Throwing π at a Wall :

from Bouncing Billiards to Quantum Search

PHY312: Numerical Methods for Physicists

Project Report

Under the supervision of

Dr. Sunil Pratap Singh

Department of Physics,
Indian Institute of Science Education and Research,
Bhopal, Madhya Pradesh - 462066

2020

Abstract

Pi (π) is considered to be a Universal Constant and one can almost encounter in almost all branches of Science and Mathematics. It can be encountered in Planetary Motion, Fluid Dynamics, Trigonometric and various other areas. It even occurs in areas where there is no sense of Circle at all, like Quantum Mechanics, Series summation etc. There are various ways of obtaining the value of Pi like the Oldest Archimedes Polygon Method, Monte Carlo Method, Pin Paper Method etc.

In this project, we demonstrate a unique way to obtain the value of Pi from a purely physical perspective. We have modelled a Two-Body Collision problem bounded one side by wall and the other at infinity. The model explains how by taking different ratios of masses and with different initial conditions, we can obtain any number of digits of Pi. First we have built the model by considering tools of physics and then numerical simulation has been done for different cases. It was found that our result matches with the theoretical result with negligible error.

When we compared a ray optics setup with the dynamics of our model, we were surprised to know that some of the fascinating concepts like reflection between the two inclined plane mirrors can also be explained through this model. It was found that the number of reflections occuring between the mirrors turn out to be successive digits of Pi provided the condition mentioned above is followed.

This model developed, finds a link with the Grover's Algorithm which is one of the two most sophisticated Quantum Information search algorithms used so far. We have shown how such a complex algorithm can be seen by merely viewing it as a Two-Body problem .

Academic Integrity and Declaration

We hereby declare that the entire work embodied in the project report entitled **"Throwing π at a Wall : From Bouncing Billiards to Quantum Search"** has been carried out by Group The Jones Brothers under the supervision of **Dr Sunil Pratap Singh** Department Of Physics, Indian Institute of Science Education and Research, Bhopal.We certify that all copyrighted material incorporated into this project is in compliance with the Indian Copyright (Amendment) Act (2012). The matter presented in this report incorporates the result of investigations and research carried out by the group.

Name	Roll No.
Aditya Sharma	17013
Himanshu Pandey	17105
Priyam Srivastava	17194
Shashikesh Kumar Thakur	17256
Yash Vikas Mandlecha	17327

June 2020,

Department of Physics,

Indian Institute of Science Education and Research,

(Bhopal)

Acknowledgements

We are grateful to Dr. Sunil Pratap Singh for giving us this opportunity to collectively work on this project and being a continuous guidance throughout the course. It was an enriching learning experience.

The Jones Brothers
Department of Physics,
IISER Bhopal

Contents

List of Figures

Chapter 1

Introduction

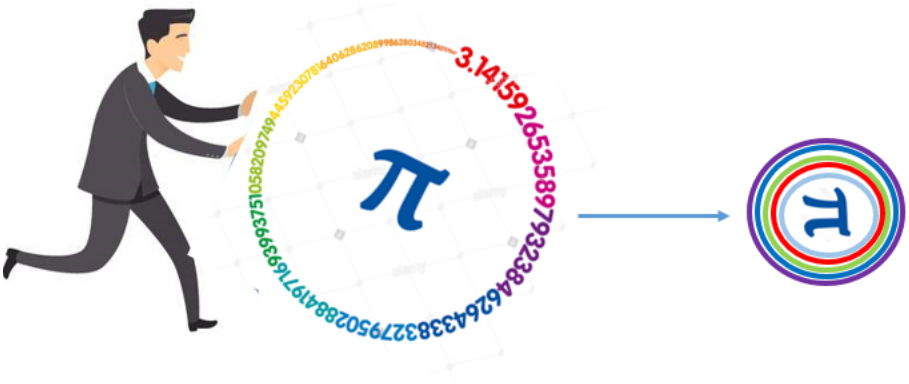
Consider a circle. If we take the ratio of its circumference to its diameter, it always turns out to be a constant. This constant has been known and is being used by humanity since ancient times. There are many ways to obtain the value of π. The oldest known algorithm for calculating π is attributed placing upper and lower bounds on its value by inscribing and circumscribing n-sided regular polygons on a **circle**. Some of the recent approaches include the Monte Carlo Simulation which is probabilistic in nature and requires modern electronic devices.
By keeping in mind the previous and modern ways of obtaining π one thing is common, that is *whenever π appears there is some link with the circle.* This make sense because π is defined with respect to circle. But the problem starts when it started showing in the area of physics and mathematics where there is no sense of circle at all. For example how can you think it's appearance in Quantum Mechanics (involvement in Schrodinger Equation), the Riemann-zeta function ($\zeta(s = 2)$ results in $\pi^2/6$. *There are various field where its presence is just unexplained and researcher have started thinking it as just a universal constant and it's presence in circle is also special case.*

Our group after going through various research papers and article come to conclusion that presence of π in any problem is not random but it always signals that there is some cyclic nature associated in that problem. We have developed a model in which a completely physical problem of Billiard balls turns out to act like π gen-

erator. This model will also give new and **Deterministic** nature of obtaining π instead of probabilistic nature as in most of the previous methods. Using the same approach of *"π relates to a circle"* we have also tried to explain and link some of the area where it is found mainly the Grover's Algorithm for Quantum Search and Planetary Motion.

1.1 Basis of the Model: Two body elastic collision

Consider an experiment where you launch a dynamical system consisting of just two arbitrary balls towards and an obstacle(a wall). If we assume that the collisions are purely elastic. Then just by counting the number of collision within the balls and wall, the integer will turn out to be 3141592653.....!



1.2 Procedure

Consider elastic collisions between two blocks of mass M and m that slides on a friction less surface, as illustrated in Figure 1. When mass m hits the wall at the right it rebounds elastically.

All collisions are elastic and there is no friction, so the energy is conserved at all times. When the two blocks collide with each other, momentum is conserved during the collision whereas there is no conservation of momentum during the block-wall collision(the velocity of the smaller block is simply reversed).

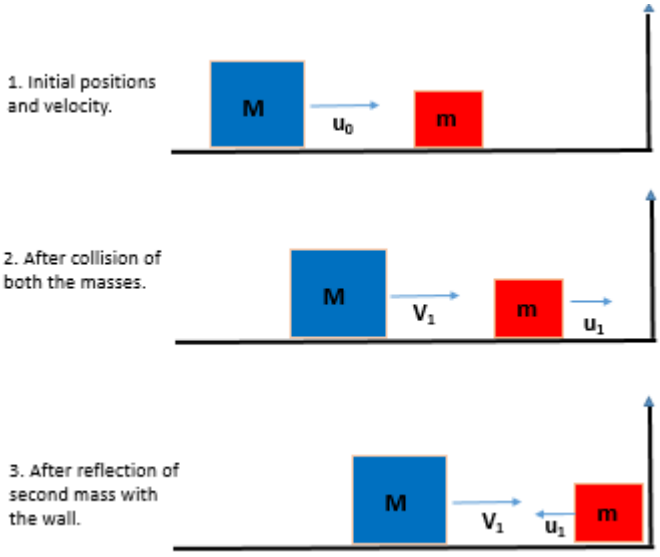


Figure 1.1: Elastic Collision

We will assume that the mass m ,placed at a small distance from the wall, is initially stationary (i.e, v = 0), and mass M is moving towards it from the left with unit velocity (V = 1). If the two masses are equal then when they collide, mass M will stop and transfer its momentum completely to the smaller mass m, which will move to the right with unit velocity. Mass m will now collide with the wall and return with the same speed (but the velocity will have opposite sign) to collide again with mass M. Now mass m will stop and mass M will move off to negative infinity.So,there will be three collisions in total.

Suppose we repeat the previous scenario with mass M now 100 times that of m. In this case m will again move off towards the wall, but M not transfer its momentum completely to the smaller block,rather a part of its momentum will be transferred and both the blocks would continue moving even after the collision . After this,mass m bounces off the fixed wall and collide with M again, slowing it a little more, but still not stopping it, while itself shooting off towards the wall again. In fact m will bounce between the wall and M many times before the bigger mass changes its direction and moves to negative infinity with the velocity greater than that of the smaller mass m so that there would be no more collisions between them.In this scenario,mass m will undergo a total of 31 collisions .

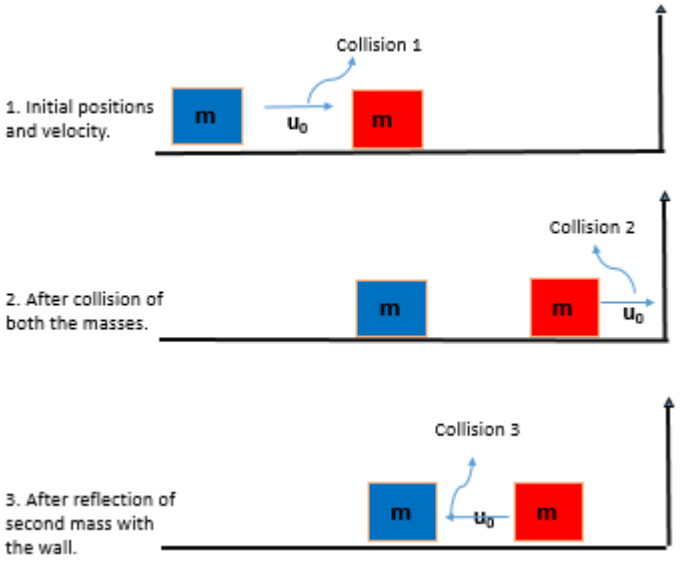


Figure 1.2: Same Mass Collision

Again we repeat the process, but with M now 10000 (100^2) times the mass of m. Again there will be multiple collisions before M moves away to negative infinity faster than m. This time m will undergo 314 collisions in total. Repeat with M 100^3 times the mass of m and m will undergo 3141 collisions. If M is 100^4 times the mass of m then m will experience 31415 collisions. Note the pattern in the number of collisions: 3, 31, 314, 3141, 31415. These are the digits of π, and the pattern continues! For M = 100^Nm, then the number of collisions will be given by the first (N + 1) digits of π !

1.3 Mathematical formulation

We can calculate the velocities of the masses after each collision by making use of conservation of energy, conservation of momentum for the two mass collisions and velocity reversal at the wall, as follows:
Conservation of energy:

$$\frac{1}{2}MV^2 + \frac{1}{2}mv^2 = E. \tag{1.3.1}$$

Conservation of momentum:

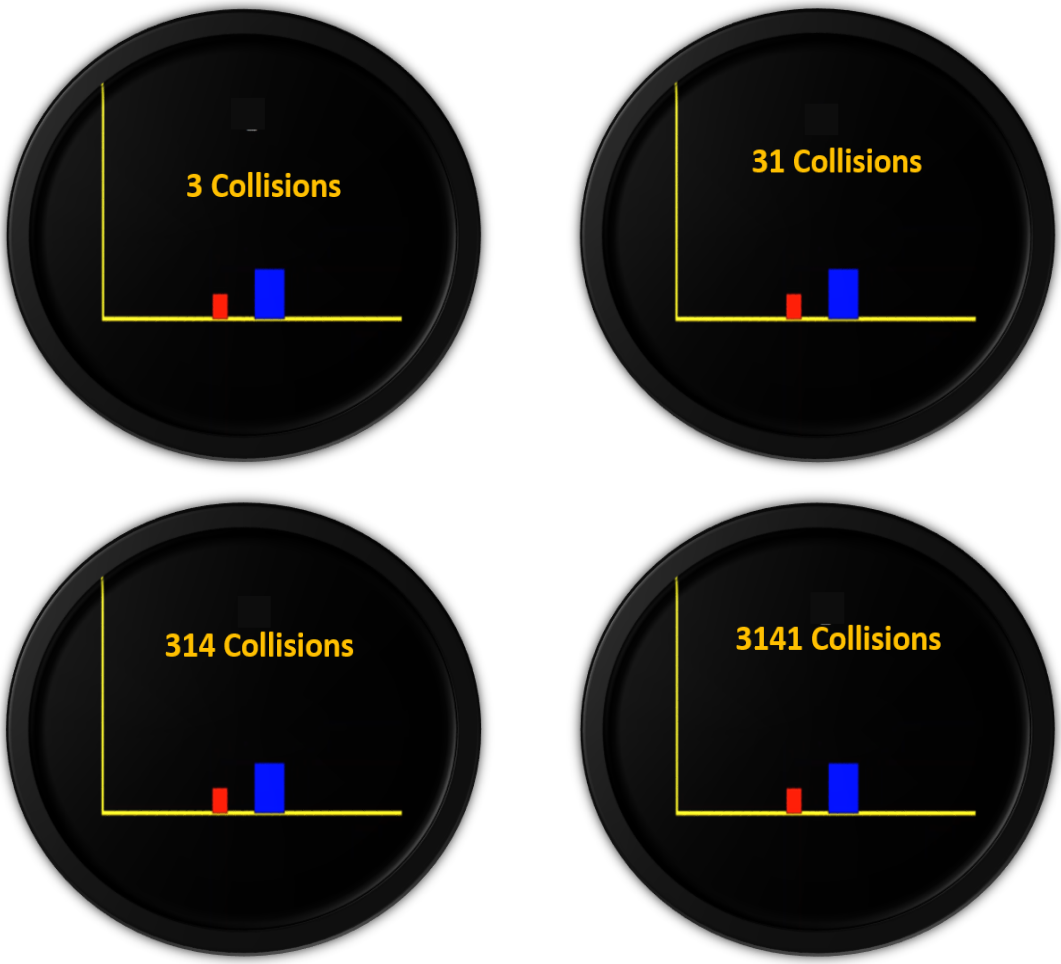


Figure 1.3: Click on the respective figure to see the collisions and click on **Simulation Code** for above Code and click on **Pi Code** for code of π calculation

$$MV + mv = P. \tag{1.3.2}$$

At the wall

$$v \to -v. \tag{1.3.3}$$

E is the total (kinetic) energy of the system and P is the momentum immediately before and after the two blocks collide (P is different for each such collision). It is convenient to multiply equation (1.3.1) by 2/M and divide equation (1.3.2) by M to get:

Conservation of energy:

$$V^2 + rv^2 = \frac{2E}{M}. \tag{1.3.4}$$

Conservation of momentum:

$$V + rv = \frac{P}{M}, \tag{1.3.5}$$

where

$$r = \frac{m}{M}. \tag{1.3.6}$$

Consider the kth collision between the two masses. From equations (4) and (5) we can see that the pre- and post-collision velocities are related by:

$$V_k^2 + rv_k^2 = V_{k-1}^2 + rv_{k-1}^2$$
$$V_k + rv_k = V_{k-1} + rv_{k-1}.$$

From these we obtain:

$$\boxed{v_k = \frac{(r-1)v_{k-1} + 2V_{k-1}}{1+r}} \tag{1.3.7a}$$
$$\boxed{V_k = \frac{2rv_{k-1} + (1-r)V_{k-1}}{1+r}} \tag{1.3.7b}$$

Chapter 2

Numerical Techniques and Simulation

Using the boxed equation from previous chapter we build a code to simulate the collision. The equations are:

$$\boxed{\begin{aligned} v_k &= \frac{(r-1)v_{k-1} + 2V_{k-1}}{1+r} \\ V_k &= \frac{2rv_{k-1} + (1-r)V_{k-1}}{1+r}, \qquad r = m/M \end{aligned}}$$

The algorithm that calculates the number of collisions, k is shown below. Observe that during collision with the wall we just change the sign of velocity (using equation 1.3.3).

2.1 Algorithm

1. Set a value for r (say, r = 100^{-1})
2. initialise velocities: v = 0, V = 1
3. initialise collision counter: k = 0
4. **while** $V > v$ **do**
5. increment counter k = k + 1

6. store pre-collision velocities

   $v_{\text{old}} = v, V_{\text{old}} = \text{V}$

7. update velocities when M and m collide:

$$\boxed{\begin{aligned} v_k &= \frac{(r-1)v_{k-1} + 2V_{k-1}}{1+r} \\ V_k &= \frac{2rv_{k-1} + (1-r)V_{k-1}}{1+r} \end{aligned}}$$

8. if m collides with wall (i.e. if $v > 0$) then

9. increment counter k = k + 1

10. update velocity v = -v

11. **end if**

12. **end while**

13. number of collisions = value of k

2.2 Results

After implementing the code, we have obtained the *phase plot of collision where X-axis represents velocity of ball with moving mass and Y-axis represents velocity of stationary mass at different stages of collision.* Phase plots for some of the cases is presented here and for some video has been attached.

2.2.1 Case 1: $M = m$

The phase plot for $r = 1$ shows there are 3 collisions in total

Observe from figure 2.2 - 2.5 that the number of collision obtained for different cases as shown in the figure is actually π. Moreover as we increase the factor of masses we see that collision traces the circular path.

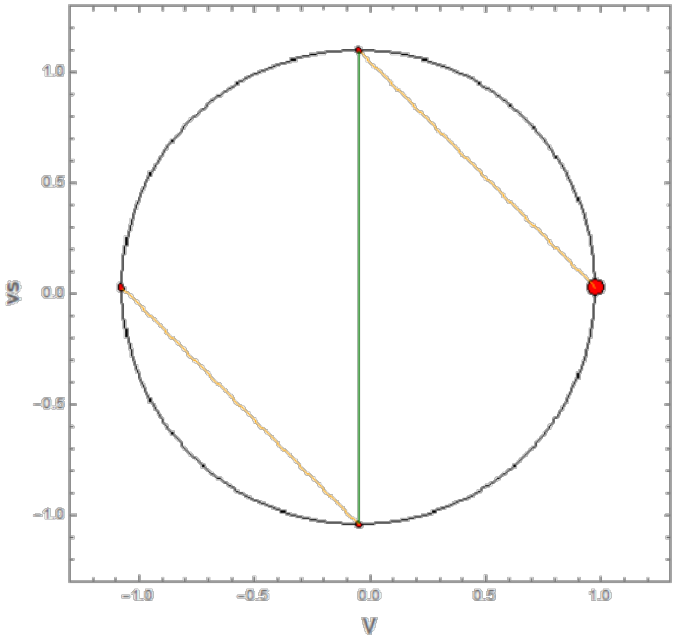


Figure 2.1: Click on the image to see the 3 collisions

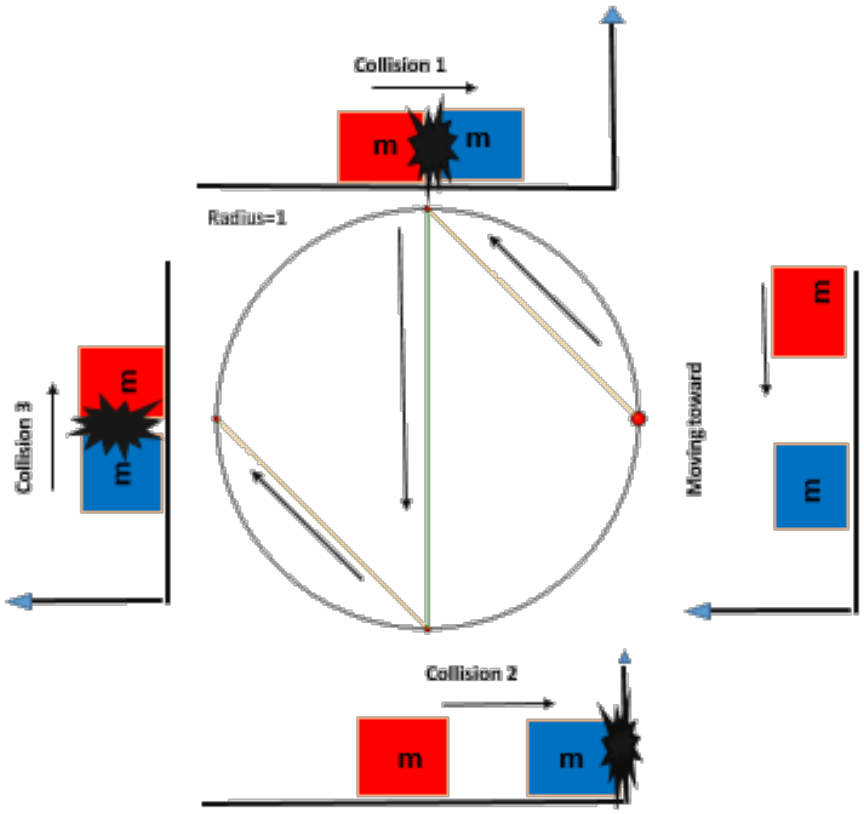


Figure 2.2: Tracking Dynamics of block collision with Phase Plot, click **here** for the simulation of 31 collisions and click **here** for code

2.2.2 Case 2: $M = 10^2 m$

The output of the code that is no. of collision k = 31.

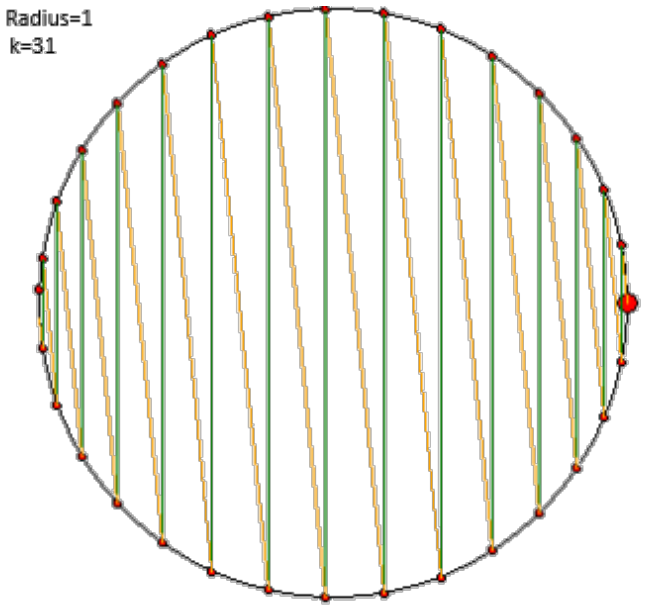


Figure 2.3: Click on the image to see the 31 collisions

2.2.3 $M = 10^4 m$, $M = 10^6 m$

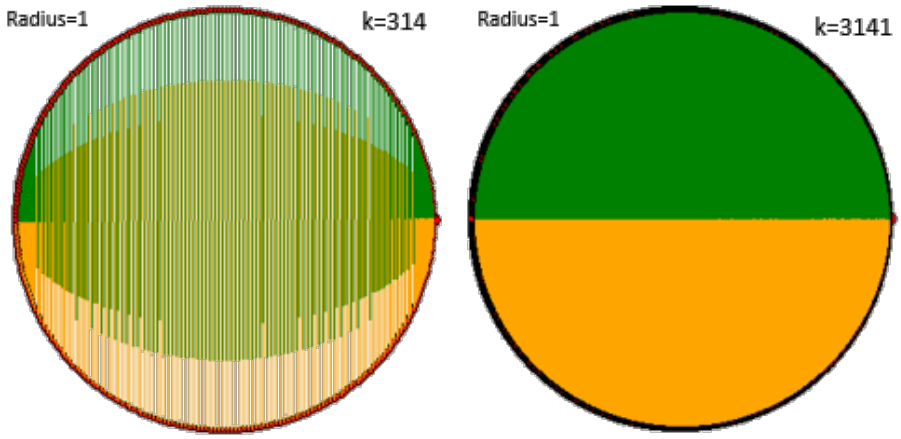


Figure 2.4: Click on the image for 314 collisions and click **here** for 3141 collisions. For the python code, click **here**

2.3 Discussion

After going through the numerical results and searching for possible explanation we found if the ratio of masses, $r = \frac{m}{M}$**, is of the form** $10^{2(1-d)}$ **where d is a positive integer, then the number of collisions is an integer consisting of the first d digits of** π. We also realised that as the we approach for larger digits of π the phase plot traces the circular nature.

2.3.1 π : It's irrational but well-rounded

We wonder why does a Circle appear in this case. Although, our understanding that "π relates to a circle" is demonstrated numerically above. Now we shall show why this billiard collision results in a circle which thereby reproduces the digits of π to a finite order. **For most of the explanation we shall be referring to 31 collision case phase plot.(Figure 2.3)*

Since our phase plot approaches to circle when we seek larger digits of π, there has to be some equation of circle hidden somewhere in our algorithm. It turns out, that there is. If we look more closely how we got our iterative equation from the conservation of energy, equation (1.3.1) and We can write this as $V^2 + (\sqrt{r}v)^2 = 2E/M$. Noting that initially $V = 1$ and $v = 0$ we have that $2E/M = 1$, and if we define a scaled velocity:

$$v_s = \sqrt{r}v, \tag{2.3.3}$$

we can write the conservation of energy as:

$$V^2 + v_s^2 = 1. \tag{2.3.4}$$

As a result of this equation we are getting circular plot in phase plot shown above. Equation (2.3.4) is that of a unit circle. By suitable modification of the list in the algorithm we obtain the velocities, v and V, at every collision as well as the number of collisions. Calculating the corresponding values of v_s from equation (2.3.3), and then plotting v_s against V, together with a unit circle, we see the collisions in configuration space or phase space, as illustrated in Figure 2.3 (where $r = 100^{-1}$, or

$M = 100m$). The collisions are marked with the red dots, which we can see all lie on the black unit circle. *The collisions between the two masses are represented by the dots on or below the horizontal axis, those between m and the wall by the dots above the axis. The solid red lines not only connect the collisions in sequence, starting with the rightmost collision at (1, 0), but the slanted lines code the conservation of momentum, and the vertical lines code changes in momentum.*

Now to see how the slanted and vertical lines joins with the dot on the unit circle we use the momentum equation (1.3.5). We rewrite momentum equation, by making use of equation (2.3.3), and rearrange to get:

$$v_s = \frac{P}{\sqrt{r}M} - \frac{1}{\sqrt{r}}V. \tag{2.3.5}$$

This represents the slanted lines, all of which have slope $-1/\sqrt{r}$. These intersect the unit circle at the collision points. The momentum is different for each slanted line of course. It decreases by the equivalent of the current value of $2v_s$ (its coded value in Figure 2.3) at each wall collision.

2.3.2 Why do the digits of π appear?

To see this we shall plot the 31 collision case (Figure 2.3) again without the momentum lines that is not slant and vertical line. Instead we will join radially outward line from the centre to each of the collision dots as shown in the figure.

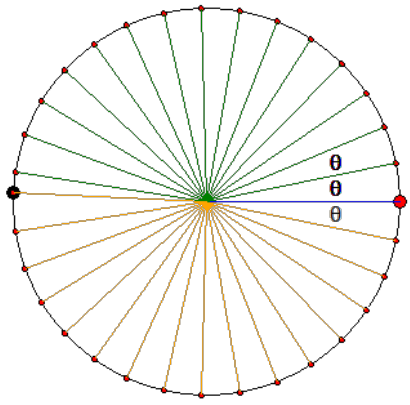


Figure 2.5: Click on image for simulation and **here** for Code

As in Figure 2.5 (the black dot represents a point where the final momentum line intersects the conservation of energy circle, but it does not represent a collision

because at that point, both masses have negative velocities, with the large mass having the more negative velocity, thereby drawing away from the small mass).

All the angles between consecutive radial lines, apart from the angle of the sector immediately below the black dot, are the same.

How are the angles equal?

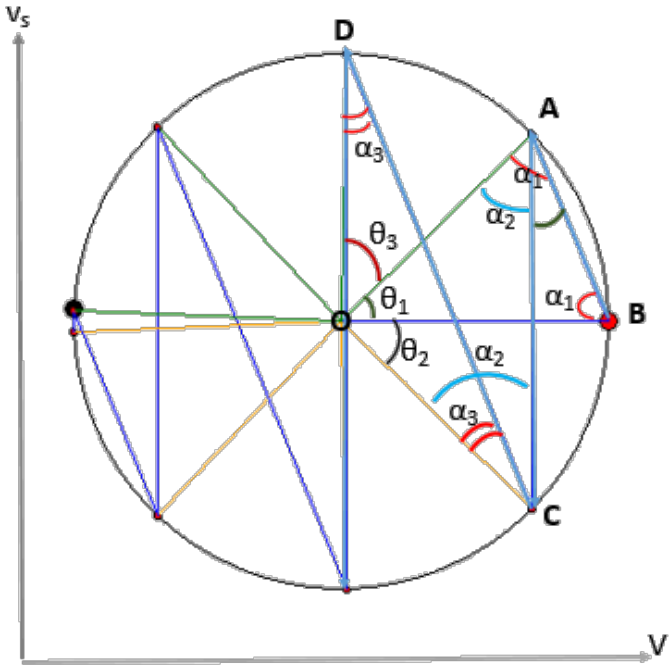


Figure 2.6: Sector with Equal angle

We use plain geometry of the phase plot to show that $\theta_1 = \theta_2 = \theta_3$. So, In the $\triangle$OAB, using the property of bilateral triangle,

$$2\alpha_1 + \theta_1 = \pi \rightarrow \alpha_1 = \frac{\pi - \theta_1}{2} \tag{2.3.6}$$

From Bilateral $\triangle$OAC we have,

$$2\alpha_2 + \theta_1 + \theta_2 = \pi \rightarrow \alpha_2 = \frac{\pi - \theta_1 - \theta_2}{2} \tag{2.3.7}$$

Similary, from Bilateral $\triangle$ODC we have,

$$2\alpha_3 + \theta_1 + \theta_2 + \theta_3 = \pi \rightarrow \alpha_3 = \frac{\pi - \theta_1 - \theta_2 - \theta_3}{2}$$

Now let $\beta = \angle$CAB which implies $\beta = \angle$DCA, Using the 3 equation derived above we have

$$\beta = \alpha_2 - \alpha_3 = \alpha_1 - \alpha_2$$

then it implies

$$\theta_1 = \theta_3 = \theta_2 = \theta$$

Calculation of the angle θ

To calculate the angle θ, let's home in on the first sector (that is involving the first two collisions) as in Figure 2.7

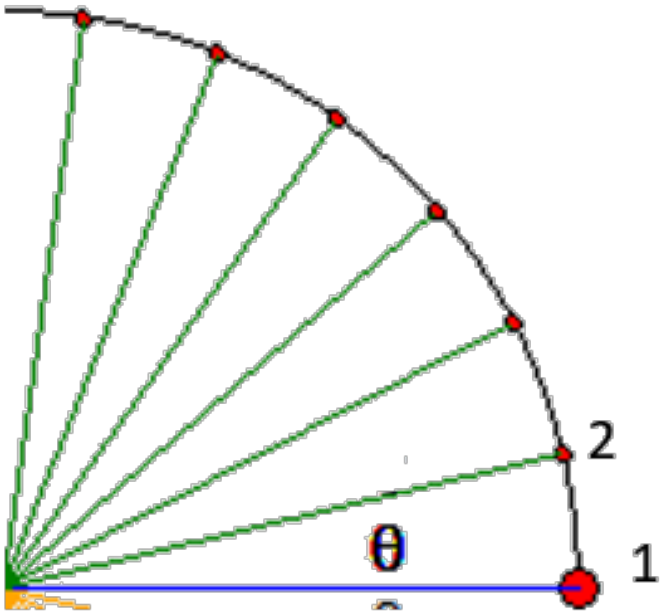


Figure 2.7: Sector containing first few collisions

Prior to the first we have $V = 1$ and $v = 0$ so from equations (1.3.7a) and (1.3.8a) we find the velocities after the first collision to be

Why not choose 99, or 2, or any other number? We can see why by looking carefully at equation (2.3.10). Generally, with r = 100^{-N} and $N > 0$, we have

$$\sqrt{\frac{r}{(1+r)}} \approx \sqrt{r} = 10^{-N}. \tag{2.3.8}$$

With small values for its argument

$$\sin^{-1}\alpha \approx \alpha, \tag{2.3.9}$$

so equation (2.3.10) becomes

$$s \approx \pi/10^{-N} or s \approx 10^{N}\pi. \tag{2.3.10}$$

In other words, in our normal base ten notation, s is approximately π with the decimal point shifted to the right by N places

```
3
31
314
3141
31415
314159
3141592
31415926
314159265
3141592653
```

Figure 2.8: Generating digits of π using above $\arcsin \alpha$. **Click here for Code**

Chapter 3

Colliding Balls, Beam of Light and π

In this chapter we introduce one more perspective of what is going on in the collision problem. It will have more connection to optics than as usual classical collision problem. What we are doing is seeking an alternative solution to the same problem. In the previous problem we had a coordinate plane where a point is represented by a pair of velocities. Now we will be dealing the same problem with a pair of positions. That is each point on the coordinate plane represents a position of the two balls from the wall. Such coordinate plane are usually known as **Configuration Space**. On doing so problems in **Dynamics** $\rightarrow$ **Geometry** which hopefully is more solvable.

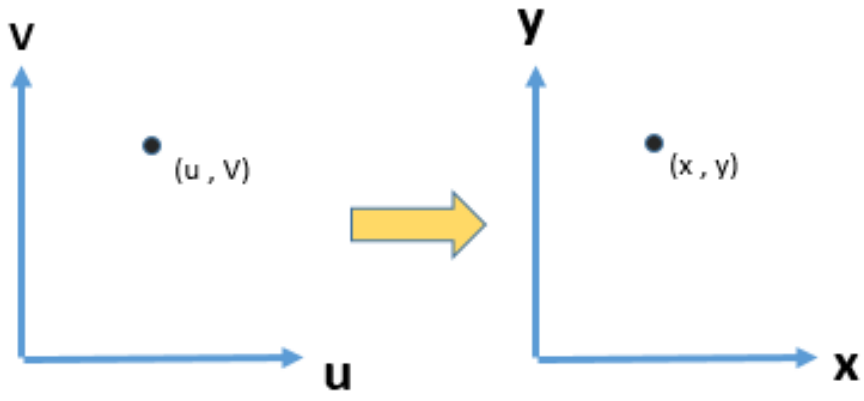


Figure 3.1: From Phase space to Configuration Space

3.1 Configuration Space of the system

Let the distance from the wall to the left edge of the fist ball(mass m) be "$x = d1$" and "$y = d2$" for left edge of second ball (mass M) as shown in the figure.

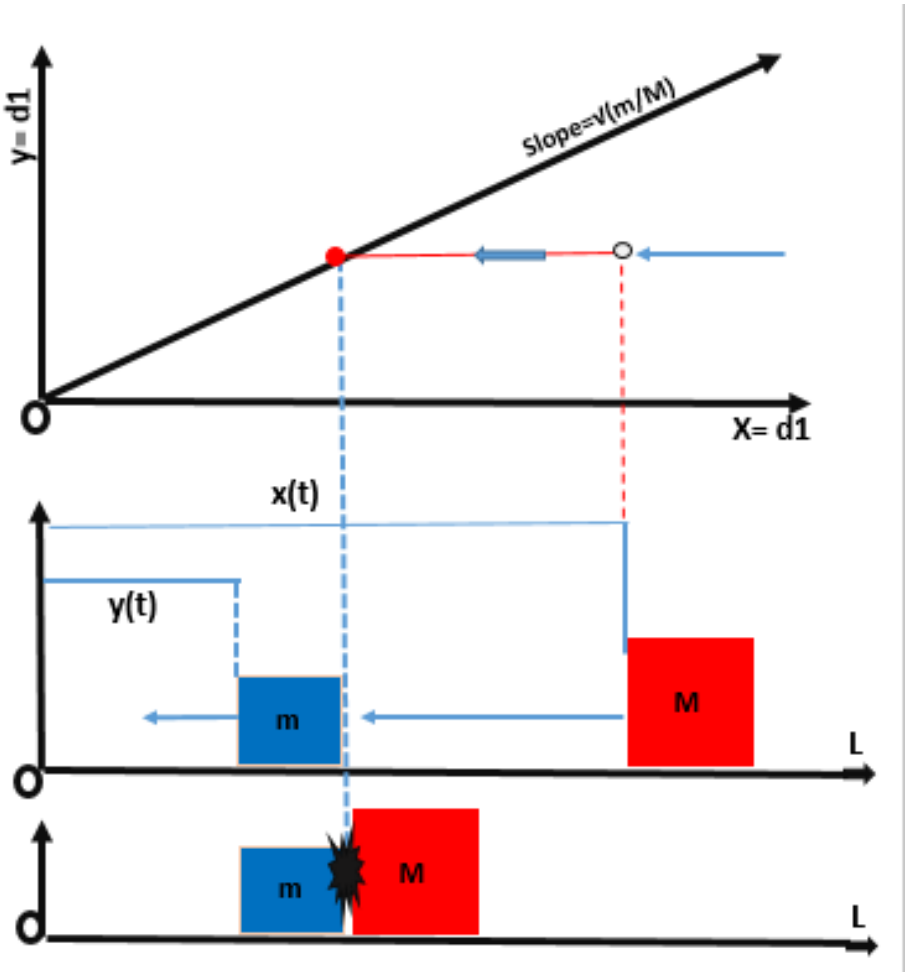


Figure 3.2: Coordinates of the Balls and Position in configuration space

3.1.1 How does it work? How it is connected to optics?

So as from the previous chapter let the balls M and m be located at points x_0 and y_0 on the horizontal line L at the initial moment of time t = 0. While moving, the ball's coordinates x and y changes in time t, so $x = x(t)$ and $y = y(t)$, t $\geq$ 0.
In particular, $x(0) = x_0$ and $y(0) = y_0$. Note that at each moment t, the small ball is situated between the wall and the big ball M. Therefore 0 $\leq$ y(t)$\leq$ x(t), at all time t. Now let us denote y(t_{ref})=0, as the position of small ball when it hit

the wall and let P(x,y) be the coordinate in configuration space x-y. Now we will consider a change of axis using the following transformation (the reason for which will be made clear in coming section):

$$\begin{aligned} x-axis : x = d1 \rightarrow x = \sqrt{M}d1 \\ y-axis : y = d2 \rightarrow y = \sqrt{m}d2 \\ \theta = \arctan(r) \rightarrow \theta = \arctan\left(\sqrt{\frac{m}{M}}\right) \end{aligned}$$

So the configuration point P(x,y) becomes P($\sqrt{M}x$,$\sqrt{m}y$) which for simplicity can still be denoted by P(x,y).

Behaviour of Coordinate P(x,y)

1. **Step 0:(before the first collision)**. At the initial moment $t = 0$, the configuration point P is located on the plane at the geometric point $P_0 = (x_0, y_0)$. The time changes and point P = P(t), starts moving. The small ball is fixed before the first collision, therefore the y-coordinate of the moving point P does not change. On the other hand, the big ball M moves towards the small ball m, and hence its coordinate x decreases, remaining however, bigger than y_0 during the entire time period before the first collision with m. Consequently, the coordinate point P moves directly left toward the y-axis (parallel to the x-axis) until the first collision as shown in the figure 3.2.

2. **Step 1:(first collision)** Thereupon the big ball collides with the small one; at this moment, t_1, $x(t1) = y_0$. Then the balls bounce off each other instantaneously, and the next step begins as shown in figure 3.2.

3. **Step 2:(between the first collision and the first reflection)** At the moment t_1, both balls begin to move along the vertical line L. The small ball moves with some velocity u and the big ball with velocity v, so that the laws of conservation of momentum and energy hold:

$$mu + Mv = MV;$$

$$\frac{1}{2}mu^2 + \frac{1}{2}Mv^2 = \frac{1}{2}MV^2$$

where V is the initial velocity of the ball M. Using the system of equations above, we can conclude that, after the first collision, the ball m will move very fast towards the wall (since the big ball gives it a big momentum) and the ball M also continues to move, a little bit slower than before, towards the wall. Both coordinates x(t) and y(t) are decreasing on the time interval after the first collision but before the reflection of the ball m from the wall. Therefore the configuration point P(x,y) moves along a straight line segment inside the $\angle$AOB, where O is the origin, OA is the positive x-axis, and OB is the ray $y = rx$ outgoing from the origin in the first quadrant. Point P(x,y) travels from the side OB to the side OA, approaching the origin O.

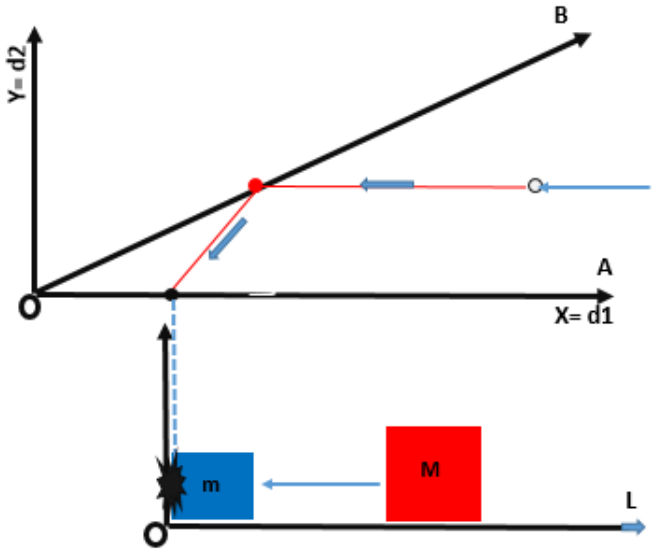


Figure 3.3: Step 2: Collision with wall

4. **Step 3:.(reflection from the wall)**. It is not hard to check that the small ball m moves faster than the big ball after the first collision, that is $u \to v$. Therefore, the ball m reaches the wall at some moment t_2 at which the ball M is still moving toward the wall. At t_2, the ball reflects off the wall, its velocity instantaneously jumps from u to $-$u, the momentum of the small ball becomes $-$mu whereas its energy remains the same: $\frac{1}{2}$m(u)^2 = $\frac{1}{2}mu^2$. The angle of incidence and the angle of reflection formed by the segments of the trajectory with the y-axis is equal.

5. **Step 4: (after the first reflection off the wall).** Point P(x,y) moves after the first reflection from the x-axis along a straight line segment towards the line $y = rx$ approaching the origin O: its y-coordinate increases and the x-coordinate decreases. This corresponds to the balls approaching each other when the ball m bounces off the wall.

6. **Step 5 (the second collision of the balls).** At the moment when P reaches the side $y = rx$ of the configuration angle AOB, the second collision of the balls occurs. Let it happen at point $x_1 \geq 0$ on line L. The balls change their velocities after the collision. If the new velocities are u_1 and v_1, then

$$mu_1 + Mv_1 = mu + Mv$$

;

$$\frac{1}{2}mu^2 + \frac{1}{2}Mv^2 = \frac{1}{2}MV^2$$

By an argument similar to the given above, we have $|(u_1)| \geq |(v)| \geq |(v_1)|$.

7. **Note:** It is important to note that, since the wall is not considered part of the ball system, the momentum of the system after the wall reflection has been changed from $mu + Mv = MV$ to $mu + Mv$ whereas the energy remains the same, $\frac{1}{2}MV^2$. The momentum does not change between two successive reflections off the wall.

8. **Step 6,7.** Starting from Step 5, the situation repeats but with final velocities becoming the initial velocities for the next process and so on.

Why Transformation of x and y axis?

Observe that the x-axis and the line inclined to both axis forms the boundary for the colliding blocks. On tracking their collisions for different cases we found that the angle of incident was not equal to the angle of reflection on the inclined axis. Thus our analogy of conservation of momentum as shown in table doesn't work. We will prove why the laws of reflection holds as we proceeds.

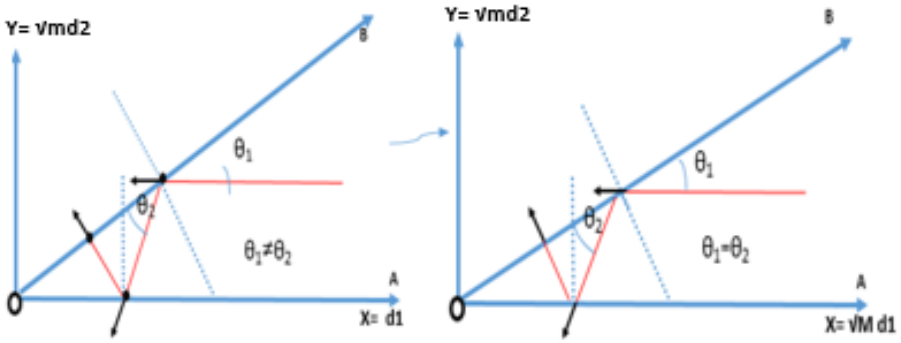


Figure 3.4: Transformation of axis and laws of reflection

3.1.2 Analogy

Kinematics	Optics
Conservation of Energy: $\frac{1}{2}(MV^2 + mv^2) = \text{E}$	Constant speed of light
Conservation of Momentum : $MV + mv = P$.	Angle of Incidence = Angle of reflection

So specifically we have translated conservation of energy into little points in space moving with constant speed which in our analogy means constant speed of light and translates conservation of momentum into law of reflection that is angle of incidence equals the angle of reflection. **Laws of Kinematics → Laws of Optics**

3.1.3 Mathematical Formulation

We reduce the problem on the motion of the configuration point P in the 45 degree angle AOB ($y = x$ line) to a billiard problem in some other angle α (line $y = rx$). *A billiard system is a dynamical system that consists of a domain (closed, like a circle or a square, or open, like an angle) and a moving point inside the domain.* The point moves along a straight line with constant speed and reflects off the domain's boundary by the billiard (= optics) law: the angle of incidence equals the angle of reflection. In other words, the point in a billiard system behaves as a ray of light in a room (domain) with mirror walls (boundary).

3.1.4 Geometrical proof of laws of reflection

Consider the following two cases: **Case 1**, in which point P(x,y) reflects from the horizontal side y = 0 (the X -axis) of the angle θ and **Case 2**, in which P(x, y) reflects off the side y =$\sqrt{\frac{m}{M}}$x. At any time the velocity vector in coordinate plane $\overline{V}$=($\sqrt{M}\dot{d}1$,$\sqrt{m}\dot{d}2$) =($\sqrt{M}V$,$\sqrt{m}u$). ($\rightarrow$ is velocity vector in figure 3.4)

Velocity Vector($\rightarrow$)

The velocity equation given above is an abstract way of representing collision balls velocity in two dimensional plane which take care of all possible velocity of the balls. In the figure given below some few steps are shown.

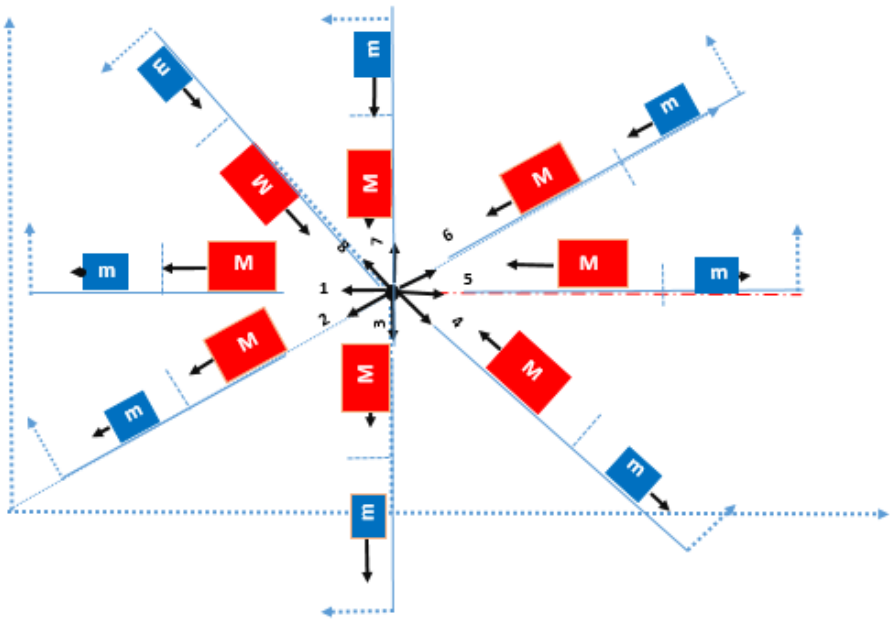


Figure 3.5: Velocity vector and direction of balls at different time

Case 1:Reflection from the X -axis

When the small ball reflects from the wall, its velocity u changes to (-u). Then velocity vector $\overline{V}$ converts into vector $\overline{V'}$=($\sqrt{M}V$,$\sqrt{m}$(-u)) , which means θ_1 =θ_2 — the billiard reflection law (Figure 3.5).

Case 2: Reflection from the side y $=\sqrt{\frac{m}{M}}$x

This reflection corresponds to the ball collision. We will consider an interval of time in which only this collision occurs (that is the interval between two successive reflections of the ball m from the wall). The system of moving balls has unchanging momentum during this interval of time, and the collision of the balls does not change it. The energy also doesn't change (it is always constant during the whole process). For the sake of convenience, denote the momentum, p, by const1 and twice the energy, 2E, by const2. Suppose the small ball has velocity u and the big ball has velocity v. The system can be written as follows:

$$mu + Mv = const_1 \tag{3.1.1}$$

$$mu^2 + Mv^2 = const_2 \tag{3.1.2}$$

Now consider the vector $\overline{m}$=($\sqrt{M}$,$\sqrt{m}$),in the xy-plane (it goes along the line Y $=\sqrt{\frac{m}{M}}$*x, and the time-variant vector $\overline{V}$=($\sqrt{M}V$,$\sqrt{m}u$).Then system of equation (3.1.1 & 3.1.2) can be rewritten as follows:

$$\overline{m} \cdot \overline{V} = const_1 \tag{3.1.3}$$

$$|\overline{V}| = const_2 \tag{3.1.4}$$

from above equation we obtain cosθ_1 $=\frac{const_1}{const_2}*(m+M)^{-\frac{1}{1}} = const_3$,

After reflection, point P moves with a new velocity, $\overline{v'}$, satisfying the same system. Therefore, the same reasoning for the angle θ_2 of P's reflection from the side Y $=\sqrt{\frac{m}{M}}$ x shows that cos$\theta_1 = const_3$. So

$$\theta_1 = \theta_2$$

3.1.5 Numerical Simulation, Results and Conclusion

Observe in the figure 3.6 that when the collision happens with the wall that is on the axis then in its next collision the angle changes by twice of $\angle AOB$ following a even angle pattern and similarly for each successive collision between the ball the angle changes by twice and follow a odd angle pattern and this keeps on happening

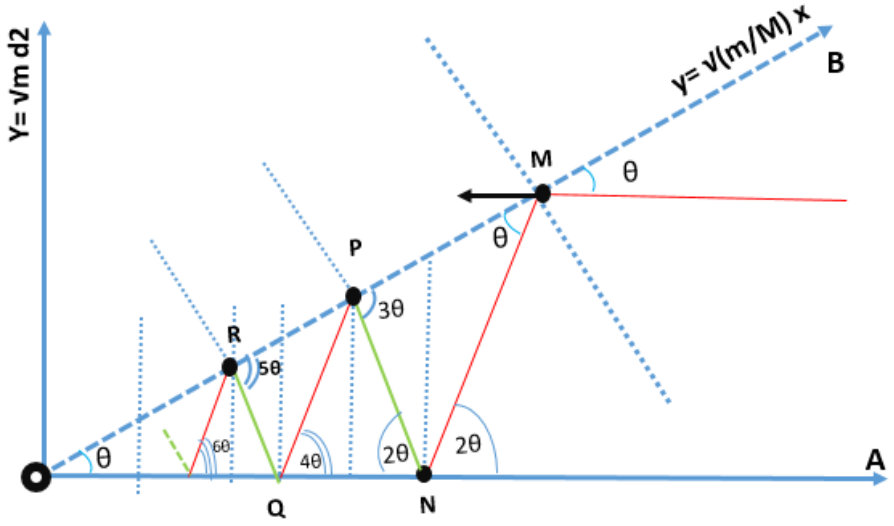


Figure 3.6: Reflection between two mirror OB & OA

upto its maximum value π. So using the above schematic diagram we have come up with algorithm to implement.

3.2 Algorithm for reflection in inclined mirror

1. Set a value for r (say, r = 100^{-1})
2. initialise velocities & position: (v=0 , V=1) ,(d1=200 , d2=200r)
3. initialise collision counter: k = 0
4. initialise angle: sl = 2θ ; $\theta = \tan r$
5. **while** $V > v$ **do**
6. increment counter k = k + 1
7. store pre-collision velocities and previous position
   $v_{\text{old}} = v, V_{\text{old}}$ = V , b1 = d1 , b2 = d2

8. update velocities when M and m collide:

$$\boxed{\begin{aligned} v_k &= \frac{(r-1)v_{k-1} + 2V_{k-1}}{1+r} \\ V_k &= \frac{2rv_{k-1} + (1-r)V_{k-1}}{1+r} \end{aligned}}$$

9. update Positions when M and m collide:

$$\boxed{\begin{aligned} x_{i+1} &= \frac{(x_i * \tan sl)}{r - \tan sl} \\ y_{i+1} &= r * \frac{(x_i * \tan sl)}{r - \tan sl} \end{aligned}}$$

10. increment angle, sl = π-sl+2θ

11. if m collides with wall (i.e. if $v > 0$) then

12. increment angle sl = π-sl

$$\boxed{\begin{aligned} x &= d1 - \frac{d2}{\tan sl} \\ d1 &= x; d2 = 0; \end{aligned}}$$

13. increment counter k = k + 1

14. update velocity v = -v

15. **end if**

16. **end while**

17. number of collisions = value of k

18. configuration space plot

Results and Discussion

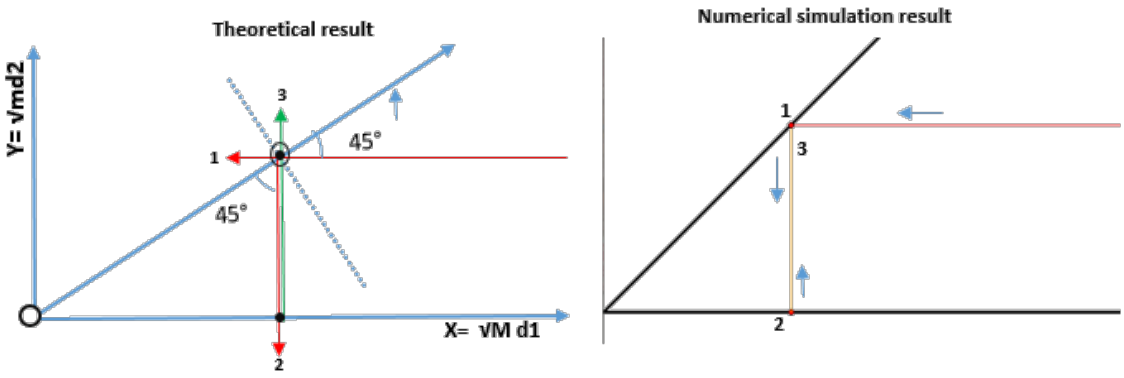


Figure 3.7: Case 1: $M = m$, Click on image to see 3 Reflection between the mirror

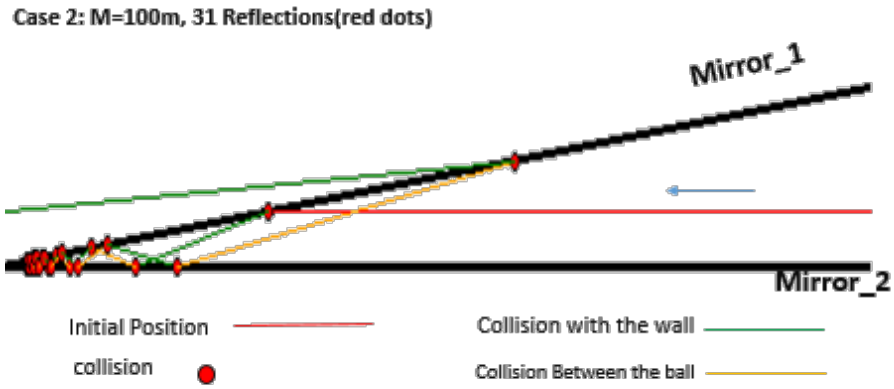


Figure 3.8: Case 2: $M = 100m$, Click on the image to see 31 Reflection between the mirror and for Code **Click Here**

Observe that the number of reflection follows same pattern as the number of collision between the balls which subsequently results in digits of π.

Why does this reflection gives digits of π?

So let's show this geometrically. We will use our basic trick which we call principle *whenever there is π there is circle*. So to show this we use a basic law of reflection. Consider shining a beam of light on two mirror inclined at angle θ(which can be completely determine from r). Illusion is that whenever we shine the beam it just passes straight through the window and the window acting as reflection axis produces

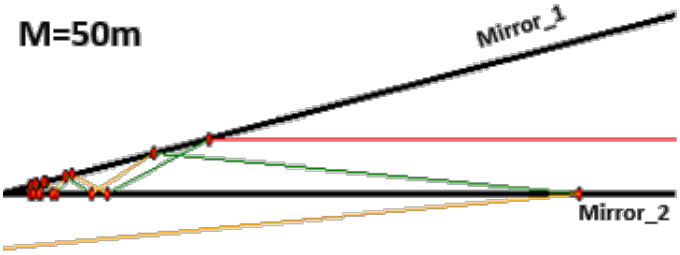


Figure 3.9: Case 0: M=50m , Click on image to see Reflections

Figure 3.10: Case 3: M=10^4m , Click on image to see 314 Reflections

reflected mirror. This happens again until the beam passes away. Step wise diagram is shown. So the question of counting number of reflection reduces to how many such reflected mirrors are required.

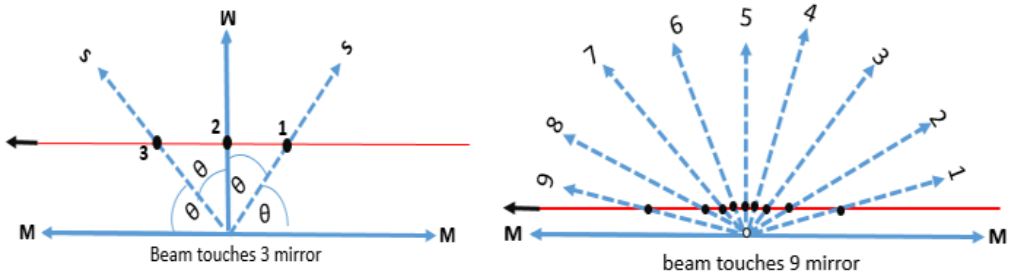


Figure 3.11: Reflection about the mirror where beam strikes

How many such mirrors are required in total?

So the answer is add as much θ before we get just more than half of θ around the circle. So let n number of mirrors are attached,then

$$n\theta = \pi$$

$$n = \lceil \frac{\pi}{\theta} \rfloor$$

But as we know that $\theta = \arctan(r)$ so substituting this in above equation reduces

to equation we derived in collision kinematics problem.

$$n = [\frac{\pi}{\arctan(r)}] \tag{3.2.8}$$

where $r = \sqrt{\frac{m}{M}}$, for small angle that is when $M >> m$ then $\arctan(r)$ reduces to r,

$$n = \frac{\pi}{r}$$

$$n = \frac{\pi}{\sqrt{100^{-N}}}$$

so for $M = 100^N$

$$\boxed{n = \pi * \sqrt{M}} \tag{3.2.9}$$

Chapter 4

The Grover's Algorithm for Quantum Search

We will now turn our attention to the field of quantum query complexity. One of the most famous algorithms in all of quantum mechanics is that of L. Grover. Grover's algorithm provides a way to " find a needle in a quantum haystack". In this chapter,specifically we demonstrate an exact isomorphism between Galperin's bouncing billiards and Grover's algorithm for quantum search. This provides an illuminating way to visualize what Grover's algorithm is actually doing. We have seen an impractical but picturesque way to determine the digits of π that is to hurl a heavy ball towards a light ball that has its back to a wall, as in Fig. 1.2 , and then count the ensuing elastic collisions.

4.1 Quantum Search

You have likely heard that one of the many advantages a quantum computer has over a classical computer is its superior speed searching databases. Grover's algorithm demonstrates this capability. This algorithm can speed up an unstructured search problem quadratically, but its uses extend beyond that; it can serve as a general trick or subroutine to obtain quadratic run time improvements for a variety of other algorithms. We start with the wave function of a N-state system-also called a 'quNit', which may in general be written as in the N-dimensional vector space :

$$|\psi\rangle = v_1 |1\rangle + v_2 |2\rangle + ... + v_{N-1} |N-1\rangle + v_N |N\rangle \tag{4.1.1}$$

and conservation of probability means that throughout its evolution the state will always have :

$$\langle\psi|\psi\rangle = \sum_{i=1}^{n} |v_i|^2 = 1 \tag{4.1.2}$$

4.1.1 The Search Problem

Given a Bulb and $\boldsymbol{n}$ switches attached to it, there exists only one specific arrangement of switches to light the Bulb up. Therefore, the total possibilities of these combinations is $\mathbf{2}^n = N$ chances (treated as items) , among these items there is one item with a unique property that it lights the bulb and we wish to locate; we will call this one the winner, $\boldsymbol{w}$. Think of each item in the list as a box of a particular characteristic.

To find the winner $\boldsymbol{w}$, which is the marked item using classical computation, one would have to check on average $\frac{\boldsymbol{N}}{2}$ of these boxes, and in the worst case, all $\boldsymbol{N}$ of them. On a Quantum Computer, however, we can find the marked item in roughly $\sqrt{\boldsymbol{N}}$ steps with Grover's amplitude amplification trick. A quadratic speedup is indeed a substantial time-saver for finding marked items in long lists. Additionally, the algorithm does not use the list's internal structure, which makes it generic; this is why it immediately provides a quadratic quantum speed-up for many classical problems.

This method applies quantum computing to a mundane problem in information processing and presents an algorithm that is significantly faster than any classical algorithm can be. The problem is this: there is an unsorted database containing $\boldsymbol{N}$ items out of which just one item satisfies a given condition - that one item has to be retrieved. Once an item is examined, it is possible to tell whether or not it satisfies the condition in one step. However, there does not exist any sorting on the database that would aid its selection. The most efficient classical algorithm for this is to examine the items in the database one by one. If an item satisfies the required condition stop; if it does not, keep track of this item so that it is not examined again.

It is easily seen that this algorithm will need to look at an average of $\frac{N}{2}$ items before finding the desired item.

4.1.2 The Black Box

An oracle (at least in this context) is simply an operation that has some property that you don't know, and are trying to find out. These are frequently used in Quantum Algorithms. The term "black box" is used equivalently, to convey the idea that it's just a box that you can't see inside, and hence you don't know what it's doing. All you know is that you can supply inputs and receive outputs.

But, how will the list of items be provided to the quantum computer? A common way to encode such a list is in terms of a function **f** which returns **f(x)** = 0 for all unmarked items w and f(w) = 1 for the winner. To use a quantum computer for this problem, we must provide the items in superposition to this function, so we encode the function into a unitary matrix called an oracle. First we choose a binary encoding of the items $x, w \in \{0,1\}$ $\forall$ n so that $N = 2^n$; now we can represent it in terms of qubits on a quantum computer. Then we define the oracle matrix $\hat{\mathbf{U}}_\mathbf{f}$ to act on any of the simple, standard basis states $|\mathbf{x}\rangle$ by : $\hat{U}_f |x\rangle = (-1)^{f(x)} |x\rangle$. Geometrically, this unitary matrix corresponds to a reflection about the origin for the marked item in an N dimensional vector space.

We see that if x is an unmarked item, the oracle does nothing to the state. However, when we apply the oracle to the basis state $|w\rangle$, it maps $\hat{U}_w |w\rangle = -|w\rangle$. That is, the Grover task imagines we are given a black box that implements the transformation:

$$\hat{U}_w \equiv \mathbf{1} - 2 |w\rangle \langle w| . \tag{4.1.3}$$

This unitary acts on the quNit by flipping the sign of the w^{th} amplitude while leaving all the others unchanged. For example, for $w = 7$, we'd have,

$$\hat{U}_7 = v_1 |1\rangle + ... + v_6 |6\rangle - v_7 |7\rangle + v_8 |8\rangle + ... + v_d |d\rangle .$$

Our job is to figure out which $\hat{U}_w$ we have been given, i.e. to determine the value of w. A simple strategy to determine w is to feed states in one-by-one, first $|1\rangle$, then $|2\rangle$, then $|3\rangle$, and so on until we come to a state whose sign is flipped. While

this strategy succeeds, it succeeds only slowly. So, on average we will need to use the box $\frac{1}{2}N$ times. That is how long it takes to find a classical needle in a classical haystack. The surprising fact that L. K Grover discovered is that we can do much better. Grover's algorithm allows us to find w using only O($\sqrt{N}$) calls.

4.1.3 Amplitude Amplification

Given that we do not know the value of w, and any guess of its location is as good as any other, which leads us to start with the most democratic option is to start with, that is an uniform superposition also called as the symmetrized state, and indeed that is where Grover's algorithm begins,

$$|s\rangle = \frac{1}{\sqrt{N}}(|1\rangle + |2\rangle + ... + |N-1\rangle + |N\rangle). \tag{4.1.4}$$

As we feed this into the black box, yielding $\hat{U}_w |s\rangle$, which will have: $v_w = -\frac{1}{\sqrt{N}}$ and $v_{i\neq w} = \frac{1}{\sqrt{N}}$. Note that this wavefunction already 'knows' the value of w, since $U_w |s\rangle \neq U_{w'} |s\rangle$ when $w \neq w'$, and so if we could determine the wave-function directly we could solve the Grover problem in one step.But, this is not how Quantum Mechanics behaves. Since we are dealing with amplitudes and not probabilities, the vector space's dimension enters as a square root. Therefore it is the amplitude, and not just the probability, that is being amplified in this procedure.

If at this point we were to measure in the standard basis $|x\rangle$, this superposition would collapse, to any one of the basis states with the same probability of $\frac{1}{N} = \frac{1}{2^n}$. Our chances of guessing the right value w is therefore 1 in 2^n , as could be expected. Hence, on average we would need to try about $N = 2^n$ times to guess the correct item. This procedure, called Amplitude Amplification, is how a Quantum Computer significantly enhances this probability. This procedure stretches out or 'amplifies' the amplitude of the marked item, which shrinks the other items' amplitude, so that measuring the final state will return the right item with near-certainty.

Since for large N the states $\hat{U}_w |s\rangle$ and $\hat{U}_{w'} |s\rangle$ are far from orthogonal, in order to determine the value of w we must amplify the difference. Note that if we plug the output back into the black box again, it is counter productive since it takes us back to square one, that is ${U_w}^2 = 1$. Instead, Grover showed that our next step should

be to act with:

$$\hat{U}_s = 2\,|s\rangle\langle s| - \mathbf{1}. \tag{4.1.5}$$

We can construct $\hat{U}_s$ without using the black box, as it makes no reference to w and indeed treats all basis states the same. Grover's algorithm is then just to repeatedly iterate $\hat{U}_s$ and $\hat{U}_w$. This algorithm has a nice geometrical interpretation in terms of two reflections, which generate a rotation in a two-dimensional plane.

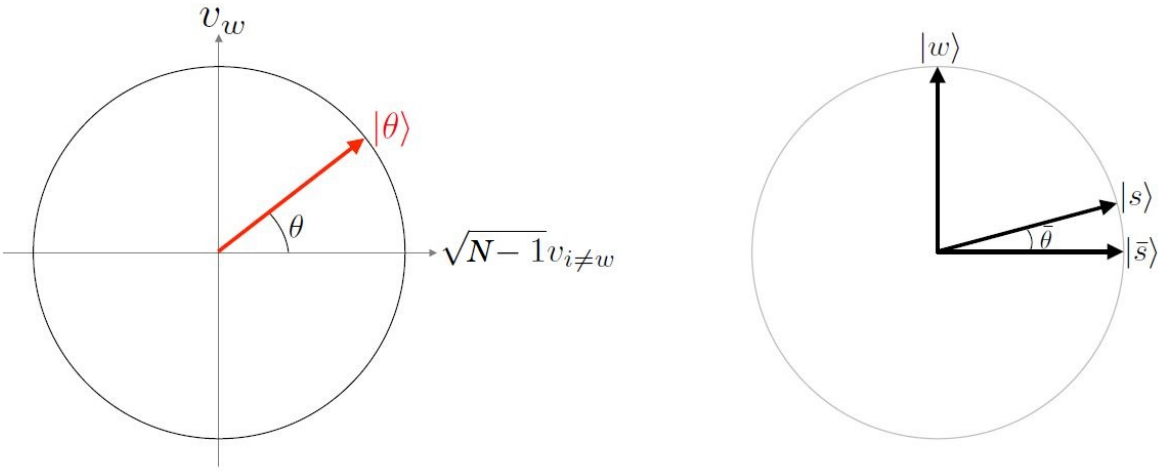


Figure 4.1: All operations keep us on the circle defined by $|\theta\rangle \equiv \cos\theta\,|\bar{s}\rangle + \sin\theta\,|w\rangle$. The starting state $|s\rangle$ and the target state $|w\rangle$ are not exactly orthogonal, but we can define another state $|\bar{s}\rangle$ such that $\langle\bar{s}|w\rangle = 0$.

The only two special states we need to consider are the winner $|w\rangle$ and the uniform superposition $|s\rangle$. These two vectors span a two-dimensional plane in the vector space $\mathbb{C}^N$. They are not quite perpendicular because $|w\rangle$ occurs in the superposition with amplitude $\frac{1}{\sqrt{N}}$ as well. We can, however, introduce an additional state $|\bar{s}\rangle$ that is in the span of these two vectors, which is perpendicular to $|w\rangle$ and is obtained from $|s\rangle$ by removing $|w\rangle$ and rescaling it.

In order to analyze the effect of this iteration, it will be helpful to use an orthonormal coordinate system. The state $|s\rangle$ is not exactly orthogonal to $|w\rangle$ as,

$$\sin\bar{\theta} \equiv \langle s|w\rangle = \frac{1}{\sqrt{N}} \tag{4.1.6}$$

but we can find a state $|\bar{s}\rangle$ such that $\langle w|\bar{s}\rangle = 0$ by defining

$$|\bar{s}\rangle = \frac{1}{\sqrt{N-1}} \sum_{i\neq w} |i\rangle \tag{4.1.7}$$

Both $\hat{U}_s$ and $\hat{U}_w$ keep us on the circle defined by:

$$|\theta\rangle \equiv \cos\theta\,|\overline{s}\rangle + \sin\theta\,|w\rangle\,. \tag{4.1.8}$$

On this circle,$\hat{U}_s$ reflects about the $|s\rangle$-axis and $\hat{U}_w$ reacts about the $|\overline{s}\rangle$-axis,

$$\hat{U}_w\,|\theta\rangle = (\mathbf{1} - 2\,|w\rangle\,\langle w|)\,|\theta\rangle = (2\,|\overline{s}\rangle\,\langle\overline{s}| - \mathbf{1})\,|\theta\rangle \tag{4.1.9}$$

Two reflections always correspond to a rotation. The two consecutive non-parallel reactions combine to give a rotation by $2\overline{\theta}$,

$$\hat{U}_s\hat{U}_w\,|\theta\rangle = \left|\theta + 2\overline{\theta}\right\rangle \tag{4.1.10}$$

We will use repeated applications of $\hat{U}_s\hat{U}_w$ to rotate the state from $|s\rangle$ (i.e. $\theta = \overline{\theta}$) to $|w\rangle$ (i.e. $\theta = \frac{\pi}{2}$) in steps of size $2\overline{\theta}$. This is called the *Grover Iterate*. The transformation $\hat{U}_s\hat{U}_w$ rotates the initial state $|s\rangle$ closer towards the winner $|w\rangle$. The action of the reflection $\hat{U}_s$ in the amplitude bar diagram can be understood as a reflection about the average amplitude. Since the average amplitude has been lowered by the first reflection, this transformation boosts the negative amplitude of $|w\rangle$ to roughly three times its original value, while it decreases the other amplitudes. For almost all N, and in particular for all $N - 1 = 100^k$ (which is relevant for comparison with Galperin's result, $k \in \mathbf{N}$), the integer closest to $(\frac{\pi}{2} - \overline{\theta})/2\overline{\theta}$ is $[\frac{\pi}{4}\sqrt{N-1}]$, and so Grover's algorithm calls for that many steps.

$$(\hat{U}_s\hat{U}_w)^{[\frac{\pi}{4}\sqrt{N-1}]}\,|s\rangle = \left|\overline{\theta} + 2[\frac{\pi}{4}\sqrt{N-1}]\overline{\theta}\right\rangle = |w\rangle + O(\frac{1}{\sqrt{N}}) \tag{4.1.11}$$

Since $\langle w|w'\rangle = 0$ for $w \neq w'$, with high probability we can now just measure the qunit to learn the value of w and successfully complete the Grover task.

4.2 Balls to the Wall

We saw in the previous chapters that if the left ball is much heavier than the right, then it is harder to slow down and reverse. The heavier the left ball, the more collisions are needed,

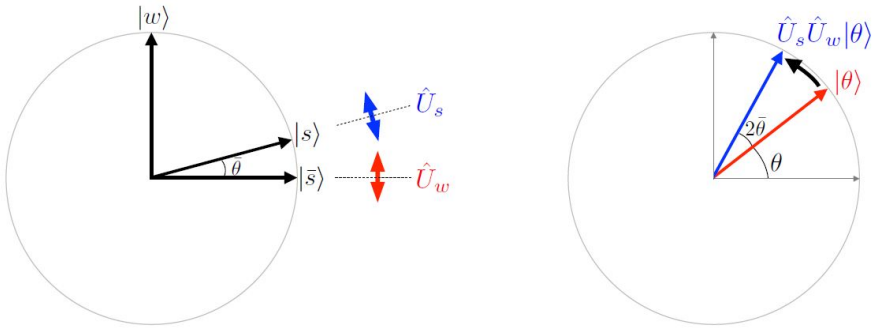


Figure 4.2: $\hat{U}_w$ reflects about $|s\rangle$, and $\hat{(}U)_w$ reflects about $|s\rangle$. In combination, $\hat{U}_s\hat{U}_w$ gives a rotation by $2\overline{\theta}$.

$$M = 100 \Rightarrow \text{collisions} = 31$$
$$M = 10^6 \Rightarrow \text{collisions} = 3141$$
$$M = 10^{20} \Rightarrow \text{collisions} = 31415926535$$

G. Galperin proved that this algorithm really is spitting out the digits of π, since for $M = 100^k$

$$\text{number of collisions} = [\pi\sqrt{M}] \tag{4.2.12}$$

From this expression,making an abrupt shift and now turning our attention to L. K Grover's algorithm, which provides a way to find a needle in a quantum haystack"- or more precisely to determine which out of the N mystery functions are being implemented by a black-box quantum oracle. For $N - 1 = 100^k$ the runtime is

$$\text{number of oracle cells} = [\frac{1}{4}\pi\sqrt{N-1}] \tag{4.2.13}$$

Indeed, we shall now argue that there is a precise isomorphism between bouncing billiard balls and Grover's algorithm of quantum search. The velocity of N billiards moving on a line may be described by a N-dimensional vector, v_i. To make this analogy with quantum mechanics more visceral, we could write this vector in the Dirac notation as given in the equation (4.1.1) which is:

$$|velocity\rangle = v_1 |1\rangle + v_2 |2\rangle + ... + v_{N-1} |N-1\rangle + v_N |N\rangle .$$

Elastic collisions conserve kinetic energy. If all the billiards have mass 1, and if they start with total kinetic energy $\frac{1}{2}$, then throughout their evolution they will preserve

conservation of kinetic energy given in the equation (4.1.2) which is :

$$\langle velocity|velocity\rangle = \sum_{i=1} N|v_i|^2 = 1.$$

To replicate the π-calculating set-up from Sec. 1, we should separate off one billiard (let us say the w^{th}) to be the light ball, and glue together all the other billiards to form one big heavy ball of mass

$$\boldsymbol{M = N - 1}. \tag{4.2.14}$$

The glue constrains all the $v_i \neq w$ to be the same, and **then conservation of energy confines the velocity vector to lie on the circle given by** $\sin\theta = v_w$ and $\cos\theta = \sqrt{M}\, v_{i\neq w} = \sqrt{N-1}\, v_{i\neq w}$,

$$|\theta\rangle = \cos\theta\,|\bar{s}\rangle + \sin\theta\,|w\rangle\,. \tag{4.2.15}$$

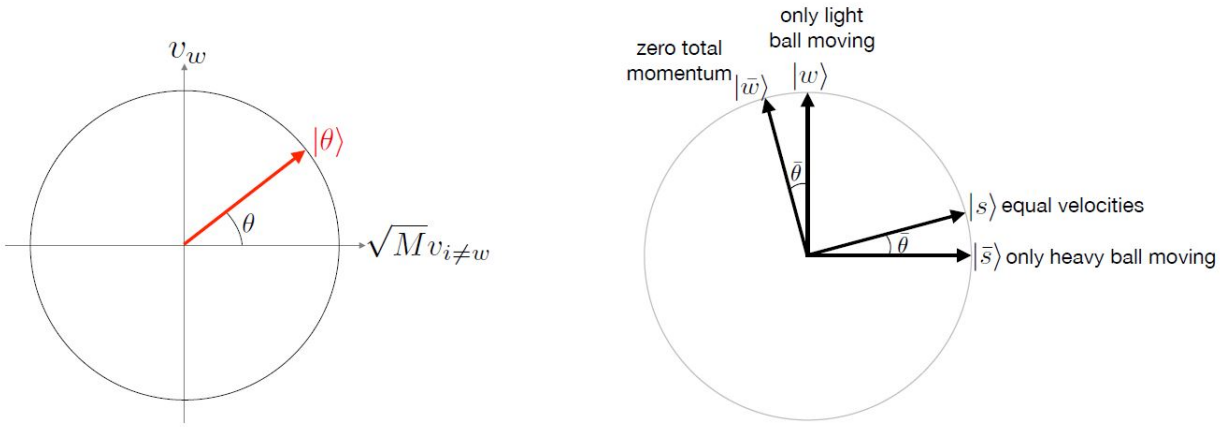


Figure 4.3: Conservation of energy guarantees that the velocities lies on the circle with equation : $\frac{1}{2}v_w^2 + \frac{1}{2}Mv_{i\neq w^2}^2 = \frac{1}{2}$

Every collision map this circle into itself. The exact form of the map can be determined by considering which momentum each collision conserves. Argueing by symmetry, the linear maps from a circle to itself are described by the group O(2), which has two components: the rotations ($det = +1$); and the reflections ($det = -1$). Rotations leave no vectors invariant, whereas reflections leave invariant the vector that points down the axis being reflected about. The conserved quantities thus exactly fix the forms of the orthogonal matrices that describe the collisions. Thus

any O(2) map that conserves the inner product with $|\phi\rangle$ is either 1 or the reflection $\widehat{O}_{|\phi\rangle} \equiv 2\,|\phi\rangle\,\langle\phi| - \mathbf{1}$.

Grover's and collision block Brief comparision table:	
BOUNCING BILLIARDS	GROVER SEARCH
all kinetic energy in light ball	$\|w\rangle$
both balls equal energy	$\|s\rangle = \frac{1}{\sqrt{N}} \Sigma_i \ \|i\rangle$
all kinetic energy in heavy ball	$\|\overline{s}\rangle = \frac{1}{\sqrt{N-1}} \Sigma_{i \neq w} \ \|i\rangle$
total momentum zero	$\|\overline{w}\rangle = \sqrt{\frac{N-1}{N}}\,\|w\rangle - \frac{1}{\sqrt{N(N-1)}} \Sigma_{i \neq w} \ \|i\rangle$
the $M = N-1$ billiards in big ball	$N-1$ wrong answers
$\widehat{O}_{ball}$	$\widehat{U}_s = 2\|s\rangle\,\langle s\|$ - 1
$\widehat{O}_{wall}$	1 - $\widehat{U}_s = 2\|w\rangle\,\langle w\|$
small ball bounces back and forth	alternate $\widehat{U}_s$ and $\widehat{U}_w$
velocity v_i of i^{th} billiard ball	amplitude v_i of i^{th} eigenstate
2 $\times$ kinetic energy of i^{th} billiard	probability $\|v_i\|^2$ of i^{th} eigenstate
conservation of kinetic energy	conservation of probability
conservation of phase space	unitarity
motion purely horizontal	wavefunction purely real
collision order matters	operators don't commute
$\widehat{O}_{ball}$ conserves total momentum	$\left[\|s\rangle\,\langle s\|\,, \widehat{U}_s\right] = 0$
$\widehat{O}_{wall}$ conserves big-ball momentum	$\left[\|s\rangle\,\langle s\|\,, \widehat{U}_w\right] = 0$

- **Collisions between the light ball and the wall, $\widehat{O}_{wall}$.**

  These collisions reverse the velocity of the light ball while leaving all other velocities unchanged $v_i \to (-1)^{\delta_{iw}} v_i$; alternatively we can say that they conserve the momentum of the large ball,

$$\text{momentum of the large ball} = \sum_{i \neq w} v_i = \sqrt{N-1}\,\langle \overline{s}|\theta\rangle \tag{4.2.16}$$

Since the collision conserves $\langle\overline{s}|\theta\rangle$, symmetry tells us it must enact

$$\hat{O}_{wall}\,|\theta\rangle = (\mathbf{1} - 2\,|w\rangle\,\langle w|)\,|\theta\rangle = (2\,|\overline{s}\rangle\,\langle\overline{s}| - \mathbf{1})\,|\theta\rangle = (|\overline{s}\rangle\,\langle\overline{s}| - |w\rangle\,\langle w|)\,|\theta\rangle \tag{4.2.17}$$

- **Collisions between the heavy ball and the light ball, $\hat{O_{ball}}$.**
  In the center-of-mass rest frame both velocities get reversed; alternatively we can say these collisions conserve the total momentum of the balls,

$$\text{total momentum} = \sum_i v_i = \sqrt{N}\,\langle s|\theta\rangle \tag{4.2.18}$$

Since the collision conserves $\langle s|\theta\rangle$, symmetry tells us it must enact

$$\hat{O}_{ball}\,|\theta\rangle = (\mathbf{1} - 2\,|\overline{w}\rangle\,\langle\overline{w}|)\,|\theta\rangle = (2\,|s\rangle\,\langle s| - \mathbf{1})\,|\theta\rangle = (|s\rangle\,\langle s| - |\overline{w}\rangle\,\langle\overline{w}|)\,|\theta\rangle \tag{4.2.19}$$

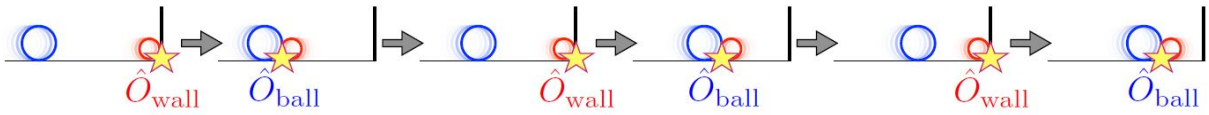


Figure 4.4: The billiard bouncing back and forth between the wall and the large ball alternates $\hat{O}_{wall}$ and $\hat{O}_{ball}$.

The small ball bounces back and forth between the wall ($\hat{O}_{wall}$). and the large ball ($\hat{O}_{ball}$). Each lap, the two reflections combine to make a rotation by 2θ, exactly as in (Equation 4.1.10):

$$\hat{O}_{ball}\hat{O}_{wall}\,|\theta\rangle = \left|\theta + 2\overline{\theta}\right\rangle.$$

What remains is to specify the initial state. Galperin's original π-counting plan starts in $|\overline{s}\rangle$: the light ball begins at rest. To make the analogy with Grover's algorithm perfect and evident, we should instead start in $|s\rangle$: the two balls begin with the same velocity. With this minor tweak, the isomorphism between billiards and Quantum Search becomes exact.

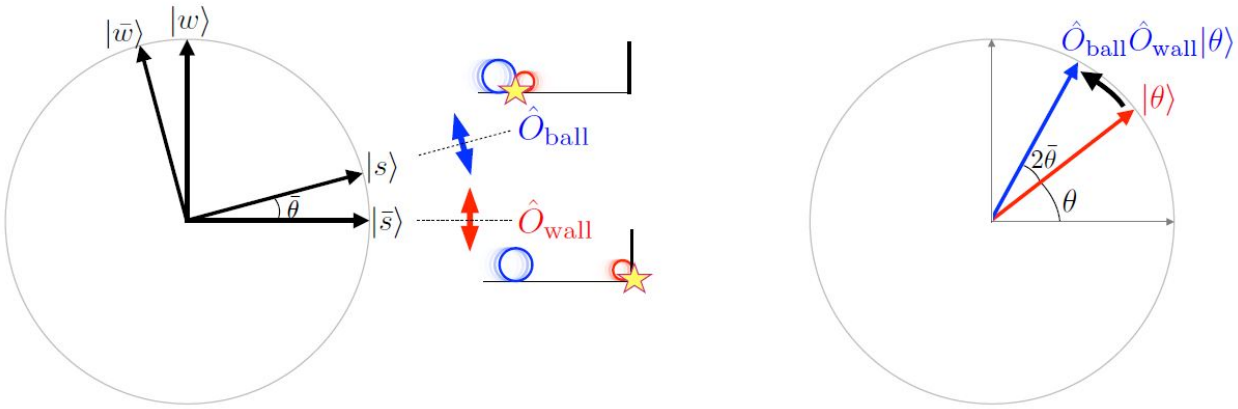


Figure 4.5: $\hat{O}_{wall}$ reflects about $|\bar{s}\rangle$, and $\hat{O}_{ball}$ reflects about $|s\rangle$. In combination, $\hat{O}_{ball}\hat{O}_{ball}$ gives a rotation by $2\bar{\theta}$.

4.3 Discussion

The isomorphism between Grover search Algorithm and the bouncing of billiards is a duality between a discrete quantum system and a continuous classical system. Here we discuss and examine some aspects of the duality in more detail:

- **the Factor of Four**

  There is a factor of 4 discrepancy between the collision-counting (Eq. 4.2.12) and the query- counting (Eq. 4.2.13) . The factor of four is really two factors of two. One factor of 2 comes from how we count. The billiard problem counts every collision; by contrast Grover's problem treats $\hat{U}_s$ as free and only charges for applications of $\hat{U}_w$. The other factor of 2 comes from when we stop. Grover's algorithm stops when the heavy ball has transferred all its energy to the small ball to reach $|w\rangle = \left|\theta = \frac{\pi}{2}\right\rangle$; by contrast the billiard algorithm keeps going twice as long, until the small ball has retransferred all its energy back to the heavy ball again to reach $-|\bar{s}\rangle = |\theta = \pi\rangle$

- **Real Wave function**

  Throughout the Grover process, the wave-function stays real. The v_i are always real, and operators $\hat{U}_s$ and $\hat{U}_w$ are not just unitary but also orthogonal. In the billiard problem, the reality of the wave-function corresponds to the billiards moving only horizontally.

- **the Squareroot**

  The crowd-pleaser in the collision-counting (*Equation*4.2.12) is the "π", but the square root is somewhat surprising. After all, the first collision transfers only a single unit of kinetic energy, and there are M units to be transferred in total, so you might think it is going to take O(M) collisions. The bouncing algorithm nevertheless gets the job done in O($\sqrt{M}$) steps because the momentum of the light ball grows linearly, so the kinetic energy grows quadratically.

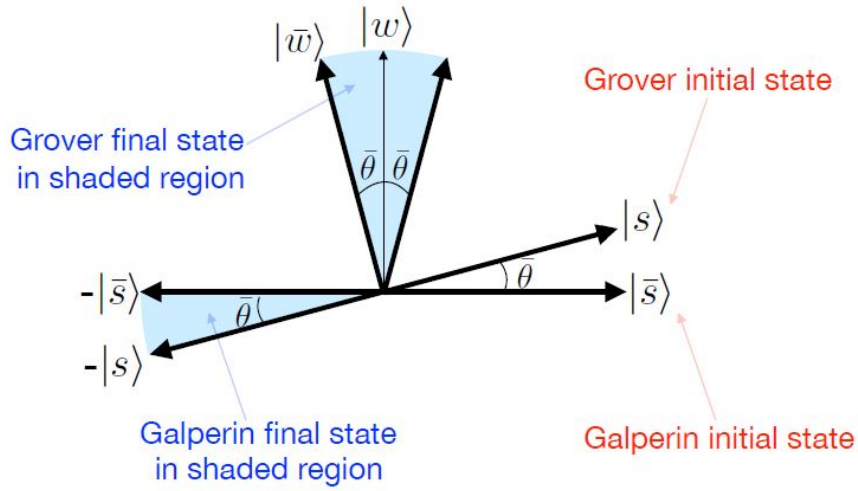


Figure 4.6: Galperin's protocol starts with the small ball stationary,$|\bar{s}\rangle$, and ends once $v_{i\neq w}v_w0$. Grover's algorithm starts in $|s\rangle$, with all the velocities equal, and ends at the closest approach to $|w\rangle$.

The dual of this surprise occurs for Grover's algorithm. The first iteration of $\hat{U}_s\hat{U}_w$ only increases the probability of measuring $|w\rangle$ by O($\frac{1}{N}$), but nevertheless only O($\sqrt{N}$) steps are needed to distinguish with near certainty. Grover's algorithm gets the job done in O($\sqrt{N}$) steps because the amplitude v_w grows linearly, so the probability grows quadratically.

- **Multiple Searches**

  It is clear that if you double the mass of *both* balls, the number of collisions does not change. Only the ratio matters. This is dual to the slightly less obvious fact that if you repeat the Grover task with more than one correct

answer (more than one needle in the haystack), the number of required queries only depends on the ratio of the number of needles, n, to the number of non-needles, $N - n$ so that the number of queries $= [\frac{\pi}{4}\sqrt{(N-n)/n}]$

Note :

Galperin's method requires that we overcome friction, overcome inelasticities, overcome the blurring effects of quantum mechanics.Although,this is a picturesquely quixotic way to seek π in the $|\psi\rangle$.

4.3.1 Simulation

The algorithm consists of the following steps:

1. The Grover's search algorithm searches the target value from a set by calculating the *Mean amplitude* and the *Grover's amplitude*.

2. GetOracle method takes the value and returns the hex digest of that value. This method is referred to as the Oracle function.

3. Execute Grover's algorithm takes input target, objects, nvalue (strength of the data, N) and rounds. Rounds are given by :$[\frac{\pi}{4}\sqrt{N-1}]$. The amplitude is retrieved based on the $1/\sqrt{N}$.

4. The goal of the algorithm is to search for a target item in the set of objects. The amplitude is searched from the dictionary based on the value $1/\sqrt{N}$.

5. The mean of the amplitudes is calculated and printed. The Grover amplitude is calculated based on the average amplitude and the Grover's operator. The Grover's operator is defined as $(2|x\rangle\langle x|$ – average amplitude) * Oracle function.

6. The plot of the graph is calculated from the amplitudes derived from the Grover's algorithm. The plot is supposed to show the target value with the highest amplitude.

7. The amplitudes are plotted after running the Grover's algorithm.

Chapter 5

Summary of Codes

1. Block Collisions Simulations: **Click Here**
2. Pi Calculation Using Energy and Momentum Conservation equations : **Click Here**
3. Velocity Phase Space Simulation : **Click Here**
4. Combined Block Collisions and Velocity Phase Space Simulation : **Click Here**
5. Equal Angle Sector Plot : **Click Here**
6. Pi calculation using solved expressions(equation (2.3.9) & (2.3.10) : **Click Here**
7. Position Phase Space(Configuration space) Simulation : **Click Here**
8. Grover's Algorithm Code : **Click Here**

Bibliography

[1] Adam R. Brown (2019), *"Playing Pool with $|\psi\rangle$: from Bouncing Billiards to Quantum Search"*
arXiv:1912.02207v1.

[2] G. Galperin (2003), *"Playing pool with π", Regular and Chaotic Dynamics" v. 8, no. 4*
DOI: 10.1070/RD2003v008n04ABEH000252

[3] L. K. Grover (1969), *"A Fast quantum mechanical algorithm for database search"*
arXiv:quant-ph/9605043.

[4] Grant Sanderson (2019), *"The most unexpected answer to a counting puzzle", https://youtu.be/HEfHFsfGXjs.*

[5] Pi in our Daily Life
https://amazingarchimedes.weebly.com/real-life-application-of-pi.html.