Directed Graphs

Algorithms & Data Structures
ITCS 6114/8114

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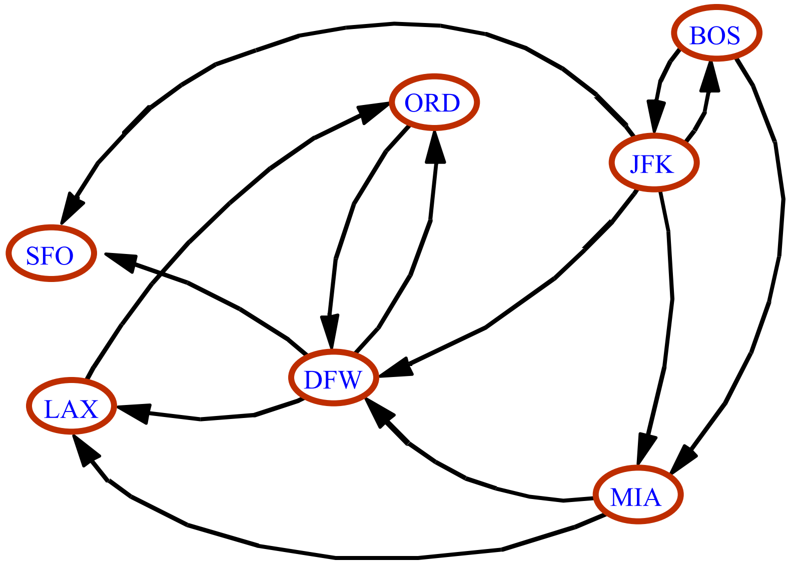
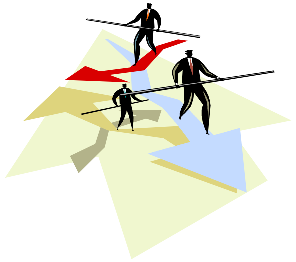
Directed Graphs



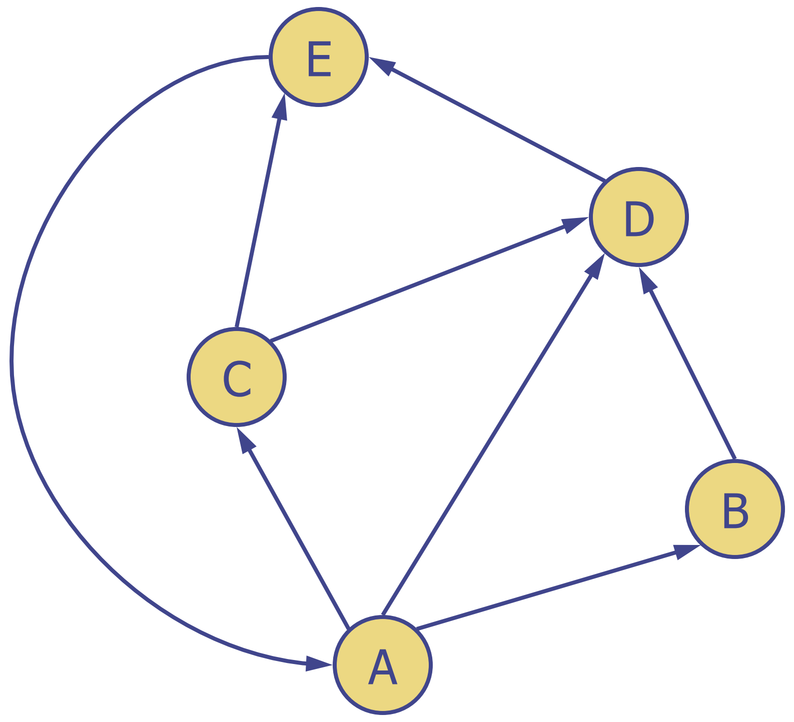
Outline and Reading (§6.4)



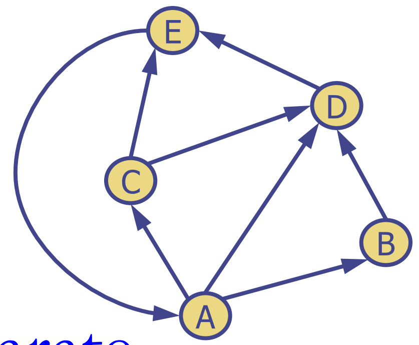
- Reachability (§6.4.1)
  - Directed DFS
  - Strong connectivity
  - Connected Components [Cormen]
- Transitive closure (§6.4.2)
  - The Floyd-Warshall Algorithm
- Directed Acyclic Graphs (DAG's) [Cormen]
  - Topological Sorting

Digraphs

- A **digraph** is a graph whose edges are all directed
  - Short for "directed graph"
- Applications
  - one-way streets
  - flights
  - task scheduling



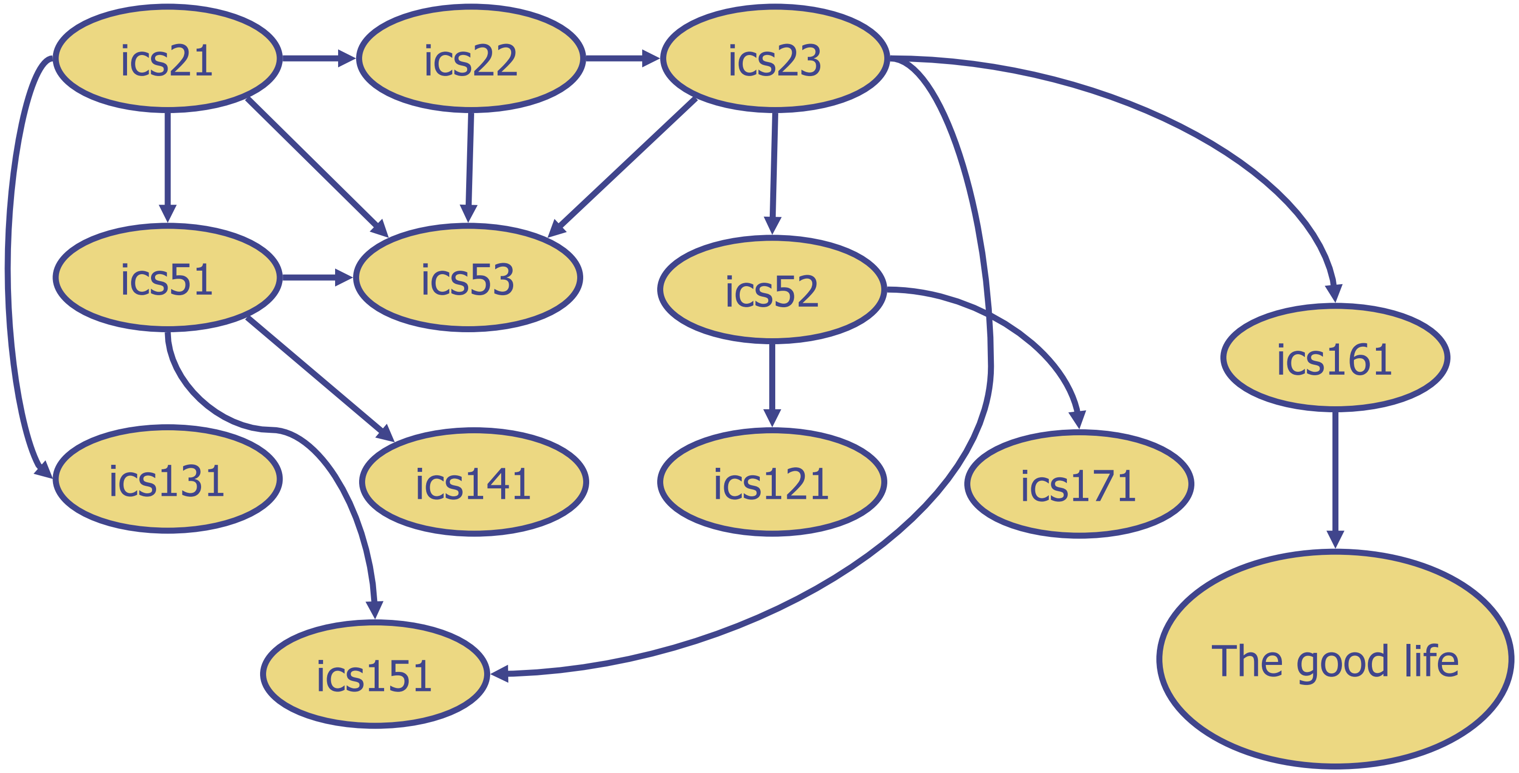
Digraph Properties



- Each edge goes in one direction:
  - edge (a,b) goes from a to b, but not b to a.
- If G is simple, $m \leq n(n-1)$.
- If we keep in-edges and out-edges in separate adjacency lists, we can perform listing of the sets of in-edges and out-edges in time proportional to their size.

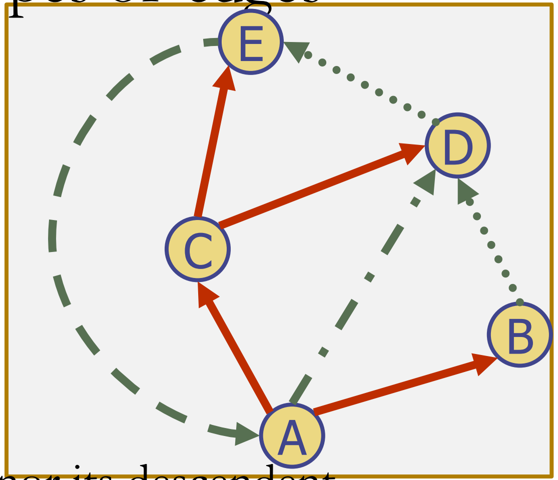
Digraph Application

- Scheduling: edge (a, b) means task a must be completed before b can be started



Directed DFS

- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - Tree/discovery edges
  - back edges
    - Connect a vertex to an ancestor in the DFS tree
  - forward edges
    - Connect a vertex to a descendent in the DFS tree
  - cross edges
    - Connect a vertex to a vertex that is neither its ancestor nor its descendent



- A directed DFS starting at a vertex ***s*** determines the vertices reachable from ***s***

Reachability

- Theorem 6.20:

  Let $\vec{G}$ be a digraph. Depth-first search on $\vec{G}$ starting at a vertex s visits all the vertices of $\vec{G}$ that are reachable from s.

  Also, the DFS tree contains directed paths from s to every vertex reachable from s.

Reachability

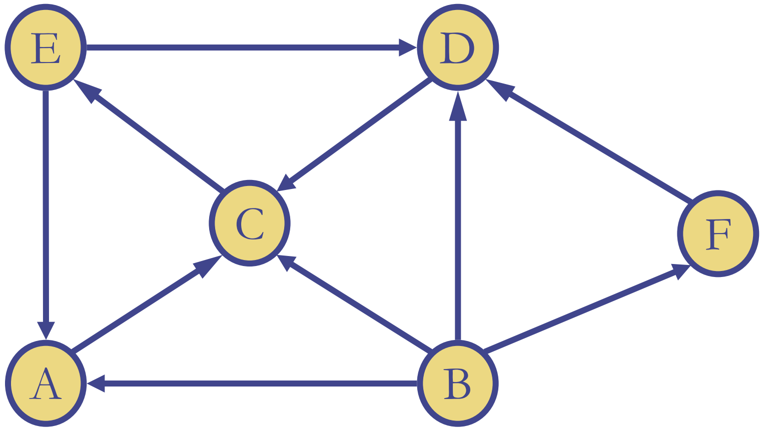
- Theorem 6.21:

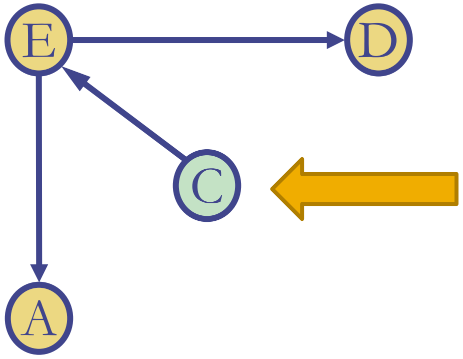
  Let be a $\vec{G}$ digraph with n vertices and m edges. The following problems can be solved by an algorithm that traverses n times using DFS, runs in $O(n(n+m))$ time, and uses $O(n)$ auxiliary space:

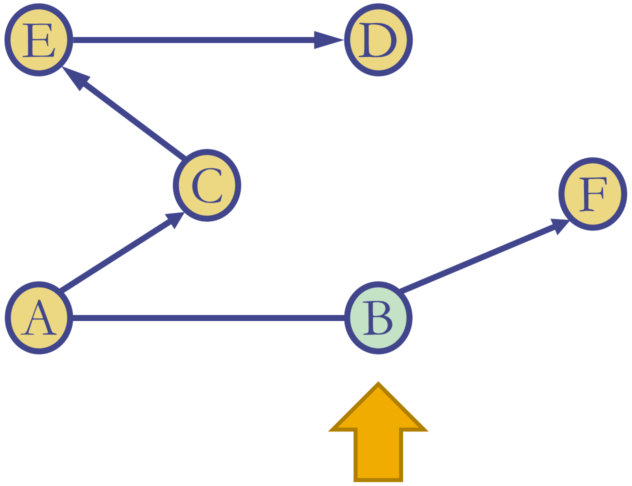
  - Computing, for each vertex v of $\vec{G}$, the subgraph reachable from v
  - Testing whether $\vec{G}$ is strongly connected
  - Computing the transitive closure $\vec{G}^*$ of $\vec{G}$.

Reachability

- DFS tree rooted at v: vertices reachable from v via directed paths

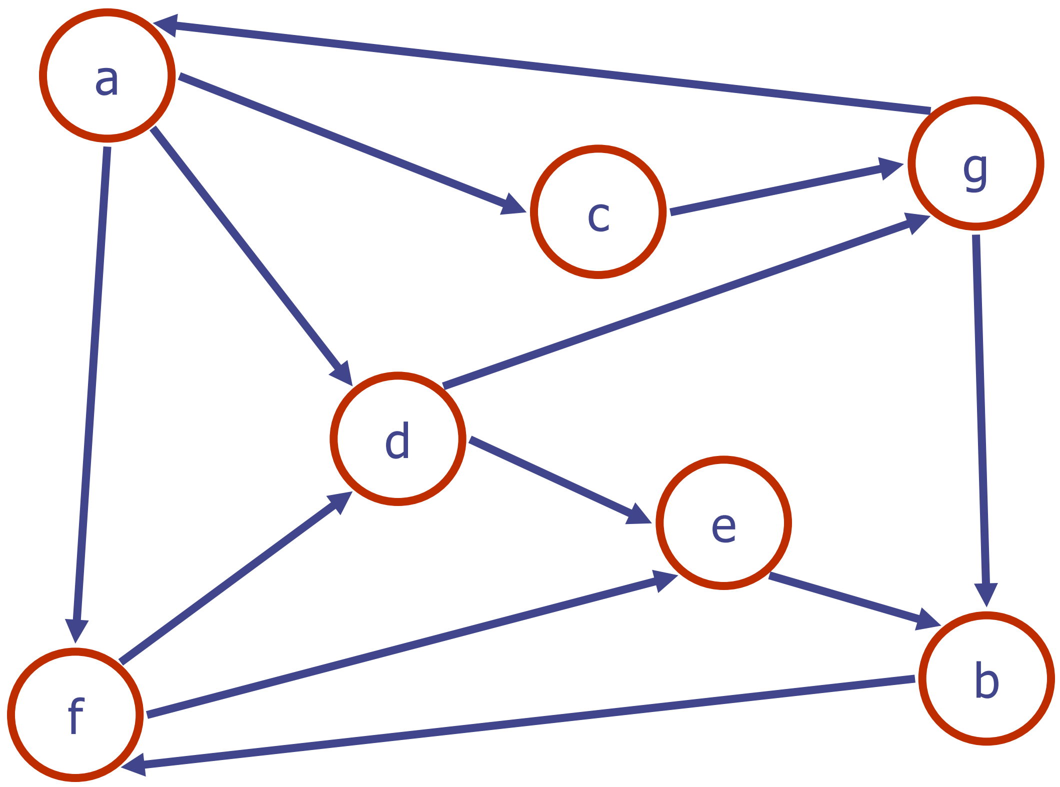




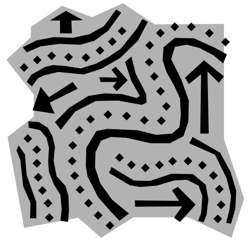


Strong Connectivity

- Each vertex can reach all other vertices



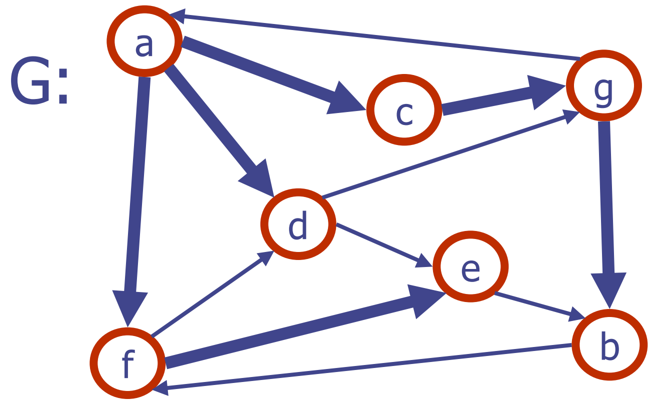
Strong Connectivity Algorithm

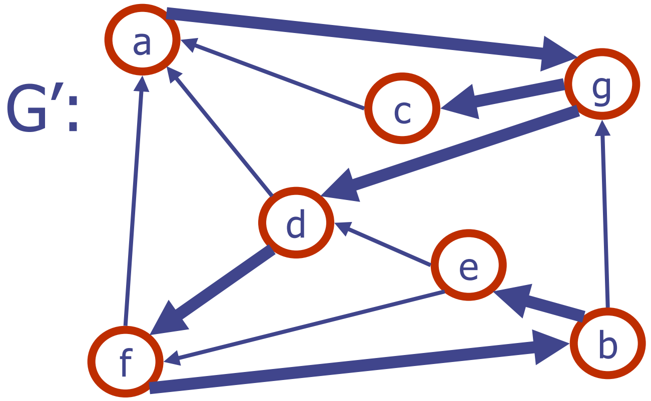


- Pick a vertex v in G.
- Perform a DFS from v in G.
  - If there's a w not visited, print "no".
- Let G' be G with edges reversed.
- Perform a DFS from v in G'.
  - If there's a w not visited, print "no".
  - else, print "yes".

If every vertex of *G'* is visited by this second DFS, then the graph is strongly connected, for each of the vertices visited in this DFS can reach s.

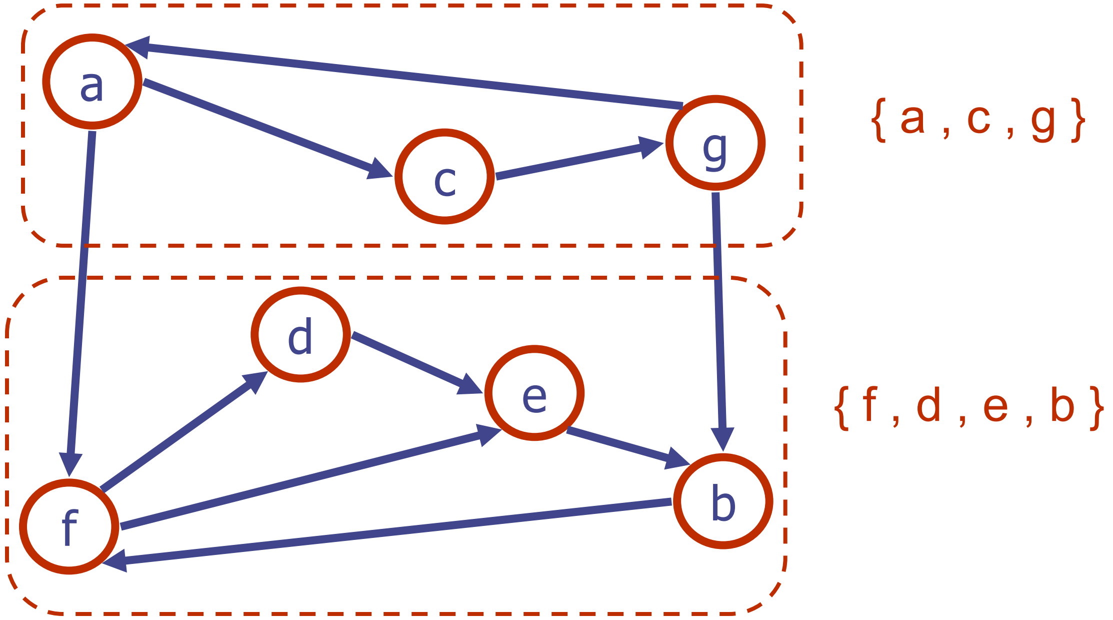
- Running time: $O(n + m)$.



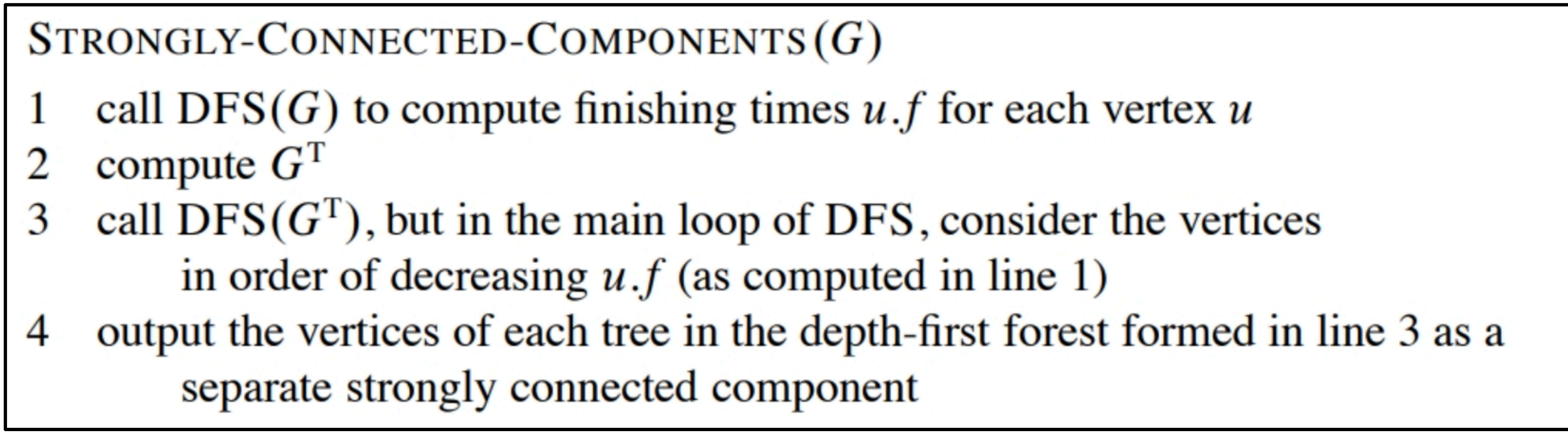


Strongly Connected Components [22.5]

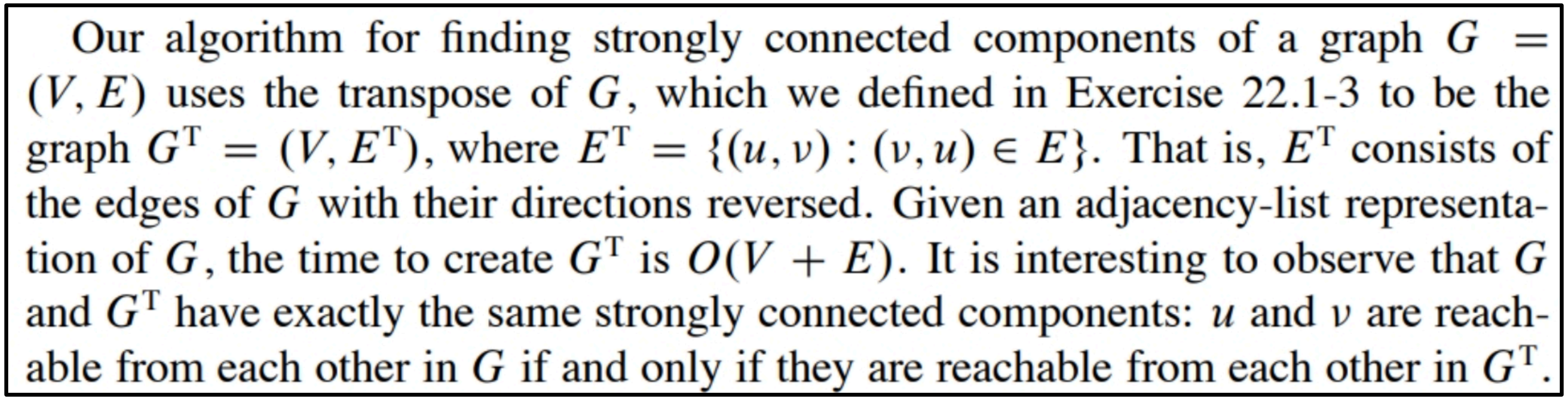
- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in $O(n+m)$ time using DFS, but is more complicated (similar to biconnectivity).



Strongly connected components



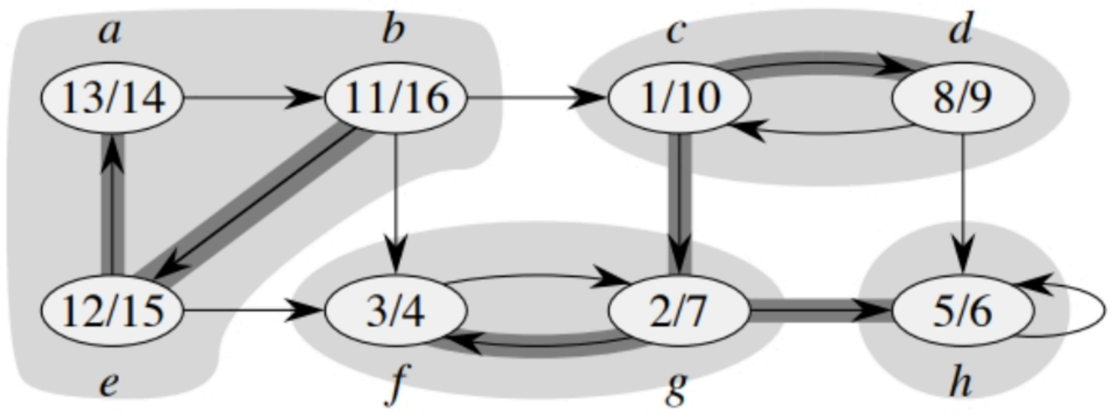
```
STRONGLY-CONNECTED-COMPONENTS(G)
1  call DFS(G) to compute finishing times u.f for each vertex u
2  compute G^T
3  call DFS(G^T), but in the main loop of DFS, consider the vertices
       in order of decreasing u.f (as computed in line 1)
4  output the vertices of each tree in the depth-first forest formed in line 3 as a
       separate strongly connected component
```



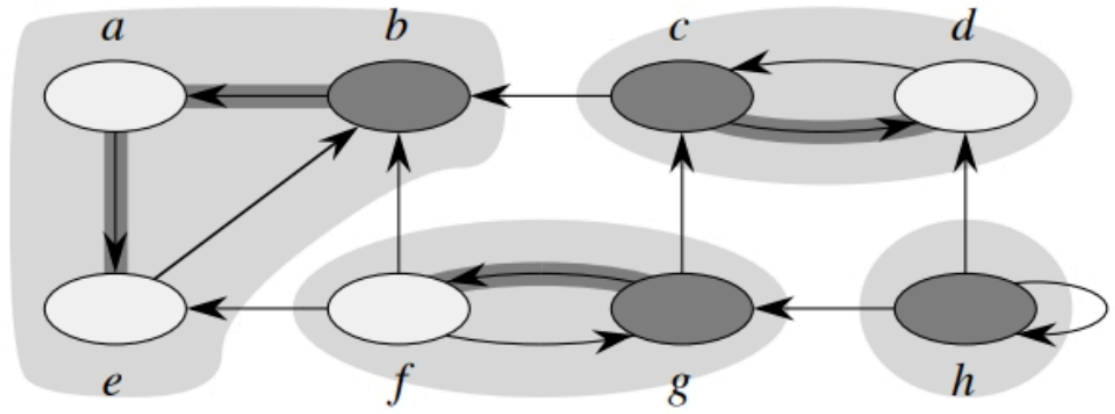
Our algorithm for finding strongly connected components of a graph $G = (V, E)$ uses the transpose of G, which we defined in Exercise 22.1-3 to be the graph $G^{\mathrm{T}} = (V, E^{\mathrm{T}})$, where $E^{\mathrm{T}} = \{(u, v) : (v, u) \in E\}$. That is, E^{T} consists of the edges of G with their directions reversed. Given an adjacency-list representation of G, the time to create G^{T} is $O(V + E)$. It is interesting to observe that G and G^{T} have exactly the same strongly connected components: u and v are reachable from each other in G if and only if they are reachable from each other in G^{T}.

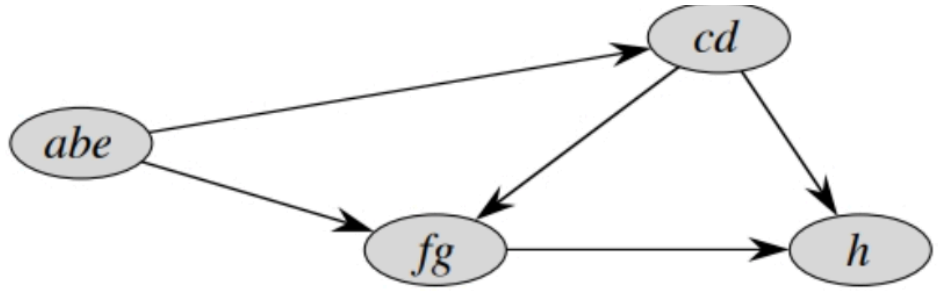
Strongly connected components

(a)



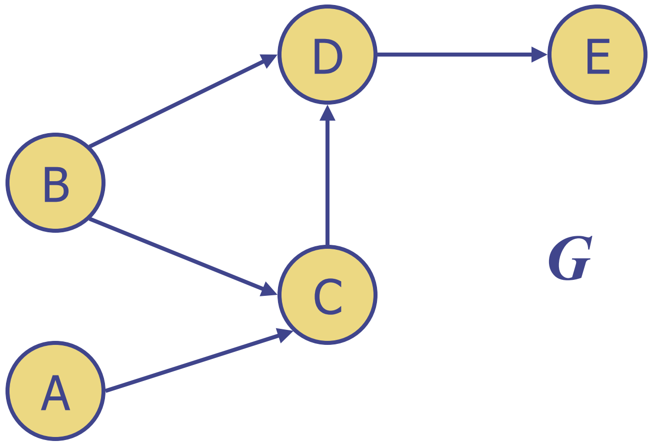
(b)

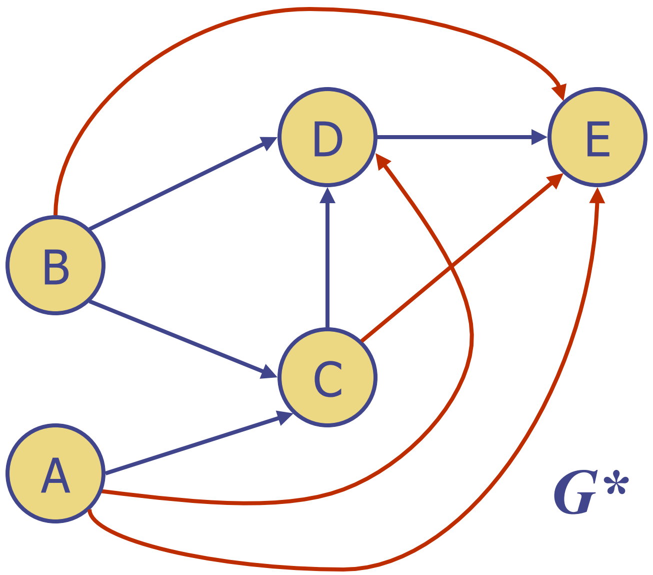




Transitive Closure

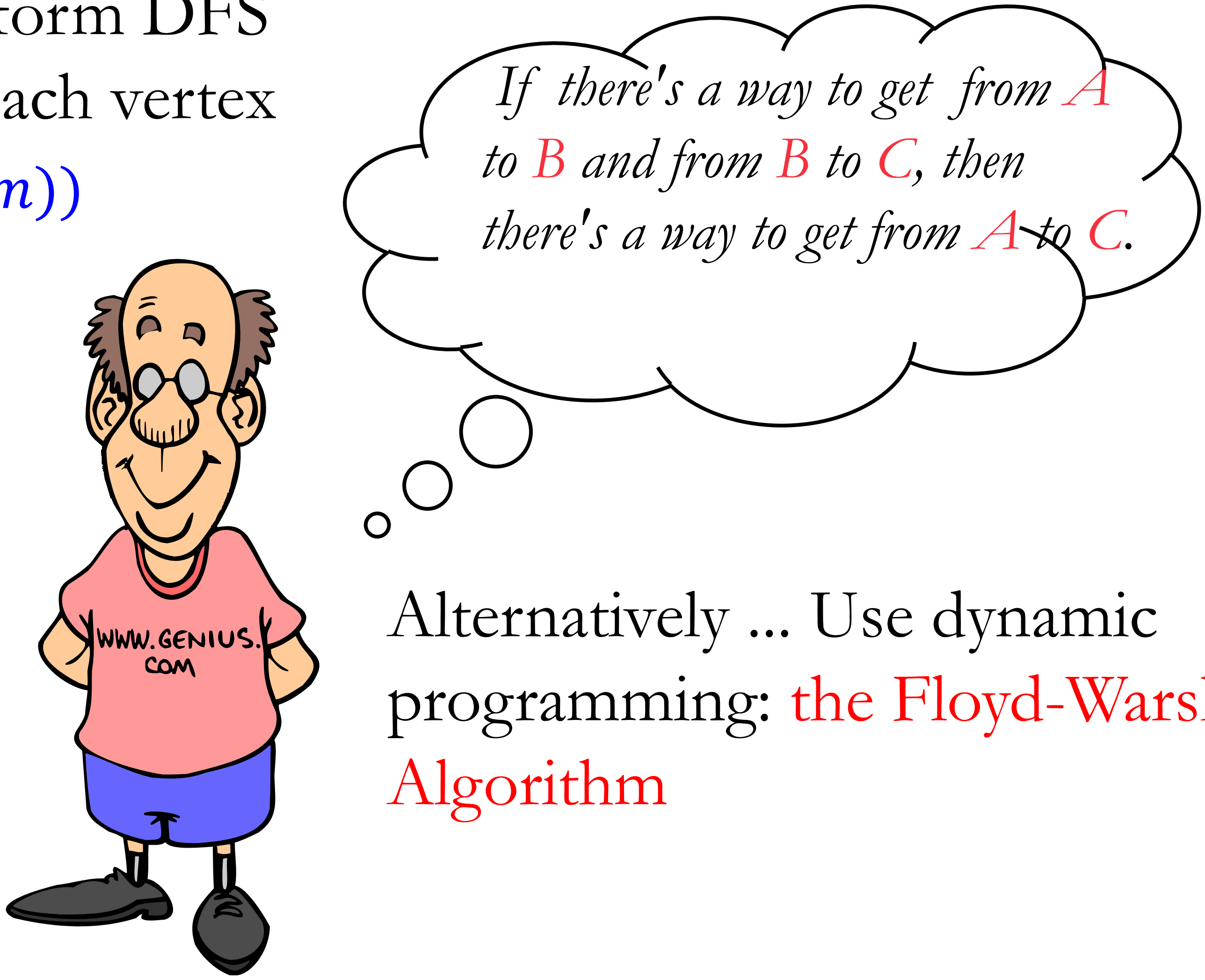
- Given a digraph ***G***, the transitive closure of ***G*** is the digraph ***G**** such that
  - ***G**** has the same vertices as ***G***
  - if ***G*** has a directed path from ***u*** to ***v*** ($u \neq v$), ***G**** has a directed edge from ***u*** to ***v***
- The transitive closure provides reachability information about a digraph





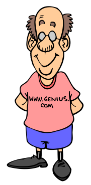
Computing the Transitive Closure

- We can perform DFS starting at each vertex
  - $O(n(n+m))$

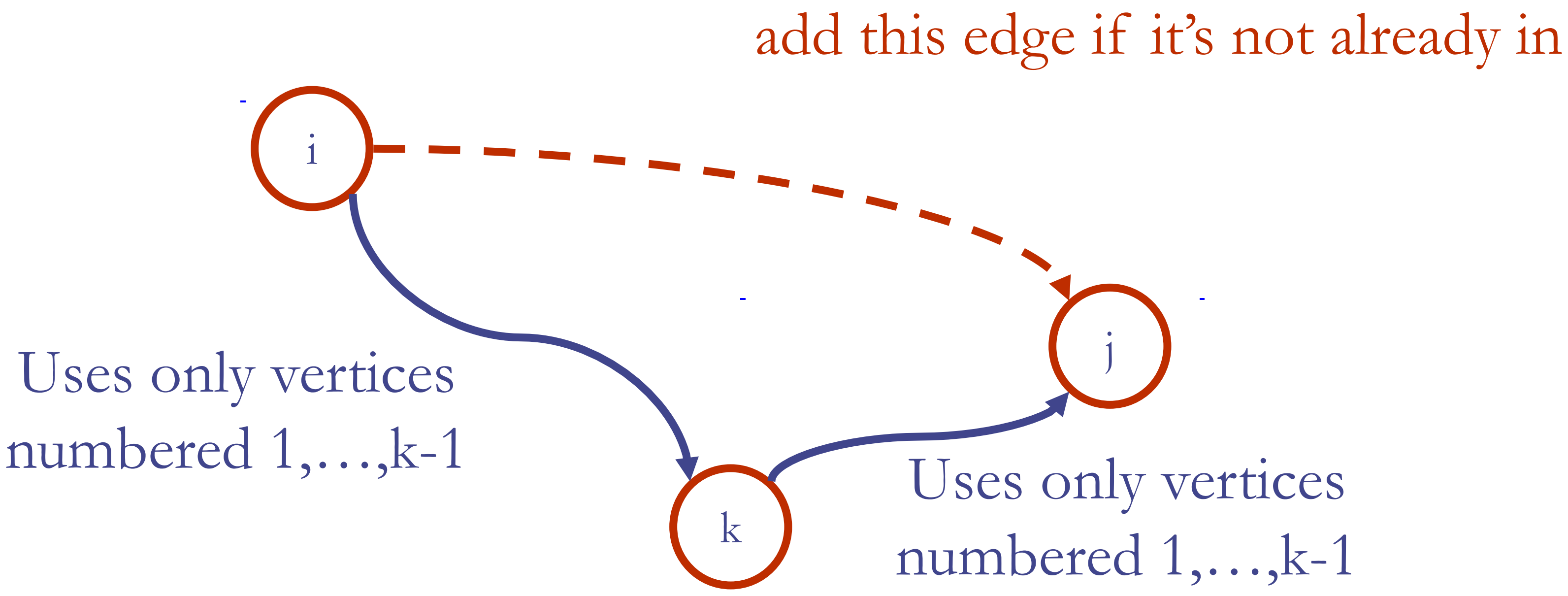

If there's a way to get from A to B and from B to C, then there's a way to get from A to C.
WWW.GENIUS.COM

Alternatively ... Use dynamic programming: the Floyd-Warshall Algorithm

Floyd-Warshall Transitive Closure



- Number the vertices $1, 2, \ldots, n$.
- Consider paths that use only vertices numbered 1, 2, …, k, as intermediate vertices:


add this edge if it's not already in
i
j
k
Uses only vertices numbered 1,…,k-1
Uses only vertices numbered 1,…,k-1

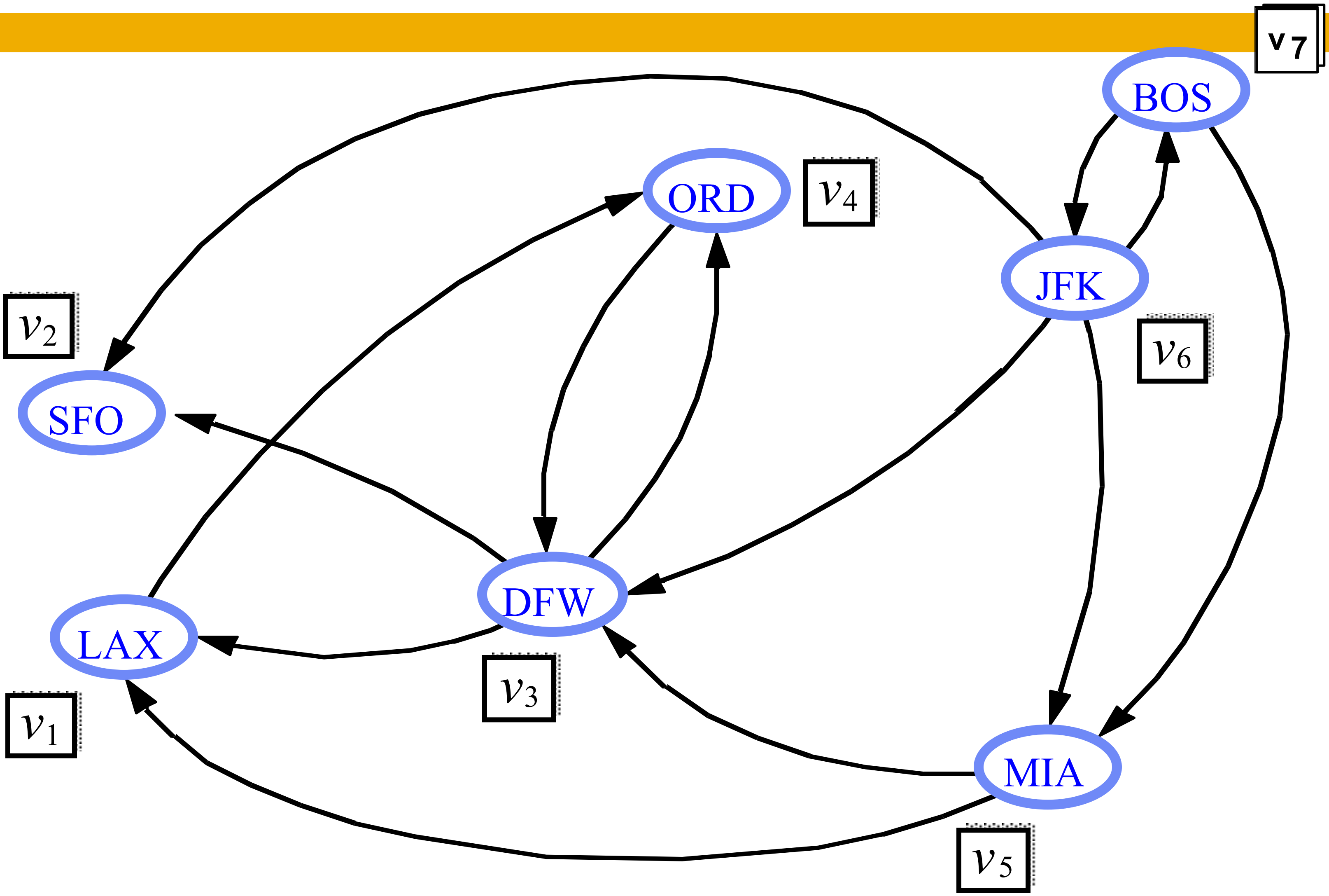
Floyd-Warshall's Algorithm

- Floyd-Warshall's algorithm numbers the vertices of G as $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{\boldsymbol{n}}$ and computes a series of digraphs $\boldsymbol{G}_0, \dots, \boldsymbol{G}_{\boldsymbol{n}}$
  - $\boldsymbol{G}_0 = \boldsymbol{G}$
  - $\boldsymbol{G}_{\boldsymbol{k}}$ has a directed edge $(\boldsymbol{v}_{\boldsymbol{i}}, \boldsymbol{v}_{\boldsymbol{j}})$ if $\boldsymbol{G}$ has a directed path from $\boldsymbol{v}_{\boldsymbol{i}}$ to $\boldsymbol{v}_{\boldsymbol{j}}$ with intermediate vertices in the set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{\boldsymbol{k}}\}$
- We have that $\boldsymbol{G}_{\boldsymbol{n}} = G^*$
- In phase k, digraph G_k is computed from G_{k-1}
  - Running time: $O(n^3)$, assuming areAdjacent is $O(1)$ (e.g., adjacency matrix)

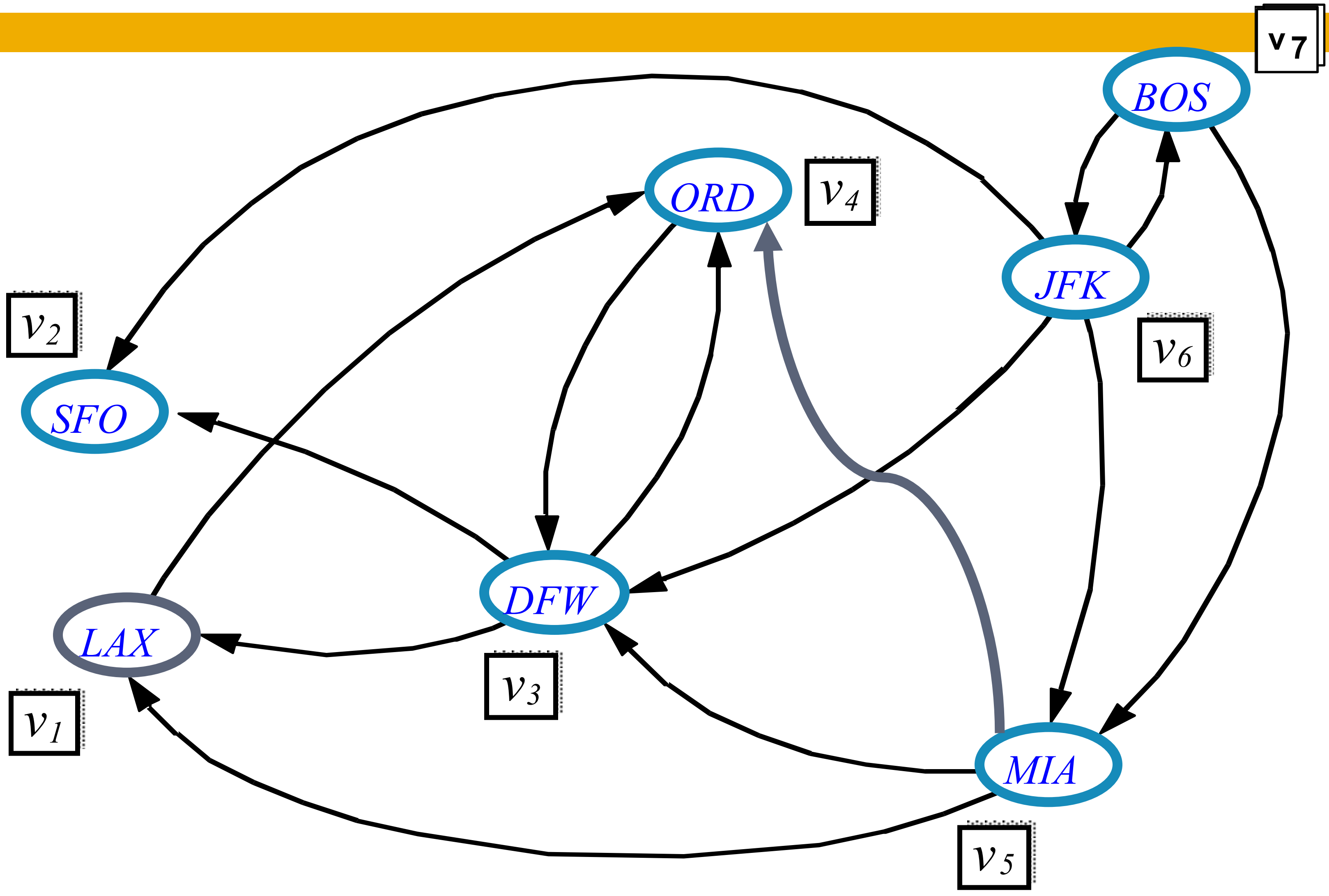
Floyd-Warshall's Algorithm

```
Algorithm FloydWarshall(G)
  Input digraph G
  Output transitive closure G* of G
  let $v_1, \dots, v_n$ be an arbitrary numbering of the vertices of G
  $G_0 \leftarrow G$
  for k ← 1 to n do
    $G_k \leftarrow G_{k-1}$
    for i ← 1 to n, $(i \neq k)$ do
      for j ← 1 to n, $(j \neq i, k)$ do
        if $G_{k-1}$.areAdjacent($v_i, v_k$) $\wedge$ $G_{k-1}$.areAdjacent($v_k, v_j$)
            if !$G_k$.areAdjacent($v_i$, $v_j$)
              $G_k$.insertDirectedEdge($v_i, v_j, k$)
    return $G_n$
```

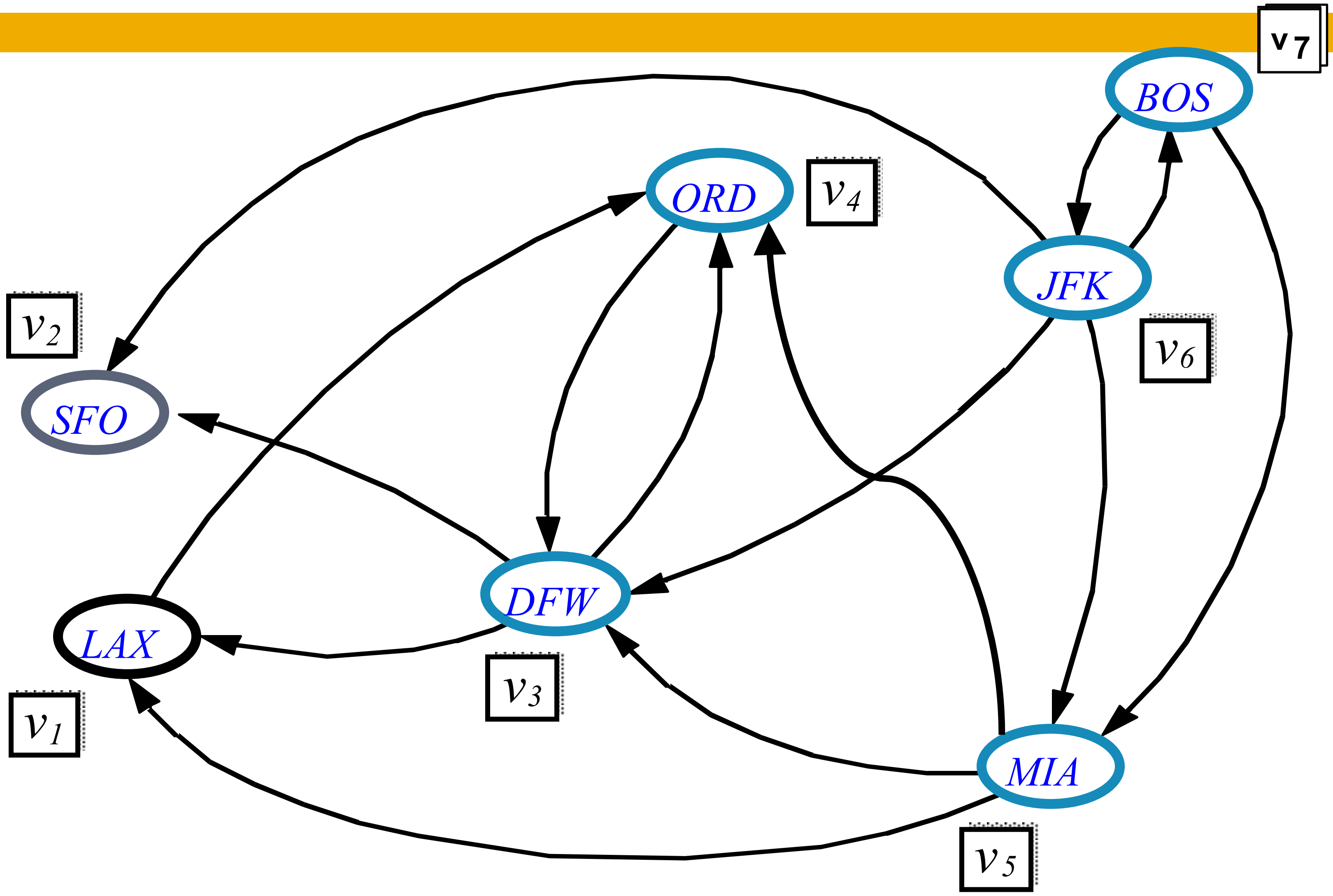
Floyd-Warshall Example



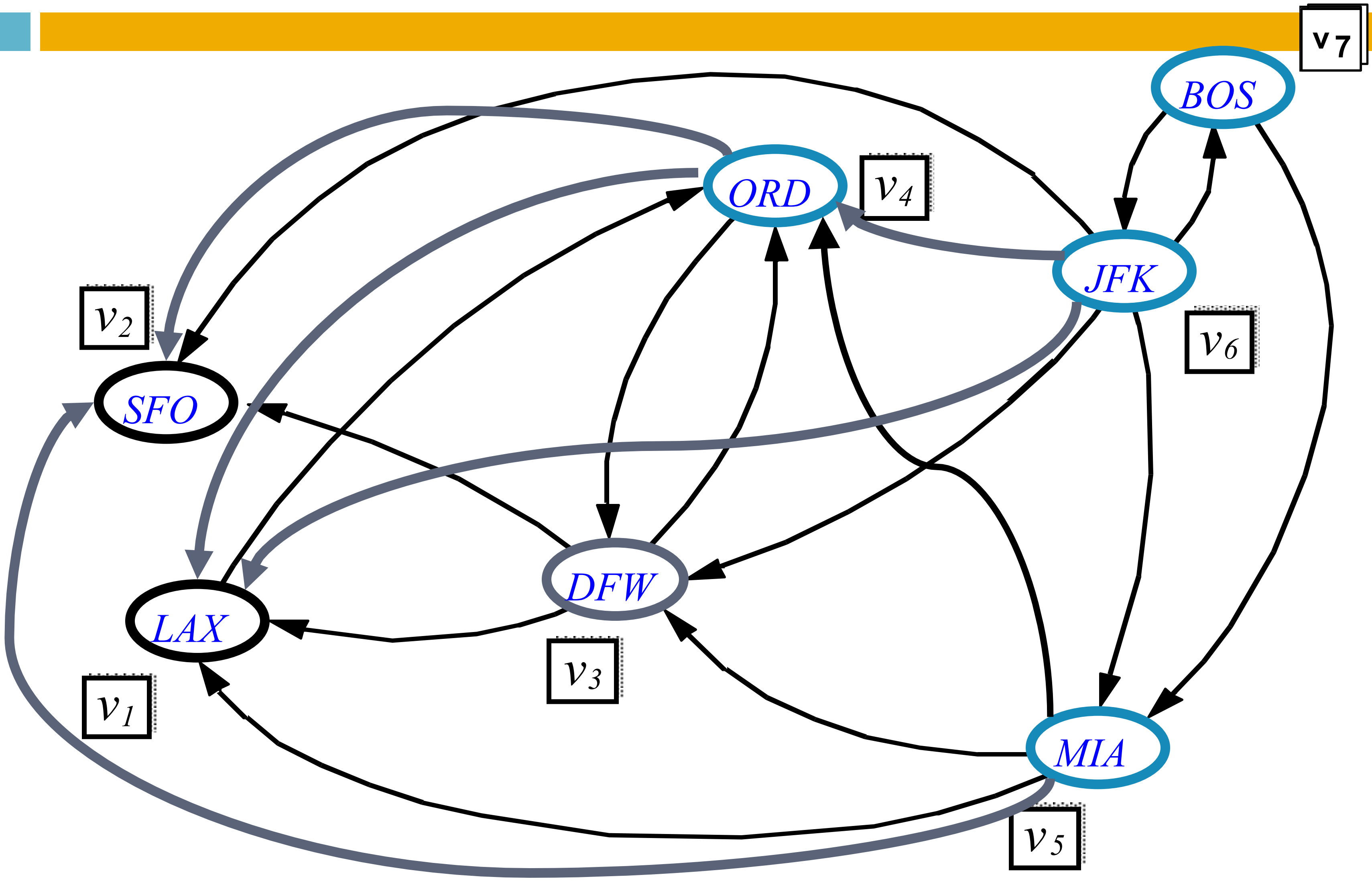
Floyd-Warshall, Iteration 1



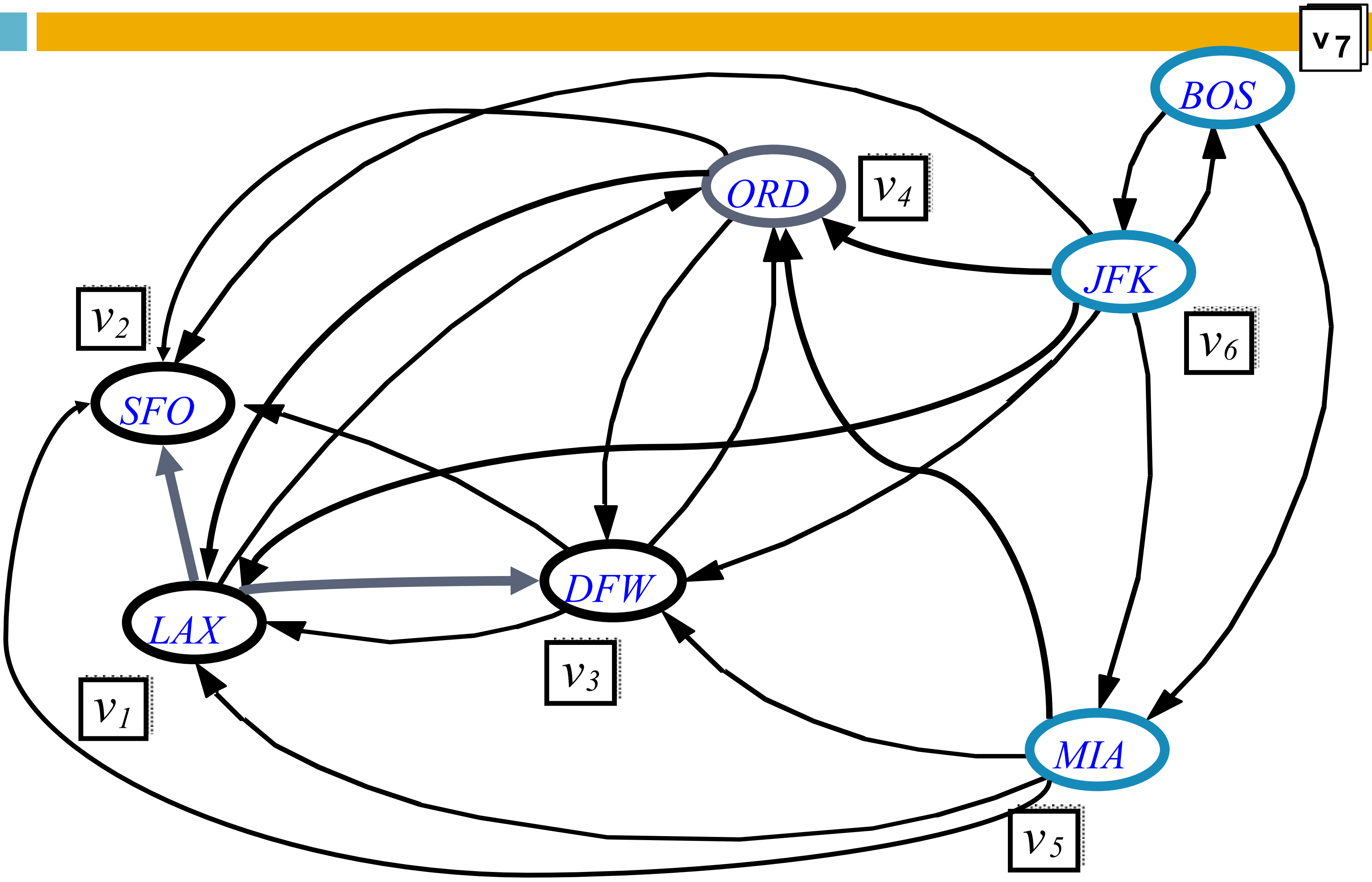
Floyd-Warshall, Iteration 2



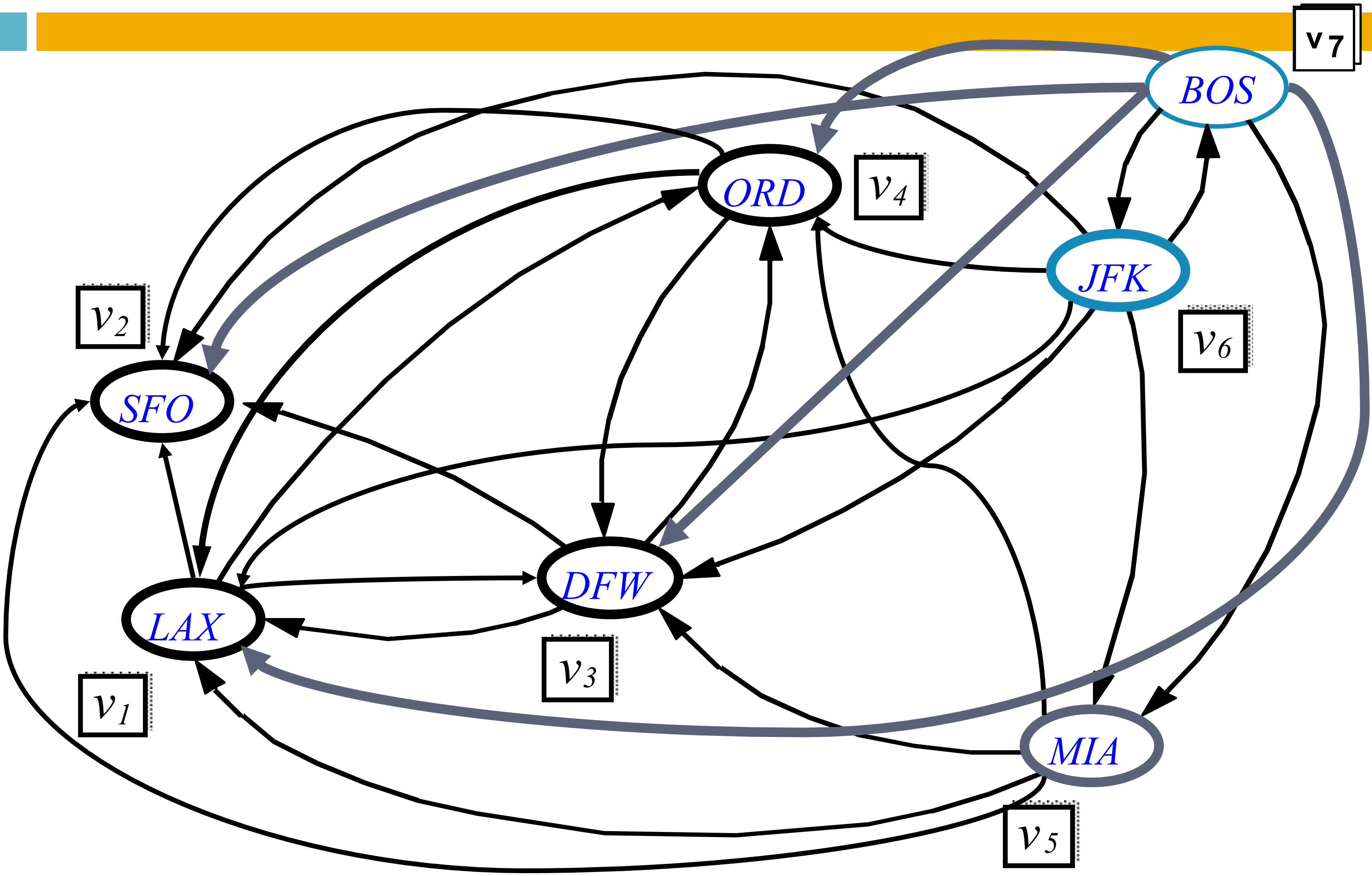
Floyd-Warshall, Iteration 3



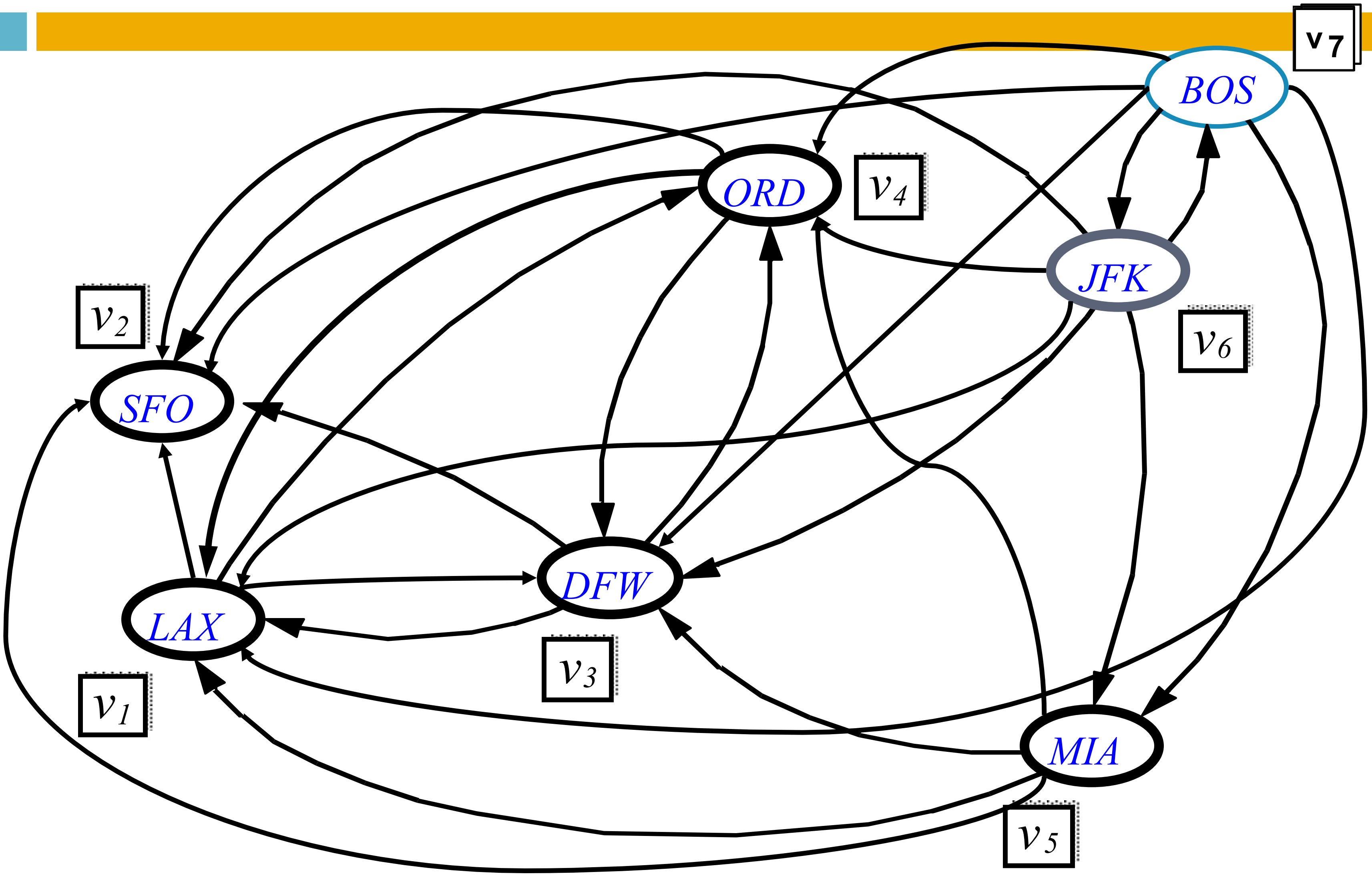
Floyd-Warshall, Iteration 4



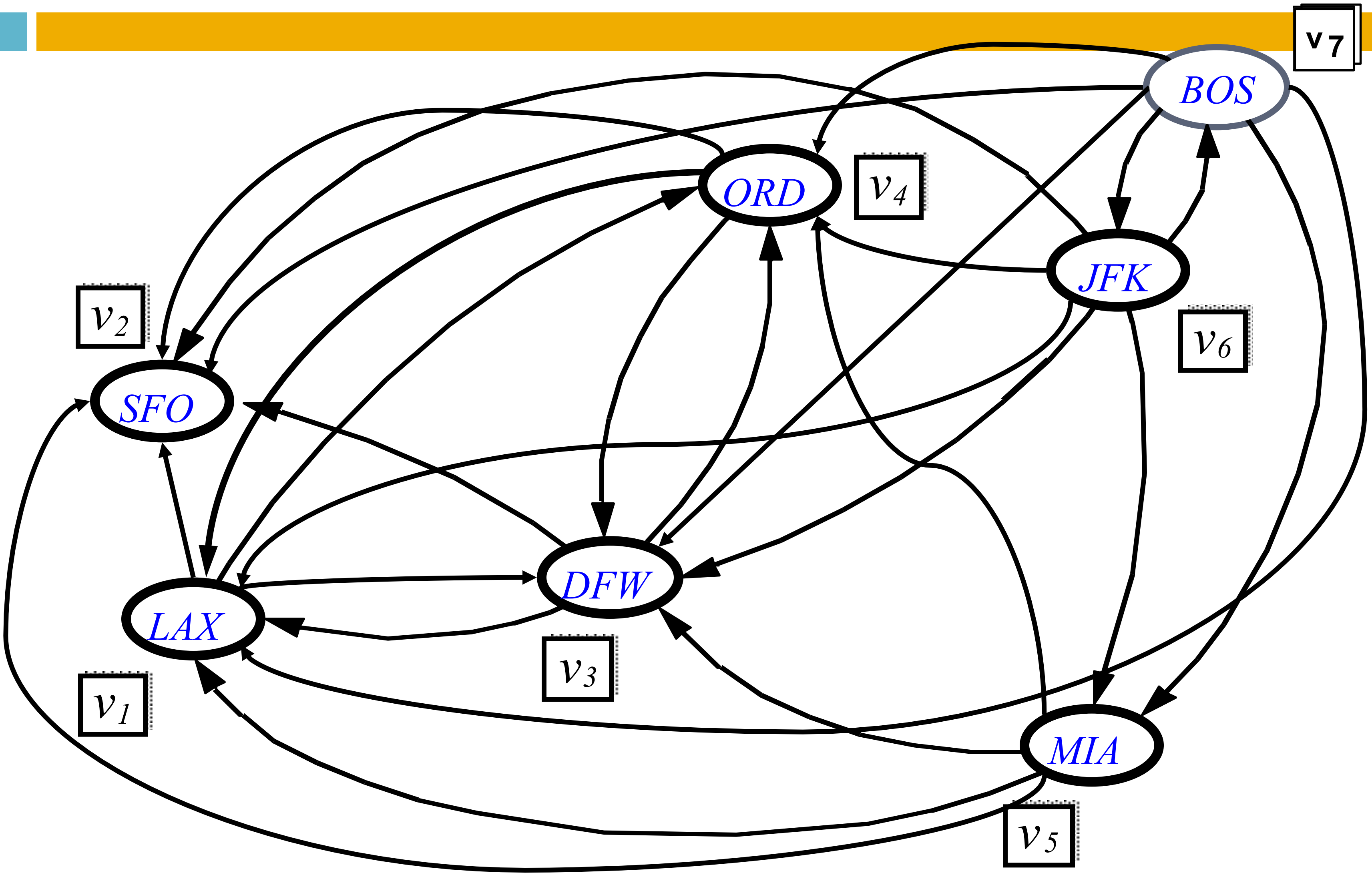
Floyd-Warshall, Iteration 5



Floyd-Warshall, Iteration 6



Floyd-Warshall, Conclusion



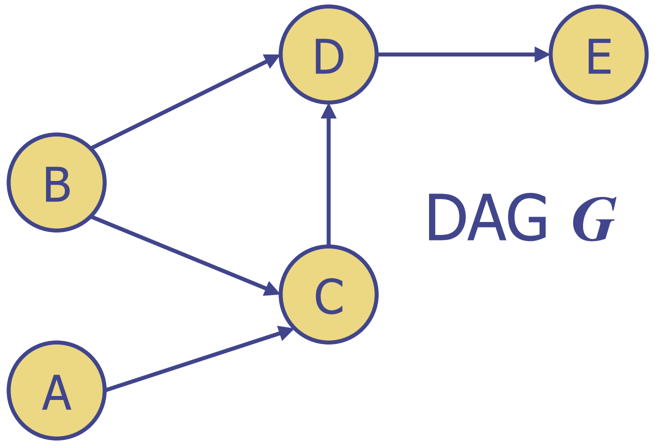
Topological Sort [22.4]

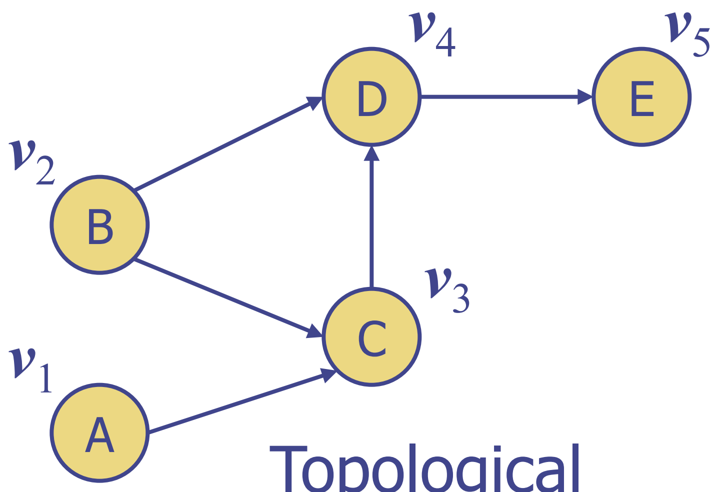
- A topological sort of a dag G = (V, E) is a linear ordering of all its vertices such that if G contains an edge (u, v), then u appears before v in the ordering.
- If the graph contains a cycle, then no linear ordering is possible.
- We can view a topological sort of a graph as an ordering of its vertices along a horizontal line so that all directed edges go from left to right. T

DAGs and Topological Ordering

- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering $v_1, \ldots, v_n$ of the vertices such that for every edge (v_i, v_j), we have $i < j$
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints

Theorem: A digraph admits a topological ordering if and only if it is a DAG

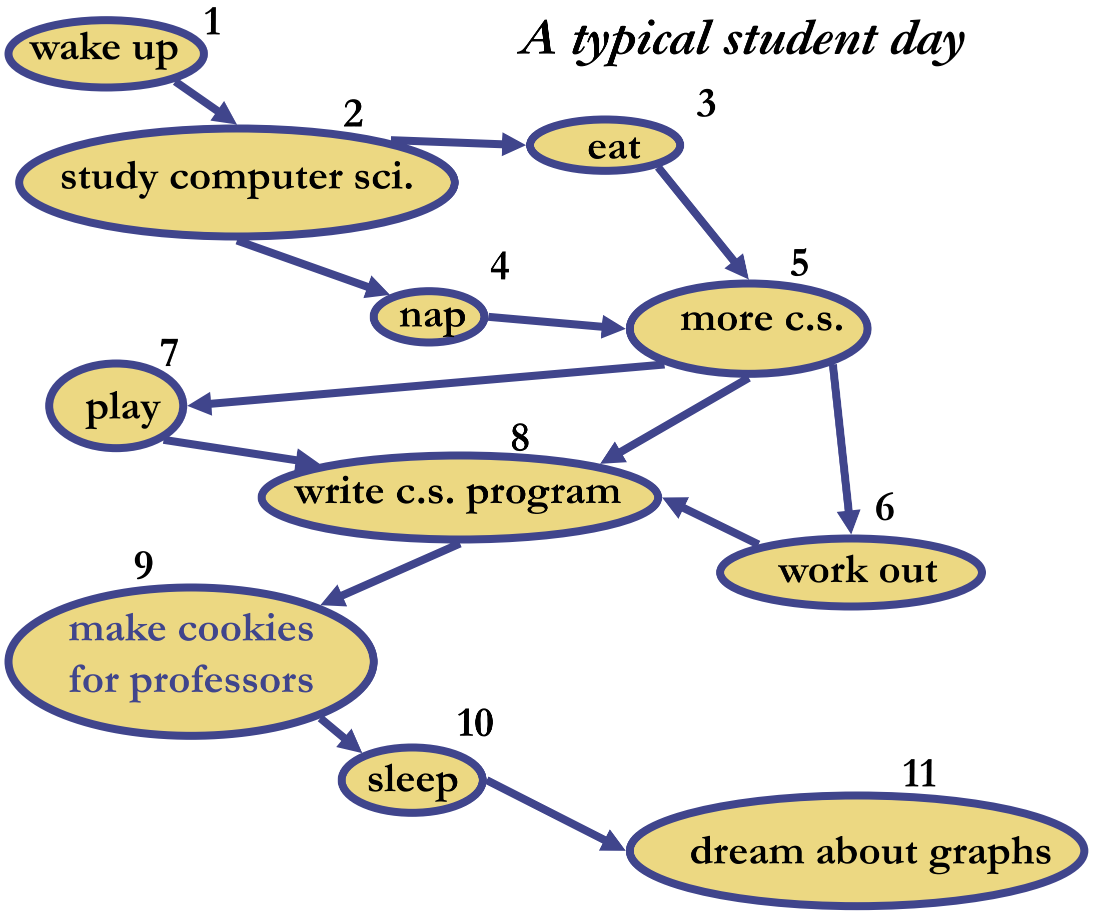




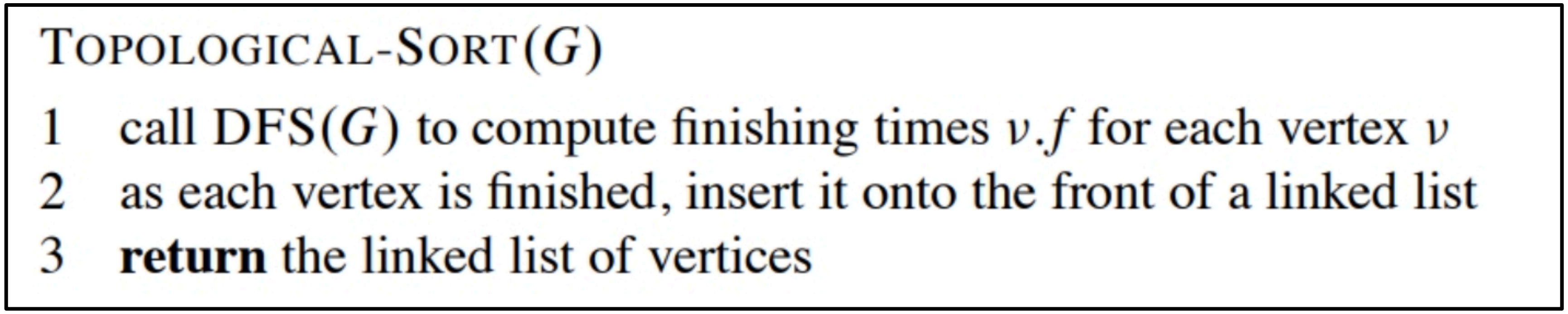
Topological ordering of G

Topological Sorting

- Number vertices, so that (u, v) in E implies $u < v$

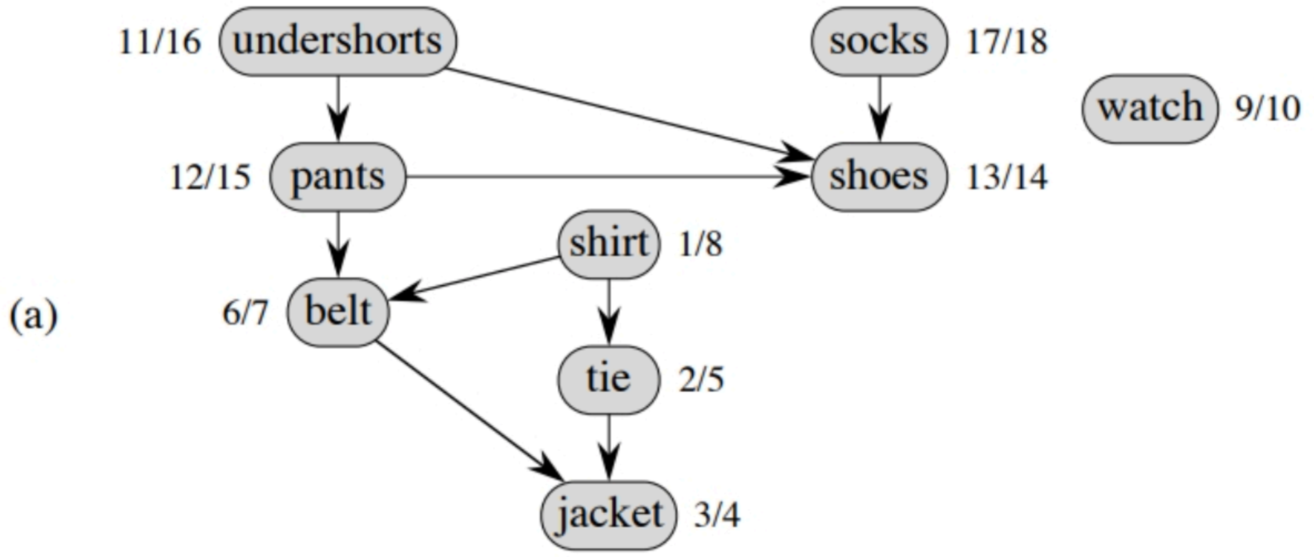


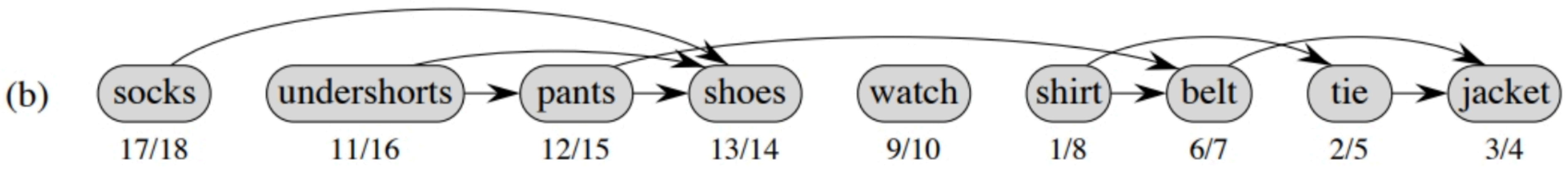
Topological Sorting



```
TOPOLOGICAL-SORT(G)
1  call DFS(G) to compute finishing times v.f for each vertex v
2  as each vertex is finished, insert it onto the front of a linked list
3  return the linked list of vertices
```

Figure 22.7 **(a)** Professor Bumstead topologically sorts his clothing when getting dressed. Each directed edge (u, v) means that garment u must be put on before garment v. The discovery and finishing times from a depth-first search are shown next to each vertex. **(b)** The same graph shown topologically sorted, with its vertices arranged from left to right in order of decreasing finishing time. All directed edges go from left to right.





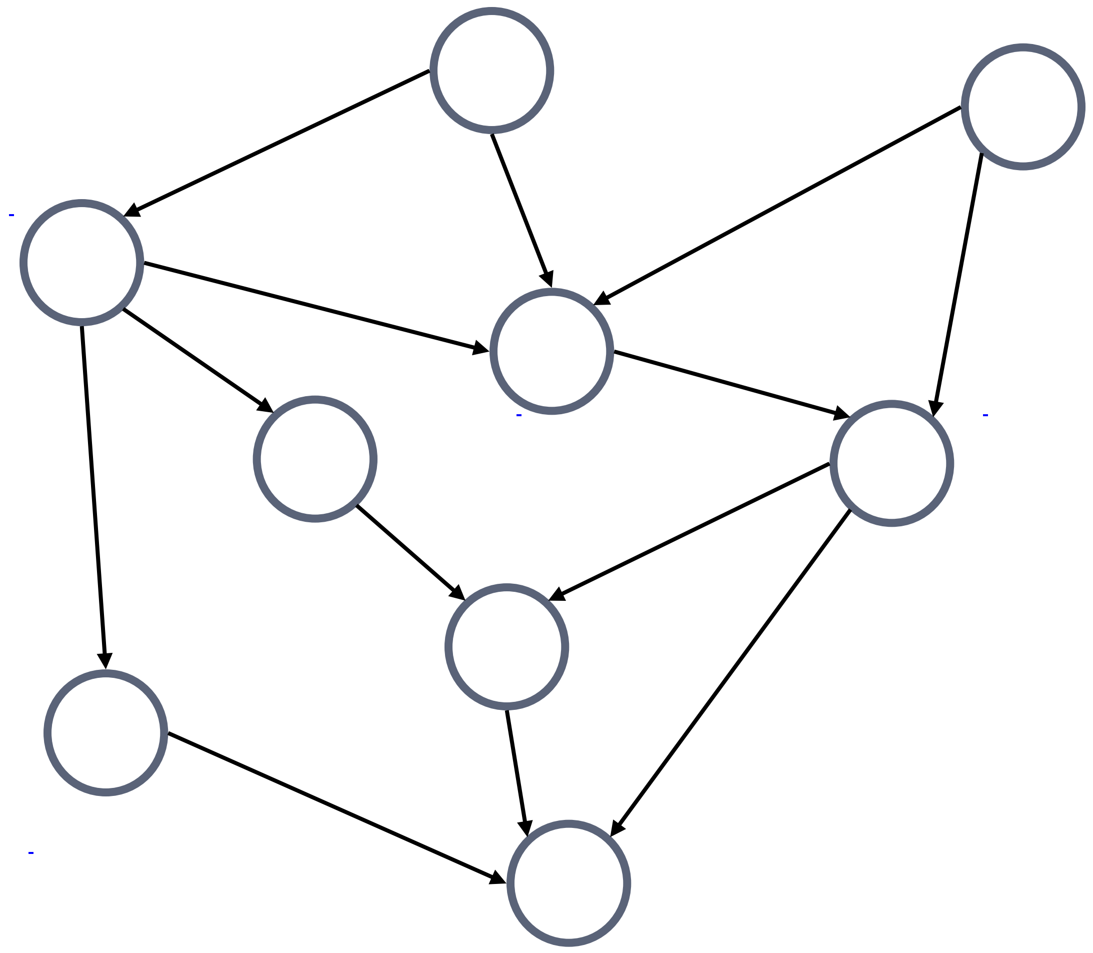
Topological Sorting

- Another way of thinking
- Note: This algorithm is different than the one in

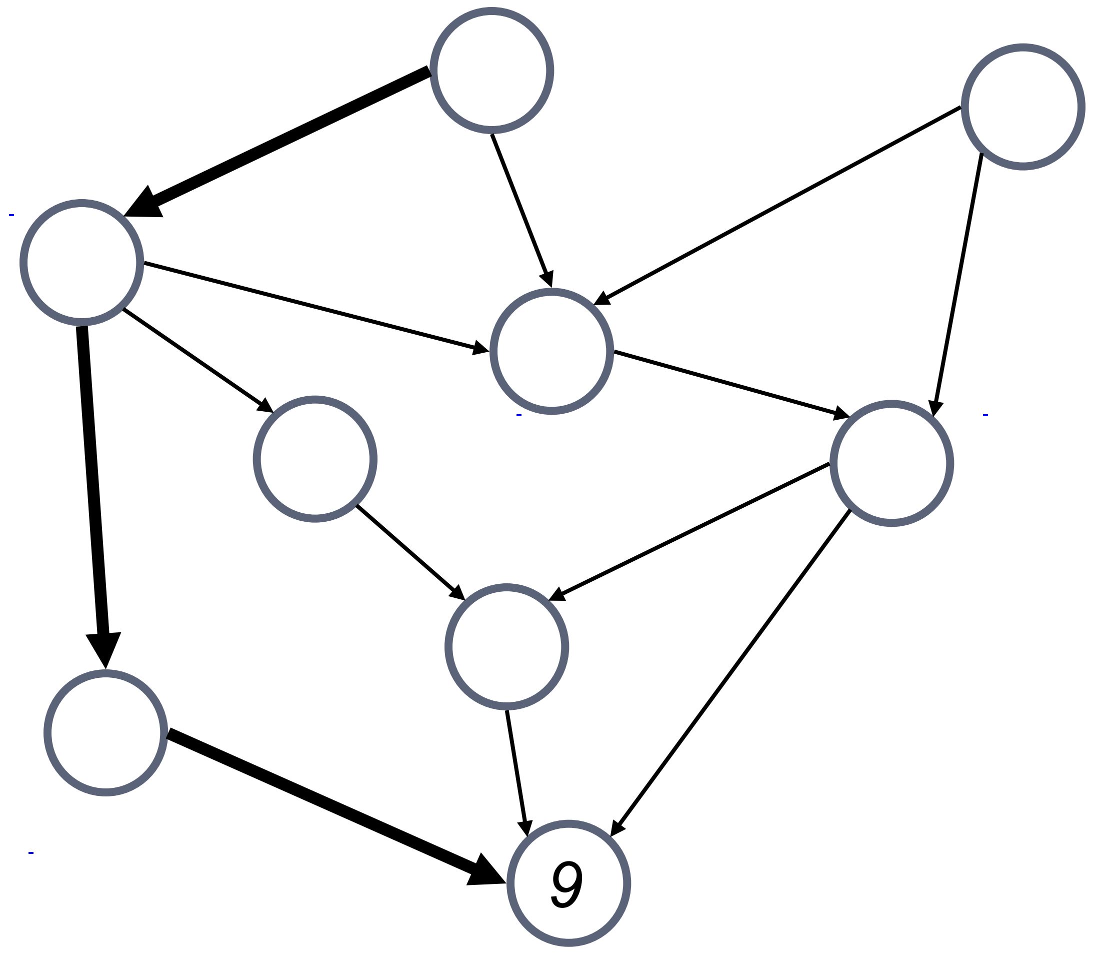
```
Method TopologicalSort(G)
    H ← G    // Temporary copy of G
    n ← G.numVertices()
    while H is not empty do
        Let v be a vertex with no outgoing edges
        Label v ← n
        n ← n - 1
        Remove v from H
```

- Running time: O(n + m). How…?

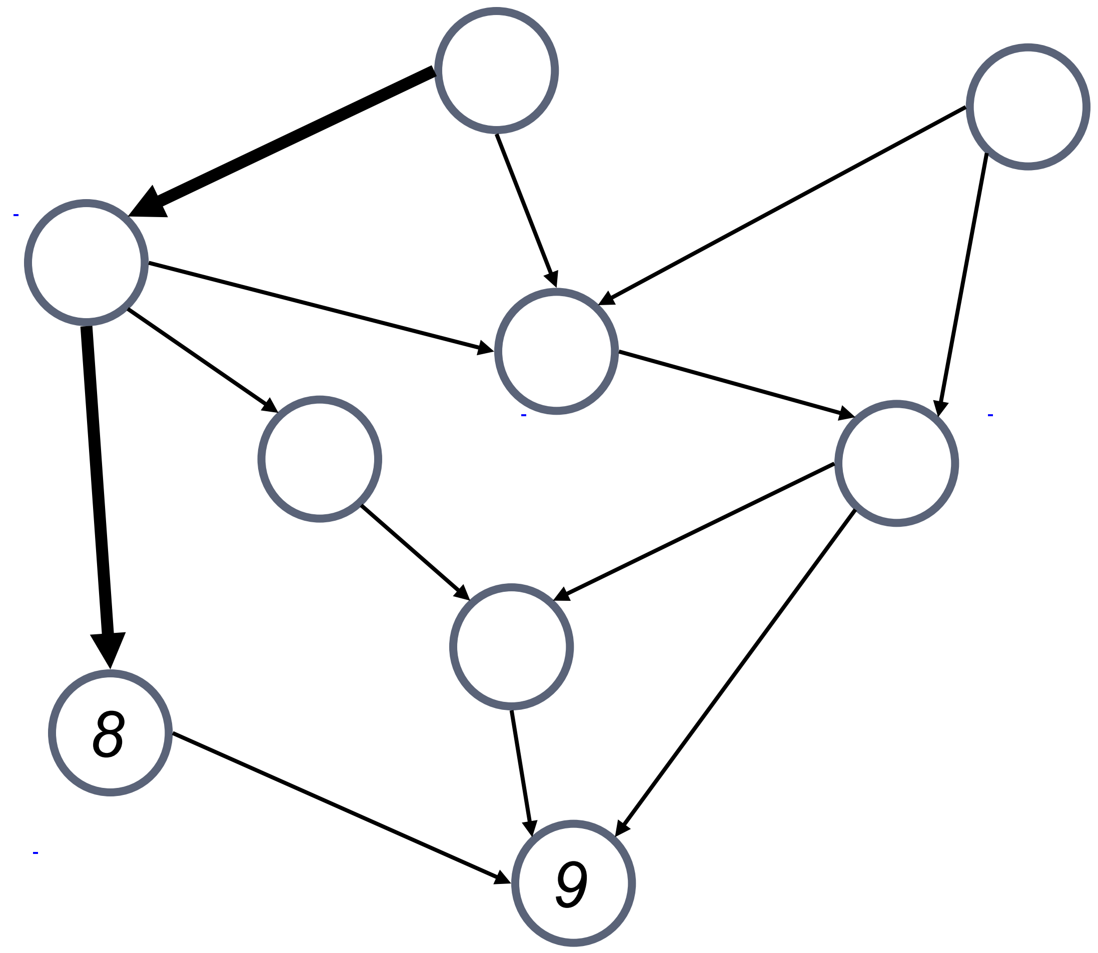
Topological Sorting Example



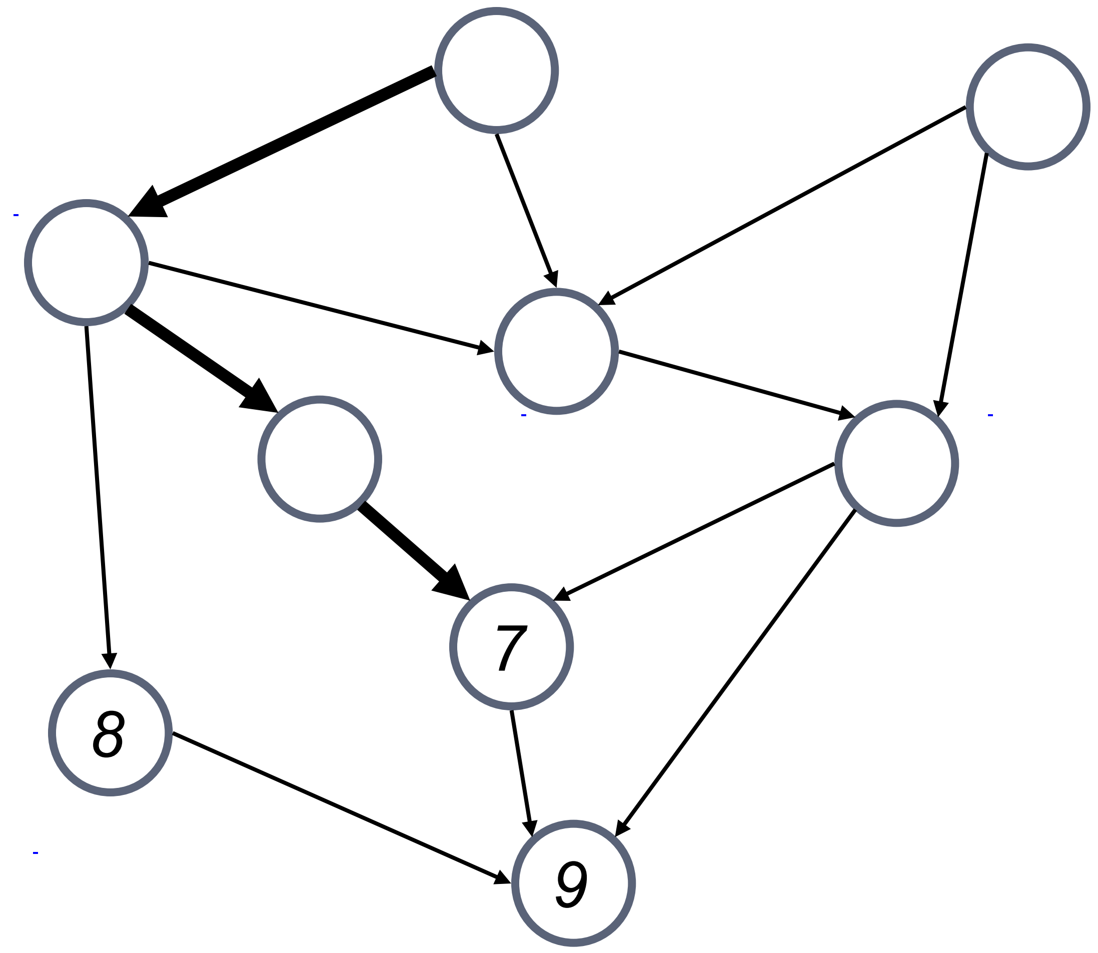
Topological Sorting Example



Topological Sorting Example



Topological Sorting Example



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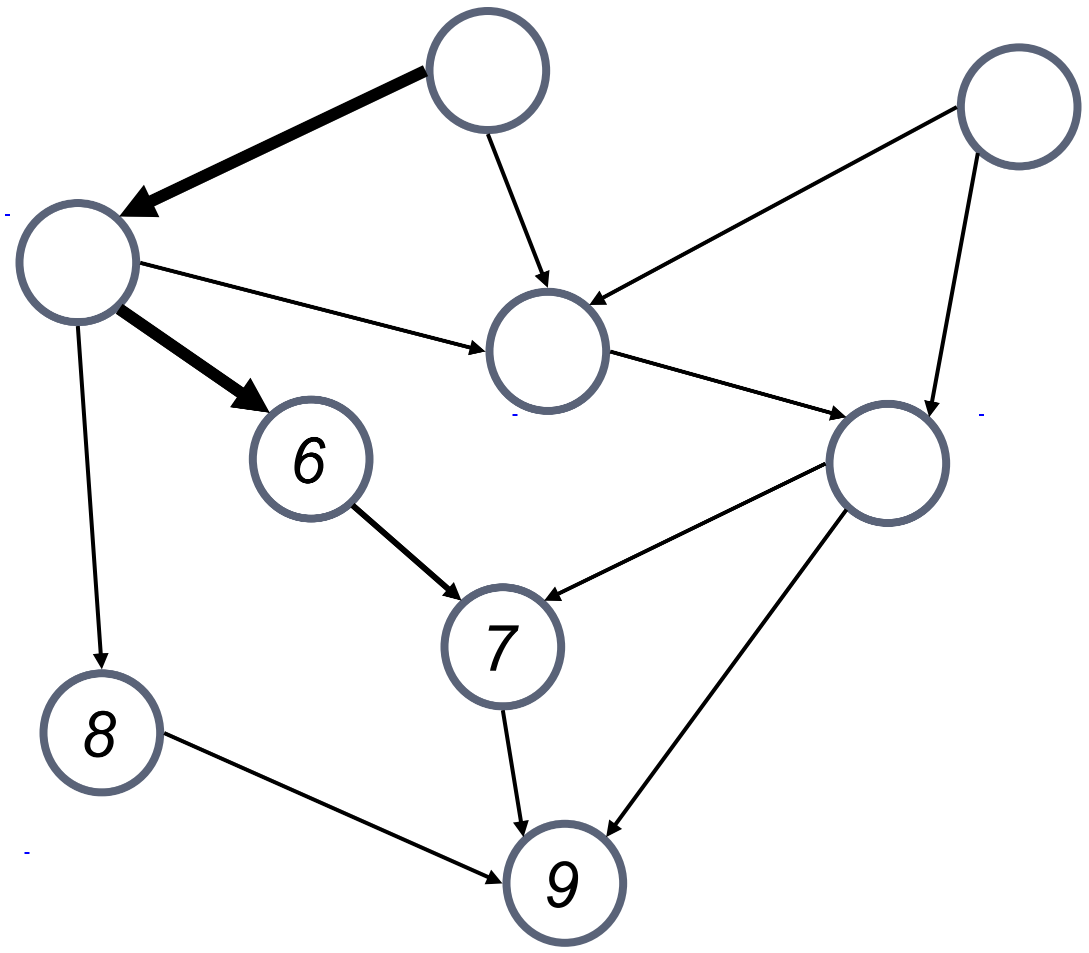
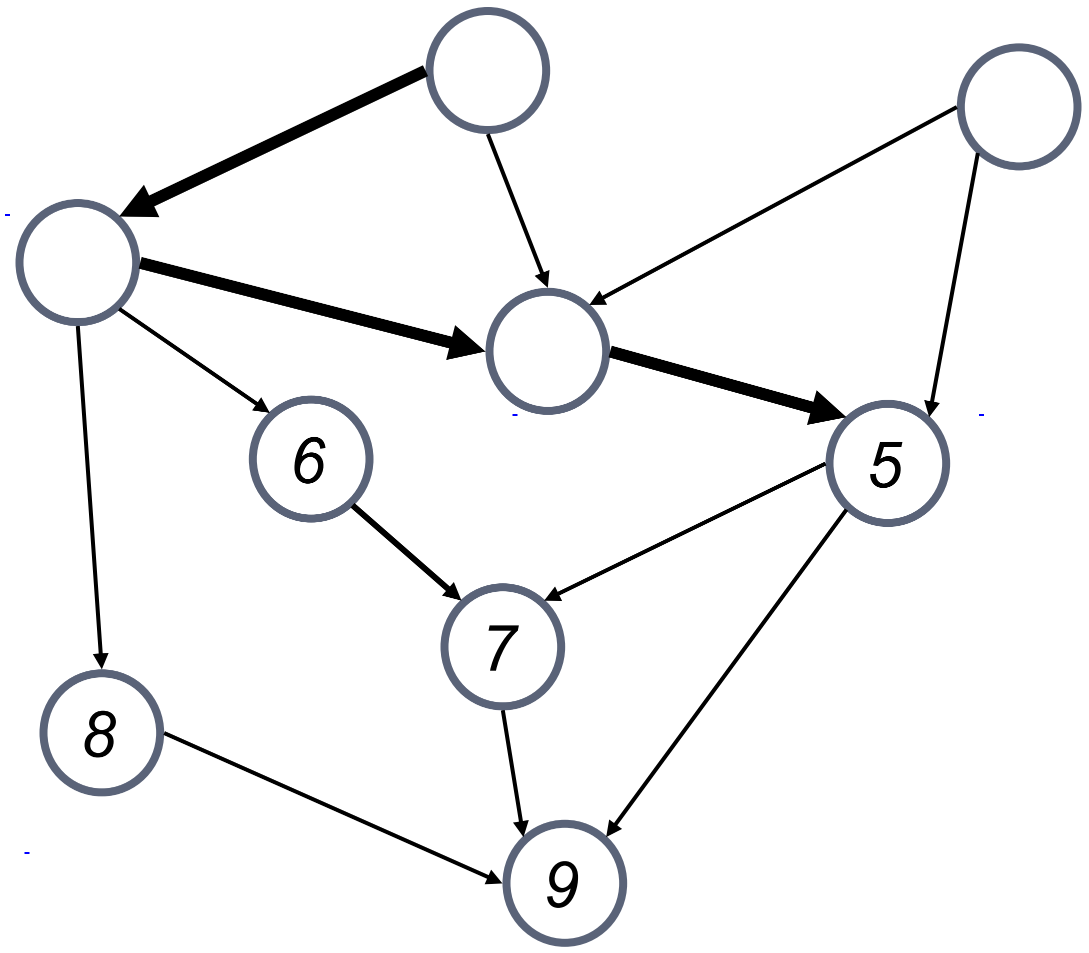


Topological Sorting Example



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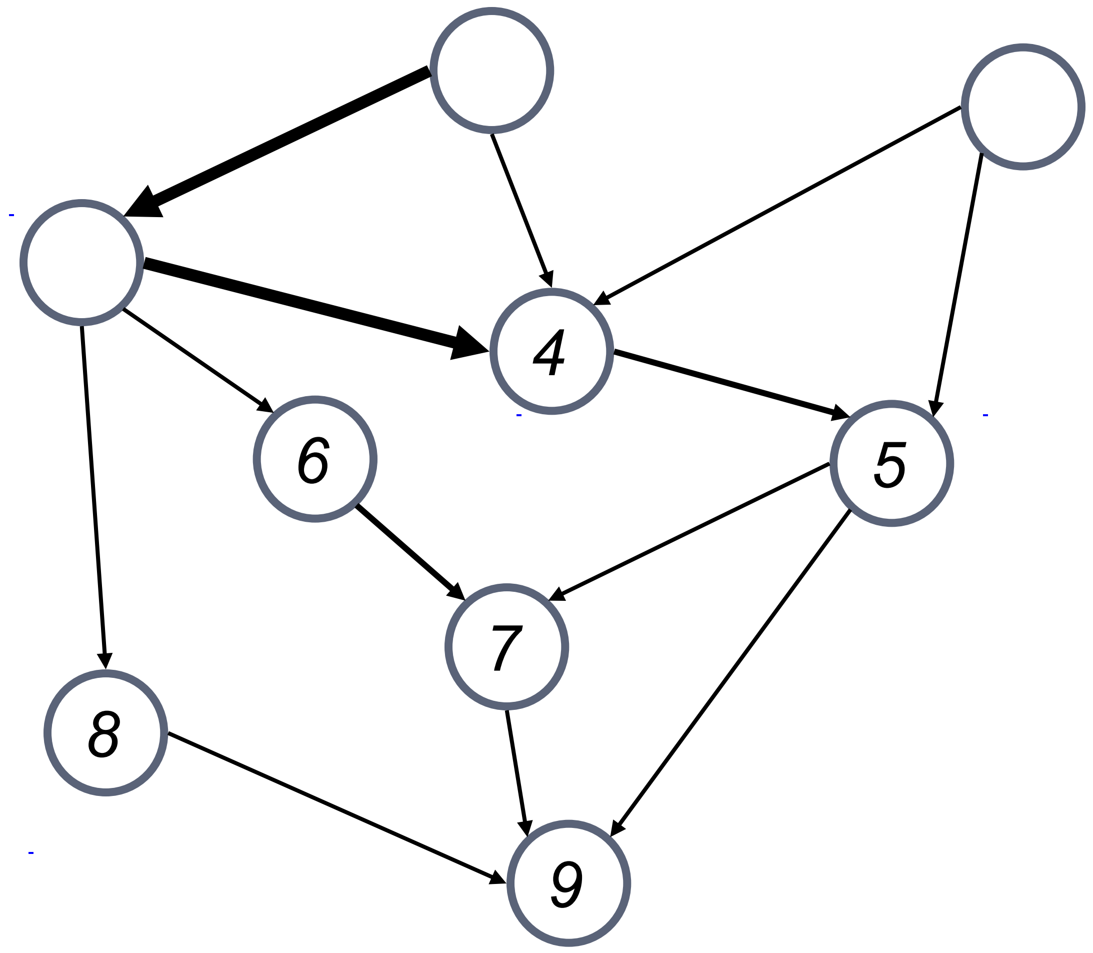

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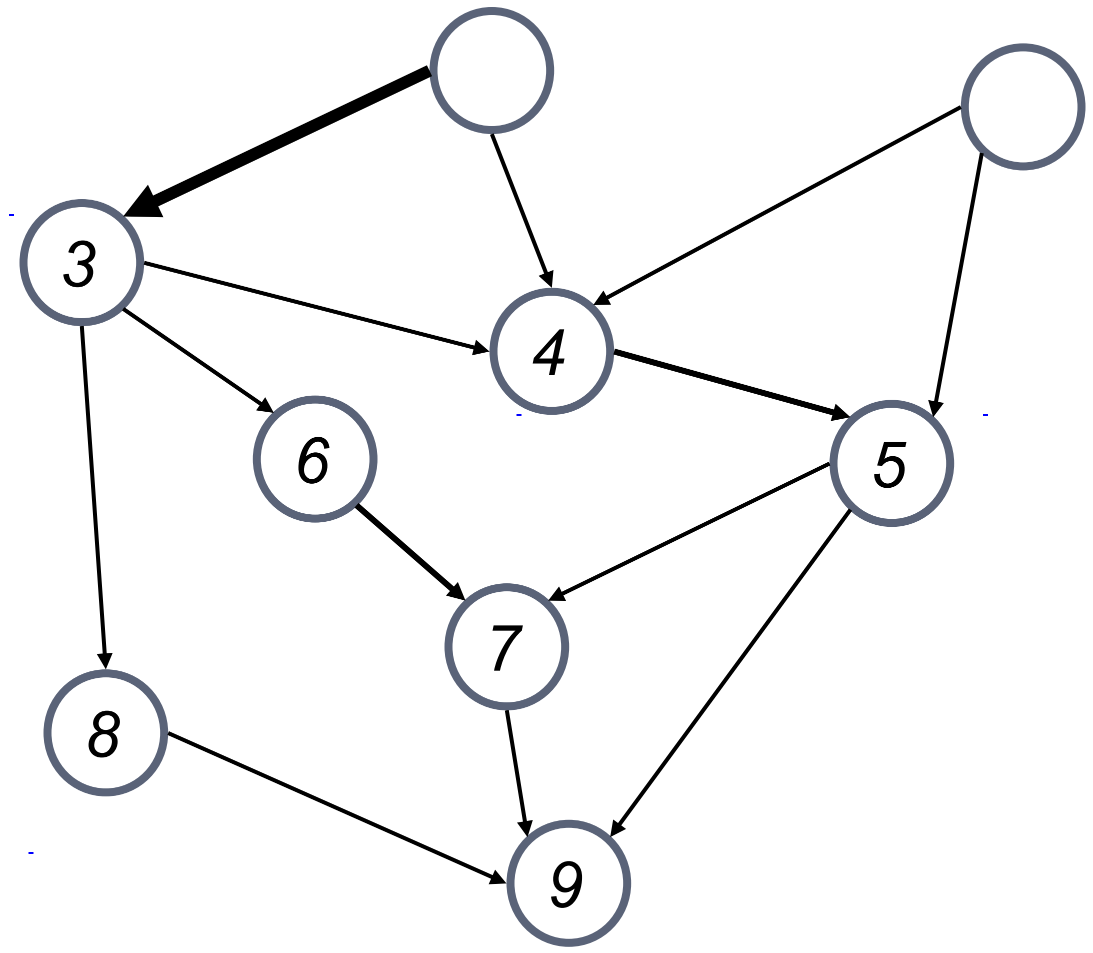

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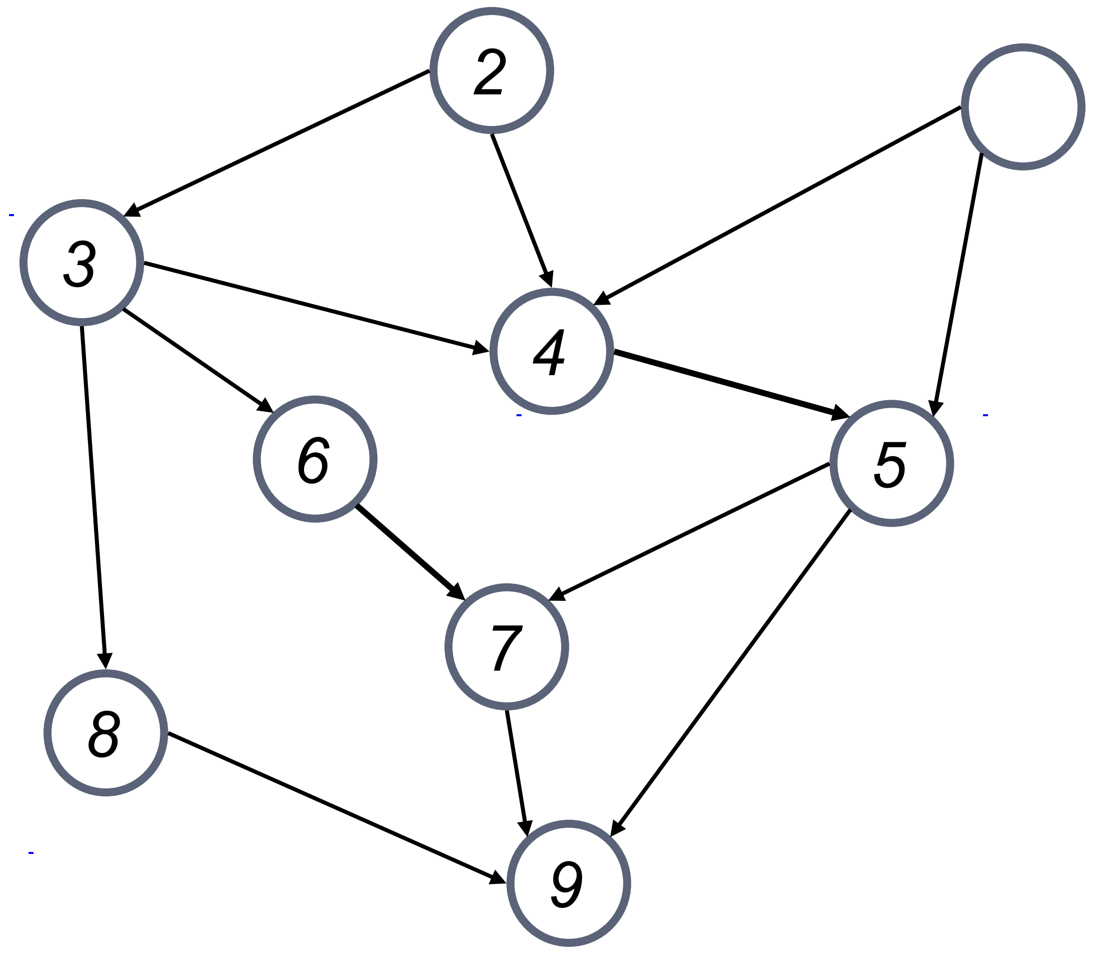
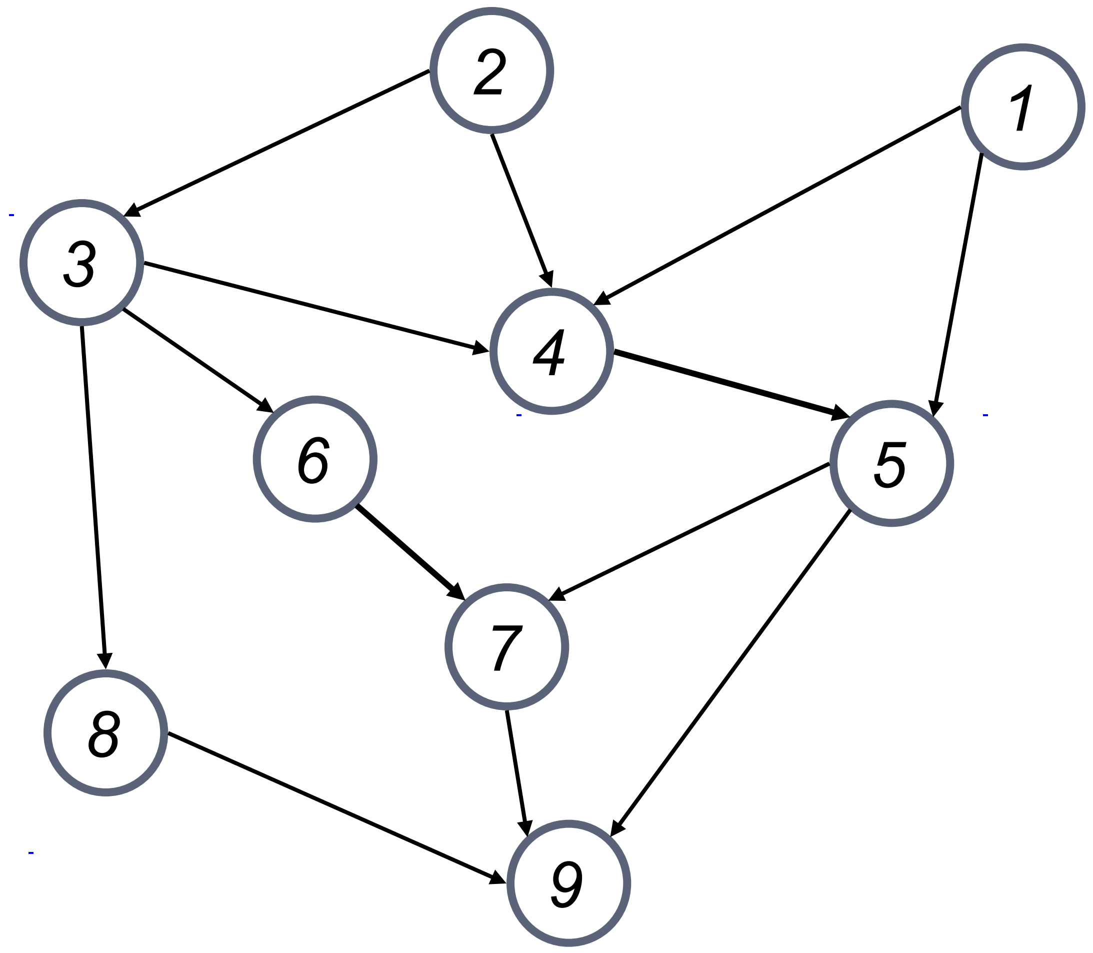


Topological Sorting Example



References

- Algorithm Design: Foundations, Analysis, and Internet Examples. Michael T. Goodrich and Roberto Tamassia. John Wiley & Sons.
- Introduction to Algorithms. Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein.

Thank you!