Circuits and Transforms

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

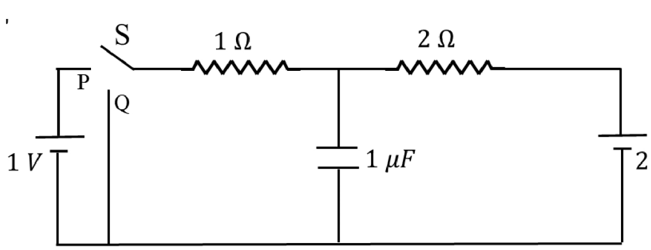

Fig. 2.1

2. Draw the circuit using latex-tikz. **Solution:** See fig. 2.2

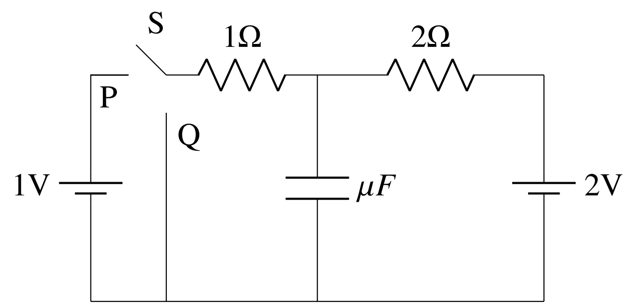

Fig. 2.2: Given Circuit

3. Find q_1. **Solution:** On connecting S to P for a long time capacitor get fully charged behave like disconnected wire. So, circuit will look like 2.3 On applying KVL on circuit in fig.

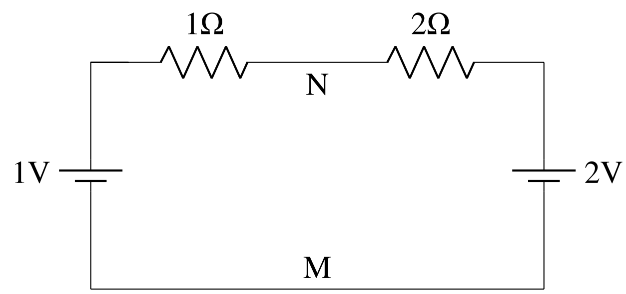

Fig. 2.3: circuit when conected to P for long time

2.3 we get current as,

$$i = 1/3 \tag{2.1}$$

in counter-clockwise direction So, PD across the capacitor i.e. across M and N is given by,

$$v_{C_0} = 1 + \frac{1}{3} \times 1 \tag{2.2}$$
$$= \frac{4}{3} \tag{2.3}$$

So, charge on capacitor will be

$$q_1 = v_{C_0} C_0 \tag{2.4}$$
$$= \frac{4}{3}\mu C \tag{2.5}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$

and find the ROC. **Solution:** Laplace of u(t) is given by

$$U(t) = \int_{-\infty}^{\infty} u(t)e^{-st}\,dt \tag{2.6}$$
$$= \int_{0}^{\infty} e^{-st}\,dt \tag{2.7}$$
$$= \left.\frac{e^{-st}}{-s}\right]_0^{\infty} \tag{2.8}$$
$$= \frac{1}{s} \tag{2.9}$$

Since e^{-st} is defined at $t \to \infty$ only for $s > 0$ So, ROC will be $s > 0$

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{H}} L\frac{1}{s+a}, \quad a > 0 \tag{2.10}$$

and find the ROC. **Solution:**

$$\mathcal{L}\left[e^{-at}u(t)\right] = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt \tag{2.11}$$
$$= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt \tag{2.12}$$
$$= \int_{0}^{\infty} e^{-(a+s)t}\,dt \tag{2.13}$$
$$= \left.\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty} \tag{2.14}$$
$$= \frac{1}{s+a} \tag{2.15}$$

Since $e^{-(s+a)t}$ is defined at $t \to \infty$ only for $s > -a$ So, ROC will be $s > -a$

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

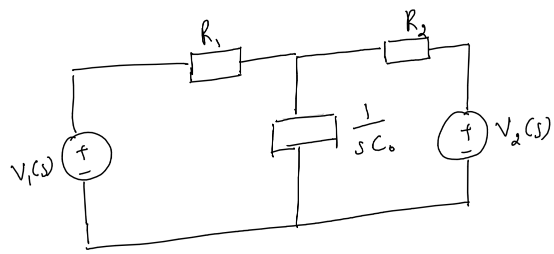

Fig. 2.4: hbhb

$$u(t) \xleftrightarrow{\mathcal{H}} LV_1(s) \tag{2.16}$$
$$2u(t) \xleftrightarrow{\mathcal{H}} LV_2(s) \tag{2.17}$$

Find the voltage across the capacitor $V_{C_0}(s)$. **Solution:** from above 2 equations we can

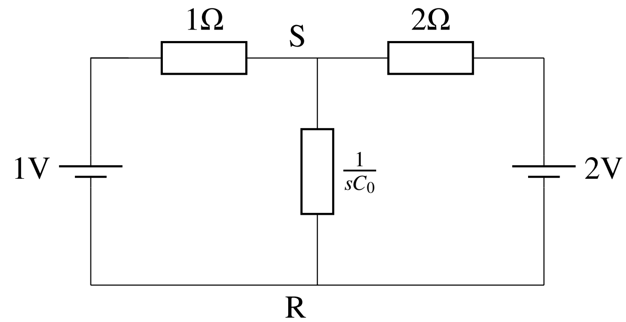

Fig. 2.5: s-domain resistive circuit

conclude that $V_1(s) = \frac{1}{s}$ and $V_2(s) = \frac{2}{s}$ Now, let voltage at point R is 0. Applying KCL at point S i.e. junction of 3 resistors we get

$$\frac{V_S(s) - 0}{1/sC_0} + \frac{V_S(s) - 1/s}{1} + \frac{V_S(s) - 2/s}{2} = 0 \tag{2.18}$$
$$\implies V_S(s) = \frac{4}{s(3+2s)} \tag{2.19}$$

Now, the voltage across capacitor is given by

$$V_{C_0}(s) = V_S - V_R \tag{2.20}$$
$$= \frac{4}{s(3+2s)} \tag{2.21}$$

7. Find $v_{C_0}(t)$. Plot using python. **Solution:** $v_{C_0}(t)$ is given by

$$v_{C_0}(t) = \mathcal{L}^{-1}\left[V_{C_0}(s)\right] \tag{2.22}$$
$$= \mathcal{L}^{-1}\left[\frac{4}{s(3+2s)}\right] \tag{2.23}$$
$$= \frac{4}{3}\left(\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+3/2}\right]\right) \tag{2.24}$$
$$= \frac{4}{3}\left(1 - e^{-\frac{3}{2}}\right)u(t) \tag{2.25}$$

Run below python code to get fig. **??** which is of $v_{C_0}(t)$

8. Verify your result using ngspice. **Solution:** Run the below ngspice code to get data and then use given python file to plot fig. 2.7 In fig. 2.7, we can see that both results are coinciding. So, our answer is correct

```
https://github.com/himanshukumargupta11012/
    EE3900_assignments/blob/main/cktsig/
    ques_2/2.8.cir
https://github.com/himanshukumargupta11012/
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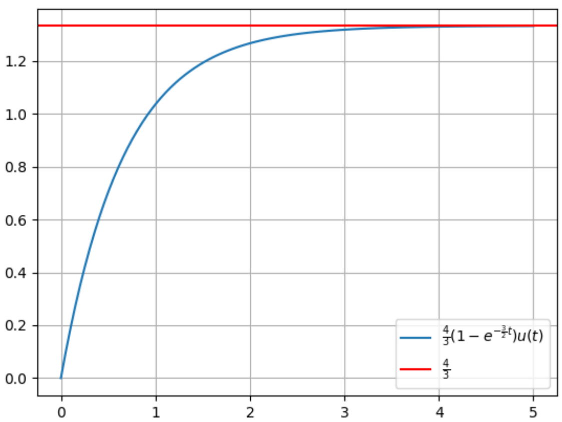

Fig. 2.6: $v_{C_0}(t)$ wrt t

EE3900_assignments/blob/main/cktsig/ques_2/2.8.py

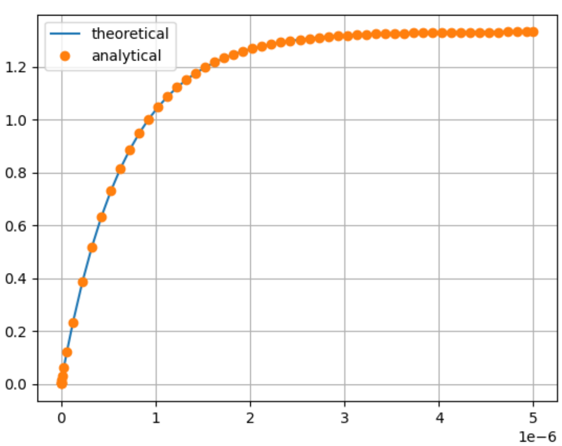

Fig. 2.7: theoretical v/s analytical

9. Obtain Fig. 2.4 using the equivalent differential equation. **Solution:** Using KCL at the junction of resistors and capacitor, we get

$$\frac{v_C(t) - v_1(t)}{R_1} + \frac{v_C(t) - v_2(t)}{R_2} + \frac{dq(t)}{dt} = 0 \tag{2.26}$$

$$\tag{2.27}$$

where $q(t)$ is charge on capacitor Taking laplace both sides, we get

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sQ(s) - q(0) = 0 \tag{2.28}$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0\frac{Q(s)}{C_0} - 0 = 0 \tag{2.29}$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0V_C = 0 \tag{2.30}$$

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + \frac{V_C(s)}{\frac{1}{sC_0}} = 0 \tag{2.31}$$

$$\tag{2.32}$$

The above equation is same as Fig. 2.4

3 Initial Conditions

1. Find q_2 in Fig. 2.1. **Solution:** Since S is connected to Q for a long time so capacitor get fully charged and behave as a disconnected wire. So, the circuit will look like this

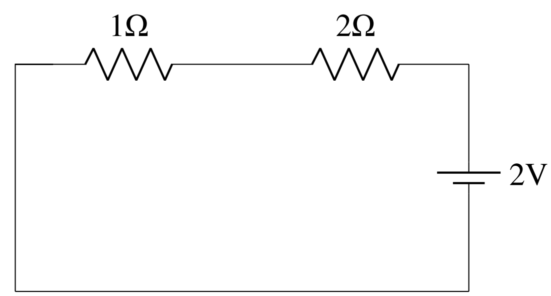

Fig. 3.1: when connected to Q for long time

So, current in the circuit is given by

$$i = \frac{2}{3}A \tag{3.1}$$

in counter-clockwise direction So, voltage across capacitor is given by

$$v_{C_0} = \frac{2}{3} \times 1 = \frac{2}{3} \tag{3.2}$$

So, q_2 will be

$$q_2 = V_{C_0}C_0 \tag{3.3}$$

$$= \frac{2}{3}\mu C \tag{3.4}$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables

R_1, R_2, C_0 for the passive elements. Use latex-tikz. **Solution:** s-domain resistive circuit would be similar to previous one except a difference that here we have to add a potential difference of $\frac{4}{3}$ because there is initial charge present in cpacitor. So, the circuit looks like fig. 3.2

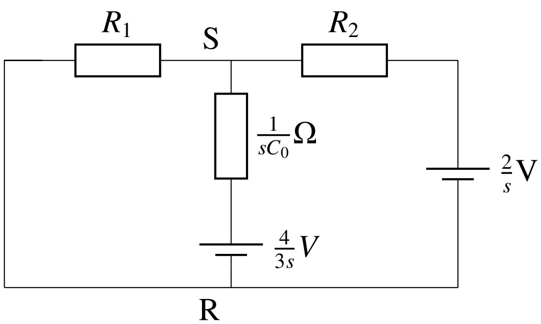

Fig. 3.2: s-domain resistive circuit

3. $V_{C_0}(s) = ?$ **Solution:** In fig. 3.2, let voltage at point R is 0. Applying KCL at point S i.e. junction of 3 resistors we get

$$\frac{V_S(s) - \frac{4}{3}}{1/sC_0} + \frac{V_S(s) - 0}{1} + \frac{V_S(s) - 2/s}{2} = 0 \tag{3.5}$$

$$V_S(s) = \frac{\frac{4}{3}s^2 + \frac{1}{C_0}}{s(s + \frac{3}{2C_0})} \tag{3.6}$$

Now, the voltage across capacitor is given by

$$V_{C_0}(s) = V_S - V_R \tag{3.7}$$
$$= \frac{\frac{4}{3}s^2 + \frac{1}{C_0}}{s(s + \frac{3}{2C_0})} \tag{3.8}$$

4. $v_{C_0}(t) = ?$ Plot using python. **Solution:** $v_{C_0}(t)$ is given by

$$v_{C_0}(t) = \mathcal{L}^{-1}\left[V_{C_0}(s)\right] \tag{3.9}$$
$$= \mathcal{L}^{-1}\left[\frac{\frac{4}{3}s^2 + \frac{1}{C_0}}{s(s + \frac{3}{2C_0})}\right] \tag{3.10}$$
$$= \frac{4}{3}\mathcal{L}^{-1}[1] + \mathcal{L}^{-1}\left[\frac{2}{3s}\right] - \mathcal{L}^{-1}\left[-\frac{8}{3(s+3/2)}\right] \tag{3.11}$$
$$= \frac{2}{3}\left(1 + e^{-\frac{3}{2}}\right)u(t) \tag{3.12}$$

Run below python code to get fig. 3.3 which is of $v_{C_0}(t)$

https://github.com/himanshukumargupta11012/EE3900_assignments/blob/main/cktsig/ques_3/3.4.py

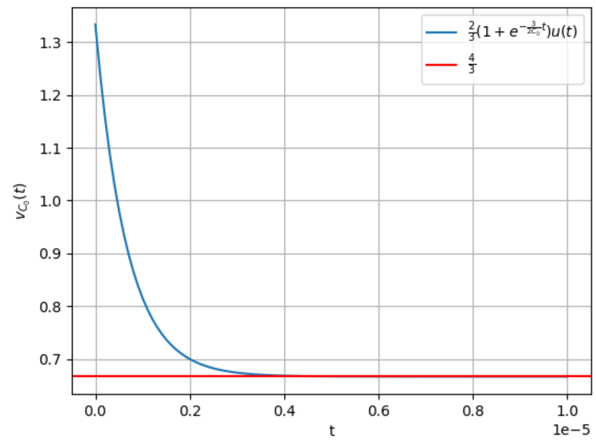

Fig. 3.3: $v_{C_0}(t)$ wrt t

5. Verify your result using ngspice. **Solution:** Run the below ngspice code to get data and then use given python file to plot fig. 3.4 In fig. 3.4, we can see that both results are coinciding. So, our answer is correct

https://github.com/himanshukumargupta11012/EE3900_assignments/blob/main/cktsig/ques_3/3.5.cir
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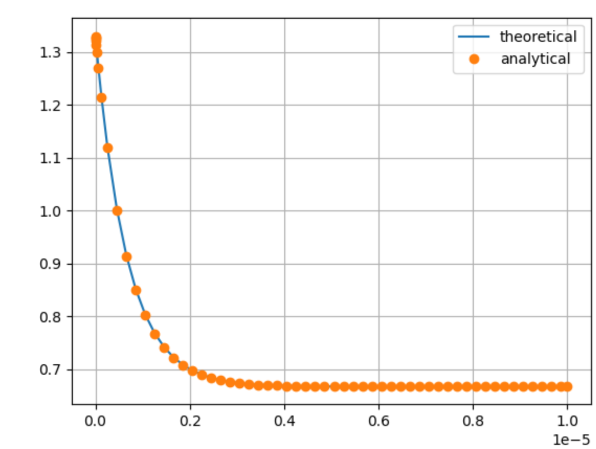

Fig. 3.4: theoretical v/s analytical

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$. **Solution:** Charge not changes abruptly.So,

$$v_C(0^-) = v_C(0^+) = v_C(0) \tag{3.13}$$
$$= \frac{2}{3}\left(1 + e^{-\frac{3\times10^6\times0}{2}}\right)u(0) \tag{3.14}$$
$$= \frac{4}{3} \tag{3.15}$$

and

$$v_C(\infty) = \lim_{t\to\infty} \frac{2}{3}\left(1 + e^{-\frac{3\times10^6 t}{2}}\right)u(t) \tag{3.16}$$
$$= \frac{2}{3} \tag{3.17}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation. **Solution:** Using KCL at the junction of resistors and capacitor after joining S to Q in fig. 2.2, we get

$$\frac{v_C(t) - 0}{R_1} + \frac{v_C(t) - v_2(t)}{R_2} + \frac{dq(t)}{dt} = 0 \tag{3.18}$$
$$\tag{3.19}$$

where $q(t)$ is charge on capacitor Taking laplace both sides, we get

$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sQ(s) - q(0) = 0 \tag{3.20}$$
$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0\frac{Q(s)}{C_0} - q(0) = 0 \tag{3.21}$$
$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + sC_0V_C - \frac{4}{3}C_0 = 0 \tag{3.22}$$
$$\frac{V_C(s) - 0}{R_1} + \frac{V_C(s) - V_2(s)}{R_2} + \frac{V_C(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \tag{3.23}$$
$$\tag{3.24}$$

The above equation is same as equation of Fig. in problem 3.2

4 Bilinear Transform

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.

Solution: Applyling KCL after switching S to Q, we get differential equation

$$\frac{v_{C_0}(t) - 0}{R_1} + \frac{v_{C_0}(t) - v_2(t)}{R_2} + \frac{dq(t)}{dt} = 0 \tag{4.1}$$
$$\frac{v_{C_0}(t) - 0}{R_1} + \frac{v_{C_0}(t) - v_2(t)}{R_2} + C_0\frac{d\left\{v_{C_0}(t)\right\}}{dt} = 0 \tag{4.2}$$

2. Find $H(s)$ considering the ouput voltage at the capacitor.
Solution: Taking laplace of eqn (4.1), we get

$$\frac{V_{C_0}(s) - 0}{R_1} + \frac{V_{C_0}(s) - V_2(s)}{R_2} + sQ(s) - q(0) = 0 \tag{4.3}$$
$$\frac{V_{C_0}(s) - 0}{R_1} + \frac{V_{C_0}(s) - V_2(s)}{R_2} + sC_0V_{C_0}(s) = 0 \tag{4.4}$$
$$\implies \frac{V_{C_0}(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{4.5}$$
$$\implies H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad \because H(s) = \frac{O}{I} \tag{4.6}$$

Since at starting, capacitor is uncharged so $q(0) = 0$ in above equation.
We know that

$$H(s) = \frac{\text{output}}{\text{input}} \tag{4.7}$$
$$= \frac{V_{C_0}(s)}{V_2(s)} \tag{4.8}$$
$$= \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad \text{from (4.5)} \tag{4.9}$$

3. Plot $H(s)$. What kind of filter is it?
Solution: Run the following code to get fig 4.1

```
https://github.com/himanshukumargupta11012/
    EE3900_assignments/blob/main/cktsig/
    ques_4/4.3.py
```

As ω increases, s increases since $s = j\omega$. And as s increases H(s) decreases as we can see from fig. 4.1 or equation (4.9).
So, conclusively as ω increses, $H(s)$ decreases. As high frequency passes from this it becomes negligible which is equivalent to removing high frequency. Hence, it is low-pass filter.

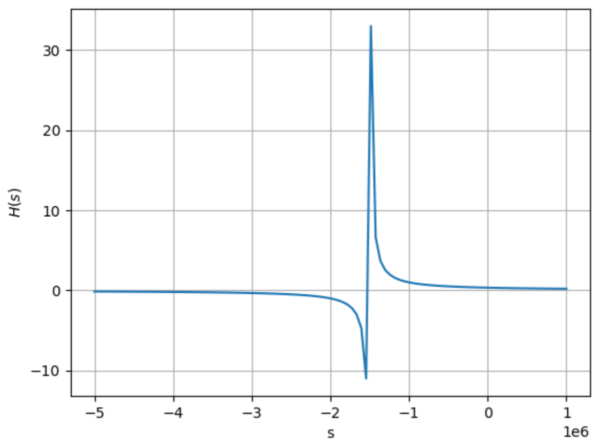

Fig. 4.1: $H(s)$ wrt s

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.10}$$

Solution: From equation (4.2)

$$\frac{v_{C_0}(t) - 0}{R_1} + \frac{v_{C_0}(t) - v_2(t)}{R_2} + C_0 \frac{d\{v_{C_0}(t)\}}{dt} = 0 \tag{4.11}$$

$$\implies \frac{d\{v_{C_0}(t)\}}{dt} = \frac{2u(t) - v_{C_0}(t)}{R_2 C_0} - \frac{v_{C_0}(t)}{R_1 C_0} \tag{4.12}$$

$$v_{C_0}(t)\Big]_n^{n+1} = \int_n^{n+1} \left(\frac{2u(t) - v_{C_0}(t)}{R_2 C_0} - \frac{v_{C_0}(t)}{R_1 C_0}\right) dt \tag{4.13}$$

From trapezoidal rule of integration

$$\int_a^b f(x)dx = (b-a)\,\frac{f(b) - f(a)}{2} \tag{4.14}$$

Using this rule in RHS of equation (4.13), we get

$$v_{C_0}(n+1) - v_{C_0}(n) = \frac{u(N+1+u(n)}{R_2 C_0} \tag{4.15}$$

$$-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{v_{C_0}(n+1) + v_{C_0}(n)}{2C_0} \tag{4.16}$$

Since v_{C_0} is output so $y(n) = v_{C_0}(n)$.

So, difference equation will be

$$y(n+1)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = \tag{4.17}$$

$$y(n)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{u(n+1) + u(n)}{R_2C_0} \tag{4.18}$$

5. Find $H(z)$.
Solution: Taking z-transform of difference equation we get,

$$zY(z)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) \tag{4.19}$$

$$= Y(z)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + U(z)\frac{z+1}{R_2C_0} \tag{4.20}$$

$$\implies Y(z)\left(z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) \tag{4.21}$$

$$= U(z)\frac{z+1}{R_2C_0} \tag{4.22}$$

$$\implies \frac{Y(z)}{2U(z)} = \frac{\frac{z+1}{2R_2C_0}}{\left(z\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)} \tag{4.23}$$

$$\implies H(z) = \frac{\frac{z+1}{2R_2C_0}}{\left(z\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)} \tag{4.24}$$

$$\because H(s) = \frac{O}{I} \tag{4.25}$$

6. How can you obtain $H(z)$ from $H(s)$?
Solution: The z-transfrom can be obtained from laplace by substituting

$$s = \frac{2}{L}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{4.26}$$

where T is sampling time period or in simple words length of interval taken in trapezoidal equation. In our case $T = (n+1) - n = 1$
Putting this in equation (4.9), we get

$$H(z) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + 2\frac{1-z^{-1}}{1+z^{-1}}C_0} \tag{4.27}$$

$$= \frac{\frac{z+1}{2R_2C_0}}{\left(z\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)} \tag{4.28}$$