Linear Equations Of Second Order With Variable Coeff.

Is an eqn of the form

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$

can be solved by following methods:

Change of the dependent variable when a part of the CF is known

Method for solving

Method for Finding one integral (soln) in CF by inspection i.e. one soln $u(x)$ of $(D^2 + P(x)D + Q(x))y = 0$

Condition Satisfied	one soln of CF
$a^2 + aP + Q = 0$	$u = e^{ax}$
$1 + P + Q = 0$	$u = e^x$
$1 - P + Q = 0$	$u = e^{-x}$
$m(m-1) + Pmx + Qx^2$	$u = x^m (m \geq 2)$
$P + Qx = 0$	$u = x$
$2 + 2Px + Qx^2$	$u = x^2$

Now assume the GS of given eqn is of the form $y = uv$ where u is obtained as above, now v can be obtained by solving:

$\frac{d^2v}{dx^2} + (P + \frac{2}{u}\frac{du}{dx})\frac{dv}{dx} = \frac{R(x)}{u}$

Examples:

- $xy'' - (2x-1)y' + (x-1)y = 0$

  $\rightarrow y'' - (2 - 1/x)y' + (1 - 1/x)y = 0$

  $\rightarrow u = e^x$

  $\rightarrow \frac{d^2v}{dx^2} + (-2 + 1/x + 2e^{-x}e^x)\frac{dv}{dx} = 0$

$\rightarrow \frac{dt}{dx} + t/x = 0$

$\rightarrow log(t) = -log(x) + c$

$\rightarrow tx = c_1$

$\rightarrow v = c_1 log(x) + c_2$

:::warning Warning Here from "part of soln" means that $u(x)$ is a soln of the corresponding homogeneous eqn. Thus if in general we are given $y = u(x)v(x)$ where $u(x)$ is given, we **cannot** apply this method unless corresponding homogeneous eqn turns out to be zero when substituting $y = u(x)$ in it. :::

Changing the dependent variable and removal of the first order derivative

i.e. Reduce $y'' + P(x)y' + Q(x)y = R(x)$ to the form $\frac{d^2v}{dx^2} + Iv = S$ which is called as the **normal form** of the given eqn.

Method for solving

1. Write the given eqn in the standard form $y'' + P(x)y' + Q(x)y = R(x)$
2. To remove the first order derivative we choose $u = e^{\frac{-1}{2}\int Pdx}$
3. Assume the GS is $y = uv$, where v is given by the normal form $\frac{d^2v}{dx^2} + Iv = S$ where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$ and $S = \frac{R}{u}$

Examples:

- $y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2} sin(2x)$

  $\Rightarrow u = e^{x^2}$

  $\Rightarrow I = 4x^2 - 1 - \frac{1}{4}16x^2 - \frac{1}{2}(-4) = 1$

  $\Rightarrow S = -3sin(2x)$

  $\Rightarrow \frac{d^2v}{dx^2} + v = -3sin(2x)$

  $\Rightarrow PI = -3sin(2x)/(D^2 + 1) = sin(2x)$

  $\Rightarrow CF = c_1 cos(x) + c_2 sin(x)$

  $\Rightarrow y = e^{x^2}(CF + PI)$

- Make use of the transformation $y(x) = v(x)sec(x)$ to obtain the soln of $y'' - 2tanxy' + 5y = 0$, where $y(0) = 0, y'(0) = \sqrt{6}$

  Here note that $e^{\frac{-1}{2}\int -2tanxdx} = e^{-log(cosx)} = secx$ which is our given u. Thus we can apply our method.

Soln by changing independent variable

Let $z = f(x)$ then after a bit of work,

$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ where:

$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{(\frac{dz}{dx})^2}$

$Q_1 = \frac{Q}{(\frac{dz}{dx})^2}$

$R_1 = \frac{R}{(\frac{dz}{dx})^2}$

Case 1

Choose z to make $P_1 = 0$ i.e., $\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0$

$\rightarrow z = \int e^{-\int Pdx}dx$

Now the eqn reduces to $\frac{d^2y}{dz^2} + Q_1 y = R_1$

which can be easily solved if Q_1 turns out to be a constant or a constant multiplied by $\frac{1}{z^2}$

Case 2

Choose z such that $Q_1 = a^2$

$\rightarrow a\int dz = \int \sqrt{\pm Q}dx$

Take appropriate sign to make expression under radical positive.

Now the eqn reduces to $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + a^2y = R_1$

which can be easily solved provided P_1 comes out to be a constant.

Examples:

- $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3sin(x^2)$

  $\Rightarrow \frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2sin(x^2)$

  $\Rightarrow z = \int e^{-\int Pdx}dx = x^2/2$

  $\Rightarrow Q_1 = -4, R_1 = 8sin(x^2) = 8sin(2z)$

  $\Rightarrow PI = \frac{8sin(2z)}{D^2-4} = -sin(2z) = -sin(x^2)$

  $\Rightarrow y = c_1e^{x^2} + c_2e^{-x^2} - sin(x^2)$

- Transform the DE $xy'' - y' + 4x^3y = x^5$ into z as independent variable where $z = x^2$ and solve it.

  $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 2x\frac{dy}{dz}$

$\frac{d^2y}{dx^2} = 2\frac{dy}{dz} + 2x\frac{d^2y}{dz^2}\frac{dz}{dx}$

$= 2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2}$

Now using this, it will reduce in a good solvable form.

Method of variation of parameters

1. Write the given equation in the standard form $y'' + Py' + Qy = R$.
2. Find the soln of corresponding homogeneous eqn. Let it be $y_c = c_1u(x) + c_2v(x)$ by using methods discussed before.
3. Let the PI of the given eqn be $y_p = A(x)u + B(x)v$ where $A = -\int \frac{vR}{W(u,v)}dx$ and $B = \int \frac{uR}{W(u,v)}dx$ are functions of x.
4. GS of the given eqn is $y = y_c + y_p$

Examples:

- $((x-1)D^2 - xD + 1)y = (x-1)^2$

  $\rightarrow (D^2 - \frac{x}{x-1}D + \frac{1}{x-1})y = x - 1$

  It can be seen by inspection that $y = e^x$ and $y = x$ are soln of corresponding homogeneous eqn. Therefore

  $\rightarrow y_c = c_1e^x + c_2x$

  And after some calculation

  $y_p = -(1 + x + x^2)$

  $GS = y_c + y_p$