

Assignment 9

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Question: Ex. 6.62 , Papoulis

Suppose X represents the inverse of a chi-square random variable with one degree of freedom, and the conditional p.d.f. of Y given X is $N(0, x)$. Show that Y has a Cauchy distribution.

Solution

Let

$$X = 1/Z$$

where Z represents a chi-square random variable. So,

$$f_Z(z) = \frac{z^{-1/2}}{\sqrt{2}\Gamma(1/2)} e^{-z/2} = \frac{z^{-1/2}}{\sqrt{2\pi}} e^{-z/2} \quad (1)$$

$$f_X(x) = \frac{1}{\left|\frac{dx}{dz}\right|} f_Z(1/x) = \frac{1}{x^2} \frac{x^{1/2}}{\sqrt{2\pi}} e^{-1/2x} = \frac{1}{\sqrt{2\pi} x^{3/2}} e^{-1/2x}, \quad x > 0 \quad (2)$$

It is also given that

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi x}} e^{-y^2/2x} \quad (3)$$

so

$$\begin{aligned} f_{XY}(x, y) &= f_{Y|X}(y|x) f_X(x) \\ &= \frac{1}{\sqrt{2\pi x}} e^{-y^2/2x} \times \frac{1}{\sqrt{2\pi x^{3/2}}} e^{-1/2x} \quad \text{..from eq}^n (2) \text{ and } (3) \\ &= \frac{1}{2\pi x^2} e^{-(1+y^2)/2x} \end{aligned} \quad (4)$$

We know that

$$\begin{aligned}
 f_Y(y) &= \int_0^{\infty} f_{XY}(x, y) dx \\
 &= \frac{1}{2\pi} \int_0^{\infty} \frac{1}{x^2} e^{-(1+y^2)/2x} dx \quad \text{..from (4)} \quad (5)
 \end{aligned}$$

Let

$$u = \frac{1 + y^2}{2x}$$

$$du = \frac{1 + y^2}{2} \frac{(-1)}{x^2}$$

putting this in equation (5)

$$f_Y(y) = \frac{1}{2\pi} \frac{2}{1 + y^2} \int_0^\infty e^{-u} du = \frac{1/\pi}{1 + y^2}, \quad -\infty < y < \infty$$

Thus Y represents a Cauchy random variable.

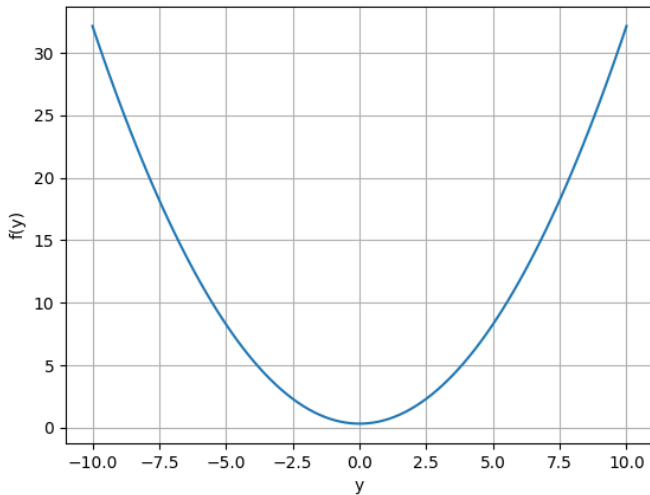


Figure: $f(y)$ function