Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_2/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#reading .wav file
# input_signal,fs=sf.read("Sound_Noise.wav
    ")
input_signal,fs = sf.read('ques_2/
    Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('ques_2/Sound_Without_Noise.wav'
    , output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_Without_Noise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as back-

ground noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: The following code yields Fig. 3.1.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_3/3.1_2.py
```

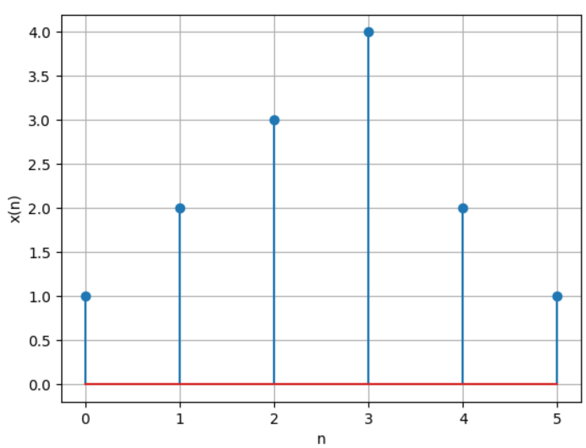

Fig. 3.1: $x(n)$ wrt n

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_3/3.1_2.py
```

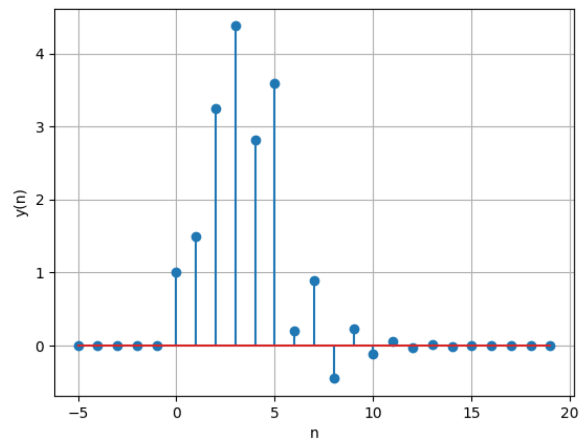

Fig. 3.2: y(n) wrt n

3.3 Repeat the above exercise using a C code.
Solution: Download and run the following code

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_3/3.3.c
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1)

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

Let $m = n - 1$.Then

$$\mathcal{Z}\{x(n-1)\} = \sum_{m+1=-\infty}^{\infty} x(m)z^{-m-1} \tag{4.5}$$
$$= z^{-1}\sum_{m=-\infty}^{\infty} x(m)z^{-m} \tag{4.6}$$
$$= z^{-1}X(z) \tag{4.7}$$

Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.8}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.
Solution: Z-transform of x(n),$X(z)$ is given by

$$X(n) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.9}$$
$$= \sum_{n=0}^{5} x(n)z^{-n} \tag{4.10}$$
$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.11}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.12}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.8) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.13}$$
$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.14}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.17}$$

Solution: Z-transform of $\delta(n)$ is given by

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \tag{4.18}$$
$$= 1 \tag{4.19}$$

and from (4.16),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.20}$$
$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.21}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.22}$$

Solution: Z-transform is given by

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n)z^{-n} \tag{4.23}$$
$$= \sum_{n=0}^{\infty} (az^{-1})^n \tag{4.24}$$
$$= \frac{1}{1 - az^{-1}}, \quad |z| > |a| \tag{4.25}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.26}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Comment.Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.
Solution: $H(e^{jw})$ is given by

$$H(e^{jw}) = \frac{1 + (e^{jw})^{-2}}{1 + \frac{1}{2}(e^{jw})^{-1}} \tag{4.27}$$
$$= 2\frac{1 + \cos(-2\omega) + j\sin(-2\omega)}{2 + \cos(-\omega) + j\sin(-\omega)} \tag{4.28}$$
$$= 2\frac{1 + \cos(2\omega) - j\sin(2\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.29}$$
$$= 2\frac{2\cos^2(\omega) - 2j\sin(\omega)\cos(\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.30}$$
$$= 4\cos(\omega)\frac{\cos(\omega) - j\sin(\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.31}$$

So,

$$|H(e^{jw})| = \frac{4|\cos(\omega)|}{\sqrt{5 + 4\cos(\omega)}} \tag{4.32}$$

Yes it is periodic because cos is periodic function. Period of numerator is π and period of denominator is 2π So,period of $|H(e^{jw})|$ would be LCM of π and 2π which is 2π The following code plots Fig. 4.6.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_4/4.5.py
```

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$.

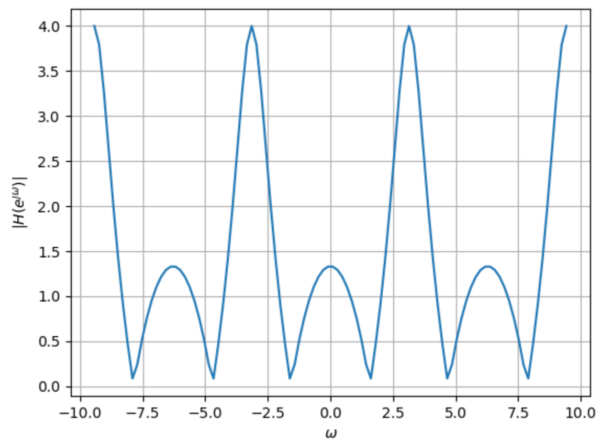

Fig. 4.6: Discret Time Fourier Transform

Solution: We know that

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \tag{4.33}$$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.34}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.35}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n}d\omega \tag{4.36}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \tag{4.37}$$

$$\tag{4.38}$$

case-1:If$(n \neq k)$

$$= \frac{1}{2\pi}\sum_{k\neq n} h(k)\frac{e^{j\omega(n-k)}}{j(n-k)}\Bigg]_{-\pi}^{\pi} \tag{4.39}$$

$$= \frac{1}{2\pi}\sum_{k\neq n} h(k)\frac{2\sin(\pi(n-k))}{(n-k)} \tag{4.40}$$

$$= 0 \tag{4.41}$$

case-2:If$(n = k)$

$$= \frac{1}{2\pi}h(n)\int_{-\pi}^{\pi} d\omega \tag{4.42}$$

$$= h(n) \tag{4.43}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.14).
Solution: $H(z)$ is given by

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} = \frac{2+2z^{-2}}{2+z^{-1}} \tag{5.2}$$

$$2z^{-1} - 4 \tag{5.3}$$

$$z^{-1} + 2 \;\big)\; 2z^{-2} + 2 \tag{5.4}$$

$$2z^{-2} + 4z^{-1} \tag{5.5}$$

$$-4z^{-1} + 2 \tag{5.6}$$

$$-\,4z^{-1} - 8 \tag{5.7}$$

$$10 \tag{5.8}$$

So,

$$H(z) = 2z^{-1} - 4 + \frac{10}{z^{-1}+2} \tag{5.9}$$

$$= 2z^{-1} - 4 + \frac{5}{\frac{1}{2}z^{-1}+1} \tag{5.10}$$

$$= 2z^{-1} - 4 + 5\sum_{n=0}^{\infty}\left(-\frac{z^{-1}}{2}\right)^n \tag{5.11}$$

$$= 1 - \frac{1}{2}z^{-1} + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.12}$$

So,h(n) will be given by

$$h(n) = \begin{cases} 5\times\left(-\frac{1}{2}\right)^n & n \geq 2 \\ \left(-\frac{1}{2}\right)^n & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.13}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.14}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.14),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.15}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.16}$$

using (4.22) and (4.8).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.

Solution: The following code plots Fig. 5.3.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_5/5.2.py
```

on simplfying we get h(n) as

$$\begin{cases} 5\times\left(-\frac{1}{2}\right)^n & n \geq 2 \\ \left(-\frac{1}{2}\right)^n & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.17}$$

$$\because 5\times\left(-\frac{1}{2}\right)^n \to 0 \quad \text{for} \quad n \to \infty \tag{5.18}$$

So, we can conclude that h(n) is bounded.

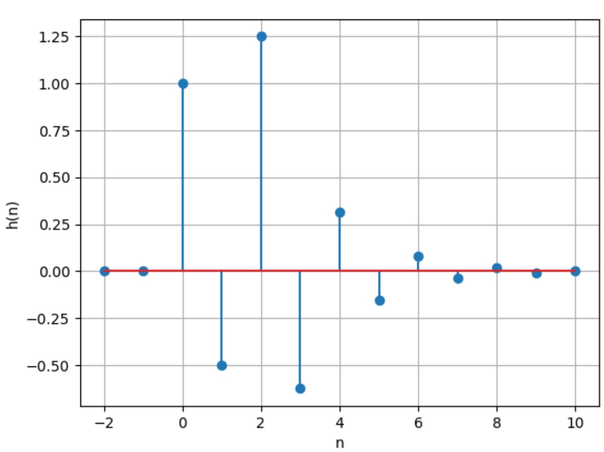

Fig. 5.3: $h(n)$ wrt n

5.4 Convergent? Justify using the ratio test.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.19}$$

Is the system defined by (3.2) stable for the impulse response in (5.14)?

Solution: For system of 3.2 ,$h(n)$ is defined in (5.17) So,

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=2}^{\infty} 5\times\left(-\frac{1}{2}\right)^n + \sum_{n=0}^{1}\left(-\frac{1}{2}\right)^n + \sum_{n=-\infty}^{-1} 0 \tag{5.20}$$

$$= 5\times\frac{1}{6} + \frac{1}{2} \tag{5.21}$$

$$= \frac{4}{3} \tag{5.22}$$

Since the sum is finite so the system is stable for impulsive response

5.6 Verify the above result using a python code.

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.23}$$

This is the definition of $h(n)$.

Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_5/5.4.py
```

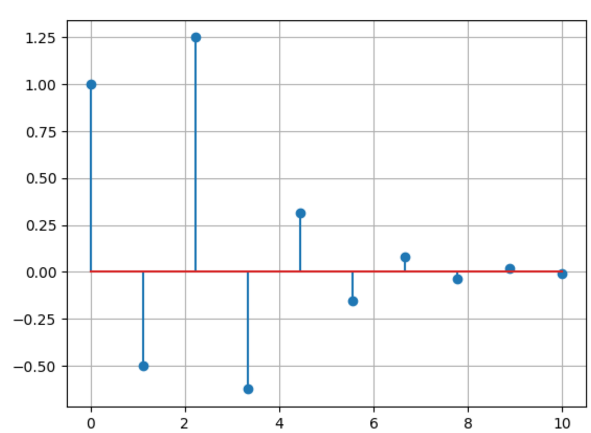

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.24}$$

Comment. The operation in (5.24) is known as

convolution.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_5/5.5.py
```

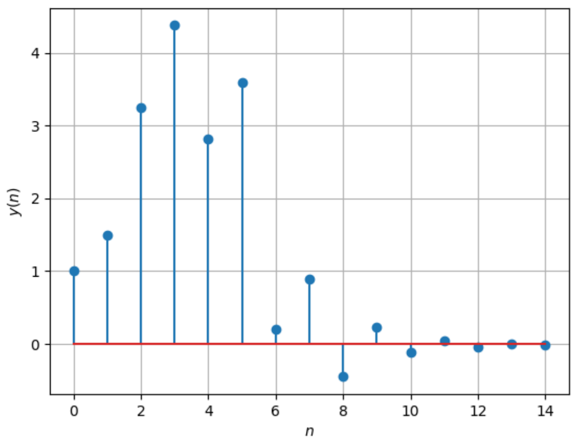

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.
Solution: We will . For finding the above convolution using topleitz matrix we have to find topleitz matrix of h(n).
h(n) is tending to 0 for large n. So,we take upto some n only. So,

$$h(n) = \begin{pmatrix} 1 \\ -.5 \\ . \\ . \end{pmatrix} \quad for \quad n = 0, 2...9 \tag{5.25}$$

So,topleitz matrix of $h(n)$ will be

$$top\{h(n)\} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -.5 & 1 & 0 & 0 & 0 \\ . & . & . & . & . & . \end{pmatrix} \tag{5.26}$$

and

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.27}$$

So,

$$x(n) * h(n) = top\{h(n)\}x(n) \tag{5.28}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \end{pmatrix} \tag{5.29}$$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.30}$$

Solution: From (5.24)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.31}$$

Replacing n-k with l,we get

$$y(n) = \sum_{n-l=-\infty}^{\infty} x(n-l)h(l) \tag{5.32}$$

$$= \sum_{-l=-\infty}^{\infty} x(n-l)h(l) \tag{5.33}$$

$$= \sum_{l=-\infty}^{\infty} x(n-l)h(l) \tag{5.34}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. **??**. Note that this is the same as $y(n)$ in Fig. **??**.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/yndft.
    py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
6.5 Wherever possible, express all the above equations as matrix equations.

7 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

7.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (7.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

7.2 Repeat all the exercises in the previous sections for the above a and b.

7.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

7.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

7.5 Modifying the code with different input parameters and to get the best possible output.