EE5606 Final Project Report

Heterogeneous 2-facility location games with Minimum Distance Requirement

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Contents

Introduction to facility location games(FLG)

Jargons used:

Planner: In facility location games, a planner is an individual or government responsible for selecting a facility's location (such as a factory or store) within a geographic region. The planner acts like a function that takes input as the location vector of the agents and gives the output the optimal location vector of the facilities.

Facility: In facility location games, a facility is one which produces or provides goods or services to the agents. And based on the goods or services it produces, it charges some price. And agents will have preference for each of these facilities.

Agent: In facility location games, agents refer to the person who uses or being provided the goods or services by the facilities and agent may want to be near or far from a facility and can represent this by his preference for each facility.

Facility location games are a type of game theory model that deals with the strategic decision-making of multiple facilities when selecting the location of their facilies (e.g., factories, stores, or distribution centers) in a geographic region. In these games, facilities aim to maximize their profits while considering their competitors' actions and strategies.

Examples:

- "A market is commonly subdivided into regions within each of which one seller is in a quasi-monopolistic position." - Piero Sraffa. The seller has significant regional market power and can influence market outcomes. So, by this, the seller has some monopoly in a particular region. That means that the locations of the seller matter.
- Assumption: the market is inelastic. N sellers(immovable) in a market, if one seller decreases the price(identical) of a particular good, then one can increase the profit by n times, but that doesn't happen in real. Buyers will still prefer the previous seller. For example, If a gasolene seller increases the price, all his customers will not shift to the one having a lower price. It can be the case that the gasolene station is nearer to some agents and the others are too far, so the cost of transportation is high.
- A market of sellers(movable facility) tends to cluster at a single point despite being distributed in the socially optimum manner. For example, two ice cream parlors are present at a distance of a and b from each end of a closed street of fixed total length. There will always be the tendency of each ice cream parlor to shift toward the other one to capture its market. As a result, they will come in the middle forming a cluster.
- The tendency of a slight change in the product when a new merchant comes into a market(to avoid price wars) is a reason for standardization of everything with a bit of change in the name of improvement to attract as many consumers as the old ones. If a new merchant wants to enter a market, he must sell identical goods that a seller is selling, but there is no incentive for a buyer to choose the new seller. So, to be in the market, he must produce a similar product with a slight notion of improvement to capture the market. Here also, the idea of location matters.

Homogeneous, Heterogeneous facility location games and examples

Facility location games can be mainly classified into two:

Homogenous Facility Location Games: In Homogenous Facility Location Games, facilities are of the same type, and all agents have the same preference for facilities.

Heterogenous Facility Location Games: In Heterogeneous Facility Location Games, facilities are of different types, or agents have different preferences over facilities or both.

Examples:

1. An example of a homogeneous facility location game is the problem of the Gasolene station. Everyone wants gasoline having the same preference, and every gasoline station sells gasoline at the same price. So, all agents have the same preference for all facilities.

2. An example of a heterogeneous facility location game is the problem of location of a school and an industry. Here different agents have different preferences for facilities. One agent working in a factory will want to live nearer to the industry. The second agent not working will want to live far away from the industry to avoid pollution. And another agent having children will want to live close to school but far from the industry.

Minimum distance requirement(MDR) and real life application/motivation

There is often a case when the government or the planner wants to build some facilities, let's say, a dumping yard and water reservoir. Both facilities are necessary, and everybody(agents) needs them, but these facilities shouldn't be closed because the dumping yard will pollute the water reservoirs. So the government or the planner will apply some restrictions that these locations must be at least d distance apart to prevent the polluting of the water reservoir. Another limitation is that the facilities should be close to the agents(consumers). We want to find the locations for the facilities means where should these facilities be placed so that we can achieve the objectives.

Given two facilities, we want to minimize the overall distance of facilities from the agents, but there must be some minimum distance between the locations. So, the planner will be given the set(domain) of locations where the facilities can be present(as facilities can't be present anywhere) and the agents' locations and the job of the planner is to find the optimal facility location so that it minimizes the overall distance between the agents and facilities with at least d distance between the facilities.

Heterogenous 2-FLG with MDR Settings

- In this problem we have n-agents, the set of agents is defined as $N = \{1, \ldots, n\}$. Each agent is located at a point x_i in the interval $I = [0,1]$
- Without losing generality we consider that all agents from 1 to n are located on I in the ascending order. That is, $x_1 \leq x_2 \leq \ldots \leq x_{n-1} \leq x_n$
- $x = (x_1, x_2, \ldots, x_{n-1}, x_n)$ be the agents location profile which is known to the planner.
- We consider 2-facilities those will be located at locations y_1 and y_2. Say, $y = (y_1, y_2)$ and $y_1, y_2 \in I$. The planner need to locate these 2-facilities at some optimal positions.
- Without losing generality we consider $y_1 \leq y_2$
- The distance d is the minimum separation we want between the two facilities y_1 and y_2
- Mechanism: It is a function that maps from the agents location profile space to the facility location space. Mechanism can be mathematically written as,
  $f: I^n \rightarrow I^2$ and $y = f(x)$

Description of problem (1/2)

COST OF AGENT: We assume that every agent wants the facility to be nearer to them (We may also assume that agents do not like the facilities (example: factory due to pollution), this particular assumption is made since, the final problem in this case will be a convex optimization). Also, the game is heterogenous. This, implies the agent want both the facilities. So, the cost of agent-i can be written as below,

$$c_i(f(x),x_i) = c_i(y,x_i) = |x_i-y_1| + |x_i-y_2|$$

The cost of agent-i is the summation of distances from agent's location to the facility's location.

SOCIAL COST: It is sum of costs of all agents. Social cost can be written as below,

$$SC(y,x) = SC(f(x),x) = \sum_i c_i(f(x),x_i)$$

Description of problem (2/2)

Problem: minimize $SC(y,x)$

such that 1) $y_1, y_2 \in [0,1]$
2) $y_1 \leq y_2$
3) $y_2 - y_1 \geq d$

The objective is to minimize the sum of costs of all agents (1 to n), in other words minimizing the social cost. Subjected to the above three constraints.

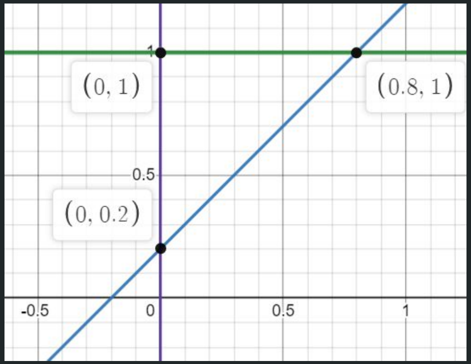
Explanation of the constraints:

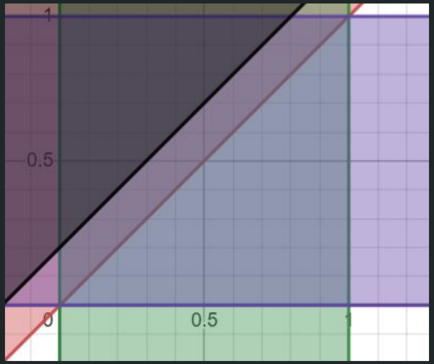
1) We want the facilities also to locate in the [0,1] interval
2) Without losing generality, it has been considered that $y_1 \leq y_2$
3) Minimum distance requirement constraint(d also lies in [0,1], which is known)

It is convex optimization problem?(1/2)

The below is the proof that the above described problem is a convex optimization problem. That is we prove that Constraint set and objective function are convex.

(i) Constraint set(say D) is convex: From the constraints it can be noticed that the constraint set is the triangle that is enclosed by the three points (0,d),(0,1) and (1-d,1) in the (y_1,y_2) plane. The constraint set for d = 0.2 is visualized below. Since it is a triangle, it is a convex set.





It is convex optimization problem?(2/2)

(ii) Objective function is convex: The stepwise proof is given below,

(a) The function $|x_i-y_1|$ is convex in (y_1,y_2) given x_i (Since, it is a 2-norm w.r.t y_1 and in (y_1,y_2) plane it has no dependency on second variable y_2, the plot looks like a v-shaped grove)
(b) Similarly $|x_i-y_2|$ is also convex in (y_1,y_2) given x_i
(c) Sum of convex functions is also convex. So, cost of agent-i $|x_i-y_1| + |x_i-y_2|$ is also convex
(d) Similarly sum of costs of all agents is also convex
(e) Implies objective function, $SC(y,x)$ is convex in y given x

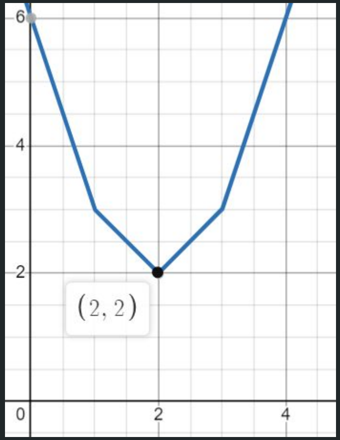
As both the constraint set and objective function are convex we can say that the optimization problem described above is also convex.

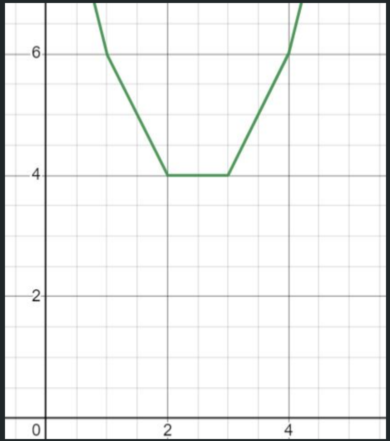
Results

Result 1.1: $\sum_i |x_i - y_1|$ occurs minimum at $x_{(n+1)/2}$ if n is odd

Result 1.2: $\sum_i |x_i - y_1|$ occurs minimum in the entire interval $[x_{(n/2)}, x_{(n/2)+1}]$ if n is even

These two results are very intuitive it can be simply visualised using the functions $|x-1|+|x-2|+|x-3|$ (n-odd) and $|x-1|+|x-2|+|x-3|+|x-4|$ (n-even)





Results

Result 2: Social cost obtains minimum in the segment $y_2 - y_1 = d$
(This proof is followed by the Results 1.1 and 2.2)

Since SC obtains minimum on segment $y_2 - y_1 = d$, we can pose the problem as follows,

$$y^* = \text{argmin}_{y \in D} \sum_i (|x_i - y_1| + |x_i - y_2|)$$

$$\text{s.t} \quad D = \{ y : y_1, y_2 \in I,\ y_1 \le y_2,\ y_2 - y_1 \ge d \}$$

$$y^* = \text{argmin}_{y \in D} \sum_i (|x_i - y_1| + |x_i - y_2|)$$

$$\text{s.t} \quad D = \{ y : y_1, y_2 \in I,\ y_1 \le y_2,\ y_2 - y_1 = d \}$$

Results

$$y_1^* = \text{argmin}_{y1}\ \Sigma_i(|y_1 - x_i| + |y_1 - x_i + d|)$$

$$\text{s. t } y_1 \in [0,1\text{-}d]$$

This can be written as,

$$y_1^* = \text{argmin}_{y1}\ \Sigma_i|y_1 - x_i'|,\ \text{where } x_i' \in \{x\text{-}d, x\}$$

$$\text{s. t } y_1 \in [0,1\text{-}d]$$

The set {x-d,x} has even cardinality. So, using result 1.2 we can get the value of y_1 where minimum occurs. But, before using result 1.2 we need to sort the set {x-d,x}.
Say, z = sort{x-d,x} in ascending order. Using, result 1.2 the result 3 is given below.

Results

Result 3: y_1* lies in the interval $[z_n, z_{n+1}]$, where z_i is i-th order statistic of set $\{x\text{-}d, x\} \in I^{2n}$

Note that we already had constraint on y_1 s.t $y_1 \in [0, 1-d]$

But, the interval $[z_n, z_{n+1}]$ need not lie entirely in $[0, 1-d]$. So,

$$y_1^* = [z_n, z_{n+1}] \cap [0, 1-d]$$

$$= [\max\{0, z_n\}, \min\{1-d, z_{n+1}\}]$$

Result 4: $y_1^* = [\max\{0, z_n\}, \min\{1-d, z_{n+1}\}]$
(This is the theoretically calculated optimal location of facility)
(It is compared with the practical result using cvxpy library in the code file submitted)
(Both coincide)

Final Solution

$$\min_{y \in D} \sum_i (|x_i - y_1| + |x_i - y_2|)$$

$$D = \{\, y : y_1, y_2 \in I,\ y_1 \le y_2\ ,\ y_2 - y_1 \ge d \}$$

This problem has the solution set S,

Result 5: The optimal facility location is a set $S = \{\, y^* : y^* = (y_1^*, y_1^* + d),$

$y_1^* \in [\max\{0, z_n\}, \min\{1-d, z_{n+1}\}]\,\}$

(This result follows the result 4)

Any point y from the set S, gives the minimum Social Cost given agents profile(x) and d

Conclusion

- In this report, we designed a Mechanism Function that takes the input of the n agents location profile and outputs the optimal locations of 2-facilities with some minimum distance between them. The problem formed was a convex optimization problem which has been solved. We found the final solution set S.
- Implementation is done using numpy and cvxpy libraries. The theoretical results and the solution given by cvxpy are coinciding (have been displayed in the code submitted)
- Interesting result from Game Theory is the lower limit in the set y_1^* gives strategyproof mechanism.
- But this is not the case in the case of obnoxious facilities, there doesn't exist any optimal solution for the strategyproofness. Only some γ-approximate optimal solutions are strategyproof. Reducing the γ value and making it nearer to 1 is an open problem.

References

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