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1) Let X be a random variable with probability density function

$$f(x) = \begin{cases} e^{-x} & if x \geq 0 \\ 0 & otherwise \end{cases}$$

For $a < b$, if $U(a, b)$ denotes the uniform distribution over the interval (a, b), then which of the following is/are true?

a) e^{-x} follows $U(-1, 0)$ distribution
b) $1 - e^{-x}$ follows $U(0, 2)$ distribution
c) $2e^{-x} - 1$ follows $U(-1, 1)$ distribution
d) The probability mass function of $Y = [X]$ is

$$P(Y = k) = \left(1 - e^{-1}\right) e^{-k} \text{ for } k = 0, 1, 2, \cdots$$

where [x] denotes the largest integer not exceeding x

2) Suppose that X is a discrete random variable with the following probability mass function

$$P(X = 0) = \frac{1}{2}\left(1 + e^{-1}\right) \tag{1}$$

$$P(X = k) = \frac{e^{-1}}{2k!} \text{ for } k = 0, 1, 2, 3, \cdots \tag{2}$$

Which one of the following is/are true?

a) $E(X) = 1$
b) $E(X) < 1$
c) $E(X|X > 0) < \frac{1}{2}$
d) $E(X|X > 0) > \frac{1}{2}$

3) Suppose that U and V are two independent and identically distributed random variables each having probability density function

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & if x > 0 \\ 0 & otherwise \end{cases}$$

where $\lambda > 0$. Which of the following statements is/are true?

a) The distribution of $U - V$ is symmetric about 0
b) The distribution of UV does not depend on λ
c) The distribution of $\frac{U}{V}$ does not depend on λ
d) The distribution of $\frac{U}{V}$ is symmetric about 1.

4) Let (X, Y) have joint probability mass function

$$p(x) = \begin{cases} \frac{e^{-2}}{x!(y-x)!} & \text{if} x = 0, 1, 2 \cdots, y; \text{ y} = 0, 1, 2, \cdots \\ 0 & otherwise \end{cases}$$

Then which of the following is/are true?

a) $E(X|Y = 4) = 2$
b) The moment generating function of Y is $e^{2(e^v - 1)}$ for all $v \in \mathbb{R}$

c) E(X)=2

d) The joint moment generating the function of (X,Y) is $e^{-2+(1+e^{u})e^{v}}$ or all $(u, v) \in \mathbb{R}^2$

5) Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1, all of them defined on the same probability space. For $n = 1, 2, 3, \cdots$, let

$$Y_n = \frac{1}{n}\left(X_1X_2 + X_3X_4 + \cdots + X_{2n-1}X_{2n}\right)$$

Then which of the following is/are true?

a) $\left\{\sqrt{n}Y_n\right\}_{n\geq 1}$ converges in distribution to a standard normal random variable

b) $\{Y_n\}_{n\geq 1}$ converges in 2^{nd} mean to 0.

c) $\left\{Y_n + \frac{1}{n}\right\}_{n\geq 1}$ converges in probability 0

d) $\{X_n\}_{n\geq 1}$ converges almost surely to 0

6) Consider the following regression model

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \epsilon_t,\ \ t = 1, 2, \cdots, 100$$

where α_0, α_1 and α_2 are unknown parameters and $\epsilon_t' s$ are independent and identically distributed random variables each having $N(\mu, 1)$ distribution with $\mu \in \mathbb{R}$ unknown. Then which of the following statements is/are true?

a) There exists an unbiased estimator of α_1

b) There exists an unbiased estimator of α_2

c) There exists an unbiased estimator of α_0

d) There exists an unbiased estimator of μ

7) Consider the orthonormal set

$$\left\{\mathbf{v_1} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}\right\}$$

with respect to the standard inner product on $\mathbb{R}^3$. If $\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the vector such that inner products of u with v_1, v_2 and v_3 are 1,2, and 3, respectively, then $a^2 + b^2 + c^2$(in integer) equals _____ .

8) Consider the probability space (Ω, G, P), where $\Omega = \{1, 2, 3, 4\}, G = \{\phi, \Omega, \{1\}, \{4\}, \{2, 3\}, \{1, 4\}, \{1, 2, 3\}, \{2, 3, 4\}$ and $P(\{1\}) = \frac{1}{4}$. Let X be the random variable defined on the above probability space as $X(1) = 1, X(2) = X(3) = 2$ and $X(4) = 3$. If $P(X \leq 2) = \frac{3}{4}$, then $P(\{1, 4\})$(rounded off to two decimal places) equals _____ .

9) Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having probability density function

$$f(x) = \begin{cases} e^{-x} & if x > 0 \\ 0 & otherwise \end{cases}$$

For $n \geq 1$, let $Y_n = |X_{2n} - X_{2n-1}|$. If $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ for $n \geq 1$ and $\left\{\sqrt{n}\left(e^{-\bar{Y}_n} - e^{-1}\right)\right\}_{n\geq 1}$ converges in distribution to a normal random variable with mean 0 and variance σ^2, then σ^2(rounded off to two decimal places) equals _____ .

10) Consider a birth-death process on the state space $\{1, 2, 3, 4\}$. The birth rates are given by $\lambda_0 = 1, \lambda_1 = 1, \lambda_2 = 2$ and $\lambda_3 = 0$. The death rates are given by $\mu_0 = 0, \mu_1 = 1, \mu_2 = 1$ and $\mu_3 = 1$. If $[\pi_0, \pi_1, \pi_2, \pi_4]$ is the unique stationary distribution, then $\pi_0 + 2\pi_1 + 3\pi_2 + 4\pi_3$(rounded off to two decimal places) equals _____ .

11) Let $\left\{-1, -\frac{1}{2}, 1, \frac{5}{2}, 3\right\}$ be a realization of a random sample of size 5 from a population having $N\left(\frac{1}{2}, \sigma^2\right)$ distribution, where $\sigma > 0$ is an unknown parameter. Let T be an unbiased estimator of σ^2 whose variance attains the Camer-Rao lower bound. Then based on the above data, the realized value of T (rounded off to two decimal places) equals

12) Let X be a random sample of size 1 from a population with cumulative distribution functiion

$$F(x) = \begin{cases} 0 & if x < 0 \\ 1 - (1-x)^{\theta} & if 0 \leq x \leq 1 \\ 1 & if x \geq 1 \end{cases}$$

where $\theta > 0$ is an unknown parameter. To test $H_0 : \theta = 1$ against $H_1 : \theta = 2$, consider using the critical region $\{x \in \mathbb{R} : x < 0.5\}$. if α and β denote the level and power of the test, respectively, then $\alpha + \beta$(rounded off to two decimal places) equals _____ .

13) Let $\{0.13, 0.12, 0.78, 0.51\}$ be a realization of a random sample of size 4 from a population with cumulative distribution function $F(\cdot)$. Consider testing

$$H_0 : F = F_0 \text{ against } H_1 : F \neq F_0,$$

where

$$F_0(x) = \begin{cases} 0 & if x < 0 \\ x & if 0 \leq x < 1 \\ 1 & if x \geq 1 \end{cases}$$

Let D denote the Kolmorgov-smirnov test statistics . If $P(D > 0.669) = 0.01$ under H_0 and

$$\psi = \begin{cases} 1 & \text{if } H_0 \text{ is accepted at level 0,01} \\ 0 & otherwise, \end{cases}$$

then based on the given data, the observed value of $D + \psi$(rounded off to two decimal places) equals _____ .