



## 1 UNIFORM RANDOM NUMBERS

Let  $U$  be a uniform random variable between 0 and 1.

- 1.1 Generate  $10^6$  samples of  $U$  using a C program and save into a file called uni.dat .

**Solution:** Download the following files and execute the C program.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_1/1.1.c
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/coeffs.h
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of  $U$  using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

**Solution:** The CDF of  $U$  is plotted in Fig. 0 using the below python code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_1/1.2.py
```

- 1.3 Find a theoretical expression for  $F_U(x)$ .

**Solution:** The CDF of  $U$  is given by

$$F_U(x) = \Pr(U \leq x) \quad (1.2)$$

$$= \int_{-\infty}^x f_U(x) dx \quad (1.3)$$

$$(1.4)$$

Now, for  $x < 0$

$$f_U(x) = 0 \quad (1.5)$$

$$\therefore F_U(x) = 0 \quad (1.6)$$

,for  $0 \leq x \leq 1$

$$f_U(x) = 1 \quad (1.7)$$

$$\therefore F_U(x) = \int_{-\infty}^x dx \quad (1.8)$$

$$= 0 + \int_0^x dx \quad (1.9)$$

$$= x \quad (1.10)$$

and for  $x > 1$

$$f_U(x) = 0 \quad (1.11)$$

$$\therefore F_U(x) = \int_{-\infty}^x dx \quad (1.12)$$

$$= 0 + \int_0^1 dx + 0 \quad (1.13)$$

$$= 1 \quad (1.14)$$

$$\text{So, } F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

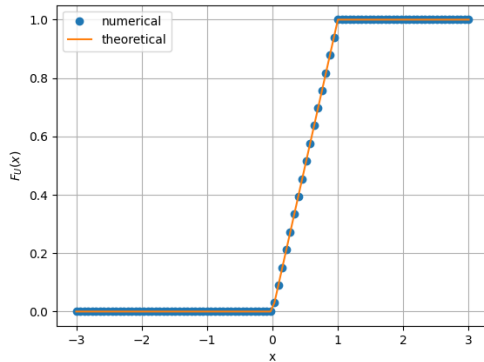


Fig. 0: The CDF of  $U$

1.4 The mean of  $U$  is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.15)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.16)$$

Write a C program to find the mean and variance of  $U$ .

**Solution:** Download and run the following C program.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_1/1.4.c
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.17)$$

**Solution:** The mean of  $U$  is given by

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.18)$$

$$= 0 + \int_0^1 x dx + 0 \quad (1.19)$$

$$= \left[ \frac{x^2}{2} \right]_0^1 \quad (1.20)$$

$$= 1/2 \quad (1.21)$$

and variance is given by,

$$E[U^2] - E[U]^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \left( \frac{1}{2} \right)^2 \quad (1.22)$$

$$= 0 + \int_0^1 x^2 dx + 0 - \left( \frac{1}{2} \right)^2 \quad (1.23)$$

$$= \left[ \frac{x^3}{3} \right]_0^1 - \left( \frac{1}{2} \right)^2 \quad (1.24)$$

$$= 1/12 \quad (1.25)$$

## 2 CENTRAL LIMIT THEOREM

2.1 Generate  $10^6$  samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where  $U_i, i = 1, 2, \dots, 12$  are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

**Solution:** Download the following files and run the C program.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
```

```
blob/master/ques
_2/2.1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of  $X$  using the samples in gau.dat. What properties does a CDF have?

**Solution:** The CDF of  $X$  is plotted in Fig. 0 using the below python code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_2/2.2.py
```

Properties of CDF:

- It is always continuous.
- It is monotonic increasing.
- It is bounded between 0 to 1.
- At infinity, it tends to 1.
- At negative infinity, it tends to 0.
- Since it is bounded and have limit so it is convergent also.
- Q function is defined as:

$$Q_X(x) = \Pr(x < X) \quad (2.2)$$

$$\therefore F_X(x) = 1 - Q_X(x) \quad (2.3)$$

2.3 Load gau.dat in python and plot the empirical PDF of  $X$  using the samples in gau.dat. The PDF of  $X$  is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.4)$$

What properties does the PDF have?

**Solution:** The PDF of  $X$  is plotted in Fig. 0 using the below python code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
```

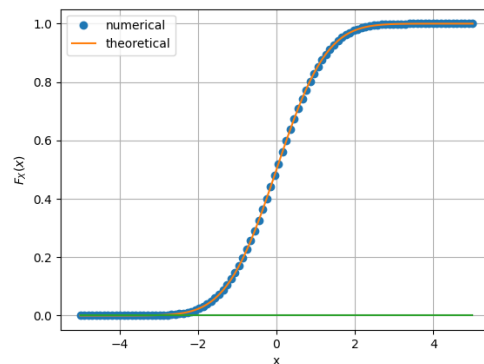


Fig. 0: The CDF of  $X$

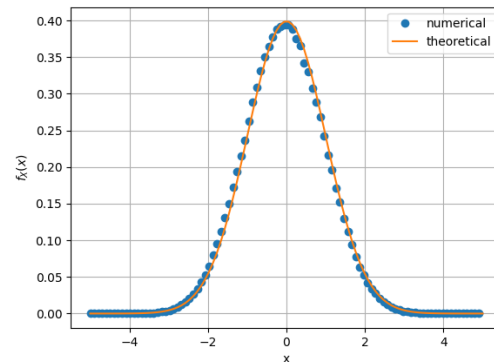


Fig. 0: The PDF of  $X$

```
blob/master/ques
_2/2.2.py
```

Properties of PDF:

- It is always non-negative
- It is always bounded between 0 to 1
- Always symmetric about the mean.
- Maxima is observed at mean
- Area under the curve of PDF is always equal to 1

2.4 Find the mean and variance of  $X$  by writing a C program.

**Solution:** download the following files and run the C program.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_2/2.4.c
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/coeffs.h
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.

**Solution:** The mean of  $X$  is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.6)$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.7)$$

$$= \left[ -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \right]_{-\infty}^{+\infty} = 0 \quad (2.8)$$

and variance is given by

$$E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx \quad (2.9)$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.10)$$

$$= 2 \int_0^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.11)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{-1/2} \exp(-t) dt \quad (2.12)$$

$$= \frac{2}{\sqrt{\pi}} \Gamma(1 + 1/2) \quad (2.13)$$

$$= 1 \quad (2.14)$$

### 3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

**Solution:** Download and run the python and C file where you get samples of  $V$  and fig. ??

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_3/3.1.py
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_3/3.1.c
```

3.2 Find a theoretical expression for  $F_V(x)$ .

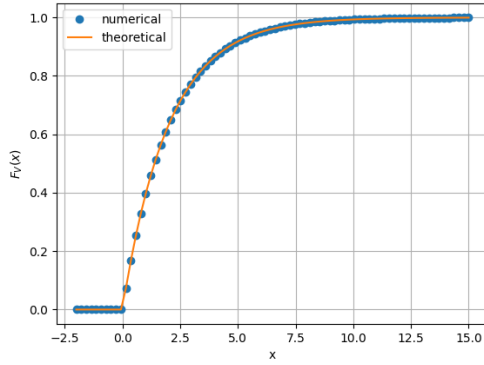


Fig. 0: The CDF of  $V$

**Solution:** The CDF of  $V$  is given by

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.4)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

let  $a = \exp(-x/2)$ . So,

$$F_V(x) = \begin{cases} 0 & 1 - a < 0 \\ 1 - a & 0 \leq 1 - a < 1 \\ 1 & 1 - a \geq 1 \end{cases}$$

on solving, we get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases}$$

#### 4 TRIANGULAR DISTRIBUTION

##### 4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

**Solution:** Download the following code and run the C file

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_4/4.1.c
```

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/coeffs.h
```

##### 4.2 Find the CDF of $T$ .

**Solution:** The CDF of  $T$  is given by

$$F_T(t) = \Pr(T \leq t) \quad (4.2)$$

Since  $T$  is sum of 2 uniform random variable between 0 and 1 So, any  $T$  will be always less than 2 and greater than 0 So,

$$F_T(t) = 0 \quad \text{for } X < 0 \quad (4.3)$$

$$F_T(t) = 1 \quad \text{for } X \geq 0 \quad (4.4)$$

Now, for  $0 \leq t < 2$ ,

let  $U_1$  as a constant  $x$  and later we integrate it for all possible values of  $x$  So,

$$F_T(t) = \Pr(T \leq t) \quad (4.5)$$

$$= \Pr(U_1 + U_2 \leq t) \quad (4.6)$$

$$= \Pr(u_2 \leq t - x) \quad (4.7)$$

$$= \int_0^t F_{U_2}(t - x) p_{u_1}(x) dx \quad (4.8)$$

Now, we make 2 cases

**case-1:**  $0 \leq t < 1$

$$F_T(t) = \int_0^t F_{U_2}(t-x) dx \quad (4.9)$$

$$= \int_0^t F_{U_2}(m) dm \quad (4.10)$$

$$= \int_0^t m dm \quad (4.11)$$

$$= \left[ \frac{m^2}{2} \right]_0^t \quad (4.12)$$

$$= \frac{t^2}{2} \quad (4.13)$$

**case-2:**  $1 \leq t < 2$

$$F_T(t) = \int_0^1 F_{U_2}(t-x) dx \quad (4.14)$$

$$= \int_0^1 F_{U_2}(t-x) dx \quad (4.15)$$

$$= \int_{t-1}^t F_{U_2}(m) dm \quad (4.16)$$

$$= \int_{t-1}^1 m dm + \int_1^t dm \quad (4.17)$$

$$= \left[ \frac{m^2}{2} \right]_{t-1}^1 + [1]_1^t \quad (4.18)$$

$$= \frac{-t^2}{2} + 2t - 1 \quad (4.19)$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t < 1 \\ \frac{-t^2}{2} + 2t - 1 & 1 \leq t < 2 \\ 1 & 2 \leq t \end{cases}$$

4.3 Find the PDF of  $T$ .

**Solution:** The PDF of  $T$  is given by

$$p_T(t) = \frac{d}{dt} F_T(t) \quad (4.20)$$

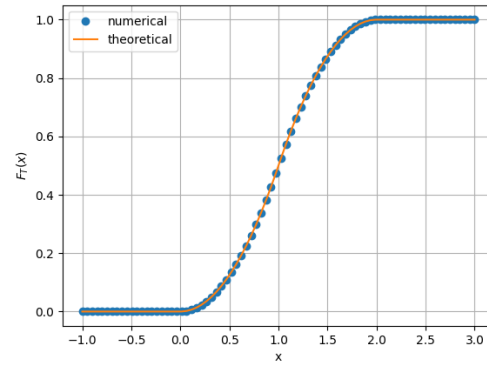


Fig. 0: The CDF of  $T$

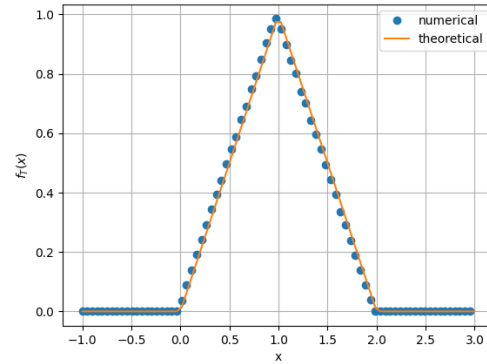


Fig. 0: The PDF of  $T$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2 - t & 1 \leq t < 2 \\ 0 & 2 \leq t \end{cases}$$

4.4 Find the theoretical expressions for the PDF and CDF of  $T$ .

**Solution:** Already done in 4.2 and 4.3.

4.5 Verify your results through a plot.

**Solution:** Plot of CDF and PDF is plotted in fig.

(0) and (0) respectively using following python code.

## 5 MAXIMUM LIKELIHOOD

5.1 Generate equiprobable  $X \in \{1, -1\}$ .

**Solution:** Download the following files and execute the C file

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_5/5.1.c
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/coeffs.h
```

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

where  $A = 5$  dB, and  $N \sim \mathcal{N}(0, 1)$ .

**Solution:** Download the following files and run the C code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_5/5.2.c
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/coeffs.h
```

5.3 Plot  $Y$  using a scatter plot.

**Solution:** The scatter plot of  $Y$  is plotted in fig. 0 using below python code.

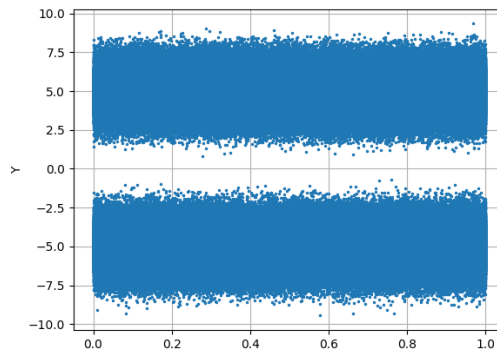


Fig. 0: The scatter plot of  $Y$

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_5/5.3.py
```

5.4 Guess how to estimate  $X$  from  $Y$ .

**Solution:** After seeing the scatter plot we can conclude that if  $Y$  is positive then  $X$  will be 1 and if  $Y$  is negative then  $X$  will be -1. This conclusion also depends on value of  $A$  because if we decrease  $A$  the two parts of graph intermix and then we can't say anything. If  $A$  is approximately greater than 3 then only we can conclude this.

5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.3)$$

**Solution:**  $\hat{X}$  is defined as

$$\hat{X} = \begin{cases} 1 & Y \geq 0 \\ -1 & Y < 0 \end{cases} \quad (5.4)$$

So,

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.5)$$

$$= \Pr(Y < 0|X = 1) \quad (5.6)$$

$$= \Pr(A \times 1 + N < 0) \quad (5.7)$$

$$= \Pr(N < -A) \quad (5.8)$$

$$= F_N(-A) \quad (5.9)$$

$$= 0 \quad (5.10)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.11)$$

$$= \Pr(Y \geq 0|X = -1) \quad (5.12)$$

$$= \Pr(A \times -1 + N \geq 0) \quad (5.13)$$

$$= \Pr(N \geq A) \quad (5.14)$$

$$= 1 - \Pr(N < A) \quad (5.15)$$

$$= 1 - F_N(A) \quad (5.16)$$

$$= 0 \quad (5.17)$$

Download and run the following file to get coding proof

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_5/5.5.py
```

5.6 Find  $P_e$  assuming that  $X$  has equiprobable symbols.

**Solution:**  $P_e$  can be defined as

$$P_e = \Pr(X = -1) P_{e|1} + \Pr(X = 1) P_{e|0} \quad (5.18)$$

$$\because P_{e|1} = 0, P_{e|0} = 0 \quad (5.19)$$

$$\therefore P_e = 0 \quad (5.20)$$

5.7 Verify by plotting the theoretical  $P_e$  with respect to  $A$  from 0 to 10 dB.

**Solution:**  $X$  is equiprobable. So

$$\because \Pr(X = 1) = \Pr(X = -1), \quad (5.21)$$

$$\Pr(X = 1) + \Pr(X = -1) = 1 \quad (5.22)$$

$$\therefore \Pr(X = 1) = \Pr(X = -1) = 1/2 \quad (5.23)$$

Now,  $P_e$  is given by

$$P_e = \Pr(X = -1) P_{e|1} + \Pr(X = 1) P_{e|0} \quad (5.24)$$

$$P_e = \frac{1}{2} (1 - F_N(A)) + \frac{1}{2} (F_N(-A)) \quad (5.25)$$

Now, the plot of  $P_e$  w.r.t.  $A$  is in fig. 0 by plotting the above function using following code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_5/5.7.py
```

5.8 Now, consider a threshold  $\delta$  while estimating  $X$  from  $Y$ . Find the value of  $\delta$  that maximizes the theoretical  $P_e$ .

**Solution:** Taking threshold as  $\delta$  we



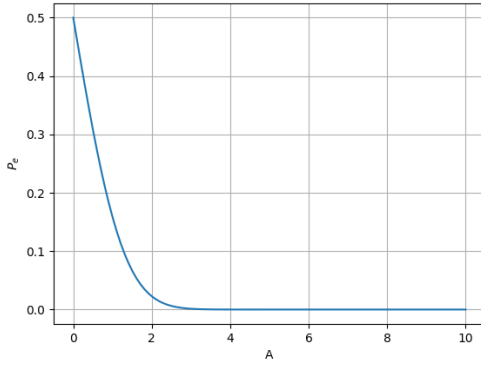


Fig. 0:  $P_e$  graph wrt A

get

$$P_{e|0} = \Pr(Y < \delta | X = 1) \quad (5.26)$$

$$= \Pr(A \times 1 + N < \delta) \quad (5.27)$$

$$= \Pr(N < -A + \delta) \quad (5.28)$$

$$= F_N(-A + \delta) \quad (5.29)$$

and

$$P_{e|1} = \Pr(Y \geq \delta | X = -1) \quad (5.30)$$

$$= \Pr(A \times -1 + N \geq \delta) \quad (5.31)$$

$$= \Pr(N \geq A + \delta) \quad (5.32)$$

$$= 1 - \Pr(N < A + \delta) \quad (5.33)$$

$$= 1 - F_N(A + \delta) \quad (5.34)$$

and  $P_e$  is defined as

$$P_e = \Pr(X = -1) P_{e|1} + \Pr(X = 1) P_{e|0} \quad (5.35)$$

$$= \frac{1}{2} (1 - F_N(A + \delta) + F_N(-A + \delta)) \quad (5.36)$$

Now we differentiate  $P_e$  and equate it

to 0 to get  $\delta$  that minimizes the  $P_e$ . So,

$$\frac{d}{d\delta} P_e = \frac{1}{2} \left( \frac{d}{d\delta} F_N(-A + \delta) - \frac{d}{d\delta} F_N(A + \delta) \right) = 0 \quad (5.37)$$

$$\Rightarrow p_N(-A + \delta) - p_N(A + \delta) = 0 \quad (5.38)$$

$$\Rightarrow \exp\left(-\frac{(-A + \delta)^2}{2}\right) - \exp\left(-\frac{(A + \delta)^2}{2}\right) = 0 \quad (5.39)$$

$$\Rightarrow \delta = 0 \quad (5.40)$$

Now we have to check that  $\delta=0$  is for maxima or minima by double differentiating it. So,

$$\frac{d^2}{d\delta^2} \Big|_{\delta=0} = 2A \exp\left(\frac{-A^2}{2}\right) \quad (5.41)$$

$$= 10 \exp\left(\frac{-A^2}{2}\right) > 0 \quad (5.42)$$

Double derivative is positive. So,  $\delta=0$  will give minima of  $P_e$

Plot of  $P_e$  w.r.t.  $\delta$  is plotted in fig. 0 using following code

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_5/5.8.py
```

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.43)$$

**Solution:**

$$\because \Pr(X = 1) = p_X(0) = p \quad (5.44)$$

$$\therefore \Pr(X = -1) = 1 - \Pr(X = 1) \quad (5.45)$$

$$= 1 - p \quad (5.46)$$

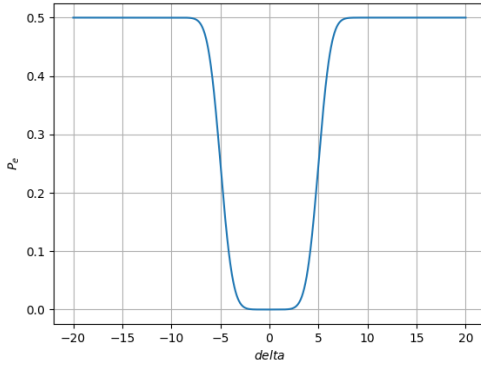


Fig. 0:  $P_e$  graph wrt  $\delta$

and  $P_e$  is given by

$$P_e = \Pr(X = -1) P_{e|1} + \Pr(X = 1) P_{e|0} \quad (5.47)$$

$$= (1 - p)(1 - F_N(A + \delta)) \quad (5.48)$$

$$+ p F_N(-A + \delta) \quad (5.49)$$

Then,

$$\frac{d}{d\delta} P_e = p \frac{d}{d\delta} F_N(-A + \delta) \quad (5.50)$$

$$- (1 - p) \frac{d}{d\delta} F_N(A + \delta) = 0 \quad (5.51)$$

$$\Rightarrow p \times p_N(-A + \delta) \quad (5.52)$$

$$- (1 - p) p_N(A + \delta) = 0 \quad (5.53)$$

$$\Rightarrow p \times \exp\left(-\frac{(-A + \delta)^2}{2}\right) \quad (5.54)$$

$$- (1 - p) \exp\left(-\frac{(A + \delta)^2}{2}\right) = 0 \quad (5.55)$$

$$\Rightarrow \delta = \frac{\log\left(\frac{1}{p} - 1\right)}{2A} \quad (5.56)$$

5.10 Repeat the above exercise using the MAP criterion.

**Solution:** From bayes theorem we can write,

$$\Pr(X = 1|Y = y) \quad (5.57)$$

$$= \frac{\Pr(Y = y|X = 1) \Pr(X = 1)}{\Pr(Y = y)} \quad (5.58)$$

$$= \frac{\Pr(N = y - A) p}{\Pr(N = y - A) p + \Pr(N = y + A) (1 - p)} \quad (5.59)$$

$$= \frac{p_N(y - A) p}{p_N(y - A) p + p_N(y + A) (1 - p)} \quad (5.60)$$

$$= \frac{p}{p + (1 - p) \exp(-2Ay)} \quad (5.61)$$

$$\Pr(X = -1|Y = 0) \quad (5.62)$$

$$= 1 - \Pr(X = 1|Y = 0) \quad (5.63)$$

$$= 1 - \frac{p}{p + (1 - p) \exp(-2Ay)} \quad (5.64)$$

$$= \frac{(1 - p) \exp(-2Ay)}{p + (1 - p) \exp(-2Ay)} \quad (5.65)$$

$$= \frac{1 - p}{p \exp(2Ay) + 1 - p} \quad (5.66)$$

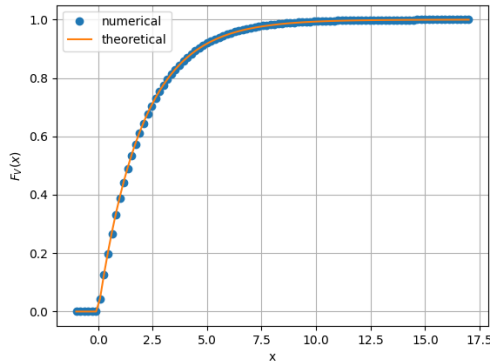
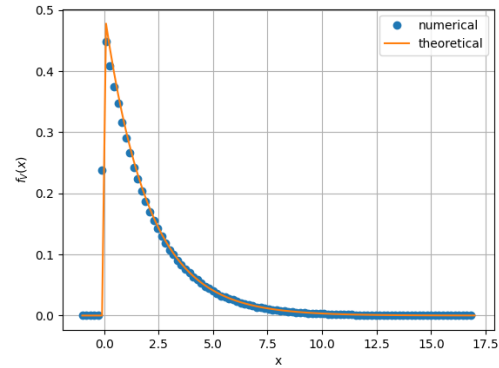
## 6 GAUSSIAN TO OTHER

6.1 Let  $X_1 \sim \mathcal{N}(0, 1)$  and  $X_2 \sim \mathcal{N}(0, 1)$ . Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

**Solution:** Download the following files and run C and python program. CDF and PDF of V is plotted in fig. 0 and 0 respectively using below python code.

```
wget https://github.com/himanshukumargupta
```

Fig. 0: The CDF of  $V$ Fig. 0: The PDF of  $V$ 

```
11012/Random-Numbers/
blob/master/coeffs.h
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_6/6.1.c
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
_6/6.1.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find  $\alpha$ .**Solution:** Let

$$X_1 = R \cos \theta, X_2 = R \sin \theta$$

So, PDF of  $R$  and  $\theta$  is given by

$$p_{R,\theta}(r, \phi) = p_{X_1, X_2}(x_1, x_2) |J| \quad (6.3)$$

where  $J$  is jacobian matrix transform-  
ing  $R, \theta$  to  $X_1, X_2$ On solving we get jacobian,  $J=R$ . So,

$$p_{R,\theta}(r, \phi) = p_{X_1, X_2}(x_1, x_2) r \quad (6.4)$$

since  $X_1$  and  $X_2$  are i.i.d. and gaussian  
random variable,  $\mathcal{N}(0, 1)$ . So,

$$p_{R,\theta}(r, \phi) = p_{X_1}(x_1) p_{X_2}(x_2) r \quad (6.5)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_2^2}{2}\right) r \quad (6.6)$$

$$= \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right) r \quad (6.7)$$

Now PDF of  $R$ ,

$$p_R(r) = \int_0^{2\pi} p_{R,\theta}(r, \phi) d\phi \quad (6.8)$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right) r d\phi \quad (6.9)$$

$$= \exp\left(-\frac{r^2}{2}\right) r \quad (6.10)$$

and

$$V = X_1^2 + X_2^2 = R^2 \quad (6.11)$$

Since  $V = R^2 > 0$

So, CDF and PDF would be zero for  $v < 0$

Then CDF of  $V$  for  $v \geq 0$ ,

$$F_V(v) = \Pr(V \leq v) \quad (6.12)$$

$$= \Pr(R^2 \leq v) \quad (6.13)$$

$$= \Pr(R \leq \sqrt{v}) \quad (6.14)$$

$$= F_R(\sqrt{v}) \quad (6.15)$$

$$= \int_{-\infty}^{\sqrt{v}} p_R(r) dr \quad (6.16)$$

$$= 0 + \int_0^{\sqrt{v}} \exp\left(-\frac{r^2}{2}\right) r dr \quad (6.17)$$

$$= 1 - \exp\left(-\frac{v}{2}\right) \quad (6.18)$$

Comparing the 2 equations we get

$$\alpha = \frac{1}{2} \quad (6.19)$$

and PDF is given by

$$p_V(v) = \frac{d}{dx} F_V(v) \quad (6.20)$$

$$\therefore p_V(v) = \begin{cases} \frac{\exp(-\frac{v}{2})}{2} & v \geq 0 \\ 0 & v < 0 \end{cases} \quad (6.21)$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.22)$$

**Solution:** CDF and PDF of  $A$  is plotted in fig. ?? and ?? respectively using below python code.

```
wget https://github.com/
himanshukumargupta
11012/Random-Numbers/
blob/master/ques
```

\_6/6.3.py

We know that

$$V = R^2 \quad (6.23)$$

$$\therefore A = \sqrt{V} = R \quad (6.24)$$

## 7 CONDITIONAL PROBABILITY

7.1 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

where  $A$  is Raleigh with  $E[A^2] = \gamma$ ,  $N \sim \mathcal{N}(0, 1)$ ,  $X \in (-1, 1)$  for  $0 \leq \gamma \leq 10$  dB.

7.2 Assuming that  $N$  is a constant, find an expression for  $P_e$ . Call this  $P_e(N)$

7.3 For a function  $g$ ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx \quad (7.3)$$

Find  $P_e = E[P_e(N)]$ .

7.4 Plot  $P_e$  in problems 7.1 and 7.3 on the same graph w.r.t  $\gamma$ . Comment.

## 8 TWO DIMENSIONS

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$\mathbf{x} \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols  $\mathbf{s}_0$  and  $\mathbf{s}_1$ .

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.5)$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for  $P_e$ . Verify this by comparing the theory and simulation plots on the same graph.