

# Circuits and Transforms

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*Abstract*—This manual provides a simple introduction to Transforms

## 1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of  $g(t)$  is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

## 2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes  $q_1 \mu C$ . Then S is switched to position Q. After a long time, the charge on the capacitor is  $q_2 \mu C$ .

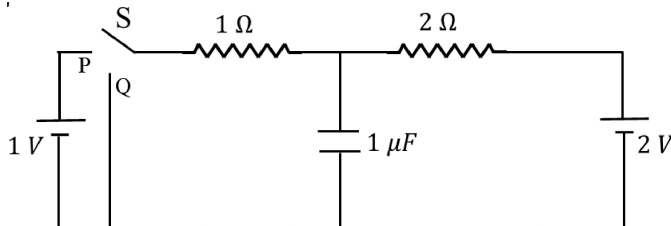


Fig. 2.1

2. Draw the circuit using latex-tikz. **Solution:** See fig. 2.2

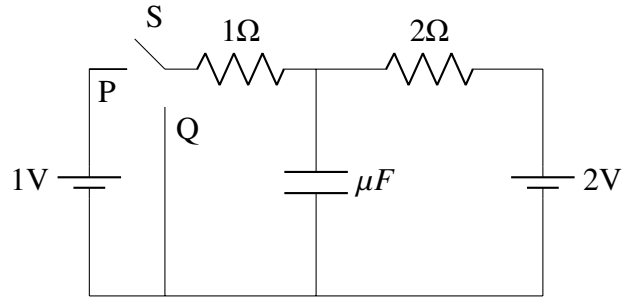


Fig. 2.2: Given Circuit

3. Find  $q_1$ . **Solution:** On connecting S to P for a long time capacitor get fully charged behave like disconnected wire. So, circuit will look like 2.3 On applying KVL on circuit in fig.

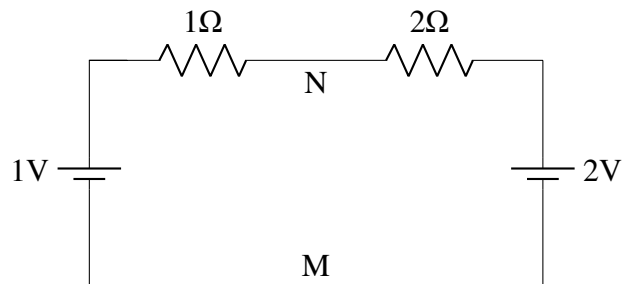


Fig. 2.3: circuit when connected to P for long time

2.3 we get current as,

$$i = 1/3 \quad (2.1)$$

in counter-clockwise direction So, PD across the capacitor i.e. across M and N is given by,

$$v_{C_0} = 1 + \frac{1}{3} \times 1 \quad (2.2)$$

$$= \frac{4}{3} \quad (2.3)$$

So, charge on capacitor will be

$$q_1 = v_{C_0} C_0 \quad (2.4)$$

$$= \frac{4}{3} \mu C \quad (2.5)$$

4. Show that the Laplace transform of  $u(t)$  is  $\frac{1}{s}$

and find the ROC. **Solution:** Laplace of  $u(t)$  is given by

$$U(s) = \int_{-\infty}^{\infty} u(t)e^{-st} dt \quad (2.6)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.7)$$

$$= \left[ \frac{e^{-st}}{-s} \right]_0^{\infty} \quad (2.8)$$

$$= \frac{1}{s} \quad (2.9)$$

Since  $e^{-st}$  is defined at  $t \rightarrow \infty$  only for  $s > 0$   
So, ROC will be  $s > 0$

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{H}} L \frac{1}{s+a}, \quad a > 0 \quad (2.10)$$

and find the ROC. **Solution:**

$$\mathcal{L}[e^{-at}u(t)] = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st} dt \quad (2.11)$$

$$= \int_0^{\infty} e^{-at}u(t)e^{-st} dt \quad (2.12)$$

$$= \int_0^{\infty} e^{-(a+s)t} dt \quad (2.13)$$

$$= \left[ \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} \quad (2.14)$$

$$= \frac{1}{s+a} \quad (2.15)$$

Since  $e^{-(s+a)t}$  is defined at  $t \rightarrow \infty$  only for  $s > -a$   
So, ROC will be  $s > -a$

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

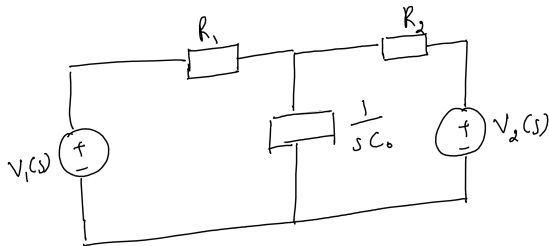


Fig. 2.4: hbhb

$$u(t) \xleftrightarrow{\mathcal{H}} LV_1(s) \quad (2.16)$$

$$2u(t) \xleftrightarrow{\mathcal{H}} LV_2(s) \quad (2.17)$$

Find the voltage across the capacitor  $V_{C_0}(s)$ .

**Solution:** from above 2 equations we can

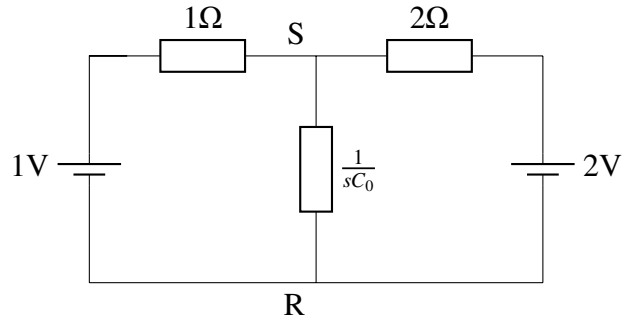


Fig. 2.5: s-domain resistive circuit

conclude that  $V_1(s) = \frac{1}{s}$  and  $V_2(s) = \frac{2}{s}$  Now, let voltage at point R is 0. Applying KCL at point S i.e. junction of 3 resistors we get

$$\frac{V_S(s) - 0}{1/sC_0} + \frac{V_S(s) - 1/s}{1} + \frac{V_S(s) - 2/s}{2} = 0 \quad (2.18)$$

$$\Rightarrow V_S(s) = \frac{4}{s(3+2s)} \quad (2.19)$$

Now, the voltage across capacitor is given by

$$V_{C_0}(s) = V_S - V_R \quad (2.20)$$

$$= \frac{4}{s(3+2s)} \quad (2.21)$$

7. Find  $v_{C_0}(t)$ . Plot using python. **Solution:**  $v_{C_0}(t)$  is given by

$$v_{C_0}(t) = \mathcal{L}^{-1}[V_{C_0}(s)] \quad (2.22)$$

$$= \mathcal{L}^{-1}\left[\frac{4}{s(3+2s)}\right] \quad (2.23)$$

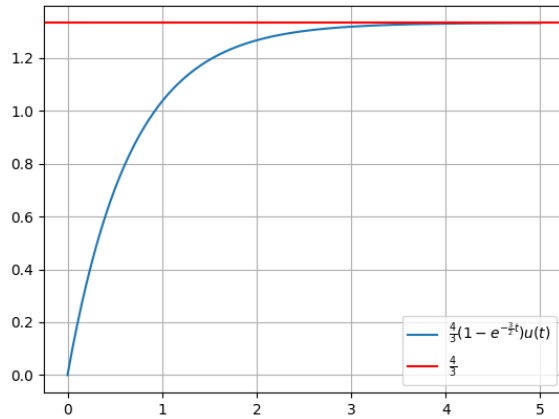
$$= \frac{4}{3} \left( \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+3/2}\right] \right) \quad (2.24)$$

$$= \frac{4}{3} (1 - e^{-\frac{3}{2}t}) u(t) \quad (2.25)$$

Run below python code to get fig. ?? which is of  $v_{C_0}(t)$

8. Verify your result using ngspice. **Solution:** Run the below ngspice code to get data and then use given python file to plot fig. 2.7 In fig. 2.7, we can see that both results are coinciding. So, our answer is correct

[https://github.com/himanshukumargupta11012/EE3900\\_assignments/blob/main/cktsig/ques\\_2/2.8.cir](https://github.com/himanshukumargupta11012/EE3900_assignments/blob/main/cktsig/ques_2/2.8.cir)  
<https://github.com/himanshukumargupta11012/>

Fig. 2.6:  $v_{C_0}(t)$  wrt  $t$ 

EE3900\_assignments/blob/main/cktsig/  
ques\_2/2.8.py

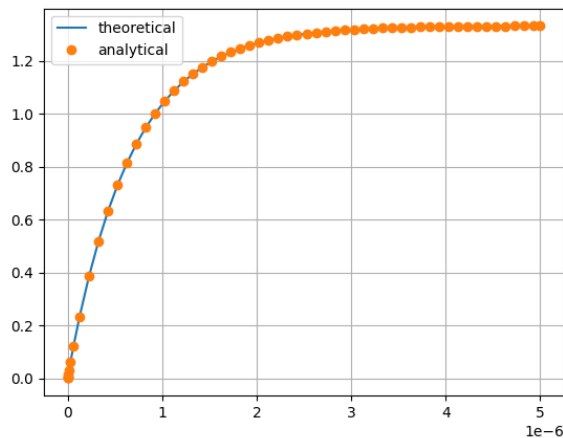


Fig. 2.7: theoretical v/s analytical

9. Obtain Fig. 2.4 using the equivalent differential equation. **Solution:** Using KCL at the junction of resistors and capacitor, we get

$$\frac{v_C(t) - v_1(t)}{R_1} + \frac{v_C(t) - v_2(t)}{R_2} + \frac{dq(t)}{dt} = 0 \quad (2.26)$$

$$(2.27)$$

where  $q(t)$  is charge on capacitor Taking

laplace both sides, we get

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sQ(s) - q(0) = 0 \quad (2.28)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0 \frac{Q(s)}{C_0} - 0 = 0 \quad (2.29)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0 V_C = 0 \quad (2.30)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + \frac{V_C(s)}{\frac{1}{sC_0}} = 0 \quad (2.31)$$

$$(2.32)$$

The above equation is same as Fig. 2.4

### 3 INITIAL CONDITIONS

- Find  $q_2$  in Fig. 2.1. **Solution:** Since S is connected to Q for a long time so capacitor get fully charged and behave as a disconnected wire. So, the circuit will look like this

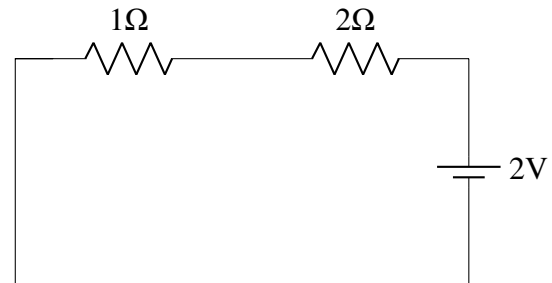


Fig. 3.1: when connected to Q for long time

So, current in the circuit is given by

$$i = \frac{2}{3}A \quad (3.1)$$

in counter-clockwise direction So, voltage across capacitor is given by

$$v_{C_0} = \frac{2}{3} \times 1 = \frac{2}{3} \quad (3.2)$$

So,  $q_2$  will be

$$q_2 = V_{C_0} C_0 \quad (3.3)$$

$$= \frac{2}{3} \mu C \quad (3.4)$$

- Draw the equivalent  $s$ -domain resistive circuit when S is switched to position Q. Use variables

$R_1, R_2, C_0$  for the passive elements. Use latex-tikz. **Solution:**  $s$ -domain resistive circuit would be similar to previous one except a difference that here we have to add a potential difference of  $\frac{4}{3}$  because there is initial charge present in capacitor. So, the circuit looks like fig. 3.2

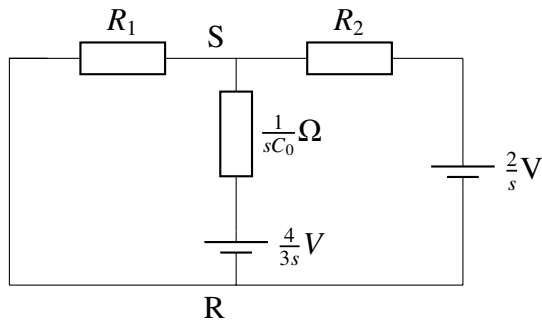


Fig. 3.2:  $s$ -domain resistive circuit

3.  $V_{C_0}(s) = ?$  **Solution:** In fig. 3.2, let voltage at point R is 0. Applying KCL at point S i.e. junction of 3 resistors we get

$$\frac{V_S(s) - \frac{4}{3}}{1/sC_0} + \frac{V_S(s) - 0}{1} + \frac{V_S(s) - 2/s}{2} = 0 \quad (3.5)$$

$$V_S(s) = \frac{\frac{4}{3}s^2 + \frac{1}{C_0}}{s(s + \frac{3}{2C_0})} \quad (3.6)$$

Now, the voltage across capacitor is given by

$$V_{C_0}(s) = V_S - V_R \quad (3.7)$$

$$= \frac{\frac{4}{3}s^2 + \frac{1}{C_0}}{s(s + \frac{3}{2C_0})} \quad (3.8)$$

4.  $v_{C_0}(t) = ?$  Plot using python. **Solution:**  $v_{C_0}(t)$  is given by

$$v_{C_0}(t) = \mathcal{L}^{-1} [V_{C_0}(s)] \quad (3.9)$$

$$= \mathcal{L}^{-1} \left[ \frac{\frac{4}{3}s^2 + \frac{1}{C_0}}{s(s + \frac{3}{2C_0})} \right] \quad (3.10)$$

$$= \frac{4}{3} \mathcal{L}^{-1} [1] + \mathcal{L}^{-1} \left[ \frac{2}{3s} \right] - \mathcal{L}^{-1} \left[ -\frac{8}{3(s + 3/2)} \right] \quad (3.11)$$

$$= \frac{2}{3} (1 + e^{-\frac{3}{2}t}) u(t) \quad (3.12)$$

Run below python code to get fig. 3.3 which is of  $v_{C_0}(t)$

[https://github.com/himanshukumargupta11012/EE3900\\_assignments/blob/main/cktsig/ques\\_3/3.4.py](https://github.com/himanshukumargupta11012/EE3900_assignments/blob/main/cktsig/ques_3/3.4.py)

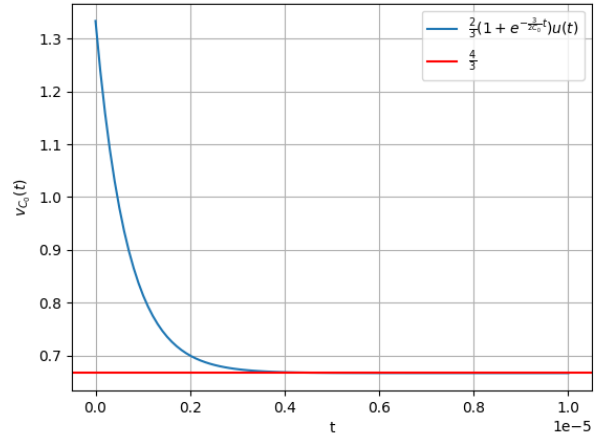


Fig. 3.3:  $v_{C_0}(t)$  wrt  $t$

5. Verify your result using ngspice. **Solution:** Run the below ngspice code to get data and then use given python file to plot fig. 3.4 In fig. 3.4, we can see that both results are coinciding. So, our answer is correct

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[https://github.com/himanshukumargupta11012/EE3900\\_assignments/blob/main/cktsig/ques\\_3/3.5.py](https://github.com/himanshukumargupta11012/EE3900_assignments/blob/main/cktsig/ques_3/3.5.py)

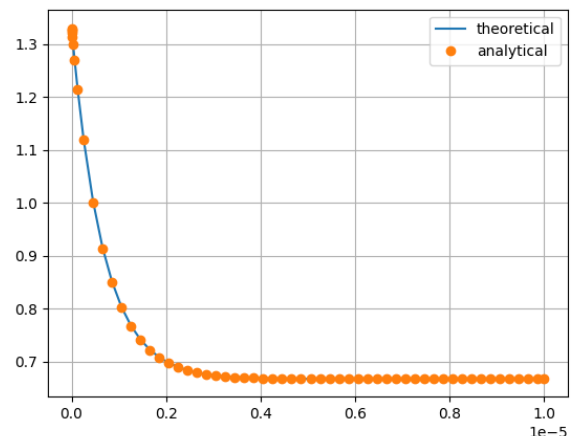


Fig. 3.4: theoretical v/s analytical

6. Find  $v_{C_0}(0-)$ ,  $v_{C_0}(0+)$  and  $v_{C_0}(\infty)$ . **Solution:** Charge not changes abruptly. So,

$$v_C(0^-) = v_C(0^+) = v_C(0) \quad (3.13)$$

$$= \frac{2}{3} \left( 1 + e^{-\frac{3 \times 10^6 \times 0}{2}} \right) u(0) \quad (3.14)$$

$$= \frac{4}{3} \quad (3.15)$$

and

$$v_C(\infty) = \lim_{t \rightarrow \infty} \frac{2}{3} \left( 1 + e^{-\frac{3 \times 10^6 t}{2}} \right) u(t) \quad (3.16)$$

$$= \frac{2}{3} \quad (3.17)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation. **Solution:** Using KCL at the junction of resistors and capacitor after joining S to Q in fig. 2.2, we get

$$\frac{v_C(t) - 0}{R_1} + \frac{v_C(t) - v_2(t)}{R_2} + \frac{dq(t)}{dt} = 0 \quad (3.18)$$

$$(3.19)$$

where  $q(t)$  is charge on capacitor Taking laplace both sides, we get

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sQ(s) - q(0) = 0 \quad (3.20)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0 \frac{Q(s)}{C_0} - q(0) = 0 \quad (3.21)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0 V_C - \frac{4}{3} C_0 = 0 \quad (3.22)$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + \frac{V_C(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \quad (3.23)$$

$$(3.24)$$

The above equation is same as equation of Fig. in problem 3.2

#### 4 BILINEAR TRANSFORM

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.

**Solution:** Applying KCL after switching S to Q, we get differential equation

$$\frac{v_{C_0}(t) - 0}{R_1} + \frac{v_{C_0}(t) - v_2(t)}{R_2} + \frac{dq(t)}{dt} = 0 \quad (4.1)$$

$$\frac{v_{C_0}(t) - 0}{R_1} + \frac{v_{C_0}(t) - v_2(t)}{R_2} + C_0 \frac{d\{v_{C_0}(t)\}}{dt} = 0 \quad (4.2)$$

2. Find  $H(s)$  considering the output voltage at the capacitor.

**Solution:** Taking laplace of eqn (4.1), we get

$$\frac{V_{C_0}(s) - 0}{R_1} + \frac{V_{C_0}(s) - V_2(s)}{R_2} + sQ(s) - q(0) = 0 \quad (4.3)$$

$$\frac{V_{C_0}(s) - 0}{R_1} + \frac{V_{C_0}(s) - V_2(s)}{R_2} + sC_0 V_{C_0}(s) = 0 \quad (4.4)$$

$$\Rightarrow \frac{V_{C_0}(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (4.5)$$

$$\Rightarrow H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad \therefore H(s) = \frac{O}{I} \quad (4.6)$$

Since at starting, capacitor is uncharged so  $q(0) = 0$  in above equation.

We know that

$$H(s) = \frac{\text{output}}{\text{input}} \quad (4.7)$$

$$= \frac{V_{C_0}(s)}{V_2(s)} \quad (4.8)$$

$$= \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad \text{from (4.5)} \quad (4.9)$$

3. Plot  $H(s)$ . What kind of filter is it?

**Solution:** Run the following code to get fig 4.1

```
https://github.com/himanshukumargupta11012/EE3900_assignments/blob/main/cktsig/ques_4/4.3.py
```

As  $\omega$  increases,  $s$  increases since  $s = j\omega$ . And as  $s$  increases  $H(s)$  decreases as we can see from fig. 4.1 or equation (4.9).

So, conclusively as  $\omega$  increases,  $H(s)$  decreases. As high frequency passes from this it becomes negligible which is equivalent to removing high frequency. Hence, it is low-pass filter.

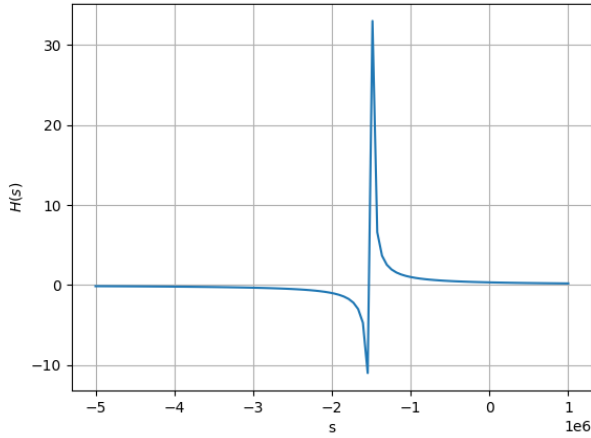


Fig. 4.1:  $H(s)$  wrt  $s$

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.10)$$

**Solution:** From equation (4.2)

$$\frac{v_{C_0}(t) - 0}{R_1} + \frac{v_{C_0}(t) - v_2(t)}{R_2} + C_0 \frac{d\{v_{C_0}(t)\}}{dt} = 0 \quad (4.11)$$

$$\Rightarrow \frac{d\{v_{C_0}(t)\}}{dt} = \frac{2u(t) - v_{C_0}(t)}{R_2 C_0} - \frac{v_{C_0}(t)}{R_1 C_0} \quad (4.12)$$

$$v_{C_0}(t) \Big|_n^{n+1} = \int_n^{n+1} \left( \frac{2u(t) - v_{C_0}(t)}{R_2 C_0} - \frac{v_{C_0}(t)}{R_1 C_0} \right) dt \quad (4.13)$$

From trapezoidal rule of integration

$$\int_a^b f(x) dx = (b - a) \frac{f(b) + f(a)}{2} \quad (4.14)$$

Using this rule in RHS of equation (4.13), we get

$$v_{C_0}(n+1) - v_{C_0}(n) = \frac{u(n+1) + u(n)}{R_2 C_0} \quad (4.15)$$

$$- \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \frac{v_{C_0}(n+1) + v_{C_0}(n)}{2C_0} \quad (4.16)$$

Since  $v_{C_0}$  is output so  $y(n) = v_{C_0}(n)$ .

So, difference equation will be

$$y(n+1) \left( 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) = \quad (4.17)$$

$$y(n) \left( 1 - \frac{1}{2R_1 C_0} - \frac{1}{2R_2 C_0} \right) + \frac{u(n+1) + u(n)}{R_2 C_0} \quad (4.18)$$

5. Find  $H(z)$ .

**Solution:** Taking z-transform of difference equation we get,

$$zY(z) \left( 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \quad (4.19)$$

$$= Y(z) \left( 1 - \frac{1}{2R_1 C_0} - \frac{1}{2R_2 C_0} \right) + U(z) \frac{z+1}{R_2 C_0} \quad (4.20)$$

$$\Rightarrow Y(z) \left( z + \frac{z}{2R_1 C_0} + \frac{z}{2R_2 C_0} - 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \quad (4.21)$$

$$= U(z) \frac{z+1}{R_2 C_0} \quad (4.22)$$

$$\Rightarrow \frac{Y(z)}{2U(z)} = \frac{\frac{z+1}{2R_2 C_0}}{\left( z \left( 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) - 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right)} \quad (4.23)$$

$$\Rightarrow H(z) = \frac{\frac{z+1}{2R_2 C_0}}{\left( z \left( 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) - 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right)} \quad (4.24)$$

$$\therefore H(s) = \frac{O}{I} \quad (4.25)$$

6. How can you obtain  $H(z)$  from  $H(s)$ ?

**Solution:** The z-transform can be obtained from laplace by substituting

$$s = \frac{2}{L} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (4.26)$$

where T is sampling time period or in simple words length of interval taken in trapezoidal equation. In our case  $T = (n+1) - n = 1$

Putting this in equation (4.9), we get

$$H(z) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + 2 \frac{1-z^{-1}}{1+z^{-1}} C_0} \quad (4.27)$$

$$= \frac{\frac{z+1}{2R_2 C_0}}{\left( z \left( 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) - 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right)} \quad (4.28)$$

7. Find  $y(n)$  from  $H(z)$  and verify whether  $y(n) = y(t)|_{t=n}$ .

**Solution:**

$$Y(z) = H(z)X(z) \quad (4.29)$$

$$= \frac{2.5 \times 10^5 (1 + z^{-1})}{z^{-1} (7.5 \times 10^5 - 1) + 7.5 \times 10^5 + 1} \frac{2}{1 - z^{-1}} \quad (4.30)$$

$$= \frac{2}{3} \left[ \frac{1}{1 - z^{-1}} - \frac{2}{3} \frac{1}{(7.5 \times 10^5 - 1) z^{-1} + (7.5 \times 10^5 + 1)} \right] \quad (4.31)$$

Now if we apply inverse z-transform by taking ROC  $|z| > 1$ ,

$$y(n) = \frac{2}{3} u(n) - \frac{2}{3} \left( -\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1} \right)^n u(n) \quad (4.32)$$

Here we used the following properties of z-transform,

$$u(n) \xleftrightarrow{z} \frac{1}{1 - z^{-1}}, |z| > 1 \quad (4.33)$$

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, |z| > |a| \quad (4.34)$$

Here we are sampling the signal at high frequency means at small intervals of time let say for  $n = 0.5 \times 10^{-5}, 1 \times 10^{-5} \dots$ , for that we can approximate  $y(n)$  as,

$$y(n) \approx \frac{2}{3} \left( 1 - \frac{1 - 7.5 \times 10^5 n}{1 + 7.5 \times 10^5 n} \right) u(n) \quad (4.35)$$

Now we will verify it from  $y(t)$ , for that consider

$$Y(s) = H(s)X(s) \quad (4.36)$$

$$= \frac{5 \times 10^5}{s + 1.5 \times 10^6} \frac{2}{s} \quad (4.37)$$

$$= \frac{2}{3} \left( \frac{1}{s} - \frac{1}{s + 1.5 \times 10^6} \right) \quad (4.38)$$

Now we will apply inverse laplace transform with ROC being  $\mathcal{R}\{s\} > 0$  on b.s.,

$$y(t) = \frac{2}{3} \left( 1 - e^{-1.5 \times 10^6 t} \right) u(t) \quad (4.39)$$

Now

$$e^{-1.5 \times 10^6 t} = \frac{e^{-7.5 \times 10^5 t}}{e^{7.5 \times 10^5 t}} \quad (4.40)$$

$$\approx \frac{1 - 7.5 \times 10^5 t}{1 + 7.5 \times 10^5 t}, \text{ when } t \ll 10^{-6} \quad (4.41)$$

Using that,

$$y(t) = \frac{2}{3} \left( 1 - \frac{1 - 7.5 \times 10^5 t}{1 + 7.5 \times 10^5 t} \right) u(t) \quad (4.42)$$

And

$$y(t)|_{t=n} = \frac{2}{3} \left( 1 - \frac{1 - 7.5 \times 10^5 n}{1 + 7.5 \times 10^5 n} \right) u(n) \quad (4.43)$$

As you can see both are turn to be the same. Download the following codes for the simulation and plot 4.2

```
wget https://github.com/
himanshukumargupta11012/
EE3900_assignments/blob/main/cktsig/
ques_4/4.7.py
wget https://github.com/
himanshukumargupta11012/
EE3900_assignments/blob/main/cktsig/
ques_4/4.7.cir
```

Then run the following command,

```
ngspice 4.7.cir
python3 4.7.py
```

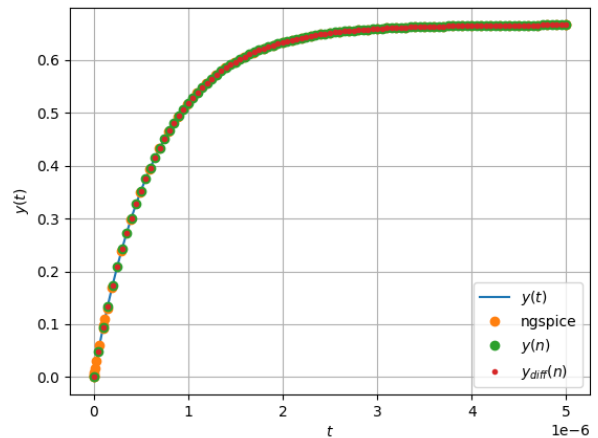


Fig. 4.2: Output of the filter vs time