1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _1/1.1.c
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques_1/
    coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The CDF of U is plotted in Fig. 0 using the below python code

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _1/1.2.py
```

1.3 Find a theoretical expression for $F_U(x)$.

Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) \tag{1.2}$$
$$= \int_{-\infty}^{x} f_U(x)dx \tag{1.3}$$
$$\tag{1.4}$$

Now, for $x < 0$

$$f_U(x) = 0 \tag{1.5}$$
$$\therefore F_U(x) = 0 \tag{1.6}$$

,for $0 \leq x \leq 1$

$$f_U(x) = 1 \tag{1.7}$$
$$\therefore F_U(x) = \int_{-\infty}^{x} dx \tag{1.8}$$
$$= 0 + \int_{0}^{x} dx \tag{1.9}$$
$$= x \tag{1.10}$$

and for $x > 1$

$$f_U(x) = 0 \tag{1.11}$$
$$\therefore F_U(x) = \int_{-\infty}^{x} dx \tag{1.12}$$
$$= 0 + \int_{0}^{1} dx + 0 \tag{1.13}$$
$$= 1 \tag{1.14}$$

$$So, F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

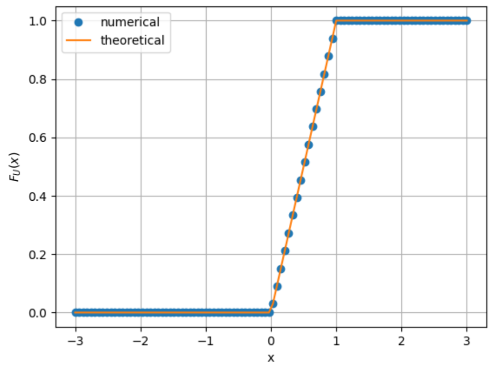

Fig. 0: The CDF of U

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.15}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.16}$$

Write a C program to find the mean and variance of U.
Solution: Download and run the following C program.

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _1/1.4.c
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.17}$$

Solution: The mean of U is given by

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.18}$$

$$= 0 + \int_0^1 x dx + 0 \tag{1.19}$$

$$= \left[\frac{x^2}{2}\right]_0^1 \tag{1.20}$$

$$= 1/2 \tag{1.21}$$

and variance is given by,

$$E[U^2] - E[U]^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \left(\frac{1}{2}\right)^2 \tag{1.22}$$

$$= 0 + \int_0^1 x^2 dx + 0 - \left(\frac{1}{2}\right)^2 \tag{1.23}$$

$$= \left[\frac{x^2}{3}\right]_0^1 - \left(\frac{1}{2}\right)^2 \tag{1.24}$$

$$= 1/12 \tag{1.25}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and run the C program.

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
```

```
blob/master/ques
_2/2.1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 0 using the below python code

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _2/2.2.py
```

Properties of CDF:

a) It is always continuous.
b) It is monotonic increasing.
c) It is bounded between 0 to 1.
d) At infinity, it tends to 1.
e) At negative infinity, it tends to 0.
f) Since it is bounded and have limit so it is convergent also.
g) Q function is defined as:

$$Q_X(x) = \Pr(x < X) \tag{2.2}$$
$$\therefore F_X(x) = 1 - Q_X(x) \tag{2.3}$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.4}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 0 using the below python code

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
```

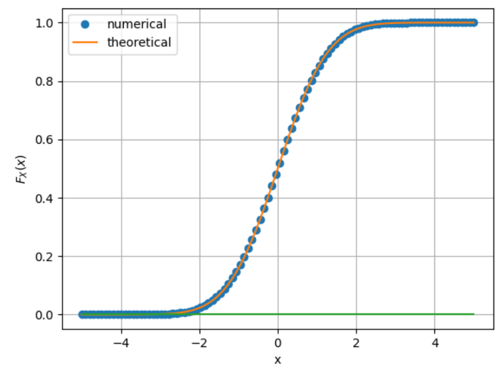

Fig. 0: The CDF of X

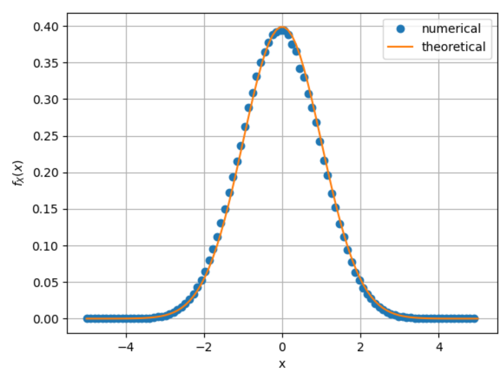

Fig. 0: The PDF of X

```
    blob/master/ques
    _2/2.2.py
```

Properties of PDF:

a) It is always non-negative
b) It is always bounded between 0 to 1
c) Always symmetric about the mean.
d) Maxima is observed at mean
e) Area under the curve of PDF is always equal to 1

2.4 Find the mean and variance of X by writing a C program.
Solution: download the following files and run the C program.

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _2/2.4.c
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/coeffs.h
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{2.6}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.7}$$

$$= \left[-\frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{+\infty} = 0 \tag{2.8}$$

Fig. 0: The CDF of V

and variance is given by

$$E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx \tag{2.9}$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.10}$$

$$= 2\int_{0}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.11}$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} t^{-1/2} \exp(-t)\, dt \tag{2.12}$$

$$= \frac{2}{\sqrt{\pi}} \Gamma(1 + 1/2) \tag{2.13}$$

$$= 1 \tag{2.14}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution: Download and run the python code where you get samples of V and fig.

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _3/3.1.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution: The CDF of V is given by

$$F_V(x) = \Pr(V \leq x) \tag{3.2}$$
$$= \Pr(-2\ln(1-U) \leq x) \tag{3.3}$$
$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$
$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

let $a = \exp(-x/2)$ .So,

$$F_V(x) = \begin{cases} 0 & 1-a < 0 \\ 1-a & 0 \leq 1-a < 1 \\ 1 & 1-a \geq 1 \end{cases}$$

on solving, we get

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following code and run the C file

```
wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/ques
    _4/4.1.c

wget https://github.com/
    himanshukumargupta
    11012/Random-Numbers/
    blob/master/coeffs.h
```

4.2 Find the CDF of T.

Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \leq t) \tag{4.2}$$

Since T is sum of 2 uniform random variable between 0 and 1 So,any T will be always less than 2 and greater than 0 So,

$$F_T(t) = 0 \qquad \text{for } X < 0 \tag{4.3}$$
$$F_T(t) = 1 \qquad \text{for } X \geq 0 \tag{4.4}$$

Now,for $0 \leq t < 2$,
let U_1 as a constant x and later we integrate it for all possible values of x So,

$$F_T(t) = \Pr(T \leq t) \tag{4.5}$$
$$= \Pr(U_1 + U_2 \leq t) \tag{4.6}$$
$$= \Pr(u_2 \leq t - x) \tag{4.7}$$
$$= \int_0^t F_{U_2}(t-x)\, p_{u_1}(x)\, dx \tag{4.8}$$

Now, we make 2 cases
case-1: $0 \leq t < 1$

$$F_T(t) = \int_0^t F_{U_2}(t-x)\, dx \tag{4.9}$$
$$= \int_0^t F_{U_2}(m)\, dm \tag{4.10}$$
$$= \int_0^t m dm \tag{4.11}$$
$$= \left[\frac{m^2}{2}\right]_0^t \tag{4.12}$$
$$= \frac{t^2}{2} \tag{4.13}$$

case-2: $1 \le t < 2$

$$F_T(t) = \int_0^1 F_{U_2}(t-x)\,dx \tag{4.14}$$

$$= \int_0^1 F_{U_2}(t-x)\,dx \tag{4.15}$$

$$= \int_{t-1}^{t} F_{U_2}(m)\,dm \tag{4.16}$$

$$= \int_{t-1}^{1} mdm + \int_{1}^{t} dm \tag{4.17}$$

$$= \left[\frac{m^2}{2}\right]_{t-1}^{1} + [1]_1^t \tag{4.18}$$

$$= \frac{-t^2}{2} + 2t - 1 \tag{4.19}$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t_2}{2} & 0 \le t < 1 \\ \frac{-t^2}{2} + 2t - 1 & 1 \le t < 2 \\ 1 & 2 \le t \end{cases}$$

4.3 Find the PDF of T.
Solution: The PDF of T is given by

$$p_T(t) = \frac{d}{dt} F_T(t) \tag{4.20}$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \\ 0 & 2 \le t \end{cases}$$

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Already done in 4.2 and 4.3.

4.5 Verify your results through a plot.
Solution: Plot of CDF and PDF is plotted in fig. 0 and 0 respectively using following python code.

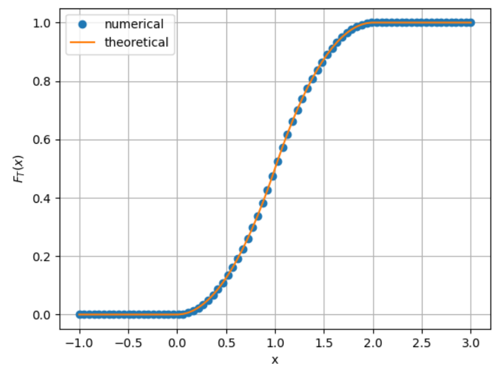

Fig. 0: The CDF of T

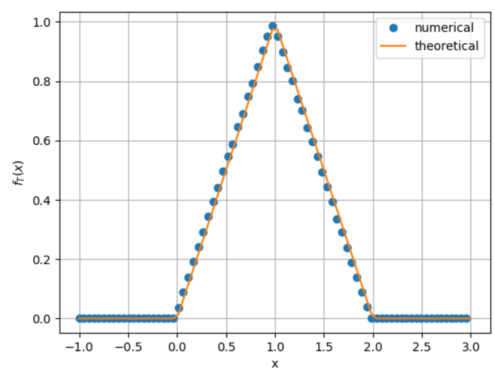

Fig. 0: The PDF of T

5 Maximul Likelihood

5.1 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, $X1\{1,-1\}$, is Bernoulli and $N \sim \mathcal{N}(0,1)$.
Solution: Download the following files and execute the C code

```
wget https://github.com/
    himanshukumargupta
```

```
    11012/Random−Numbers/
    blob/master/ques
    _5/5.1.c
wget https://github.com/
    himanshukumargupta
    11012/Random−Numbers/
    blob/master/coeffs.h
```

5.2 Plot Y.
Solution: The plot of Y is plotted in fig. using below python code

```
wget https://github.com/
    himanshukumargupta
    11012/Random−Numbers/
    blob/master/ques
    _5/5.2.py
```

5.3 Guess how to estimate X from Y.
5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.3}$$

5.5 Find P_e.
5.6 Verify by plotting the theoretical P_e.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution:
6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.
6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$

7 Conditional Probability

7.1
7.2 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.
7.3 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$
7.4 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.
7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.