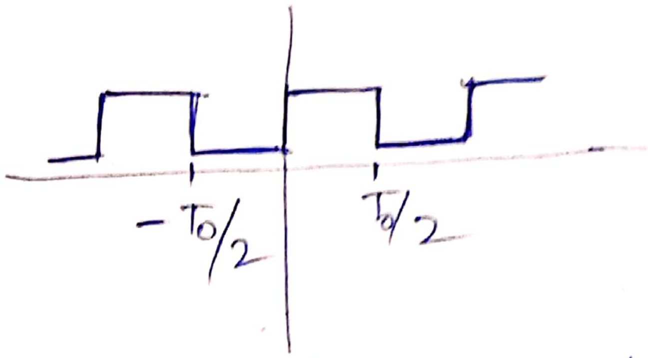
Half wave symmetry :- A function is said to have half wave symmetry if

$$-f\left(t - \frac{T_0}{2}\right) = f(t)$$

i.e. the half cycle is an inverted version of the adjacent half cycle.
i.e. if the waveform from $-T_0/2$ to 0 is inverted then it becomes identical to the waveform from 0 to $T_0/2$.







Mathematically it can be shown that by changing the first variable of the Fourier coefficient expression for $t = x + T_0/2$ and

$$f\left(t - T_0/2\right) = -f(t)$$

$a_0 = 0$, $a_n = 0$ & $b_n = 0$ when n is even.

and $a_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\cos(n\omega_0 t)\,dt$ when n is odd

$b_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\sin(n\omega_0 t)\,dt$ when n is odd.