Linear Mappings

Matrix Mappings

Let A be any $m \times n$ matrix over K. Then A determines a mapping $F_A : K^n \to K^m$ by

$F_A(u) = Au$

where the vectors in K_n and K_m are written as columns.

For notational convenience, we will frequently denote the mapping F_A by the letter A, the same symbol as used for the matrix.

Linear Mappings (Linear Transformations)

Let V and U be vector spaces over the same field K. A mapping $F : V \to U$ is called a linear mapping or linear transformation if it satisfies the following two conditions:

1. For any vectors $v, w \in V, F(v + w) = F(v) + F(w)$
2. For any scalar k and vector $v \in V, F(kv) = kF(v)$

Substituting $k = 0$ into condition (2), we obtain $F(0) = 0$. Thus, every linear mapping takes the zero vector into the zero vector.

For any scalars $a_i \in K$ and any vectors $v_i \in V$, we obtain the following basic property of linear mappings:

$F(a_1 v_1 + a_2 v_2 + \cdots + a_m v_m) = a_1 F(v_1) + a_2 F(v_2) + \cdots + a_m F(v_m)$

Remark: A linear mapping $F : V \to U$ is completely characterized by the condition

$F(av + bw) = aF(v) + bF(w)$

Theorem 4.1: Let V and U be vector spaces over a field K. Let $v_1, v_2, \ldots, v_n$ be a basis of V and let $u_1, u_2, \ldots, u_n$ be any vectors in U. Then there exists a unique linear mapping $F : V \to U$ such that $F(v_1) = u_1, F(v_2) = u_2, \ldots, F(v_n) = u_n$.

Note that the vectors $u_1, u_2, \ldots, u_n$ are completely arbitrary; they may be linearly dependent or they may even be equal to each other.

Proof: Consider $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$

Define $F(v) = a_1u_1 + a_2u_2 + \cdots + a_nu_n$

Suppose $G : V \to U$ is linear and $G(v_1) = u_i, i = 1, \ldots, n$. Let

Then

$G(v) = G(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1G(v_1) + a_2G(v_2) + \cdots + a_nG(v_n)$

$= a_1u_1 + a_2u_2 + \cdots + a_nu_n = F(v)$

Because $G(v) = F(v)$ for every $v \in V, G = F$. Thus, F is unique and the theorem is proved.

Matrices as Linear Mappings

$F_A(v + w) = A(v + w) = Av + Aw = F_A(v) + F_A(w)$

$F_A(kv) = A(kv) = k(Av) = kF_A(v)$

In other words, using A to represent the mapping, we have,

$A(v + w) = Av + Aw \quad \text{and} \quad A(kv) = k(Av)$

Vector Space Isomorphism

Two vector spaces V and U over K are isomorphic, written $V \cong U$, if there exists a bijective (one-to-one and onto) linear mapping $F : V \to U$. The mapping F is then called an isomorphism between V and U.

Consider any vector space V of dimension n and let S be any basis of V. Then the mapping

$v \mapsto [v]_S$

which maps each vector $v \in V$ into its coordinate vector $[v]_S$, is an isomorphism between V and K^n

Kernal and Image of a Linear Mapping

Let $F : V \to U$ be a linear mapping. The kernel of F, written $\operatorname{Ker} F$, is:

Ker $F =$
$v \in V : F(v) = 0$

And image or range of F is defined as:

Im $F =$
$u \in U :$ there exists $v \in V$ for which $F(v) = u$

Theorem 4.2 Let $F : V \to U$ be a linear mapping. Then the kernel of F is a subspace of V and the image of F is a subspace of U. (easy to see)

Theorem 4.3 Suppose $v_1, v_2, \ldots, v_m$ span a vector space V, and suppose $F : V \to U$ is linear. Then $F(v_1), F(v_2), \ldots, F(v_m)$ span $\operatorname{Im} F$. (easy to see)

Thus one can either use this, or theorem 4.4 to find the dimension of $\operatorname{Im} F$

Examples:

- Let $F : R^3 \to R^3$ be the projection of a vector v into the xy-plane that is, $F(x, y, z) = (x, y, 0)$ Clearly the image of F is the entire xy-plane—that is, points of the form $(x, y, 0)$. Moreover, the kernel of F is the z-axis—that is, points of the form $(0, 0, c)$.

Kernal and Image of Matrix Mappings

Let A by any $m \times n$ matrix over a field K viewed as a linear map $A : K^n \times K^m$. Then

1. $\operatorname{Im} A = \operatorname{colsp}(A)$ as consider usual basis e_i of K^n, $Ae_1, Ae_2, \ldots, Ae^n$ are respectively the columns of A.
2. Kernel of A consists of all vectors v for which $Av = 0$. This means that the kernel of A is the solution space of the homogeneous system $AX = 0$, called the null space of A.

If we have computed the dimension of column space, which is same as rank of the matrix, r. And hence the dimension of kernal of A is $n - r$.

Examples:

- Find a linear map $F : R^3 \to R^4$ whose image is spanned by $(1, 2, 0, -4)$ and $(2, 0, 1, -3)$.

  $u = (x + 2y, 2x, -y, -4x - 3y)$

$$\therefore A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 0 \\ 0 & -1 & -1 \\ -4 & -3 & -3 \end{bmatrix}$$

Also recall that A determines a linear map $A : R^3 \to R^4$ whose image is spanned by the columns of A. Thus, A satisfies the required condition

Rank and Nullity of a Linear Mapping

Let $F : V \to U$ be a linear mapping.

$\text{rank}(F) = \dim(\text{Im}\, F)$ and $\text{nullity}\ (F) = \dim(\text{Ker}\, F)$

Theorem 4.4 Let V be of finite dimension, and let $F : V \to U$ be linear. Then

$\dim V = \dim(\text{Ker}\, F) + \dim(\text{Im}\, F) = \text{nullity}\ (F) + \text{rank}(F)$

Singular and Nonsingular Linear Mappings, Isomorphisms

Let $F : V \to U$ be a linear mapping. Recall that $F(0) = 0$. F is said to be singular if the image of some nonzero vector v is 0. Thus, $F : V \to U$ is nonsingular if the zero vector 0 is the only vector whose image under F is 0 or, in other words, if $\text{Ker}\, F =$
0

Theorem 4.5: Let $F : V \to U$ be a nonsingular linear mapping. Then the image of any linearly independent set is linearly independent.

Theorem 4.6: A linear mapping $F : V \to U$ is one-to-one if and only if F is nonsingular. (easy to see)

Theorem 4.7: Suppose V has finite dimension and $dimV = dimU$. Suppose $F : V \to U$ is linear. Then F is an isomorphism if and only if F is nonsingular.

Operations with Linear Mappings

Let $F : V \to U$ and $G : V \to U$ be linear mappings over a field K. The sum $F + G$ and the scalar product kF, where $k \in K$, are defined to be the following mappings from V into U:

$$(F+G)(v) \equiv F(v)+G(v) \quad \text{and} \quad (kF)(v) \equiv kF(v)$$

It is easy to see that if F and G are linear, then $F+G$ and kF are also linear.

Theorem 4.8: Let V and U be vector spaces over a field K. Then the collection of all linear mappings from V into U with the above operations of addition and scalar multiplication forms a vector space over K.

TODO

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