

Random Numbers

Mannem Charan - AI21BTECH11019

CONTENTS

1	Uniform Random Numbers	1
2	Central Limit Theorem	3
3	From Uniform to Other	5
4	Triangular Distribution	5
5	Maximul Likelihood	7
6	Gaussian to Other	8
7	Conditional Probability	9
8	Two Dimensions	9

Abstract—This manual provides solutions for random numbers assignment.

CONTENTS

Abstract—This manual provides solutions for random numbers assignment.



1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_1/1.1.c
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The CDF of U is plotted in Fig. 1.3 using the below python code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_1/1.2.py
```

AI1110

1.3 Find a theoretical expression for $F_U(x)$.

Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) \quad (1.2)$$

$$= \int_{-\infty}^x f_U(x) dx \quad (1.3)$$

$$(1.4)$$

Now, for $x < 0$

$$f_U(x) = 0 \quad (1.5)$$

$$\therefore F_U(x) = 0 \quad (1.6)$$

,for $0 \leq x \leq 1$

$$f_U(x) = 1 \quad (1.7)$$

$$\therefore F_U(x) = \int_{-\infty}^x dx \quad (1.8)$$

$$= 0 + \int_0^x dx \quad (1.9)$$

$$= x \quad (1.10)$$

and for $x > 1$

$$f_U(x) = 0 \quad (1.11)$$

$$\therefore F_U(x) = \int_{-\infty}^x dx \quad (1.12)$$

$$= 0 + \int_0^1 dx + 0 \quad (1.13)$$

$$= 1 \quad (1.14)$$

$$\text{So, } F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

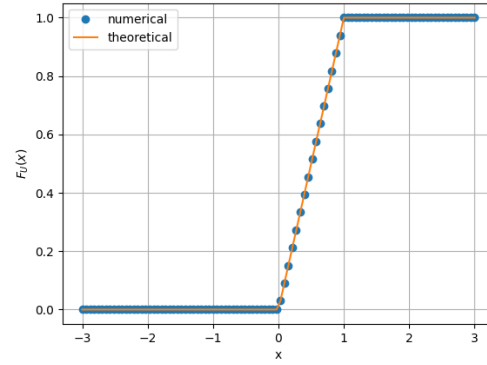


Fig. 1.3: The CDF of U

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.15)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.16)$$

Write a C program to find the mean and variance of U .

Solution: Download and run the following C program.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_1/1.4.c
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.17)$$

Solution: The mean of U is given by

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.18)$$

$$= 0 + \int_0^1 x dx + 0 \quad (1.19)$$

$$= \left[\frac{x^2}{2} \right]_0^1 \quad (1.20)$$

$$= 1/2 \quad (1.21)$$

and variance is given by,

$$E[U^2] - E[U]^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \left(\frac{1}{2}\right)^2 \quad (1.22)$$

$$= 0 + \int_0^1 x^2 dx + 0 - \left(\frac{1}{2}\right)^2 \quad (1.23)$$

$$= \left[\frac{x^3}{3} \right]_0^1 - \left(\frac{1}{2}\right)^2 \quad (1.24)$$

$$= 1/12 \quad (1.25)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and run the C program.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
```

```
blob/master/ques
_2/2.1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.3 using the below python code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_2/2.2.py
```

Properties of CDF:

- It is always continuous.
- It is monotonic increasing.
- It is bounded between 0 to 1.
- At infinity, it tends to 1.
- At negative infinity, it tends to 0.
- Since it is bounded and have limit so it is convergent also.
- Q function is defined as:

$$Q_X(x) = \Pr(x < X) \quad (2.2)$$

$$\therefore F_X(x) = 1 - Q_X(x) \quad (2.3)$$

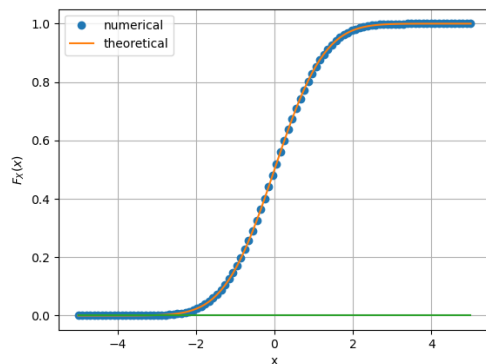
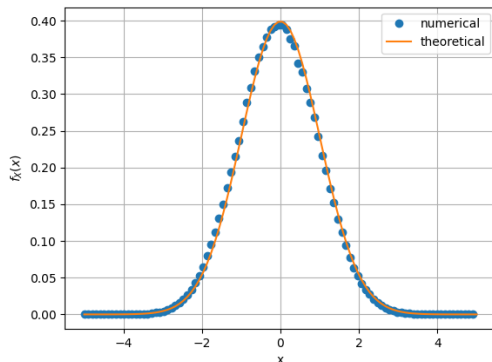
2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.4)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the below python code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
```

Fig. 2.3: The CDF of X Fig. 2.3: The PDF of X

```
blob/master/ques
_2/2.2.py
```

Properties of PDF:

- It is always non-negative
- Always symmetric about the mean.
- Maxima is observed at mean
- Area under the curve of PDF is always equal to 1

2.4 Find the mean and variance of X by writing a C program.

Solution: download the following files and run the C program.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_2/2.4.c
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/coeffs.h
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.

Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.6)$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.7)$$

$$= \left[-\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \right]_{-\infty}^{+\infty} = 0 \quad (2.8)$$

and variance is given by

$$E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx \quad (2.9)$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.10)$$

$$= 2 \int_0^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.11)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{-1/2} \exp(-t) dt \quad (2.12)$$

$$= \frac{2}{\sqrt{\pi}} \Gamma(1 + 1/2) \quad (2.13)$$

$$= 1 \quad (2.14)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download and run the python and C file where you get samples of V and fig. ??

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_3/3.1.py
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_3/3.1.c
```

3.2 Find a theoretical expression for $F_V(x)$.

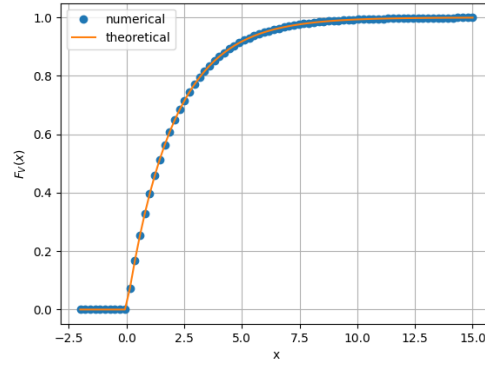


Fig. 3.1: The CDF of V

Solution: The CDF of V is given by

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.4)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

let $a = \exp(-x/2)$. So,

$$F_V(x) = \begin{cases} 0 & 1 - a < 0 \\ 1 - a & 0 \leq 1 - a < 1 \\ 1 & 1 - a \geq 1 \end{cases}$$

on solving, we get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases}$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following code and run the C file

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_4/4.1.c

wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/coeffs.h
```

4.2 Find the CDF of T .

Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \leq t) \quad (4.2)$$

Since T is sum of 2 uniform random variable between 0 and 1 So, any T will be always less than 2 and greater than 0 So,

$$F_T(t) = 0 \quad \text{for } X < 0 \quad (4.3)$$

$$F_T(t) = 1 \quad \text{for } X \geq 0 \quad (4.4)$$

Now, for $0 \leq t < 2$,

let U_1 as a constant x and later we integrate it for all possible values of x So,

$$F_T(t) = \Pr(T \leq t) \quad (4.5)$$

$$= \Pr(U_1 + U_2 \leq t) \quad (4.6)$$

$$= \Pr(u_2 \leq t - x) \quad (4.7)$$

$$= \int_0^t F_{U_2}(t - x) p_{u_1}(x) dx \quad (4.8)$$

Now, we make 2 cases

case-1: $0 \leq t < 1$

$$F_T(t) = \int_0^t F_{U_2}(t - x) dx \quad (4.9)$$

$$= \int_0^t F_{U_2}(m) dm \quad (4.10)$$

$$= \int_0^t m dm \quad (4.11)$$

$$= \left[\frac{m^2}{2} \right]_0^t \quad (4.12)$$

$$= \frac{t^2}{2} \quad (4.13)$$

case-2: $1 \leq t < 2$

$$F_T(t) = \int_0^1 F_{U_2}(t - x) dx \quad (4.14)$$

$$= \int_0^1 F_{U_2}(t - x) dx \quad (4.15)$$

$$= \int_{t-1}^t F_{U_2}(m) dm \quad (4.16)$$

$$= \int_{t-1}^1 m dm + \int_1^t dm \quad (4.17)$$

$$= \left[\frac{m^2}{2} \right]_{t-1}^1 + [1]_1^t \quad (4.18)$$

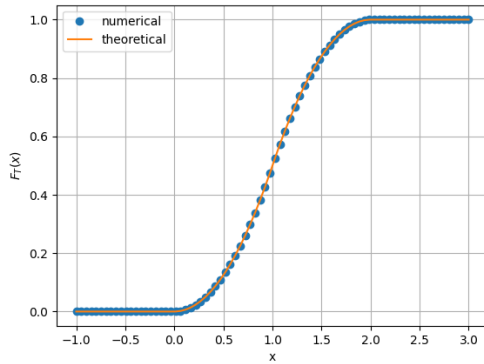
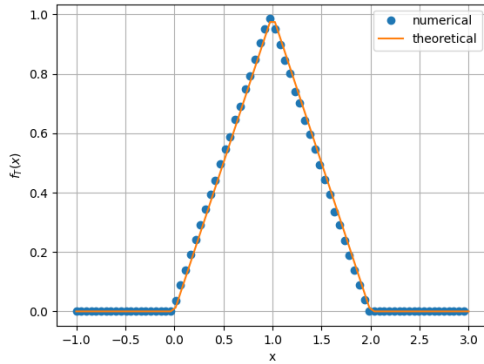
$$= \frac{-t^2}{2} + 2t - 1 \quad (4.19)$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t < 1 \\ \frac{-t^2}{2} + 2t - 1 & 1 \leq t < 2 \\ 1 & 2 \leq t \end{cases}$$

4.3 Find the PDF of T .

Solution: The PDF of T is given by

$$p_T(t) = \frac{d}{dt} F_T(t) \quad (4.20)$$

Fig. 4.5: The CDF of T Fig. 4.5: The PDF of T

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2 - t & 1 \leq t < 2 \\ 0 & 2 \leq t \end{cases}$$

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution: Already done in 4.2 and 4.3.

4.5 Verify your results through a plot.

Solution: Plot of CDF and PDF is plotted in fig.

(4.5) and (4.5) respectively using following python code.

5 MAXIMUM LIKELIHOOD

5.1 Generate

$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, $X \in \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.

Solution: Download the following files and execute the C code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_5/5.1.c
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/coeffs.h
```

5.2 Plot Y .

Solution: The plot of Y is plotted in fig. using below python code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_5/5.2.py
```

5.3 Guess how to estimate X from Y .

5.4 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.3)$$

5.5 Find P_e .

5.6 Verify by plotting the theoretical P_e .

6 GAUSSIAN TO OTHER

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

Solution: Download the following files and run C and python program. CDF and PDF of V is plotted in fig. 6.1 and 6.1 respectively using below python code.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/coeffs.h
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_6/6.1.c
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_6/6.1.py
```

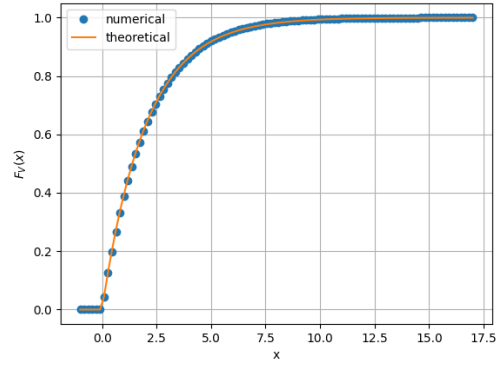


Fig. 6.1: The CDF of V

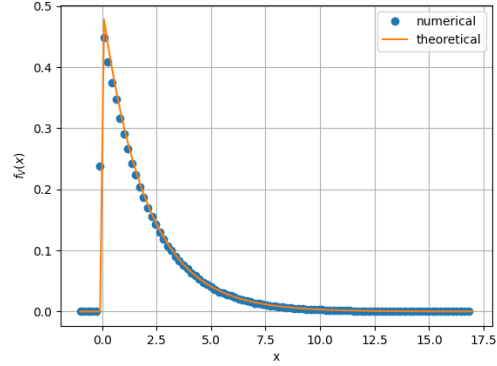


Fig. 6.1: The PDF of V

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find α .

Solution: Let

$$X_1 = R \cos \theta, \quad X_2 = R \sin \theta$$

Now PDF of R, θ is given by

$$p_{R,\theta}(r, \phi) = p_{X_1, X_2}(x_1, x_2) |J| \quad (6.3)$$

where J is jacobian matrix for transforming R, θ to X_1, X_2

On solving, we get

$$J = R \quad (6.4)$$

and since X_1 and X_2 are i.i.d.

$$\therefore p_{X_1, X_2}(x_1, x_2) = p_{X_1}(x_1) \times p_{X_2}(x_2) \quad (6.5)$$

6.3 Plot the CDF and Pdf of

$$A = \sqrt{V} \quad (6.6)$$

Solution: CDF and PDF of A is plot-

ted in fig. ?? and ?? respectively using below python code.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_6/6.3.py
```

7 CONDITIONAL PROBABILITY

7.1

7.2 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

where A is Rayleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$

7.4 For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)dx \quad (7.3)$$

Find $P_e = E[P_e(N)]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ . Comment.

8 Two DIMENSIONS

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1 .

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.5)$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e . Verify this by comparing the theory and simulation plots on the same graph.