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gnal System (EC : Sem-4)
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CONTENTS

KEC 403 : SIGNAL SYSTEM

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Q. 9. What are the linear and non-linear systems ?

Ans. Refer Q. 1.15.

Q. 10. Check whether the system is :

$$y(t) = E_v\,[x(t)]$$

i. **Static or dynamic.**
ii. **Linear or non-linear.**
iii. **Causal or non-causal.**
iv. **Time-variant or invariant.**

Ans. Refer Q. 1.21.

Q. 11. Check whether the system with impulse response is :

$$y[n] = \sum_{k=-\infty}^{n+5} x[k]$$

i. **Causal/non-causal**
ii. **Stable/unstable**

Ans. Refer Q. 1.24.

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2 UNIT

Linear Shift-Invariant (LSI) System

CONTENTS

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PART-1

Linear Shift-Invariant (LSI) Systems, Impulse Response and Step Response, Convolution, Input-Output Behaviour With Aperiodic Convergent Inputs.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. What are linear shift-invariant (LSI) systems ?

Answer

LSI systems are a class of systems used in signals and systems that ar both linear and shift-invariant.

1. Linear systems are systems whose outputs for a linear combination o inputs are the same as a linear combination of individual responses t those inputs.
2. Shift-invariant systems are systems where the output does not depen on when an input was applied.

Que 2.2. Define the following terms :

i. Impulse response function of an LTI system.
ii. Unit-sample response sequence of an LSI system.

Answer

i. Impulse response of systems :

The impulse response, $h(t)$, of a linear time-invariant continuous-tim system is defined as the response of the system to a unit impulse give to it as input when the system is in ground state.

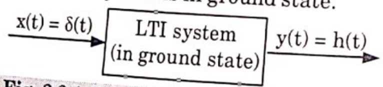

Fig. 2.2.1. Impulse response of a system.

ii. Unit-sample response sequence :

1. The unit sample response sequence, $h[n]$, of a linear shift-invarian (LSI) discrete-time system is defined as the response of the system i ground state, to a unit-sample sequence applied to it as input.
2. A discrete-time system is said to be in ground state if all its memor elements contain only zeros.

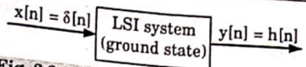

Fig. 2.2.2: Unit-sample response sequence.

Que 2.3. Explain impulse response characterization of LTI systems ?

OR

Write a short note on convolution integral.

Answer

1. An LTI system could be completely characterized by its impulse response function, $h(t)$.
2. We know that

$$x(t) \xrightarrow{T} y(t) = T[x(t)]$$

but,
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

so,
$$y(t) = T\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right]$$

$$= \int_{-\infty}^{\infty} x(\tau)T[\delta(t-\tau)]d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

3. Therefore we have,

$$x(t) \xrightarrow{T} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = y(t) = h(t) \otimes x(t) \quad ...(2.3.1)$$

4. From eq. (2.3.1), we find that the output $y(t)$, of an LTI system in ground state to an input $x(t)$ is given by the convolution of the input signal $x(t)$ with the impulse response, $h(t)$, of the system.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad ...(2.3.2)$$

5. By a change of the variable, $\lambda = (t - \tau)$, we may write eq. (2.3.2) as

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \quad ...(2.3.3)$$

hence,
$$y(t) = x(t) \otimes h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

6. Integral on the RHS of eq. (2.3.3) and (2.3.2) are called convolution integrals, or the superposition integrals.

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7. From eq. (2.3.1), we find that a knowledge of the $h(t)$, the impulse response of the system would enable us to calculate the output, $y(t)$, of the system for any given input signal, $x(t)$. It is in this sense that we say that $h(t)$, the impulse response, completely characterizes an LTI system.

Que 2.4. **Explain unit-sample response characterization of an LSI system.**

OR

Write a short note on convolution sum.

Answer

1. An LSI system can be completely characterized by its unit-sample response sequence, $h(n)$.
   The sequence $x[n]$ can be written as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k]\} \quad ...(2.4.1)$$

2. If the discrete-time signal $x[n]$ is given as input to an LSI system T with $h[n]$ as its unit-sample response sequence, and if the output is $y[n]$, we can write

$$y[n] = T\{x[n]\} = T\left[\sum_{k=-\infty}^{\infty} x[k]\{\delta[n-k]\}\right]$$

$$= \sum_{k=-\infty}^{\infty} x[k]T[\{\delta[n-k]\}]$$

i.e.,
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\{h[n-k]\}$$

or,
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad ...(2.4.2)$$

3. RHS of eq. (2.4.2) is called the convolution sum of the two sequences $\{x[n]\}$ and $\{h[n]\}$. Knowledge of $\{h[n]\}$ will enable one to determine the output for any input using eq. (2.4.2). The unit-sample response, $\{h[n]\}$, thus completely characterizes the LSI system.

Que 2.5. **What are the properties of convolution ?**

Answer

1. Let us consider two signals $x_1(t)$ and $x_2(t)$. The convolution of two signals $x_1(t)$ and $x_2(t)$ is given by

$$x_1(t) \otimes x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau) = \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)d\tau$$

2. The properties of convolution are as follows :

i. **Commutative property :** The commutative property of convolution states that

$$x_1(t) \otimes x_2(t) = x_2(t) \otimes x_1(t)$$

ii. **Distributive property :** The distributive property of convolution states that

$$x_1(t) \otimes [x_2(t) + x_3(t)] = [x_1(t) \otimes x_2(t)] + [x_1(t) \otimes x_3(t)]$$

iii. **Associative property :** The associative property of convolution states that

$$x_1(t) \otimes [x_2(t) \otimes x_3(t)] = [x_1(t) \otimes x_2(t)] \otimes x_3(t)$$

iv. **Shift property :** The shift property of convolution states that
   if $\quad x_1(t) \otimes x_2(t) = z(t)$
   then $\quad x_1(t) \otimes x_2(t-T) = z(t-T)$
   $\quad x_1(t-T) \otimes x_2(t) = z(t-T)$
   and $\quad x_1(t-T_1) \otimes x_2(t-T_2) = z(t-T_1-T_2)$

v. **Convolution with an impulse :** Convolution of a signal $x(t)$ with a unit impulse is the signal itself, *i.e.*,

$$x(t) \otimes \delta(t) = x(t)$$

vi. **Width property :** Let the duration of $x_1(t)$ and $x_2(t)$ be T_1 and T_2 respectively. Then the duration of the signal obtained by convolving $x_1(t)$ and $x_2(t)$ is $T_1 + T_2$.

vii. **Differentiation property :**
   According to the differentiation property,
   if $\quad x(t) \otimes h(t) = y(t)$
   then $\quad \dfrac{dx(t)}{dt} \otimes h(t) = x(t) \otimes \dfrac{dh(t)}{dt} = \dfrac{dy}{dt}(t)$

viii. **Time-scaling property :**
   If $\quad x(t) \otimes h(t) = y(t),$
   then $\quad x(at) \otimes h(at) = \dfrac{1}{|a|}y(at)$

Que 2.6. **Obtain the convolution of**

$x(t) = u(t)$ and $h(t) = 1$ for $-1 \le t \le 1$ **AKTU 2014-15, Marks 05**

Answer

1. We can also write $h(t) = u(t+1) - u(t-1)$
2. We know that $x(t) \otimes h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$$= \int_{-\infty}^{\infty} [u(\tau+1) - u(\tau-1)]\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau+1)\, u(t-\tau)\, d\tau - \int_{-\infty}^{\infty} u(\tau-1)\, u(t-\tau)\, d\tau$$

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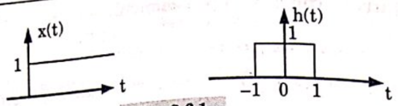

Fig. 2.6.1.

3. In the 1st part, $u(\tau + 1) = 1$ for $\tau > -1$ and $u(t - \tau) = 1$ for $\tau < t$, and in 2nd part, $u(\tau - 1) = 1$ for $\tau > 1$.

4. $$x(t) \otimes h(t) = \int_{-1}^{t} 1\, d\tau - \int_{1}^{t} 1\, d\tau = [\tau]_{-1}^{t} - [\tau]_{1}^{t}$$

$$= (t + 1)\, u\,(t + 1) - (t - 1)\, u\,(t - 1)$$

Que 2.7. **Derive the response of LTI system to arbitrary inputs.**

Answer

Here we are considering two arbitrary inputs :

i. **The unit step response :** The unit step response $s(t)$ of an LTI system is the output of the system for a unit step input, $u(t)$.

The unit step response can be obtained by the transfer functions as follows,

$$H(s) = \frac{Y(s)}{X(s)}$$

where, $x(t) = \text{input} = u(t) \rightarrow X(s) = 1/s$

$$Y(s) = H(s)\, X(s) = \frac{1}{s} H(s)$$

taking inverse Laplace transform

$$y(t) = h(t) \otimes u(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

ii. **The unit impulse response :** It is the output of the system for a unit impulse input $\delta(t)$.

$$H(s) = \frac{Y(s)}{X(s)}$$

$$Y(s) = H(s)\, X(s)$$

Since, $x(t) = \delta(t) \Rightarrow X(s) = 1$

So,
$$Y(s) = H(s)$$
$$y(t) = h(t)$$

Que 2.8. **Determine the impulse response and step response of the circuit is shown in Fig. 2.8.1.**

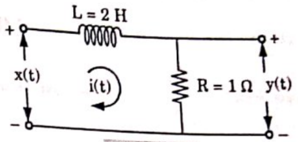

Fig. 2.8.1.

Answer

1. Writing KVL equation, we have

$$L \frac{di(t)}{dt} + R\, i(t) = x(t)$$

and
$$R\, i(t) = y(t)$$

$$\therefore \quad \frac{L}{R} \frac{dy(t)}{dt} + y(t) = x(t)$$

2. We will find the step response first and then determine the impulse response.

3. For step response :

$$x(t) = u(t)$$

$$\frac{L}{R} \frac{dy(t)}{dt} + y(t) = u(t)\,;\, t > 0$$

putting values of L and R and taking Laplace transform on both sides,

$$2[sY(s) - y(0)] + Y(s) = \frac{1}{s}$$

$$Y(s)\,[2s + 1] = \frac{1}{s} \qquad (\because \text{ Zero initial condition})$$

$$Y(s) = \frac{1}{s(2s+1)} = \frac{A}{s} + \frac{B}{(2s+1)}$$

so,
$$Y(s) = \frac{1}{s} - \frac{2}{(2s+1)} = \frac{1}{s} - \frac{1}{s + 1/2}$$

taking the inverse Laplace transform, we have

$$y(t) = g(t) = (1 - e^{-t/2})u(t)$$

4. Impulse response :

$$h(t) = \frac{d}{dt} g(t)$$

$$= \frac{d}{dt}[1 - e^{-t/2}]\, u(t) = 0 + \frac{1}{2} e^{-t/2}\, u(t) = \frac{1}{2} e^{-t/2}\, u(t)$$

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Que 2.9. **Determine the output sequence of the system with impulse response $h[n] = \left(\frac{1}{4}\right)^n u[n]$; when input is complex exponential sequence $x[n] = Ae^{\frac{j\pi n}{2}}$.**

AKTU 2014-15, Marks 05

Answer

1. Given, $h[n] = \left(\frac{1}{4}\right)^n u[n]$

   and $x[n] = Ae^{\frac{j\pi n}{2}}$

2. We know, $y[n] = h[n] \otimes x[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} Ae^{\frac{j\pi k}{2}} \left(\frac{1}{4}\right)^{(n-k)} u[n-k]$$

$$= \sum_{k=-\infty}^{n} Ae^{\frac{j\pi k}{2}} \left(\frac{1}{4}\right)^{n-k} = \sum_{k=-n}^{\infty} Ae^{\frac{-j\pi k}{2}} \left(\frac{1}{4}\right)^{n+k}$$

$$= \sum_{k=-n}^{\infty} A\left(\frac{1}{4}\right)^n \left(\frac{e^{-j\pi/2}}{4}\right)^k = A\left(\frac{1}{4}\right)^n \sum_{k=-n}^{\infty} \left(\frac{1}{4}e^{-j\pi/2}\right)^k$$

$$= A\left(\frac{1}{4}\right)^n \frac{\left(\frac{1}{4}e^{-j\pi/2}\right)^{-n}}{1-\left(\frac{1}{4}e^{-j\pi/2}\right)} = \frac{A\left(\frac{1}{4}\right)^n \left(\frac{1}{4}\right)^{-n} e^{\frac{j\pi n}{2}}}{1-\frac{1}{4}e^{-j\pi/2}}$$

so, $$y[n] = \frac{Ae^{\frac{j\pi n}{2}}}{1-\frac{1}{4}e^{-j\pi/2}}$$

Que 2.10. **Calculate the convolution for given sequence**

$$x[n] = \begin{cases} 1 & ;\ \text{for } n = -2, 0, 1 \\ 2 & ;\ \text{for } n = -1 \\ 0 & ;\ \text{else} \end{cases}$$

$$h[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3]$$

AKTU 2015-16, Marks 7.5

OR

Find the convolution for given sequence

$x[n] = 1$ for $n = -2, 0, 1$

2 for $n = -1$

0 else

and $h[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3]$

AKTU 2017-18, Marks 07

Answer

1. Given, $x[n] = \begin{cases} 1 & ;\ \text{for } n = -2, 0, 1 \\ 2 & ;\ \text{for } n = -1 \\ 0 & ;\ \text{else} \end{cases}$

   i.e., $x[n] = \{1, 2, \underset{\uparrow}{1}, 1\}$

   $h[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3]$

   i.e., $h[n] = \{\underset{\uparrow}{1}, -1, 1, -1\}$

2. Using multiplication method the convolution can be obtained as,

$x[n] \Rightarrow$				1	2	1	1
$h[n] \Rightarrow$				1	–1	1	–1
				–1	–2	–1	–1
			1	2	1	1	×
		–1	–2	–1	–1	×	×
	1	2	1	1	×	×	×
	1	1	0	1	–2	0	–1

3. The output is,

   $x[n] \otimes h[n] = \{1, 1, \underset{\uparrow}{0}, 1, -2, 0, -1\}$

Que 2.11. **An interconnection of LTI system is :**

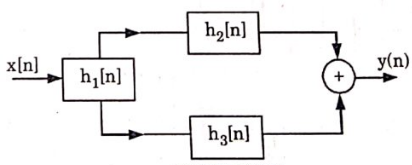

Fig. 2.11.1.

The impulse response are :

i. $h_1[n] = \left(\frac{1}{2}\right)^n \{u[n] - u[n-4]\}$

ii. $h_2[n] = \delta[n]$

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iii. $h_3[n] = u[n-2]$

Let impulse response of overall system from $x[n]$ to $y[n]$ be $h[n]$.

i. Express $h[n]$ in term of $h_1[n]$, $h_2[n]$ and $h_3[n]$.

ii. Evaluate $h[n]$.

AKTU 2015-16, Marks 7.5

Answer

1. Let,

$$h[n] = h_1[n] \otimes \{h_2[n] + h_3[n]\}$$
$$h[n] = h_1[n] \otimes h_2[n] + h_1[n] \otimes h_3[n]$$

2. $$h_1[n] = \left(\frac{1}{2}\right)^n .\{u[n] - u[n-4]\}$$

so, $$h_1[n] = \left[1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right]$$

and, $h_2[n] + h_3[n] = \delta[n] + u\ [n-2]$

hence, $h_2[n] + h_3[n] = [1, 0, 1, 1, 1, 1, ...]$

3. Let us consider,

$$h_2[n] + h_3[n] = h_4[n]$$

therefore, $$h[n] = h_1[n] \otimes h_4[n]$$

$$h[n] = \sum_{k=-\infty}^{\infty} h_1[k].h_4[-k+n]$$

$$h[n] = \sum_{k=-\infty}^{\infty} h_4[k].h_1[-k+n]$$

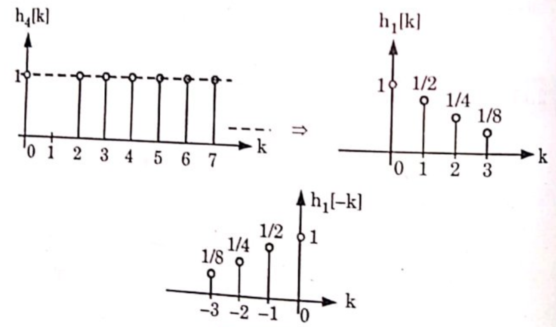
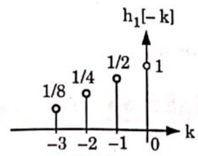


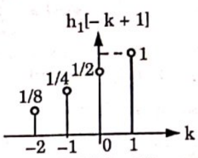
Fig. 2.11.2.

4. For $n = 0$



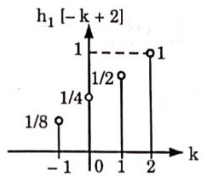
(a) h[0] = 1

For $n = 1$



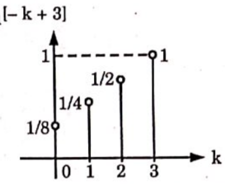
(b) h[1] = 1/2

5. For $n = 2$



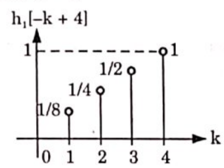
(c) h[2] = 1 + 1/4 = 5/4

For $n = 3$



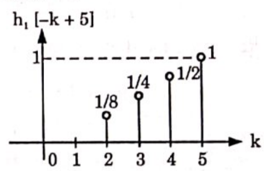
(d) h[3] = 1 + 1/2 + 1/8 = 13/8

6. For $n = 4$



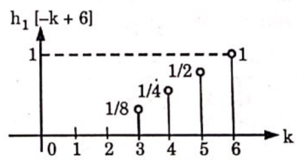
(e) h[4] = 1 + 1/2 + 1/4 + 0 = 7/4

For $n = 5$



(f) h[5] = 1 + 1/2 + 1/4 + 1/8 = 15/8

7. For $n = 6$



(g) h[6] = 1 + 1/2 + 1/4 + 1/8 = 15/8

Fig. 2.11.3. Impulse response of overall system.

8. Hence, the overall impulse response is,

$$h[n] = \left[1, \frac{1}{2}, \frac{5}{4}, \frac{13}{8}, \frac{7}{4}, \frac{15}{8}, \frac{15}{8} .....\right]$$

Que 2.12. **The accumulator is excited by the sequence $x[n] = nu[n]$. Accumulator can be defined by following input and output relationship.**

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$$y[n] = \sum_{k=-\infty}^{n} x[n]$$

Determine its output under the condition :

i. It is initially relaxed.

ii. Initially $y[-1] = 1$.

AKTU 2016-17, Marks 15

Answer

1. The output of the system is defined as

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{-1} x[k] + \sum_{k=0}^{n} x[k]$$

$$= y[-1] + \sum_{k=0}^{n} x[k]$$

$$= y[-1] + \frac{n(n+1)}{2}$$

2. If the system is initially relaxed, $y[-1] = 0$ and hence

$$y[n] = \frac{n(n+1)}{2}, n \geq 0$$

3. On the other hand, if the initial condition is $y[-1] = 1$, then

$$y[n] = 1 + \frac{n(n+1)}{2} = \frac{n^2+n+2}{2}, n \geq 0$$

PART-2

Characterization of Causality and Stability of Linear Shift Invariant Systems, System Representation Through Differential Equations and Difference Equations.

Questions-Answers
Long Answer Type and Medium Answer Type Questions

Que 2.13. Explain in detail about the causality of LSI systems.

Answer

1. The output of a causal system depends only on the present and past values of the input to the system.

2. The impulse response of a causal discrete-time LSI system satisfy the condition

$$h[n] = 0 \quad \text{for } n < 0 \qquad ...(2.13.1)$$

3. According to eq. (2.13.1) the impulse response of a causal LSI system must be zero before the impulse occurs, which is consistent with the intuitive concept of causality.
4. Causality for a linear system is equivalent to the condition of initial rest, *i.e.*, if the input to a causal system is 0 upto some point in time, then the output must also be 0 upto that time.
4. The equivalent of causality and the condition of initial rest applies only to linear system.
5. For a causal discrete-time LSI system, the condition in eq. (2.13.1) implies that the convolution sum representation becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] \qquad ...(2.13.2)$$

and the alternative equivalent form,

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] \qquad ...(2.13.3)$$

6. Similarly, a continuous-time LSI system is causal if

$$h(t) = 0 \quad \text{for } t < 0 \qquad ...(2.13.4)$$

and in this case, the convolution integral is given by

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

7. Causality of an LSI system is equivalent to its impulse response being a causal signal.

Que 2.14. Discuss about the stability of LSI systems.

Answer

1. A system is stable if every bounded input produced a bounded output. In order to determine conditions under which LSI system are stable, consider an input $x[n]$ that is bounded in magnitude :

$$|x[n]| < B \quad \text{for all } n \qquad ...(2.14.1)$$

2. Suppose that we apply this input to an LSI system with unit impulse response $h[n]$. Then, using the convolution sum, we obtain an expression for the magnitude of the output :

$$|y[n]| = \left|\sum_{k=-\infty}^{+\infty} h[k]x[n-k]\right| \qquad ...(2.14.2)$$

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3. Since the magnitude of the sum of a set of numbers is not larger th the sum of the magnitudes of the numbers, it follows from eq. (2.14 that

$$|y[n]| \le \sum_{k=-\infty}^{+\infty} |h[k]|\,|x[n-k]| \quad ...(2.14$$

4. From eq. (2.14.1) $|x[n-k]| < B$ for all values of k and n. Together wi eq. (2.14.3), this implies that

$$|y[n]| \le B \sum_{k=-\infty}^{+\infty} |h[k]| \quad \text{for all } n \quad ...(2.14$$

5. From eq. (2.14.4) we can conclude that if the impulse response is absolut summable, that is, if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty \quad ...(2.14$$

the $y[n]$ is bounded in magnitude, and hence, the system is stabl Therefore, eq. (2.14.5) is a sufficient condition to guarantee the stabili of a discrete-time LSI system.

6. In continuous time, we obtain an analogous characterization of stabili in terms of the impulse response of an LTI system. Specifically, i $|x(t)| < B$ for all t, then in analogy with eq. (2.14.2), (2.14.3) an eq. (2.14.4), it follows that.

$$|y(t)| = \left|\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau\right|$$

$$\le \int_{-\infty}^{+\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau$$

$$\le B\int_{-\infty}^{+\infty} |h(\tau)|\,d\tau$$

7. Therefore, the system is stable if the impulse response is absolutel integrable *i.e.*, if

$$\int_{-\infty}^{+\infty} |h(\tau)|\,d\tau < \infty \quad ...(2.14.6$$

Que 2.15. **Perform the stability check for the following system,**

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

Answer

1. Given, $h[n] = \left(\frac{1}{2}\right)^n u[n]$

2. The condition for a system to be stable

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

taking left hand side

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{2}\right)^n u[n]\right|$$

$$= \sum_{n=0}^{\infty} \left|\left(\frac{1}{2}\right)^n\right| = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + ...$$

$$= \frac{1}{1-\frac{1}{2}} = 2 < \infty$$

Hence, the system is stable.

Que 2.16. **Discuss about the linear constant-coefficient differential equation and linear constant-coefficient difference equations.**

Answer

Linear constant-coefficient differential equation :

1. A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad ...(2.16.1)$$

2. The order refers to the highest derivative of the output $y(t)$ appearing in the equation. In the case, when $N = 0$, eq. (2.16.1) reduces to

$$y(t) = \frac{1}{a_0}\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad ...(2.16.2)$$

3. In this case, $y(t)$ is an explicit function of the input $x(t)$ and its derivatives. For $N \ge 1$, eq. (2.16.1) specifies the output implicitly in terms of the input.

4. The solution $y(t)$ consists of two parts : a particular solution to eq. (2.16.1) plus a solution to the homogeneous differential equation.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0 \quad ...(2.16.3)$$

The solutions to this eq. (2.16.3) are referred to as the natural responses of the system.

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5. If $x(t) = 0$ for $t \le t_0$, we assume that $y(t) = 0$ for $t \le t_0$, and therefore, t response for $t > t_0$ can be calculated from the differential eq. (2.16 with the initial conditions,

$$y(t_0) = \frac{dy(t_0)}{dt} = \dots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0 \quad ...(2.16.$$

6. Under the condition of initial rest, the system described by eq. (2.16.1) causal and LTI.

Linear constant-coefficient difference equations :

1. The discrete-time counterpart of eq. (2.16.1) is the Nth-order line constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad ...(2.16.$$

2. An equation of this type can be solved in a manner exactly analogous t that for differential equations. Specifically, the solution $y[n]$ can b written as the sum of a particular solution to eq. (2.16.5) and a solutio to the homogeneous equation

$$\sum_{k=0}^{N} a_k y[n-k] = 0 \quad ...(2.16.6$$

3. The solutions to this homogeneous equation are often referred to as th natural responses of the system described by eq. (2.16.5).
4. If $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$ as well. With initial rest, th system described by eq. (2.16.5) is LSI and causal.

Que 2.17. **Write a differential equation description relating the output to the input of the circuit shown in Fig. 2.17.1.**

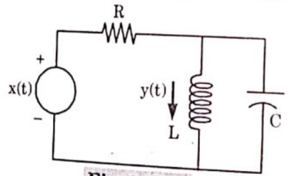

Fig. 2.17.1.

Answer

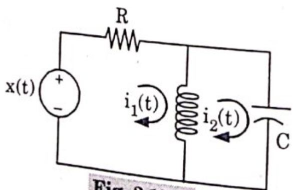

Fig. 2.17.2.

Now taking KVL in loop 1.

$$Ri_1(t) + L\frac{d}{dt}[i_1(t) - i_2(t)] = x(t) \quad ...(2.17.1)$$

Taking KVL in loop 2.

$$L\frac{d}{dt}[i_2(t) - i_1(t)] + \frac{1}{C}\int_{-\infty}^{t} i_2(t)dt = 0 \quad ...(2.17.2)$$

$$\therefore \quad y(t) = L\frac{d}{dt}[i_1(t) - i_2(t)] \quad ...(2.17.3)$$

Taking integration of eq. (2.17.3)

$$\int_{-\infty}^{t} y(t) = L[i_1(t) - i_2(t)]$$

or,

$$\frac{1}{L}\int_{-\infty}^{t} y(t) = [i_1(t) - i_2(t)] \quad ...(2.17.4)$$

4. From eq. (2.17.1) and eq. (2.17.3) then we get

$$Ri_1(t) + y(t) = x(t)$$

$$\therefore \quad i_1(t) = \frac{x(t) - y(t)}{R} \quad ...(2.17.5)$$

5. From eq. (2.17.2) and eq. (2.17.3) then we get

$$-y(t) + \frac{1}{C}\int_{-\infty}^{t} i_2(t)\,dt = 0$$

or,

$$-\frac{dy(t)}{dt} + \frac{1}{C}i_2(t) = 0$$

or,

$$i_2(t) = +C\frac{dy(t)}{dt} \quad ...(2.17.6)$$

6. Putting eq. (2.17.5) and (2.17.6) in eq. (2.17.4)

$$\frac{1}{L}\int_{-\infty}^{t} y(t) = \frac{x(t) - y(t)}{R} - \left(+C\frac{dy(t)}{dt}\right)$$

or

$$\frac{R}{L}\int_{-\infty}^{t} y(t) = x(t) - y(t) - RC\frac{dy(t)}{dt}$$

7. Now, differentiating both sides

$$\frac{R}{L}y(t) = \frac{dx(t)}{dt} - \frac{dy(t)}{dt} - RC\frac{d^2y(t)}{dt^2}$$

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$$\text{or,}\quad RC\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+\frac{R}{L}y(t) = \frac{dx(t)}{dt}$$

$$\text{or,}\quad \frac{d^2y(t)}{dt^2}+\frac{1}{RC}\frac{dy(t)}{dt}+\frac{1}{LC}y(t) = +\frac{1}{RC}\frac{dx(t)}{dt}$$

Que 2.18. **Find differential equation description for the gi system depicted in Fig. 2.18.1.**

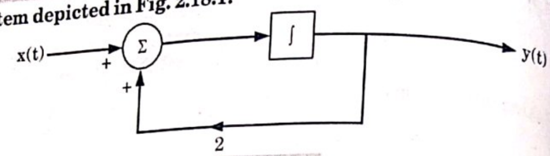

Fig. 2.18.1.

Answer

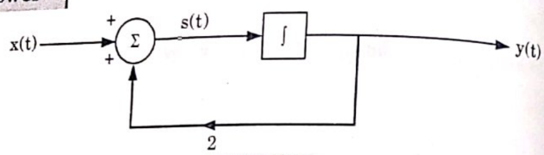

Fig. 2.18.2.

1. At the summing point,

$$s(t) = [x(t) + 2y(t)] \text{ and, } y(t) = \int_{-\infty}^{t} s(t)\, dt$$

2. Differentiating both sides, we have

$$\frac{dy(t)}{dt} = x(t) + 2y(t)$$

$$\frac{dy(t)}{dt} - 2y(t) = x(t)$$

Que 2.19. **Elaborate the system representation of differen equations.**

Answer

1. We can represent the general form of linear constant-coeffici difference equations as

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k) \qquad \text{...(2.19}$$

where a_k, b_k are scalars

2. There are only three operations involved in eq. (2.19.1) *i.e.*, multiplication by a scalar, addition, and shifting in time.
3. Shifting in time is done via the unit delay component. They are graphically represented as :

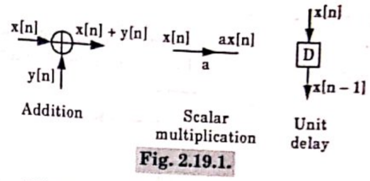

Fig. 2.19.1.

4. Assume the difference equation is,

$$y[n] = x[n] + 2x[n-1] - 3y[n-1]$$

5. The difference equation which is represented graphically as shown in Fig. 2.19.2.

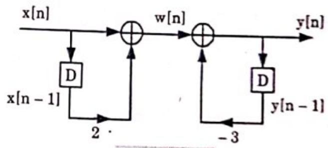

Fig. 2.19.2.

PART-3

Periodic and Semi-Periodic Inputs to an LSI System, The Notion of a Frequency Response and Its Relation to the Impulse Response.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.20. **Explain the complex exponentials response of LSI system.**

Answer

Complex exponentials :

Complex exponentials are an extremely useful class of functions for representing signals in LSI systems.

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1. A very wide class of real world signals can be represented, to virtually any desired level of accuracy, by complex exponentials.
2. The responses of LSI systems to this broad class of signals can be represented and analyzed quite effectively using complex exponentials.

Response of LSI system :

1. The response of an LSI system, to a complex exponential, is that same complex exponential multiplied by another complex variable.

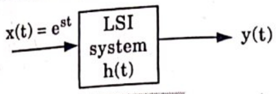

Fig. 2.20.1.

$$x(t) = e^{st} = \text{Complex exponential input}$$
$$s = \text{Complex variable} = \sigma + jw$$
$$y(t) = \text{Output}$$
$$h(t) = \text{LTI system impulse response}$$

where the output is obtained by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, dt = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$

2. Now, the exponential of the difference can be written as a product, so

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{st}\, e^{-s\tau}\, d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

and we define

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

This integral is the transfer function of the LTI system.

2. Substituting back into the equation for the output we obtain a simple expression for the output in terms of the input

$$y(t) = H(s)\, e^{st}$$

3. Such signals, for which the system output is simply a multiplication of the input by another complex variable, are called eigen functions and the multiplicative factor is the called the eigen value. Thus complex exponentials are eigen functions of LSI systems.

Linear combination of response :

1. If the input to a system is made up of the sum of two signals, then the total output is the sum of the outputs that result from the system operating on each of the two inputs separately.
2. In particular, the system

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

3. Hence if the input signals are complex exponentials, where C_1 and C_2 are constants. Then

$$y(t) = H(s_1)\, C_1 e^{s_1 t} + H(s_2)\, C_2 e^{s_2 t}$$

Que 2.21. **Find the sinusoidal response of LSI system? Discuss about frequency response and its relation to impulse response.**

Answer

Sinusoidal response of LSI system :

1. The signal is a tone (sinusoid) that we designate as u(t), so

$$u(t) = A \sin \omega_o t$$

2. In order to analyze the effects of $u(t)$, we will use a complex exponential representation of $u(t)$. We can write Euler's Equation as

$$e^{j\theta} = \cos\theta + j \sin\theta$$

and if we define

$$s_1 = +j\omega_0 \quad s_2 = -j\omega_0$$

then $u(t)$ can be written as the difference of two complex exponentials

$$u(t) = A\left(\frac{e^{s_1 t} - e^{s_2 t}}{2j}\right)$$

3. Also, the transmission phenomena can be modeled as a LSI system

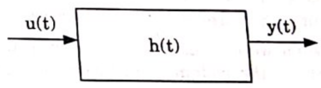

Fig. 2.21.1.

Let the impulse response of the transmission is

$$h(t) = k\delta(t - T)$$

4. To determine the output y(t) we need the system transfer function. From above the system transfer function is

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau = \int_{-\infty}^{\infty} k\delta(\tau - T) e^{-s\tau}\, d\tau$$
$$= ke^{-sT}$$

5. The sinusoidal input is the difference of two complex exponentials,

$$u(t) = \frac{Ae^{s_1 t}}{2j} - \frac{Ae^{s_2 t}}{2j}$$

6. The output $y(t)$,

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$$y(t) = \frac{Ae^{s_1 t}}{2j} H(s_1) - \frac{Ae^{s_2 t}}{2j} H(s_2)$$

$$= \frac{Ae^{s_1 t}}{2j} ke^{-s_1 T} - \frac{Ae^{s_2 t}}{2j} ke^{-s_2 T}$$

$$= kA\left[\frac{e^{j\omega_0 t}e^{-j\omega_0 T} - e^{-j\omega_0 t}e^{j\omega_0 T}}{2j}\right]$$

$$= kA\left[\frac{e^{j\omega_0 (t-T)} - e^{-j\omega_0 (t-T)}}{2j}\right]$$

$$= kA \sin \omega_0 (t - T)$$

Frequency response and its relation to impulse response :

1. The transfer function of the LSI system given as :

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

2. By setting $s = \pm j\omega_0$ in the transfer function we have ignored the real part of the complex variable s.
3. In doing so we have eliminated the transient (homogeneous) part of the solution.
4. When we do this we are ignoring any startup process. Physically this means that the stable system has been operating long enough with the sinusoidal input so that all effects of the startup process have disappeared.
5. In general, when we do this substitution into the system transfer function, we obtain the system frequency response.

$$H(s)|_{s = j\omega} = H(j\omega) = \text{Frequency response}$$

$$|H(j\omega)| = M(\omega)$$

$$\angle H(j\omega) = \phi(\omega)$$

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. Explain impulse response characterization of LTI systems?

Ans. Refer Q. 2.2.

Q. 2. What are the properties of convolution ?

Ans. Refer Q. 2.5.

Q. 3. Obtain the convolution of $x(t) = u(t)$ and $h(t) = 1$ for $-1 \leq t \leq 1$

Ans. Refer Q. 2.6.

Q. 4. Determine the output sequence of the system with impulse response $h[n] = \left(\frac{1}{4}\right)^n u[n]$; when input is complex exponential sequence $x[n] = Ae^{\frac{j\pi n}{2}}$.

Ans. Refer Q. 2.9.

Q. 5. An interconnection of LTI system is :

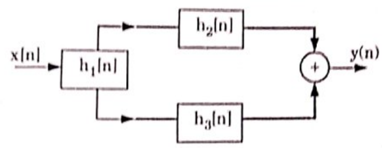

Fig. 1.

The impulse response are :

i. $h_1[n] = \left(\frac{1}{2}\right)^n \{u[n] - u[n-4]\}$

ii. $h_2[n] = \delta[n]$

iii. $h_3[n] = u[n-2]$

Let impulse response of overall system from $x[n]$ to $y[n]$ be $h[n]$.

i. **Express $h[n]$ in term of $h_1[n]$, $h_2[n]$ and $h_3[n]$.**

ii. **Evaluate $h[n]$.**

Ans. Refer Q. 2.11.

Q. 6. The accumulator is excited by the sequence $x[n] = nu[n]$. Accumulator can be defined by following input and output relationship.

$$y[n] = \sum_{k=-\infty}^{n} x[n]$$

Determine its output under the condition :

i. **It is initially relaxed.**

ii. **Initially $y[-1] = 1$.**

Ans. Refer Q. 2.12.

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Q. 7. Discuss about the stability of LSI systems.

Ans. *Refer Q. 2.14.*

Q. 8. Discuss about the linear constant-coefficient differe equation and linear constant-coefficient differe equations.

Ans. *Refer Q. 2.16.*

Q. 9. Find the sinusoidal response of LSI system? Discuss a frequency response and its relation to impulse respo

Ans. *Refer Q. 2.21.*

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3 UNIT

Transforms

CONTENTS

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PART-1

Fourier Series Representation.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.1. **Briefly explain the three forms of Fourier series.**

Answer

Three forms of Fourier series are :

i. **Trigonometric Fourier series :** The trigonometric Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 is given by :

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where, $\omega_0 = \frac{2\pi}{T_0}$

a_0, a_k and b_k are the Fourier coefficients given by :

$$a_0 = \frac{2}{T_0}\int_{T_0} x(t)\, dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t) \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t) \sin n\omega_0 t\, dt$$

ii. **Harmonic Fourier series :** Another form of Fourier series representation of a real periodic signal $x(t)$ with fundamental period T_0 is :

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos (n\omega_0 t - \theta_n),$$

where, $\omega_0 = \frac{2\pi}{T_0}$

c_0 = DC component

$c_n \cos (n\omega_0 t - \theta_n) = n^{\text{th}}$ harmonic component of $x(t)$

$$c_0 = \frac{a_0}{2},\quad c_n = \sqrt{a_n^2 + b_n^2},\quad \theta_n = \tan^{-1}\frac{b_n}{a_n}$$

iii. **Exponential Fourier series :**

1. The exponential Fourier series is the most widely used form of Fourier series.
2. In this, the function $x(t)$ is expressed as a weighted sum of the complex exponential functions.

$$x(t) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{jn\omega_0 t}$$

where, $$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t}\, dt$$

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, dt$$

Que 3.2. **Discuss waveform symmetries.**

OR

Define odd and even function. Also find Fourier coefficient for odd and even function.

Answer

1. **Even symmetry :**
   i. A function $f(x)$ is said to have even symmetry if $f(x) = f(-x)$.
   ii. Even function shows even symmetry.
   iii. Even nature is preserved on addition of a constant.
   iv. Sum of even functions remains even.

   Example :

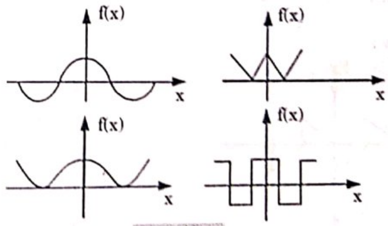

Fig. 3.2.1.

Fourier constants for even function :

$$a_0 = \frac{4}{T}\int_0^{T/2} f(t)\, dt$$

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$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\, dt$$

$$b_n = 0$$

2. **Odd symmetry :**
   i. A function $f(x)$ is said to have odd symmetry, if $f(x) = -f(-x)$.
   ii. If $f(x) = -f(-x)$ function is an odd function.
   iii. Addition of a constant removes odd nature of the function.
   iv. Sum of odd functions remains odd.

   Example :

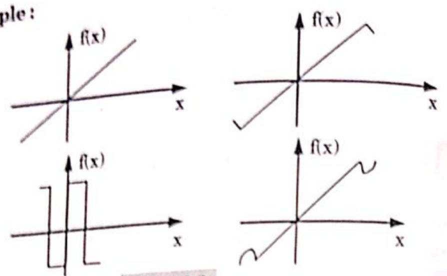

Fig. 3.2.2.

Fourier constants for odd function :

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt$$

3. **Half wave symmetry :**
   A periodic function is said to have half wave symmetry if,
   $$f(x) = -f(x \pm T/2)$$

   Example :

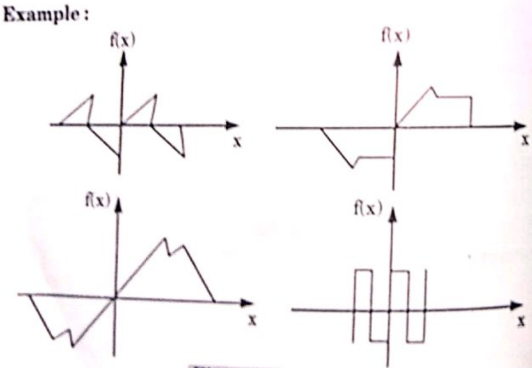

Fig. 3.2.3.

Fourier constants for half wave symmetry :

	n odd	**n even**
	$a_0 = 0$	$a_0 = 0$
	$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\, dt$	$a_n = 0$
	$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt$	$b_n = 0$

4. **Quarter wave symmetry :**
   If signal has following both properties, it is said to have quarter wave symmetry :
   i. It is half wave symmetric.
   ii. It has symmetry (odd or even) about the quarter period point.

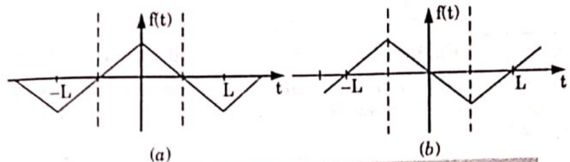

Fig. 3.2.4. (*a*) Even signal with Quarter wave symmetry.
(*b*) Odd signal with Quarter wave symmetry.

Que 3.3. **Give the properties of continuous-time Fourier series.**

Answer

1. **Linearity property :**
   The linearity property states that, if
   $$x_1(t) \xleftrightarrow{FS} C_n \text{ and } x_2(t) \xleftrightarrow{FS} D_n$$
   then $$Ax_1(t) + Bx_2(t) \xleftrightarrow{FS} AC_n + BD_n$$
2. **Time shifting property :**
   The time shifting property states that, if
   $$x(t) \xleftrightarrow{FS} C_n$$
   then $$x(t - t_0) \xleftrightarrow{FS} e^{-jn\omega_0 t_0} C_n$$
3. **Time reversal property :**
   The time reversal property states that, if
   $$x(t) \xleftrightarrow{FS} C_n$$
   then $$x(-t) \xleftrightarrow{FS} C_{-n}$$
4. **Time scaling property :**
   The time scaling property states that, if

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$$x(t) \xleftrightarrow{FS} C_n$$

then
$$x(at) \xleftrightarrow{FS} C_n \text{ with } \omega_0 \to \alpha\omega_0$$

5. **Time differentiation property :**
The time differentiation property states that, if
$$x(t) \xleftrightarrow{FS} C_n$$

then
$$\frac{dx(t)}{dt} \xleftrightarrow{FS} jn\omega_0\, C_n$$

6. **Time integration property :**
The time integration property states that, if
$$x(t) \xleftrightarrow{FS} C_n$$

then
$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FS} \frac{C_n}{jn\omega_0} \qquad (\text{if } C_0 = 0)$$

Que 3.4. Explain the trigonometric and exponential form Fourier series representation of periodic signals. Find trigonometric Fourier series for the periodic signal shown Fig. 3.4.1.

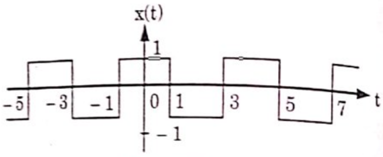

Fig. 3.4.1.

Answer

Trigonometric and exponential forms : Refer Q. 3.1, Page 3 Unit-3.

Numerical :

Assumption : Shift the signal upward by 1 unit.

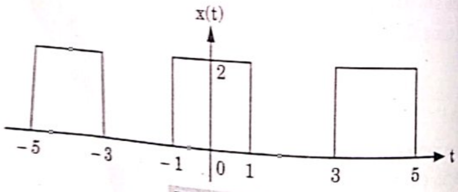

Fig. 3.4.2.

1.
$$x(t) = \begin{cases} 2 \ ; & -1 < t < 1 \\ 0 \ ; & 1 < t < 3 \end{cases}$$

2. Fundamental time period = 4
$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

3.
$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\, dt = \frac{4}{4}\left(\int_0^1 2dt + \int_1^2 0\, dt\right) = 2$$

4.
$$a_n = \frac{4}{T}\int_0^{T/2} x(t) \cos n\omega_0 t\, dt$$
$$= \frac{4}{T}\left[\int_0^1 2\cos n\omega_0 t\, dt + \int_1^2 0 \cos n\omega_0 t\, dt\right]$$
$$= 2\int_0^1 \cos n\omega_0 t\, dt + 0 = 2\left[\frac{\sin n\omega_0 t}{n\omega_0}\right]_0^1$$
$$= 2\left[\frac{\sin n\omega_0 - \sin n\omega_0 . 0}{n\omega_0}\right] = \frac{2 \sin n\omega_0}{n\omega_0}$$

Putting
$$\omega_0 = \frac{\pi}{2}$$
$$a_n = \frac{2\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} = \frac{4}{n\pi}\sin\frac{n\pi}{2}$$
$$= \begin{cases} 0 \ ; & n \text{ even} \\ \dfrac{4(-1)^n}{n\pi} \ ; & n \text{ odd} \end{cases}$$

5. Since, given signal is even,
$$b_n = 0$$

6.
$$x(t) = 1 + \sum_{n = \text{odd}}^{\infty} \frac{4(-1)^n}{n\pi} \cos n\frac{\pi}{2} t$$

PART-2

Fourier Transform, Convolution / Multiplication and their Effect in the Frequency Domain, Magnitude and Phase Response, Fourier Domain Duality.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

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Que 3.5. Define Fourier transform (FT) and inverse Fourier transform (IFT).

Answer

1. The Fourier transform of signal $x(t)$ is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

2. The inverse Fourier transform is given by,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \;;\; \text{for} -\infty < t < \infty$$

Que 3.6. State the conditions for the existence of Fourier transform.

OR

Explain Dirichlet's condition.

Answer

1. The Fourier transform does not exist for all aperiodic functions. The conditions for a function $x(t)$ to have Fourier transform, called Dirichlet's conditions which are as follows :

i. $x(t)$ is absolutely integrable over the interval $-\infty$ to ∞, *i.e.*,

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

ii. $x(t)$ has a finite number of discontinuities in every finite time interval. Further, each of these discontinuities must be finite.

iii. $x(t)$ has a finite number of maxima and minima in every finite time interval.

2. Dirichlet's condition is a sufficient condition but not necessary condition for the signal to be Fourier transformable.

Que 3.7. What are the different properties of Fourier transform?

OR

Explain the following properties of Fourier transform : time scaling, conjugate functions. **AKTU 2018-19, Marks 07**

Answer

Properties of Fourier transform : The Fourier transform (FT) has following properties :

1. **Linearity :**

If $\quad x_1(t) \xleftrightarrow{FT} X_1(\omega)$

and $\quad x_2(t) \xleftrightarrow{FT} X_2(\omega)$

then according to this property,

$$a_1x_1(t) + a_2x_2(t) \xleftrightarrow{FT} a_1X_1(\omega) + a_2X_2(\omega)$$

2. **Symmetry or duality property :**

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$

3. **Time-shifting :** This property states that a shift in the time-domain by an amount of a is equivalent to multiplication by $e^{-j\omega a}$ in the frequency-domain.

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad x(t-a) \xleftrightarrow{FT} e^{-j\omega a}\,X(\omega)$

4. **Frequency-shifting :** This property states that a shift in frequency domain by an amount 'ω_0' is equivalent to multiplication by '$e^{j\omega_0 t}$' to time-domain function $x(t)$.

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad e^{j\omega_0 t}x(t) \xleftrightarrow{FT} X(\omega-\omega_0)$

5. **Differentiation in time-domain :**

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad \dfrac{dx(t)}{dt} \xleftrightarrow{FT} j\omega\,.\,X(\omega)$

6. **Differentiation in frequency-domain :**

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad t.x(t) \xleftrightarrow{FT} j\dfrac{d}{d\omega}X(\omega)$

7. **Time-scaling :** This property states that "Time compression of a signal results in its spectral (frequency-domain) expansion, whereas, time expansion of a signal results in its spectral (frequency-domain) compression".

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad x(at) \xleftrightarrow{FT} \dfrac{1}{|a|}X(\omega / a)$

where $a \rightarrow$ real constant.

8. **Time-integration :**

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$$x(t) \xleftrightarrow{FT} X(\omega)$$

If

$$\int_{-\infty}^{t} x(t)dt \xleftrightarrow{FT} \frac{1}{j\omega}X(\omega)$$

then,

Assuming all initial conditions equal to zero.

9. **Time-reversal :**

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad x(-t) \xleftrightarrow{FT} X(-\omega)$

10. **Conjugation :**

If $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad x^*(t) \xleftrightarrow{FT} X^*(-\omega)$

11. **Multiplication property (frequency-convolution) :**

If $\quad x_1(t) \xleftrightarrow{FT} X_1(\omega)$

and $\quad x_2(t) \xleftrightarrow{FT} X_2(\omega)$

then, $\quad x_1(t) . x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}\left[X_1(\omega) \otimes X_2(\omega)\right]$

12. **Convolution (time-convolution) :**

If $\quad x_1(t) \xleftrightarrow{FT} X_1(\omega)$

and $\quad x_2(t) \xleftrightarrow{FT} X_2(\omega)$

then, $\quad x_1(t) \otimes x_2(t) \xleftrightarrow{FT} X_1(\omega) . X_2(\omega)$

Que 3.8. **Describe the magnitude and phase representation Fourier transform.**

Answer

1. The magnitude and phase representation of the Fourier transform the tool used to analyse the transformed signal.
2. In general, $X(\omega)$ is a complex valued function of ω.
$$X(\omega) = X_R(\omega) + jX_I(\omega)$$
where, $X_R(\omega)$ is the real part of $X(\omega)$ and $X_I(\omega)$ is the imaginary part $X(\omega)$.
3. The magnitude of $X(\omega)$ is given by
$$|X(\omega)| = \sqrt{|X_R(\omega)|^2 + |X_I(\omega)|^2}$$
and the phase of $X(\omega)$ is given by

$$\angle X(\omega) = \tan^{-1}\frac{X_I(\omega)}{X_R(\omega)}$$

4. The plot of $|X(\omega)|$ versus ω is known as amplitude spectrum, and the plot of $\angle X(\omega)$ versus ω is known as phase spectrum. The amplitude spectrum and phase spectrum together is called frequency spectrum.

Que 3.9. **Show that the convolution in time-domain is same as product in frequency-domain.**

OR

State and prove the convolution theorem.

Answer

i. **Time-convolution theorem :** It states that the convolution of two signals in time-domain is equivalent to the product of their spectra in frequency-domain. *i.e.,*

if $\quad x_1(t) \xleftrightarrow{FT} X_1(\omega)$ and $x_2(t) \xleftrightarrow{FT} X_2(\omega)$

then, $\quad x_1(t) \otimes x_2(t) \xleftrightarrow{FT} X_1(\omega)\, X_2(\omega)$

Proof : We know, $x_1(t) \otimes x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$

$$F[x_1(t) \otimes x_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\right] e^{-j\omega t}dt$$

Interchanging the order of integration, we have

$$F[x_1(t) \otimes x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt\right] d\tau$$

Substituting $\quad t - \tau = p$

$t = \tau + p$ and $dt = dp$

$$\therefore \quad F[x_1(t) \otimes x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(p)\, e^{-j\omega(\tau+p)}\, dp\right] d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(p)\, e^{-j\omega p}\, dp\right] e^{-j\omega\tau}\, d\tau$$

$$= \left[\int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau\right] X_2(\omega) = X_1(\omega)\, X_2(\omega)$$

Hence, $\quad x_1(t) \otimes x_2(t) \xleftrightarrow{FT} X_1(\omega)\, X_2(\omega)$

ii. **Frequency-convolution theorem or multiplication theorem :** It states that multiplication of two functions in time-domain is equivalent to the convolution of their spectra in the frequency-domain. *i.e.,*

if $\quad x_1(t) \xleftrightarrow{FT} X_1(\omega)$ and $x_2(t) \xleftrightarrow{FT} X_2(\omega)$

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then $x_1(t)x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}[X_1(\omega) \otimes X_2(\omega)]$

Proof : We know, $F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

and $F^{-1}[X(\omega)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

Now, $F[x_1(t)\, x_2(t)] = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-j\omega t}dt$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)e^{j\lambda t}d\lambda\right] x_2(t)\, e^{-j\omega t}dt$$

Interchanging the order of integration, we have

$$F[x_1(t)\, x_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)\, e^{j\lambda t}\, e^{-j\omega t}\, dt\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)\, e^{-j(\omega-\lambda)t}dt\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, X_2\,(\omega-\lambda)\, d\lambda$$

$$= \frac{1}{2\pi}\,[X_1(\omega) \otimes X_2(\omega)]$$

Hence, $x_1(t)\, .\, x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}\,[X_1(\omega) \otimes X_2(\omega)]$

Que 3.10. Find the Fourier transform of the following function using the properties of Fourier transform.

$$y(t) = \frac{d}{dt}te^{-3t}u(t) \otimes e^{-2t}u(t)$$

AKTU 2014-15, Marks 05

Answer

1. The function $y(t)$ can be written as,

$$y(t) = \underbrace{\frac{d}{dt}te^{-3t}\, u(t)}_{x_1(t)} \otimes \underbrace{e^{-2t}\, u(t)}_{x_2(t)}$$

$$y(t) = x_1(t) \otimes x_2(t)$$

2. By Fourier transform property of convolution,

$$Y(\omega) = X_1(\omega).\, X_2\,(\omega) \quad ...(3.10.1)$$

3. Now, if $x_1(t) = \frac{d}{dt}te^{-3t}\, u(t)$

$$F[e^{-3t}\, u(t)] = \frac{1}{3+j\omega}$$

using the FT property of differentiation in frequency, we get

$$F[te^{-3t}\, u(t)] = \frac{jd}{d\omega}\left[\frac{1}{3+j\omega}\right] = \frac{1}{(3+j\omega)^2} \quad ...(3.10.2)$$

now using the FT property of differentiation in time, we get

$$F\left[\frac{d}{dt}\, t\, e^{-3t}\, u(t)\right] = F\,[x_1(t)] = j\omega\, F[te^{-3t}u(t)] = \frac{j\omega}{(3+j\omega)^2} \quad ...(3.10.3)$$

and, $x_2(t) = e^{-2t}\, u(t)$

so, $$X_2(\omega) = \frac{1}{2+j\omega} \quad ...(3.10.4)$$

4. Substituting eq. (3.10.3) and (3.10.4) in eq. (3.10.1), we get

$$Y(\omega) = \frac{j\omega}{(3+j\omega)^2}\cdot\frac{1}{(2+j\omega)} = \frac{j\omega}{(3+j\omega)^2\,(2+j\omega)}$$

Que 3.11. A signal, $x(t)$ has a Fourier transform given by $X(\omega) = \frac{1}{(1+\omega^2)}$. Write down the Fourier transform of $x\left(\frac{3t}{2}-1\right)$.

AKTU 2014-15, Marks 2.5

Answer

1. Given, $$F[x(t)] = X(\omega) = \frac{1}{1+\omega^2} \quad ...(3.11.1)$$

We have to find out FT of $x\left(\frac{3t}{2}-1\right)$, we have to apply two theorems :

(i) Time shifting and (ii) Time scaling.

2. Now, $F[x(t-1)] = e^{-j\omega}\, X(\omega)$ $\quad [\because F\,[x(t-a)] = e^{-j\omega a}\, X(\omega)]$

$$F\left[x\left(\frac{3}{2}t-1\right)\right] = \frac{1}{|3/2|}e^{-(j\omega/(3/2))}\, X\left(\frac{\omega}{3/2}\right)$$

$$\left[\because\ F\,[x(at)] = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)\right]$$

Thus, $$F\left[x\left(\frac{3}{2}t-1\right)\right] = \frac{2}{3}e^{\frac{-j2\omega}{3}}\ \frac{1}{1+\left(\frac{2\omega}{3}\right)^2}$$

$$= \frac{2}{3}e^{\frac{-2j\omega}{3}}\ \frac{9}{9+4\,\omega^2} = \frac{6e^{-2j\omega/3}}{9+4\,\omega^2}$$

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Que 3.12. Determine the Fourier transform of the rectangu pulse :

$$x(t) = \text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } \frac{-\tau}{2} < t < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

OR

Find the continuous-time Fourier transform of the gate rectangu signal. Also plot its magnitude response. **AKTU 2015-16, Marks**

Answer

1. The Fourier transform is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}.dt$$

$$= \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right)e^{-j\omega t}.dt = \int_{-\tau/2}^{\tau/2} 1.e^{-j\omega t}.dt$$

$$X(\omega) = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2} = \left[\frac{e^{-j\omega\tau/2} - e^{+j\omega\tau/2}}{-j\omega}\right]$$

$$= \left[\frac{-2j\sin\left(\frac{\omega\tau}{2}\right)}{-j\omega}\right] = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\omega/2} = \tau.\frac{\sin\left(\frac{\omega\tau}{2}\right)}{(\omega\tau/2)}$$

$$X(\omega) = \tau \text{ sinc } (\omega\tau/2)$$

2. Since, sinc $(x) = 0$ for $x = \pm n\pi$

and sinc(0) = 1

Therefore, sinc $(\omega\tau/2) = 0$ for $(\omega\tau/2) = \pm n\pi$

or $$\omega = \frac{\pm 2n\pi}{\tau} \qquad n = 1, 2, 3, ...$$

or $$\omega = \frac{\pm 2\pi}{\tau}, \frac{\pm 4\pi}{\tau}, \frac{\pm 6\pi}{\tau}, \frac{\pm 8\pi}{\tau}, ...$$

3. The plot of the $X(\omega)$ is given in Fig. 3.12.1.

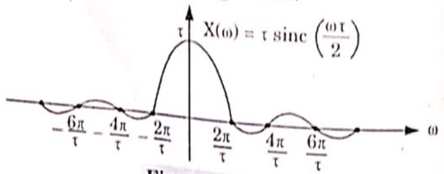

Fig. 3.12.1.

Que 3.13. **Using Fourier transform, find the convolution of :**

$$x_1(t) = e^{-2t}.u(t)$$

$$x_2(t) = e^{-3t}.u(t)$$

AKTU 2015-16, Marks 05

Answer

1. Given, $$x_1(t) = e^{-2t}u(t)$$

$\therefore$ $$X_1(\omega) = \frac{1}{j\omega + 2}$$

and $$x_2(t) = e^{-3t}u(t)$$

$\therefore$ $$X_2(\omega) = \frac{1}{j\omega + 3}$$

2. Using convolution property of Fourier transform

$$x_1(t) \otimes x_2(t) = F^{-1}[X_1(\omega).X_2(\omega)]$$

$$= F^{-1}\left[\frac{1}{(j\omega + 2)(j\omega + 3)}\right]$$

$$= F^{-1}\left[\frac{A}{j\omega + 2} + \frac{B}{j\omega + 3}\right]$$

$$= F^{-1}\left[\frac{1}{j\omega + 2} - \frac{1}{j\omega + 3}\right]$$

$$x_1(t) \otimes x_2(t) = e^{-2t}u(t) - e^{-3t}u(t)$$

Que 3.14. **Show that if $x_3(t) = ax_1(t) + bx_2(t)$, then $X_3(\omega) = aX_1(\omega) + bX_2(\omega)$** **AKTU 2016-17, Marks 10**

Answer

1. Given, $$x_3(t) = ax_1(t) + bx_2(t) \qquad ...(3.14.1)$$

2. Taking the Fourier transform of eq. (3.14.1)

$$X_3(\omega) = F[ax_1(t)] + F[bx_2(t)]$$

$$= \int_{-\infty}^{\infty} ax_1(t).e^{-j\omega t}dt + \int_{-\infty}^{\infty} bx_2(t).e^{-j\omega t}dt$$

$$= a\int_{-\infty}^{\infty} x_1(t).e^{-j\omega t}dt + b\int_{-\infty}^{\infty} x_2(t).e^{-j\omega t}dt$$

Thus, $$X_3(\omega) = aX_1(\omega) + bX_2(\omega)$$

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Que 3.15. Given $x(t) = 5 \cos t$, $y(t) = 2e^{-|t|}$, find the convolution of $x(t)$ and $y(t)$ using Fourier transform. **AKTU 2016-17, Marks**

Answer

1. Given, $x(t) = 5 \cos t$ and $y(t) = 2\,e^{-|t|}$
2. The convolution of $x(t)$ and $y(t)$ is $z(t) = x(t) * y(t)$

then $$Z(f) = X(f)\,.\,Y(f) \quad ...(3.1$$

3. Now, $x(t) = 5 \cos t\,;\, -\infty < t < \infty$

$$X(f) = \frac{5}{2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \quad ...(3.1$$

so, $y(t) = 2e^{-|t|}\,;\, -\infty < t < \infty$

and $$Y(f) = 2\int_{-\infty}^{\infty} e^{-|t|} e^{-j2\pi ft} dt$$

so, $$= 2\left[\int_{-\infty}^{0} e^{t} e^{-j2\pi ft} dt + \int_{0}^{\infty} e^{-t} e^{-j2\pi ft} dt\right] = \frac{4}{1+\omega^2} \quad ...(3.15$$

4. Put the values of $X(f)$ and $Y(f)$ in eq. (3.15.1)

$$Z(f) = X(f).Y(f) = \frac{10}{1+\omega^2}\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \quad ...(3.15$$

5. From the sampling property of impulses, we know that

$$f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau)$$

$$Z(f) = \left[\frac{10}{1+4\pi^2 f^2}\bigg|_{f=1/2\pi}\right]\delta(f - 1/2\pi) + \left[\frac{10}{1+4\pi^2 f^2}\bigg|_{f=-1/2\pi}\right]\delta(f + 1/2\pi)$$

$$= 5\left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right] \quad ...(3.15$$

Taking the inverse Fourier transform on both sides,

$$z(t) = x(t) * y(t) = 10 \cos t\,;\, -\infty < t < \infty$$

Que 3.16. Explain Fourier transform of single sided exponential pulse. **AKTU 2018-19, Marks 07**

OR

Find the Fourier transform of the signals given below :

i. $x(t) = \begin{cases} A, & |t| < T_0 \\ 0, & |t| > T_0 \end{cases}$

ii. $x(t) = e^{-at} u(t)$

Draw the magnitude and phase response of the transformed signal. **AKTU 2017-18, Marks 07**

OR

Find the Fourier transform of the signal $x(t) = e^{-at} u(t)$ and plot its magnitude and phase spectrum. **AKTU 2017-18, Marks 07**

Answer

i. Given, $x(t) = \begin{cases} A, & |t| < T_0 \\ 0, & |t| > T_0 \end{cases}$

Taking Fourier transform

So, $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$= A\int_{-T_0}^{T_0} e^{-j\omega t} dt = A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_0}^{T_0}$$

$$= -\frac{A}{j\omega}\left[e^{-j\omega T_0} - e^{j\omega T_0}\right] = \frac{A}{j\omega}\left[e^{j\omega T_0} - e^{-j\omega T_0}\right]$$

$$X(\omega) = \frac{2A}{\omega}\left[\frac{e^{j\omega T_0} - e^{-j\omega T_0}}{2j}\right] = \frac{2A}{\omega}\sin \omega T_0$$

$$= 2AT_0 \frac{\sin \omega T_0}{\omega T_0} = 2AT_0 \operatorname{sinc} \omega T_0$$

Magnitude and phase response of $X(\omega)$:

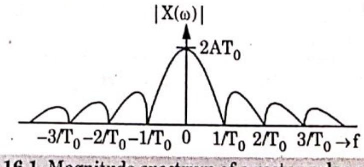

Fig. 3.16.1. Magnitude spectrum of a rectangular pulse $x(t)$.

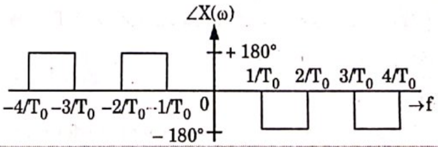

Fig. 3.16.1. Phase spectrum of rectangular pulse $x(t)$.

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$$x(t) = e^{-at}\,u(t)$$

ii. Given,

$$x(t) = \begin{cases} e^{-at} & , t > 0 \\ 0 & , t < 0 \end{cases}$$

The Fourier transform of the given signal is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{-1}{(a+j\omega)}[0-1] = \frac{1}{a+j\omega}$$

Magnitude and phase response :

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

and $$\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

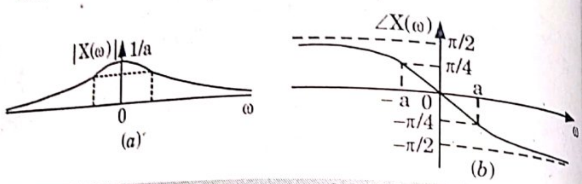

Fig. 3.16.1.(a). Magnitude spectrum and (b). Phase spectrum of $x(t)$.

PART-3

Discrete-Time Fourier Transform (DTFT).

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.17. What do you understand by DTFT ? What is the condition for existence of DTFT ?

Answer

DTFT : It stands for discrete-time Fourier transform which gives the description of $x[n]$ in frequency-domain. It is given by,

$$F[x[n]] = X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Condition for existence of DTFT :

We know that

$$|e^{-j\omega n}| = 1$$

The $X(\omega)$ can exist only if $x[n]$ is absolutely summable, *i.e.*,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Que 3.18. What are the different properties of DTFT ?

Answer

The DTFT has following properties :

1. **Linearity :** The DTFT is linear in nature.

   If $x_1[n] \xleftrightarrow{FT} X_1(\omega)$

   and $x_2[n] \xleftrightarrow{FT} X_2(\omega)$

   then $a_1x_1[n] + a_2x_2[n] \xleftrightarrow{FT} a_1X_1(\omega) + a_2X_2(\omega)$

2. **Periodicity :** The DTFT is always periodic with period 2π.

   If $x[n] \xleftrightarrow{FT} X(\omega)$

   then $X(\omega + 2\pi k) = X(\omega)$

3. **Time-shifting :** This property states that if a discrete-time signal is shifted in the time-domain by an amount of 'n_0', a factor '$e^{-j\omega n_0}$' is multiplied with $X(\omega)$.

   If $x[n] \xleftrightarrow{FT} X(\omega)$

   then, $x[n-n_0] \xleftrightarrow{FT} e^{-j\omega n_0}\,.\,X(\omega)$

4. **Frequency-shifting :** This property states that the multiplication by a factor $e^{j\omega_0 n}$ in time-domain sequence $x[n]$ results shift in frequency in $X(\omega)$ by an amount of ω_0.

   If $x[n] \xleftrightarrow{FT} X(\omega)$

   then, $e^{j\omega_0 n}\,.\,x[n] \xleftrightarrow{FT} X(\omega - \omega_0)$

5. **Time-reversal :** This property states that if a discrete-time signal is folded about the origin in time, its magnitude spectrum remains unchanged while the phase spectrum of this signal undergoes just change in sign.

   If $x[n] \xleftrightarrow{FT} X(\omega)$

   then $x[-n] \xleftrightarrow{FT} X(-\omega)$

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6. **Scaling :**

If $x[n] \xleftrightarrow{FT} X(\omega)$

then $y[n] = x[kn] \xleftrightarrow{FT} = X\left(\frac{\omega}{k}\right) = Y(\omega)$

7. **Differentiation in frequency-domain :**

If $x[n] \xleftrightarrow{FT} X(\omega)$

then $nx[n] \xleftrightarrow{FT} j\frac{dX(\omega)}{d\omega}$

8. **Multiplication in time-domain :** This property states that multiplication of two time-domain sequence is equivalent to convolution of their FT.

If $x_1[n] \xleftrightarrow{FT} X_1(\omega)$

and $x_2[n] \xleftrightarrow{FT} X_2(\omega)$

then $y[n] = x_1[n]\, x_2[n] \xleftrightarrow{FT} Y(\omega) = [X_1(\omega) \otimes X_2(\omega)]$

9. **Conjugation and conjugate symmetry :**

If $x[n] \xleftrightarrow{FT} X(\omega)$

then $x^*[n] \xleftrightarrow{FT} X^*(-\omega)$

10. **Convolution :**

If $x_1[n] \xleftrightarrow{FT} X_1(\omega)$

and $x_2[n] \xleftrightarrow{FT} X_2(\omega)$

then $y[n] = x_1[n] \otimes x_2[n] \xleftrightarrow{FT} Y(\omega) = X_1(\omega)\,.\,X_2(\omega)$

Que 3.19. Obtain the discrete-time Fourier transform $x[n] = a^n u[n] + a^{-n} u[-n-1]$

AKTU 2014-15, Marks

Answer

1. Given, $x[n] = a^n u[n] + a^{-n} u[-n-1]$
2. DTFT, $X(\omega) = F\{x[n]\}$

$$= \sum_{n=-\infty}^{\infty} \left[a^n u[n] + a^{-n}\, u[-n-1]\right] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} a^{-n} u[-n-1] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{n=1}^{\infty} (ae^{j\omega})^n$$

$$= \frac{1-a^2}{1-a(e^{j\omega}+e^{-j\omega})+a^2} = \frac{1-a^2}{1-2a\cos\omega + a^2}$$

Que 3.20. **Obtain DTFT of a signal $x[n] = r^n \sin(\omega_0 n) u[n]$, $r < 1$.**

AKTU 2014-15, Marks 05

Answer

1. Given, $x[n] = r^n \sin(\omega_0 n)\, u[n]$, $r < 1$
2. DTFT, $X(\omega) = \sum_{n=0}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} r^n \sin(\omega_0 n) u[n] e^{-j\omega n}$

$$= \sum_{n=0}^{\infty} r^n \sin(\omega_0 n) e^{-j\omega n} = \sum_{n=0}^{\infty} r^n \left[\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right] e^{-j\omega n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} r^n e^{j\omega_0 n} e^{-j\omega n} - \sum_{n=0}^{\infty} r^n e^{-j\omega_0 n} e^{-j\omega n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} r^n e^{jn(\omega_0-\omega)} - \sum_{n=0}^{\infty} r^n e^{-jn(\omega_0+\omega)}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-re^{j(\omega_0-\omega)}} - \frac{1}{1-re^{-j(\omega_0+\omega)}}\right]$$

$$= \frac{1}{2j}\left[\frac{1-re^{-j\omega_0}e^{-j\omega} - 1 + re^{j\omega_0}e^{-j\omega}}{1-re^{-j\omega_0}e^{-j\omega} - re^{j\omega_0}e^{-j\omega} + r^2e^{-2j\omega}}\right]$$

$$= \frac{1}{2j}\left[\frac{re^{-j\omega}(e^{j\omega_0} - e^{-j\omega_0})}{1+r^2e^{-2j\omega} - re^{-j\omega}(e^{-j\omega_0} + e^{j\omega_0})}\right]$$

$$= \frac{re^{-j\omega}\left[\dfrac{e^{j\omega_0} - e^{-j\omega_0}}{2j}\right]}{1+r^2e^{-2j\omega} - 2re^{-j\omega}\left[\dfrac{e^{j\omega_0} + e^{-j\omega_0}}{2}\right]}$$

$$= \frac{re^{-j\omega}\sin\omega_0}{1-2re^{-j\omega}\cos\omega_0 + r^2e^{-2j\omega}}$$

Que 3.21. **For a linear time invariant system**

$$h[n] = u[n-1] + u[n-2] + u[n-3].$$

Find the frequency response $H(\omega)$, and plot the magnitude and pha response.

AKTU 2014-15, Marks

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Answer

1. Given, $h[n] = u[n-1] + u[n-2] + u[n-3]$

Taking DTFT on both sides,

$$H(\omega) = \frac{e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega}}{1 - e^{-j\omega}}$$

$$= \frac{\cos\omega - j\sin\omega + \cos 2\omega - j\sin 2\omega + \cos 3\omega - j\sin 3\omega}{1 - \cos\omega + j\sin\omega}$$

$$H(\omega) = \frac{[(\cos\omega + \cos 2\omega + \cos 3\omega) - j(\sin\omega + \sin 2\omega + \sin 3\omega)]}{(1 - \cos\omega) + j(\sin\omega)}$$

2. The magnitude response of the system is given by

$$|H(\omega)| = \sqrt{\frac{(\cos\omega + \cos 2\omega + \cos 3\omega)^2 + (\sin\omega + \sin 2\omega + \sin 3\omega)^2}{(1 - \cos\omega)^2 + \sin^2\omega}}$$

3. The phase response of the system is given by

$$\angle H(\omega) = \tan^{-1}\left(-\frac{\sin\omega + \sin 2\omega + \sin 3\omega}{\cos\omega + \cos 2\omega + \cos 3\omega}\right) - \tan^{-1}\left(\frac{\sin\omega}{1 - \cos\omega}\right)$$

$$= -\tan^{-1}\left(\tan\left(\frac{3+1}{2}\right)\omega\right) - \tan^{-1}\left(\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{1 - 1 + 2\sin^2\frac{\omega}{2}}\right)$$

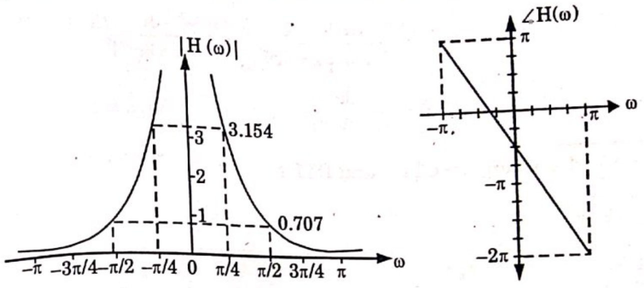
$$\left(\because \frac{\sin\omega + \sin 2\omega + \ldots + \sin n\omega}{\cos\omega + \cos 2\omega + \ldots \cos n\omega} = \tan\left(\frac{n+1}{2}\right)\omega\right)$$

$$= -\tan^{-1}(\tan 2\omega) - \tan^{-1}\left(\cot\frac{\omega}{2}\right)$$

$$= -2\omega - \tan^{-1}\left(\tan\left(90° - \frac{\omega}{2}\right)\right)$$

$$= -2\omega - 90° + \frac{\omega}{2} = -\frac{3\omega}{2} - 90° = -\frac{3\omega}{2} - \frac{\pi}{2}$$

4. The plots for magnitude response and phase response are shown i Fig. 3.21.1(a) and (b) respectively.



(a) (b)

Fig. 3.21.1.

Que 3.22. **Calculate the DTFT of the following using properties of DTFT :**

$$x[n] = u[n+3] - u[n-3]$$ **AKTU 2015-16, Marks 05**

Answer

1. Given, $x[n] = u[n+3] - u[n-3]$
2. DTFT, $X(\omega) = F\{u[n+3] - u[n-3]\}$

$$= \sum_{n=-\infty}^{\infty} \{u[n+3] - u[n-3]\}\, e^{-j\omega n}$$

$$= \sum_{n=-3}^{\infty} (1)\, e^{-j\omega n} - \sum_{n=3}^{\infty} (1)\, e^{-j\omega n}$$

$$= e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} + \ldots - e^{-j3\omega} - e^{-j4\omega} - \ldots$$

$$= e^{j3\omega} + e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}$$

Que 3.23. **Find DTFT of the signal :**

$$x[n] = n.3^{-n}.u[-n]$$ **AKTU 2015-16, Marks 05**

Answer

1. Given, $x[n] = n3^{-n}\, u[-n]$
2. Using differentiation in frequency-domain and time reversal properties, we have,

$$F\{n3^{-n}\, u[-n]\} = j\frac{d}{d\omega}[F\{3^{-n}\, u[-n]\}]$$

$$= j\frac{d}{d\omega}[F\{3^{n}\, u[n]\}]_{\omega=-\omega}$$

$$= j\frac{d}{d\omega}\left[\frac{1}{1 - 3e^{-j\omega}}\right]$$

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$$= j\frac{d}{d\omega}\left[\frac{1}{1-3e^{j\omega}}\right] = j\frac{-[-3e^{j\omega}(j)]}{[1-3e^{j\omega}]^2}$$

$$X(\omega) = \frac{-3e^{j\omega}}{(1-3e^{j\omega})^2}$$

Que 3.24. **Compare CTFT and DTFT.**

Answer

S. No.	CTFT	DTFT
1.	The CTFT, $X(f)$ or $X(j\omega)$ is not periodic in ω.	The DTFT, $X(\omega)$ is periodic in ω.
2.	The inverse CTFT involves integration over all frequencies from $-\infty$ to $+\infty$.	The inverse DTFT involves integration over a frequency interval of only 2π.
3.	Scaling theorem : $F[x(at)] = \frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$	Scaling theorem : $F[x(an)] = X\left(\frac{\omega}{a}\right)$
4.	In the case of CTFT, we observed a symmetry or duality between the Fourier transform and inverse Fourier transform expressions.	In the case of DTFT, no corresponding duality exists between the DTFT and IDFT expressions.

Que 3.25. **An LTI system with impulse response $h_1(n)$**

$$h_1(n) = \left(\frac{1}{3}\right)^n u(n)$$

is connected in parallel with another causal LTI system with impulse response $h_2(n)$. The resulting parallel interconnection has the frequency response.

$$H(e^{j\omega}) = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-2j\omega}}$$

Find the impulse response $h_2(n)$.

AKTU 2017-18, Marks 07

Answer

1. When two LTI systems are connected in parallel, the impulse response of overall system is the sum of the impulse response of the individual system. Therefore,

$$h[n] = h_1[n] + h_2[n]$$

So, the Fourier transform of $h[n]$ is given by

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

2. Given that $h_1[n] = \left(\frac{1}{3}\right)^n u(n)$

So, the Fourier transform of $h_1[n]$ is ;

$$H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}} = \frac{3}{3-e^{-j\omega}}$$

3. Therefore,

$$H_2(e^{j\omega}) = H(e^{j\omega}) - H_1(e^{j\omega})$$

$$= \frac{-12+5e^{-j\omega}}{(12-7e^{-j\omega}+e^{-2j\omega})} - \frac{3}{(3-e^{-j\omega})}$$

$$= \frac{-12+5e^{-j\omega}}{(4-e^{-j\omega})(3-e^{-j\omega})} - \frac{3}{(3-e^{-j\omega})}$$

$$= \frac{-12+5e^{-j\omega}-3(4-e^{-j\omega})}{(4-e^{-j\omega})(3-e^{-j\omega})}$$

$$= \frac{-24+8e^{-j\omega}}{(4-e^{-j\omega})(3-e^{-j\omega})} = \frac{-8(3-e^{-j\omega})}{(4-e^{-j\omega})(3-e^{-j\omega})}$$

$$H_2(e^{j\omega}) = \frac{-8}{4-e^{-j\omega}} = -2\cdot\frac{1}{1-\left(\frac{1}{4}\right)e^{-j\omega}}$$

So,

$$h_2(n) = -2\left(\frac{1}{4}\right)^n u(n)$$

Que 3.26. **State and prove frequency shifting theorem of DTFT.**

AKTU 2018-19, Marks 07

Answer

A. **Statement :** This property states that the multiplication by a factor $e^{j\omega_0 n}$ in time-domain sequence $x[n]$ results shift in frequency in $X(\omega)$ by an amount of ω_0.

If $x[n] \xleftrightarrow{FT} X(\omega)$

then, $e^{j\omega_0 n}x[n] \xleftrightarrow{FT} X(\omega-\omega_0)$

B. **Prove :**

1. Fourier transform of signal $x[n]$

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$$F[x[n]] = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

2. Multiplying $e^{j\omega_0 n}$ in the signal $x[n]$ and taking Fourier transform

$$F[x[n]\, e^{j\omega_0 n}] = \sum_{n=-\infty}^{\infty} x[n]\, e^{j\omega_0 n}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega-\omega_0)n}$$

$$= X(\omega - \omega_0)$$

PART-4

Discrete Fourier Transform (DFT).

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.27. **Define discrete Fourier transform (DFT). Mention some important features of DFT.**

Answer

1. Let $x[n]$ be a finite-length sequence of length N, that is,

$$x[n] = 0 \quad \text{outside the range } 0 \le n \le N-1 \quad ...(3.27.1)$$

2. The DFT of $x[n]$, denoted as $X[k]$, is defined by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad k = 0, 1, ..., N-1 \quad ...(3.27.2)$$

where W_N is the Nth root of unity given by

$$W_N = e^{-j(2\pi/N)} \quad ...(3.27.3)$$

3. The inverse DFT (IDFT) is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad n = 0, 1, ..., N-1 \quad ...(3.27.4)$$

4. The DFT pair is denoted by

$$x[n] \leftrightarrow X[k]$$

Important features of the DFT are the following :

1. There is a one-to-one correspondence between $x[n]$ and $X[k]$.
2. There is an extremely fast algorithm, called the fast Fourier transform (FFT) for its calculation.
3. The DFT is closely related to the discrete Fourier series and the Fourier transform.
4. The DFT is the appropriate Fourier representation for digital computer realization because it is discrete and of finite length in both time and frequency domains.

Que 3.28. **Write relationship between DFT and the Fourier transform.**

Answer

1. The Fourier transform of $x[n]$ defined can be expressed as

$$X(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \quad ...(3.28.1)$$

2. Comparing eq. (3.28.1) with eq. (3.27.2), we see that

$$X[k] = X(\omega)\big|_{\omega = k2\pi/N} = X\left(\frac{k2\pi}{N}\right) \quad ...(3.28.2)$$

3. Thus, $X[k]$ corresponds to the sampled $X(\omega)$ at the uniformly spaced frequencies $\omega = k2\pi/N$ for integer k.

Que 3.29. **Explain the properties of DFT.**

Answer

1. **Linearity :**

$$a_1 x_1[n] + a_2 x_2[n] \leftrightarrow a_1 X_1[k] + a_2 X_2[k] \quad ...(3.29.1)$$

2. **Time shifting :**

$$x[n - n_0]_{\text{mod } N} \leftrightarrow W_N^{kn_0} X[k], \; W_N = e^{-j(2\pi/N)} \quad ...(3.29.2)$$

3. **Frequency Shifting :**

$$W_N^{k_0 n} x[n] \leftrightarrow X[k - k_0]_{\text{mod } N} \quad ...(3.29.3)$$

4. **Conjugation :**

$$x^*[n] \leftrightarrow X^*[-k]_{\text{mod } N} \quad ...(3.29.4)$$

where * denotes the complex conjugate.

5. **Time Reversal :**

$$x[-n_0]_{\text{mod } N} \leftrightarrow X[-k]_{\text{mod } N} \quad ...(3.29.5)$$

6. **Duality :** If $x[n] \leftrightarrow X[k]$

$$\text{then } X[n] \leftrightarrow Nx[-k]_{\text{mod } N} \quad ...(3.29.6)$$

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7. **Circular convolution :**

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1[k]X_2[k] \quad ...(3.29$$

$$\text{where } x_1[n] \otimes x_2[n] = \sum_{i=0}^{N-1} x_1[i]x_2[n-i]_{\text{mod } N} \quad ...(3.29$$

The convolution sum in eq. (3.29.8) is known as the circular convolu of $x_1[n]$ and $x_2[n]$.

8. **Multiplication :**

$$x_1[n]x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \otimes X_2[k] \quad ...(3.29$$

$$\text{where } X_1[k] \otimes X_2[k] = \sum_{i=0}^{N-1} X_1[i] X_2[k-i]_{\text{mod } N} \quad ...(3.29.$$

9. **Additional Properties :** When $x[n]$ is real, let

$$x[n] = x_e[n] + x_o[n]$$

where $x_e[n]$ and $x_o[n]$ are the even and odd components of $x[n]$ respectively. Let

$$x[n] \leftrightarrow X[k] = A[k] + jB[k] = |X[k]|\, e^{j\theta[k]}$$

Then $\quad X[-k]_{\text{mod } N} = X^*[k]$

$$x_e[n] \leftrightarrow \text{Re}\{X[k]\} = A[k]$$

$$x_o[n] \leftrightarrow j\,\text{Im}\{X[k]\} = jB[k]$$

$$A[-k]_{\text{mod } N} = A[k], \quad B[-k]_{\text{mod } N} = -B[k]$$

$$|X[-k]|_{\text{mod } N} = |X[k]| \quad \theta[-k]_{\text{mod } N} = -\theta[k]$$

10. **Parseval's relation :**

$$\sum_{n=0}^{N-1} |x[n]^2| = \frac{1}{N}\sum_{n=0}^{N-1} |X[k]|^2 \quad ...(3.29.11$$

Eq. (3.29.11) is known as Parseval's identity (or Parseval's theorem) for the DFT.

PART-5

Parseval's Theorem, the Idea of Signal Space and Orthogonal Bases.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.30. **State and prove Parseval's theorem and Parseval's identity.**

OR

Prove Parseval's theorem for continuous-time system.

AKTU 2014-15, Marks 05

Answer

i. **Parseval's theorem :**

Parseval's theorem states that if

$$x_1(t) \xleftrightarrow{FT} X_1(\omega) \text{ and } x_2(t) \xleftrightarrow{FT} X_2(\omega)$$

then $\displaystyle\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, X_2^*(\omega)\, d\omega$

ii. **Parseval's identity :**

If $x_1(t) = x_2(t) = x(t)$, then Parseval's identity states that the energy content of $x(t)$ is

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Proof :

1. $$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, e^{j\omega t}\, d\omega\right\} x_2^*(t)\, dt$$

Interchanging the order of integration, we have

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\left\{\int_{-\infty}^{\infty} x_2^*(t)\, e^{j\omega t}\, dt\right\} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\left\{\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt\right\}^* d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, \{X_2(\omega)\}^*\, d\omega$$

So, $\displaystyle\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, X_2^*(\omega)\, d\omega$

2. If $x_1(t) = x_2(t) = x(t)$, then the energy of the signal is given by

$$E = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, X^*(\omega)\, d\omega$$

Since, $\quad |x(t)|^2 = x(t)x^*(t)$ and $|X(\omega)|^2 = X(\omega)\, X^*(\omega)$, we get

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$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Que 3.31. **Discuss signal space and orthogonal bases.**

Answer

Signal space :

1. The basic idea is to view waveforms, *i.e.*, finite-energy functions, as vectors in a certain vector space that we call signal-space.
2. The set of waveforms in this space is the set of finite-energy complex functions $u(t)$ mapping each t on the real time axis into a complex value.
3. The energy in $u(t)$ is defined as

$$||u||^2 = \int_{-\infty}^{\infty} |u(t)|^2 \, dt$$

4. The set of functions that we are considering is the set for which this integral exists and is finite.
5. This set of waveforms is called the space of finite-energy complex functions, or, more briefly, the space of complex waveforms.

Orthogonal bases :

1. In an inner product space, a set of vectors $\phi_1, \phi_2, \ldots$ is orthonormal if

$$(\phi_j, \phi_k) = \begin{cases} 0 \text{ for } j \neq k \\ 1 \text{ for } j \neq k \end{cases}$$

2. An orthonormal set is a set of orthogonal vectors where each vector is normalized in the sense of having unit length.
3. If a set of vectors $v_1, v_2, \ldots$ is orthogonal, then the set is,

$$\phi_j = \frac{1}{||v_j||} v_j$$

4. If two vectors are orthogonal, then any scaling (including normalization) of each vector maintains the orthogonality.
5. If we project a vector u onto a normalized vector ϕ, then, the projection has the simplified form

$$u = (u, \phi)\phi$$

PART-6

The Laplace Transform, Notion of Eigen Functions of LSI Systems, a Basis of Eigen Functions, Region of Convergence, Poles and Zeroes of System.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.32. **What is Laplace transform and inverse Laplace transform ?**

Answer

i. **Laplace transform :** Let $x(t)$ is defined for $t > 0$ and has zero value for $t \le 0$.

The Laplace transform of $x(t)$ is denoted as $L[x(t)]$ and is given as,

$$L[x(t)] = X(s) = \int_0^{\infty} x(t)\, e^{-(\sigma + j\omega)t} \, dt = \int_0^{\infty} x(t)\, e^{-st} dt$$

Types of Laplace transform :

1. **Unilateral Laplace transform :** When the integration is taken from '0' to '∞', the Laplace transform is called one-sided, or unilateral Laplace transform. It is mainly used for the analysis of causal signals.
2. **Right-sided :** When integration is taken for positive side *i.e.* limits are taken from '0' to '∞'.

$$X(s) = \int_0^{\infty} x(t)\, e^{-st} dt$$

3. **Left-sided :** When integration is taken for negative side, *i.e.*, limits are taken from '– ∞' to '0'.

$$X(s) = \int_{-\infty}^{0} x(t)\, e^{-st} dt$$

4. **Bilateral (two sided) Laplace transform :** When integration is taken from $-\infty$ to ∞, then it is called bilateral or two sided Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$

5. **Laplace transform pair :** The functions $x(t)$ and $X(s)$ form a Laplace transform pair. It is expressed as,

$$x(t) \xleftrightarrow{LT} X(s)$$

here, $x(t)$ = Continuous-time signal

$X(s)$ = Laplace transform (LT) of $x(t)$

and $s = \sigma + j\omega$ (complex variable)

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σ = Real part of the complex variable

ω = Imaginary part of the complex variable

ii. **Inverse Laplace transform :** It is used to convert frequency-domain signal $X(s)$ to the time-domain signal $x(t)$. It is given as,

$$x(t) = L^{-1}[X(s)]$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

Que 3.33. **Find the unilateral Laplace transform of the following signals :**

i. $x(t) = \delta(t)$ **ii.** $x(t) = u(t)$

Answer

i. Given, $x(t) = \delta(t)$

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty \delta(t)\, e^{-st}\, dt$$

since, $\delta(t) = \begin{cases} 1 & \text{at } t = 0 \\ 0 & \text{otherwise} \end{cases}$

thus, $X(s) = e^{-0} = 1$

ii. Given, $x(t) = u(t)$

$$X(s) = \int_0^\infty x(t)\, e^{-st}dt = \int_0^\infty u(t)\, e^{-st}dt$$

since, $u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

therefore, $X(s) = \frac{-1}{s}e^{-st}\Big|_0^\infty = \frac{-1}{s}[e^{-\infty} - e^0]$

$$X(s) = \frac{1}{s}$$

Que 3.34. **Find the Laplace transform of the signal :**

i. $\cos(\omega_0 t)$ **ii.** $\sin(\omega_0 t)$

Answer

i. Given, $x(t) = \cos(\omega_0 t)$

$$X(s) = \int_0^\infty \cos(\omega_0 t)e^{-st}dt$$

$$X(s) = \int_0^\infty \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]e^{-st}dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{j\omega_0 t}e^{-st}dt + \int_0^\infty e^{-j\omega_0 t}e^{-st}dt\right]$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-j\omega_0)t}\,dt + \int_0^\infty e^{-(s+j\omega_0)t}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{-1}{(s-j\omega_0)}e^{-(s-j\omega_0)t}\Big|_0^\infty + \frac{-1}{(s+j\omega_0)}e^{-(s+j\omega_0)t}\Big|_0^\infty\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right] = \frac{1}{2}\left[\frac{s+j\omega_0+s-j\omega_0}{s^2-(j)^2\omega_0^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2+\omega_0^2}\right]$$

$$X(s) = \frac{s}{s^2+\omega_0^2}$$

ii. Given, $x(t) = \sin(\omega_0 t)$

$$X(s) = \int_0^\infty \sin(\omega_0 t)dt = \int_0^\infty \frac{1}{2j}\left[e^{+j\omega_0 t} - e^{-j\omega_0 t}\right]e^{-st}dt$$

$$= \frac{1}{2j}\left[\int_0^\infty \left[e^{-(s-j\omega_0)t} - e^{-(s+j\omega_0)t}\right]dt\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right] = \frac{1}{2j}\left[\frac{s+j\omega_0-s+j\omega_0}{s^2+\omega_0^2}\right]$$

$$X(s) = \frac{\omega_0}{s^2+\omega_0^2}$$

Que 3.35. **Find the unilateral Laplace transform of the following signals :**

i. $x(t) = tu(t)$ **ii.** $e^{at}u(t)$

Answer

i. **Ramp function, $x(t) = tu(t)$:**

$$L[tu(t)] = X(s) = \int_0^\infty tu(t)e^{-st}\,dt = \int_0^\infty te^{-st}dt$$

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$$= \left[t\frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} 1dt = 0 - \left[\frac{e^{-st}}{s^2}\right]_0^{\infty}$$

$$= 0 - \left(0 - \frac{1}{s^2}\right) = \frac{1}{s^2}$$

ii. **Real exponential function, $x(t) = e^{at}\,u(t)$:**

$$X(s) = L[e^{at}\,u(t)] = \int_0^{\infty} e^{at} u(t) e^{-st} dt = \int_0^{\infty} e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} = -\frac{[0-1]}{(s-a)} = \frac{1}{s-a}$$

Que 3.36. What are the different properties of unilateral Laplace transform ?

OR

Show that if $x_3(t) = ax_1(t) + bx_2(t)$ then $X_3(s) = aX_1(s) + bX_2(s)$.

AKTU 2017-18, Marks 3.5

Answer

There are following properties of unilateral Laplace transform :

1. **Linearity :** Considering two signals $x_1(t)$ and $x_2(t)$ with Laplace transforms $X_1(s)$ and $X_2(s)$ respectively.

   If $x_1(t) \xleftrightarrow{LT} X_1(s)$

   and $x_2(t) \xleftrightarrow{LT} X_2(s)$

   then, $L[a_1x_1(t) + a_2x_2(t)] = a_1X_1(s) + a_2X_2(s)$

   where, a_1 and a_2 are constants.

2. **Time shifting :**

   If the $x(t)$ signal is delayed in time by t_0 unit then its Laplace transform is multiplied by e^{-st_0} factor.

   if $x(t) \xleftrightarrow{LT} X(s)$

   then, $x(t-t_0) \xleftrightarrow{LT} e^{-st_0}X(s)$

   where, t_0 is a constant.

3. **Frequency shifting :** If there is any shift in $X(s)$ by a unit, it will cause a multiplication factor e^{at} in signal $x(t)$.

   if $x(t) \xleftrightarrow{LT} X(s)$

   then, $e^{at}\,x(t) \xleftrightarrow{LT} X(s-a)$

4. **Differentiation theorem :**

   If $x(t) \xleftrightarrow{LT} X(s)$

   then differentiation theorem gives,

$$L\left[\frac{d}{dt}x(t)\right] \xleftrightarrow{LT} sX(s) - x(0^-)$$

where, $x(0^-) \rightarrow$ value of $x(t)$ at $t = 0^-$

$(t = 0^-) \rightarrow$ indicates the time first before $t = 0$

5. **Differentiation by s :** The differentiation in complex frequency-domain corresponds to multiplication by t in the time-domain.

   If $x(t) \xleftrightarrow{LT} X(s)$

   then, $tx(t) \xleftrightarrow{LT} -\frac{d}{ds}X(s)$

   then $t^n x(t) \xleftrightarrow{LT} (-1)^n \frac{d^n}{ds^n} X(s)$

5. **Integration theorem :** This theorem states that,

   If $x(t) \xleftrightarrow{LT} X(s)$

   then, $\int_0^t x(t)dt \xleftrightarrow{LT} \frac{X(s)}{s}$

6. **Scaling property :** This theorem states that,

   If $x(t) \xleftrightarrow{LT} X(s)$

   then, $x(at) \xleftrightarrow{LT} \frac{1}{|a|} X\left(\frac{s}{a}\right)$

7. **Convolution theorem :** The convolution theorem can be given in two forms.

   i. **Convolution in time-domain :** According to this theorem the Laplace transform of convolution of two functions in time-domain is equivalent to the multiplication of their Laplace transforms.

   If $x_1(t) \xleftrightarrow{LT} X_1(s)$

   and $x_2(t) \xleftrightarrow{LT} X_2(s)$

   then, $x_1(t) \otimes x_2(t) \xleftrightarrow{LT} X_2(s) \,.\, X_2(s)$

   ii. **Convolution in frequency-domain :** According to this theorem, the inverse Laplace transform of convolution of two functions in frequency-domain is equivalent to the linear multiplication of these functions in the time-domain.

   $$L^{-1}[X_1(s) \otimes X_2(s)] = x_1(t) \,.\, x_2(t)$$

   or, $x_1(t) \,.\, x_2(t) \xleftrightarrow{LT} X_1(s) \otimes X_2(s)$

8. **Initial value theorem :** Using initial value theorem, we can directly find out the initial value of $x(t)$, *i.e.*, $x(0)$ from $X(s)$, without finding the inverse Laplace transform of $X(s)$.

   If $x(t) \xleftrightarrow{LT} X(s)$

   then the initial value of $x(t)$ can be calculated as,

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$$x(0^+) = \lim_{t\to 0+} x(t) = \lim_{s\to\infty}[sX(s)]$$

9. **Final value theorem :** Using this theorem the final value of $x(t)$ (*i.e.*, $x(\infty)$) can be calculated directly from $X(s)$ without finding the inverse Laplace transform of $X(s)$.

If $$x(t) \xleftrightarrow{LT} X(s)$$

then the final value theorem gives,

$$x(\infty) = \lim_{t\to\infty} x(t) = \lim_{s\to 0}[sX(s)]$$

Que 3.37. **Explain basis of eigen function.**

Answer

1. Eigen functions are special functions associated with a given type of system in which the output of the system has same functional form as the input but with a possible scale factor.
2. The scale factor is called eigen value.
   For example, the derivative operator has exponential function for their eigen functions because
   $$\frac{d}{dx}e^{-\alpha x} = -\alpha e^{-\alpha x}$$
   with the decay constant α as the eigen value.
3. Digitized sinusoidal functions are eigen functions of digital LSI systems.

Que 3.38. **State and prove initial and final value theorem of Laplace transform.**

AKTU 2018-19, Marks 07

Answer

Initial value theorem : Refer Q. 3.36, Page 3–34A, Unit-3.

Proof : Using differentiation in time property for unilateral Laplace transform,

$$L\left[\frac{dx(t)}{dt}\right] = \int_{0^-}^{\infty}\frac{dx(t)}{dt}e^{-st}\,dt = sX(s) - x(0^-)$$

$$sX(s) - x(0^-) = \int_{0^-}^{0^+}\frac{dx(t)}{dt}e^{-st}\,dt + \int_{0^+}^{\infty}\frac{dx(t)}{dt}e^{-st}\,dt$$

$$= [x(t)]_{0^-}^{0^+} + \int_{0^+}^{\infty}\frac{dx(t)}{dt}e^{-st}\,dt$$

$$= x(0^+) - x(0^-) + \int_{0^+}^{\infty}\frac{dx(t)}{dt}e^{-st}\,dt$$

so, $$sX(s) = x(0^+) + \int_{0^+}^{\infty}\frac{dx(t)}{dt}e^{-st}\,dt$$

and $$\lim_{s\to\infty}[sX(s)] = x(0^+) + \lim_{s\to\infty}\int_{0^+}^{\infty}\frac{dx(t)}{dt}e^{-st}\,dt$$

$$= x(0^+) + \int_{0^+}^{\infty}\frac{dx(t)}{dt}\left(\lim_{s\to\infty}e^{-st}\right)dt = x(0^+)$$

hence, $$x(0^+) = \lim_{s\to\infty}[sX(s)]$$

Final value theorem : Refer Q. 3.36, Page 3–34A, Unit-3.

Proof : Using differentiation in time property, for unilateral Laplace transform,

$$L\left[\frac{dx(t)}{dt}\right] = \int_{0^-}^{\infty}\frac{dx(t)}{dt}e^{-st}\,dt = sX(s) - x(0^-)$$

$$\therefore \lim_{s\to 0}[sX(s) - x(0^-)] = \int_{0^-}^{\infty}\frac{dx(t)}{dt}(\lim_{s\to 0}e^{-st})\,dt$$

$$= \int_{0^-}^{\infty}\frac{dx(t)}{dt}\,dt = [x(t)]_{0^-}^{\infty} = x(\infty) - x(0^-)$$

hence, $$x(\infty) = \lim_{s\to 0}[sX(s)]$$

Que 3.39. **Give the verification of convolution property,**

$$x_1(t) \otimes x_2(t) \longleftrightarrow X_1(s)X_2(s),\ R' \supset R_1 \cap R_2.$$

Answer

1. Let $$y(t) = x_1(t) \otimes x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau \quad ...(3.39.1)$$
2. Taking Laplace transform of eq. (3.39.1) we get,
   $$Y(s) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau\right]e^{-st}\,dt$$
   $$= \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(t-\tau)\,e^{-st}\,dt\right]d\tau$$
   According to time-shifting property,
   $$x(t-t_0) \longleftrightarrow e^{-st_0}X(s)$$
   So, $$Y(s) = \int_{-\infty}^{\infty} x_1(\tau)\,e^{-s\tau}\,X_2(s)\,d\tau$$

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$$= \left[\int_{-\infty} x_1(\tau)\, e^{-s\tau}\, d\tau\right] X_2(s) = X_1(s)\, X_2(s)$$

Hence, $x_1(t) \otimes x_2(t) \xleftrightarrow{LT} X_1(s)\, X_2(s);\ R' \supset R_1 \cap R_2$

Que 3.40. **What is the condition of existence (convergence) of the Laplace transform ? Give the relationship between Laplace transform and Fourier transform.**

Answer

Conditions of existence of Laplace transform :

1. The Laplace transform of $x(t)$ is defined as,

$$X(s) = L[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$

2. The condition for existence of Laplace transform for some finite value of σ, is given as

$$\int_{-\infty}^{\infty} \left| x(t)\, e^{-st} \right| dt < \infty$$

3. Due to the convergence factor $e^{-\sigma t}$, the ramp, parabolic signals etc. are Laplace transformable.

Relationship between Laplace and Fourier transform :

1. The Fourier transform of signal $x(t)$ is,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad ...(3.40.1)$$

2. The Laplace transform of the signal $x(t)$ is,

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad ...(3.40.2)$$

3. Putting $s = \sigma + j\omega$ in eq. (3.40.2), we get

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} x(t) \,.\, e^{-\sigma t} e^{-j\omega t} dt$$

$$X(s) = \int_{-\infty}^{\infty} \left[x(t)\, e^{-\sigma t} \right] e^{-j\omega t} dt \quad ...(3.40.3)$$

4. The eq. (3.40.3) shows that $X(s)$ is basically the Fourier transform of $x(t)e^{-\sigma t}$. Thus, the Laplace transform of any signal may be assumed as the Fourier transform of that signal after multiplication by a real exponential signal $e^{-\sigma t}$.

5. Now assuming, $\sigma = 0$

i.e., $s = 0 + j\omega = j\omega$

Thus, the Laplace transform equation will become,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$\therefore \quad X(s) = X(\omega)$

Thus, the whole process can be summarized as

$$\text{CTFT } [x(t)] = \text{LT } [x(t)]_{s = j\omega}$$

Que 3.41. **Describe the region of convergence of Laplace transform.**

Answer

Region of convergence (ROC) :

1. The ROC is defined as the region where the value of $X(s)$ is finite.
2. For example :

$$X(s) = L[e^{at}] = \frac{1}{(s-a)}$$

Here, the Laplace transform is defined only for $\text{Re}[X(s)] > a$. The region $\text{Re}[X(s)] > a$, *i.e.*, $\sigma > a$ is called the region of convergence (ROC) as shown in Fig. 3.41.1.

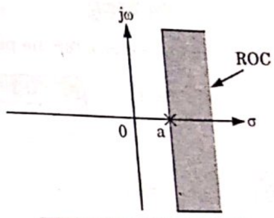

Fig. 3.41.1. ROC of $e^{at} u(t)$.

3. ROC is required to compute the Laplace transform and inverse Laplace transform. If the ROC is not specified, the inverse Laplace transform is not unique.

Que 3.42. **Explain poles and zeroes of the system.**

Answer

1. The most commonly encountered form of the Laplace transform in engineering is a ratio of two polynomials in s; that is

$$X(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + ... + b_0}{s^N + a_{N-1} s^{N-1} + ... + a_1 s + a_0}$$

2. Factor $X(s)$ as a product of terms involving the roots of the denominator and numerator polynomials :

$$X(s) = \frac{b_M \prod_{k=1}^{M} (s - c_k)}{\prod_{k=1}^{N} (s - d_k)}$$

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3. The c_k are the roots of the numerator polynomial and are termed the zeros of $X(s)$.
4. The d_k are the roots of the denominator polynomial and are termed the poles of $X(s)$.
5. We denote the locations of zeros in the s-plane with the "O" symbol and the locations of poles with the "X" symbol, as illustrated in Fig. 3.42.1.
6. The locations of poles and zeros in the s-plane uniquely specify $X(s)$ except for the constant gain factor b_M.

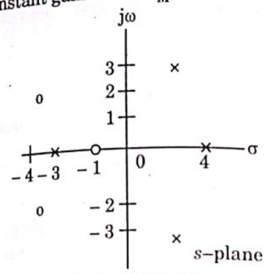

Fig. 3.42.1.

Que 3.43. Find the Laplace transform for the parabolic function

$$x(t) = t^2 e^{-3t}u(t)$$

AKTU 2015-16, Marks 05

Answer

1. Given, $x(t) = t^2 e^{-3t}u(t)$
2. $L\{x(t)\} = L\{t^2 e^{-3t}u(t)\}$
3. Since, $L[e^{-3t}] = 1/s + 3$
4. From the property of Laplace transform,

$$L\{t^2.e^{-3t}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$$

$$= \frac{d}{ds}\left[\frac{(s+3)\times 0 - 1\times 1}{(s+3)^2}\right] = \frac{-d}{ds}\frac{(1)}{(s+3)^2}$$

$$= -\left[\frac{(s+3)^2\times 0 - 1\times 2(s+3)\times 1}{(s+3)^4}\right] = \frac{2}{(s+3)^3}$$

Que 3.44. Find the ROC of the following function.

$X(s) = \dfrac{s+2}{s^2+4s+5}$ then find the Laplace of $y(t) = tx(t)$.

AKTU 2014-15, Marks 05

Answer

1. Given, $X(s) = \dfrac{s+2}{s^2+4s+5} = \dfrac{s+2}{(s+2+i)(s+2-i)}$

$$= \frac{1}{2(s+2+i)} + \frac{1}{2(s+2-i)}$$

$$= \frac{1}{2}\left[\frac{1}{(s+2+i)} + \frac{1}{(s+2-i)}\right] \quad ...(3.44.1)$$

2. The above Laplace transform converges if Re $(s) > -2$. therefore, ROC is Re $(s) > -2$
3. Now $y(t) = t\,x(t)$
   Taking Laplace transform on both sides and from the property of Laplace transform,

$$Y(s) = -\frac{d}{ds}X(s) = -\frac{d}{ds}\left[\frac{(s+2)}{s^2+4s+5}\right]$$

$$= -\left[\frac{(s^2+4s+5)1-(s+2)(2s+4)}{(s^2+4s+5)^2}\right]$$

$$= -\left[\frac{s^2+4s+5-2s^2-8s-8}{(s^2+4s+5)^2}\right] = \frac{s^2+4s+3}{(s^2+4s+5)^2}$$

Que 3.45. Find Laplace transform for

$$x(t) = \cos^3 2t\, u(t)$$

AKTU 2015-16, Marks 05

Answer

1. Given, $x(t) = \cos^3 2t\, u(t)$

$$= \frac{[\cos 6t + 3\cos 2t]}{4}u(t) \quad ...(3.45.1)$$

2. Taking Laplace transform of eq. (3.45.1)

$$X(s) = L\left[\left(\frac{\cos 6t + 3\cos 2t}{4}\right)u(t)\right]$$

$$= \frac{1}{4}\{L[\cos 6t\, u(t)] + 3L[\cos 2t\, u(t)]\}$$

$$= \frac{1}{4}\left(\frac{s}{s^2+(6)^2} + 3\frac{s}{s^2+(2)^2}\right) = \frac{1}{4}\left\{\frac{4s^3+112s}{(s^2+(6)^2)(s^2+(2)^2)}\right\}$$

$$= \frac{s(s^2+28)}{(s^2+36)(s^2+4)}$$

Que 3.46. Find inverse Laplace transform for

$$X(s) = \frac{s}{s^2a^2 + b^2}$$

AKTU 2015-16, Marks 05

Answer

1. Given, $$X(s) = \frac{s}{s^2a^2 + b^2} = \frac{1}{a^2} \cdot \frac{s}{s^2 + (b/a)^2} \quad ...(3.46.1)$$
2. We know that $$L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos(at)\, u(t)$$
3. So, taking inverse Laplace transform of eq. (3.46.1), $$x(t) = \frac{1}{a^2} \cdot \cos\left(\frac{b}{a} t\right) u(t)$$

Que 3.47. Evaluate the convolution integral of $x(t) \otimes x(2-t)$, where $x(t)$ is shown in Fig. 3.47.1.

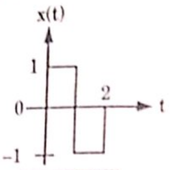

Fig. 3.47.1.

AKTU 2014-15, Marks 10

Answer

1. We can write, $x(t) = u(t) - 2\,u(t-1) + u(t-2)$ ...(3.47.1)

   and $x(2-t) = -u(t) + 2\,u(t-1) - u(t-2) = -x(t)$. ...(3.47.2)

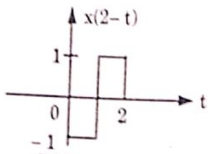

Fig. 3.47.2.

   Therefore, we have to find convolution of $x(t)$ with $[-x(t)]$.
2. Let $$y(t) = x(t) \otimes [-x(t)]$$ $$L[x(t) \otimes [-x(t)]] = Y(s) = -[X(s)]^2$$
3. From eq. (3.47.1), $$X(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1}{s}[1 - 2e^{-s} + e^{-2s}]$$

$$Y(s) = -\frac{1}{s^2}[1 + 4e^{-2s} + e^{-4s} - 4e^{-s} - 4e^{-3s} + 2e^{-2s}]$$

$$= -\frac{1}{s^2}[1 + 6e^{-2s} - 4e^{-3s} - 4e^{-s} + e^{-4s}] \quad ...(3.47.3)$$

4. Taking inverse Laplace of eq. (3.47.3) $$y(t) = x(t) \otimes x(2-t)$$ $$= -[tu(t) - 4(t-1)\,u(t-1) + 6(t-2)\,u(t-2) - 4(t-3)\,u(t-3) + (t-4)\,u(t-4)]$$

Que 3.48. If Laplace transform of $x(t)$ is $\frac{(s+2)}{(s^2+4s+5)}$. Determine Laplace transform of $y(t) = x(2t-1)u(2t-1)$.

AKTU 2016-17, Marks 7.5

AKTU 2017-18, Marks 3.5

Answer

1. The Laplace transform of $x(t)\,u(t)$ is $$x(t)\,u(t) \xleftrightarrow{\text{LT}} \frac{(s+2)}{(s^2+4s+5)}$$
2. Applying time shifting property. $$x(t-1)\,u(t-1) \xleftrightarrow{\text{LT}} e^{-s}\frac{(s+2)}{(s^2+4s+5)}$$
3. Now, applying time scaling property. $$x(2t-1)\,u(2t-1) \xleftrightarrow{\text{LT}} \frac{1}{2}e^{-\frac{s}{2}} \frac{\left(\frac{s}{2}+2\right)}{\left[\left(\frac{s}{2}\right)^2 + 4\left(\frac{s}{2}\right) + 5\right]}$$ $$= e^{-s/2}\frac{(s+4)}{(s^2+8s+20)}$$
4. Hence, the Laplace transform of $y(t) = x(2t-1)\,u(2t-1)$ is, $$Y(s) = \frac{e^{\frac{-s}{2}}(s+4)}{(s^2+8s+20)}$$

Que 3.49. Use the convolution theorem to find the Laplace transform of $y(t) = x_1(t) \otimes x_2(t)$, if $x_1(t) = e^{-3t}\,u(t)$ and $x_2(t) = u(t-2)$.

AKTU 2016-17, Marks 7.5

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Answer

1. From the convolution property, the convolution of two signals in time-domain results in the multiplication of those signals in frequency-domain.
2. So taking Laplace transform of $x_1(t)$ and $x_2(t)$.

$$L\{x_1(t)\} = \int_{-\infty}^{\infty} e^{-3t}u(t)e^{-st}dt$$

$$= \int_0^{\infty} e^{-(3+s)t}dt = \left[\frac{e^{-(3+s)t}}{-(3+s)}\right]_0^{\infty}$$

$$X_1(s) = \frac{1}{3+s}$$

3. $$L\{x_2(t)\} = \int_{-\infty}^{\infty} u(t-2)e^{-st}dt$$

$$= \int_2^{\infty} e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_2^{\infty}$$

$$X_2(s) = \left[0 - \frac{e^{-2s}}{-s}\right] = \frac{e^{-2s}}{s}$$

4. So, $$Y(s) = X_1(s).X_2(s) = \frac{e^{-2s}}{s(s+3)}$$

Que 3.50. If $X(s) = \dfrac{5s-7}{(s-1)(s+2)}$ with $-2 < R\{s\} < -1$. Find $x(t)$.

AKTU 2017-18, Marks 07

Answer

1. Given, $$X(s) = \frac{5s-7}{(s-1)(s+2)}, -2 < R\{s\} < -1$$

Now, $$\frac{5s-7}{(s-1)(s+2)} = \frac{k_1}{(s-1)} + \frac{k_2}{(s+2)}$$

$$k_1 = \left.\frac{5s-7}{s+2}\right|_{s=1} = \frac{-2}{3}$$

$$k_2 = \left.\frac{5s-7}{s-1}\right|_{s=-2} = \frac{-17}{-3} = \frac{17}{3}$$

2. So, $$X(s) = \frac{-2}{3(s-1)} + \frac{17}{3(s+2)} = X_1(s) + X_2(s)$$

The given ROC is : $-2 < R\{s\} < -1$

Since, the pole $s = -2$ is to the left of the ROC, so,

$$X_2(s) = \frac{17}{3(s+2)}$$

then, $$x_2(t) = \frac{17}{3}e^{-2t}u(t)$$

3. And since, the pole $s = 1$ lies right to the ROC and so giving non-causal term, hence

$$X_1(-s) = \frac{-2}{3(-s-1)} = \frac{2}{3(s+1)}$$

then, $$x_1(t) = \frac{2}{3}e^{-t}u(t)$$

So, $$x(t) = L^{-1}[X(s)] = x_1(t) + x_2(t)$$

$$x(t) = \frac{2}{3}e^{-t}u(t) + \frac{17}{3}e^{-2t}u(t)$$

Que 3.51. For a LTI system with unit impulse response $h(t) = e^{-2t}u(t)$. Determine the output to the input $x(t) = e^{-t}u(t)$.

AKTU 2018-19, Marks 07

Answer

1. For an LTI system,

$$y(t) = h(t) * x(t)$$

$$\therefore \quad Y(s) = H(s)X(s)$$

2. $$h(t) = e^{-2t}u(t)$$

$$L[h(t)] = H(s) = \frac{1}{s+2}$$

3. $$x(t) = e^{-t}u(t)$$

$$L[x(t)] = X(s) = \frac{1}{s+1}$$

4. $$y(t) = L^{-1}\left[\frac{1}{s+1} \times \frac{1}{s+2}\right]$$

$$y(t) = L^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

$$y(t) = L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right)$$

$$y(t) = (e^{-t} - e^{-2t})\,u(t)$$

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Que 3.52. **Find impulse response of system described by the equation $2y'(t) + 3y(t) = x(t)$.**

AKTU 2018-19, Marks 07

Answer

1. We have, $\dfrac{2\,dy(t)}{dt} + 3\,y(t) = x(t)$
2. Taking Laplace on both sides by considering zero initial conditions
$$2\,sY(s) + 3Y(s) = X(s)$$
$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{2s+3} = \frac{1/2}{s+3/2}$$
3. Taking Inverse Laplace transform
$$h(t) = \frac{1}{2}e^{-\frac{3}{2}t}u(t)$$

Que 3.53. **Find Laplace transform of following signal and draw ROC $x(t) = \cos(3t + \pi/4)\,u(t)$.**

AKTU 2018-19, Marks 07

Answer

1.
$$x(t) = \cot(3t + \pi/4)\,u(t)$$
$$x(t) = (\cos 3t \cos \pi/4 - \sin 3t \sin \pi/4)u(t)$$
$$= \left(\frac{1}{\sqrt{2}}\cos 3t - \frac{1}{\sqrt{2}}\sin 3t\right)u(t)$$
$$x(t) = \frac{1}{\sqrt{2}}(\cos 3t - \sin 3t)\;u(t)$$
2. Taking Laplace transform on both sides
$$L[x(t)] = \frac{1}{\sqrt{2}}\left[\frac{s}{s^2+3^2} - \frac{3}{s^2+3^2}\right]$$
$$X(s) = \frac{1}{\sqrt{2}}\left[\frac{s-3}{s^2+3^2}\right]$$
The location of the poles are $(3j)$ and $(-3j)$.
3. The ROC and location of poles of $X(s)$ is shown in Fig. 3.53.1.

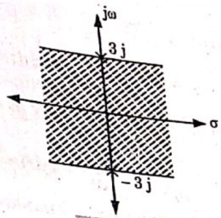
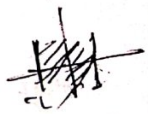


Fig. 3.53.1.



PART-7

Laplace Domain Analysis, Solution of Differential Equations and System Behavirour.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.54. **Discuss the relation between the Laplace domain and time domain of the differentiation and integration.**

Answer

1. **Differentiation in the time domain :**
$$\frac{dx(t)}{dt} \leftrightarrow sX(s) - x(0^-) \qquad ...(3.54.1)$$
provided that $\lim_{t\to\infty} x(t)e^{-st} = 0$. Repeated application of this property yields,
$$\frac{d^2x(t)}{dt} \leftrightarrow s^2X(s) - sx(0^-) - x'(0^-) \qquad ...(3.54.2)$$
$$\frac{d^nx(t)}{dt^n} \leftrightarrow s^nX(s) - s^{n-1}x(0^-) - s^{n-2}x'(0^-) - ... - x^{(n-1)}(0^-) \qquad ...(3.54.3)$$
where $\quad x^r(0^-) = \left.\dfrac{d^r\,x(t)}{dt^r}\right|_{t=0^-}$

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$$\int_{0^-}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X(s)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)\,d\tau$$

Que 3.55. **LTI System, which is initially at rest is described by differential equation, $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$. Calculate system transfer function and impulse response.**

AKTU 2014-15, Marks 10

Answer

1. Given, $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$
2. Taking Laplace transform on both sides,

$$s^2Y(s) + 3sY(s) + 2Y(s) = s\,X(s)$$
$$[s^2 + 3s + 2]\;Y(s) = s\,X(s)$$

3. Transfer function, $H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s^2+3s+2}$ ...(3.55.1)
4. Now impulse response can be calculated for

$$x(t) = \delta(t) \Rightarrow X(s) = 1$$

From eq. (3.55.1), $H(s) = \frac{s}{(s+2)(s+1)}$

5. Using partial fraction expansion,

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$A = (s+1)\left.\frac{s}{(s+2)(s+1)}\right|_{s=-1} = -1$$

$$B = (s+2)\left.\frac{s}{(s+2)(s+1)}\right|_{s=-2} = 2$$

6. $$H(s) = \frac{-1}{(s+1)} + \frac{2}{(s+2)}$$

Taking inverse Laplace transform, we get impulse response,

$$h(t) = [-e^{-t} + 2e^{-2t}]\;u(t)$$

Que 3.56. $5\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 4y(t) = 3x(t)$ **for the system described by the above differential equation, determine whether the system is underdamped, overdamped or critically damped. Also find the impulse response of the system.**

AKTU 2014-15, Marks 10

Answer

1. Given, $5\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 4y(t) = 3x(t)$ ...(3.56.1)
2. Taking Laplace transform of both sides of eq. (3.56.1)

$$5s^2\,Y(s) + 8sY(s) + 4Y(s) = 3\,X(s)$$
$$(5s^2 + 8s + 4)\;Y(s) = 3\,X(s)$$

X HW
Doing in fourier trafom

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3}{5s^2+8s+4} = \frac{3/5}{s^2+\frac{8}{5}s+\frac{4}{5}}$$

Comparing with $(s^2 + 2\xi\,\omega_n s + \omega_n^2)$

thus, $\omega_n^2 = 4/5 = 0.8,\quad 2\xi\omega_n = \frac{8}{5}$

$$\xi = 0.894$$

As, $\xi < 1$, thus system is underdamped.

3. We can also write $H(s)$ as

$$H(s) = \frac{3/5}{\left(s+\frac{4}{5}-\frac{2}{5}i\right)\left(s+\frac{4}{5}+\frac{2}{5}i\right)}$$

Doing partial fraction expansion,

$$H(s) = \frac{A}{\left(s+\frac{4}{5}-\frac{2}{5}i\right)} + \frac{B}{\left(s+\frac{4}{5}+\frac{2}{5}i\right)}$$

$$= \frac{3/4i}{\left(s+\frac{4}{5}-\frac{2}{5}i\right)} - \frac{3/4i}{\left(s+\frac{4}{5}+\frac{2}{5}i\right)}$$

$$= \frac{3}{4i}\left[\frac{1}{\left(s+\frac{4}{5}-\frac{2}{5}i\right)} - \frac{1}{\left(s+\frac{4}{5}+\frac{2}{5}i\right)}\right]$$

Taking inverse Laplace transform, the impulse response

$$h(t) = \frac{3}{4i}\left[e^{-\left(\frac{4}{5}-\frac{2}{5}i\right)t} - e^{-\left(\frac{4}{5}+\frac{2}{5}i\right)t}\right]u(t)$$

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$$= \frac{3}{4i}\left[e^{\frac{-4}{5}t}\left(e^{\frac{2}{5}it} - e^{\frac{-2}{5}it}\right)\right]u(t)$$

$$= \frac{-3i}{4}e^{-\frac{4}{5}t}\left(2i\sin\left(\frac{2}{5}t\right)\right)u(t) = \frac{3}{2}e^{-\frac{4}{5}t}\sin\left(\frac{2}{5}t\right)u(t)$$

Important Question

Que 3.57. **If $X(s) = \frac{2s+3}{(s+1)(s+2)}$, Find $x(t)$ for**

i. System is stable.

ii. System is causal.

iii. System is non-causal.

AKTU 2014-15, 2016-17; Marks 10

Answer

Given, $X(s) = \frac{2s+3}{(s+1)(s+2)}$

Using partial fraction expansion,

$$X(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

on solving partial fraction, $A = 1, B = 1$

so,
$$X(s) = \frac{1}{(s+1)} + \frac{1}{(s+2)} \quad \text{...(3.57.1)}$$

Causal mean ROC

non causal L

If system is stable, its ROC should include $j\omega$ axis, thus ROC is $\text{Re}(s) > -1$.

Now taking inverse Laplace transform of eq. (3.57.1)

$$x(t) = e^{-t}u(t) + e^{-2t}u(t)$$

ii. If system is causal then ROC is $\text{Re}(s) > -1$, hence taking inverse Laplace transform of eq. (3.57.1),

$$x(t) = e^{-t}u(t) + e^{-2t}u(t)$$

iii. If system is non-causal, then ROC is $\text{Re}(s) < -2$. Hence taking inverse Laplace transform of eq. (3.57.1)

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(-t)$$

Que 3.58. **Determine the total response of the differential equation : $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$ where $y(0) = 3$, $y'(0) = 4$, $x(t) = 4e^{-2t}$ and $t \geq 0$.**

AKTU 2015-16, Marks 10

Answer

1. Given, $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$
2. Taking Laplace transform

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = X(s)$$

$$Y(s)[s^2 + 3s + 2] - s.3 - 4 - 3 \times 3 = X(s)$$

$$Y(s)[s^2 + 3s + 2] - 3s - 13 = X(s) = \frac{4}{s+2}$$

3.
$$Y(s) = \frac{\frac{4}{s+2} + 3s + 13}{s^2 + 3s + 2}$$

$$Y(s) = \frac{4 + (3s+13)(s+2)}{(s+2)(s^2+3s+2)} = \frac{4 + 3s^2 + 19s + 26}{(s+2)(s^2+3s+2)}$$

4.
$$Y(s) = \frac{3s^2 + 19s + 30}{(s+2)^2(s+1)}$$

5.
$$Y(s) = \frac{3s^2 + 19s + 30}{(s+2)^2(s+1)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+1)}$$

6. On solving partial fraction, we get

$$A = -11, B = -4, C = 14$$

7.
$$Y(s) = \frac{-11}{s+2} + \frac{(-4)}{(s+2)^2} + \frac{14}{(s+1)}$$

$$y(t) = -11e^{-2t}u(t) - 4t\,e^{-2t}u(t) + 14e^{-t}u(t)$$

$$y(t) = -e^{-2t}(11 + 4t)u(t) + 14e^{-t}u(t)$$

Que 3.59. **Find the energy spectral density of $f(t) = e^{-st}u(t)$.**

AKTU 2018-19, Marks 07

Answer

Taking Fourier transform of

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-st}u(t)e^{-j\omega t}\,dt$$

$$= \int_{0}^{\infty} e^{-(s+j\omega)t}dt = \frac{-1}{s+j\omega}\left[e^{-(s+j\omega)t}\right]_0^{\infty}$$

$$X(\omega) = \frac{1}{s+j\omega}$$

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$$EDS = |X(\omega)|^2_{s=j\omega}$$

$$EDS = \left|\frac{1}{2j\omega}\right|^2 = \frac{1}{4\omega^2}$$

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. What are the different properties of Fourier transform?
Ans. Refer Q. 3.7.

Q. 2. Find the Fourier transform of the following function using the properties of Fourier transform.

$$y(t) = \frac{d}{dt} te^{-3t}u(t) \otimes e^{-2t}u(t)$$

Ans. Refer Q. 3.10.

Q. 3. Determine the Fourier transform of the rectangular pulse:

$$x(t) = \text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } \frac{-\tau}{2} < t < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

Ans. Refer Q. 3.12.

Q. 4. Obtain the discrete-time Fourier transform of $x[n] = a^n u[n] + a^{-n} u[-n-1]$
Ans. Refer Q. 3.19.

Q. 5. Obtain DTFT of a signal $x[n] = r^n \sin(\omega_0 n) u[n]$, $r < 1$.
Ans. Refer Q. 3.20.

Q. 6. An LTI system with impulse response $h_1(n)$

$$h_1(n) = \left(\frac{1}{3}\right)^n u(n)$$

is connected in parallel with another causal LTI system with impulse response $h_2(n)$. The resulting parallel interconnection has the frequency response.

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}}$$

Find the impulse response $h_2(n)$.
Ans. Refer Q. 3.25.

Q. 7. State and prove frequency shifting theorem of DTFT.
Ans. Refer Q. 3.26.

Q. 8. State and prove Parseval's theorem and Parseval's identity.
Ans. Refer Q. 3.30.

Q. 9. What are the different properties of unilateral Laplace transform ?
Ans. Refer Q. 3.36.

Q. 10. State and prove initial and final value theorem of Laplace transform.
Ans. Refer Q. 3.38.

Q. 11. Find the Laplace transform for the parabolic function $x(t) = t^2 e^{-3t} u(t)$
Ans. Refer Q. 3.43.

Q. 12. Evaluate the convolution integral of $x(t) \otimes x(2-t)$, where $x(t)$ is shown in Fig. 1.

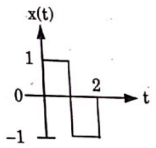

Fig. 1.

Ans. Refer Q. 3.47.

Q. 13. Use the convolution theorem to find the Laplace transform of $y(t) = x_1(t) \otimes x_2(t)$, if $x_1(t) = e^{-3t} u(t)$ and $x_2(t) = u(t-2)$.
Ans. Refer Q. 3.49.

Q. 14. Find impulse response of system described by the equation $2y'(t) + 3y(t) = x(t)$.
Ans. Refer Q. 3.52.

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Q. 15. If $X(s) = \frac{2s+3}{(s+1)(s+2)}$, Find $x(t)$ for

i. System is stable.
ii. System is causal.
iii. System is non-causal.

Ans. Refer Q. 3.57.

☺☺☺

4
UNIT

Z-Transform

CONTENTS

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PART-1

The z-Transform for Discrete Time Signals and Systems-Eigen Functions.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.1. **Explain unilateral and bilateral z-transform.**

Answer

1. The unilateral or one sided z-transform is evaluated for the non-negative values of time index ($n \geq 0$). This form of the z-transform is appropriate for problems involving causal and LTI systems.
2. The unilateral z-transform of a signal $x[n]$ is defined as,

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

which depends only on $x[n]$ for $n \geq 0$.

3. For a discrete-time signal $x[n]$, the bilateral z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

4. The bilateral and unilateral z-transform are equivalent if $x[n] = 0$ for $n < 0$.

Que 4.2. **Determine the z-transform of**

i. $x[n] = \delta[n]$

ii. $x[n] = u[n]$

Answer

i. We know that

$$\delta(n) = \begin{cases} 1 & ; \text{ for } n = 0 \\ 0 & ; \text{ for } n \neq 0 \end{cases}$$

$$\therefore \quad X(z) = Z[x(n)] = Z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n)\, z^{-n} = 1$$

ii. We know that

$$u(n) = \begin{cases} 1 & ; \text{ for } n \geq 0 \\ 0 & ; \text{ for } n < 0 \end{cases}$$

$$\therefore \quad X(z) = Z[x(n)] = Z[u(n)] = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

Que 4.3. **Determine the z-transform of**

i. $x[n] = \sin(\omega n)\, u[n]$

ii. $x[n] = \cos(\omega n)\, u[n]$

OR

Determine Z-transform of $x(n) = \sin \omega_0 n\, u(n)$.

AKTU 2018-19, Marks 07

Answer

i. Given, $x(n) = \sin(\omega n)\, u(n) = \begin{cases} \sin(\omega n) & ; \text{ for } n \geq 0 \\ 0 & ; \text{ for } n < 0 \end{cases}$

$$\therefore \quad X(z) = Z[\sin(\omega n)\, u(n)] = \sum_{n=0}^{\infty} \sin(\omega n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j} \right) z^{-n} = \frac{1}{2j} \sum_{n=0}^{\infty} (e^{j\omega n} z^{-n} - e^{-j\omega n} z^{-n})$$

$$= \sum_{n=0}^{\infty} \frac{(z^{-1}e^{j\omega})^n - (z^{-1}e^{-j\omega})^n}{2j} = \frac{1}{2j}\left[\sum_{n=0}^{\infty} (z^{-1}e^{j\omega})^n - \sum_{n=0}^{\infty} (z^{-1}e^{-j\omega})^n \right]$$

$$\therefore \quad X(z) = \frac{1}{2j}\left(\frac{1}{1 - z^{-1}e^{j\omega}} - \frac{1}{1 - z^{-1}e^{-j\omega}} \right) = \frac{1}{2j}\left(\frac{z}{z - e^{j\omega}} - \frac{z}{z - e^{-j\omega}} \right)$$

$$= \frac{1}{2j}\left[\frac{z(z - e^{-j\omega}) - z(z - e^{j\omega})}{(z - e^{j\omega})(z - e^{-j\omega})} \right] = \frac{1}{2j}\left[\frac{z(e^{j\omega} - e^{-j\omega})}{z^2 - z(e^{j\omega} + e^{-j\omega}) + 1} \right]$$

$$= \frac{z \sin \omega}{z^2 - 2z \cos \omega + 1}$$

ii. Given, $x(n) = \cos(\omega n)\, u(n) = \begin{cases} \cos(\omega n) & ; \text{ for } n \geq 0 \\ 0 & ; \text{ for } n < 0 \end{cases}$

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$$Z[\cos(\omega n)\, u(n)] = Z\left[\frac{e^{j\omega n} + e^{-j\omega n}}{2} u(n)\right]$$

$$= \frac{1}{2}\{Z[e^{j\omega n}u(n)] + Z[e^{-j\omega n}u(n)]\}$$

$$= \frac{1}{2}\left(\frac{z}{z - e^{j\omega}} + \frac{z}{z - e^{-j\omega}}\right) = \frac{1}{2}\left[\frac{z(z - e^{-j\omega}) + z(z - e^{j\omega})}{(z - e^{j\omega})(z - e^{-j\omega})}\right]$$

$$= \frac{1}{2}\left\{\frac{z[2z - (e^{j\omega} + e^{-j\omega})]}{z^2 - z(e^{j\omega} + e^{-j\omega}) + 1}\right\} = \frac{z(z - \cos\omega)}{z^2 - 2z\cos\omega + 1}$$

Que 4.4. What are the different properties of z-transform ?

Answer

1. **Linearity :** This property states that the z-transform of weighted sum of two signals is equal to the weighted sum of the individual z-transforms.

   If $\quad x_1[n] \xleftrightarrow{ZT} X_1(z)$ and $x_2[n] \xleftrightarrow{ZT} X_2(z)$

   then, $Z[a_1x_1[n] + a_2x_2[n]] \xleftrightarrow{ZT} a_1X_1(z) + a_2X_2(z)$

   where, a_1 and a_2 are constants.

2. **Time reversal :**

   If $\quad x[n] \xleftrightarrow{ZT} X(z) \qquad$ ROC : $r_1 < |z| < r_2$

   then, $\quad x[-n] \xleftrightarrow{ZT} X(z^{-1}) \qquad$ ROC : $\dfrac{1}{r_2} < |z| < \dfrac{1}{r_1}$

3. **Time shifting :** According to this property a shift of 'k' unit in time-domain signal causes a multiplication factor 'z^{-k}' in z-transform.

   If $\quad x[n] \xleftrightarrow{ZT} X(z)$

   then, $x[n-k] \xleftrightarrow{ZT} z^{-k}.X(z)$

   The ROC will remain same as that of $X(z)$ except for $n = 0$.

4. **Differentiation in z-domain of multiplication by n :**

   If $\quad x[n] \xleftrightarrow{ZT} X(z)$

   then, $\quad n\,x[n] \xleftrightarrow{ZT} -z\dfrac{dX(z)}{dz}$

5. **Accumulation property :** The accumulation property is given as,

   If $\quad x[n] \xleftrightarrow{ZT} X(z)$

   then, $\quad \displaystyle\sum_{k=-\infty}^{n} x(k) \xleftrightarrow{ZT} \frac{1}{1 - z^{-1}}X(z)$

6. **Scaling property :**

   If $\quad x[n] \xleftrightarrow{ZT} X(z) \qquad$ ROC : $r_1 < |z| < r_2$

   then, $\quad a^n x[n] \xleftrightarrow{ZT} X(a^{-1}z) \qquad$ ROC : $|a|\, r_1 < |z| < |a|\, r_2$

   Note : Constant 'a' may be real or complex quantity.

7. **Convolution :** This property gives that the convolution of two discrete-time signal can be expressed as simple multiplication of their z-transforms.

   $$x[n] = x_1[n] \otimes x_2[n] \xleftrightarrow{ZT} X(z) = X_1(z).\,X_2(z)$$

8. **Conjugate property :**

   If $\quad x[n] \xleftrightarrow{ZT} X(z) \qquad$ ROC = R

   then, $\quad x^*[n] \xleftrightarrow{ZT} X^*(z^*) \qquad$ ROC = R

   It means that if z-transform $X(z)$ has pole or zero at $z = z_0$, it must also have pole or zero at $z = z_0^*$.

9. **Initial value theorem :** For a causal discrete-time signal, the initial value may be determined by,

   $$x(0) = \lim_{n\to 0} x[n] = \lim_{z\to\infty} X(z)$$

10. **Final value theorem :** The final value of a discrete-time signal can be determined as,

    $$x(\infty) = \lim_{n\to\infty} x[n] = \lim_{z\to 1}\left[(z-1)\,X(z)\right]$$

11. **Parseval's theorem:** Let $x_1[n]$ and $x_2[n]$ are two complex valued sequences then Parseval's theorem gives,

    $$\sum_{n=-\infty}^{\infty} x_1[n]\,.\,x_2^*[n] = \frac{1}{2\pi j}\oint X_1(z).\,X_2^*[(z^*)^{-1}]z.^{-1}dz$$

Que 4.5. State and prove initial value theorem and final value theorem for z-transform.

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Answer

Initial value theorem : For a causal signal $x[n]$,

If $\quad x[n] \xleftrightarrow{ZT} X(z)$

then $\quad \displaystyle\lim_{n\to 0} x[n] = x[0] = \lim_{z\to\infty} X(z)$

Proof : We know that for a causal signal $x[n]$

$$Z\{x[n]\} = X(z)$$

$$= \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]\,z^{-1} + x[2]\,z^{-2} + ...$$

$$= x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + ...$$

Taking limit $(z \to \infty)$ on both sides, we have

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty}\left\{x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + ...\right\}$$

$$= x[0] + 0 + 0 + ... = x[0]$$

Hence, $\quad \displaystyle\lim_{n\to 0} x[n] = x[0] = \lim_{z\to\infty} X[z]$

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Final value theorem : For a causal signal $x[n]$,

if $$x[n] \xleftrightarrow{ZT} X(z)$$

and if $X(z)$ has no poles outside the unit circle and it has no double or higher order poles on the unit circle centred at the origin of the z-plane, then

$$\lim_{n \to \infty} x[n] = x[\infty] = \lim_{z \to 1} [z-1] X(z)$$

Proof : We know, for a causal signal $x[n]$,

$$Z\{x[n]\} = X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$Z\{x[n+1]\} = zX(z) - zx[0] = \sum_{n=0}^{\infty} x[n+1] z^{-n}$$

$$\therefore \quad Z\{x[n+1]\} - Z\{x[n]\} = zX(z) - zx[0] - X(z)$$

$$= \sum_{n=0}^{\infty} x[n+1] z^{-n} - \sum_{n=0}^{\infty} x[n] z^{-n}$$

or $$(z-1) X(z) - zx[0] = \sum_{n=0}^{\infty} \{x[n+1] - x[n]\} z^{-n}$$

$$= \{x[1] - x[0]\} + \{x[2] - x[1]\}z^{-1} + \{x[3] - x[2]\} z^{-2} + \ldots$$

Taking limit ($z \to 1$) on both sides, we have

$$\lim_{z \to 1} (z-1) X(z) - x[0] = x[1] - x[0] + x[2] - x[1] + x[3] - x[2] + \ldots + x[\infty]$$

Hence, $$\lim_{n \to \infty} x[n] = x[\infty] = \lim_{z \to 1} (z-1) X(z)$$

Que 4.6. **Find the unilateral z-transform of $x[n] = [a^n \cos \omega_0 n]$**

Answer

1. Given, $x[n] = [a^n \cos \{\omega_0 n\}] u[n]$
2. z-transform of $x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} [a^n \cos \omega_0 n] u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} [a^n \cos \omega_0 n] z^{-n} = \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] (az^{-1})^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} e^{j\omega_0 n} (az^{-1})^n + \frac{1}{2} \sum_{n=0}^{\infty} e^{-j\omega_0 n} (az^{-1})^n$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0} az^{-1}} + \frac{1}{1-e^{-j\omega_0} az^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j\omega_0} az^{-1} + 1 - e^{j\omega_0} az^{-1}}{(1-e^{j\omega_0} az^{-1})(1-e^{-j\omega_0} az^{-1})}\right]$$

$$= \frac{1}{2}\left[\frac{2-2\cos(\omega_0) az^{-1}}{1-e^{-j\omega_0} az^{-1} - e^{j\omega_0} az^{-1} + a^2 z^{-2}}\right]$$

$$= \frac{1 - az^{-1} \cos \omega_0}{1 - 2(\cos \omega_0) az^{-1} + a^2 z^{-2}} \qquad \text{ROC} : |z| > |a|$$

Que 4.7. **Find the convolution of $x_1[n]$ and $x_2[n]$ using z-transform.**

i. $x_1[n] = [1, 3, 4, 5]$

ii. $x_2[n] = [5, 1, 2, 6, 3, 4, 5]$

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Answer

1. Given,
$$x_1[n] = [1, 3, 4, 5] \qquad \ldots(4.7.1)$$
$$x_2[n] = [5, 1, 2, 6, 3, 4, 5] \qquad \ldots(4.7.2)$$
2. Let convolution of $x_1[n]$ and $x_2[n]$ is $y[n]$
$$y[n] = x_1[n] \otimes x_2[n]$$
or $$Y(z) = X_1(z) X_2(z) \qquad \ldots(4.7.3)$$
3. Taking z-transform of eq. (4.7.1) and (4.7.2), we get
$$X_1(z) = 1 + 3z^{-1} + 4z^{-2} + 5z^{-3}$$
$$X_2(z) = 5 + z^{-1} + 2z^{-2} + 6z^{-3} + 3z^{-4} + 4z^{-5} + 5z^{-6}$$
Now substituting in eq. (4.7.3),
$$Y(z) = X_1(z) X_2(z) = (1 + 3z^{-1} + 4z^{-2} + 5z^{-3})(5 + z^{-1} + 2z^{-2} + 6z^{-3} + 3z^{-4} + 4z^{-5} + 5z^{-6})$$
$$= 5 + (1+15)z^{-1} + (2+3+20)z^{-2} + (6+6+4+25)z^{-3} + (3+18+8+5)z^{-4} + (4+9+24+10)z^{-5} + (5+12+12+30)z^{-6} + (15+16+15)z^{-7} + (20+20)z^{-8} + 25z^{-9}$$
$$Y(z) = 5 + 16z^{-1} + 25z^{-2} + 41z^{-3} + 34z^{-4} + 47z^{-5} + 59z^{-6} + 46z^{-7} + 40z^{-8} + 25z^{-9} \qquad \ldots(4.7.4)$$
4. Thus, by taking inverse z-transform of eq. (4.7.4)
$$y[n] = [5, 16, 25, 41, 34, 47, 59, 46, 40, 25]$$

Que 4.8.

i. Find the Laplace transform of $x(t) = e^{-2t} u(t+1)$

ii. Find the z-transform of $x[n] = \begin{cases} n : & 0 \le n \le N-1 \\ N : & N \le n \end{cases}$

AKTU 2014-15, Marks 05

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Answer

i. 1. Given, $x(t) = e^{-2t}u(t+1)$

2. We know, $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-2t}\, u(t+1)\, e^{-st}\, dt$

$$= \int_{-1}^{\infty} e^{-(s+2)t}\, dt$$

$$= \left[\frac{-e^{-(s+2)t}}{(s+2)}\right]_{-1}^{\infty} = \frac{e^{(s+2)}}{(s+2)}$$

ii. Given, $x[n] = \begin{cases} n: & 0 \le n \le N-1 \\ N: & N \le n \end{cases}$

So, $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{N-1} nz^{-n} + \sum_{n=N}^{\infty} Nz^{-n}$

$$= (0 + z^{-1} + 2z^{-2} + 3z^{-3} + ... + (N-1)z^{-N+1}) + N\frac{z^{-N}}{1-z^{-1}}$$

If we take $N = 4$, then

$$X(z) = z^{-1} + 2z^{-2} + 3z^{-3} + 4\frac{z^{-4}}{1-z^{-1}}$$

Que 4.9. **Discuss about the eigen function and eigen value associated with the discrete-time LTI system.**

Answer

1. The complex exponential sequences are eigen functions of discrete-time LTI systems.
2. Suppose that an LTI system with impulse response $h[n]$ has its input sequence as

$$x[n] = z^n, \quad ...(4.9.1)$$

where z is a complex number.

3. Then the output of the system can be determined from the convolution sum as

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k} \quad ...(4.9.2)$$

4. If the input $x[n]$ is the complex exponential given by eq. (4.9.1), then, assuming that the summation on the right-hand side of eq. (4.9.2) converges, the output is the same complex exponential multiplied by a constant that depends on the value of z. That is,

$$y[n] = H(z)z^n, \quad ...(4.9.3)$$

where $H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$ ...(4.9.4)

5. Complex exponentials are eigen functions of discrete-time LTI systems. The constant $H(z)$ for a specified value of z is the eigen value associated with the eigen function z^n.
6. If the input to a discrete-time LTI system is represented as a linear combination of complex exponentials, that is, if

$$x[n] = \sum_k a_k z_k^n, \quad ...(4.9.5)$$

then the output will be

$$y[n] = \sum_k a_k H(z_k)$$

PART-2

Region of Convergence, z-domain analysis.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.10. **What is region of convergence ? What are the important properties of ROC for z-transform ?**

Answer

ROC : ROC is defined as the set of values of z in the z-plane for which the magnitude of $X(z)$ is finite.

Characteristics of ROC :

1. The condition for z-transform $X(z)$ to be finite is $|z| > 1$.
2. ROC for $X(z)$ is the area outside the unit circle in the z-plane.
3. The ROC of a rational z-transform is bounded by the location of its poles.

Properties of ROC for z-transform :

1. The ROC is a ring in the z-plane centred at the origin.
2. The ROC cannot contain any poles.
3. The ROC must be connected region.
4. If $x(n)$ is a finite duration sequence, then the ROC is the entire z-plane except possibly $z = 0$ or $z = \infty$.

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5. If $x(n)$ is right-sided sequence, then the ROC exist outward from the outermost finite pole to $z = \infty$.
6. If $x(n)$ is a left-sided sequence, then the ROC extends inward from the innermost pole to $z = 0$.
7. If two sided sequence, then the ROC will consist a ring in the z-plane, bounded on the interior and exterior by a pole and consistent with property (2).

Example : Let $X(z) = 1/(z-1)$

The $X(z)$ has a pole at $z = 1$, therefore the ROC will be $|z| > 1$ as shown in Fig. 4.10.1.

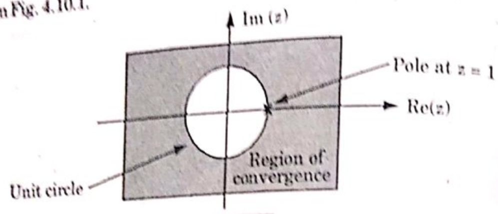

Fig. 4.10.1.

Que 4.11. Determine the z-transform and ROC of the following signals :

i. $x[n] = a^n u[n]$
ii. $x[n] = a^{|n|}, |a| < 1$

Answer

i. Given, $x[n] = a^n u[n]$

1. $$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} a^n u[n]\, z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \ldots.$$

$$= 1 + a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \ldots.$$

$$X(z) = \frac{1}{1 - az^{-1}}$$

For ROC $\quad 1 - az^{-1} > 0$

$$|z| > |a|$$

2. The ROC for $X(z)$ is shown in Fig. 4.11.1.

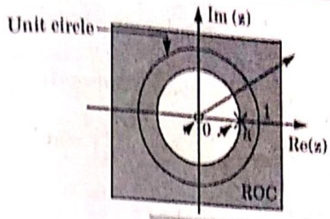

Fig. 4.11.1.

ii. Given, $x[n] = a^{|n|}, |a| < 1$

1. Now,

$$X(z) = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=1}^{\infty} (az)^n + \sum_{n=0}^{\infty} (az^{-1})^n = \frac{az}{1-az} + \frac{1}{1-az^{-1}}$$

2. $$X(z) = -\frac{z}{z - \frac{1}{a}} + \frac{z}{z-a}$$

This part converges if $|z| < \frac{1}{|a|}$ $\qquad$ This part converges if $|z| > |a|$

3. Therefore, ROC of $X(z)$ will be $|a| < |z| < \frac{1}{|a|}$

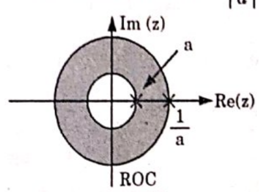

Fig. 4.11.2.

Que 4.12. When the input to an LTI system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + (2)^n u[-n-1].$$

and the corresponding $y[n] = 5\left(\frac{1}{5}\right)^n u[n] - 5\left(\frac{2}{3}\right)^n u[n]$.

i. Find the system function $H(z)$ of the system and its ROC.
ii. Find the impulse response $h(n)$ of the system.
iii. Is system stable and causal ?

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Answer

1. Given,

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + (2)^n u[-n-1] \qquad ...(4.12.1)$$

$$y[n] = 5\left(\frac{1}{5}\right)^n u[n] - 5\left(\frac{2}{3}\right)^n u[n]. \qquad ...(4.12.2)$$

2. System function, $H(z) = \dfrac{Y(z)}{X(z)}$ ...(4.12.3)

Taking z-transform of eq. (4.12.1) and (4.12.2)

$$X(z) = \frac{z}{z-1/3} - \frac{z}{z-2}, \qquad \text{ROC} : \frac{1}{3} < |z| < 2$$

$$Y(z) = \frac{5z}{z-1/5} - \frac{5z}{z-2/3}, \qquad \text{ROC} : |z| > 2/3$$

so,

$$H(z) = \frac{5z\left(\dfrac{1}{z-1/5} - \dfrac{1}{z-2/3}\right)}{z\left(\dfrac{1}{z-1/3} - \dfrac{1}{z-2}\right)}$$

$$= \frac{5z(z-2/3-z+1/5)(z-1/3)(z-2)}{z(z-2-z+1/3)(z-1/5)(z-2/3)}$$

$$H(z) = \frac{7z(z-1/3)(z-2)}{5z(z-1/5)(z-2/3)}$$

3. The combined ROC for $H(z)$ will be $2/3 < |z| < 2$.

$$\frac{H(z)}{z} = \frac{7(z-1/3)(z-2)}{5z(z-1/5)(z-2/3)} \qquad ...(4.12.4)$$

4. Doing partial fraction expansion of eq. (4.12.4)

$$\frac{H(z)}{z} = \frac{A}{z} + \frac{B}{(z-1/5)} + \frac{C}{(z-2/3)}$$

$$= \frac{7}{z} - \frac{18}{5(z-1/5)} - \frac{2}{(z-2/3)}$$

$$H(z) = 7 - \frac{18}{5}\frac{z}{(z-1/5)} - \frac{2z}{(z-2/3)} \qquad ...(4.12.5)$$

7. Taking inverse z-transform of eq. (4.12.5)

$$h(n) = 7\delta[n] - \frac{18}{5}\left(\frac{1}{5}\right)^n u[n] - 2\left(\frac{2}{3}\right)^n u[n]$$

8. As ROC consists of the unit circle, thus system is both stable and causal.

Que 4.13. **Determine inverse z-transform of the following function.**

$$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2} + z^{-3}}$$

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Answer

1. Given,

$$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2} + z^{-3}} \qquad ...(4.13.1)$$

2. From eq. (4.13.1), $H(z) = \dfrac{3z^3 + 3.6z^2 + 0.6z}{z^3 + 0.1z^2 - 0.2z + 1}$

3. We can find inverse z-transform by long division method.

$$\begin{array}{l}
z^3 + 0.1z^2 - 0.2z + 1 \,\big)\, 3z^3 + 3.6z^2 + 0.6z \,\big(\, 3 + 3.3z^{-1} + 0.87z^{-2} - 2.253z^{-3} - 3.3513z^{-4} \\
\qquad\qquad 3z^3 + 0.3z^2 - 0.6z + 3 \\
\qquad\qquad \underline{- \quad - \quad + \quad -} \\
\qquad\qquad\quad 3.3z^2 + 1.2z - 3 \\
\qquad\qquad\quad 3.3z^2 + 0.33z - 0.66 + 3.3z^{-1} \\
\qquad\qquad\quad \underline{- \quad - \quad + \quad -} \\
\qquad\qquad\qquad 0.87z - 2.34 - 3.3z^{-1} \\
\qquad\qquad\qquad 0.87z - 0.087 - 0.174z^{-1} + 0.87z^{-2} \\
\qquad\qquad\qquad \underline{- \quad + \quad + \quad -} \\
\qquad\qquad\qquad\quad - 2.253 - 3.126z^{-1} - 0.87z^{-2} \\
\qquad\qquad\qquad\quad - 2.253 - 0.2253z^{-1} + 0.4506z^{-2} - 2.253z^{-3} \\
\qquad\qquad\qquad\quad \underline{+ \quad - \quad - \quad +} \\
\qquad\qquad\qquad\qquad - 3.3513z^{-1} \ldots\ldots\ldots
\end{array}$$

so,

$$H(z) = 3 + 3.3z^{-1} + 0.87z^{-2} - 2.253z^{-3} - 3.3513z^{-4} \qquad ...(4.13.2)$$

4. Taking inverse z-transform of eq. (4.13.2)

$$h[n] = [3, 3.3, 0.87, -2.253, -3.3513, \ldots]$$

Que 4.14. **Determine inverse z-transform of the following signal**

$$X(z) = \frac{z^3 - z^2 + z}{(z-0.5)(z-2)(z-1)}; \text{ROC} : 1 < |z| < 2$$

AKTU 2014-15, Marks 05

Answer

1. Given, $X(z) = \dfrac{z^3 - z^2 + z}{(z-0.5)(z-2)(z-1)}$; ROC : $1 < |z| < 2$ ...(4.14.1)

2. Now, eq. (4.14.1) can be written as

$$\frac{X(z)}{z} = \frac{z^2 - z + 1}{(z-0.5)(z-2)(z-1)} = X_1(z)$$

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3. Doing partial fraction expansion,

$$\frac{X(z)}{z} = \frac{A}{(z-0.5)} + \frac{B}{(z-2)} + \frac{C}{(z-1)}$$

On solving, $A = 1, B = 2, C = -2$

So, $$\frac{X(z)}{z} = \frac{1}{(z-0.5)} + \frac{2}{(z-2)} - \frac{2}{(z-1)}$$

4. Thus the terms with pole at $z = 0.5$ and 1 must be causal and term with pole $z = 2$ must be non-causal, therefore

$$X(z) = \frac{z}{(z-0.5)} + \frac{2z}{(z-2)} - \frac{2z}{(z-1)} \quad ...(4.14.2)$$

5. Taking inverse z-transform, of eq. (4.14.2)

$$x[n] = (0.5)^n u[n] - 2(2)^n u[-n-1] - 2\, u[n]$$

Que 4.15. **Determine the inverse z-transform using partial fraction method for**

$$X(z) = \frac{\left(\frac{1}{4}\right)z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

i. $|z| > \frac{1}{2}$

ii. $|z| < \frac{1}{4}$

iii. $\frac{1}{4} < |z| < \frac{1}{2}$

AKTU 2015-16, Marks 10

Answer

1. Given, $$X(z) = \frac{\left(\frac{1}{4}\right)z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} \quad ...(4.15.1)$$

2. Multiplying the numerator and denominator with z^2 in eq. (4.15.1), we obtain

$$X(z) = \frac{(1/4)z}{[z-(1/2)][z-(1/4)]}$$

3. Now, $X(z)$ has two poles, one at $z = (1/2)$ and the other at $z = (1/4)$ as shown in Fig. 4.15.1.

or $$X(z) = \frac{z}{z-(1/2)} - \frac{z}{z-(1/4)}$$

i. **For ROC : $|z| > \frac{1}{2}$:**

Here both the poles, *i.e.*, $z = (1/2)$ and $z = (1/4)$ correspond to causal terms.

So, $$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

ii. **For ROC : $|z| < \frac{1}{4}$:**

Here both the poles must correspond to anticausal terms.

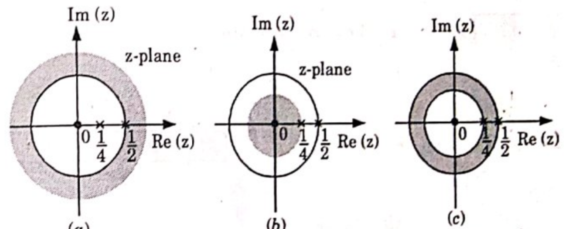
So, $$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(\frac{1}{4}\right)^n u[-n-1]$$

iii. **For ROC : $\frac{1}{4} < |z| < \frac{1}{2}$:**

Here the pole at $z = (1/4)$ must correspond to causal term and the pole at $z = (1/2)$ must correspond to non-causal term.

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{1}{4}\right)^n u[n]$$

4. The ROCs are shown in Fig. 4.15.1.



(*a*) ROC : $|z| > \frac{1}{2}$ (*b*) ROC : $|z| < \frac{1}{4}$ (*c*) ROC : $\frac{1}{4} < |z| < \frac{1}{2}$

Fig. 4.15.1.

Que 4.16. **Using properties of z-transform, find z-transform and ROC of signal.**

$$x[n] = n2^n \sin\left(\frac{n\pi}{2}\right) u[n]$$

AKTU 2015-16, Marks 7.5

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Answer

1. Given, $x[n] = n2^n \sin\left(\frac{n\pi}{2}\right)u[n]$
2. We know that
$$Z\left[\sin\left(\frac{\pi}{2}n\right)u[n]\right] = \frac{z\sin(\pi/2)}{z^2 - 2z\cos(\pi/2) + 1} = \frac{z}{z^2+1}, \; |z| > 1$$
3. Using the multiplication by an exponential property, we have
$$Z\left[2^n \sin\left(\frac{\pi}{2}n\right)u[n]\right] = Z\left[\sin\left(\frac{\pi}{2}n\right)u[n]\right]\Bigg|_{z\to(z/2)}$$
$$= \frac{z}{z^2+1}\Bigg|_{z\to(z/2)} = \frac{z/2}{(z/2)^2+1} = \frac{2z}{z^2+4}, \; |z| > 2$$
4. Using differentiation in z-domain property, we have
$$X(z) = Z\left[n2^n \sin\left(\frac{\pi}{2}n\right)u[n]\right] = -z\frac{dZ}{dz}\left[2^n \sin\left(\frac{\pi}{2}n\right)u[n]\right]$$
$$X(z) = -z\frac{d}{dz}\left(\frac{2z}{z^2+4}\right) = -z\left[\frac{(z^2+4)(2) - 2z(2z)}{(z^2+4)^2}\right], \; |z| > 2$$
$$X(z) = -z\left[\frac{-2z^2+8}{(z^2+4)^2}\right] = \frac{2z(z^2-4)}{(z^2+4)^2} \text{ and ROC : } |z| > 2.$$

Que 4.17. Determine the z-transform of following sequences with ROC :

i. $u[n]$
ii. $-u[-n-1]$
iii. $x[n] = a^n u[n] - b^n u[-n-1]$

AKTU 2016-17, Marks 10

Answer

i. Given, $x[n] = u[n]$

then, $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=0}^{+\infty} 1.z^{-n}$
$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1} \text{ provided } |z^{-1}| < 1, \text{ i.e.; } |z| > 1$$

hence, as shown in Fig. 4.17.1, the ROC of $X(z)$ is the exterior of the unit circle.

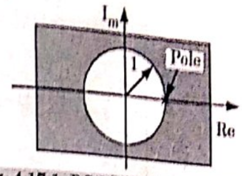

Fig. 4.17.1. ROC of $X(z)$ when $x[n] = u[n]$.

ii. Given, $x[n] = -u[-n-1]$
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = -\sum_{n=-\infty}^{-1} z^{-n}$$

put $k = -n$
$$X(z) = -\sum_{k=1}^{\infty} z^k = 1 - \sum_{k=0}^{\infty} z^k = 1 - \frac{1}{1-z} = \frac{z}{z-1}; \text{ provided } |z| < 1$$

so, $X(z) = \frac{z}{z-1}; |z| < 1$

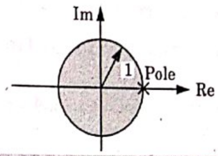

Fig. 4.17.2. ROC and pole location of $X(z)$ when $x[n] = -u[-n-1]$.

iii. Given, $x[n] = a^n u[n] - b^n u[-n-1]$
$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n].z^{-n} - \sum_{n=-\infty}^{\infty} b^n u[-n-1].z^{-n}$$
$$= \sum_{n=0}^{\infty} (az^{-1})^n - \sum_{n=-\infty}^{-1} (bz^{-1})^n$$
$$= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} \quad ; \text{ROC} : |a| < |z| < |b|$$
$$X(z) = \frac{z}{z-a} + \frac{z}{z-b} \quad ; \text{ROC} : |a| < |z| < |b|$$

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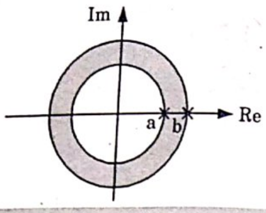

Fig. 4.17.3. ROC and pole location of $X(z)$ when $x[n] = a^n u[n] - b^n u[-n-1]$.

Que 4.18. **Determine the total response of the differential equation**

$$y[n] + 4y[n-1] + 4y[n-2] = (-2)^n u[n]$$

where $y[-1] = 0$ and $y[-2] = 1$ **AKTU 2015-16, Marks 10**

Answer

1. Given, $y[n] + 4y[n-1] + 4y[n-2] = (-2)^n u[n]$
   $y[-1] = 0, \; y[-2] = 1$
2. Taking Z-transform, we get

$$Y(z) + 4[z^{-1}Y(z) + y(-1)] + 4[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = \frac{z}{z-(-2)}$$

$$Y(z)[1 + 4z^{-1} + 4z^{-2}] + 4 \times 0 + 4 \times z^{-1} \times 0 + 4 \times 1 = \frac{z}{z+2}$$

3. $Y(z)[1 + 4z^{-1} + 4z^{-2}] + 4 = \dfrac{z}{z+2}$

$$Y(z) = \frac{\dfrac{z}{z+2} - 4}{1 + 4z^{-1} + 4z^{-2}} = \frac{(z - 4z - 8)z^2}{(z+2)(z^2 + 4z + 4)}$$

$$Y(z) = \frac{-z^2(3z+8)}{(z+2)(z^2+4z+4)}$$

$$\frac{Y(z)}{z} = \frac{-z(3z+8)}{(z+2)(z^2+4z+4)}$$

$$\frac{Y(z)}{z} = \frac{-3z^2 - 8z}{(z+2)^3} = \frac{A}{z+2} + \frac{B}{(z+2)^2} + \frac{C}{(z+2)^3}$$

4. On solving partial fraction, we get
   $A = -3, B = 4, C = 4$
5. $\dfrac{Y(z)}{z} = \dfrac{-3}{(z+2)} + \dfrac{4}{(z+2)^2} + \dfrac{4}{(z+2)^3}$
6. Taking inverse Z-transform
   $y[n] = -3(-2)^n u[n] + n(-2)^{n+1} u[n] + (-1)^n 2^{n-1}(n-1)n\, u[n]$

Que 4.19. **A causal LTI system is described by difference equation.**

$$y(n) = y(n-1) + y(n-2) + x(n-1)$$

Find the system function $H(z)$ for this system. Plot the poles zeros of $H(z)$ and indicate the region of convergence. **AKTU 2017-18, Marks 07**

Answer

1. Given, $y(n) = y(n-1) + y(n-2) + x(n-1)$
2. Taking z-transform

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$$

$$Y(z)(1 - z^{-1} - z^{-2}) = z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$$

or $$H(z) = \frac{z}{z^2 - z - 1} = \frac{z}{\left(z - \left(\frac{1+\sqrt{5}}{2}\right)\right)\left(z - \left(\frac{1-\sqrt{5}}{2}\right)\right)}$$

Poles : $z = \left(\dfrac{1+\sqrt{5}}{2}\right)$ and $z = \left(\dfrac{1-\sqrt{5}}{2}\right)$

Zeros : $z = 0$

ROC : $|z| < \left(\dfrac{1+\sqrt{5}}{2}\right)$ and $|z| < \left(\dfrac{-1+\sqrt{5}}{2}\right)$

Since, ROC should not contain any pole, hence, ROC will be

$$|z| < \left(\frac{-1+\sqrt{5}}{2}\right)$$

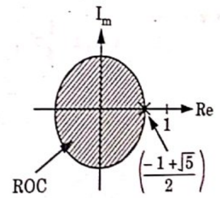

Fig. 4.19.1.

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

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Q. 1. Determine the z-transform of
i. $x[n] = \sin(\omega n)\, u[n]$
ii. $x[n] = \cos(\omega n)\, u[n]$

Ans. Refer Q. 4.3.

Q. 2. State and prove initial value theorem and final value theorem for z-transform.

Ans. Refer Q. 4.5.

Q. 3. Find the convolution of $x_1[n]$ and $x_2[n]$ using z-transform.
i. $x_1[n] = [1, 3, 4, 5]$
ii. $x_2[n] = [5, 1, 2, 6, 3, 4, 5]$

Ans. Refer Q. 4.7.

Q. 4. When the input to an LTI system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + (2)^n u[-n-1]$$

and the corresponding $y[n] = 5\left(\frac{1}{5}\right)^n u[n] - 5\left(\frac{2}{3}\right)^n u[n]$.

i. **Find the system function $H(z)$ of the system and its ROC.**
ii. **Find the impulse response $h(n)$ of the system.**
iii. **Is system stable and causal ?**

Ans. Refer Q. 4.12.

Q. 5. Determine the inverse z-transform using partial fraction method for

$$X(z) = \frac{\left(\frac{1}{4}\right)z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

i. $|z| > \frac{1}{2}$

ii. $|z| < \frac{1}{4}$

iii. $\frac{1}{4} < |z| < \frac{1}{2}$

Ans. Refer Q. 4.15.

Q. 6. Using properties of z-transform, find z-transform and ROC of signal.

$$x[n] = n2^n \sin\left(\frac{n\pi}{2}\right) u[n]$$

Ans. Refer Q. 4.16.

Q. 7. Determine the z-transform of following sequences with ROC :
i. $u[n]$
ii. $-u[-n-1]$
iii. $x[n] = a^n u[n] - b^n u[-n-1]$

Ans. Refer Q. 4.17.

Q. 8. Determine the total response of the differential equation $y[n] + 4y[n-1] + 4y[n-2] = (-2)^n u[n]$ **where** $y[-1] = 0$ **and** $y[-2] = 1$

Ans. Refer Q. 4.18.

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5
UNIT

Sampling of Time Signals

CONTENTS

PART-1

The Sampling Theorem and its Implications-Spectra of Sampled Signals.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.1. **What do you mean by sampling of a signal ? Also explain Nyquist rate of sampling.**

Answer

Sampling :

Sampling is the process of converting a continuous-time signal $x(t)$ into a discrete-time signal $x[n]$ by measuring the amplitudes of the continuous-time signal $x(t)$ at integer multiples of a sampling interval T as shown in Fig. 5.1.1.

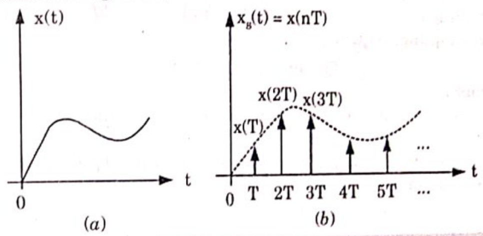

Fig. 5.1.1. (*a*) Continuous-time signal and (*b*) Its sampled version.

Nyquist rate of sampling :

1. Nyquist rate of sampling is the theoretical minimum sampling rate at which a signal can be sampled and still be reconstructed from its samples without any distortion.
2. It is theoretically by minimum, because when the Nyquist rate of sampling is used, only an ideal LPF can be used to extract $X(\omega)$ from $X_s(\omega)$, *i.e.*, to recover $x(t)$ from $x_s(t)$.
3. It is always equal to $2f_m$ where, f_m is the maximum frequency component present in the signal.
4. The minimum sampling rate, or minimum sampling frequency $\omega_s = 2\omega_m$ (or $f_s = 2f_m$), is referred as the Nyquist rate; its reciprocal $T_s = 2\pi/\omega_s = 1/f_s \leq 1/2f_m$; (measured in seconds) is called the Nyquist interval.

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5. Sampling a signal at a rate less than the Nyquist rate is referred as under sampling and greater than the Nyquist rate is referred as oversampling.

Nyquist interval :

1. Nyquist interval is the time interval between any two adjacent samples when sampling rate is Nyquist rate.
2. Nyquist rate, $f_s = 2f_m$ Hz
3. Nyquist interval, $T_s = \frac{1}{2f_m}$ sec

Que 5.2. State and prove sampling theorem. Also give some applications of sampling theorem.

OR

State and prove sampling theorem. **AKTU 2018-19, Marks 07**

Answer

Statement :

Sampling theorem states that a band-limited signal of finite energy, which has no frequency components higher than ω_m rad/s (or f_m Hz), may be completely recovered from a knowledge of its samples if the sampling frequency $\omega_s \geq 2\omega_m$ samples/rad/s (or $f_s \geq 2f_m$ samples/s). Thus, the sampling frequency

$$\omega_s \geq 2\omega_m \text{ or } f_s \geq 2f_m$$

Proof :

1. Let $x(t)$ is a continuous-time band limited signal and it has $X(\omega) = 0$ for $\omega > \omega_m$.
2. $\delta_T(t)$ is an impulse train which samples at a rate of f_s Hz and $x_s(t)$ is the sampled signal.

$$x_s(t) = x(t) \cdot \delta_T(t)$$

where, $$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

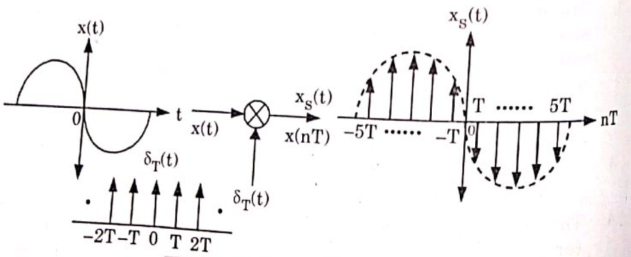

Fig. 5.2.1. Sampling operation.

3. Since, the exponential form of Fourier series of $\delta_T(t)$ is

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$\therefore \quad x_s(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)\, e^{jn\omega_s t} \quad ...(5.2.1)$$

4. Taking Fourier transform on both sides of eq. (5.2.1), we have

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T}n\right)$$

or $$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

5. Thus, the spectrum of the sampled signal $X_s(\omega)$ is the sum of shifted replicas $\frac{1}{T}X(\omega)$ of centering at $\left(\frac{2\pi}{T}\right)n$, where $n = 0, \pm 1, \pm 2, ...$

6. For signal recovery, $\omega_s - \omega_m \geq \omega_m$

or $\omega_s \geq 2\omega_m$

or $f_s \geq 2f_m$

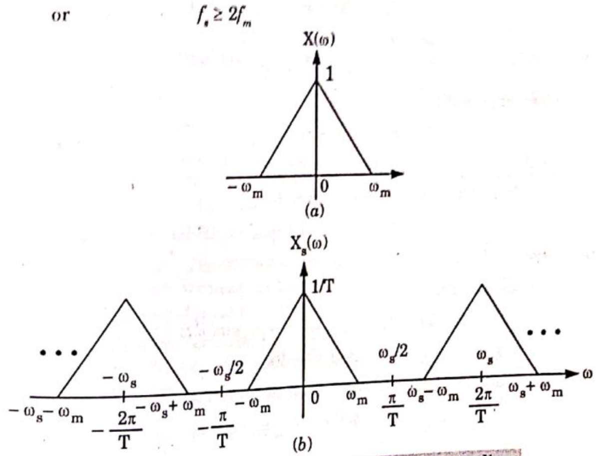

Fig. 5.2.2. Frequency-domain representation of sampling.

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Applications of sampling theorem :

1. The sampling theorem is very important in signal analysis, processing and transmission because it allows us to replace a continuous-time signal by a discrete sequence of numbers. This leads us directly into the area of digital filtering.
2. This opens doors to many new techniques of communicating continuous time signals by pulse train. In this we may vary the amplitude, widths or positions. We can have pulse amplitude modulation (PAM), pulse width modulation (PWM) and pulse position modulation (PPM).

Que 5.3. **Determine the Nyquist rate corresponding to each of the following signals :**

i. $x(t) = \sin(200\pi t)$
ii. $x(t) = \sin^2(200\pi t)$
iii. $x(t) = 1 + \cos(200\pi t) + \sin(400\pi t)$

Answer

i. Given, $x(t) = \sin(200\pi t)$

Comparing with

$$x(t) = \sin(\omega_1 t)$$

we have $\omega_1 = 200\pi$ or $f_1 = 100$

Therefore, the Nyquist rate is given by

$$\omega_s = 2\omega_1 = 400\pi \text{ or } f_s = 2f_1 = 200 \text{ Hz}$$

ii. Given, $x(t) = \sin^2(200\pi t) = \frac{1}{2} - \frac{1}{2}\cos(400\pi t)$

Comparing with

$$x(t) = A_0 - A_1 \cos(\omega_1 t)$$

we have $\omega_1 = 400\pi$ or $f_1 = 200$

Therefore, the Nyquist rate is given by

$$\omega_s = 2\omega_1 = 800\pi \text{ or } f_s = 2f_1 = 400 \text{ Hz}$$

iii. Given, $x(t) = 1 + \cos(200\pi t) + \sin(400\pi t)$

Comparing with

$$x(t) = A_0 + A_1 \cos(\omega_1 t) + A_2 \sin(\omega_2 t)$$

we have $\omega_1 = 200\pi$ and $\omega_2 = 400\pi$

Maximum frequency present in the signal $x(t)$ is

$$\omega_{max} = \omega_2 = 400\pi \text{ or } f_{max} = 200$$

therefore, the Nyquist rate is given by

$$\omega_s = 2\omega_{max} = 800\pi \text{ or } f_s = 2f_{max} = 400 \text{ Hz}$$

Que 5.4. **Explain the sampling theorem. What are the practical difficulties in signal reconstruction? Also give some applications of sampling theorem.**

Answer

Sampling theorem and its application : Refer Q. 5.2, Page 5–3A, Unit-5.

Practical difficulties in signal reconstruction :

1. $x(t)$ should be band limited to frequency ω_m.
2. The sampling frequency ω_s should be at least twice the band-limiting frequency ω_m [*i.e.*, $\omega_s \geq 2\omega_m$].

Que 5.5. **What are the different techniques of sampling? Discuss any one of them in detail.**

OR

Explain the impulse train sampling of discrete time signals.

AKTU 2018-19, Marks 07

Answer

Basically there are three types of sampling techniques :

i. Instantaneous sampling or impulse sampling
ii. Natural sampling
iii. Flat-top sampling.

Out of these three methods, instantaneous or impulse sampling is also called ideal sampling, whereas the natural sampling and flat-top sampling are called practical sampling methods.

Ideal or impulse sampling :

1. Ideally, sampling should be done instantaneously so that the k^{th} element of the sequence obtained by sampling represents the value of $x(t)$ at $t = kT$. The operation is shown in Fig. 5.5.1(*a*).
2. Assume that the fictitious sampler closes almost for zero time once in every T sec. It is equivalent to transmitting the input signal to the output for a very very short time (almost zero time) once in every T sec.

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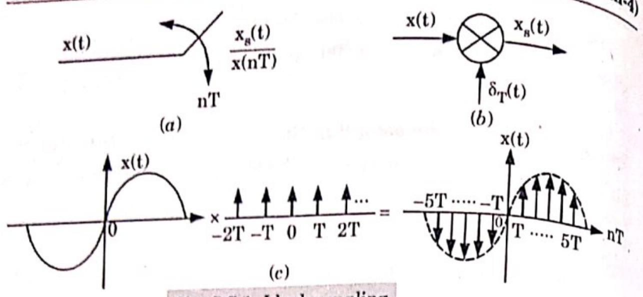

Fig. 5.5.1. Ideal sampling.

3. Now, the operation is equivalent to multiplying the input signal $x(t)$ by an impulse train $\delta_T(t)$ as shown in Fig. 5.5.1(*b*). So the output of the sampler is a train of impulses of height equal to the instantaneous value of the input signal at the sampling instant.
4. The impulse train, also called the sampling function is represented as :

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

5. The sampled signal is given by

$$x_s(t) = x(t)\ \delta_T(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

$$\therefore \qquad X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

or

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad ...(5.5.1)$$

6. This eq. (5.5.1) gives the spectrum of ideally sampled signal. It shows that the spectrum $X_s(\omega)$ is an infinite sum of shifted replicas of $X(\omega)$ spaced $n\omega_s$ apart, where $n = \pm 1, \pm 2$, etc., and scaled by a factor $1/T$.

Que 5.6. **Explain natural sampling in detail.**

Answer

1. Natural sampling, is achieved by multiplying the signal $x(t)$ with a pulse train $p_T(t)$ as shown in Fig. 5.6.1.
2. Each pulse of $p_T(t)$ is of short duration τ and occurs at a sampling period of T sec. The output of the sampler is same as the input during that short duration τ. Hence it is termed as natural sampling.

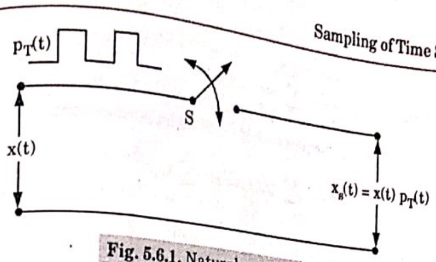

Fig. 5.6.1. Natural sampler.

3. Fig. 5.6.2 explains the process of natural sampling. Fig. 5.6.2(*a*) is the signal $x(t)$ to be sampled, and Fig. 5.6.2(*b*) is its spectrum $X(f)$. Fig. 5.6.2(*c*) is the pulse train $p_T(t)$, and Fig. 5.6.2(*d*) is its spectrum $P(f)$. Fig. 5.6.2(*e*) is the output of the sampler $x_s(t)$, and Fig. 5.6.2(*f*) is the output spectrum $X_s(f)$.

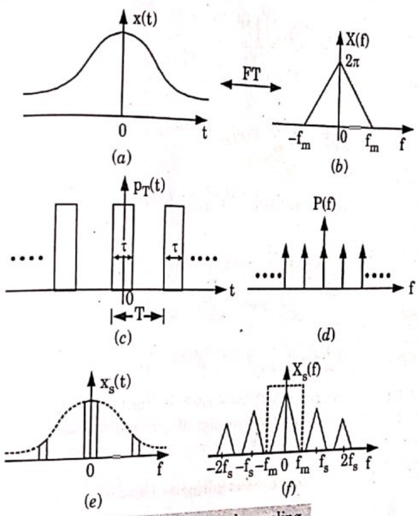

Fig. 5.6.2. Natural sampling.

4. From Fig. 5.6.2(*f*), it is clear that $X(f)$ can be recovered from $X_s(f)$, *i.e.*, $x(t)$ can be recovered from $x_s(t)$, if $f_s > 2f_m$ by using an LPF whose gain is constant at least upto $f = f_m$ and whose cut-off frequency B is such that $f_m < B < f_s - f_m$.
5. The output of the sampler is

$$x_s(t) = x(t)\ p_T(t)$$

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where $$p_T(t) = \sum_{n=-\infty}^{\infty} p(t-nT)$$

6. As $p_T(t)$ is a periodic pulse train, its Fourier series expansion will be

$$p_T(t) = \sum_{n=-\infty}^{\infty} p(t-nT) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_s t}$$

where, $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} p_T(t)e^{-j2\pi n f_s t}dt$$

7. Since τ the width of $p(t)$, single pulse in $p_T(t)$ is very much less than T and $p(t) = 0$ for $|t| \geq \tau/2$, we may write

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} p_T(t)e^{-j2\pi n f_s t}dt = \frac{1}{T}\int_{-\infty}^{\infty} p(t)e^{-j2\pi n f_s t}dt$$

$$C_n = f_s P(nf_s)$$

where, $$P(nf_s) = F[p(t)]|_{f=nf_s}$$

$\therefore$ $$p_T(t) = f_s \sum_{n=-\infty}^{\infty} P(nf_s)e^{j2\pi n f_s t}$$

and $$X_s(f) = F[x_s(t)] = F\left[f_s \sum_{n=-\infty}^{\infty} P(nf_s)x(t)e^{j2\pi n f_s t}\right]$$

$$= f_s \sum_{n=-\infty}^{\infty} P(nf_s)X(f)\delta(f-nf_s)$$

Since $$F[e^{j2\pi n f_s t}] = \delta(f-nf_s)$$

Hence, $$X_s(f) = f_s \sum_{n=-\infty}^{\infty} P(nf_s)\, X(f-nf_s)$$

8. If $x(t)$ has a spectrum $X(f)$, as shown in Fig. 5.6.2(*b*), then $X_s(f)$, the spectrum of the sampled version of $x(t)$ will appear as shown in Fig. 5.6.2(*f*).

Que 5.7. **Explain flat-top sampling method.**

Answer

1. This is the simplest and most popular sampling method that uses the sample and hold (*S/H*) circuit with flat-top samples. This is also called practical sampling.
2. Here the top of the samples remain constant which is equal to the instantaneous value of the baseband signal $x(t)$ at the beginning of sampling.

3. The duration or width of each sample is τ and the sampling rate, $f_s = 1/T$.
4. The schematic of a 'sample and hold' (*S/H*) circuit is shown in Fig. 5.7.1(*a*), and a typical output waveform from an *S/H* circuit is shown in Fig. 5.7.1(*b*).
5. The *S/H* circuit essentially consists of two switches S_1 and S_2 and capacitor C connected as shown in Fig. 5.7.1(*a*).
6. With S_2 open, S_1 is closed for a very brief period at each sampling instant. The capacitor C then gets charged to a voltage equal to the value of the input signal $x(t)$ at the sampling instant and holds it for a period τ at the end of which S_2 is closed to allow the capacitor to discharge.
7. This sequence of operations is repeated at the next and all subsequent sampling instant.
8. The switches S_1 and S_2 are generally FET switches and are operated by giving appropriate pulses to their gates. The voltage across C appears as $x_s(t)$ and is sketched in Fig. 5.7.1(*b*).

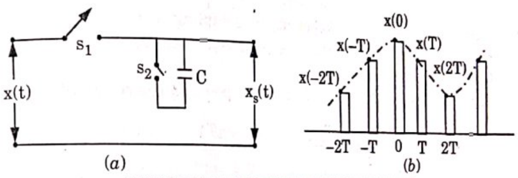

Fig. 5.7.1. (*a*) Schematic of an *S/H* circuit, (*b*) Signal $x(t)$ and output of *S/H* circuit.

9. From the Fig. 5.7.1, it is the obvious that the sampled version, $x_s(t)$ consists of a sequence of rectangular pulses, the leading edge of the kth pulse being at $t = kT$ and the amplitude of the pulse being the value of $x(t)$ at $t = kT$, *i.e.*, $x(kT)$.
10. The sampled signal $x_s(t)$ is the convolution of rectangular pulses $p(t)$ and the ideally sampled version of $x(t)$, *i.e.*, of $x_s(t)$

$\therefore$ $$X_s(f) = P(f)\, X_s(f)$$

PART-2

Reconstruction : Ideal Interpolator, Zero-Order Hold, First-Order Hold, and so on, Aliasing and its Effects.

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Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.8. **How a band limited signal can be reconstructed from its sampled signal ? Explain in detail.**

Answer

1. The process of obtaining the analog signal $x(t)$ from the sampled signal $x_s(t)$ is called data reconstruction or interpolation. We know that

$$x_s(t) = x(t)\,\delta_T(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

or $$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

2. Since $\delta(t-nT)$ is zero except at the sampling instants $t = nT$. The reconstruction filter, which is assumed to be linear and time-invariant, has unit impulse response $h(t)$.
3. The reconstruction filter output, $y(t)$ is given by the convolution,

$$y(t) = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(nT)\delta(\lambda-nT)\,h(t-\lambda)\,d\lambda$$

or, upon changing the order of summation and integration,

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\int_{-\infty}^{\infty}\delta(\lambda-nT)\,h(t-\lambda)\,d\lambda$$

i.e., $$y(t) = \sum_{n=-\infty}^{\infty} x(nT)\,h(t-nT) \qquad \ldots(5.8.1)$$

Ideal reconstruction filter :

1. If $x(t)$ is sampled at a frequency exceeding the Nyquist rate and if the sampled signal $x_s(t)$ is passed through an ideal LPF, with bandwidth greater than f_m but less than $f_s - f_m$ and a pass band amplitude response of T, the filter output is $x(t)$.
2. We choose the bandwidth of the ideal reconstruction filter to be $0.5f_s$. The transfer function of this ideal reconstruction filter is, therefore,

$$H(f) = \begin{cases} T & ; \ |f| < 0.5f_s \\ 0 & ; \ \text{otherwise}\end{cases}$$

as shown in Fig. 5.8.1.

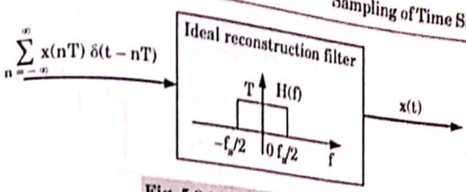

Fig. 5.8.1. Reconstruction filtering.

3. The impulse response of the ideal reconstruction filter is given by

$$h(t) = \int_{-f_s/2}^{f_s/2} T e^{j2\pi ft}\,df$$

$$h(t) = T\left[\frac{e^{j2\pi ft}}{j2\pi t}\right]_{-f_s/2}^{f_s/2} = \frac{T}{j2\pi t}\left[e^{j\pi f_s t} - e^{-j\pi f_s t}\right]$$

$$= \frac{1}{\pi f_s t}\left(\frac{e^{j\pi f_s t} - e^{-j\pi f_s t}}{2j}\right) = \frac{\sin \pi f_s t}{\pi f_s t}$$

or $$h(t) = \text{sinc}\, f_s t$$

4. Substituting this value of $h(t)$ in the eq. (5.8.1), we get

$$y(t) = x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\text{sinc}\, f_s(t-nT)$$

5. A more convenient form for this expression, which is often referred to as an interpolation formula is

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\text{sinc}\left(\frac{t}{T} - n\right)$$

This shows that the original data signal can be reconstructed by weighing each sample by a sinc function, centred at the sample time and summing.

Que 5.9. **What is zero order hold ? Obtain the transfer function of zero order hold.**

Answer

1. One of the most widely used interpolator is the zero order hold (ZOH). The ZOH reconstructs the continuous-time signal from its samples by holding the given sample for an interval until the next sample is received as shown in Fig. 5.9.1. So the ZOH generates step approximations.
2. Mathematically, $\tilde{x}_a(t) = x(n)$ for $nT \le n \le (n+1)\,T$

In particular, $\tilde{x}_a(t) = x(0)$ for $0 \le t \le T$
$= x(T)$ for $T \le t \le 2T$
$= x(2T)$ for $2T \le T \le 3\,T$
$\vdots$

3. The impulse response of a zero order hold is given by

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$$h(t) = \begin{cases} 1 \ ; & 0 \le t \le T \\ 0 \ ; & \text{otherwise} \end{cases}$$

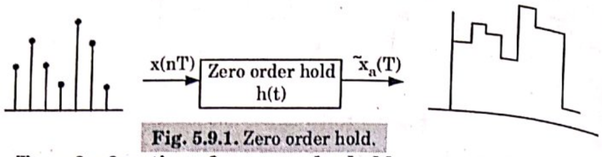


Fig. 5.9.1. Zero order hold.

Transfer function of a zero order hold :

1. The output $\tilde{x}_a(t)$ of a zero order hold is the convolution of its input $x(nT)$ and its impulse response $h(t)$, *i.e.*,

$$\tilde{x}_a(t) = x(nT) \otimes h(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t - nT)$$

2. For zero order hold,

$$h(t) = u(t) - u(t - T)$$

$$\therefore \quad h(t - nT) = u(t - nT) - u[t - (n + 1)T]$$

$$\therefore \quad \tilde{x}_a(t) = \sum_{n=-\infty}^{\infty} x(nT)[u(t - nT) - u[t - (n+1)T]$$

Taking Laplace transform on both sides, we have

$$L[\tilde{x}_a(t)] = \tilde{X}_a(s) = L\left[\sum_{n=-\infty}^{\infty} x(nT)\{u(t - nT) - u(t - (n+1)T)\}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\left(\frac{e^{-nTs}}{s} - \frac{e^{-(n+1)Ts}}{s}\right)$$

$$= \frac{1 - e^{-Ts}}{s} \sum_{n=-\infty}^{\infty} x(nT)e^{-nTs} = \left(\frac{1 - e^{-Ts}}{s}\right) X^*(s)$$

$\therefore$ Transfer function of zero order hold $= \dfrac{\tilde{X}_a(s)}{X^*(s)} = \dfrac{1 - e^{-Ts}}{s}$

3. Since the output of the ZOH consists of steps, it consists of higher order harmonics. To remove these harmonics, the output of ZOH is applied to an LPF. This filter tends to smooth the corners on the step approximations generated by the ZOH. Hence this filter is often called a smoothing filter.

Que 5.10. Discuss the first order hold interpolation (or linear interpolation).

Answer

1. The first order hold can be viewed as a form of interpolation between sample values in which the impulse response $h(t)$ of the interpolating filter is a triangular pulse, as depicted in Fig. 5.10.1(*c*).

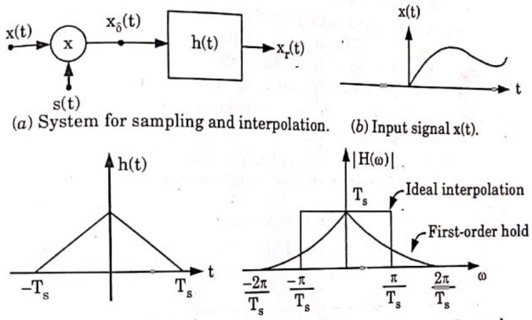
$$h(t) = r(t + T_s) - 2r(t) + r(t - T_s)$$

$$= \Delta\left(\frac{t}{T_s}\right) = \begin{cases} \left(1 + \dfrac{t}{T_s}\right), & -T_s \le t \le 0 \\ \left(1 - \dfrac{t}{T_s}\right), & 0 \le t \le T_s \\ 0, & \text{otherwise} \end{cases} \quad ...(5.10.1)$$

2. When the sampled signal $x_\delta(t)$ is applied to this filter, the output is $x_r(t)$.
3. Each sample in $x_\delta(t)$, being an impulse, generates a triangular pulse of height equal to the strength of the sample.
4. The output of the filter $x_r(t)$ is obtained by the convolution of the input $x_\delta(t)$ with the impulse response $h(t)$.

$$x_r(t) = x_\delta(t) \otimes h(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s) \otimes h(t)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t - nT_s) \quad ...(5.10.2)$$



(*a*) System for sampling and interpolation. (*b*) Input signal x(t).

(*c*) Impulse response of first order interpolating filter. (*d*) Frequency response for the first order hold and for the ideal interpolating filter.

Fig. 5.10.1.

5. Substitution of $h(t)$ from eq. (5.10.1) in eq. (5.10.2) yields

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\Delta\left(\frac{t - nT_s}{T_s}\right) \quad ...(5.10.3)$$

6. The filter output is a linear approximation of $x(t)$. Fig. 5.10.1(*d*) shows the magnitude of the transfer function of the first order hold interpolating filter, superimposed on the desired transfer function of the exact interpolating filter.

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Que 5.11. **Write a short note on aliasing effect.**

OR

State and prove sampling theorem and discuss the effect of under sampling.

AKTU 2017-18, Marks 07

OR

What is Shannon's sampling theorem ? Also discuss aliasing by taking an example.

AKTU 2018-19, Marks 07

Answer

Shannon's sampling theorem (Sampling theorem) : Refer Q. 5.2, Page 5–3A, Unit-5.

Aliasing effect :

1. When $\omega_s < 2\omega_m$ (*i.e.*, undersampling), there is an overlap between the shifted replicas of $X(\omega)$.
2. Consequently $x(t)$ cannot be recovered exactly from $x_\delta(t)$ even by means of an ideal low-pass filter.
3. Overlap in the shifted replicas of the original spectrum is termed as aliasing, which refers to the phenomenon of a high frequency component in the spectrum of the signal seemingly taking on the identity of a lower frequency in the spectrum of its sampled version.
4. Overlap between the replicas of $X(\omega)$ centred at $\omega = 0$ and at $\omega = \omega_s$ occurs for frequencies between $\omega_s - \omega_m$ and ω_m.
5. These replicas add, and thus the basic shape of the spectrum changes from portions of a triangle to a constant, as shown in Fig. 5.11.1.

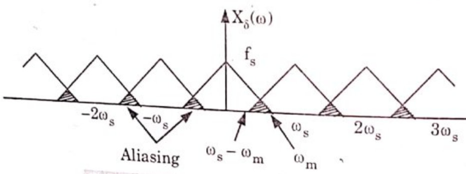

Fig. 5.11.1. Aliasing or spectrum folding.

6. The spectra cross at frequency $\omega_s/2$. This frequency is called the folding frequency. The spectrum, therefore, folds onto itself at the folding frequency.
7. The components of frequencies above $\omega_s/2$ reappear as components of frequencies below $\omega_s/2$. This tail inversion is known as spectral folding or aliasing.
8. To combat the effect of aliasing we may use two corrective measures as described here :

i. Prior to sampling, a low-pass pre-alias filter is used to attenuate those high frequency components of the signal that are not essential to the information conveyed by the signal.

ii. The filtered signal is sampled at a rate slightly higher than the Nyquist rate.

PART-3

Relation between Continuous and Discrete Time Systems.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.12. **Discuss the relationship between the continuous and discrete time systems.**

Answer

1. The relationship between the continuous-time signal $x_c(t)$ and its discrete-time representation $x_d[n]$ is helpful to represent *C/D* as a process of periodic sampling followed by a mapping of the impulse train to sequence.
2. These two steps are illustrated in Fig. 5.12.1. In the first step, representing the sampling process, the impulse train $x_p(t)$ corresponds to a sequence of impulses with amplitudes corresponding to the samples of $x_c(t)$ and with a time spacing equal to the sampling period *T*.
3. In the conversion from the impulse train to the discrete-time sequence, we obtain $x_d[n]$, corresponding to the same sequence of samples of $x_c(t)$, but with unity spacing in terms of the new independent variable *n*.
4. Thus, in effect, the conversion from the impulse train sequence of samples to the discrete-time sequence of samples can be thought of as normalization in time.
5. This normalization in converting $x_p(t)$ to $x_d[n]$ is evident in Fig. 5.10.1(b) and (c), in which $x_p(t)$ and $x_d[n]$ are respectively illustrated for sampling rates of $T = T_1$ and $T = 2T_1$.
6. We distinguish the continuous-time and discrete-time frequency variables by using ω in continuous time and Ω in discrete time.
7. The continuous-time Fourier transforms of $x_c(t)$ and $y_c(t)$ are $X_c(j\omega)$ and $Y_c(j\omega)$, respectively, while the discrete-time Fourier transforms of $x_d[n]$ and $y_d[n]$ are $X_d(e^{j\Omega})$ and $Y_d(e^{j\Omega})$, respectively.

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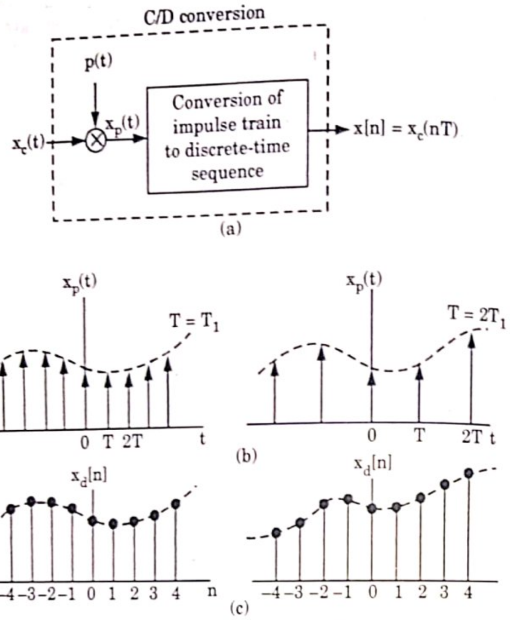

Fig. 5.12.1. Sampling with a periodic impulse train followed by conversion to a discrete-time sequence : (a) overall system; (b) $x_p(t)$ for two sampling rates. The dashed envelope represents $x_c(t)$ (c) the output sequence for the two different sampling rates.

8. The continuous-time Fourier transform of $x_p(t)$, in terms of the sample value of $x_c(t)$ by applying the Fourier transform is,

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT), \quad ...(5.12.1)$$

and since the transform of $\delta(t - nT)$ is $e^{-j\omega nT}$, it follows that

$$X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\omega nT} \quad ...(5.12.2)$$

9. Now consider the discrete-time Fourier transform of $x_d[n]$, that is,

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d(n)e^{-j\Omega n} \quad ...(5.12.3)$$

or

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n} \quad ...(5.12.3)$$

10. Comparing eq. (5.12.2) and (5.12.3), we see that $X_d(e^{j\Omega})$ and $X_p(j\omega)$ are related through

$$X_d(e^{j\Omega}) = X_p(j\Omega/T). \quad ...(5.12.4)$$

Also,

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) \quad ...(5.12.5)$$

Consequently,

$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\Omega - 2\pi k)/T) \quad ...(5.12.6)$$

11. The relationship among $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ is illustrated in Fig. 5.12.2 for two different sampling rates. Here, $X_d(e^{j\Omega})$ is a frequency-scaled version of $X_p(j\omega)$ and, in particular, is periodic in Ω with period 2π.

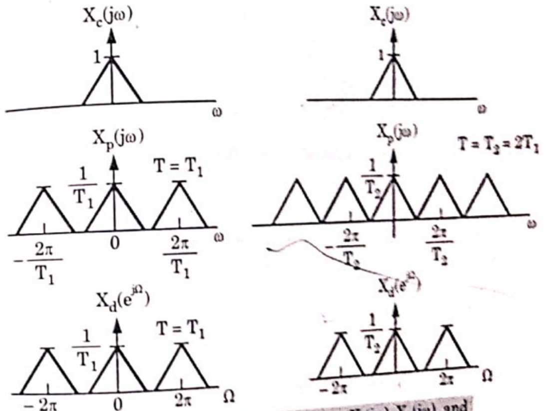

Fig. 5.12.2. Relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for two different sampling rates.

12. The relationship $\Omega = \omega T$ exist between continuous and discrete-time variables, which we can get by comparing eq. (5.12.5) and (5.12.6).

Que 5.13. **Explain the sampling of sinusoidal signals.**

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Answer

1. Consider a continuous-time sinusoidal signal of the form

$$x(t) = A\cos(2\pi ft) \quad ...(5.13.1)$$

which, when sampled periodically at a rate $f_s = 1/T_s$ samples/s, yields

$$x(t)|_{t=nTs} = x(nT_s) = x(n) = A\cos(2\pi fnT_s)$$

$$x(n) = A\cos\left(2\pi n\frac{f}{f_s}\right) \quad ...(5.13.2)$$

$$x(n) = A\cos(2\pi nf_d) \quad ...(5.13.4)$$

where,

$$f_d = \frac{f}{f_s} = \text{frequency for discrete-time signal} \quad ...(5.13.5)$$

or, equivalently, as

$$2\pi f_d = \frac{2\pi f}{f_s}$$

$$\omega_d = \frac{\omega}{f_s} = \omega T_s \quad ...(5.13.6)$$

2. The range of the frequency variable f or ω for continuous-time sinusoids are

$$-\infty < f < \infty \text{ or } -\infty < \omega < \infty \quad ...(5.13.7)$$

3. According to the sampling theorem, the sampling frequency $f_s \geq 2f$. Substituting this value of f_s in eq. (5.13.5), we obtain

$$f_d \leq \frac{1}{2}$$

or,

$$\omega_d \leq \pi$$

Therefore, the range of the frequency variable f_d or ω_d for discrete-time sinusoids are

$$-\frac{1}{2} \leq f_d \leq \frac{1}{2} \text{ or } -\pi \leq \omega_d \leq \pi \quad ...(5.13.8)$$

4. By substituting $f_d = f/f_s$ from eq. (5.13.5) and $\omega_d = \omega/f_s$ from eq. (5.13.6) into eq. (5.13.8), we find that the frequency of the continuous-time sinusoids when sampled at a rate $f_s = 1/T_s$ must fall in the range

$$-\frac{1}{2T_s} = -\frac{f_s}{2} \leq f \leq \frac{f_s}{2} = \frac{1}{2T_s} \quad ...(5.13.9)$$

or,

$$-\frac{\pi}{T_s} = -\pi f_s \leq \omega \leq \pi f_s = \frac{\pi}{T_s} \quad ...(5.13.10)$$

5. Periodic sampling of a continuous-time signal implies a mapping of the infinite frequency range for the variable f (or ω) into a finite frequency range for the variable f_d (or ω_d).

6. Since the highest frequency in a discrete-time signal is $\omega_d = \pi$ or $f_d = 1/2$, it follows that, with a sampling rate f_s, the corresponding highest values of f and ω are

$$f_{max} = \frac{f_s}{2} = \frac{1}{2T_s} \quad ...(5.13.11)$$

$$\omega_{max} = \pi f_s = \frac{\pi}{T_s}$$

Que 5.14. **The signals $x_1(t) = 10\cos(100\pi t)$ and $x_2(t) = 10\cos(50\pi t)$ are both sampled with $f_s = 75$ Hz. Show that the two sequences of samples obtained are identical.**

Answer

1. If the signal is sampled at $f_s = 75$ Hz, the discrete-time signal is

$$x_1(t)|_{t=nT_s} = x_1(nT_s) = x_1(n) = 10\cos(100\pi nT_s)$$

$$= 10\cos\left(\frac{100\pi}{75}n\right)$$

$$x_1(n) = 10\cos\left(\frac{4\pi}{3}n\right)$$

$$= 10\cos\left(2n\pi - \frac{2\pi}{3}n\right)$$

$$x_1(n) = 10\cos\left(\frac{2\pi}{3}n\right)$$

2. Similarly, we have

$$x_2(t)|_{t=nT_s} = x_2(nT_s) = x_2(n) = 10\cos(50\pi nT_s)$$

$$= 10\cos\left(\frac{50\pi}{75}n\right)$$

$$x_2(n) = 10\cos\left(\frac{2\pi}{3}n\right) = x_1(n)$$

Que 5.15. **Consider the two continuous-time sinusoidal signals $x_1(t) = \cos(20\pi t)$ and $x_2(t) = \cos(100\pi t)$ which are sampled at a rate $f_s = 40$ Hz. Find the corresponding discrete time signals.**

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Answer

1. Given, $x_1(t) = \cos(20\pi t)$
   $x_2(t) = \cos(100\pi t)$
2. The two sinusoidal signals can be written in standard for $\cos(2\pi f t)$ as follows :
   $x_1(t) = \cos 2\pi(10)t$
   and $x_2(t) = \cos 2\pi(50)t$
   Sampling rate, $f_s = 40$ Hz
3. The corresponding discrete time signal can be given as :

$$x_1(n) = \cos 2\pi\left(\frac{10}{40}\right)n$$

$$x_1(n) = \cos \frac{\pi}{2} n$$

and

$$x_2(n) = \cos 2\pi\left(\frac{50}{40}\right)n$$

$$x_2(n) = \cos \frac{5\pi}{2} n$$

W.2.(t.

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. State and prove sampling theorem. Also give some applications of sampling theorem.

Ans. Refer Q. 5.2.

Q. 2. Determine the Nyquist rate corresponding to each of the following signals :

i. $x(t) = \sin(200\pi t)$
ii. $x(t) = \sin^2(200\pi t)$
iii. $x(t) = 1 + \cos(200\pi t) + \sin(400\pi t)$

Ans. Refer Q. 5.3.

Q. 3. What are the different techniques of sampling ? Discuss any one of them in detail.

Ans. Refer Q. 5.5.

Q. 4. How a band limited signal can be reconstructed from its sampled signal ? Explain in detail.

Ans. Refer Q. 5.8.

Q. 5. Write a short note on aliasing effect.

Ans. Refer Q. 5.11.

Q. 6. Discuss the relationship between the continuous and discrete time systems.

Ans. Refer Q. 5.12.

Q. 7. Consider the two continuous-time sinusoidal signals $x_1(t) = \cos(20\pi t)$ and $x_2(t) = \cos(100\pi t)$ which are sampled at a rate $f_s = 40$ Hz. Find the corresponding discrete time signals.

Ans. Refer Q. 5.15.

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