Random Numbers

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Contents

***Abstract*—This manual provides solutions for random numbers assignment.**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/ques_1/1.1.c
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The CDF of U is plotted in Fig. 1.1 using the below python code

```
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/ques_1/1.2.py
```

1.3 Find a theoretical expression for $F_U(x)$.

Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \le x) \tag{1.2}$$
$$= \int_{-\infty}^{x} f_U(x)dx \tag{1.3}$$
$$\tag{1.4}$$

Now, for $x < 0$

$$f_U(x) = 0 \tag{1.5}$$
$$\therefore F_U(x) = 0 \tag{1.6}$$

,for $0 \le x \le 1$

$$f_U(x) = 1 \tag{1.7}$$
$$\therefore F_U(x) = \int_{-\infty}^{x} dx \tag{1.8}$$
$$= 0 + \int_{0}^{x} dx \tag{1.9}$$
$$= x \tag{1.10}$$

and for $x > 1$

$$f_U(x) = 0 \tag{1.11}$$
$$\therefore F_U(x) = \int_{-\infty}^{x} dx \tag{1.12}$$
$$= 0 + \int_{0}^{1} dx + 0 \tag{1.13}$$
$$= 1 \tag{1.14}$$

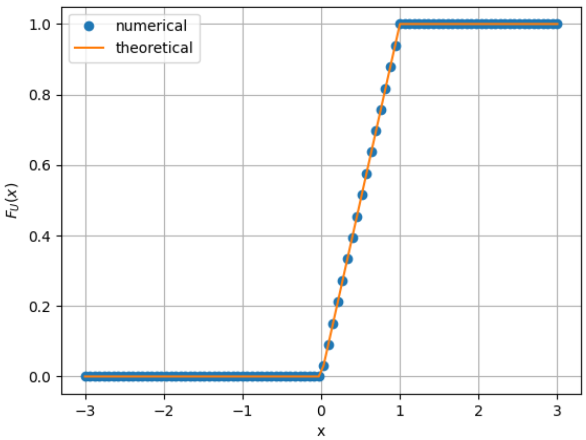

Fig. 1.1: The CDF of U

$$So, F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.15}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.16}$$

Write a C program to find the mean and variance of U.
Solution: Download and run the following C program.

```
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/ques_1/1.4.c
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.17}$$

Solution: The mean of U is given by

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.18}$$
$$= 0 + \int_{0}^{1} x dx + 0 \tag{1.19}$$
$$= \left[\frac{x^2}{2}\right]_0^1 \tag{1.20}$$
$$= 1/2 \tag{1.21}$$

and variance is given by,

$$E[U^2] - E[U]^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \left(\frac{1}{2}\right)^2 \tag{1.22}$$
$$= 0 + \int_{0}^{1} x^2 dx + 0 - \left(\frac{1}{2}\right)^2 \tag{1.23}$$
$$= \left[\frac{x^2}{3}\right]_0^1 - \left(\frac{1}{2}\right)^2 \tag{1.24}$$
$$= 1/12 \tag{1.25}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat **Solution:** Download the following files and run the C program.

```
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/ques_2/2.1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.1 using the below python code

```
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/ques_2/2.2.py
```

Properties of CDF:

a) It is always continuous.
b) It is monotonic increasing.

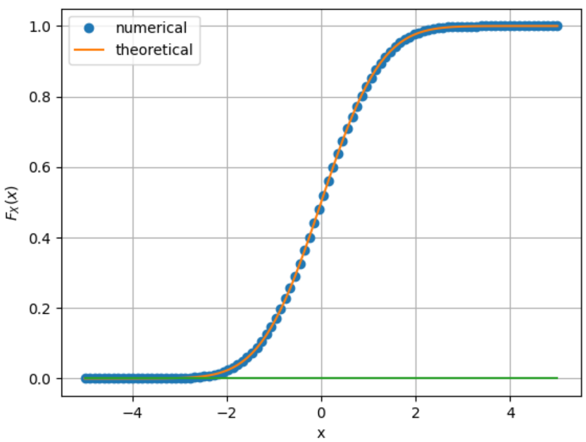

Fig. 2.1: The CDF of X

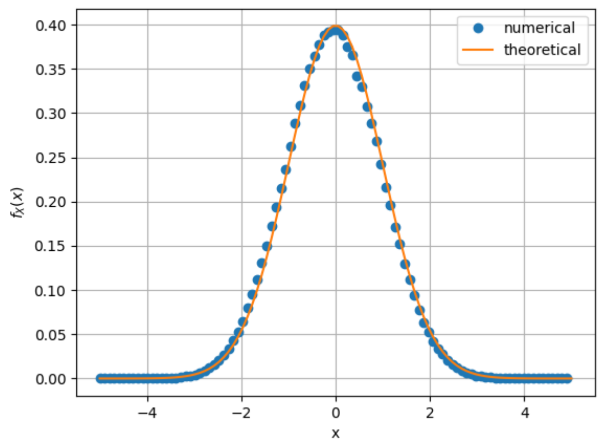

Fig. 2.2: The PDF of X

c) It is bounded between 0 to 1.
d) At infinity, it tends to 1.
e) At negative infinity, it tends to 0.
f) Since it is bounded and have limit so it is convergent also.
g) Q function is defined as:

$$Q_X(x) = \Pr(x < X) \tag{2.2}$$
$$\therefore F_X(x) = 1 - Q_X(x) \tag{2.3}$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.4}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.2 using the below python code

```
wget
https://github.com/himanshukumargupta11012/
    Random−Numbers/blob/master/ques_2
    /2.2.py
```

Properties of PDF:

a) It is always non-negative
b) It is always bounded between 0 to 1
c) Always symmetric about the mean.
d) Maxima is observed at mean
e) Area under the curve of PDF is always equal to 1

2.4 Find the mean and variance of X by writing a C program.

Solution: download the following files and run the C program.

```
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/ques_2/2.4.c
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/coeffs.h
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically. **Solution:** The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{2.6}$$
$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.7}$$
$$= \left[-\frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) \right]_{-\infty}^{+\infty} = 0 \tag{2.8}$$

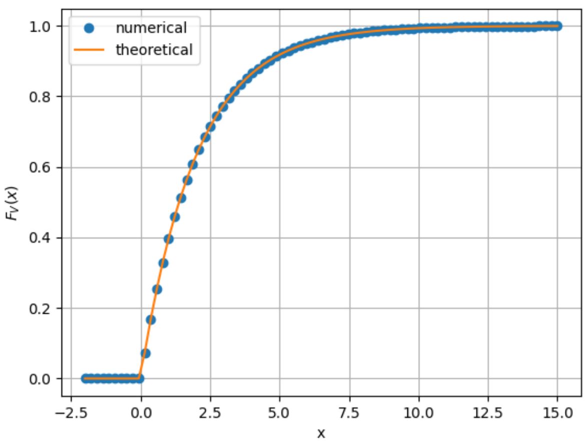

Fig. 3.1: The CDF of V

and variance is given by

$$E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx \tag{2.9}$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.10}$$

$$= 2\int_{0}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.11}$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} t^{-1/2} \exp(-t)\, dt \tag{2.12}$$

$$= \frac{2}{\sqrt{\pi}} \Gamma(1 + 1/2) \tag{2.13}$$

$$= 1 \tag{2.14}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution: Download and run the python and C file where you get samples of V and fig. **??**

```
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/ques_3/3.1.py
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/ques_3/3.1.c
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution: The CDF of V is given by

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$

$$= \Pr(-2\ln(1 - U) \le x) \tag{3.3}$$

$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

let $a = \exp(-x/2)$ .So,

$$F_V(x) = \begin{cases} 0 & 1 - a < 0 \\ 1 - a & 0 \le 1 - a < 1 \\ 1 & 1 - a \ge 1 \end{cases}$$

on solving, we get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \ge 0 \end{cases}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following code and run the C file

```
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/ques_4/4.1.c
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/coeffs.h
```

4.2 Find the CDF of T.
Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \le t) \tag{4.2}$$

Since T is sum of 2 uniform random variable between 0 and 1 So,any T will be always less than 2 and greater than 0 So,

$$F_T(t) = 0 \qquad \text{for } X < 0 \tag{4.3}$$

$$F_T(t) = 1 \qquad \text{for } X \ge 0 \tag{4.4}$$

Now,for $0 \le t < 2$,
let U_1 as a constant x and later we integrate it

for all possible values of x So,

$$F_T(t) = \Pr(T \le t) \tag{4.5}$$
$$= \Pr(U_1 + U_2 \le t) \tag{4.6}$$
$$= \Pr(U_2 \le t - U_1) \tag{4.7}$$
$$= \int_0^t F_{U_2}(t-x)\, p_{U_1}(x)\, dx \tag{4.8}$$

Now, we make 2 cases
case-1: $0 \le t < 1$

$$F_T(t) = \int_0^t F_{U_2}(t-x)\, dx \tag{4.9}$$
$$= \int_0^t F_{U_2}(m)\, dm \tag{4.10}$$
$$= \int_0^t m dm \tag{4.11}$$
$$= \left[\frac{m^2}{2}\right]_0^t \tag{4.12}$$
$$= \frac{t^2}{2} \tag{4.13}$$

case-2: $1 \le t < 2$

$$F_T(t) = \int_0^1 F_{U_2}(t-x)\, dx \tag{4.14}$$
$$= \int_0^1 F_{U_2}(t-x)\, dx \tag{4.15}$$
$$= \int_{t-1}^t F_{U_2}(m)\, dm \tag{4.16}$$
$$= \int_{t-1}^1 m dm + \int_1^t dm \tag{4.17}$$
$$= \left[\frac{m^2}{2}\right]_{t-1}^1 + [1]_1^t \tag{4.18}$$
$$= \frac{-t^2}{2} + 2t - 1 \tag{4.19}$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t_2}{2} & 0 \le t < 1 \\ \frac{-t^2}{2} + 2t - 1 & 1 \le t < 2 \\ 1 & 2 \le t \end{cases}$$

4.3 Find the PDF of T.
Solution: The PDF of T is given by

$$p_T(t) = \frac{d}{dt} F_T(t) \tag{4.20}$$

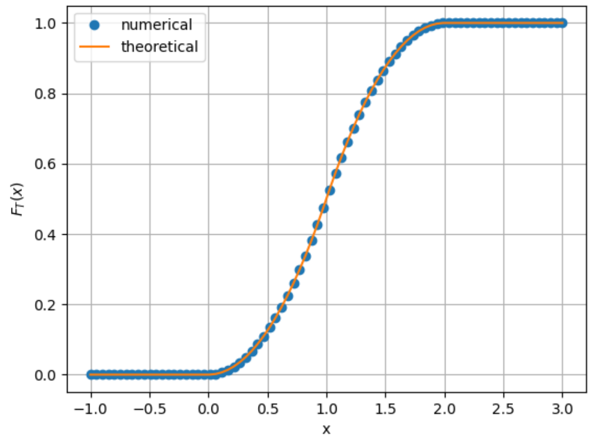

Fig. 4.1: The CDF of T

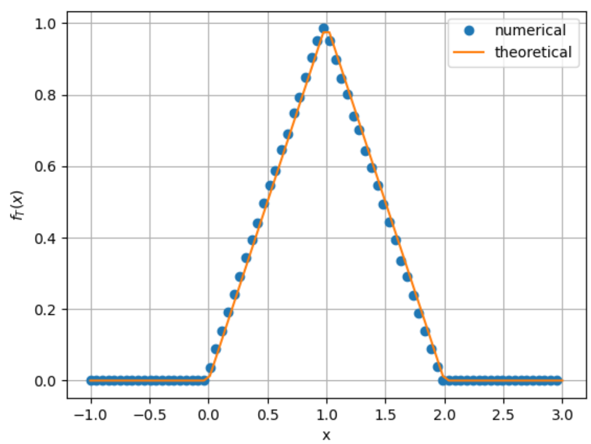

Fig. 4.2: The PDF of T

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \\ 0 & 2 \le t \end{cases}$$

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Already done in 4.2 and 4.3.

4.5 Verify your results through a plot.
Solution: Plot of CDF and PDF is plotted in fig.
(4.1) and (4.2) respectively using following python code.

5 Maximul Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.

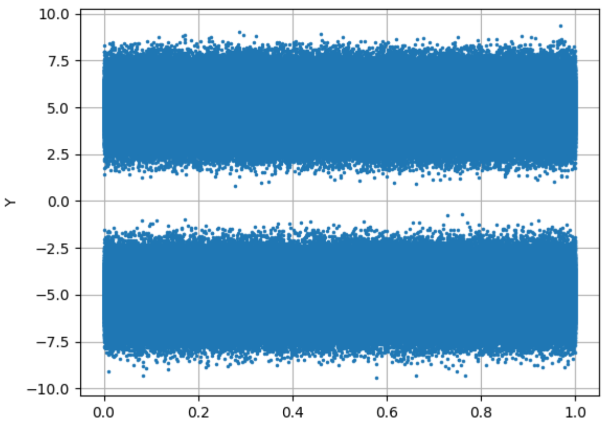

Fig. 5.1: The scatter plot of Y

Solution: Download the following files and execute the C file

```
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/ques_5/5.1.c
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/coeffs.h
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Download the following files and run the C code

```
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/ques_5/5.2.c
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/coeffs.h
```

5.3 Plot Y using a scatter plot.
Solution: The scatter plot of Y is plotted in fig. 5.1 using below python code.

```
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/ques_5/5.3.py
```

5.4 Guess how to estimate X from Y.
Solution: After seeing the scatter plot we can conclude that if Y is positive then X will be 1 and if Y is negative then X will be -1.This conclusion also depends on value of A because if we decrease A the two parts of graph intermix and then we can't say anything.If A is approximately greater than 3 then only we can conclude this.

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.3}$$

Solution: $\hat{X}$ is defined as

$$\hat{X} = \begin{cases} 1 & Y \geq 0 \\ -1 & Y < 0 \end{cases} \tag{5.4}$$

So,

$$\begin{aligned} P_{e|0} &= \Pr\left(\hat{X} = -1|X = 1\right) && (5.5)\\ &= \Pr\left(Y < 0|X = 1\right) && (5.6)\\ &= \Pr\left(A \times 1 + N < 0\right) && (5.7)\\ &= \Pr\left(N < -A\right) && (5.8)\\ &= F_N\left(-A\right) && (5.9)\\ &= 1 - Q_N\left(-5\right) && (5.10)\\ &= 2.866515719235352 \times 10^{-07} && (5.11) \end{aligned}$$

and

$$\begin{aligned} P_{e|1} &= \Pr\left(\hat{X} = 1|X = -1\right) && (5.12)\\ &= \Pr\left(Y \geq 0|X = -1\right) && (5.13)\\ &= \Pr\left(A \times -1 + N \geq 0\right) && (5.14)\\ &= \Pr\left(N \geq A\right) && (5.15)\\ &= 1 - \Pr\left(N < A\right) && (5.16)\\ &= 1 - F_N\left(A\right) && (5.17)\\ &= Q_N\left(5\right) && (5.18)\\ &= 2.866515719235352 \times 10^{-07} && (5.19) \end{aligned}$$

Download and run the following file to get coding proof

```
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/ques_5/5.5.py
```

5.6 Find P_e assuming that X has equiprobable symbols.

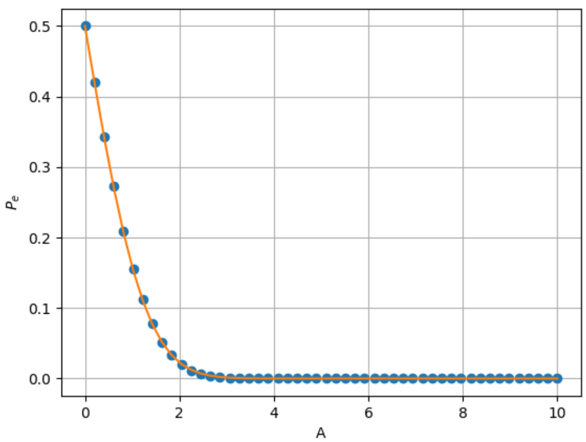

Fig. 5.2: P_e graph wrt A

Solution: X is equiprobable.So

$$\because \Pr(X=1) = \Pr(X=-1), \tag{5.20}$$
$$\Pr(X=1) + \Pr(X=-1) = 1 \tag{5.21}$$
$$\therefore \Pr(X=1) = \Pr(X=-1) = 1/2 \tag{5.22}$$

P_e can be defined as

$$P_e = \Pr(X=-1) P_{e|1} + \Pr(X=1) P_{e|0} \tag{5.23}$$
$$= 2.866515719235352 \times 10^{-7} \tag{5.24}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: X is equiprobable.So

$$\because \Pr(X=1) = \Pr(X=-1), \tag{5.25}$$
$$\Pr(X=1) + \Pr(X=-1) = 1 \tag{5.26}$$
$$\therefore \Pr(X=1) = \Pr(X=-1) = 1/2 \tag{5.27}$$

Now,P_e is given by

$$P_e = \Pr(X=-1) P_{e|1} + \Pr(X=1) P_{e|0} \tag{5.28}$$
$$P_e = \frac{1}{2}(1 - F_N(A)) + \frac{1}{2}(F_N(-A)) \tag{5.29}$$

Now, the plot of P_e w.r.t. A is in fig. 5.2 by plotting the above function using following code

```
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/ques_5/5.7.py
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

Solution: Taking threshold as δ we get

$$P_{e|0} = \Pr(Y < \delta | X = 1) \tag{5.30}$$
$$= \Pr(A \times 1 + N < \delta) \tag{5.31}$$
$$= \Pr(N < -A + \delta) \tag{5.32}$$
$$= F_N(-A + \delta) \tag{5.33}$$

and

$$P_{e|1} = \Pr(Y \geq \delta | X = -1) \tag{5.34}$$
$$= \Pr(A \times -1 + N \geq \delta) \tag{5.35}$$
$$= \Pr(N \geq A + \delta) \tag{5.36}$$
$$= 1 - \Pr(N < A + \delta) \tag{5.37}$$
$$= 1 - F_N(A + \delta) \tag{5.38}$$

and P_e is defined as

$$P_e = \Pr(X=-1) P_{e|1} + \Pr(X=1) P_{e|0} \tag{5.39}$$
$$= \frac{1}{2}(1 - F_N(A+\delta) + F_N(-A+\delta)) \tag{5.40}$$

Now we differentiate P_e and equate it to 0 to get δ that minimizes the P_e.So,

$$\frac{d}{d\delta} P_e = \frac{1}{2}\left(\frac{d}{d\delta} F_N(-A+\delta) - \frac{d}{d\delta} F_N(A+\delta)\right) = 0 \tag{5.41}$$
$$\implies p_N(-A+\delta) - p_N(A+\delta) = 0 \tag{5.42}$$
$$\implies \exp\left(-\frac{(-A+\delta)^2}{2}\right) - \exp\left(-\frac{(A+\delta)^2}{2}\right) = 0 \tag{5.43}$$
$$\implies \delta = 0 \tag{5.44}$$

Now we have to check that δ=0 is for maxima or minima by double differentiating it. So,

$$\frac{d^2}{d\delta^2}|_{\delta=0} = 2A \exp\left(\frac{-A^2}{2}\right) \tag{5.45}$$
$$= 10 \exp\left(\frac{-A^2}{2}\right) \quad > 0 \tag{5.46}$$

Double derivative is positive. So,δ=0 will give minima of P_e
Plot of P_e w.r.t. δ is plotted in fig. 5.3 using following code

```
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/ques_5/5.8.py
```

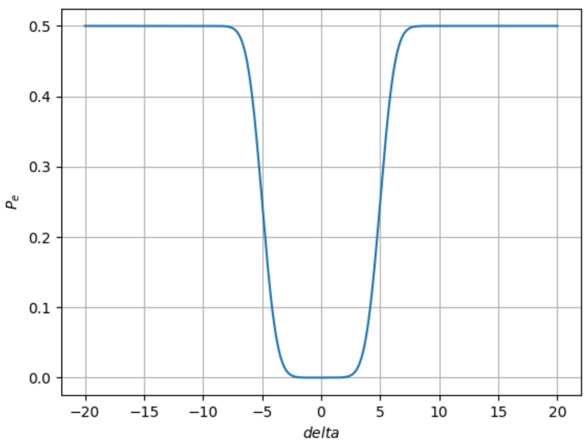

Fig. 5.3: P_e graph wrt δ

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.47}$$

Solution:

$$\because \Pr(X = 1) = p_X(0) = p \tag{5.48}$$

$$\therefore \Pr(X = -1) = 1 - \Pr(X = 1) \tag{5.49}$$

$$= 1 - p \tag{5.50}$$

and P_e is given by

$$P_e = \Pr(X = -1) P_{e|1} + \Pr(X = 1) P_{e|0} \tag{5.51}$$

$$= (1 - p)(1 - F_N(A + \delta)) \tag{5.52}$$

$$+ pF_N(-A + \delta) \tag{5.53}$$

Then,

$$\frac{d}{d\delta} P_e = p \frac{d}{d\delta} F_N(-A + \delta) \tag{5.54}$$

$$- (1 - p) \frac{d}{d\delta} F_N(A + \delta) = 0 \tag{5.55}$$

$$\implies p \times p_N(-A + \delta) \tag{5.56}$$

$$- (1 - p) p_N(A + \delta) = 0 \tag{5.57}$$

$$\implies p \times \exp\left(-\frac{(-A + \delta)^2}{2}\right) \tag{5.58}$$

$$- (1 - p) \exp\left(-\frac{(A + \delta)^2}{2}\right) = 0 \tag{5.59}$$

$$\implies \delta = \frac{\log\left(\frac{1}{p} - 1\right)}{2A} \tag{5.60}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution: From bayes theorem we can write,

$$\Pr(X = 1|Y = y) \tag{5.61}$$

$$= \frac{\Pr(Y = y|X = 1) \Pr(X = 1)}{\Pr(Y = y)} \tag{5.62}$$

$$= \frac{\Pr(N = y - A) p}{\Pr(N = y - A) p + \Pr(N = y + A)(1 - p)} \tag{5.63}$$

$$= \frac{p_N(y - A) p}{p_N(y - A) p + p_N(y + A)(1 - p)} \tag{5.64}$$

$$= \frac{p}{p + (1 - p) \exp(-2Ay)} \tag{5.65}$$

and

$$\Pr(X = -1|Y = y) \tag{5.66}$$

$$= 1 - \Pr(X = 1|Y = y) \tag{5.67}$$

$$= 1 - \frac{p}{p + (1 - p) \exp(-2Ay)} \tag{5.68}$$

$$= \frac{(1 - p) \exp(-2Ay)}{p + (1 - p) \exp(-2Ay)} \tag{5.69}$$

$$= \frac{1 - p}{p \exp(2Ay) + 1 - p} \tag{5.70}$$

Now, if $X = 1$ is more likely than $X = -1$ for a y. Then,

$$\Pr(X = 1|Y = y) > \Pr(X = -1|Y = y) \tag{5.71}$$

$$\frac{p}{p + (1 - p) \exp(-2Ay)} > \frac{1 - p}{p \exp(2Ay) + 1 - p} \tag{5.72}$$

$$y > \frac{\log\left(\frac{1-p}{p}\right)}{2A} \tag{5.73}$$

and if $X = -1$ is more lileky for a y. Then,

$$\Pr(X = 1|Y = y) < \Pr(X = -1|Y = y) \tag{5.74}$$

$$\frac{p}{p + (1 - p) \exp(-2Ay)} \tag{5.75}$$

$$< \frac{1 - p}{p \exp(2Ay) + 1 - p} \tag{5.76}$$

$$y < \frac{\log\left(\frac{1-p}{p}\right)}{2A} \tag{5.77}$$

So,we can see that optimal value of Y for P_e to be minimum is

$$\frac{\log\left(\frac{1-p}{p}\right)}{2A} \tag{5.78}$$

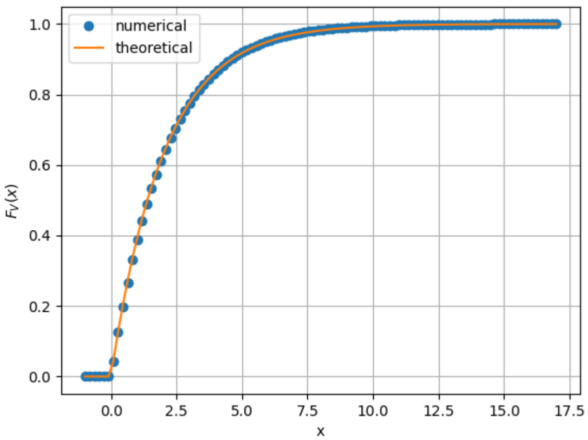

Fig. 6.1: The CDF of V

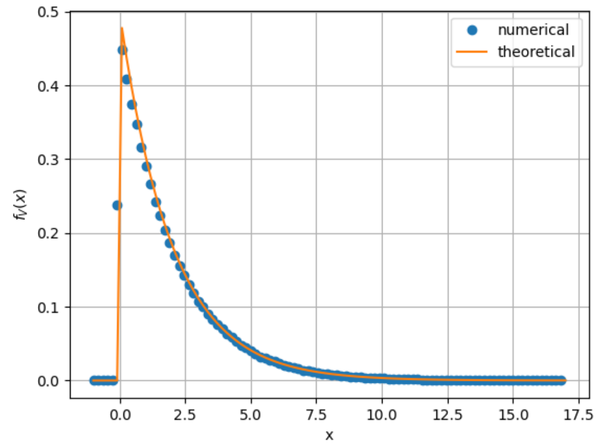

Fig. 6.2: The PDF of V

So,

$$\delta = \frac{\log\left(\frac{1-p}{p}\right)}{2A} \tag{5.79}$$

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: Download the following files and run C and python program. CDF and PDF of V is plotted in fig. 6.1 and 6.2 respectively using below python code.

```
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/coeffs.h
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/ques_6/6.1.c
wget https://github.com/
    himanshukumargupta11012/Random-
    Numbers/blob/master/ques_6/6.1.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.
Solution: Let
$X_1 = R\cos\theta$,$X_2 = R\sin\theta$
So, PDF of R and θ is given by

$$p_{R,\theta}(r,\phi) = p_{X_1,X_2}(x_1,x_2)\,|J| \tag{6.3}$$

where J is jacobian matrix transforming R,θ to X_1, X_2
On soving we get jacobian, $J = R$. So,

$$p_{R,\theta}(r,\phi) = p_{X_1,X_2}(x_1,x_2)\,r \tag{6.4}$$

since X_1 and X_2 are i.i.d. and gaussian random variable,$\mathcal{N}(0,1)$.So,

$$p_{R,\theta}(r,\phi) = p_{X_1}(x_1)\,p_{X_2}(x_2)\,r \tag{6.5}$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x_1^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x_2^2}{2}\right)r \tag{6.6}$$

$$= \frac{1}{2\pi}\exp\left(-\frac{r^2}{2}\right)r \tag{6.7}$$

Now PDF of R,

$$p_R(r) = \int_0^{2\pi} p_{R,\theta}(r,\phi)\,d\phi \tag{6.8}$$

$$= \int_0^{2\pi} \frac{1}{2\pi}\exp\left(-\frac{r^2}{2}\right)r\,d\phi \tag{6.9}$$

$$= \exp\left(-\frac{r^2}{2}\right)r \tag{6.10}$$

and

$$V = X_1^2 + X_2^2 = R^2 \tag{6.11}$$

Since $V = R^2 > 0$
So,CDF and PDF would be zero for v<0

Then CDF of V for v≥0,

$$\begin{aligned} F_V(v) &= \Pr(V \le v) && (6.12)\\ &= \Pr\left(R^2 \le v\right) && (6.13)\\ &= \Pr\left(R \le \sqrt{v}\right) && (6.14)\\ &= F_R\left(\sqrt{v}\right) && (6.15)\\ &= \int_{-\infty}^{\sqrt{v}} p_R(r)\,dr && (6.16)\\ &= 0 + \int_0^{\sqrt{v}} \exp\left(-\frac{r^2}{2}\right) rdr && (6.17)\\ &= 1 - \exp\left(-\frac{v}{2}\right) && (6.18) \end{aligned}$$

Comparing the 2 equations we get

$$\alpha = \frac{1}{2} \tag{6.19}$$

and PDF is given by

$$p_V(v) = \frac{d}{dx} F_V(v) \tag{6.20}$$

$$\therefore p_V(v) = \begin{cases} \frac{\exp\left(-\frac{v}{2}\right)}{2} & v \ge 0 \\ 0 & v < 0 \end{cases} \tag{6.21}$$

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.22}$$

Solution: CDF and PDF of A is plotted in fig. 6.3 and 6.4 respectively using below python code.

```
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/ques_6/6.3.py
```

We know that

$$V = R^2 \tag{6.23}$$

$$\therefore A = \sqrt{V} = R \tag{6.24}$$

Since $A = R > 0$. So PDF and CDF will be 0 for $a < 0$

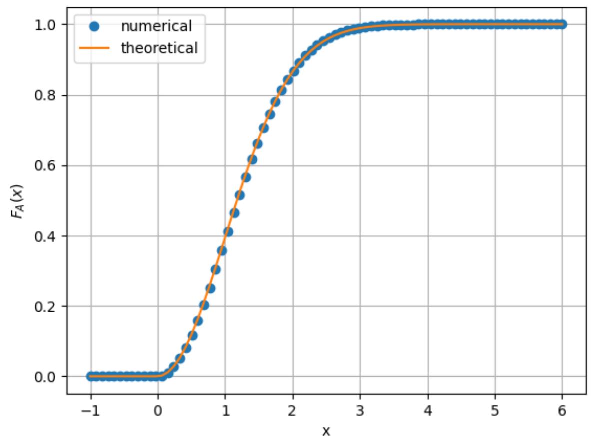

Fig. 6.3: The CDF of A

Then,PDF of A for $a \ge 0$,

$$\begin{aligned} F_A(a) &= \Pr(A \le a) && (6.25)\\ &= \Pr(R \le a) && (6.26)\\ &= F_R(a) && (6.27)\\ &= \int_{-\infty}^{a} p_R(r)\,dr && (6.28)\\ &= 0 + \int_0^{a} \exp\left(-\frac{r^2}{2}\right) rdr && (6.29)\\ &= 1 - \exp\left(-\frac{a^2}{2}\right) && (6.30) \end{aligned}$$

So,CDF of A,

$$F_A(a) = \begin{cases} 1 - \exp\left(-\frac{a^2}{2}\right) & a \ge 0 \\ 0 & a < 0 \end{cases} \tag{6.31}$$

PDF of A is given by

$$p_A(a) = \frac{d}{dx} F_A(a) \tag{6.32}$$

$$\therefore p_A(a) = \begin{cases} \exp\left(-\frac{a^2}{2}\right) a & a \ge 0 \\ 0 & a < 0 \end{cases} \tag{6.33}$$

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \le \gamma \le 10$ dB.

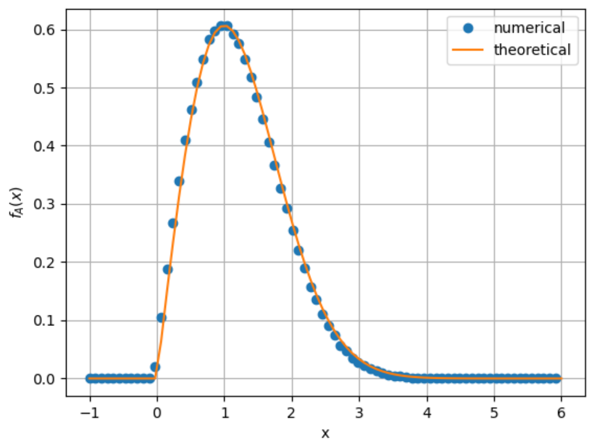

Fig. 6.4: The PDF of A

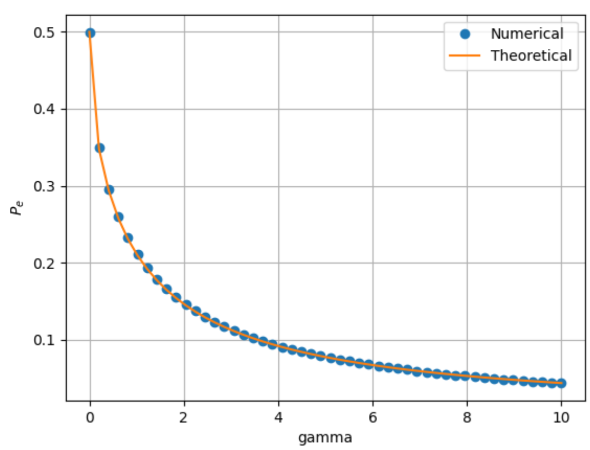

Fig. 7.1: P_e wrt γ

Solution: Plot of P_e wrt γ is given in fig. 7.1 using below python code

```
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/ques_7/7.1.py
```

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$
Solution: Since A is rayleish distribution.So,mean,

$$E[A] = \sigma\sqrt{\frac{\pi}{2}} \tag{7.3}$$

$$\tag{7.4}$$

and variance,

$$E[A - E[A]^2] = \frac{4-\pi}{2}\sigma^2 \tag{7.5}$$

$$\tag{7.6}$$

where σ is parameter and we know that

$$E[X - E[X]^2] = E[X^2] - E[X]^2 \tag{7.7}$$

$$\implies E[X]^2 = \frac{4-\pi}{2}\sigma^2 - \left(\sigma\sqrt{\frac{\pi}{2}}\right)^2 \tag{7.8}$$

$$\gamma = 2\sigma^2 \tag{7.9}$$

Now,PDF of A can be defined as

$$F_A(a) = \begin{cases} 1 - \exp\left(-\frac{a^2}{2\sigma^2}\right) & a \geq 0 \\ 0 & a < 0 \end{cases} \tag{7.10}$$

P_e is given by

$$P_e[N] = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.11}$$

$$= \Pr(Y < 0 | X = 1) \tag{7.12}$$

$$= \Pr(A \times 1 + N < 0) \tag{7.13}$$

$$= \Pr(A < -N) \tag{7.14}$$

$$\text{Since N is assumed as constant} \tag{7.15}$$

$$= F_A(-N) \tag{7.16}$$

$$P_e[N] = \begin{cases} 1 - \exp\left(-\frac{N^2}{\gamma}\right) & N \geq 0 \\ 0 & N < 0 \end{cases} \tag{7.17}$$

7.3 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.18}$$

Find $P_e = E[P_e(N)]$.
Solution: P_e is given by

$$P_e = E[P_e(N)] \tag{7.19}$$

$$= \int_{-\infty}^{\infty} P_e(n) p_N(n)\, dn \tag{7.20}$$

$$= 0 + \int_0^{\infty} \left(1 - \exp\left(-\frac{n^2}{\gamma}\right)\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{n^2}{2}\right) dn \tag{7.21}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(-\frac{n^2}{2}\right) - \exp\left(-n^2\left(\frac{1}{\gamma} + \frac{1}{2}\right)\right) dn \tag{7.22}$$

$$= erfc(0)\left\{\frac{1}{2} - \frac{1}{2\sqrt{\frac{2}{\gamma}+1}}\right\} \tag{7.23}$$

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.
Solution: Theoretical and numerical plot of P_e wrt γ is plotted in fig. 7.1 using below python code

```
wget https://github.com/
    himanshukumargupta11012/Random−
    Numbers/blob/master/ques_7/7.1.py
```

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.