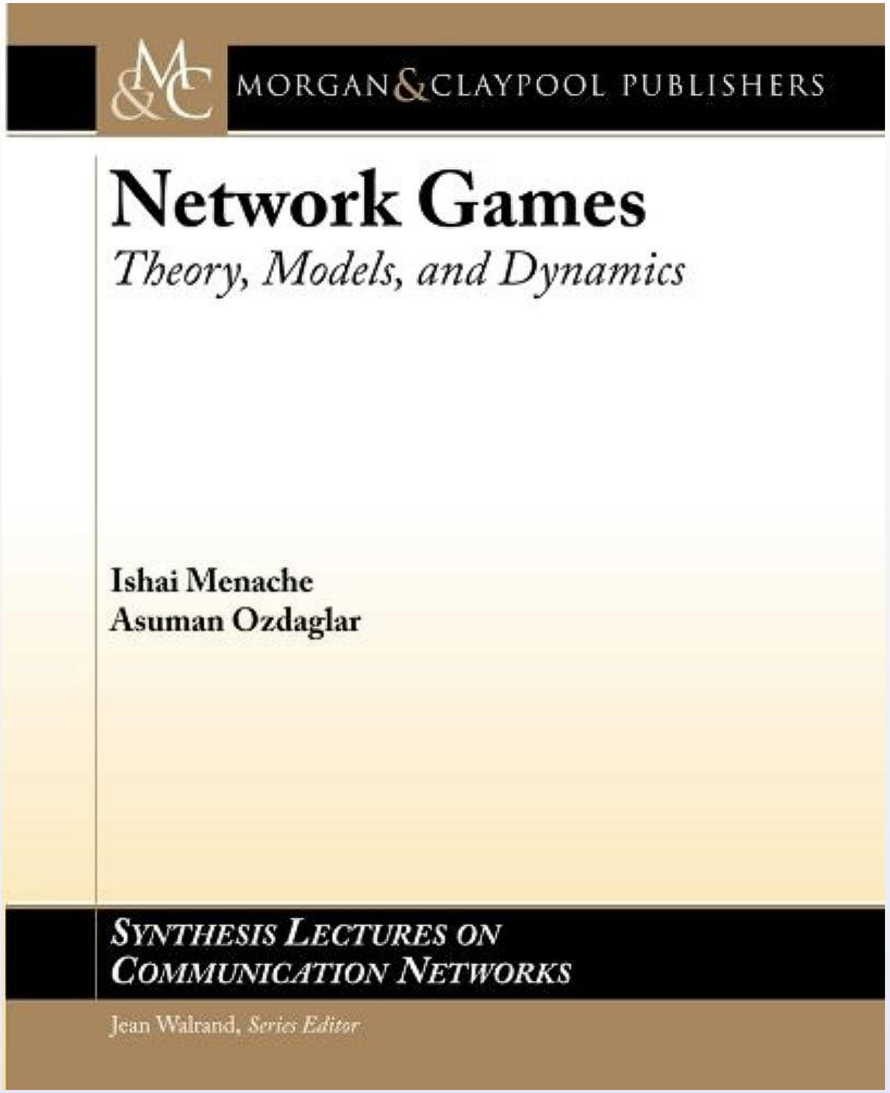
VI. System Resource Management

Reference



- Ozdaglar, Asu, and Ishai Menache. *Network Games*. Springer Nature, 2011.

Introduction

- System resource management from the viewpoint of game theory
  - Most of today's network system works under the individual decision making
    - e.g., In a P2P network, each peer aims to exchange the piece of data with each other.
      A peer wants to download a desired piece faster while supressing the upload data size.
      If any peers do not want to use their own network resources, the P2P network does not work.
      What incentives should be given to individual peers to make the P2P network work in such a situation?
- Game theory
  - Mathematical model of strategic interaction among rational/selfish players
  - Game theory can design an incentive mechanism for system resource management under the situation where decision makers' actions affect the system resources and the actions of others.

Strategic Form Games

- A strategic form game is $\langle \mathcal{N}, \{\mathcal{S}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}} \rangle$
  - $\mathcal{N}$ denotes a set of players, $\mathcal{N} = \{1, \ldots N\}$
  - $\mathcal{S}_i$ denotes a set of actions (strategies) for player i
  - $u_i : \mathcal{S} \to \mathbb{R}$ is the payoff (utility) function of player i where $\mathcal{S} = \prod_i \mathcal{S}_i$
  - $\mathcal{S}_{-i} = \prod_{j\neq i} \mathcal{S}_j$ is the set of actions of all players except i

Zero-Sum Game

- A two-player game can be represented in matrix form
  - This class of games is reffered to as bimatrix games

Player choice

		Player 2	
		Heads	Tails
Player 1	Heads	-1, 1	1, -1
	Tails	1, -1	-1, 1

Matching Pennies

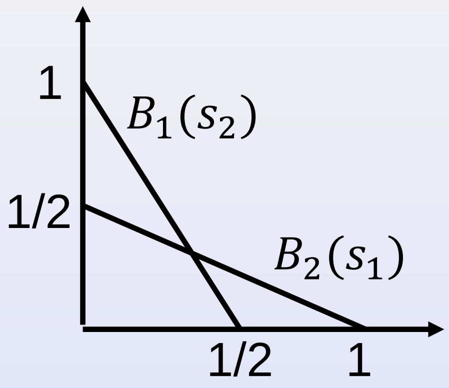
Player 1’s utility   Player 2’s utility

Zero-sum Game: The sum of utilities for both players at each outcome is zero

Cournot Competition Game

- Assumption
  - Two firms produce the same homogeneous good and seek to maximize their profits
  - A set of players (firms) $\mathcal{N} = \{1, 2\}$
  - A strategy set $\mathcal{S}_i = [0, \infty)$ for each player i , where $s_i \in \mathcal{S}_i$ represents the amount of good that the player produces
  - A utility function u_i for each player i is given by its total revenue minus its total cost $u_i(s_1, s_2) = s_i p(s_1 + s_2) - c_i s_i$, where $p(q)$ represents the price of good under the supply q and c_i is the unit cost for firm i
  - We assume $c_1 = c_2 = 1$ and $p(q) = \max\{0, 2 - q\}$
- Best-response correspondence
  - Definition: $B_i(s_{-i}) = \{ s_i \in \mathcal{S}_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in \mathcal{S}_i \}$.

$$B_i(s_i) = \arg\max_{s_i \geq 0} (s_i p(s_i + s_{-i}) - s_i)$$
$$= \begin{cases} \frac{1 - s_{-i}}{2} & \text{if } s_{-i} \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$



Prisoner’s Dilemma

- Prisoner’s Dilemma Game
  - Two people are arrested for a crime, placed in separate rooms, and questioned by authorities trying to extract a confession
  - If they both remain silent, they will both serve a short prison term, i.e., 2 years
  - If only one of them confesses, his term is reduced to 1 year. The another one get a sentence of 5 years
  - If they both confess, they both get a smaller sentence of 4 years
- Playing Don’t cooperate is strictly dominant
  - Playing Don’t cooperate yields higher payoff for each player, regardless of the other player choices
  - Prisoner’s Dilemma is an example of rational behavior not leading to social optimal outcome

	Cooperate	Don’t cooperate
Cooperate	-2, -2	-5, -1
Don’t cooperate	-1, -5	-4, -4

Dominant Strategy Equilibrium

- A strategy profile s_i^* is a *(strictly) dominant strategy equilibrium* if for each player i, s_i^* is a (strictly) dominant strategy
- Definition : A strategy is a *dominant strategy* $s_i \in \mathcal{S}_i$ for player i if

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}), \forall s_i \in \mathcal{S}_i, \forall s_{-i} \in \mathcal{S}_{-i}.$$

- Definition: A strategy $s_i' \in \mathcal{S}_i$ is a *strictly dominanted* for player i if there exist some $s_i \in \mathcal{S}_i$ such that

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}),\ \forall s_{-i} \in \mathcal{S}_{-i}.$$

- Definition: A strategy $s_i' \in \mathcal{S}_i$ is a *weakly dominant* for player i if there exist some $s_i \in \mathcal{S}_i$ such that

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}),\ \forall s_{-i} \in \mathcal{S}_{-i}.$$

and

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}),\ \exists s_{-i} \in \mathcal{S}_{-i}.$$

Iterated Elimination of Strictly Dominated Strategies

- Consider the following game

		Player 2		
		LEFT	MIDDLE	RIGHT
Player 1	UP	4,3	5,1	6,2
	MIDDLE	2,1	8,4	3,6
	DOWN	3,0	9,6	2,8

1. MIDDLE is strictly dominated by RIGHT for player 2
   - It is not rational for player 2 to play MIDDLE

	LEFT	RIGHT
UP	4,3	6,2
MIDDLE	2,1	3,6
DOWN	3,0	2,8

2. MIDDLE and RIGHT are strictly dominated by UP for player 1

	LEFT	RIGHT
UP	4,3	6,2

3. Player 1 has no choice while player 2 choose between LEFT and RIGHT

4. Since LEFT can maximize the utility of player 2, rational strategy profile become (UP, LEFT)

Iterated Elimination of Strictly Dominated Strategies

- Procedure of iterated elimination of strictly dominated strategies
- Step 0: For each $i \in \mathcal{N}$, let $\mathcal{S}_i^0 = \mathcal{S}_i$
- Step 1: For each $i \in \mathcal{N}$, define

$$\mathcal{S}_i^1 = \left\{ s_i \in \mathcal{S}_i^0 \,\middle|\, \text{there does not exitst } s_i' \in \mathcal{S}_i^0 \text{ such that } u(s_i', s_{-i}) > u\left(s_i \ \ , s_{-i}\right), \forall s_{-i} \in \mathcal{S}_{-i}^0 \right\}$$

- Step k: For each $i \in \mathcal{N}$, define

$$\mathcal{S}_i^k = \left\{ s_i \in \mathcal{S}_i^{k-1} \,\middle|\, \text{there does not exitst } s_i' \in \mathcal{S}_i^{k-1} \text{ such that } u(s_i', s_{-i}) > u\left(s_i \ \ , s_{-i}\right), \forall s_{-i} \in \mathcal{S}_{-i}^{k-1} \right\}$$

- For each $i \in \mathcal{N}$, define

$$\mathcal{S}_i^\infty = \bigcap_{k=0}^{\infty} \mathcal{S}_i^k$$

(Pure) Nash Equilibrium (PNE)

- Pure nash equilibrium is achieved if every player rationally chooses an action indepedently and cannot derivate its own action.

$$u_i(s_i^*, s_{-i}) \geq u_i\left(s_i \;\; , s_{-i}\right), \forall s_i \in \mathcal{S}_i, \forall i \in \mathcal{N}.$$

- Consider a two-player game with the following payoff structure

	BALLET	SOCCER
BALLET	2,1	0,0
SOCCER	0,0	1,2

- This game has two pure Nash equilibria
  - i.e., (BALLET, BALLET) and (SOCCER, SOCCER)

Mixed Nash Equilibrium (MNE)

- Consider a two-player game with the following payoff matrix
  - This game does not has a Nash equilibrium
  - Assume that every player is allowed to randomize over its choice
    - →Steady state emerges
  - Every player chooses an action with the probability p
  - $\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)$ is a steady state

		p Heads	$1-p$ Tails
p	Heads	-1, 1	1, -1
$1-p$	Tails	1, -1	-1, 1

Matching Pennies

Mixed Nash Equilibrium (MNE)

- Assumption
  - Σ_i denotes a set of probability profiles over the pure strategy set $\mathcal{S}_i$
  - $\sigma \in \Sigma(= \prod_{i \in \mathcal{N}} \Sigma_i)$ denote a mixed strategy profile
  - $\sigma_{-i} \in \Sigma_{-i}\left(= \prod_{i \in \mathcal{N}, j \neq i} \Sigma_i\right)$
  - Payoff of a mixed strategy σ is given by the expected value of pure strategy payoffs under the distribution σ

$$u_i(\sigma) = \int_s u_i(s)\, d\sigma(s).$$

- Mixed nash equilibrium is achieved if every player rationally chooses an action indepedently and cannot derivate its own action.

$$u_i(\sigma_i^*, \sigma_{-i}) \geq u_i(\sigma_i^*, \sigma_{-i}), \forall \sigma_i \in \Sigma_i, \forall i \in \mathcal{N}.$$

Mixed Nash Equilibrium (MNE)

- Let consider a game with the following matrix structure
  - Player 1 chooses the action BALLET with probability $q \in [0, 1]$
  - Player 2 chooses the action BALLET with probability $p \in [0, 1]$

Player 1's payoffs

$$\underbrace{2 \cdot q + 0 \cdot (1-q)}_{\text{Player 1 choose BALLET}} = \underbrace{0 \cdot q + 1 \cdot (1-q)}_{\text{Player 1 choose SOCCER}}$$

Player 2's payoffs

$$\underbrace{1 \cdot p + 0 \cdot (1-p)}_{\text{Player 2 choose BALLET}} = \underbrace{0 \cdot p + 2 \cdot (1-p)}_{\text{Player 2 choose SOCCER}}$$

$$q = \frac{1}{3}, \quad p = \frac{2}{3}$$

			Player 2: p BALLET	Player 2: $1-p$ SOCCER
Player 1	q	BALLET	2,1	0,0
	$1-q$	SOCCER	0,0	1,2

Correlated Equilibrium (CE)

- In a Nash equilibrium, every player choose strategies independently
- Assume that there is publicly obervable random variable, e.g., traffic light, such that no player has an incentive to deviate from the recommendation
  - Every player chooses an action under the assumption where the other players play according to the probability mass function $\pi(s, s_{-i})$
- A probability mass function $\pi(s, s_{-i})$ is a correlated equilibrium if it satisfies:

$$\sum_{s_{-i} \in \mathcal{S}_{-i}} \pi(s_i, s_{-i}) \cdot u_i\,(s_i, s_{-i}) \geq \sum_{s_{-i} \in \mathcal{S}_{-i}} \pi(s_i, s_{-i}) \cdot u_i\,(\phi_i(s_i), s_{-i})\,,$$
$$\forall s_i \in \Sigma_i, s_i \neq \phi_i(s_i), \forall i \in \mathcal{N}.$$

  - These constraints imply that no player can gain a larger utility by individually deviating from the suggestions $(s_1, s_2, \ldots, s_N)$ of the game manager
  - $\phi_i\colon \mathcal{S}_i \to \mathcal{S}_i$ means modifying the player i's action when instructed to play s_i

Correlated Equilibrium (CE)

- Traffic intersection game
  - Two players exists on the intersection
    - Both players choose GO
    - One player chooses GO and the another chooses STOP
    - Both players choose STOP
  - A correlated device (i.e., traffic light) gives the suggested actions to each player
    - e.g., (GO, STOP) or (STOP, GO)
  - MNE case
    - The expected reward becomes 0
    - (GO, GO) occurs with the probability 0.25
  - CE case
    - The expected reward becomes 0.5
    - (GO, STOP) and (STOP, GO) occur with the probability 0.5, respectively

	GO	STOP
GO	-1,-1	1,0
STOP	0,1	0,0

Coarse Correlated Equilibrium (CCE)

- The CE assumes that non-participating players still receive the suggestion from the game manager
  - A correlated device (game manager) can give a non-participating player a great deal of information about the likelihood of actions from other players
  - → The player has an option not to participate in the game
- The following simple modification is adopted for coarse correlated equilibrium
  - Non-participating player do not receive any suggestions from the game manager
- A probability mass function $\pi(s)$ is a coarse correlated equilibrium if it satisfies:

$$\sum_{s \in \mathcal{S}} \pi(s) \cdot u_i\,(s) \;\geq \sum_{s \in \mathcal{S}} \pi(s) \cdot u_i\,(\phi_i(s_i), s_{-i})\,, \forall s_i \in \Sigma_i, \forall i \in \mathcal{N}.$$

Solution Existence vs Computational Complexity

Computational Complexity

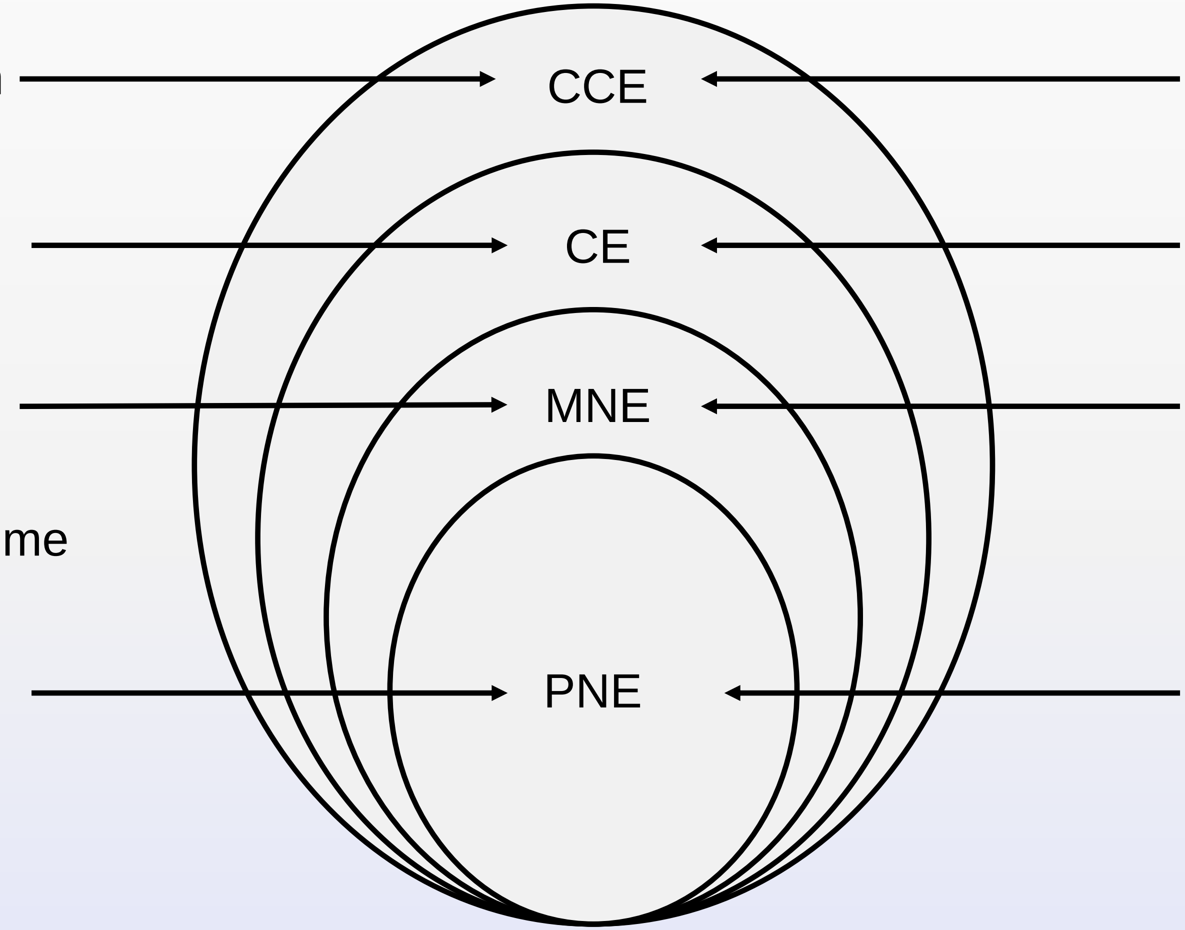
Solution Space

Solution Existence

Faster computation

Polynomial time

Non-polynomial time



The solutions includes the nash equiblirum.

At least one nash equiblirum exists.

Nash equiblirium does not always exists.

A Simple Example of MNE, CE and CCE

- Let consider a two-player game with the following matrix structure
  - There are two players: $\mathcal{A} = \{1, 2\}$
    - $s_1 = \{\alpha, \beta, \gamma\}, s_2 = \{\alpha, \beta\}$
  - Utility of player i: $u_i = \sum_{s \in \mathcal{A}} \Pr[s]\, u_i(s)$
    - $\Pr[s] \geq 0, \forall s \in \mathcal{A}$
  - Each player i aims at maximizing the utility u_i

Utility of player 1

	α	β
α	2	5
β	4	2
γ	3	5

Utility of player 2

	α	β
α	50	1
β	2	4
γ	3	0

Probabilities

	α	β
α	a	b
β	c	d
γ	e	f

A Simple Example of MNE, CE and CCE

- The MNE constraints are given by:
  - $\sum_{s\in\mathcal{A}} \Pr[s]\, u_i(s) \geq \sum_{s\in\mathcal{A}} \Pr[s]\, u_i(s_i', s_{-i})\,, \forall i \in \mathcal{N}, \forall s_i' \in \mathcal{A}_i$
    - Player 1 sees α: $2a + 5b + 4c + 2d + 3e + 5f \geq 2(a + c + e) + 5(b + d + f)$
    - Player 1 sees β: $2a + 5b + 4c + 2d + 3e + 5f \geq 4(a + c + e) + 2(b + d + f)$
    - Player 1 sees γ: $2a + 5b + 4c + 2d + 3e + 5f \geq 5(a + c + e) + 3(b + d + f)$
    - Player 2 sees α: $50a + b + 2c + 4d + 3e + 0f \geq 50(a + b) + 2(c + d) + 3(e + f)$
    - Player 2 sees β: $50a + b + 2c + 4d + 3e + 0f \geq 1(a + b) + 4(c + d) + 0(e + f)$
  - $\Pr[s] = \prod_{i=1}^{N} \Pr[s_i = s_i]$
    - $a = (a + b)(a + c + e), b = (a + b)(b + d + f), c = (c + d)(a + c + e), d = (c + d)(b + d + f), e = (e + f)(a + c + e), f = (e + f)(b + d + f)$

Utility of player 1

	α	β
α	2	5
β	4	2
γ	3	5

Utility of player 2

	α	β
α	50	1
β	2	4
γ	3	0

Probabilities

	α	β
α	a	b
β	c	d
γ	e	f

Distribution (Result)

	α	β
α	0	0
β	.45	.15
γ	.30	.10

A Simple Example of MNE, CE and CCE

- The CE constraints are given by:
  - $\sum_{s \in \mathcal{A}} \Pr[s_i, s_{-i}]\, u_i(s_i, s_{-i}) \geq \sum_{s \in \mathcal{A}} \Pr[s_i, s_{-i}]\, u_i(s_i', s_{-i})\, , \forall i \in \mathcal{N}, \forall s_i \in \mathcal{A}_i, \forall s_i' \in \mathcal{A}_i, s_i \neq s_i'$
    - Player 1 sees α: $2a + 5b \geq 4a + 2b$
    - Player 1 sees α: $2a + 5b \geq 3a + 5b$
    - Player 1 sees β: $4c + 2d \geq 2c + 5d$
    - Player 1 sees β: $4c + 2d \geq 3c + 5d$
    - Player 1 sees γ: $3e + 5f \geq 2e + 5f$
    - Player 1 sees γ: $3e + 5f \geq 4e + 2f$
    - Player 2 sees α: $50a + 2c + 3e \geq a + \ 4c + 0e$
    - Player 2 sees β: $b + 4d + 0f \geq 50b + 2d + 3f$

Utility of player 1

	α	β
α	2	5
β	4	2
γ	3	5

Utility of player 2

	α	β
α	50	1
β	2	4
γ	3	0

Probabilities

	α	β
α	a	b
β	c	d
γ	e	f

Distribution (Result)

	α	β
α	0	0
β	.45	.15
γ	.30	.10

A Simple Example of MNE, CE and CCE

- The CCE constraints are given by:
  - $\sum_{s\in\mathcal{A}} \Pr[s]\, u_i(s) \geq \sum_{s\in\mathcal{A}} \Pr[s]\, u_i(s_i', s_{-i}), \forall i \in \mathcal{N}, \forall s_i' \in \mathcal{A}_i$
    - Player 1 sees α: $2a + 5b + 4c + 2d + 3e + 5f \geq 2(a + c + e) + 5(b + d + f)$
    - Player 1 sees β: $2a + 5b + 4c + 2d + 3e + 5f \geq 4(a + c + e) + 2(b + d + f)$
    - Player 1 sees γ: $2a + 5b + 4c + 2d + 3e + 5f \geq 3(a + c + e) + 5(b + d + f)$
    - Player 2 sees α: $50a + b + 2c + 4d + 3e + 0f \geq 50(a + b) + 2(c + d) + 3(e + f)$
    - Player 2 sees β: $50a + b + 2c + 4d + 3e + 0f \geq 1(a + b) + 4(c + d) + 0(e + f)$

Utility of player 1

	α	β
α	2	5
β	4	2
γ	3	5

Utility of player 2

	α	β
α	50	1
β	2	4
γ	3	0

Probabilities

	α	β
α	a	b
β	c	d
γ	e	f

Distribution 1

	α	β
α	0	0
β	.45	.15
γ	.30	.10

Distribution 2

	α	β
α	.15	0
β	.60	.15
γ	0	.10

Distribution 3

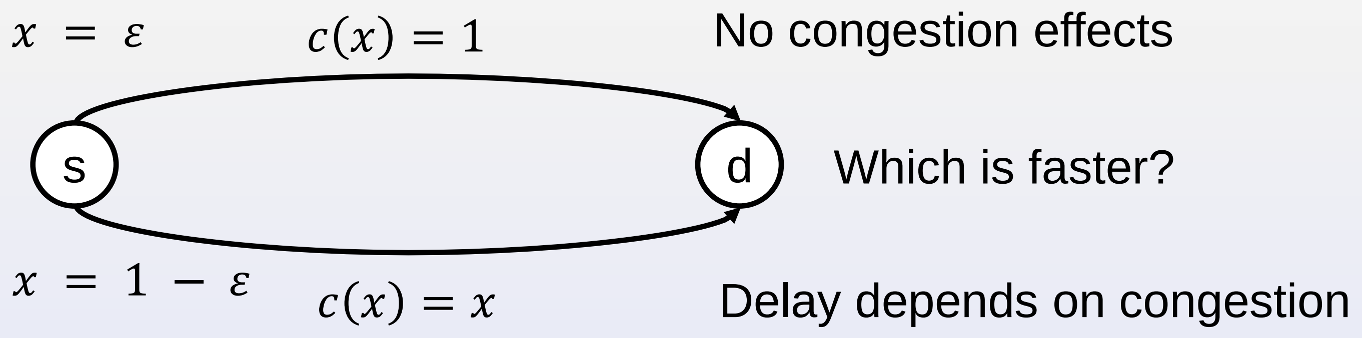
	α	β
α	.0368	0
β	.9018	.368
γ	0	.0245

Network Games

- There are N players who plays an action over networks
  - N players, $\mathcal{N} = \{1, 2, \dots, N\}$
  - Utility of player i, $u_i(\cdot)$
  - Network $G = (\mathcal{V}, \mathcal{E}, f)$
    - $\mathcal{V}$ is a set of nodes
    - $\mathcal{E}$ is a set of links
    - $c: \mathcal{E} \to \mathbb{R}$ is a mapping function that maps a link to a real value

Selfish Routing [Roughgarden+02]

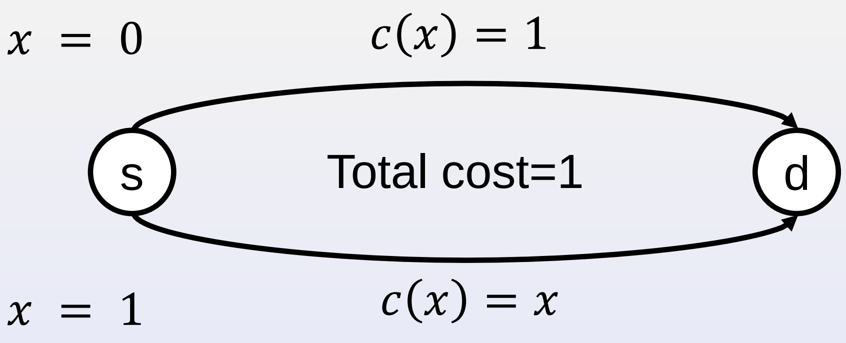
- Consider the total amount of traffic (flow) is 1, where x denotes the flow.
- Traffic cost $c(x)$ of each link is defined as follows:
  - Upper link: $c(x) \; = \; 1$
  - Lower link: $c(x) \; = \; x$
- Selfish routing aims at minimizing its own traffic cost.
  - All traffic will take the bottom link, i.e., $x = 1$.
  - Reason:
    - $\varepsilon > 0$: Traffic on the top link is envious of the bottom link
    - $\varepsilon = 0$: No envious traffic exists
      - All traffic incurs one unit of cost



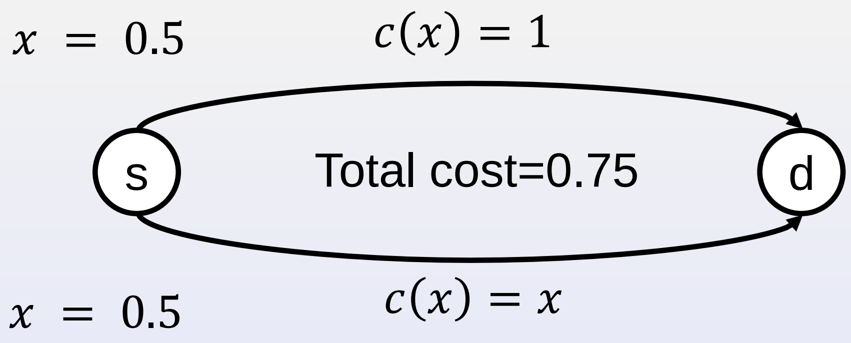
Pigou’s problem

Cooperative (Optimal) Routing

- Consider the total amount of traffic (flow) is 1, where x denotes the flow.
- Traffic cost $c(x)$ of each link is defined as follows:
  - Top link: $c(x) = 1$
  - Bottom link: $c(x) = x$
- Cooperative routing aims at minimizing the total traffic cost.
  - A half of traffic will take the top link, i.e., $x = 0.5$.
  - The rest traffic will take the bottom link, i.e., $x = 0.5$.



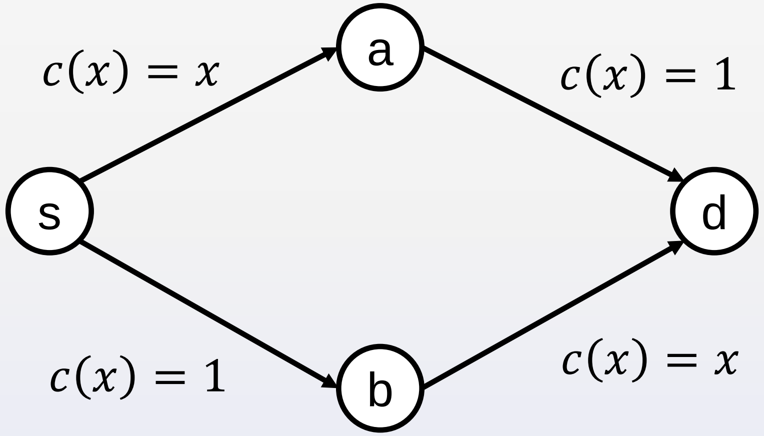
Selfish routing



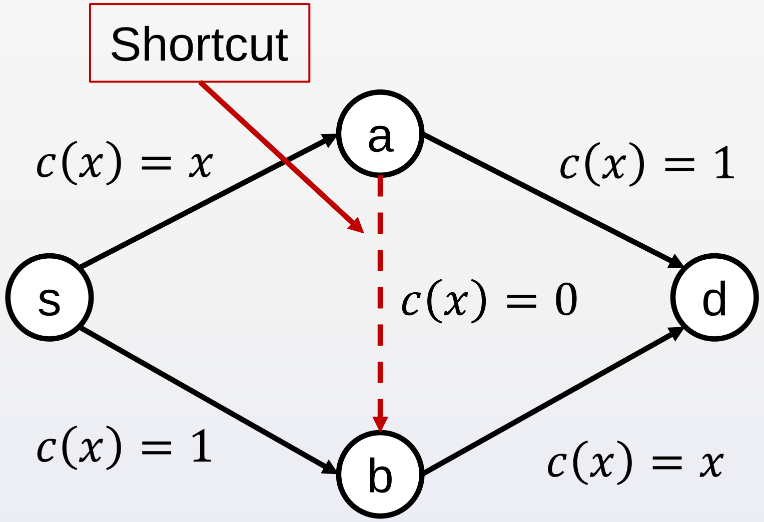
Cooperative routing

Braess's Paradox

- Selfish routing does not result in optimal routing
- In case of selfish routing, even if the network improves, the performance and individual utility deteriorate



Total cost = 1.5



Total cost = 2

Wardrop Equilibrium [Wardrop1952]

- Assumption
  - Network $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes a set of nodes and $\mathcal{E}$ denotes a set of links
  - f_e denotes flow on link $e \in \mathcal{E}$
  - f_i denotes flow on path $i \in \mathcal{P}$
  - $\mathcal{P}$ denotes all possible paths from a source to a destination
  - $c_e(f)$ denotes a cost function on the link e
- *Wardrop equilibrium* is achieved if

$$\sum_{e \in i'} c_e(f_e) = \sum_{e \in i} c_e(f_e), \quad \forall i, i' \in \mathcal{P} \text{ with } f_i, f_{i'} > 0$$

and

$$\sum_{e \in i'} c_e(f_e) \geq \sum_{e \in i} c_e(f_e), \quad \forall i, i' \in \mathcal{P} \text{ with } f_i > 0 \text{ and } f_{i'} = 0.$$

[Wardop1952] J. G. Wardrop, “Some theoretical aspects of road traffic research,” in *Proc. Inst. Civil Eng.*, vol. 1, 1952, pp. 325–362.

Wardrop Equilibrium [Wardrop1952]

- Wardrop equilibrium can be formulated as follows:
  - The optimization problem is changed according to the routing criteria

User equilibrium

$$\min \quad \sum_{e\in\mathcal{E}} \int_0^{f_e} c_e(f)df$$

$$\text{s.t} \quad \sum_{i\in\mathcal{P}|e\in i} f_i = f_e, \quad \forall e \in \mathcal{E}$$

$$\sum_{i\in\mathcal{P}} f_i = 1$$

$$f_i \geq 0, \quad \forall i \in \mathcal{P}$$

Social optimum

$$\min \quad \sum_{e\in\mathcal{E}} f_e c_e(f_e)$$

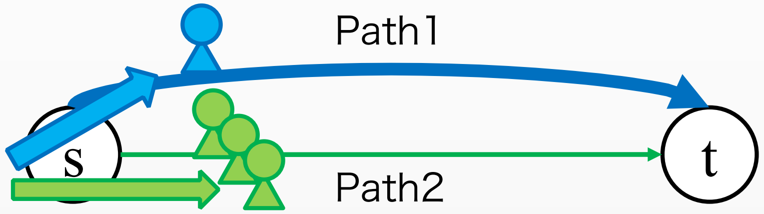
$$\text{s.t} \quad \sum_{i\in\mathcal{P}|e\in i} f_i = f_e, \quad \forall e \in \mathcal{E}$$

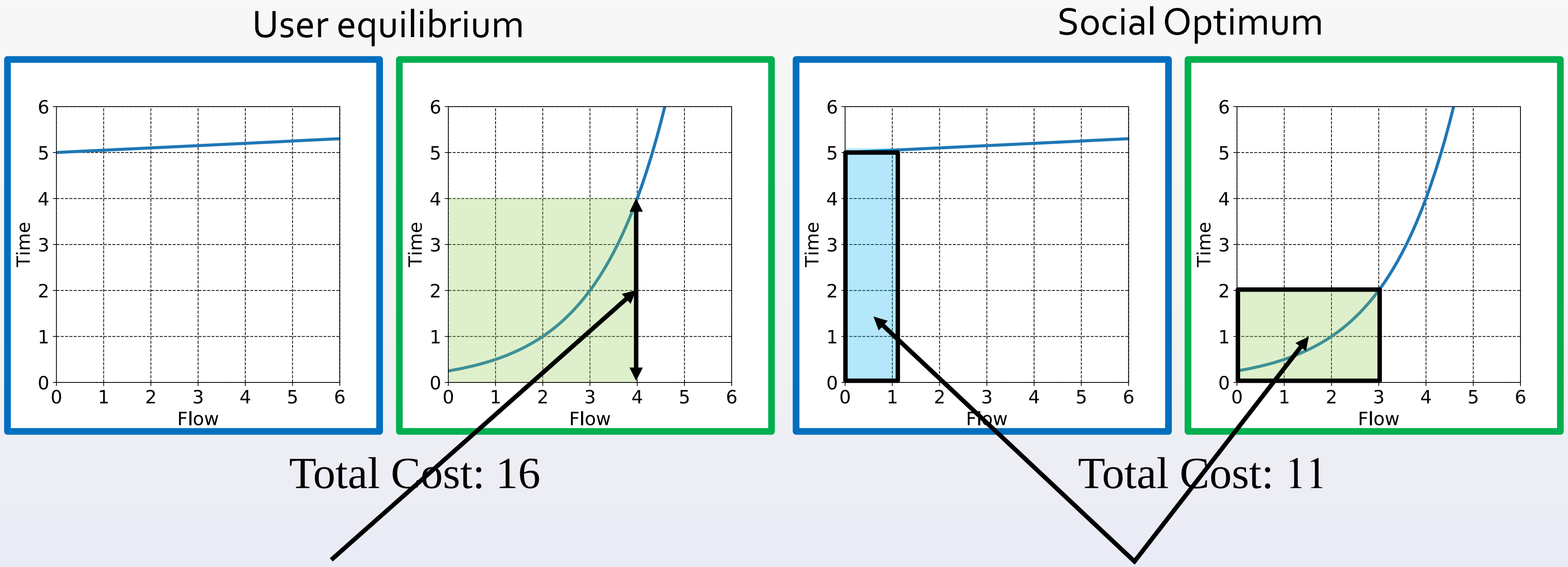
$$\sum_{i\in\mathcal{P}} f_i = 1$$

$$x_i \geq 0, \quad \forall i \in \mathcal{P}$$

[Wardop1952] J. G. Wardrop, "Some theoretical aspects of road traffic research," in *Proc. Inst. Civil Eng.*, vol. 1, 1952, pp. 325–362.

Wardrop Equilibrium: Intuitive Understanding





User equilibrium minimizes the individual user cost

Social optimum minimizes the total cost, i.e., the areas

[Wardop1952] J. G. Wardrop, “Some theoretical aspects of road traffic research,” in *Proc. Inst. Civil Eng.*, vol. 1, 1952, pp. 325–362.

Price of Anarchy (PoA)

- How does goodness of worst Nash equilibrium differ from social optimum?

$$PoA = \frac{\max_{s \in \mathcal{S}^{NE}} C(s)}{\min_{s \in \mathcal{S}} C(s)} \quad \text{or} \quad PoA = \frac{\max_{s \in \mathcal{S}} W(s)}{\min_{s \in \mathcal{S}^{NE}} W(s)}$$

$W(s) = -C(s)$

Social welfare func. Cost func.

  - Social welfare function $W(s) = \sum u_i(s)$ is defined as the sum of utility u_i of player i
    - where $u_i = -\sum_{e \in \mathcal{E}} \int_0^{f_e} c_e(f)df$ (negative traffic cost)
  - Cost function $C(s)$ is defined as $C(s) = -W(s)$
- PoA becomes 4/3 if the link cost linearly increases with the flow over the link
- PoA is the performance ratio of the cost of selfish routing to that of cooperative routing
- PoA is greater than or equal to one ($1 \leq PoA$)
- Smaller PoA indicates that the selfish routing has the competitive performance with the cooperative routing

Price of Stability (PoS)

- How does goodness of best Nash equilibrium differ from social optimum?

$$PoS = \frac{\max_{s \in \mathcal{S}^{NE}} C(s)}{\max_{s \in \mathcal{S}} C(s)} \quad \text{or} \quad PoS = \frac{\max_{s \in \mathcal{S}} W(s)}{\max_{s \in \mathcal{S}^{NE}} W(s)}$$

$W(s) = -C(s)$

Social welfare func. Cost func.

- $1 \leq PoS \leq PoA$
- Smaller PoS indicates that the selfish routing has the competitive performance with the cooperative routing (social optimum)

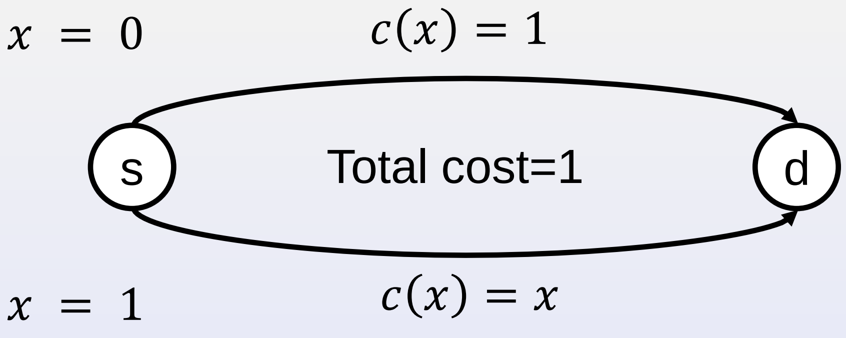
Overcoming Selfish Routing

- Ways to overcome selfish routing [Roughgarden+02]
  - Increasing the road capacity
  - Routing (part of) users in a central manner
    - Cooperative Routing
    - Stackelberg Routing
  - Internalizing the externalities into the user cost
    - Congestion Pricing
    - Selfish Yet Optimal Routing

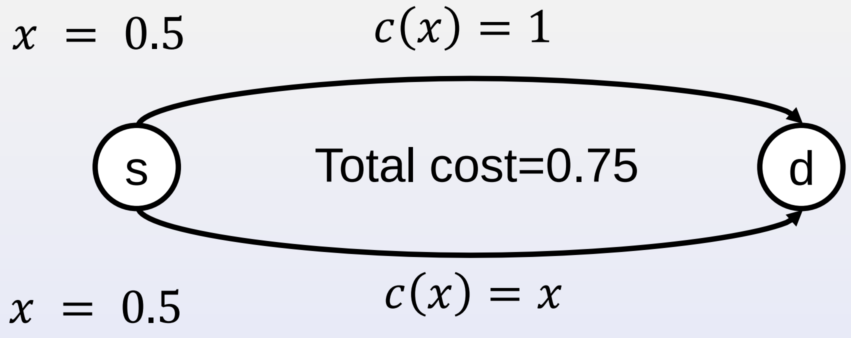
[Roughgarden+02] T. Roughgarden and E. Tardos, "How Bad is Selfish Routing?" *J. ACM*, vol. 49, no. 2, pp. 236–259, Mar. 2002.

Stackelberg Routing [Korilis+1997]

- Consider the total amount of traffic (flow) is 1, where x denotes the flow.
- Traffic cost $c(x)$ of each link is defined as follows:
  - Top link: $c(x) = 1$
  - Bottom link: $c(x) = x$
- Stackelberg routing consists of two stages:
  - Leader (cooperative user) first takes the top link
  - Follower (selfish user) then takes the bottom link after the leader's choice



Selfish routing

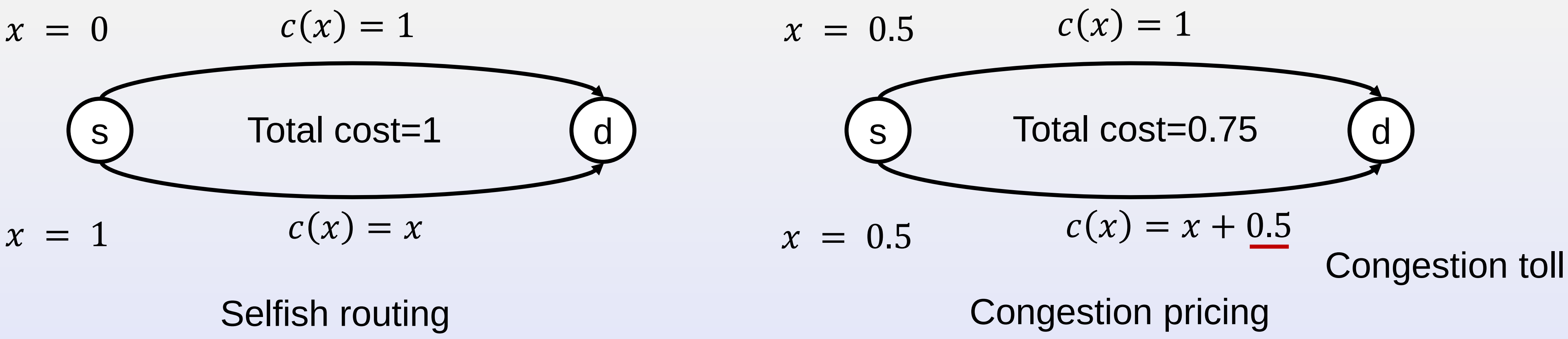


Stackelberg routing

[Korilis+1997] Y. A. Korilis, A. A. Lazar, and A. Orda, "Achieving network optima using Stackelberg routing strategies," *IEEE/ACM Trans. Netw.*, vol. 5, no. 1, pp. 161–173, Feb. 1997.

Congestion and Provider Price Competition [Cole+03]

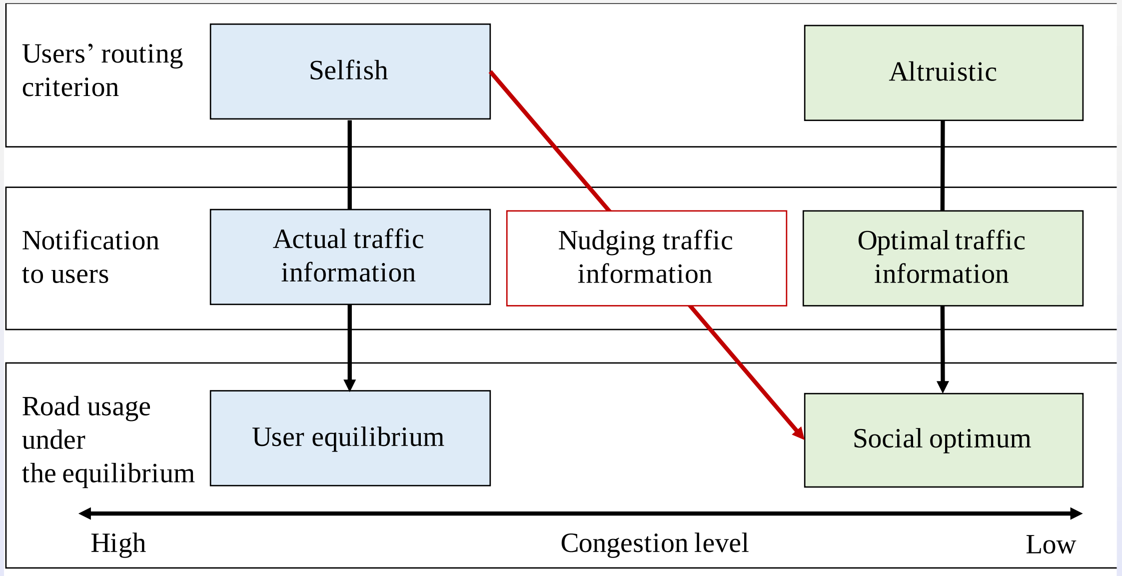
- Consider the total amount of traffic (flow) is 1, where x denotes the flow.
- Traffic cost $c(x)$ of each link is defined as follows:
  - Top link: $c(x) = 1$
  - Bottom link: $c(x) = x$
- Congestion pricing internalizes the externalities by introducing congestion toll
  - The congestion toll is imposed on the bottom link



[Cole+03] R. Cole, Y. Dodis, and T. Roughgarden, "Pricing network edges for heterogeneous selfish users," in *Proc. 35th Annu. ACM Symp. Theory Comput.*, 2003, pp. 521–530.

Selfish Yet Optimal Routing [Hara+20]

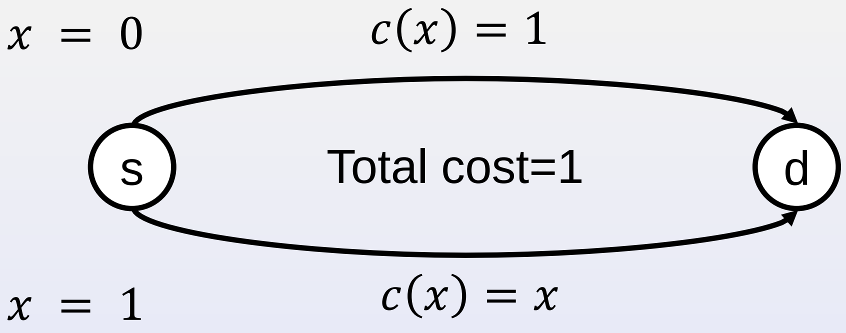
- In selfish yet optimal routing, social optimum emerges under the inidividual rational decision making
- The marginal cost is internalized into the personalized traffic information
  - The social optimum assignment for each user is equivalent to one of the Wardrop equilibria under the selfish routing criterion
  - Any Wardrop equilibrium for each user can be equivalent to the social optimum assignment under the selfish routing criterion



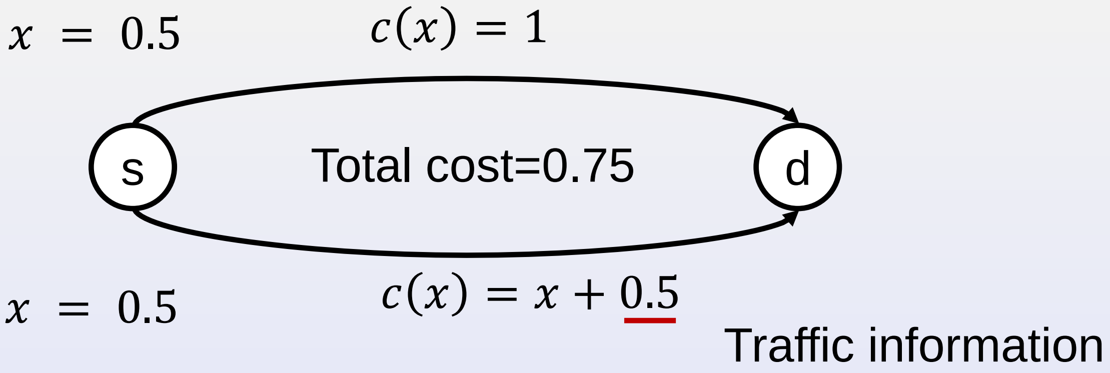
[Hara+20] T. Hara, M. Sasabe, and S. Kasahara, "Selfish Yet Optimal Routing by Adjusting Perceived Traffic Information of Road Networks," *IEEE Open J. Intell. Transp. Syst.*, vol. 1, pp. 120–133, 2020,

Selfish Yet Optimal Routing [Hara+20]

- Consider the total amount of traffic (flow) is 1, where x denotes the flow.
- Traffic cost $c(x)$ of each link is defined as follows:
  - Top link: $c(x) = 1$
  - Bottom link: $c(x) = x$
- Selfish yet optimal routing internalizes the marginal cost into the perceived traffic information
  - Selfish routing results in the social optimum by the personalized traffic information



Selfish routing



Selfish yet optimal routing

[Hara+20] T. Hara, M. Sasabe, and S. Kasahara, “Selfish Yet Optimal Routing by Adjusting Perceived Traffic Information of Road Networks,” *IEEE Open J. Intell. Transp. Syst.*, vol. 1, pp. 120–133, 2020,

Distributed Path Selection Scheme [Lim+14]

- Assumption
  - N users ($\mathcal{N} = \{1, \dots, N\}$) exist on the network $G = (\mathcal{V}, \mathcal{E})$
  - $\mathcal{V}$ denotes a set of nodes and $\mathcal{E}$ denotes a set of links
  - Each user i has candidate paths $\boldsymbol{\pi}_i = (\pi_{i,1}, \pi_{i,2}, \dots, \pi_{i,K_i})$
    - The kth path $\pi_{i,k} = (e_1, \dots e_L)$ is defined as a sequence of links
    - K_i is the number of path candidates, $\mathcal{K}_i = \{1, \dots, K_i\}$
  - Each user selects a path from the candidate according to the path choice probability $\boldsymbol{p}_i$
  - Each user adjusts the path choice probabilities $\boldsymbol{p}_i = (p_{i,1}, p_{i,2}, \dots, p_{i,K_i})$ to minimize the its cost in a distributed manner
- Applications
  - Route selection on the road network
  - Service Function Chaining

[Lim+14] S. Lim and D. Rus, "Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications," *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Distributed Path Selection Scheme [Lim+14]

- Flow on link e (under the large sample approximation)

$$f_e = \sum_{j \in \mathcal{N}} \sum_{k \in \mathcal{K}_j} \underbrace{\mathbb{I}(e \in \pi_{j,k})}_{\text{Indicator function}} p_{j,k}.$$

- Nonnegative cost function of link e relative to the flow f_e is defined as $c_e(f_e)$
  - Cost function is changed depending on the routing criteria

Selfish criteria

$$c_e(f_e) = t_e(f_e)$$

Cooperative criteria

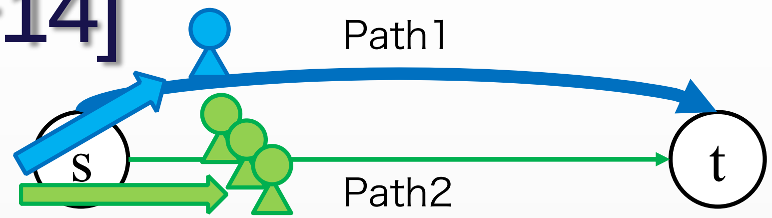
$$c(f_e) = t(f_e) + \underbrace{f_e \cdot \left.\frac{\partial t_e(f)}{\partial f}\right|_{f=f_e}}_{\text{Marginal cost}}$$

- Cost of path $\pi_{i,k}$

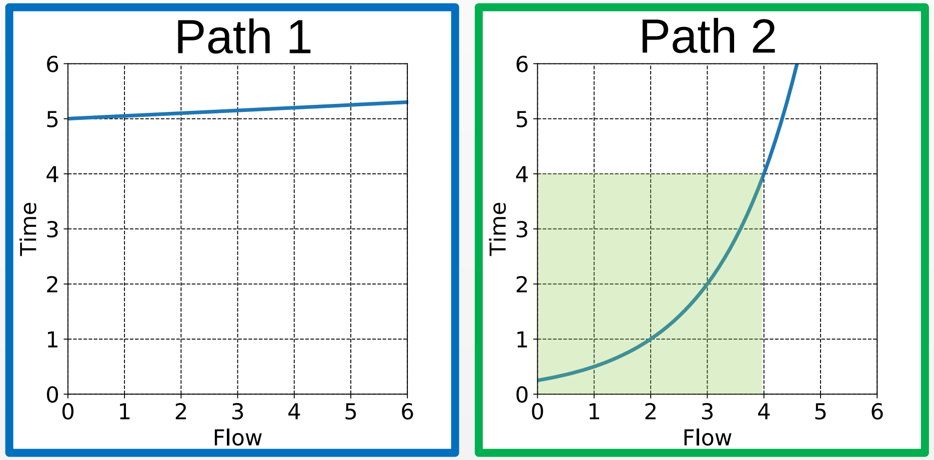
$$c_{i,k} = \sum_{e \in \pi_{i,k}} c_e(f_e).$$

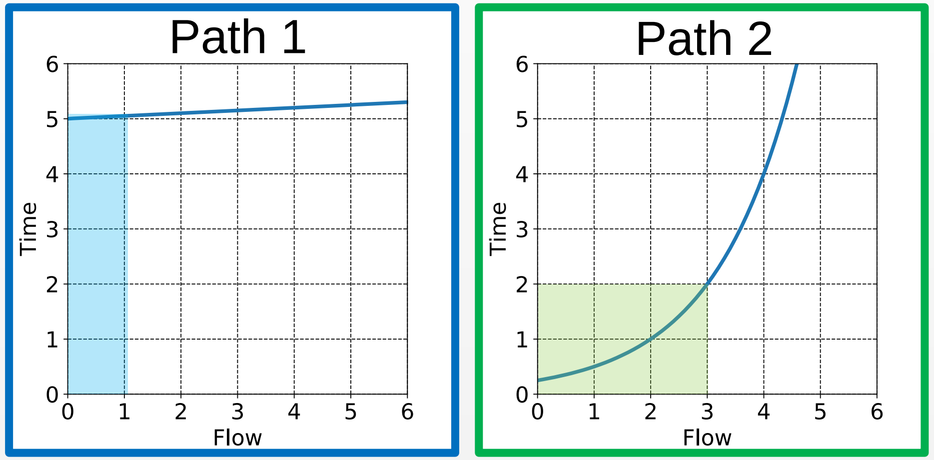
[Lim+14] S. Lim and D. Rus, "Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications," *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Distributed Path Selection Scheme [Lim+14]



- Routing Criateria





Total Cost: 16

Total Cost: 11

Selfish Criteria

Cooperative Criteria

$$c_e(f_e) = t_e(f_e)$$

$$c(f_e) = t(f_e) + f_e \cdot \left. \frac{\partial t_e(f)}{\partial f} \right|_{f=f_e}$$

Travel cost on link *e*

Total cost increase by the slightly increase of flow on link e

[Lim+14] S. Lim and D. Rus, "Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications," *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Distributed Path Selection Scheme [Lim+14]

- Global cost V is defined as the sum of the local cost V_i of all users

$$V = \sum_{i \in \mathcal{N}} V_i .$$

- Local cost V_i is the goodness of the current path choice probabilities $\boldsymbol{p}_i$.

$$V_i = \sum_{k \in \mathcal{K}_i} \underbrace{p_{i,k}}_{\text{Probability choosing the } k\text{th path}} \underbrace{\left(\underbrace{c_{i,k}}_{\text{Cost of the } k\text{th path}} - \underbrace{c_{i,k_i^*}}_{\text{Minimum path cost}} \right) \geq 0}_{\text{Difference between the average path cost and minimum one}}$$

[Lim+14] S. Lim and D. Rus, "Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications," *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Distributed Path Selection Scheme [Lim+14]

- Global cost V is defined as the sum of the local cost V_i of all users

$$V = \sum_{i \in \mathcal{N}} V_i. = 0 \rightarrow \text{Wardrop equilibrium is achieved}$$

- Local cost V_i is the goodness of the current path choice probabilities $\boldsymbol{p}_i$.

$$V_i = \sum_{k \in \mathcal{K}_i} \underline{p_{i,k}} (\underline{c_{i,k}} - \underline{c_{i,k_i^*}}) = 0$$

Probability choosing the kth path	Minimum path cost	Cost of the kth path

$p_{ik} = 0$	Not choosing the kth path
$p_{ik} \neq 0$	The kth path cost is equivalent to the minimum cost

[Lim+14] S. Lim and D. Rus, "Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications," *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Distributed Path Selection Scheme [Lim+14]

- Total cost increase by a small change in $p_{i,k}$

$$w_{ik} = \sum_{\forall j \in \mathcal{N}} \frac{\partial V_j}{\partial p_{ik}}$$

- From the local cost function, we have

$$\frac{\partial V_j}{\partial p_{ik}} = \frac{\partial}{\partial p_{ik}} \sum_{\forall l \in \mathcal{K}_j \setminus k_j^*} p_{jl}(c_{jl} - c_{jk_j^*})$$

$$= \begin{cases} \sum_{\forall l \in \mathcal{K}_j \setminus k_j^*} p_{jl}\left(\frac{\partial c_{jl}}{\partial p_{ik}} - \frac{\partial c_{jd_j}}{\partial p_{ik}}\right) & (\text{for } i \neq j), \\ c_{ik} - c_{id_i} + \sum_{\forall l \in \mathcal{K}_i \setminus k_i^*} p_{il}\left(\frac{\partial c_{il}}{\partial p_{ik}} - \frac{\partial c_{ik_i^*}}{\partial p_{ik}}\right) & (\text{for } i = j). \end{cases}$$

[Lim+14] S. Lim and D. Rus, “Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications,” *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Distributed Path Selection Scheme [Lim+14]

- Cost derivative of user j's lth path by user i's kth path flow

$$\frac{\partial c_{j,l}}{\partial p_{i,k}} = \sum_{\forall e \pi_{i,k} \cap \pi_{j,l}} \frac{\partial c_e(f_e)}{\partial f_e} - \sum_{\forall e \pi_{i,k_i^*} \cap \pi_{j,l}} \frac{\partial c_e(f_e)}{\partial f_e}$$

The edges that are shared by $\pi_{i,k}$ and $\pi_{j,l}$ The edges that are shared by π_{i,k_i^*} and $\pi_{j,l}$

  - We just consider the edges that are shared by two paths

- Cost derivative of flow is changed depending on the routing criteria

$$\text{User Equilibrium:} \quad \frac{\partial c_e(f)}{\partial f} = \left.\frac{\partial t_e(f)}{\partial f}\right|_{f=f_e}$$

$$\text{Social Optimum:} \quad \frac{\partial c_e(f)}{\partial f} = 2\left.\frac{\partial t_e(f)}{\partial f}\right|_{f=f_e} + \left.\frac{\partial^2 t_e(f)}{\partial f^2}\right|_{f=f_e}$$

[Lim+14] S. Lim and D. Rus, "Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications," *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Distributed Path Selection Scheme [Lim+14]

- Let define $\boldsymbol{y}_i$ as follows:

$$y_{ik} = \begin{cases} 0 & (\text{if } p_{ik} = 0 \text{ and } w_{ik} > 0 \text{ or } p_{ik} = 1 \text{ and } w_{ik} < 0), \\ w_{ik} & (\text{otherwise}). \end{cases}$$

- The distributed controller is defined as follows:

$$\frac{dp_{ik}}{d\tau} = \begin{cases} -\underbrace{\gamma}_{\text{Learning rate}} V_i \frac{y_{ik}}{||y||^2} & (\text{for } k \neq k_i^*) \\ -\sum_{k \neq k_i^*} \frac{dp_{ik}}{d\tau} & (\text{for } k = k_i^*) \end{cases}$$

$$\text{Control Law:} \quad \frac{dp_i}{d\tau} = -\gamma V_i \frac{y_i - (\mathbf{1}_{K_i}^{\mathrm{T}} y_i) e_{k_i^*}}{||y_i||^2}.$$

[Lim+14] S. Lim and D. Rus, "Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications," *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Convergence Analysis [Lim+14]

- Theorem: The global cost $V(t)$ exponentially decrease at the convergence rate inversely proportional to the product of number of users and that of individual path candidates (i.e., NK) under the distributed controller.
- Proof: The time derivative of V can be expressed by

$$\frac{dV}{d\tau} = \frac{1}{NK} \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} \frac{\partial V}{\partial p_{i,k}} \cdot \frac{\partial p_{i,k}}{\partial \tau}$$

$$= \frac{1}{NK} \sum_{i \in \mathcal{N}} \left(\frac{\partial V}{\partial \boldsymbol{p}_i} \right)^{\mathsf{T}} \frac{d\boldsymbol{p}_i}{d\tau}.$$

[Lim+14] S. Lim and D. Rus, “Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications,” *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Convergence Analysis [Lim+14]

- Proof (Cont.): This can be rewritten as follows:

$$\frac{dV}{d\tau} = \frac{1}{NK}\sum_{i\in\mathcal{N}}\left(\frac{\partial V}{\partial \boldsymbol{p}_i}\right)^{\mathsf{T}}\left(-\gamma V_i \frac{\boldsymbol{y}_i - (\mathbf{1}_{K_i}^{\mathsf{T}}\boldsymbol{y}_i)\boldsymbol{e}_{k_i^*}}{||\boldsymbol{y}_i||^2}\right)$$

$$= -\frac{\gamma}{NK}\sum_{i\in\mathcal{N}} V_i \left(\sum_{j\in\mathcal{N}}\frac{\partial V_j}{\partial \boldsymbol{p}_i}\right)^{\mathsf{T}}\left(\frac{\boldsymbol{y}_i - (\mathbf{1}_{K_i}^{\mathsf{T}}\boldsymbol{y}_i)\boldsymbol{e}_{k_i^*}}{||\boldsymbol{y}_i||^2}\right).$$

  - Nonzero elements of vector $\boldsymbol{y}_i$ except for k_i^*th element are equal to $\sum_{j\in\mathcal{N}}\frac{\partial V_j}{\partial \boldsymbol{p}_i}$
  - k_i^*th element of $\sum_{j\in\mathcal{N}}\frac{\partial V_j}{\partial \boldsymbol{p}_i}$ becomes 0

- We have the differential equation

$$\frac{dV}{d\tau} = -\frac{\gamma}{NK}\sum_{i\in\mathcal{N}} V_i\left(\frac{\boldsymbol{y}_i^{\mathsf{T}}\boldsymbol{y}_i}{||\boldsymbol{y}_i||^2}\right) = -\frac{\gamma}{NK}V.$$

[Lim+14] S. Lim and D. Rus, “Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications,” *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Convergence Analysis [Lim+14]

- Proof (Cont.): We have

$$\frac{dV}{d\tau} = -\frac{\gamma}{NK} \sum_{i \in \mathcal{N}} V_i \left(\frac{\boldsymbol{y}_i^\top \boldsymbol{y}_i}{||\boldsymbol{y}_i||^2} \right) = -\frac{\gamma}{NK} V.$$

- Solving the differential equation in terms of τ, we have

$$V(\tau) = V(0) \exp\left(-\frac{\gamma\tau}{NK} \right),$$

The global cost $V(t)$ exponentially decreases

- where $V(0)$ is the initial global cost.

[Lim+14] S. Lim and D. Rus, "Congestion-Aware Multi-Agent Path Planning: Distributed Algorithm and Applications," *The Computer Journal*, vol. 57, no. 6, pp. 825–839, Jun. 2014.

Conclusion

- Fundamental game theory
  - Dominant Strategy Equilibrium
  - Pure/Mixed Nash Equilibrium
  - (Coarse) Correlated Equilibirum
  - Wardrop Equilibirum
- Network resource management from the viewpoint of game theory
  - Selfish routing
  - Cooperative routing
  - Stackelberg routing
  - Congestion Pricing
  - Selfish yet optimal routing
  - Distributed Path Selection Scheme

Report Assignments

- Answer the following 2 questions related to the lectures:
  1. Assume the network illustrated in Fig. 1. x denotes the amount of flow and $c(x)$ is a link cost. Find a value of ε under selfish routing and that under optimal routing, respectively
  2. Proof that PoA is no more than 4/3 if the link cost linearly increases with the flow over the link.

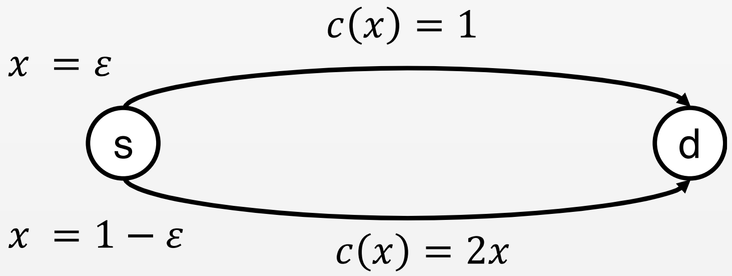


Fig. 1

Submission: Edu Portal
Deadline: No later than 23:59 (JST), Jan. 8, 2025
Format: PDF (English or Japanese)
If the following requirement is not satisfied, the report is denied.
1. Students must fill in your student id, grade, full name on the report.