Digital Signal Processing

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Contents

Abstract—**This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_2/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#reading .wav file
# input_signal,fs=sf.read("Sound_Noise.wav
    ")
input_signal,fs = sf.read('ques_2/
    Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('ques_2/Sound_Without_Noise.wav'
    , output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_Without_Noise.wav. Play the file in the spectrogram in Problem 2.2.

What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: The following code yields Fig. 3.1.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_3/3.1_2.py
```

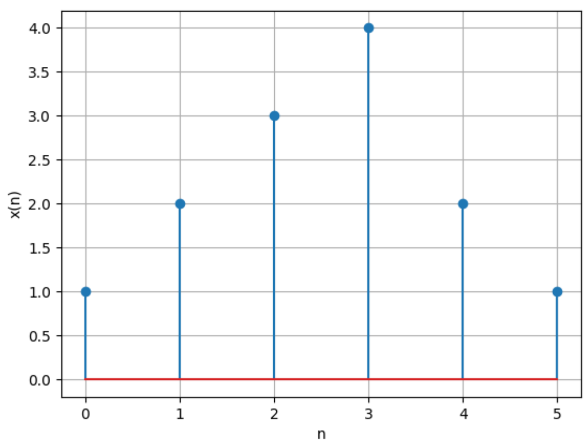


Fig. 3.1: $x(n)$ wrt n

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_3/3.1_2.py
```

3.3 Repeat the above exercise using a C code.
Solution: Download and run the following code

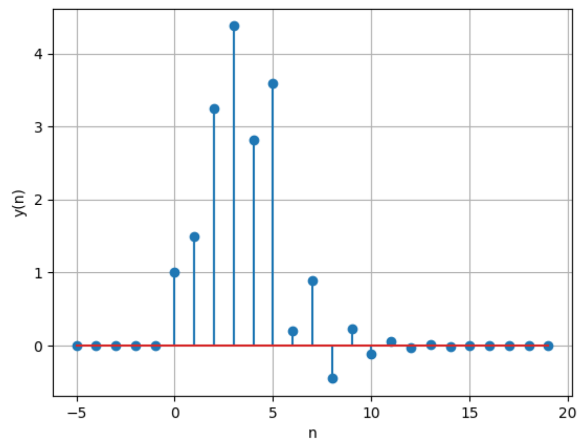


Fig. 3.2: y(n) wrt n

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_3/3.3.c
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1)

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

Let $m = n - 1$.Then

$$\mathcal{Z}\{x(n-1)\} = \sum_{m+1=-\infty}^{\infty} x(m)z^{-m-1} \tag{4.5}$$
$$= z^{-1} \sum_{m=-\infty}^{\infty} x(m)z^{-m} \tag{4.6}$$
$$= z^{-1}X(z) \tag{4.7}$$

Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.8}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.
Solution: Z-transform of x(n),$X(z)$ is given by

$$X(n) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.9}$$
$$= \sum_{n=0}^{5} x(n)z^{-n} \tag{4.10}$$
$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.11}$$

4.3 Find
$$H(z) = \frac{Y(z)}{X(z)} \tag{4.12}$$
from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.8) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.13}$$
$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.14}$$

4.4 Find the Z transform of
$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$
and show that the Z-transform of
$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$
is
$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.17}$$
Solution: Z-transform of $\delta(n)$ is given by

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \tag{4.18}$$
$$= 1 \tag{4.19}$$

and from (4.16),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.20}$$
$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.21}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that
$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.22}$$
Solution: Z-transform is given by

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n)z^{-n} \tag{4.23}$$
$$= \sum_{n=0}^{\infty} (az^{-1})^n \tag{4.24}$$
$$= \frac{1}{1 - az^{-1}}, \quad |z| > |a| \tag{4.25}$$

4.6 Let
$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{4.26}$$
Plot $|H\left(e^{j\omega}\right)|$. Comment.Is it periodic? If so, find the period. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.
Solution: $H(e^{jw})$ is given by

$$H(e^{j\omega}) = \frac{1 + (e^{j\omega})^{-2}}{1 + \frac{1}{2}(e^{j\omega})^{-1}} \tag{4.27}$$
$$= 2\frac{1 + \cos(-2\omega) + j\sin(-2\omega)}{2 + \cos(-\omega) + j\sin(-\omega)} \tag{4.28}$$
$$= 2\frac{1 + \cos(2\omega) - j\sin(2\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.29}$$
$$= 2\frac{2\cos^2(\omega) - 2j\sin(\omega)\cos(\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.30}$$
$$= 4\cos(\omega)\frac{\cos(\omega) - j\sin(\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.31}$$

So,

$$|H(e^{j\omega})| = \frac{4|\cos(\omega)|}{\sqrt{5 + 4\cos(\omega)}} \tag{4.32}$$

$$|H(e^{j(\omega+2\pi)})| = \frac{4|\cos(\omega + 2\pi)|}{\sqrt{5 + 4\cos(\omega + 2\pi)}} \tag{4.33}$$
$$= \frac{4|\cos(\omega)|}{\sqrt{5 + 4\cos(\omega)}} \tag{4.34}$$
$$= |H(e^{j\omega})| \tag{4.35}$$

Yes it is periodic because cos is periodic function. Period of numerator is π and period of denominator is 2π So,period of $|H(e^{jw})|$ would be LCM of π and 2π which is 2π.
Same you can see graphcally also.

The following code plots Fig. 4.6.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_4/4.5.py
```

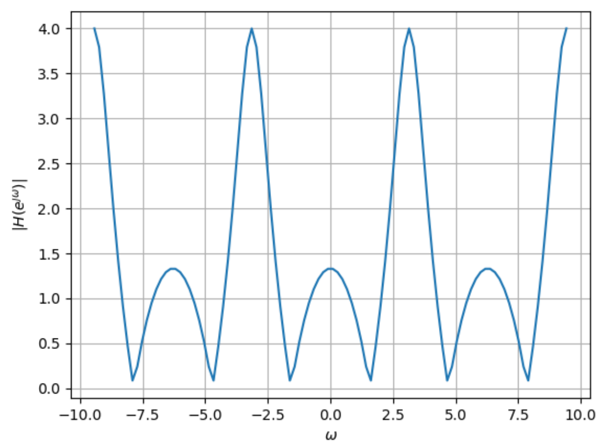


Fig. 4.6: Discret Time Fourier Transform

4.7 Express $h(n)$ in terms of $H\left(e^{j\omega}\right)$.
Solution: We know that

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \tag{4.36}$$

and

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.37}$$

Now,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.38}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n}d\omega \tag{4.39}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \tag{4.40}$$

$$= \frac{1}{2\pi}\sum_{k\neq n} h(k)\frac{e^{j\omega(n-k)}}{j(n-k)}\Bigg]_{-\pi}^{\pi} \tag{4.41}$$

$$+ \frac{1}{2\pi}h(n)\int_{-\pi}^{\pi} d\omega \tag{4.42}$$

$$= \frac{0 + 2\pi h(n)}{2\pi} \tag{4.43}$$

$$= h(n) \tag{4.44}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.14).
Solution: $H(z)$ is given by

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} = \frac{2 + 2z^{-2}}{2 + z^{-1}} \tag{5.2}$$

Now, on doing long division,

$$\begin{array}{r|l} & 2z^{-1} - 4 \\ \hline z^{-1} + 2 & 2z^{-2} + 2 \\ & 2z^{-2} + 4z^{-1} \\ \hline & -4z^{-1} + 2 \\ & -4z^{-1} - 8 \\ \hline & 10 \end{array}$$

So,

$$H(z) = 2z^{-1} - 4 + \frac{10}{z^{-1} + 2} \tag{5.3}$$

$$= 2z^{-1} - 4 + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.4}$$

$$= 2z^{-1} - 4 + 5\sum_{n=0}^{\infty}\left(-\frac{z^{-1}}{2}\right)^n \tag{5.5}$$

$$= 1 - \frac{1}{2}z^{-1} + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.6}$$

So,h(n) will be given by

$$h(n) = \begin{cases} 5 \times \left(-\frac{1}{2}\right)^n & n \geq 2 \\ \left(-\frac{1}{2}\right)^n & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.7}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.8}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.14),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.9}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.10}$$

using (4.22) and (4.8).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_5/5.2.py
```

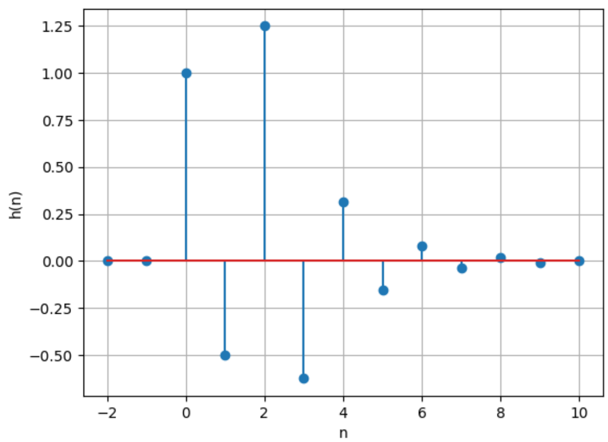


Fig. 5.3: $h(n)$ wrt n

Graphically we can see that $h(n)$ is bounded at starting and its decreasing as n increases.So,we can conclude that it is bounded.
$h(n)$ is given by

$$h(n) = \begin{cases} 5\times\left(-\frac{1}{2}\right)^n & n \geq 2 \\ \left(-\frac{1}{2}\right)^n & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.11}$$

A sequence $\{x_n\}$ is said to be bounded if and only if there exist a positive real number N such that

$$|x_i| \leq N \quad \forall i \in \mathbb{N} \tag{5.12}$$

for $n < 0$:

$$|h(n)| \leq 0 \tag{5.13}$$

for $0 \leq n < 2$:

$$|h(n)| = \left|-\frac{1}{2}\right|^n = \left(\frac{1}{2}\right)^n \leq 1 \tag{5.14}$$

for $2 \leq n$

$$|h(n)| = 5\left|-\frac{1}{2}\right|^n = 5\left(\frac{1}{2}\right)^n \leq \frac{5}{4} \tag{5.15}$$

Combining all we get

$$|h(n)| \leq \frac{5}{4} \tag{5.16}$$

So,we can conclude that h(n) is bounded

5.4 Convergent? Justify using the ratio test.
Solution: According to ratio test, a sequence $\{x_n\}$ is convergent if

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| < 1 \tag{5.17}$$

$$\because \lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \left|\frac{5\times\left(-\frac{1}{2}\right)^{(n+1)}}{5\times\left(-\frac{1}{2}\right)^n}\right| \tag{5.18}$$

$$= \frac{1}{2} \tag{5.19}$$

Therefore,$h(n)$ is convergent

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.20}$$

Is the system defined by (3.2) stable for the impulse response in (5.8)?
Solution: For system of 3.2 ,$h(n)$ is defined in (5.11) So,

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=2}^{\infty} 5\times\left(-\frac{1}{2}\right)^n + \sum_{n=0}^{1}\left(-\frac{1}{2}\right)^n + \sum_{n=-\infty}^{-1} 0 \tag{5.21}$$

$$= 5\times\frac{1}{6} + \frac{1}{2} \tag{5.22}$$

$$= \frac{4}{3} \tag{5.23}$$

Since the sum is finite so the system is stable for impulsive response

5.6 Verify the above result using a python code.
Solution: Download and run the below python

code .

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_5/stable.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.24)$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_5/5.4.py
```

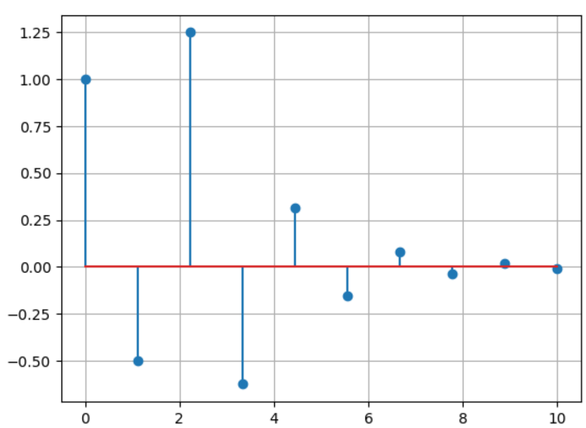


Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.25)$$

Comment. The operation in (5.25) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_5/5.5.py
```

5.9 Express the above convolution using a Teoplitz matrix.

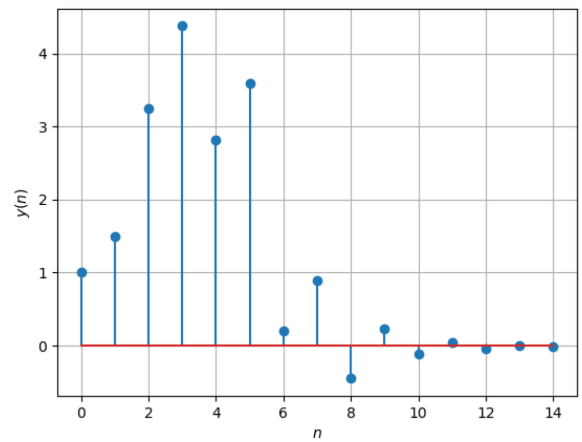


Fig. 5.8: $y(n)$ from the definition of convolution

Solution: For finding the above convolution using topleitz matrix we have to find topleitz matrix of h(n).
h(n) is tending to 0 for large n. So,we take upto some n only. So,

$$h(n) = \begin{pmatrix} 1 \\ -.5 \\ . \\ . \end{pmatrix} \quad for \quad n = 0, 2...9 \quad (5.26)$$

So,topleitz matrix of $h(n)$ will be

$$top\{h(n)\} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -.5 & 1 & 0 & 0 & 0 \\ . & . & . & . & . & . \end{pmatrix} \quad (5.27)$$

and

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad (5.28)$$

So,

$$x(n) * h(n) = top\{h(n)\}x(n) \quad (5.29)$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \end{pmatrix} \quad (5.30)$$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.31}$$

Solution: From (5.25)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.32}$$

Replacing n-k with l,we get

$$y(n) = \sum_{n-l=-\infty}^{\infty} x(n-l)h(l) \tag{5.33}$$

$$= \sum_{-l=-\infty}^{\infty} x(n-l)h(l) \tag{5.34}$$

$$= \sum_{l=-\infty}^{\infty} x(n-l)h(l) \tag{5.35}$$

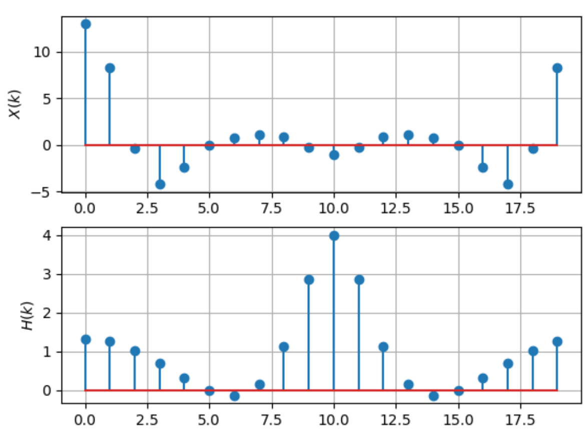
6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution: Download and run the following code.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_6/dtft_sum.py
```



6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

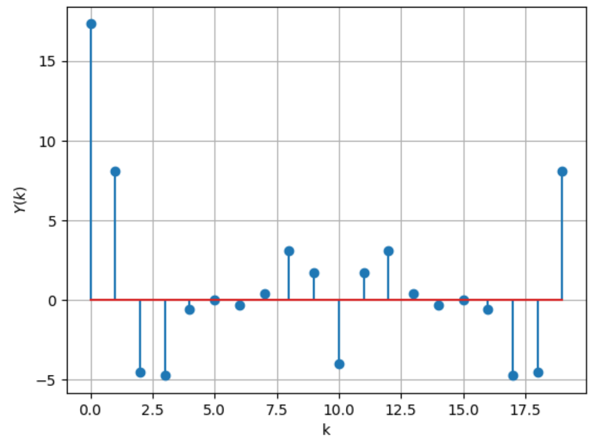


Fig. 6.2: $Y(k)$ using $X(k)$ and $H(k)$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 6.3. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_6/y_n_idtft.py
```

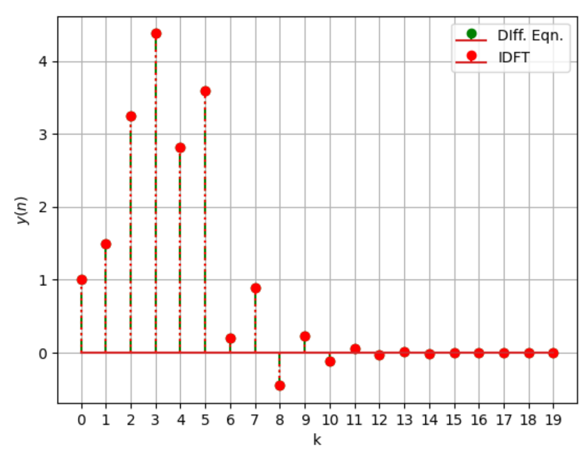


Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

Solution: Download and rum the following code.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_6/fft_ifft.py
```

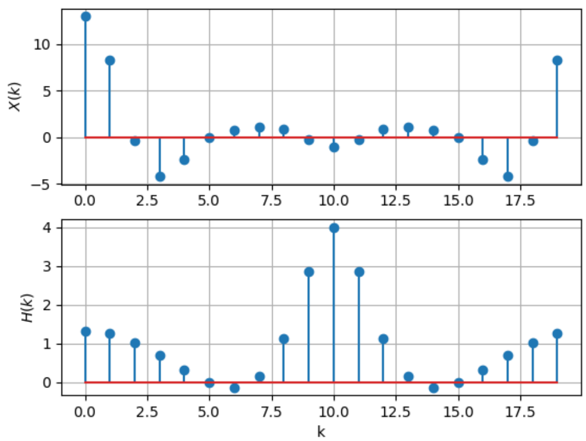


Fig. 6.4: $H(k)$ and $X(k)$ using FFT

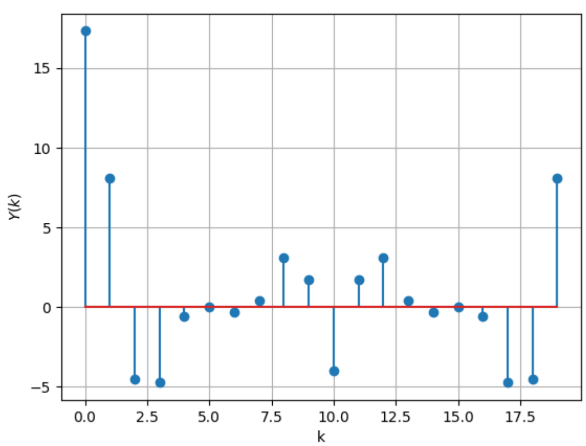


Fig. 6.4: $Y(k)$ using $X(k)$ and $H(k)$

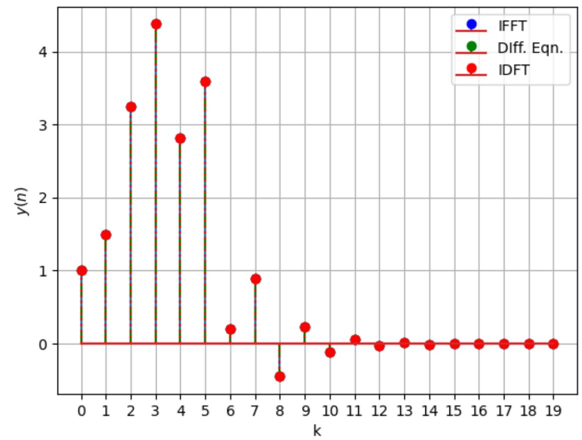


Fig. 6.4: comparison between $y(n)$ coming by different methods

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

5. Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:

$$W_N^2 = \left(e^{-j2\pi/N}\right)^2 \tag{7.8}$$
$$= e^{-j4\pi/N} \tag{7.9}$$
$$= e^{-\frac{j2\pi}{N/2}} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$

6. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.12}$$

Solution:

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.13}$$

$$\implies \mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4^2 \tag{7.14}$$

$$\implies \mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \tag{7.15}$$

Now,on multiplying $\mathbf{F}_4$ and $\mathbf{P}_4$, we get

$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix} W_4^{0\times0} & W_4^{0\times2} & W_4^{0\times1} & W_4^{0\times3} \\ W_4^{1\times0} & W_4^{1\times2} & W_4^{1\times1} & W_4^{1\times3} \\ W_4^{2\times0} & W_4^{2\times2} & W_4^{2\times1} & W_4^{2\times3} \\ W_4^{3\times0} & W_4^{3\times2} & W_4^{3\times1} & W_4^{3\times3} \end{bmatrix} \tag{7.16}$$

$$= \begin{bmatrix} \left(W_4^{0\times0}\right)^2 & \left(W_4^{0\times1}\right)^2 & \\ \left(W_4^{1\times0}\right)^2 & \left(W_4^{1\times1}\right)^2 & \mathbf{M} \\ \left(W_4^{2\times0}\right)^2 & \left(W_4^{2\times1}\right)^2 & \\ \left(W_4^{3\times0}\right)^2 & \left(W_4^{3\times1}\right)^2 & \end{bmatrix} \tag{7.17}$$

$$= \begin{bmatrix} \left(W_4^{0\times0}\right)^2 & \left(W_4^{0\times1}\right)^2 & \\ \left(W_4^{1\times0}\right)^2 & \left(W_4^{1\times1}\right)^2 & \mathbf{M} \\ \left((-1)^0 W_4^{(2-\frac{4}{2})\times0}\right)^2 & \left((-1)^1 W_4^{(2-\frac{4}{2})\times1}\right)^2 & \\ \left((-1)^0 W_4^{(3-\frac{4}{2})\times0}\right)^2 & \left((-1)^1 W_4^{(3-\frac{4}{2})\times1}\right)^2 & \end{bmatrix} \tag{7.18}$$

$$= \begin{bmatrix} \left(W_4^{0\times0}\right)^2 & \left(W_4^{0\times1}\right)^2 & W_4^0\left(W_4^{0\times0}\right) & W_4^0\left(W_4^{0\times2}\right) \\ \left(W_4^{1\times0}\right)^2 & \left(W_4^{1\times1}\right)^2 & W_4^1\left(W_4^{1\times0}\right) & W_4^1\left(W_4^{1\times2}\right) \\ \left(W_4^{0\times0}\right)^2 & \left(W_4^{0\times1}\right)^2 & W_4^2\left(W_4^{2\times0}\right) & W_4^2\left(W_4^{2\times2}\right) \\ \left(W_4^{1\times0}\right)^2 & \left(W_4^{1\times1}\right)^2 & W_4^3\left(W_4^{3\times0}\right) & W_4^3\left(W_4^{3\times2}\right) \end{bmatrix} \tag{7.19}$$

$$= \begin{bmatrix} W_2^{0\times0} & W_2^{0\times1} & W_4^0\left(W_4^{0\times0}\right) & W_4^0\left(W_4^{0\times2}\right) \\ W_2^{1\times0} & W_2^{1\times1} & W_4^1\left(W_4^{1\times0}\right) & W_4^1\left(W_4^{1\times2}\right) \\ W_2^{0\times0} & W_2^{0\times1} & -W_4^{2-\frac{4}{2}}\left(W_4^{2\times0}\right) & -W_4^{2-\frac{4}{2}}\left(W_4^{2\times2}\right) \\ W_2^{1\times0} & W_2^{1\times1} & -W_4^{3-\frac{4}{2}}\left(W_4^{3\times0}\right) & -W_4^{3-\frac{4}{2}}\left(W_4^{3\times2}\right) \end{bmatrix} \tag{7.20}$$

$$= \begin{bmatrix} \mathbf{F}_2 & W_4^0W_2^{0\times0} & W_4^0W_2^{0\times1} \\ & W_4^1W_2^{1\times0} & W_4^1W_2^{1\times1} \\ \mathbf{F}_2 & -W_4^0W_2^{0\times0} & -W_4^0W_2^{0\times1} \\ & -W_4^1W_2^{1\times0} & -W_4^1W_2^{1\times1} \end{bmatrix} \tag{7.21}$$

$$= \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \tag{7.22}$$

7. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.23}$$

Solution: For N as an even number,

$$\mathbf{F}_N\mathbf{P}_N = \begin{bmatrix} W_N^{0\times0} & W_N^{0\times2} & .. & W_N^{0\times1} & W_N^{0\times3} & .. \\ W_N^{1\times0} & W_N^{1\times2} & .. & W_N^{1\times1} & W_N^{1\times3} & .. \\ .. & .. & .. & .. & .. & .. \\ W_N^{N/2\times0} & W_N^{N/2\times2} & .. & W_N^{N/2\times1} & W_N^{N/2\times3} & .. \\ .. & .. & .. & .. & .. & .. \\ W_N^{N-1\times0} & W_N^{0\times2} & .. & W_N^{N-1\times1} & W_N^{N-1\times3} & .. \end{bmatrix} \tag{7.24}$$

$$= \begin{bmatrix} \left[W_N^{n\times2m}\right] & \left[W_N^{n\times(2m+1)}\right] \\ \left[W_N^{(n+N/2)\times2m}\right] & \left[W_N^{(n+N/2)\times(2m+1)}\right] \end{bmatrix} \tag{7.25}$$

$$\text{where } 0 \le m,n \le \frac{N}{2}-1$$

$$= \begin{bmatrix} \left[W_N^{n\times2m}\right] & \left[W_N^{n\times(2m+1)}\right] \\ \left[W_N^{n\times2m+\frac{N}{2}\times2m}\right] & \left[W_N^{n\times(2m+1)+\frac{N}{2}\times(2m+1)}\right] \end{bmatrix} \tag{7.26}$$

$$= \begin{bmatrix} \left[W_N^{n\times2m}\right] & \left[W_N^{n\times(2m+1)}\right] \\ \left[W_N^{n\times2m}\right] & \left[-W_N^{n\times(2m+1)}\right] \end{bmatrix} \tag{7.27}$$

$$= \begin{bmatrix} \left[\left(W_N^{n\times m}\right)^2\right] & \left[W_N^n\left(W_N^{n\times m}\right)^2\right] \\ \left[\left(W_N^{n\times m}\right)^2\right] & \left[-W_N^n\left(W_N^{n\times m}\right)^2\right] \end{bmatrix} \tag{7.28}$$

$$= \begin{bmatrix} \left[W_{N/2}^{n\times m}\right] & \left[W_N^nW_{N/2}^{n\times m}\right] \\ \left[W_{N/2}^{n\times m}\right] & \left[-W_N^nW_{N/2}^{n\times m}\right] \end{bmatrix} \tag{7.29}$$

$$= \begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix} \tag{7.30}$$

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.31}$$

Solution: Since $\mathbf{P_4}$ is 4×4 matrix and $\mathbf{x}$ is 6×1, so for making them compatible for multiplication we have to remove last 2 component of $\mathbf{x}$ So, we get $\mathbf{x}$ as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \tag{7.32}$$

$$\mathbf{P}_4\mathbf{x} = \mathbf{P}\mathbf{x} \tag{7.33}$$

$$= \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} \tag{7.34}$$

$$= \begin{pmatrix}1\\3\\2\\4\end{pmatrix} \tag{7.35}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.36}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution: Since $\mathbf{F}_N$ and $\mathbf{x}$ may not be compatible for matrix product. So for making them compatible either we have to remove some last component of $\mathbf{x}$ or pad zeros at the end of $\mathbf{x}$ Let $\mathbf{x}_1$ be transformed vector.
So,

$$\mathbf{F}_N\mathbf{x} = \mathbf{F}_N\mathbf{x}_1 \tag{7.37}$$

$$= \begin{bmatrix} W_N^{0\times0} & W_N^{0\times1} & .. & W_N^{0\times N-1}\\ .. & .. & .. & ..\\ W_N^{N-1\times0} & W_N^{N-1\times1} & .. & W_N^{N-1\times N-1}\end{bmatrix}\begin{pmatrix}x(0)\\x(1)\\..\\x(N-1)\end{pmatrix} \tag{7.38}$$

$$= \begin{pmatrix}\sum_{n=0}^{N-1} W_N^{0\times n}x(n)\\ \sum_{n=0}^{N-1} W_N^{1\times n}x(n)\\ ..\\ \sum_{n=0}^{N-1} W_N^{N-1\times n}x(n)\end{pmatrix} \tag{7.39}$$

$$= \begin{pmatrix}\sum_{n=0}^{N-1} e^{\frac{-j2\pi0\times n}{N}}x(n)\\ \sum_{n=0}^{N-1} e^{\frac{-j2\pi1\times n}{N}}x(n)\\ ..\\ \sum_{n=0}^{N-1} e^{\frac{-j2\pi(N-1)\times n}{N}}x(n)\end{pmatrix} \tag{7.40}$$

$$= \mathbf{X} \tag{7.41}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix}X(0)\\X(1)\\X(2)\\X(3)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} + \begin{bmatrix}W_8^0 & 0 & 0 & 0\\0 & W_8^1 & 0 & 0\\0 & 0 & W_8^2 & 0\\0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.42}$$

$$\begin{bmatrix}X(4)\\X(5)\\X(6)\\X(7)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} - \begin{bmatrix}W_8^0 & 0 & 0 & 0\\0 & W_8^1 & 0 & 0\\0 & 0 & W_8^2 & 0\\0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.43}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix}X_1(0)\\X_1(1)\end{bmatrix} = \begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} \tag{7.44}$$

$$\begin{bmatrix}X_1(2)\\X_1(3)\end{bmatrix} = \begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} - \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} \tag{7.45}$$

$$\begin{bmatrix}X_2(0)\\X_2(1)\end{bmatrix} = \begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} \tag{7.46}$$

$$\begin{bmatrix}X_2(2)\\X_2(3)\end{bmatrix} = \begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} - \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} \tag{7.47}$$

$$P_8\begin{bmatrix}x(0)\\x(1)\\x(2)\\x(3)\\x(4)\\x(5)\\x(6)\\x(7)\end{bmatrix} = \begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\\x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} \tag{7.48}$$

$$P_4\begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\end{bmatrix} = \begin{bmatrix}x(0)\\x(4)\\x(2)\\x(6)\end{bmatrix} \tag{7.49}$$

$$P_4\begin{bmatrix}x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} = \begin{bmatrix}x(1)\\x(5)\\x(3)\\x(7)\end{bmatrix} \tag{7.50}$$

Therefore,

$$\begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} = F_2\begin{bmatrix}x(0)\\x(4)\end{bmatrix} \tag{7.51}$$

$$\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} = F_2\begin{bmatrix}x(2)\\x(6)\end{bmatrix} \tag{7.52}$$

$$\begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} = F_2\begin{bmatrix}x(1)\\x(5)\end{bmatrix} \tag{7.53}$$

$$\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} = F_2\begin{bmatrix}x(3)\\x(7)\end{bmatrix} \tag{7.54}$$

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.55}$$

compte the DFT using (7.36)
Solution: Let N=6. So,

$$\mathbf{X} = \mathbf{F}_6\mathbf{x} \tag{7.56}$$

$$= \left[W_6^{mn}\right]\mathbf{x} \tag{7.57}$$

$$= \begin{bmatrix} W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 \\ W_6^0 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ W_6^0 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10} \\ W_6^0 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15} \\ W_6^0 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20} \\ W_6^0 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{bmatrix}\mathbf{x} \tag{7.58}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{\pi}{3}} & e^{-j\frac{2\pi}{3}} & e^{-j\pi} & e^{-j\frac{4\pi}{3}} & e^{\frac{5\pi}{3}} \\ 1 & e^{-j\frac{2\pi}{3}} & e^{-j\frac{4\pi}{3}} & e^{-j2\pi} & e^{-j\frac{8\pi}{3}} & e^{\frac{10\pi}{3}} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j\frac{4\pi}{3}} & e^{-j\frac{8\pi}{3}} & e^{-j4\pi} & e^{-j\frac{16\pi}{3}} & e^{\frac{20\pi}{3}} \\ 1 & e^{-j\frac{5\pi}{3}} & e^{-j\frac{10\pi}{3}} & e^{-j5\pi} & e^{-j\frac{20\pi}{3}} & e^{\frac{25\pi}{3}} \end{bmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.59}$$

$$= \begin{pmatrix} 13 + 0j \\ -4 - 1.732j \\ 1 + 0j \\ -1 + 0j \\ 1 + 0j \\ -4 + 1.732j \end{pmatrix} \tag{7.60}$$

Download and run the following python program.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_7/dft_Fx.py
```

12. Repeat the above exercise using the FFT after zero padding **x**.
Solution: We know that FFT works for N which is of the form 2^n where $n \in \mathcal{N}$. So, we have to pad **x** with zeros to its nearest 2^n length. So,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{7.61}$$

We know that if N is even then

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix}\mathbf{P}_N \tag{7.62}$$

So,

$$\mathbf{F}_2 = \begin{bmatrix} \mathbf{F}_1 & \mathbf{D}_1\mathbf{F}_1 \\ \mathbf{F}_1 & -\mathbf{D}_1\mathbf{F}_1 \end{bmatrix}\mathbf{P}_2 \tag{7.63}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.64}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.65}$$

Now,

$$\mathbf{D}_2\mathbf{F}_2 = diag\left(W_4^0 \quad W_4^1\right)\mathbf{F}_2 \tag{7.66}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.67}$$

$$= \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \tag{7.68}$$

So,

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix}\mathbf{P}_4 \tag{7.69}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.70}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \tag{7.71}$$

Now,

$$\mathbf{D}_4\mathbf{F}_4 = diag\left(W_8^0 \quad W_8^1 \quad W_8^2 \quad W_8^3\right)\mathbf{F}_4 \tag{7.72}$$

$$= \tag{7.73}$$

13. Write a C program to compute the 8-point FFT.
Solution: Download and run the following C

program.

```
wget https://github.com/
    himanshukumargupta11012/
    EE3900_assignments/blob/master/
    assignment_1/ques_7/8pnt_fft.c
```

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

8.2 Repeat all the exercises in the previous sections for the above a and b.

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.