

Circuits and Transforms

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CONTENTS

1	Definitions	1
2	Laplace Transform	1
3	Initial Conditions	3
4	Bilinear Transform	5

Abstract—This manual provides a simple introduction to Transforms

1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1 \mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2 \mu C$.

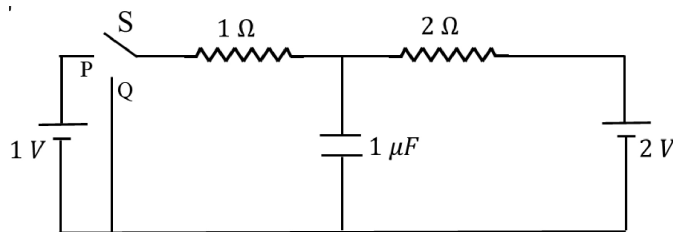


Fig. 2.1

2. Draw the circuit using latex-tikz.

Solution: See fig. 2.2

3. Find q_1 .

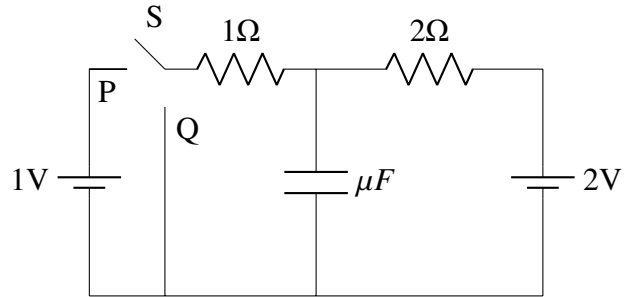


Fig. 2.2: Given Circuit

Solution: On connecting S to P for a long time capacitor get fully charged behave like disconnected wire. So, circuit will look like 2.3

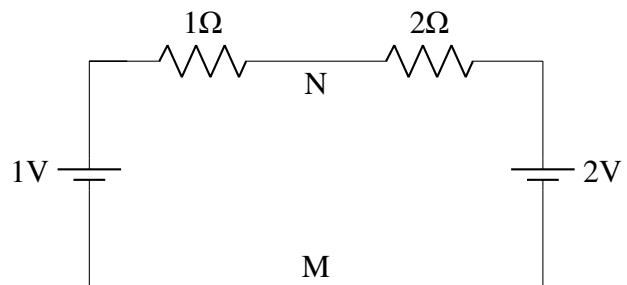


Fig. 2.3: circuit when conected to P for long time

On applying KVL on circuit in fig. 2.3 we get current as,

$$i = 1/3 \quad (2.1)$$

in counter-clockwise direction

So, PD across the capacitor i.e. across M and N is given by,

$$v_{C_0} = 1 + \frac{1}{3} \times 1 \quad (2.2)$$

$$= \frac{4}{3} \quad (2.3)$$

So, charge on capacitor will be

$$q_1 = v_{C_0} C_0 \quad (2.4)$$

$$= \frac{4}{3} \mu C \quad (2.5)$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution: Laplace of $u(t)$ is given by

$$U(s) = \int_{-\infty}^{\infty} u(t) e^{-st} dt \quad (2.6)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.7)$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^{\infty} \quad (2.8)$$

$$= \frac{1}{s} \quad (2.9)$$

Since e^{-st} is defined at $t \rightarrow \infty$ only for $s > 0$

So, ROC will be $s > 0$

5. Show that

$$e^{-at} u(t) \xleftrightarrow{\mathcal{H}} L \frac{1}{s+a}, \quad a > 0 \quad (2.10)$$

and find the ROC.

Solution:

$$\mathcal{L}[e^{-at} u(t)] = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt \quad (2.11)$$

$$= \int_0^{\infty} e^{-at} u(t) e^{-st} dt \quad (2.12)$$

$$= \int_0^{\infty} e^{-(a+s)t} dt \quad (2.13)$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} \quad (2.14)$$

$$= \frac{1}{s+a} \quad (2.15)$$

Since $e^{-(s+a)t}$ is defined at $t \rightarrow \infty$ only for $s > -a$

So, ROC will be $s > -a$

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

$$u(t) \xleftrightarrow{\mathcal{H}} L V_1(s) \quad (2.16)$$

$$2u(t) \xleftrightarrow{\mathcal{H}} L V_2(s) \quad (2.17)$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution:

from above 2 equations we can conclude that $V_1(s) = \frac{1}{s}$ and $V_2(s) = \frac{2}{s}$

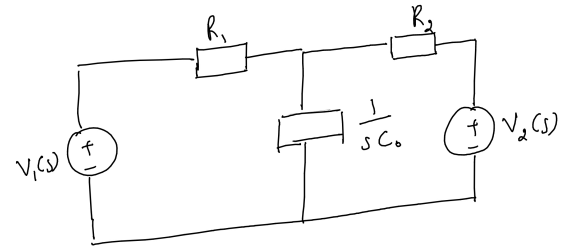


Fig. 2.4: hbbhb

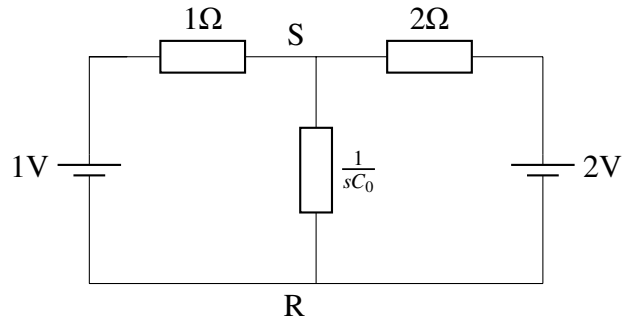


Fig. 2.5: s-domain resistive circuit

Now, let voltage at point R is 0. Applying KCL at point S i.e. junction of 3 resistors we get

$$\frac{V_S(s) - 0}{1/sC_0} + \frac{V_S(s) - 1/s}{1} + \frac{V_S(s) - 2/s}{2} = 0 \quad (2.18)$$

$$\Rightarrow V_S(s) = \frac{4}{s(3+2s)} \quad (2.19)$$

Now, the voltage across capacitor is given by

$$V_{C_0}(s) = V_S - V_R \quad (2.20)$$

$$= \frac{4}{s(3+2s)} \quad (2.21)$$

7. Find $v_{C_0}(t)$. Plot using python.

Solution: $v_{C_0}(t)$ is given by

$$v_{C_0}(t) = \mathcal{L}^{-1}[V_{C_0}(s)] \quad (2.22)$$

$$= \mathcal{L}^{-1}\left[\frac{4}{s(3+2s)}\right] \quad (2.23)$$

$$= \frac{4}{3} \left(\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+3/2}\right] \right) \quad (2.24)$$

$$= \frac{4}{3} \left(1 - e^{-\frac{3}{2}t} \right) u(t) \quad (2.25)$$

Run below python code to get fig. ?? which is of $v_{C_0}(t)$

8. Verify your result using ngspice.

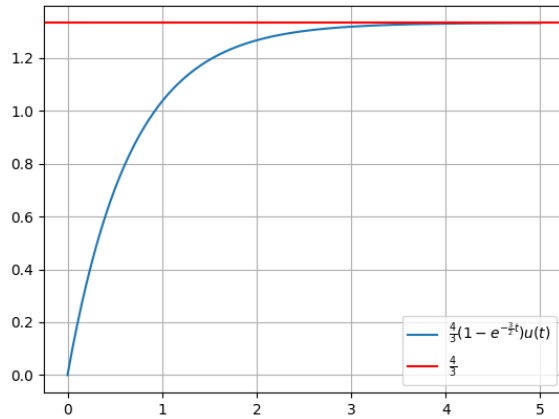


Fig. 2.6: $v_{C_0}(t)$ wrt t

Solution: Run the below ngspice code to get data and then use given python file to plot fig. 2.7

In fig. 2.7, we can see that both results are coinciding. So, our answer is correct

https://github.com/himanshukumargupta11012/EE3900_assignments/blob/main/cktsig/ques_2/2.8.cir
https://github.com/himanshukumargupta11012/EE3900_assignments/blob/main/cktsig/ques_2/2.8.py

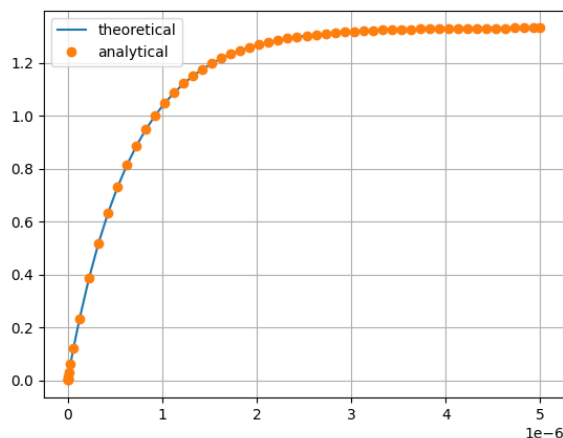


Fig. 2.7: theoretical v/s analytical

9. Obtain Fig. 2.4 using the equivalent differential equation.

Solution: Using KCL at the junction of resistors and capacitor, we get

$$\frac{v_C(t) - v_1(t)}{R_1} + \frac{v_C(t) - v_2(t)}{R_2} + \frac{dq(t)}{dt} = 0 \quad (2.26)$$

$$(2.27)$$

where $q(t)$ is charge on capacitor

Taking laplace both sides, we get

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sQ(s) - q(0) = 0 \quad (2.28)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0 \frac{Q(s)}{C_0} - 0 = 0 \quad (2.29)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0 V_C = 0 \quad (2.30)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + \frac{V_C(s)}{\frac{1}{sC_0}} = 0 \quad (2.31)$$

$$(2.32)$$

The above equation is same as Fig. 2.4

3 INITIAL CONDITIONS

1. Find q_2 in Fig. 2.1.

Solution: Since S is connected to Q for a long time so capacitor get fully charged and behave as a disconnected wire. So, the circuit will look like this

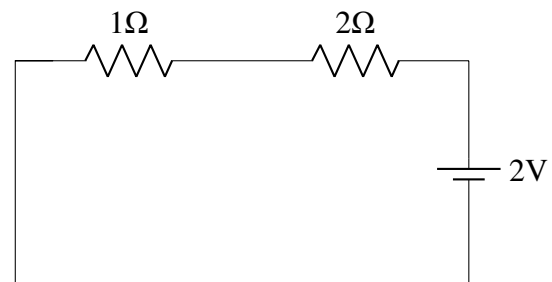


Fig. 3.1: when connected to Q for long time

So, current in the circuit is given by

$$i = \frac{2}{3}A \quad (3.1)$$

in counter-clockwise direction

So, voltage across capacitor is given by

$$v_{C_0} = \frac{2}{3} \times 1 = \frac{2}{3} \quad (3.2)$$

So, q_2 will be

$$q_2 = V_{C_0} C_0 \quad (3.3)$$

$$= \frac{2}{3} \mu C \quad (3.4)$$

2. Draw the equivalent s -domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.

Solution: s -domain resistive circuit would be similar to previous one except a difference that here we have to add a potential difference of $\frac{4}{3}$ because there is initial charge present in capacitor. So, the circuit looks like fig. 3.2

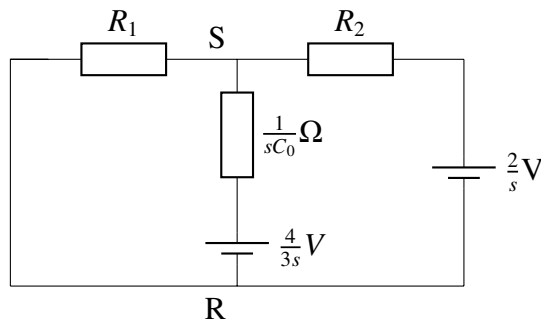


Fig. 3.2: s -domain resistive circuit

3. $V_{C_0}(s) = ?$

Solution: In fig. 3.2, let voltage at point R is 0. Applying KCL at point S i.e. junction of 3 resistors we get

$$\frac{V_S(s) - \frac{4}{3}}{1/sC_0} + \frac{V_S(s) - 0}{1} + \frac{V_S(s) - 2/s}{2} = 0 \quad (3.5)$$

$$V_S(s) = \frac{\frac{4}{3}s^2 + \frac{1}{C_0}}{s(s + \frac{3}{2C_0})} \quad (3.6)$$

Now, the voltage across capacitor is given by

$$V_{C_0}(s) = V_S - V_R \quad (3.7)$$

$$= \frac{\frac{4}{3}s^2 + \frac{1}{C_0}}{s(s + \frac{3}{2C_0})} \quad (3.8)$$

4. $v_{C_0}(t) = ?$ Plot using python.

Solution: $v_{C_0}(t)$ is given by

$$v_{C_0}(t) = \mathcal{L}^{-1} [V_{C_0}(s)] \quad (3.9)$$

$$= \mathcal{L}^{-1} \left[\frac{\frac{4}{3}s^2 + \frac{1}{C_0}}{s(s + \frac{3}{2C_0})} \right] \quad (3.10)$$

$$= \frac{4}{3} \mathcal{L}^{-1} [1] + \mathcal{L}^{-1} \left[\frac{2}{3s} \right] - \mathcal{L}^{-1} \left[-\frac{8}{3(s + 3/2)} \right] \quad (3.11)$$

$$= \frac{2}{3} (1 + e^{-\frac{3}{2}t}) u(t) \quad (3.12)$$

Run below python code to get fig. 3.3 which is of $v_{C_0}(t)$

https://github.com/himanshukumargupta11012/EE3900_assignments/blob/main/cktsig/ques_3/3.4.py

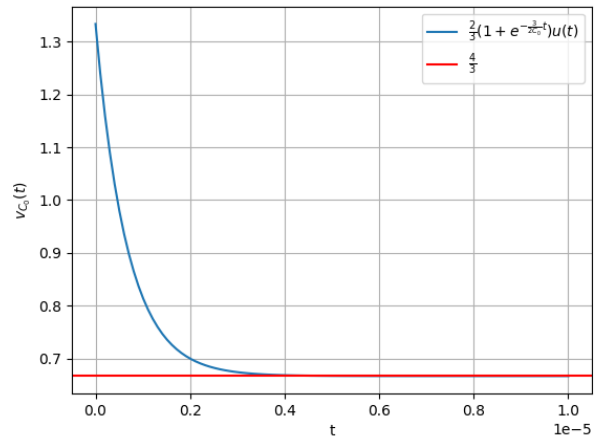


Fig. 3.3: $v_{C_0}(t)$ wrt t

5. Verify your result using ngspice.

Solution: Run the below ngspice code to get data and then use given python file to plot fig. 3.4

In fig. 3.4, we can see that both results are coinciding. So, our answer is correct

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https://github.com/himanshukumargupta11012/EE3900_assignments/blob/main/cktsig/ques_3/3.5.py

6. Find $v_{C_0}(0-)$, $v_{C_0}(0+)$ and $v_{C_0}(\infty)$.

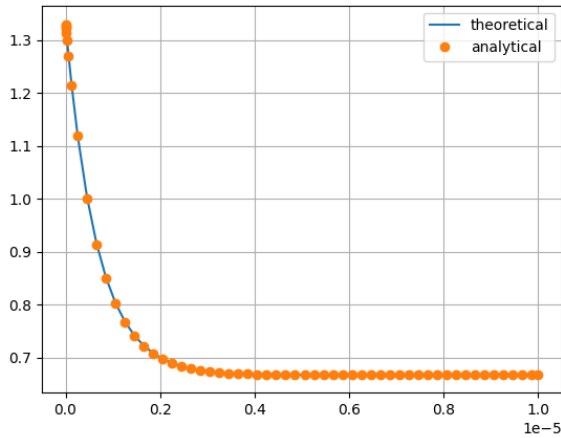


Fig. 3.4: theoretical v/s analytical

Solution: Charge not changes abruptly. So,

$$v_C(0^-) = v_C(0^+) = v_C(0) \quad (3.13)$$

$$= \frac{2}{3} \left(1 + e^{-\frac{3 \times 10^6 \times 0}{2}} \right) u(0) \quad (3.14)$$

$$= \frac{4}{3} \quad (3.15)$$

and

$$v_C(\infty) = \lim_{t \rightarrow \infty} \frac{2}{3} \left(1 + e^{-\frac{3 \times 10^6 t}{2}} \right) u(t) \quad (3.16)$$

$$= \frac{2}{3} \quad (3.17)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.

Solution: Using KCL at the junction of resistors and capacitor after joining S to Q in fig. 2.2, we get

$$\frac{v_C(t) - 0}{R_1} + \frac{v_C(t) - v_2(t)}{R_2} + \frac{dq(t)}{dt} = 0 \quad (3.18)$$

$$(3.19)$$

where $q(t)$ is charge on capacitor

Taking laplace both sides, we get

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sQ(s) - q(0) = 0 \quad (3.20)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0 \frac{Q(s)}{C_0} - q(0) = 0 \quad (3.21)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0 V_C - \frac{4}{3} C_0 = 0 \quad (3.22)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + \frac{V_C(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \quad (3.23)$$

$$(3.24)$$

The above equation is same as equation of Fig. in problem 3.2

4 BILINEAR TRANSFORM

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
2. Find $H(s)$ considering the output voltage at the capacitor.
3. Plot $H(s)$. What kind of filter is it?
4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.1)$$

5. Find $H(z)$.
6. How can you obtain $H(z)$ from $H(s)$?