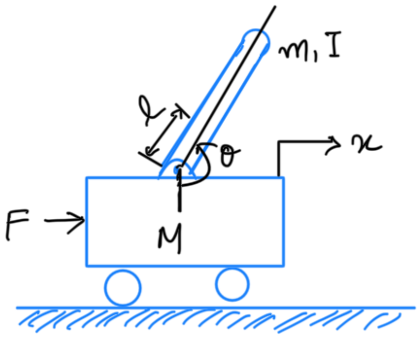
PENDULUM CART SYSTEM

i) unstable without control

ii) non linear dynamics



<u>objective</u> : Balance the inverted pendulum by applying a force to the cart the pendulum is attached to

<u>assumption</u> : 2D problem where pendulum is constrained to move in vertical plane.

<u>Given</u> : control Force (F)

<u>output</u> : angular position (θ) and horizontal position (x)

Example):

$M = 0.5\,kg$ (cart)

$m = 0.2\,kg$ (pendulum)

$b = 0.1\,N/m/s$ (coeff of friction for cart)

$l = 0.3\,m$ (length of pendulum COM)

$I = 0.006\,kgm^2$ (mass moment of Inertia of pendulum)

F = force applied

x = cart position coordinate

θ = angle from vertical

$\boxed{\text{Initial } \theta = \pi}$

↳ vertical condition

$\boxed{\text{SISO systems :}}$ Single input single output systems

↳ No design is dealing with cart's position] we have 2 outputs

controller's effect on the cart's position = ?

<u>To do</u> : design a controller that restore the pendulum to a vertically upward position

after it experiences a bump to the cart.
Impulsive

Requirements :

i) Pendulum returns to upright position within 5s
→ setting time for $\boxed{\theta < 5s}$

ii) Pendulum never moves more than 0.05 radians away from vertical after being disturbed by an impulse of 1 Nsec magnitude
→ Pendulum angle $(\theta) \not> 0.05$ radians from vertical

Our inverted pendulum system ⟹ SIMO
↳ we will use state-space design techniques

F x, θ

Example

0.2 m change in the cart's desired position

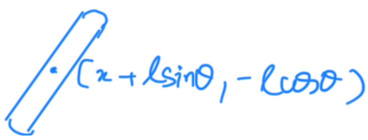
Rise time = 0.5s
Setting time = 5s
Pendulum $\theta = 0.35$ rad (20°)

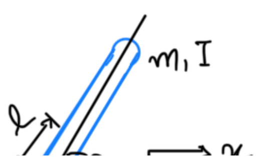
Time taken by pendulum to show effect after action is applied

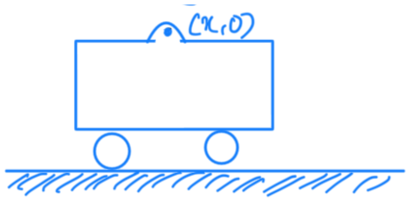
$\boxed{\text{Requirements :}}$

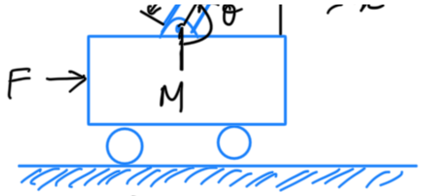
i) Setting time for x & $\theta < 5s$
ii) Rise time for $x < 0.5s$
iii) $\theta \not> 20°$ (0.35 rad)
iv) Steady error < 2% for x & θ

Analysis :









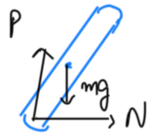
Psinθ + Ncosθ   Nsinθ − Pcosθ

$(x_c, y_c) \equiv \text{COM}\ (x + l\sin\theta,\ -l\cos\theta)$

$(v_x, v_y) \equiv (\dot{x} + l\cos\theta\,\dot{\theta},\ +l\sin\theta\,\dot{\theta})$

$(a_x, a_y) \equiv (\ddot{x} + -l\sin\theta\,\dot{\theta}^2 + l\cos\theta\,\ddot{\theta},\quad l\cos\theta\,\dot{\theta}^2 + l\sin\theta\,\ddot{\theta})$

Pendulum :

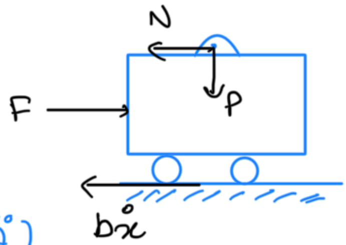


$N = m\ddot{x} \qquad P - mg = m\ddot{y}$

$N = m(\ddot{x} - l\sin\theta\,\dot{\theta}^2 + l\cos\theta\,\ddot{\theta})$

$P - mg = m(l\cos\theta\,\dot{\theta}^2 + l\sin\theta\,\ddot{\theta})$

Cart :



$F - N - b\ddot{x} = M\ddot{x}$

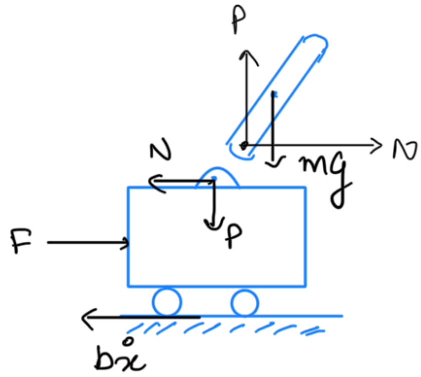
$M\ddot{x} = F - b\ddot{x} - m(\ddot{x} - l\sin\theta\,\dot{\theta}^2 + l\cos\theta\,\ddot{\theta})$

$(M+m)\ddot{x} + b\ddot{x} = F + m(l\sin\theta\,\dot{\theta}^2 - l\cos\theta\,\ddot{\theta})$

$$\boxed{F = (M+m)\ddot{x} + b\ddot{x} + m(l\cos\theta\,\ddot{\theta} - l\sin\theta\,\dot{\theta}^2)} \longleftarrow \text{eq (1)}$$

Rotation (moment) equation

(Newton Euler)



Torque = rate of change of angular momentum about pivot

$\sum \tau_{pivot} = I_p\ddot{\theta}$

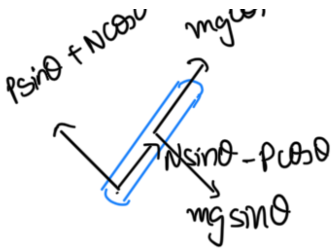
$I_p = I + mL^2$ (parallel axis theorem)

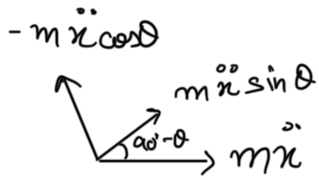
Torque contribution :

(⊥ forces)

$$P\sin\theta + N\cos\theta - mg\sin\theta = \underline{mL\ddot{\theta}} + \underline{m\ddot{x}\cos\theta}$$

angular acc^n — horizontal cart acc^n proj onto tangential dir^n





(moment about COM)

(P, N components not passing through COM)

$$-(P\sin\theta\, l) + -(N\cos\theta)l = I\ddot{\theta}$$

combining both and eliminating P, N:

$$-mgl\sin\theta = mL\ddot{\theta} + m\ddot{x}l\cos\theta + I\ddot{\theta}$$

$$\boxed{(I + ml^2)\ddot{\theta} + mgl\sin\theta = -m\ddot{x}(l\cos\theta)} \longleftarrow \text{eq}(2)$$

<u>Linearising the eq^n</u> : (about vertically upward eqb $\theta = \pi$)

Assumption): system stays within a small neighbourhood of this eqb which is valid since (<20°) is desirable.

ϕ : deviation of pendulum's position from eqb ($\theta = \pi + \phi$)

$$\cos\theta = \cos(\pi + \phi) \approx \cos(\pi) \approx -1$$
$$\sin\theta = \sin(\pi + \phi) \approx \sin(\pi) \approx -\phi$$
$$\dot{\theta}^2 = \dot{\phi}^2 \approx 0$$

substitute in eq(1) & eq(2)

$$(I + ml^2)\ddot{\phi} + mgl(-\phi) = -m\ddot{x}l(-1)$$

$$(M+m)\ddot{x} + b\dot{x} + m(l\ddot{\phi}(-1) - l(0)(-\phi)) = u$$

$$\boxed{\begin{aligned}(I + ml^2)\ddot{\phi} - mgl\phi &= m\ddot{x}l \\ (M+m)\ddot{x} + b\dot{x} - ml\ddot{\phi} &= u\end{aligned}}$$

① **Transfer function:** Laplace Transform (assuming zero initial condⁿ)

$$(I+ml^2)\,\Phi(s)\cdot s^2 - mgl\,\Phi(s) = ml\,X(s)\,s^2 \longrightarrow eq(3)$$

$$(M+m)\,X(s)\,s^2 + b\,X(s)\,s - ml\,\Phi(s)\,s^2 = U(s) \longrightarrow eq(4)$$

Transfer funcⁿ : $\boxed{\text{SISO}}$ first) : Input : $\Phi(s)$, output : $U(s)$ — eliminate $X(s)$

$$X(s) = \frac{(I+ml^2)\,\Phi(s)\cdot s^2 - mgl\,\Phi(s)}{ml\,s^2}$$

$$X(s) = \left[\frac{I+ml^2}{ml} - \frac{g}{s^2}\right]\Phi(s)$$

$$(M+m)\left(\frac{I+ml^2}{ml} - \frac{g}{s^2}\right)\Phi(s)\,s^2 - ml\,\Phi(s)\,s^2 + b\left(\frac{I+ml^2}{ml} - \frac{g}{s^2}\right)\Phi(s)\cdot s = U(s)$$

$$\left(\frac{(M+m)(I+ml^2)s^2}{ml} - \frac{(M+m)gs^2}{s^2} - ml\,s^2\right)\Phi(s)$$

$$= \left(\left(\frac{(M+m)(I+ml^2) - (ml)^2}{ml}\right)s^2 - \frac{(M+m)gs^2}{s^2}\right)\Phi(s)$$

$$\left(\left[(M+m)(I+ml^2) - (ml)^2\right]\frac{s^2}{ml} - \frac{(M+m)gs^2}{s^2}\right)\Phi(s) + b\left(\frac{I+ml^2}{ml} - \frac{g}{s^2}\right)\Phi(s)\cdot s = U(s)$$

$$\boxed{q = (M+m)(I+ml^2) - (ml)^2}$$

$$\left(q\frac{s^2}{ml} - \frac{(M+m)gs^2}{s^2} + b\left(\frac{I+ml^2}{ml} - \frac{g}{s^2}\right)s\right)\Phi(s) = U(s)$$

$$\underline{qs^4 - (M+m)mlgs^2 + b\left((I+ml^2)s^2 - g(ml)\right)s} = \underline{U(s)}$$

$$mLs^2 \qquad\qquad \Phi(s)$$

$$q\left(s^4 + \frac{(M+m)mlg}{q}s^2 + \frac{bs^3(I+ml^2)}{q} - \frac{bg\,mLs}{q}\right) = mls^2 \frac{U(s)}{\Phi(s)}$$

$$\boxed{\frac{\Phi(s)}{U(s)} = \frac{(mL/q)s^2}{s^4 + \frac{b(I+mL^2)}{q}s^3 + \frac{(M+m)mgl}{q}s^2 - \frac{bgmL}{q}s}}$$

<u>Finding pole and zero:</u>

denominator $= 0$ → $s = 0$

numerator $= 0$ → $s = 0$

Hence $s=0$ is an obvious pole and zero;

$$\boxed{P_{pend}(s) = \frac{\Phi(s)}{U(s)} = \frac{\frac{mL}{q}s}{s^3 + \frac{b(I+ml^2)}{q}s^2 + \frac{(M+m)mgl}{q}s - \frac{bgmL}{q}} \left[\frac{rad}{N}\right]}$$

$$P_{cart}(s) = \frac{X(s)}{U(s)} = \left(\frac{I+ml^2}{mL} - \frac{g}{s^2}\right)\frac{\Phi(s)}{U(s)}$$

$$= \frac{\frac{(I+ml^2)s^2 - g\,mL}{mLs^2} \cdot \frac{mLs^2}{q}}{s^4 + \frac{b(I+ml^2)}{q}s^3 + \frac{(M+m)mgl}{q}s^2 - \frac{bgmLs}{q}}$$

$$\boxed{P_{cart}(s) = \frac{\frac{(I+mL^2)s^2 - gmL}{q}}{s + \frac{b(I+ml^2)}{q}s^3 + \frac{(M+m)mgl}{q}s^2 - \frac{bgmLs}{q}} \left[\frac{m}{N}\right]}$$

<u>State-space</u> (matrix form) $\dot{x}, \ddot{x}, \dot{\Phi}, \ddot{\Phi}$

$$\boxed{\begin{aligned}(I+ml^2)\ddot{\phi} - mgl\phi &= m\ddot{x}l \\ (M+m)\ddot{x} + b\dot{x} - ml\ddot{\phi} &= u\end{aligned}}$$

$$(I+ml^2)\ddot{\phi} - ml\ddot{\phi} + (M+m)\ddot{x} - mgl\phi + b\dot{x} = m\ddot{x}l + u$$

$$(I+ml^2 - ml)\ddot{\phi} + (M+m-ml)\ddot{x} - mgl\phi + b\dot{x} = u$$

$$\underbrace{\begin{bmatrix} I+ml^2 & -ml \\ -ml & M+m \end{bmatrix}}_{M_1}\begin{bmatrix}\ddot{\phi} \\ \ddot{x}\end{bmatrix} = \begin{bmatrix} mgl\phi \\ -b\dot{x}+u\end{bmatrix}$$

now,

$$\begin{bmatrix}\ddot{\phi} \\ \ddot{x}\end{bmatrix} = (M_1^{-1})\begin{bmatrix} mgl\phi \\ -b\dot{x}+u\end{bmatrix}$$

$$(M_1)^{-1} = \frac{1}{(I+ml^2)(M+m)-(ml)^2}\begin{bmatrix} m+M & -ml \\ -ml & I+ml^2\end{bmatrix}$$

$$(M_1)^{-1} = \frac{1}{q} M_1$$

$$\begin{bmatrix}\ddot{\phi} \\ \ddot{x}\end{bmatrix} = \frac{1}{q}\begin{bmatrix} M+m & ml \\ ml & I+ml^2\end{bmatrix}\begin{bmatrix} mgl\phi \\ u-b\dot{x}\end{bmatrix}$$

$$\begin{bmatrix}\ddot{\phi} \\ \ddot{x}\end{bmatrix} = \frac{1}{q}\begin{bmatrix} (M+m)(mgl)\phi + ml(u-b\dot{x}) \\ ml(mgl)\phi + (I+ml^2)(u-b\dot{x})\end{bmatrix}$$

$$\boxed{\begin{aligned}\ddot{\phi} &= 1/q\left((M+m)mgl\phi + ml(u-b\dot{x})\right) \\ \ddot{x} &= 1/q\left(ml(mgl)\phi + (I+ml^2)(u-b\dot{x})\right)\end{aligned}}$$

$$\begin{bmatrix}\dot{x} \\ \ddot{x} \\ \vdots\end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -(I+ml^2)b & m^2l^2g & 0 \\ \vdots \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ \vdots\end{bmatrix} \quad \begin{bmatrix} 0 \\ I+ml^2 \\ \vdots\end{bmatrix}$$

$$
\begin{bmatrix} x \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{\cdots}{q} & \frac{\cdots}{q} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mlb}{q} & \frac{(M+m)mgl}{q} & 0 \end{bmatrix}}_{A} \begin{bmatrix} \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\cdots}{q} \\ 0 \\ \frac{ml}{q} \end{bmatrix}}_{B} u
$$

$$
C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

$$
D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

$$
\underbrace{y}_{\begin{bmatrix} x \\ \phi \end{bmatrix}} = C \underbrace{x}_{\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \dot{x} \end{bmatrix}} + D u
$$