Assignment 9

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Question:Ex. 6.62 , Papoulis

Suppose X represents the inverse of a chi-square random variable with one degree of freedom, and the conditional p.d.f. of Y given X is N(0,x). Show that Y has a Cauchy distribution.

Solution

Let

$$X = 1/Z$$

where Z represents a chi-square random variable.So,

$$f_Z(z) = \frac{z^{-1/2}}{\sqrt{2}\Gamma(1/2)} e^{-z/2} = \frac{z^{-1/2}}{\sqrt{2\pi}} e^{-z/2} \tag{1}$$

$$f_X(x) = \frac{1}{\left|\frac{dx}{dz}\right|} f_Z(1/x) = \frac{1}{x^2} \frac{x^{1/2}}{\sqrt{2\pi}} e^{-1/2x} = \frac{1}{\sqrt{2\pi} x^{3/2}} e^{-1/2x} \quad , x > 0 \tag{2}$$

It is also given that

$$f_{Y|X}\left(y|x\right) = \frac{1}{\sqrt{2\pi x}} e^{-y^2/2x} \tag{3}$$

so

$$\begin{aligned} f_{XY}\left(x,y\right) &= f_{Y|X}\left(y|x\right) f_X\left(x\right) \\ &= \frac{1}{\sqrt{2\pi x}} e^{-y^2/2x} \times \frac{1}{\sqrt{2\pi} x^{3/2}} e^{-1/2x} \qquad \text{..from } eq^n \text{ (2) and (3)} \\ &= \frac{1}{2\pi x^2} e^{-\left(1+y^2\right)/2x} \end{aligned} \tag{4}$$

We know that

$$
\begin{aligned}
f_Y(y) &= \int_0^\infty f_{XY}(x, y)\, dx \\
&= \frac{1}{2\pi} \int_0^\infty \frac{1}{x^2} e^{-\left(1+y^2\right)/2x} dx \qquad ..\text{from } (4) \qquad (5)
\end{aligned}
$$

Let

$$u = \frac{1+y^2}{2x}$$
$$du = \frac{1+y^2}{2}\frac{(-1)}{x^2}$$

putting this in equation (5)

$$f_Y(y) = \frac{1}{2\pi}\frac{2}{1+y^2}\int_0^{\infty} e^{-u} du = \frac{1/\pi}{1+y^2} \qquad , -\infty < y < \infty$$

Thus Y represents a Cauchy random variable.