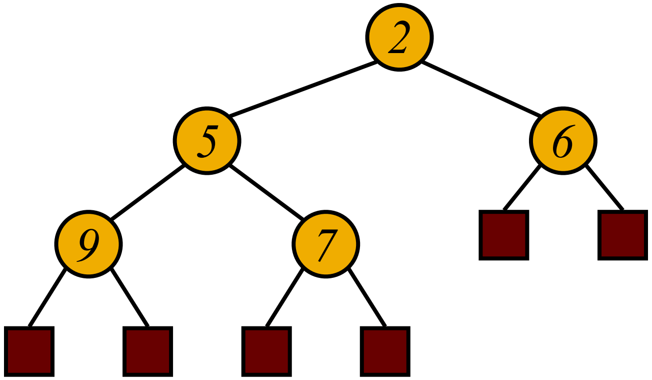
Heap

Algorithms & Data Structures
ITCS 6114/8114

Dr. Dewan Tanvir Ahmed
Department of Computer Science
University of North Carolina at Charlotte

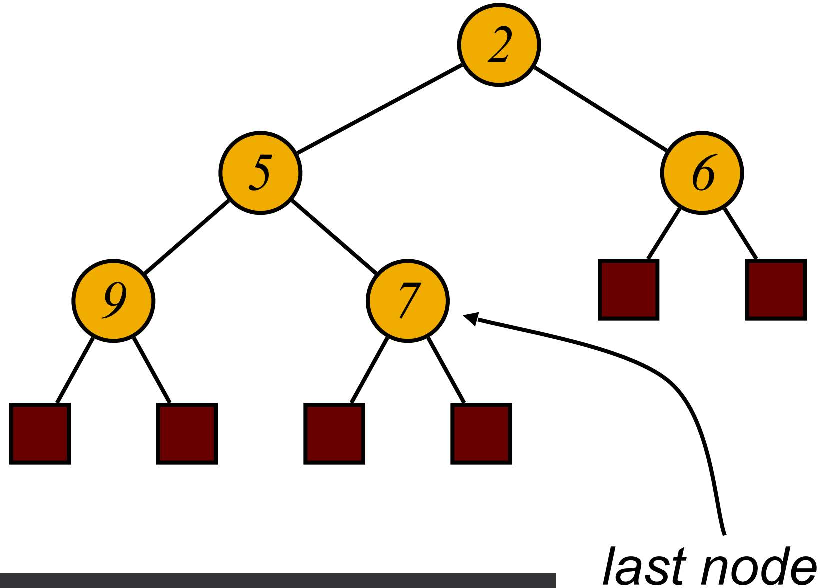
Heaps and Priority Queues



What is a heap (§2.4.3)

- A heap is a binary tree storing keys at its internal nodes and satisfying the following properties:
  - Heap-Order: for every internal node v other than the root, key(v) ≥ key(parent(v))
  - Complete Binary Tree: let h be the height of the heap
    - for i = 0, … , h − 1, there are 2^i nodes of depth i
    - at depth h − 1, the internal nodes are to the left of the external nodes

- The last node of a heap is the rightmost internal node of depth h − 1



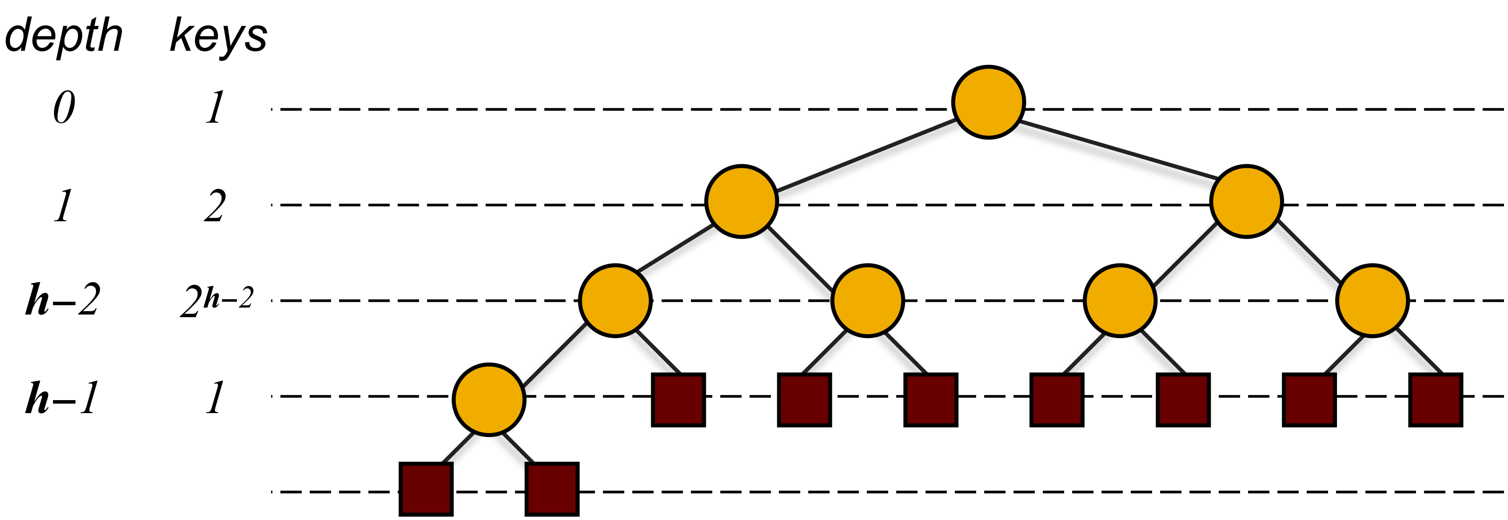
A complete binary tree is a binary tree in which every level, except possibly the last, is completely filled, and all nodes are as far left as possible

Height of a Heap (§2.4.3)

- **Theorem: A heap storing n keys has height $O(\log n)$**

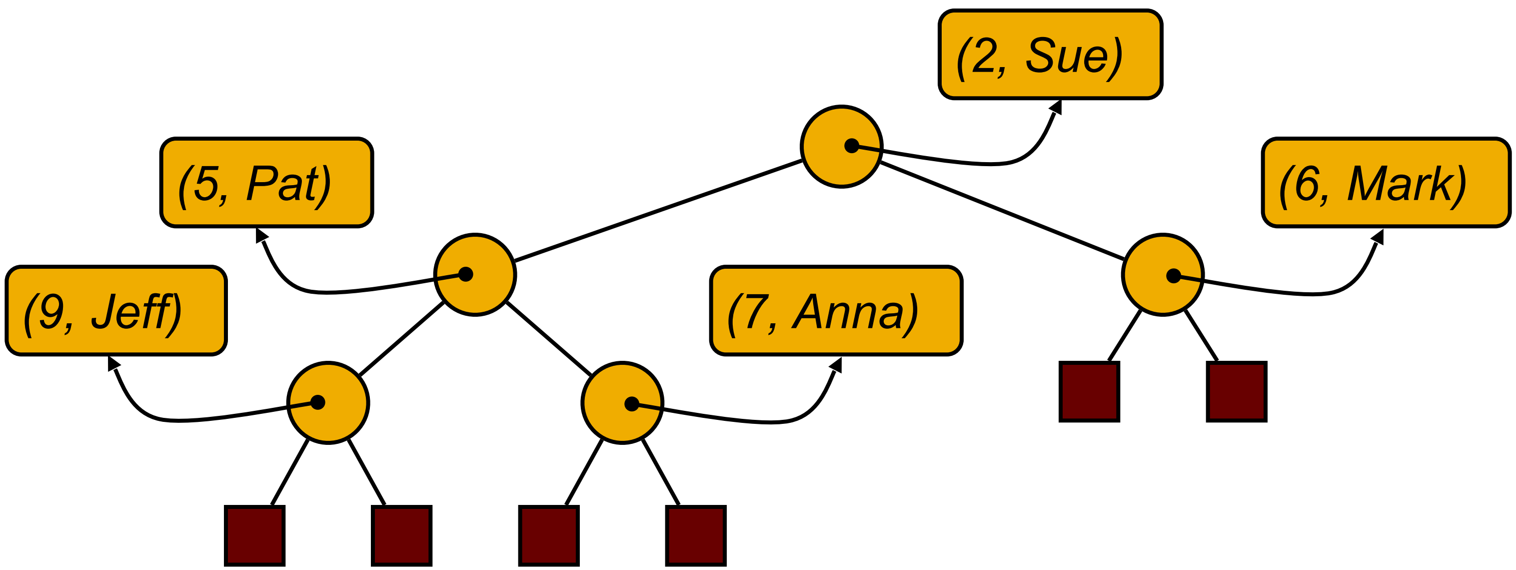
  Proof: (we apply the complete binary tree property)

  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \ldots, h-2$ and at least one key at depth $h-1$, we have $n \geq 1 + 2 + 4 + \ldots + 2^{h-2} + 1$
  - Thus, $n \geq 2^{h-1}$, i.e., $h \leq \log n + 1$



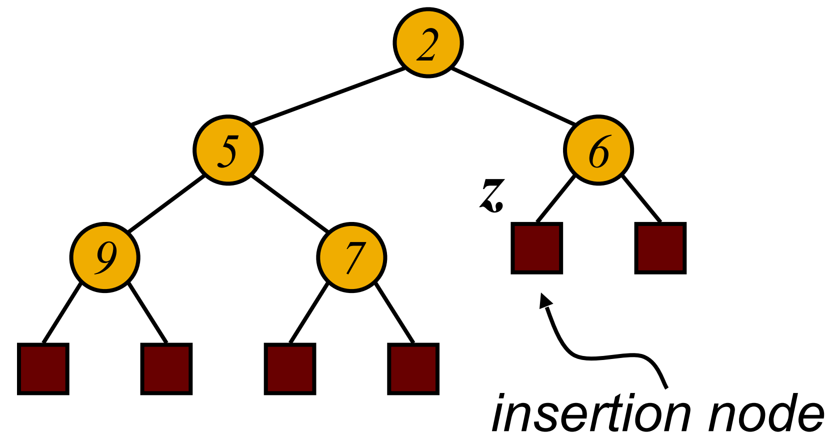
Heaps and Priority Queues

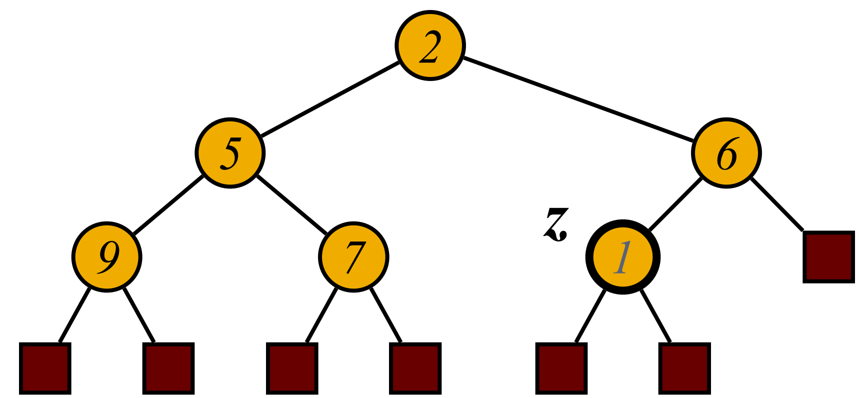
- We can use a heap to implement a priority queue
- We store an (key, element) item at each internal node
- We keep track of the position of the last node
- For simplicity, we show only the keys in the pictures



Insertion into a Heap (§2.4.3)

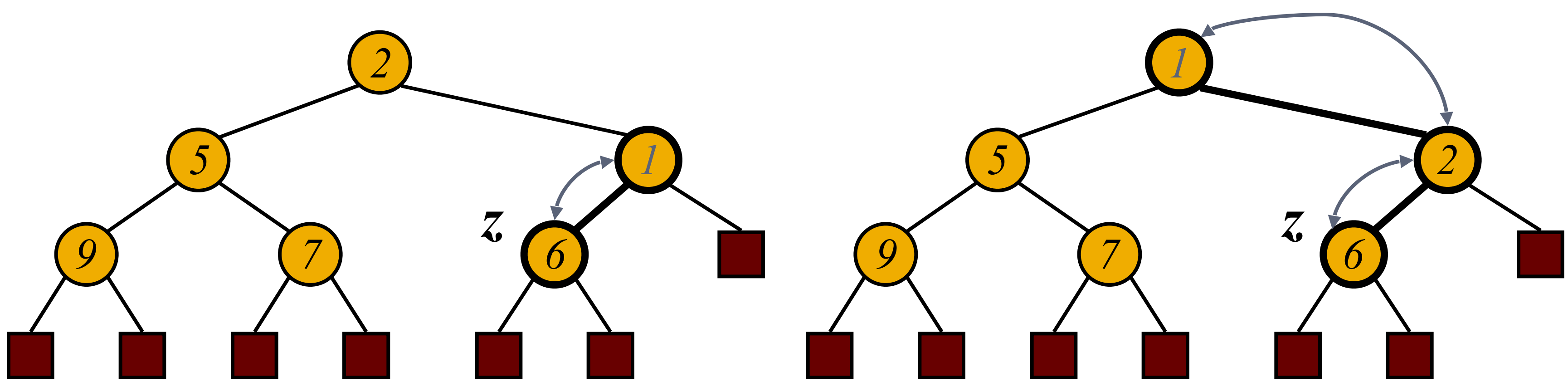
- The insertion algorithm consists of three steps
  - Find the insertion node z (the new last node)
  - Store k at z and expand z into an internal node
  - Restore the heap-order property (discussed next)





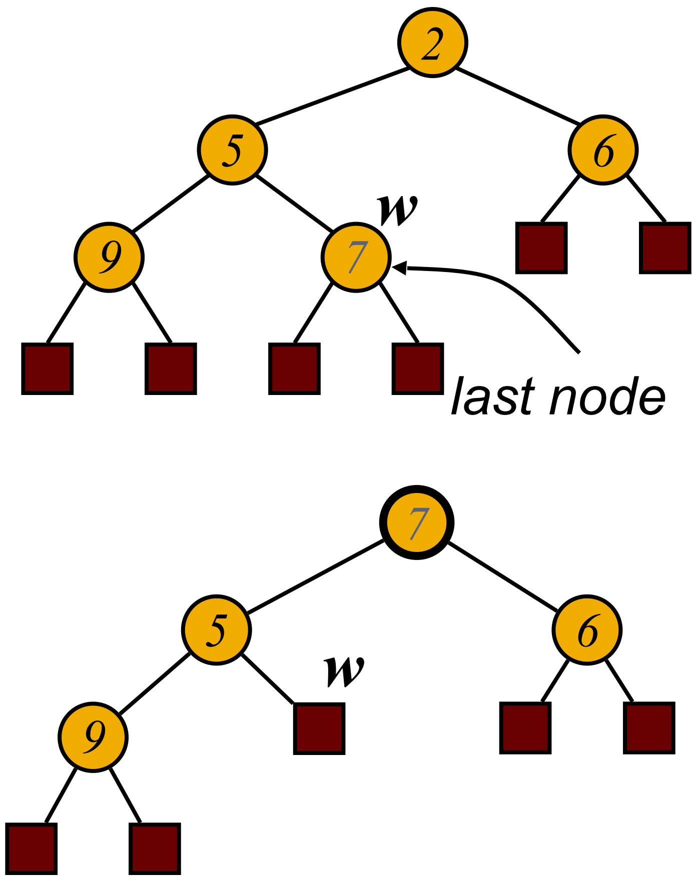
Upheap

- After the insertion of a new key k, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping k along an upward path from the insertion node
- Upheap terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time



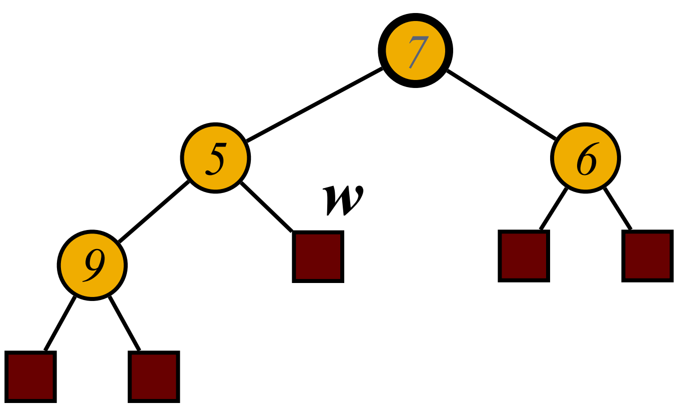
Removal from a Heap (§2.4.3)

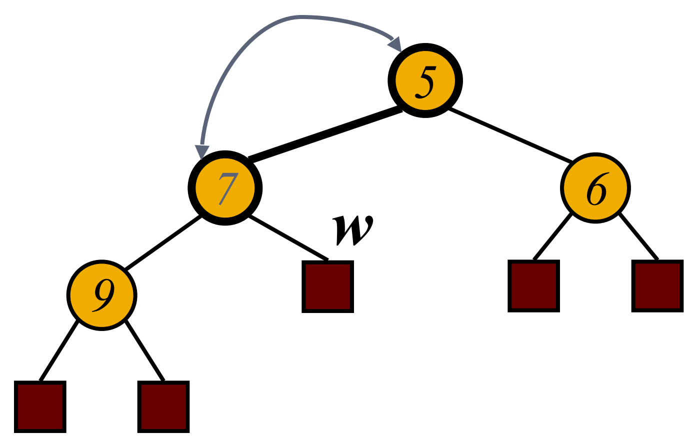
- Method removeMin of the priority queue ADT corresponds to the removal of the root key from the heap
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node w
  - Compress w and its children into a leaf
  - Restore the heap-order property (discussed next)



Downheap

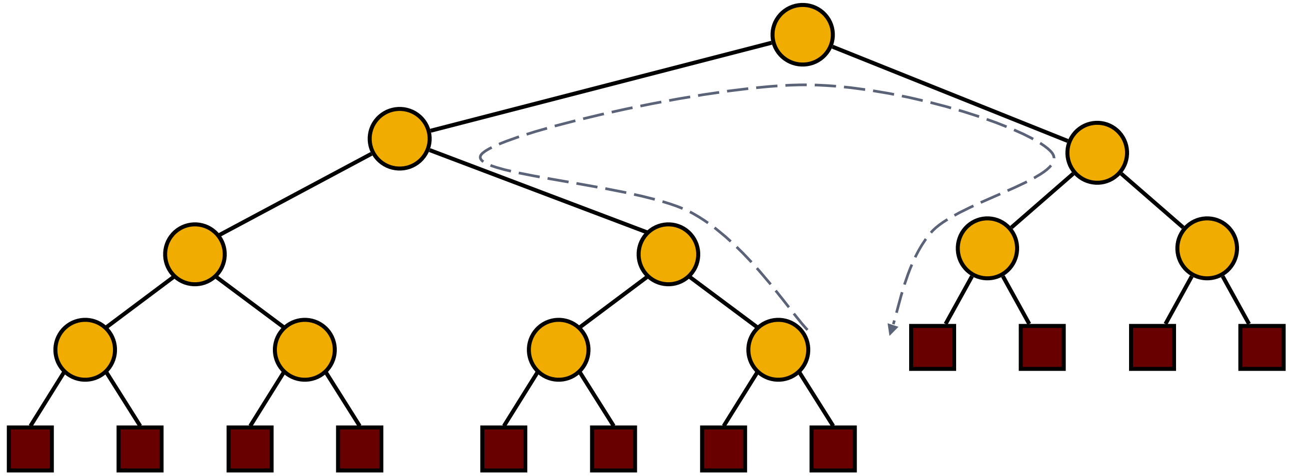
- After replacing the root key with the key $\boldsymbol{k}$ of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key $\boldsymbol{k}$ along a downward path from the root
- Downheap terminates when key $\boldsymbol{k}$ reaches a leaf or a node whose children have keys greater than or equal to $\boldsymbol{k}$
- Since a heap has height $\boldsymbol{O}(\log \boldsymbol{n})$, downheap runs in $\boldsymbol{O}(\log \boldsymbol{n})$ time



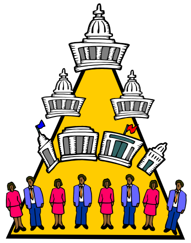


Updating the Last Node

- The insertion node can be found by traversing a path of $O(\log n)$ nodes
  - While the current node is a right child, go to the parent node
  - If the current node is a left child, go to the right child
  - While the current node is internal, go to the left child
- Similar algorithm for updating the last node after a removal



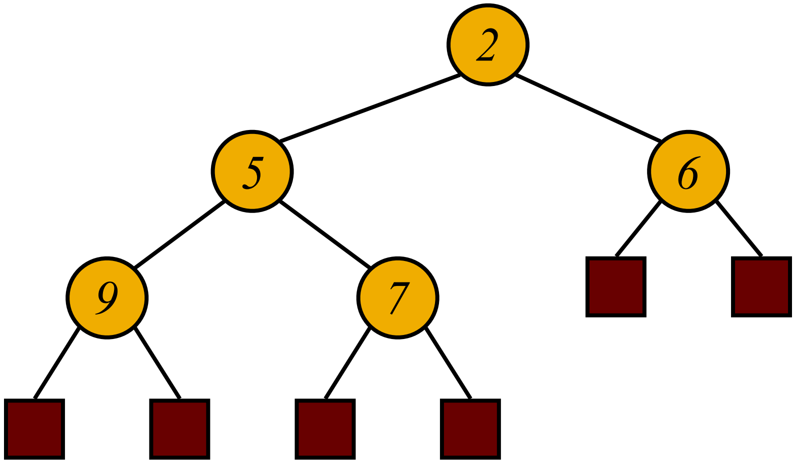
Heap-Sort (§2.4.4)

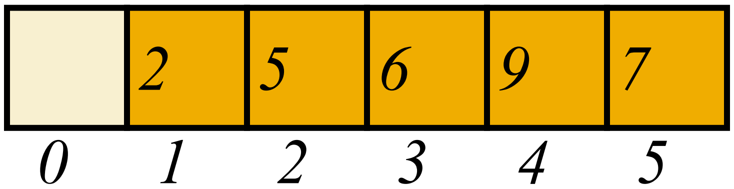


- Consider a priority queue with n items implemented by means of a heap
  - the space used is O(n)
  - methods insertItem and removeMin take O(log n) time
  - methods size, isEmpty, minKey, and minElement take time O(1) time
- Using a heap-based priority queue, we can sort a sequence of n elements in O(n log n) time
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

Vector-based Heap Implementation (§2.4.3)

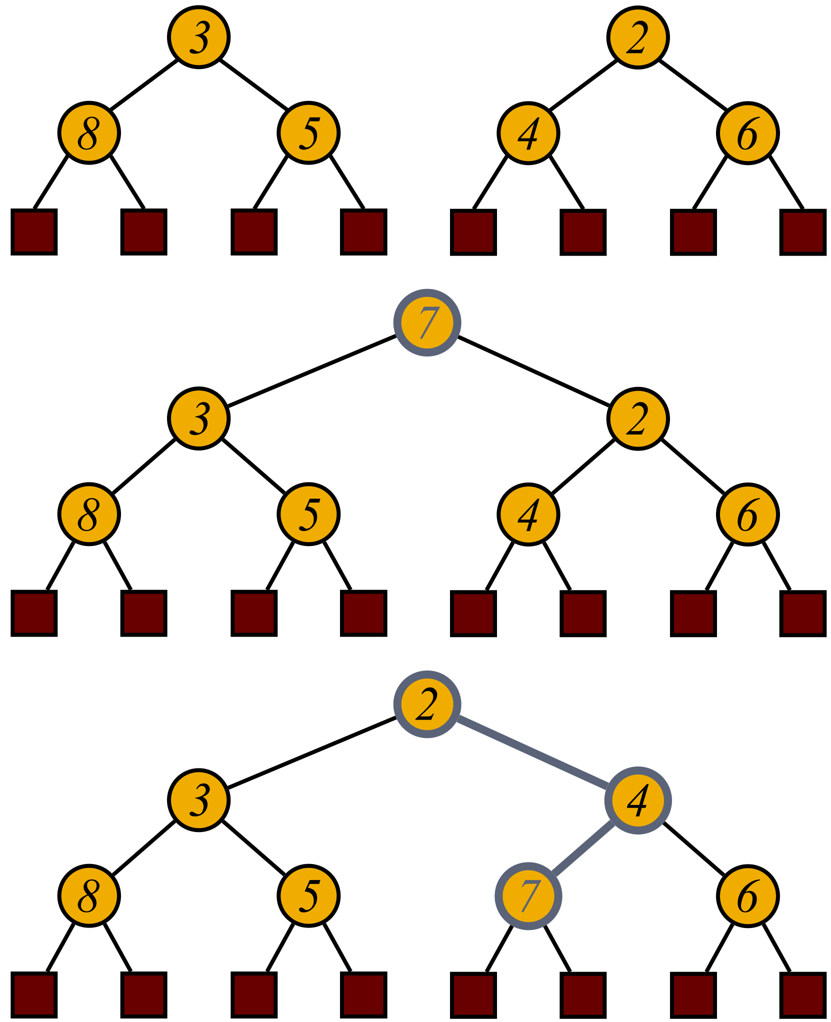
- We can represent a heap with n keys by means of a vector of length n + 1
- For the node at rank i
  - the left child is at rank 2i
  - the right child is at rank 2i + 1
- Links between nodes are not explicitly stored
- The leaves are not represented
- The cell at rank 0 is not used
- Operation insertItem corresponds to inserting at rank n + 1
- Operation removeMin corresponds to removing at rank 1
- Yields in-place heap-sort





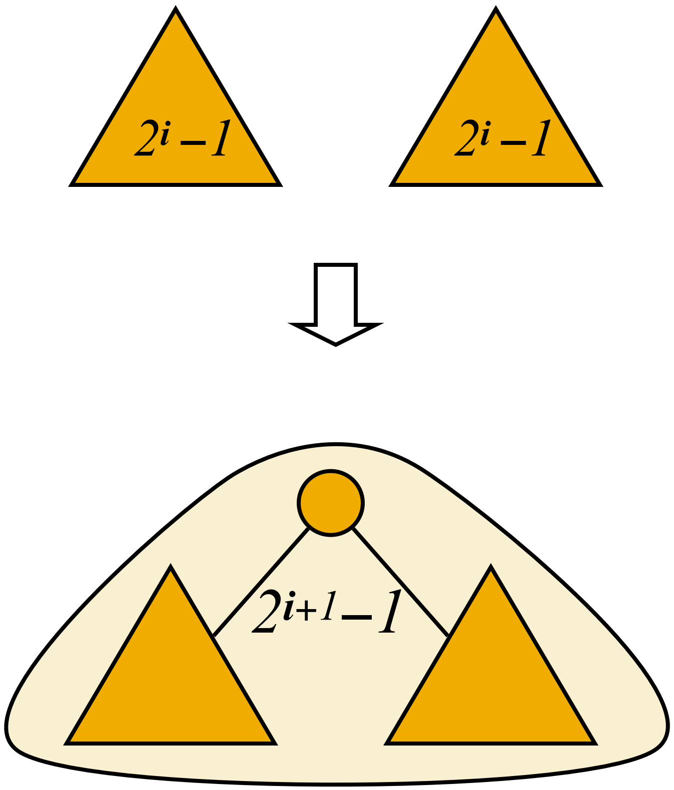
Merging Two Heaps

- We are given two heaps and a key k
- We create a new heap with the root node storing k and with the two heaps as subtrees
- We perform downheap to restore the heap-order property



Bottom-up Heap Construction (§2.4.3)

- We can construct a heap storing n given keys in using a bottom-up construction with $\log n$ phases
- In phase i, pairs of heaps with $2^i - 1$ keys are merged into heaps with $2^{i+1} - 1$ keys

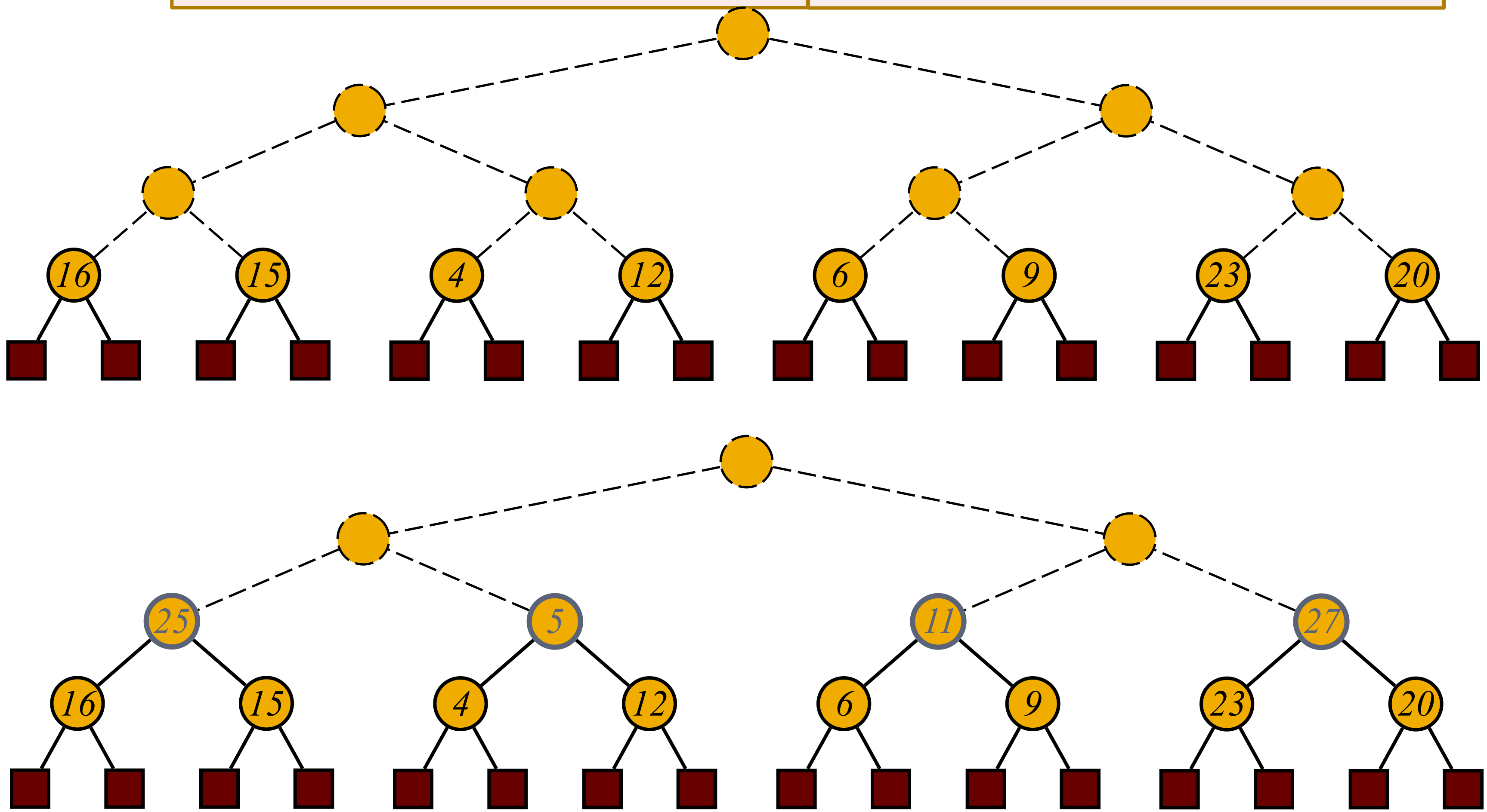


Bottom-up Heap Construction (§2.4.3)

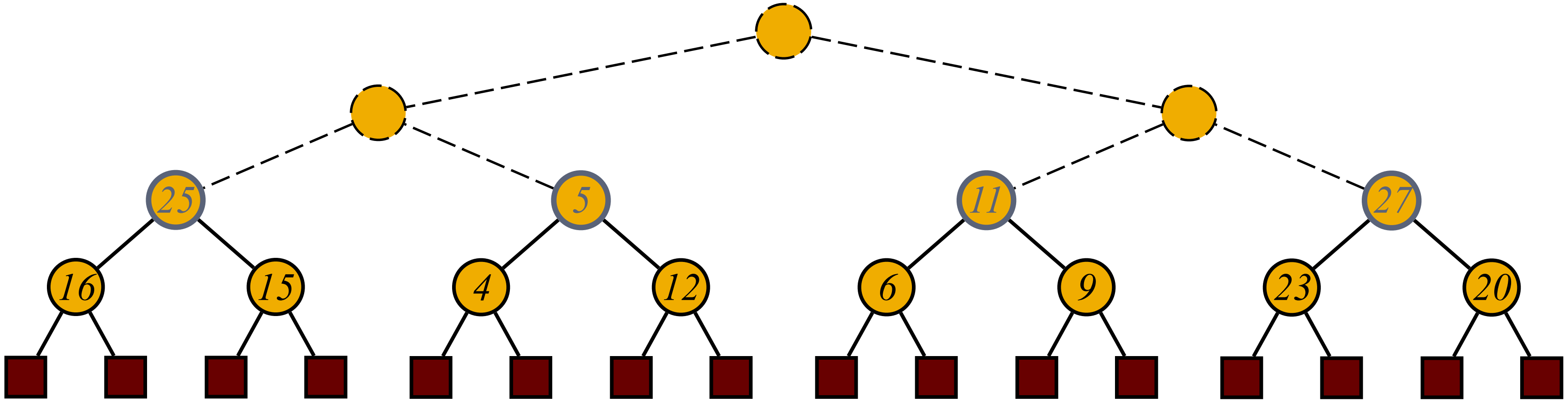
```
Algorithm BottomUpHeap(S)
Input: A sequence S storing n = 2^h - 1 keys
Output: A heap T storing the keys in S
   if S is empty then
      return an empty heap
   Remove the first key, k, from S
   Split S into two sequences, S1 and S2, each of size (n-1)/2
   T1 ← BottomUpHeap(S1)
   T2 ← BottomUpHeap(S2)

   Create binary tree T with root r storing k, left subtree
   T1, and right subtree T2.

   Perform a down-heap bubbling from the root r to T, if
   necessary
   return T
```

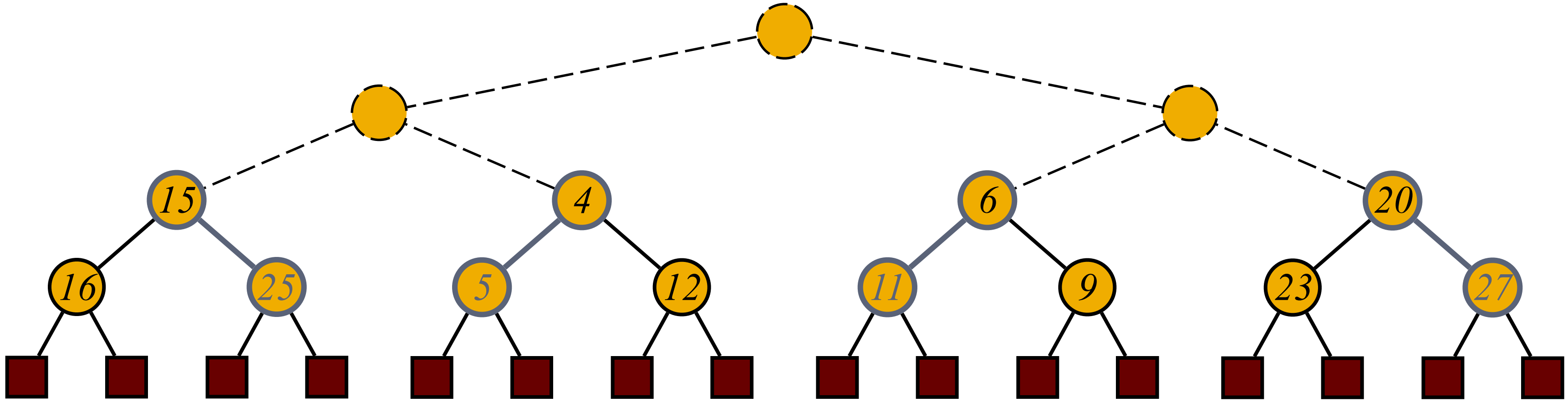
Example:

S = 10 7 25 16 15 5 4 12 8 11 6 9 27 23 20

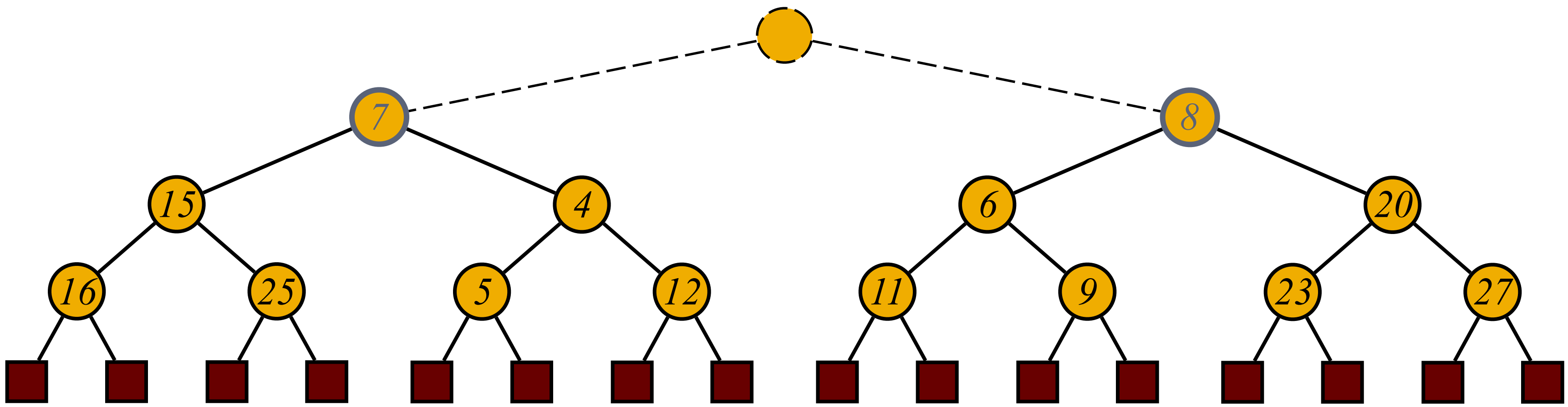


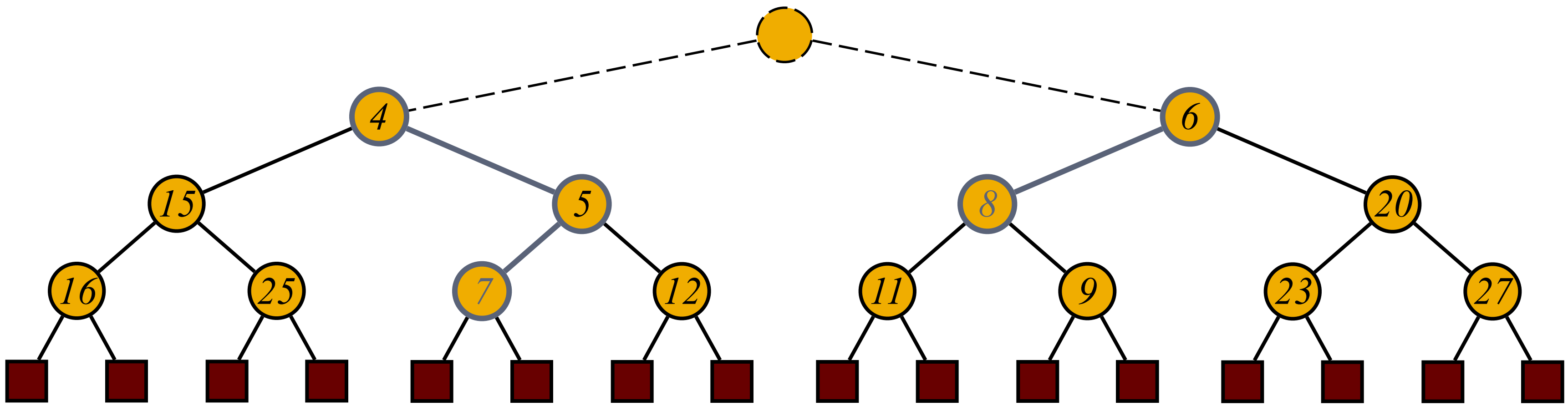
Example (contd.)



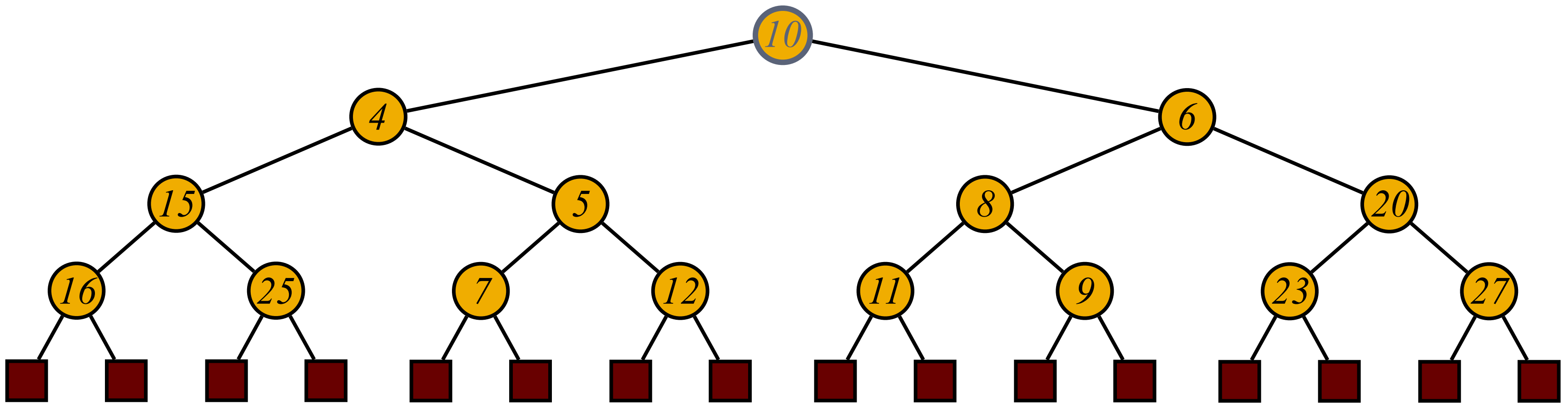


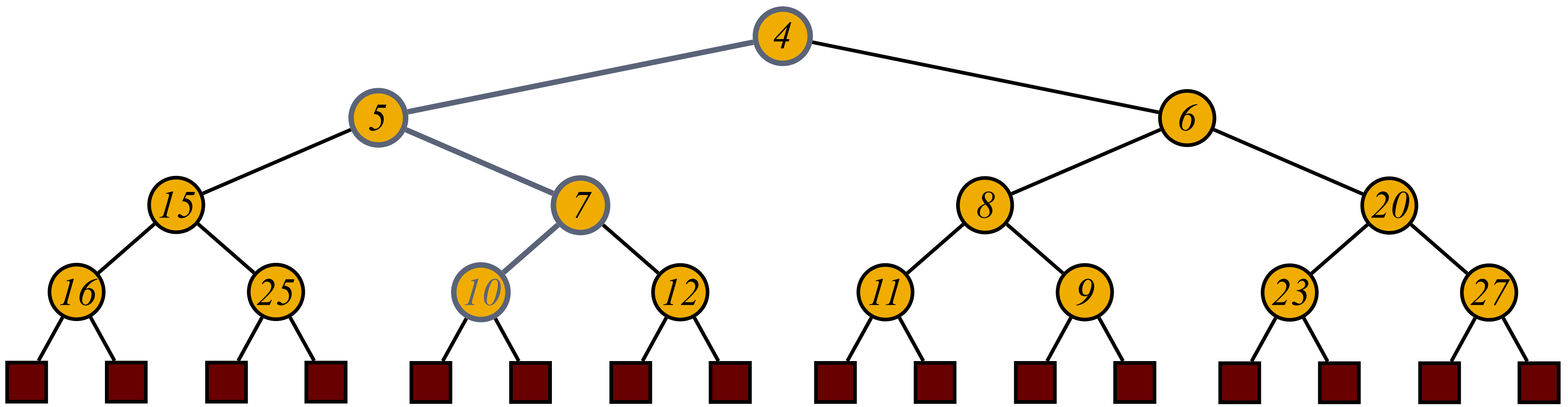
Example (contd.)





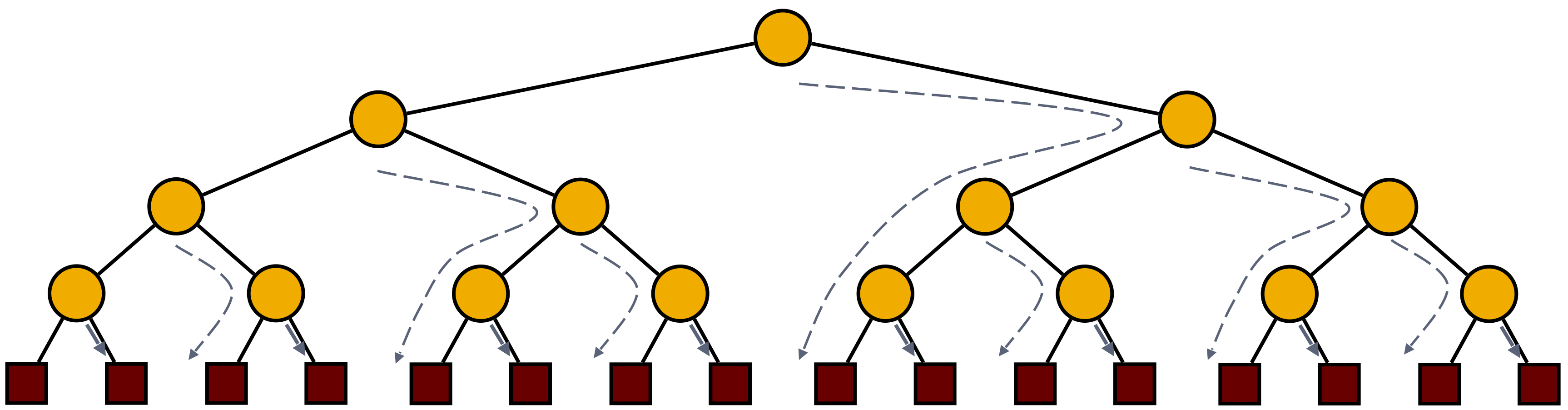
Example (end)





Analysis

- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is O(n)
- Thus, bottom-up heap construction runs in O(n) time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort



Reference

- Algorithm Design: Foundations, Analysis, and Internet Examples. Michael T. Goodrich and Roberto Tamassia. John Wiley & Sons.
- Introduction to Algorithms. Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein.

Thank you!