1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _1/1.1.c
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The CDF of U is plotted in Fig. 0 using the below python code

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _1/1.2.py
```

1.3 Find a theoretical expression for $F_U(x)$.

Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) \tag{1.2}$$
$$= \int_{-\infty}^{x} f_U(x)dx \tag{1.3}$$
$$\tag{1.4}$$

Now, for $x < 0$

$$f_U(x) = 0 \tag{1.5}$$
$$\therefore F_U(x) = 0 \tag{1.6}$$

,for $0 \leq x \leq 1$

$$f_U(x) = 1 \tag{1.7}$$
$$\therefore F_U(x) = \int_{-\infty}^{x} dx \tag{1.8}$$
$$= 0 + \int_{0}^{x} dx \tag{1.9}$$
$$= x \tag{1.10}$$

and for $x > 1$

$$f_U(x) = 0 \tag{1.11}$$
$$\therefore F_U(x) = \int_{-\infty}^{x} dx \tag{1.12}$$
$$= 0 + \int_{0}^{1} dx + 0 \tag{1.13}$$
$$= 1 \tag{1.14}$$

$$So, F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

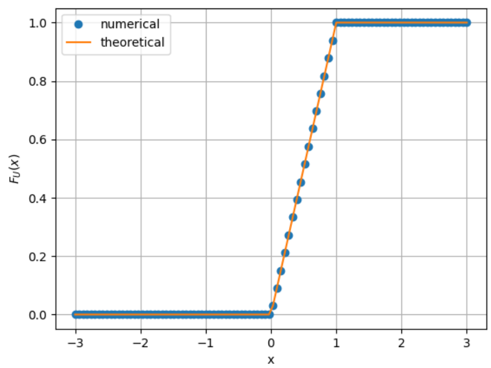

Fig. 0: The CDF of U

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.15}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.16}$$

Write a C program to find the mean and variance of U.
Solution: Download and run the following C program.

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _1/1.4.c
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.17}$$

Solution: The mean of U is given by

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.18}$$

$$= 0 + \int_0^1 x dx + 0 \tag{1.19}$$

$$= \left[\frac{x^2}{2}\right]_0^1 \tag{1.20}$$

$$= 1/2 \tag{1.21}$$

and variance is given by,

$$E[U^2] - E[U]^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \left(\frac{1}{2}\right)^2 \tag{1.22}$$

$$= 0 + \int_0^1 x^2 dx + 0 - \left(\frac{1}{2}\right)^2 \tag{1.23}$$

$$= \left[\frac{x^2}{3}\right]_0^1 - \left(\frac{1}{2}\right)^2 \tag{1.24}$$

$$= 1/12 \tag{1.25}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and run the C program.

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
```

```
    blob/master/ques
    _2/2.1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 0 using the below python code

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _2/2.2.py
```

Properties of CDF:

a) It is always continuous.
b) It is monotonic increasing.
c) It is bounded between 0 to 1.
d) At infinity, it tends to 1.
e) At negative infinity, it tends to 0.
f) Since it is bounded and have limit so it is convergent also.
g) Q function is defined as:

$$Q_X(x) = \Pr(x < X) \tag{2.2}$$
$$\therefore F_X(x) = 1 - Q_X(x) \tag{2.3}$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.4}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 0 using the below python code

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _2/2.2.py
```

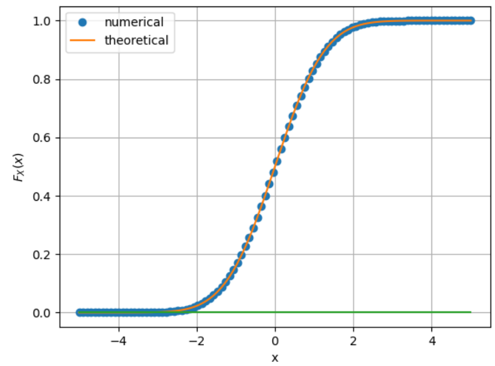

Fig. 0: The CDF of X

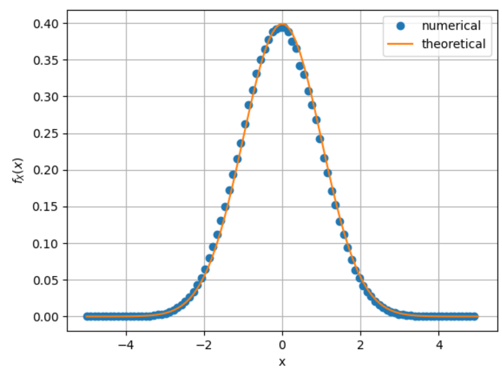

Fig. 0: The PDF of X

Properties of PDF:

a) It is always non-negative
b) It is always bounded between 0 to 1
c) Always symmetric about the mean.
d) Maxima is observed at mean
e) Area under the curve of PDF is always equal to 1

2.4 Find the mean and variance of X by writing a C program.
Solution: download the following files and run the C program.

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _2/2.4.c
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/coeffs.h
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{2.6}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.7}$$

$$= \left[-\frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{+\infty} = 0 \tag{2.8}$$

and variance is given by

$$E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx \tag{2.9}$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.10}$$

$$= 2\int_{0}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.11}$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} t^{-1/2} \exp\left(-t\right) dt \tag{2.12}$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(1 + 1/2\right) \tag{2.13}$$

$$= 1 \tag{2.14}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln\left(1 - U\right) \tag{3.1}$$

and plot its CDF.
Solution: Download and run the python and C file where you get samples of V and fig. **??**

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _3/3.1.py
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _3/3.1.c
```

3.2 Find a theoretical expression for $F_V(x)$.

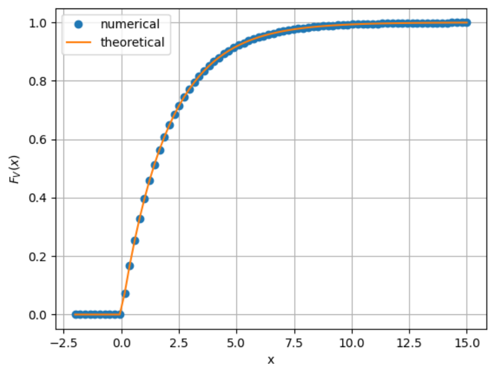

Fig. 0: The CDF of V

Solution: The CDF of V is given by

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$
$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$
$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$
$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

let $a = \exp(-x/2)$ .So,

$$F_V(x) = \begin{cases} 0 & 1-a < 0 \\ 1-a & 0 \le 1-a < 1 \\ 1 & 1-a \ge 1 \end{cases}$$

on solving, we get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \ge 0 \end{cases}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following code and run the C file

```
wget https://github.com/himanshukumargupta11012/Random-Numbers/blob/master/ques_4/4.1.c

wget https://github.com/himanshukumargupta11012/Random-Numbers/blob/master/coeffs.h
```

4.2 Find the CDF of T.
Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \le t) \tag{4.2}$$

Since T is sum of 2 uniform random variable between 0 and 1 So,any T will be always less than 2 and greater than 0 So,

$$F_T(t) = 0 \quad \text{for } X < 0 \tag{4.3}$$
$$F_T(t) = 1 \quad \text{for } X \ge 0 \tag{4.4}$$

Now,for $0 \le t < 2$,
let U_1 as a constant x and later we integrate it for all possible values of x So,

$$F_T(t) = \Pr(T \le t) \tag{4.5}$$
$$= \Pr(U_1 + U_2 \le t) \tag{4.6}$$
$$= \Pr(U_2 \le t - U_1) \tag{4.7}$$
$$= \int_0^t F_{U_2}(t-x)\, p_{U_1}(x)\, dx \tag{4.8}$$

Now, we make 2 cases

case-1: $0 \le t < 1$

$$F_T(t) = \int_0^t F_{U_2}(t-x)\,dx \tag{4.9}$$
$$= \int_0^t F_{U_2}(m)\,dm \tag{4.10}$$
$$= \int_0^t mdm \tag{4.11}$$
$$= \left[\frac{m^2}{2}\right]_0^t \tag{4.12}$$
$$= \frac{t^2}{2} \tag{4.13}$$

case-2: $1 \le t < 2$

$$F_T(t) = \int_0^1 F_{U_2}(t-x)\,dx \tag{4.14}$$
$$= \int_0^1 F_{U_2}(t-x)\,dx \tag{4.15}$$
$$= \int_{t-1}^t F_{U_2}(m)\,dm \tag{4.16}$$
$$= \int_{t-1}^1 mdm + \int_1^t dm \tag{4.17}$$
$$= \left[\frac{m^2}{2}\right]_{t-1}^1 + [1]_1^t \tag{4.18}$$
$$= \frac{-t^2}{2} + 2t - 1 \tag{4.19}$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t_2}{2} & 0 \le t < 1 \\ \frac{-t^2}{2} + 2t - 1 & 1 \le t < 2 \\ 1 & 2 \le t \end{cases}$$

4.3 Find the PDF of T.
Solution: The PDF of T is given by

$$p_T(t) = \frac{d}{dt}F_T(t) \tag{4.20}$$

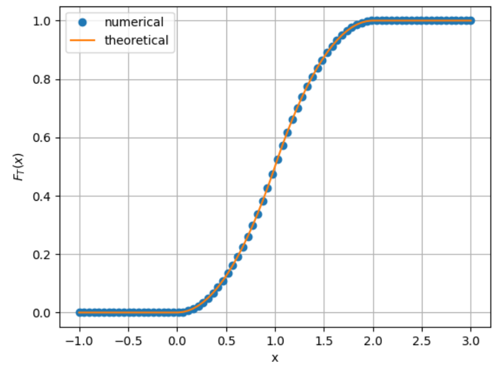

Fig. 0: The CDF of T

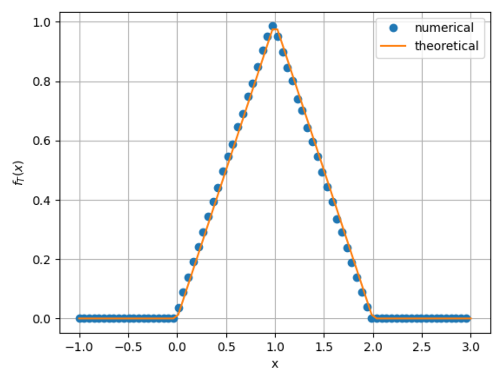

Fig. 0: The PDF of T

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \\ 0 & 2 \le t \end{cases}$$

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Already done in 4.2 and 4.3.

4.5 Verify your results through a plot.
Solution: Plot of CDF and PDF is plotted in fig.

(0) and (0) respectively using following python code.

5 Maximul Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Download the following files and execute the C file

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _5/5.1.c
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/coeffs.h
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Download the following files and run the C code

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _5/5.2.c
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/coeffs.h
```

5.3 Plot Y using a scatter plot.
Solution: The scatter plot of Y is plotted in fig. 0 using below python code.

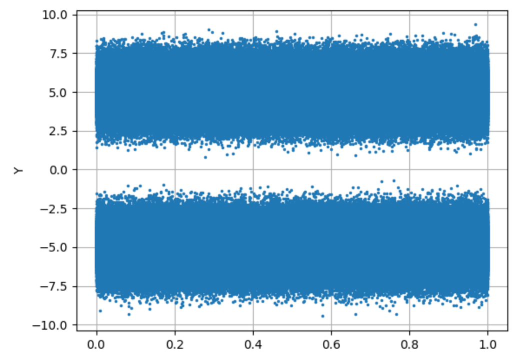

Fig. 0: The scatter plot of Y

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _5/5.3.py
```

5.4 Guess how to estimate X from Y.
Solution: After seeing the scatter plot we can conclude that if Y is positive then X will be 1 and if Y is negative then X will be -1.This conclusion also depends on value of A because if we decrease A the two parts of graph intermix and then we can't say anything.If A is approximately greater than 3 then only we can conclude this.

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.3}$$

Solution: $\hat{X}$ is defined as

$$\hat{X} = \begin{cases} 1 & Y \geq 0 \\ -1 & Y < 0 \end{cases} \tag{5.4}$$

So,

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.5}$$
$$= \Pr\left(Y < 0 | X = 1\right) \tag{5.6}$$
$$= \Pr\left(A \times 1 + N < 0\right) \tag{5.7}$$
$$= \Pr\left(N < -A\right) \tag{5.8}$$
$$= F_N\left(-A\right) \tag{5.9}$$
$$= 0 \tag{5.10}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.11}$$
$$= \Pr\left(Y \geq 0 | X = -1\right) \tag{5.12}$$
$$= \Pr\left(A \times -1 + N \geq 0\right) \tag{5.13}$$
$$= \Pr\left(N \geq A\right) \tag{5.14}$$
$$= 1 - \Pr\left(N < A\right) \tag{5.15}$$
$$= 1 - F_N\left(A\right) \tag{5.16}$$
$$= 0 \tag{5.17}$$

Download and run the following file to get coding proof

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _5/5.5.py
```

5.6 Find P_e assuming that X has equiprobable symbols.

Solution: P_e can be defined as

$$P_e = \Pr\left(X = -1\right) P_{e|1} + \Pr\left(X = 1\right) P_{e|0} \tag{5.18}$$
$$\because P_{e|1} = 0, P_{e|0} = 0 \tag{5.19}$$
$$\therefore P_e = 0 \tag{5.20}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: X is equiprobable.So

$$\because \Pr\left(X = 1\right) = \Pr\left(X = -1\right), \tag{5.21}$$
$$\Pr\left(X = 1\right) + \Pr\left(X = -1\right) = 1 \tag{5.22}$$
$$\therefore \Pr\left(X = 1\right) = \Pr\left(X = -1\right) = 1/2 \tag{5.23}$$

Now,P_e is given by

$$P_e = \Pr\left(X = -1\right) P_{e|1} + \Pr\left(X = 1\right) P_{e|0} \tag{5.24}$$

$$P_e = \frac{1}{2}\left(1 - F_N\left(A\right)\right) + \frac{1}{2}\left(F_N\left(-A\right)\right) \tag{5.25}$$

Now, the plot of P_e w.r.t. A is in fig. 0 by plotting the above function using following code

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _5/5.7.py
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution: Taking threshold as δ we

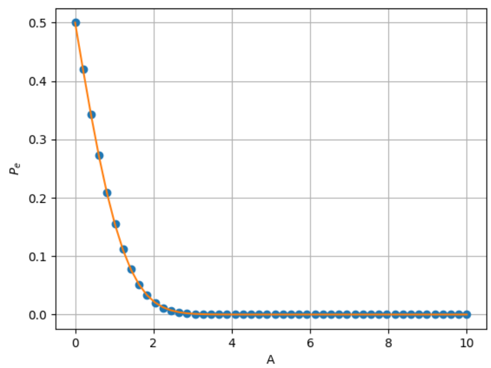

Fig. 0: P_e graph wrt A

get

$$P_{e|0} = \Pr(Y < \delta | X = 1) \quad (5.26)$$
$$= \Pr(A \times 1 + N < \delta) \quad (5.27)$$
$$= \Pr(N < -A + \delta) \quad (5.28)$$
$$= F_N(-A + \delta) \quad (5.29)$$

and

$$P_{e|1} = \Pr(Y \geq \delta | X = -1) \quad (5.30)$$
$$= \Pr(A \times -1 + N \geq \delta) \quad (5.31)$$
$$= \Pr(N \geq A + \delta) \quad (5.32)$$
$$= 1 - \Pr(N < A + \delta) \quad (5.33)$$
$$= 1 - F_N(A + \delta) \quad (5.34)$$

and P_e is defined as

$$P_e = \Pr(X = -1) P_{e|1} + \Pr(X = 1) P_{e|0} \quad (5.35)$$
$$= \frac{1}{2}(1 - F_N(A + \delta) + F_N(-A + \delta)) \quad (5.36)$$

Now we differentiate P_e and equate it to 0 to get δ that minimizes the P_e.So,

$$\frac{d}{d\delta} P_e = \frac{1}{2}\left(\frac{d}{d\delta} F_N(-A + \delta) - \frac{d}{d\delta} F_N(A + \delta)\right) = 0 \quad (5.37)$$
$$\implies p_N(-A + \delta) - p_N(A + \delta) = 0 \quad (5.38)$$
$$\implies \exp\left(-\frac{(-A + \delta)^2}{2}\right) - \exp\left(-\frac{(A + \delta)^2}{2}\right) = 0 \quad (5.39)$$
$$\implies \delta = 0 \quad (5.40)$$

Now we have to check that δ=0 is for maxima or minima by double differentiating it. So,

$$\frac{d^2}{d\delta^2}|_{\delta=0} = 2A \exp\left(\frac{-A^2}{2}\right) \quad (5.41)$$
$$= 10 \exp\left(\frac{-A^2}{2}\right) > 0 \quad (5.42)$$

Double derivative is positive. So,δ=0 will give minima of P_e
Plot of P_e w.r.t. δ is plotted in fig. 0 using following code

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _5/5.8.py
```

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.43)$$

Solution:

$$\because \Pr(X = 1) = p_X(0) = p \quad (5.44)$$
$$\therefore \Pr(X = -1) = 1 - \Pr(X = 1) \quad (5.45)$$
$$= 1 - p \quad (5.46)$$

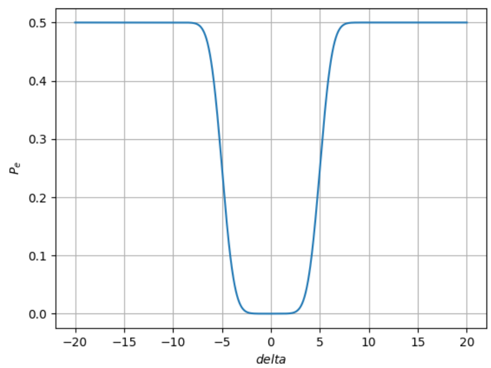

Fig. 0: P_e graph wrt δ

and P_e is given by

$$P_e = \Pr(X=-1)P_{e|1} + \Pr(X=1)P_{e|0} \tag{5.47}$$
$$= (1-p)(1-F_N(A+\delta)) \tag{5.48}$$
$$+ pF_N(-A+\delta) \tag{5.49}$$

Then,

$$\frac{d}{d\delta}P_e = p\frac{d}{d\delta}F_N(-A+\delta) \tag{5.50}$$
$$-(1-p)\frac{d}{d\delta}F_N(A+\delta) = 0 \tag{5.51}$$
$$\implies p \times p_N(-A+\delta) \tag{5.52}$$
$$-(1-p)p_N(A+\delta) = 0 \tag{5.53}$$
$$\implies p \times \exp\left(-\frac{(-A+\delta)^2}{2}\right) \tag{5.54}$$
$$-(1-p)\exp\left(-\frac{(A+\delta)^2}{2}\right) = 0 \tag{5.55}$$
$$\implies \delta = \frac{\log\left(\frac{1}{p}-1\right)}{2A} \tag{5.56}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution: From bayes theorem we can write,

$$\Pr(X=1|Y=y) \tag{5.57}$$
$$= \frac{\Pr(Y=y|X=1)\Pr(X=1)}{\Pr(Y=y)} \tag{5.58}$$
$$= \frac{\Pr(N=y-A)p}{\Pr(N=y-A)p + \Pr(N=y+A)(1-p)} \tag{5.59}$$
$$= \frac{p_N(y-A)p}{p_N(y-A)p + p_N(y+A)(1-p)} \tag{5.60}$$
$$= \frac{p}{p+(1-p)\exp(-2Ay)} \tag{5.61}$$

$$\Pr(X=-1|Y=y) \tag{5.62}$$
$$= 1-\Pr(X=1|Y=y) \tag{5.63}$$
$$= 1-\frac{p}{p+(1-p)\exp(-2Ay)} \tag{5.64}$$
$$= \frac{(1-p)\exp(-2Ay)}{p+(1-p)\exp(-2Ay)} \tag{5.65}$$
$$= \frac{1-p}{p\exp(2Ay)+1-p} \tag{5.66}$$

Now, if $X=1$ is more likely than $X=-1$ for a y. Then,

$$\Pr(X=1|Y=y) > \Pr(X=-1|Y=y) \tag{5.67}$$
$$\frac{p}{p+(1-p)\exp(-2Ay)} > \frac{1-p}{p\exp(2Ay)+1-p} \tag{5.68}$$
$$y > \frac{\log\left(\frac{1-p}{p}\right)}{2A} \tag{5.69}$$

and if $X=-1$ is more lileky for a y.

Then,

$$\Pr(X = 1|Y = y) < \Pr(X = -1|Y = y) \tag{5.70}$$

$$\frac{p}{p + (1-p)\exp(-2Ay)} \tag{5.71}$$

$$< \frac{1-p}{p\exp(2Ay) + 1 - p} \tag{5.72}$$

$$y < \frac{\log\left(\frac{1-p}{p}\right)}{2A} \tag{5.73}$$

So,we can see that optimal value of Y for P_e to be minimum is

$$\frac{\log\left(\frac{1-p}{p}\right)}{2A} \tag{5.74}$$

So,

$$\delta = \frac{\log\left(\frac{1-p}{p}\right)}{2A} \tag{5.75}$$

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: Download the following files and run C and python program. CDF and PDF of V is plotted in fig. 0 and 0 respectively using below python code.

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/coeffs.h
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _6/6.1.c
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _6/6.1.py
```

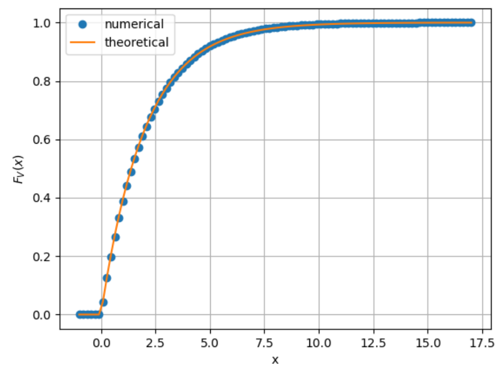

Fig. 0: The CDF of V

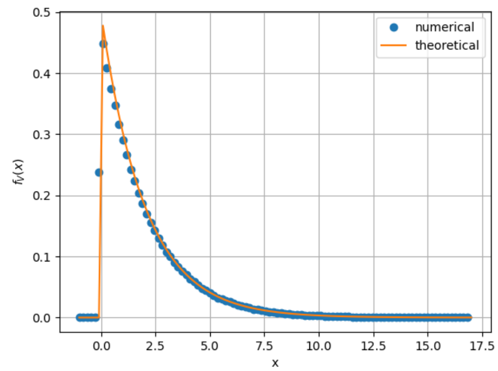

Fig. 0: The PDF of V

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.
Solution: Let
$X_1 = R\cos\theta$,$X_2 = R\sin\theta$
So, PDF of R and θ is given by

$$p_{R,\theta}(r,\phi) = p_{X_1,X_2}(x_1,x_2)\,|J| \tag{6.3}$$

where J is jacobian matrix transforming R,θ to X_1, X_2
On soving we get jacobian, $J = R$. So,

$$p_{R,\theta}(r,\phi) = p_{X_1,X_2}(x_1,x_2)\,r \tag{6.4}$$

since X_1 and X_2 are i.i.d. and gaussian random variable,$\mathcal{N}(0,1)$.So,

$$\begin{aligned} p_{R,\theta}(r,\phi) &= p_{X_1}(x_1)\,p_{X_2}(x_2)\,r && (6.5)\\ &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x_1^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x_2^2}{2}\right)r && (6.6)\\ &= \frac{1}{2\pi}\exp\left(-\frac{r^2}{2}\right)r && (6.7) \end{aligned}$$

Now PDF of R,

$$\begin{aligned} p_R(r) &= \int_0^{2\pi} p_{R,\theta}(r,\phi)\,d\phi && (6.8)\\ &= \int_0^{2\pi}\frac{1}{2\pi}\exp\left(-\frac{r^2}{2}\right)rd\phi && (6.9)\\ &= \exp\left(-\frac{r^2}{2}\right)r && (6.10) \end{aligned}$$

and

$$V = X_1^2 + X_2^2 = R^2 \tag{6.11}$$

Since $V = R^2 > 0$
So,CDF and PDF would be zero for v<0

Then CDF of V for v≥0,

$$\begin{aligned} F_V(v) &= \Pr(V \le v) && (6.12)\\ &= \Pr\left(R^2 \le v\right) && (6.13)\\ &= \Pr\left(R \le \sqrt{v}\right) && (6.14)\\ &= F_R\left(\sqrt{v}\right) && (6.15)\\ &= \int_{-\infty}^{\sqrt{v}} p_R(r)\,dr && (6.16)\\ &= 0 + \int_0^{\sqrt{v}}\exp\left(-\frac{r^2}{2}\right)rdr && (6.17)\\ &= 1 - \exp\left(-\frac{v}{2}\right) && (6.18) \end{aligned}$$

Comparing the 2 equations we get

$$\alpha = \frac{1}{2} \tag{6.19}$$

and PDF is given by

$$p_V(v) = \frac{d}{dx}F_V(v) \tag{6.20}$$

$$\therefore p_V(v) = \begin{cases} \frac{\exp\left(-\frac{v}{2}\right)}{2} & v \ge 0\\ 0 & v < 0 \end{cases} \tag{6.21}$$

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.22}$$

Solution: CDF and PDF of A is plotted in fig. 0 and 0 respectively using below python code.

```
wget https://github.com/himanshukumargupta11012/Random-Numbers/blob/master/ques_6/6.3.py
```

We know that

$$V = R^2 \tag{6.23}$$

$$\therefore A = \sqrt{V} = R \tag{6.24}$$

Since $A = R > 0$. So PDF and CDF will be 0 for $a < 0$
Then,PDF of A for $a \geq 0$,

$$F_A(a) = \Pr(A \leq a) \tag{6.25}$$

$$= \Pr(R \leq a) \tag{6.26}$$

$$= F_R(a) \tag{6.27}$$

$$= \int_{-\infty}^{a} p_R(r)\, dr \tag{6.28}$$

$$= 0 + \int_{0}^{a} \exp\left(-\frac{r^2}{2}\right) r dr \tag{6.29}$$

$$= 1 - \exp\left(-\frac{a^2}{2}\right) \tag{6.30}$$

So,CDF of A,

$$F_A(a) = \begin{cases} 1 - \exp\left(-\frac{a^2}{2}\right) & a \geq 0 \\ 0 & a < 0 \end{cases} \tag{6.31}$$

PDF of A is given by

$$p_A(a) = \frac{d}{dx} F_A(a) \tag{6.32}$$

$$\therefore p_A(a) = \begin{cases} \exp\left(-\frac{a^2}{2}\right) a & a \geq 0 \\ 0 & a < 0 \end{cases} \tag{6.33}$$

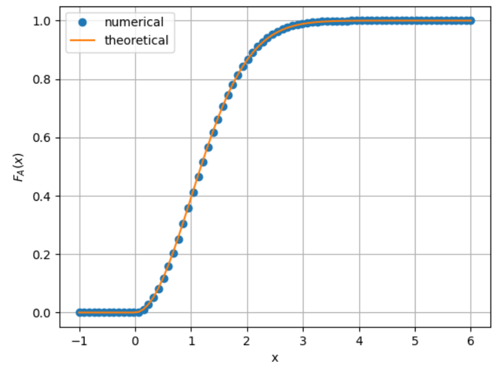

Fig. 0: The CDF of A

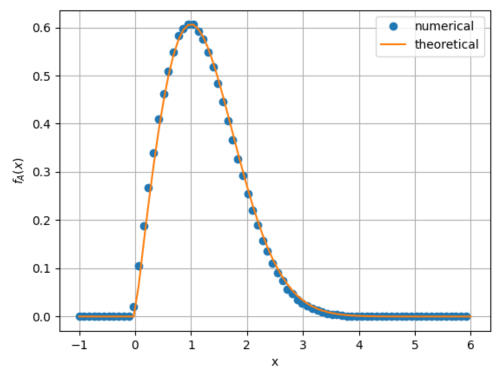

Fig. 0: The PDF of A

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E[P_e(N)]$.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.