Pingala Series

Himanshu Kumar Gupta

Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

Solution: Download and run the following python program to get fig. 1.1.

```
https://github.com/himanshukumargupta11012/
    EE3900_assignments/blob/main/pingala/
    ques_1/1.1.py
```

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

Solution: Download and run the following python program to get fig. 1.2.

```
https://github.com/himanshukumargupta11012/
    EE3900_assignments/blob/main/pingala/
    ques_1/1.1.py
```

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

Solution: Download and run the following python program to get fig. 1.3.

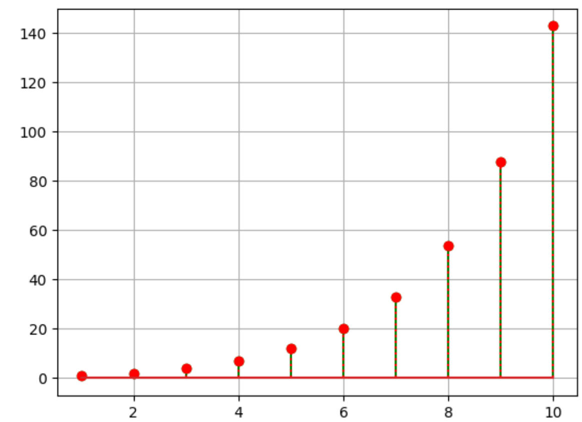

Fig. 1.1: Ques 1.1

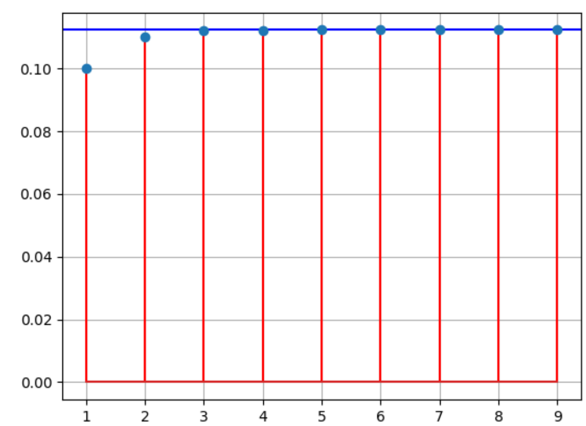

Fig. 1.2: Ques 1.2

```
https://github.com/himanshukumargupta11012/
    EE3900_assignments/blob/main/pingala/
    ques_1/1.1.py
```

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: Download and run the following

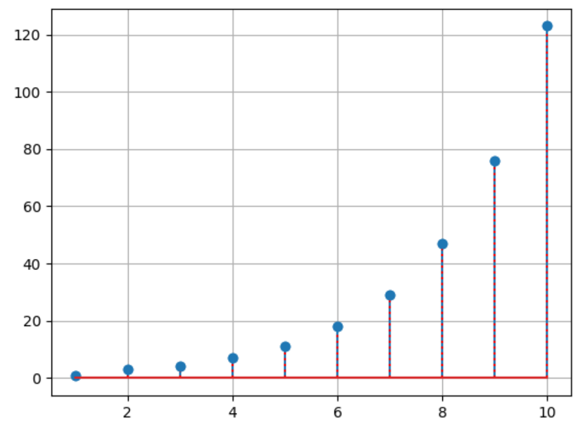

Fig. 1.3: Ques 1.3

python program to get fig. 1.4.
We can see that it is converging to $\frac{12}{89}$ instead of $\frac{8}{89}$. So the given equation is not correct.

```
https://github.com/himanshukumargupta11012/
    EE3900_assignments/blob/main/pingala/
    ques_1/1.1.py
```

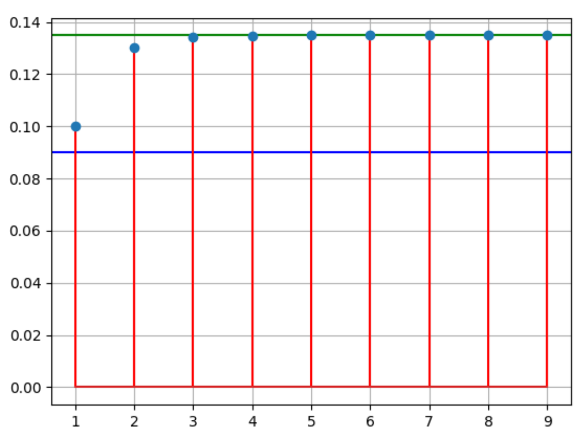

Fig. 1.4: Ques 1.4

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: Stem plot of $x(n)$ is plotted in figure 2.2 using below python code.

```
https://github.com/himanshukumargupta11012/
    EE3900_assignments/blob/main/pingala/
    ques_1/2.2.py
```

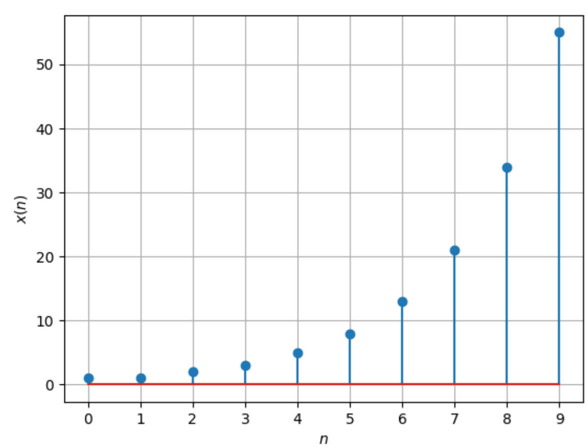

Fig. 2.2: stem plot of $x(n)$

2.3 Find $X^{+}(z)$.

Solution:

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \tag{2.3}$$

$$= x(0) + x(1) + \sum_{n=2}^{\infty}\{x(n-1) + x(n-2)\}z^{-n} \tag{2.4}$$

$$= 1 + z^{-1} + \sum_{n=2}^{\infty} x(n-1)z^{-n} + \sum_{n=2}^{\infty} x(n-2)z^{-n} \tag{2.5}$$

$$= 1 + z^{-1} + \sum_{m=1}^{\infty} x(m)z^{-m-1} + \sum_{p=0}^{\infty} x(p)z^{-p-2} \tag{2.6}$$

$$= 1 + z^{-1} + z^{-1}\{\sum_{m=0}^{\infty} x(m)z^{-m} - 1\} \tag{2.7}$$

$$= +z^{-2}\sum_{p=0}^{\infty} x(p)z^{-p} \tag{2.8}$$

$$= 1 + z^{-1} + z^{-1}\{X^{+}(z) - 1\} + z^{-2}X^{+}(z) \tag{2.9}$$

$$= 1 + z^{-1}X^{+}(z) + z^{-2}X^{+}(z) \tag{2.10}$$

$$\tag{2.11}$$

$$\implies X^{+}(z) = 1 + z^{-1}X^{+}(z) + z^{-2}X^{+}(z) \tag{2.12}$$

$$\implies X^{+}(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.13}$$

2.4 Find $x(n)$.

Solution:

$$X(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.14}$$

$$= \frac{z^2}{z^2 - z - 1} \tag{2.15}$$

$$= \frac{z^2}{(z-\alpha)(z-\beta)} \tag{2.16}$$

$$= \frac{1}{\alpha - \beta}\left(\frac{\alpha}{1 - \alpha z^{-1}} - \frac{\beta}{1 - \beta z^{-1}}\right) \tag{2.17}$$

$$\sum_{k=0}^{\infty} x(k)z^{-k} = \frac{1}{\alpha - \beta}\left(\sum_{l=0}^{\infty}\left(\alpha z^{-1}\right)^{l} - \left(\beta z^{-1}\right)^{l}\right) \tag{2.18}$$

$$\sum_{k=0}^{\infty} x(k)z^{-k} = \sum_{l=0}^{\infty}\frac{\alpha^{l} - \beta^{l}}{\alpha - \beta}z^{-l} \tag{2.19}$$

$$\tag{2.20}$$

So,from above equation we can conclude that

$$x(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 0 \tag{2.21}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.22}$$

Solution: Download and run the following python program to get fig. 2.5.

```
https://github.com/himanshukumargupta11012/
    EE3900_assignments/blob/main/pingala/
    ques_1/2.5.py
```

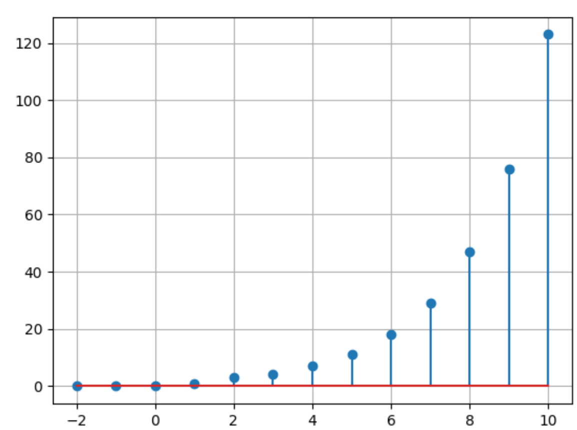

Fig. 2.5: y(n) in terms of x(k)

2.6 Find $Y^+(z)$.
Solution: We know that

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.23}$$

$$\implies \sum_{k=0}^{\infty} y(n)z^{-k} = \sum_{k=0}^{\infty} x(k-1)z^{-k} + \sum_{k=0}^{\infty} x(k+1)z^{-k} \tag{2.24}$$

$$\implies Y^+z = \sum_{n=-1}^{\infty} x(n)z^{-n-1} + \sum_{n=1}^{\infty} x(n)z^{-n+1} \tag{2.25}$$

$$= z^{-1}x(-1) + z^{-1}\sum_{n=0}^{\infty} x(n)z^{-n} + z\sum_{n=0}^{\infty} x(n)z^{-n} - zx(0) \tag{2.26}$$

$$= \left(z + z^{-1}\right)X^+(z) + z \tag{2.27}$$

$$= \frac{2z^{-1}+1}{1-z^{-1}-z^{-2}} \tag{2.28}$$

2.7 Find $y(n)$.
Solution:

$$y(n) = \mathcal{Z}^{-1}\left[Y(z)\right] \tag{2.29}$$

$$= \mathcal{Z}^{-1}\left[\frac{2z^{-1}+1}{1-z^{-1}-z^{-2}}\right] \tag{2.30}$$

$$2\mathcal{Z}^{-1}\left[\frac{z^{-1}}{1-z^{-1}-z^{-2}}\right] + \mathcal{Z}^{-1}\left[\frac{1}{1-z^{-1}-z^{-2}}\right] \tag{2.31}$$

$$= 2x(n-1) + x(n) \tag{2.32}$$

$$= 2\frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \tag{2.33}$$

$$= \alpha^{n+1} + \beta^{n+1} \tag{2.34}$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.1}$$

Solution:

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} x(k-1) \tag{3.2}$$

$$= \sum_{m=0}^{n-1} x(m) \tag{3.3}$$

First equality proved.
Now,

$$x(n) * u(n-1) = \sum_{k=-\infty}^{\infty} u((n-1)-k)x(k) \tag{3.4}$$

$$= \sum_{k=-\infty}^{n-1} x(k) \tag{3.5}$$

$$= \sum_{k=0}^{n-1} x(k) \quad \because x(n) = 0, \quad n < 0 \tag{3.6}$$

Second equality proved.
So,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.7}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.8}$$

can be expressed as

$$\left[x\left(n+1\right) - 1\right]u\left(n\right) \tag{3.9}$$

Solution:

$$a_{n+2} - 1 = x(n+1) - 1 \tag{3.10}$$

$$= \left(x(n+1) - 1\right)u(n) \tag{3.11}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{x\left(k\right)}{10^k} = \frac{1}{10}X^+\left(10\right) \tag{3.12}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=1}^{\infty} \frac{x(k-1)}{10^k} \tag{3.13}$$

$$= \frac{1}{10}\sum_{m=0}^{\infty} \frac{x(m)}{10^m} \tag{3.14}$$

First equality proved.
Now,

$$\frac{1}{10}\sum_{m=0}^{\infty} \frac{x(m)}{10^m} = \frac{1}{10}\sum_{m=0}^{\infty} x(m)10^{-m} \tag{3.15}$$

$$= \frac{1}{10}X^+\left(10\right) \tag{3.16}$$

Second equality proved.

So,

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.17)$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.18)$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \quad (3.19)$$

and find $W(z)$.
Solution: Since

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.20)$$

So,

$$\alpha^{n+1} + \beta^{n+1}, \quad n \geq 0 \quad (3.21)$$

$$= \alpha^{n+1} + \beta^{n+1} \quad (3.22)$$

Now,let

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \quad (3.23)$$

$$= y(n) \quad (3.24)$$

$$W(z) = \mathcal{Z}[w(n)] \quad (3.25)$$

$$= \mathcal{Z}[y(n)] \quad (3.26)$$

$$= Y^+(z) \quad (3.27)$$

$$= \frac{2z^{-1} + 1}{1 - z^{-1} - z^{-2}} \quad (3.28)$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.29)$$

Solution: Similar as question 3.3

3.6 Solve the JEE 2019 problem which is as follows
Let α and β be the roots of $x^2 - x - 1$ with $\alpha > \beta$. For all positive integers n. define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad , \quad n \geq 1 \quad (3.30)$$

$$b_1 = 1 \quad \text{and} \quad b_n = a_{n-1} + a_{n+1}, \quad n \geq 2 \quad (3.31)$$

then which of the following options is/are correct?

a) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$
b) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$
c) $\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1$
d) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

Solution: All parts in the question are already solved in before problems.
So, the answers are option a, b, c.