EIGENVALUES OF MATRICES WITH TREE GRAPHS

A Project Report Submitted
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MA699 Project

by

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to the

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CERTIFICATE

This is to certify that the work contained in this report entitled **"Eigenvalues of Matrices with Tree Graphs"** submitted by **Himanshu Prajapati** (**Roll No: 222123026**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

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ABSTRACT

The main aim of the project is to study signed digraph and undirected graph of a real square matrix forms a tree or forest structure, we have developed a series of finite, computable methods that provide detailed insights into the sizes and the repeated occurrences of the eigenvalues of the matrix. By using these methods on system designs that are represented as signed directed graphs, we can effectively ensure that these systems are controllable within the framework of linear dynamics.

Contents

List of notations

$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^n$	the n-dimensional real Euclidean space
$SD(A)$	Signed digraph associated with matrix A
$Q(A)$	The set comprising all matrices having the identical sign pattern as matrix A
$G(A)$	Undirected graph associated with matrix A
$det(A)$	The determinant of the matrix A

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Chapter 1

Introduction to Signed digraph

In this chapter we study signed digraph and discuss some of its properties.We work throughout with real matrices and begin by introducing some notation and definitions.

1.1 Signed digraph

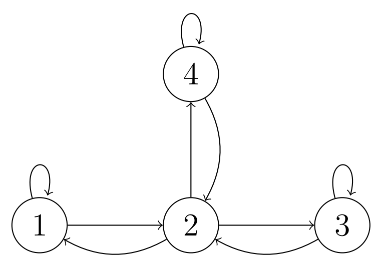
Definition 1.1.1. Let A be a real matrix of order n, its signed digraph $SD(A)$ is a directed graph with nodes labeled from $1, 2, \ldots, n$ and there is a directed edge from node i to node j iff the entry a_{ji} in A is non-zero.
If a_{ji} is positive, the edge connecting from node i to node j is labelled with positive sign and if a_{ji} is negative,the edge connecting from node i to node j is labelled with negative sign.

The set of all matrices with the same sign pattern (and thus having the same signed digraph) as A is denoted by $Q(A)$. We also use the undirected graph $G(A)$ which has the same node set as $SD(A)$ with edge set $\{\{i, j\} :$ $i \neq j$ and . An edge of $G(A)$ thus corresponds to a 2-cycle in $SD(A)$.

Example 1.1.2. Consider the matrix given by

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & -3 & 2 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -3 \end{bmatrix}$$

It's associated signed digraph is given by:



Definition 1.1.3. A 2-cycle in $SD(A)$ is positive if $a_{ij}a_{ji} > 0$ and negative if $a_{ij}a_{ji} < 0$

Example 1.1.4. In the above matrix A, $a_{12} \cdot a_{21} > 0$, indicating a positive 2-cycle in the signed digraph $SD(A)$. Conversely, $a_{23} \cdot a_{32} < 0$, indicating the presence of a negative 2-cycle in $SD(A)$.

Definition 1.1.5. A node i with a 1-cycle is called distinguished and it corresponds to a nonzero diagonal entry in A,i.e, $a_{ii} \neq 0$.

Example 1.1.6. In the above matrix A, node 1,3,4 are distinguished nodes and corresponds to non-zero diagonal entry.

Let's consider the differential equation $\dot{x}(t) = Ax(t)$ with $A \in Q(A)$, and we are going to detect the possibility of constant or sinusoidal trajectories.

Definition 1.1.7. A constant (strictly constant) trajectory $x(t) \in \mathbb{R}^n$ for $\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j$ satisfies $\dot{x}_i = 0$ and $x_i \neq 0$ for some (all) i

Definition 1.1.8. A sinusoidal (strictly sinusoidal) trajectory for our equation satisfies $\ddot{x}_i = -x_i$ and $x_i \not\equiv 0$ ("not the constant function with the value zero") for some (all) i.

Lemma 1.1.9. *Consider the differential equation $\dot{x} = \tilde{A}x$. Then it admits a constant (sinusoidal) trajectory if and only if $\tilde{A}$ has a zero (purely imaginary) eigenvalue.*

1.2 λ-consistency

Definition 1.2.1. Suppose $G(A)$ is a tree. A signed digraph $SD(A)$ is said to be λ-consistent if there exist nonzero constants $\{\lambda_1, \ldots, \lambda_n\}$ such that $\lambda_i a_{ij} = -\lambda_j a_{ji}$ for $i \neq j$; all $\lambda_i a_{ii} > 0$; and some $\lambda_i a_{ii} > 0$.

How to find whether a matrix is λ-consistent?

Consider a case where node 1 is a distinguished node in the λ-consistent $SD(A)$, having $a_{11} \neq 0$. Let's select $\lambda_1 = \pm 1$, ensuring $\lambda_1 a_{11} > 0$. Using the signs of 2-cycles along the chain, we can determine the signs of all other $\{\lambda_j\}$ values.This process can be done because due to the tree structure of $G(A)$.**Consequently, $SD(A)$ exhibits λ-consistency if and only if each $\lambda_i a_{ij} > 0$.**

Example 1.2.2. Consider the matrix given by

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & -3 & 2 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -3 \end{bmatrix}$$

In this analysis, node 1 is identified as a distinguished node where $a_{11} > 0$. Consequently, we choose λ_1 such that $\lambda_1 a_{11} > 0$, leading us to set $\lambda_1 = 1$, which indicates a positive value.

The signs of the remaining λ_j values are determined using the properties of 2-cycles:

1. The relationship $\lambda_1 a_{12} = -\lambda_2 a_{21}$, with both a_{12} and a_{21} equal to 2 and positive, implies that $\lambda_2 = -1$, which is negative.

2. Considering $\lambda_2 a_{23} = -\lambda_3 a_{32}$ where $a_{23} = -3$ (negative) and $a_{32} = 2$ (positive), we find $\lambda_3 = -\frac{3}{2}$, also negative.

3. For the cycle involving nodes 2 and 4, $\lambda_2 a_{24} = -\lambda_4 a_{42}$, with $a_{24} = 2$ (positive) and $a_{42} = -1$ (negative), leads us to conclude $\lambda_4 = -2$, again negative.

Finally, we validate the diagonal entries for λ-consistency, ensuring that $\lambda_i a_{ii} > 0$ holds for all nodes $i \in \{1, 2, 3, 4\}$.

Therefore, the matrix A is confirmed to be λ-consistent, based on these considerations.

Example 1.2.3. Consider the matrix given by:

$$B = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

In this analysis, node 1 is identified as a distinguished node where $a_{11} = -1$, which is less than zero. Therefore, we select λ_1 such that $\lambda_1 a_{11} > 0$. This requirement implies that $\lambda_1 = -1$, which is negative.

Next, we determine the signs of the remaining λ_j values using the properties of 2-cycles:

- For the cycle involving nodes 1 and 2, the relationship $\lambda_1 a_{12} = -\lambda_2 a_{21}$, where $a_{12} = 2$ (positive) and $a_{21} = 3$ (positive), implies $\lambda_2 = -\frac{2}{3}$, which is negative.

- Similarly, considering $\lambda_2 a_{23} = -\lambda_3 a_{32}$ with $a_{23} = 1$ (positive) and $a_{32} = -2$ (negative), results in $\lambda_3 = -\frac{1}{3}$, also negative.

- For the cycle involving nodes 2 and 4, $\lambda_2 a_{34} = -\lambda_4 a_{43}$, where $a_{34} = -4$ (negative) and $a_{43} = 2$ (positive), leads to $\lambda_4 = -2$, again negative.

Finally, we check the diagonal entries to verify λ-consistency. We ensure that $\lambda_i a_{ii} > 0$ for all nodes $i \in \{1, 3, 4\}$. However, node 2 is an exception in this consistency check.

Consequently, matrix B is not λ-consistent.

Definition 1.2.4. A subchain of the signed digraph $SD(A)$ is a subgraph that consists of a sequence of nodes connected by directed edges, forming a straight chain of 2-cycles. Consequently, the undirected graph representing this subchain forms a simple path (that is, an unbranched tree)

Lemma 1.2.5. *When $SD(A)$ has at least two 1-cycles, then $SD(A)$ is not λ-consistent iff some subchain of $SD(A)$ with distinguished end nodes and no other distinguished nodes, is not λ-consistent.*

Definition 1.2.6. A subchain in a signed digraph $SD(A)$ such that it has distinguished end nodes and no other distinguished nodes is called **proper subchain**.

Example 1.2.7. Consider the matrix given by

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & -3 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$$

In this case, node 1 is identified as a distinguished node with $a_{11} = 1 > 0$. Consequently, we choose λ_1 such that $\lambda_1 a_{11} > 0$. This condition dictates that $\lambda_1 = 1$, which is positive.

To determine the signs of the remaining λ_j values, we examine the information from the 2-cycles:

- Considering the cycle between nodes 1 and 2, the relationship $\lambda_1 a_{12} = -\lambda_2 a_{21}$ holds, where $a_{12} = 2$ and $a_{21} = 2$, both positive. This leads to $\lambda_2 = -1$, which is negative.

- Analyzing the cycle between nodes 2 and 3, where $\lambda_2 a_{23} = -\lambda_3 a_{32}$, with $a_{23} = -3$ (negative) and $a_{32} = 2$ (positive), it follows that $\lambda_3 = -\frac{3}{2}$, also negative.

- For the cycle involving nodes 2 and 4, given $\lambda_2 a_{24} = -\lambda_4 a_{42}$, with $a_{24} = 2$ (positive) and $a_{42} = -1$ (negative), it implies that $\lambda_4 = -2$, which is negative.

We then verify the diagonal entries for λ-consistency. Therefore, $\lambda_i a_{ii} > 0$ for $i \in \{1, 2\}$, but not for nodes 3 and 4.

Hence, matrix C is not λ-consistent.

Verifying the above mentioned lemma is straightforward, as we identify a proper subchain with the node set $\{1, 2, 3\}$ featuring distinguished end nodes. This subchain is not λ-consistent, thereby confirming that matrix C is also not λ-consistent.

Chapter 2

Strictly Constant Trajectories

2.1 Strictly Constant Trajectories

Definition 2.1.1. A strictly constant trajectory $x(t) \in \mathbb{R}^n$ for $\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j$ satisfies $\dot{x}_i = 0$ and $x_i \neq 0$ for all i

Theorem 2.1.2. *Suppose A is an irreducible matrix of order > 2 and $SD(A)$ has no k-cycle, $k > 2$. Then there exist $\tilde{A} \in Q(A)$ and a strictly constant trajectory satisfying $\dot{x} = \tilde{A}x = 0$ if and only if each end node of $SD(A)$ is distinguished and $SD(A)$ is not λ-consistent.*

Proof. Suppose x is a strictly constant trajectory. Since each component of x must be non-zero, it follows that each end node of $SD(A)$ must correspond to a distinguished node in A. Furthermore, if $SD(A)$ were λ-consistent, the existence of non-zero constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_i\tilde{a}_{ij} = -\lambda_j\tilde{a}_{ji}$ for all $i \neq j$ in $1, \ldots, n$. The subsystem

$$\dot{x}_i = \sum_{j=1}^{n} \tilde{a}_{ij}x_j \qquad (i = 1, \ldots, n) \tag{2.1}$$

then admits the positive definite Lyapunov function

$$\Lambda(x) = \sum_{i=1}^{k} \lambda_i x_i^2, \tag{2.2}$$

whose derivative

$$\dot{\Lambda}(x) = \sum_{i=1}^{n} 2\lambda_i x_i \dot{x}_i = 2\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i x_i \tilde{a}_{ij} x_j = 2\sum_{i=1}^{n} \lambda_i \tilde{a}_{ii} x_i^2$$

Hence, the derivative of $\Lambda = \sum_{i=1}^{n} \lambda_i x_i^2$ along the constant trajectory x would yield $0 = \dot{\Lambda} = 2\sum_{i=1}^{n} \lambda_i \tilde{a_{ii}} x_i^2 > 0$, which contradicts the fact that x is a strictly constant trajectory.

For the **converse**, let us assume $SD(A)$ is itself a proper subchain. Labeling the nodes in the obvious way, $\tilde{A}$ is a tridiagonal matrix; we fix all entries except $\tilde{a}_{nn}$ ($\neq 0$). Considering the disjoint cycles of A and setting $\alpha_i = \tilde{a}_{ii+1}\tilde{a}_{i+1i}$ for $i = 1, 2, \ldots, n-1$ gives

$$\det A = \begin{cases} (-1)^{(n)/2}[-\tilde{a}_{11}\alpha_2\alpha_4 \cdots \alpha_{n-2}\tilde{a}_{nn} + \alpha_1\alpha_3 \cdots \alpha_{n-1}] & \text{if } n \text{ is even,} \\ (-1)^{(n-1)/2}[\tilde{a}_{11}\alpha_2\alpha_4 \cdots \alpha_{n-1} + \alpha_1\alpha_3 \cdots \alpha_{n-2}\tilde{a}_{nn}] & \text{if } n \text{ is odd.} \end{cases} \tag{2.3}$$

The sign of $\tilde{a_{nn}}$ is either $+$ or $-$, and it is either possible to adjust the magnitude of $\tilde{a_{nn}}$ so $\det \tilde{A} = 0$ or not, depending only on that sign. If $SD(A)$ were λ-consistent, it would be impossible to have a constant trajectory with $\det \tilde{A} = 0$ because of the above argument. Since we are assuming that $SD(A)$ is not λ-consistent, $\tilde{a_{nn}}$ must be of the other sign, so for some choice of $|\tilde{a_{nn}}|$, $\det A = 0$.

Now let x be a nontrivial solution of $\tilde{A}x = 0$ with $\tilde{A}$ as above. The equations $\sum_{j=1}^{n} \tilde{a_{ij}} x_j = 0$ with $\tilde{a_{11}} \neq 0$ and $x_1 = 0$ implies $x_2 = 0$, $x_3 = 0$, and so on through the chain; therefore $x_1 \neq 0$. But $x_1 \neq 0$ implies, by the same argument, that each component of x is nonzero. So ,this ensures the existence of a non-trivial solution x to the equation $\tilde{A}x = 0$, where all components of x are non-zero.and so x is a strictly constant trajectory. (Note that if an end node is not distinguished, then the argument fails, as some component of x is zero).

Next suppose that $SD(A)$ may be partitioned into a proper subchain which is not λ-consistent and a second subchain with exactly one distinguished node, the end node (not the node of attachment); **see Figure 2.1** for an example.We can construct $\tilde{A}$ and x as before for the proper subchain, and then extend the solution x to the second subchain by recursively specifying the remaining components. The key idea is to use the equations at the attachment node to ensure that the full vector x satisfies $\tilde{A}x = 0$ with all components non-zero.

Starting at the node of attachment q, let the nodes of the second subchain be labeled $q, q+1, \ldots, q+m$, with $q+m$ the end node and $a_{q+m,q+m} \neq 0$. Let $\tilde{A}$, x be specified as above for the proper subchain, and let other $\tilde{A}$ entries be arbitrary in magnitude except $\tilde{a}_{q+1,q}$. Tentatively set $x_{q+m} = 1$. Then x_{q+m-1} can be used to specify x_{q+m-2}, which in turn can be used to specify x_{q+m-3}, and so on down the chain. Finally x_{q+1} is specified.

If there is a sign conflict in the equation at node $q+1$, start over with $x_{q+1} = -1$. Then specify $\tilde{a}_{q+1,q}$. At node q we must modify λ-values so that $\tilde{a}_{q\alpha}x_\alpha + \tilde{a}_{q\beta}x_\beta + \tilde{a}_{qq+1}x_{q+1} = 0$, where nodes α, β are neighbors of node q in the proper subchain. The first two summands are already of opposite signs, so adjustment of the magnitudes of $\tilde{a}_{q\alpha}$ and $\tilde{a}_{q\beta}$ can clearly be carried out so $\tilde{A}x = 0$, all $x_i \neq 0$, $i = q, \ldots, q+m$. The case in which the node of attachment q is also distinguished follows in a similar way, with the additional term $\tilde{a}_{qq}x_q$ in the equation at node q.

A simple extension of the above sequence shows that any number of subchains with distinguished end nodes can be accommodated. Lastly, additional nodes can acquire (small magnitude) 1-cycles by local adjustment of

$\tilde{A}$ values, since each node clearly has inputs of opposite signs. $\square$

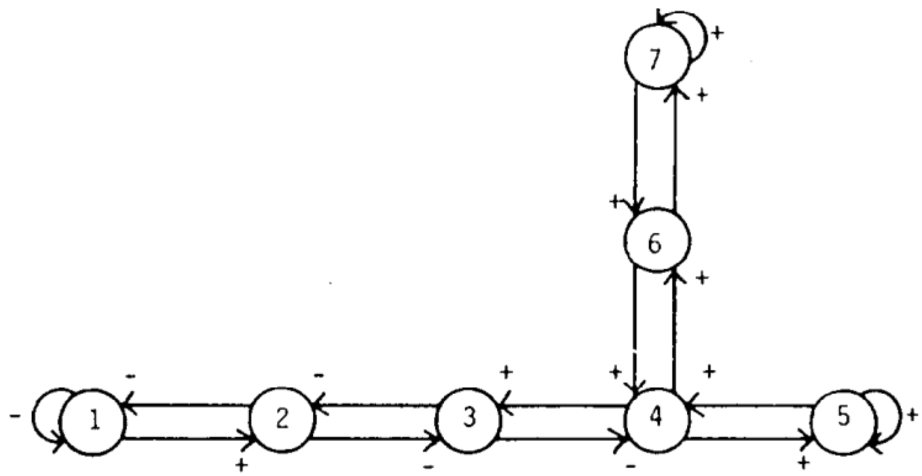


Figure 2.1: An example to illustrate method of proof of Theorem 1. Nodes $\{1,2,3,4,5\}$ are in proper subchain and nodes 4,6,7 are in a second subchain with the end end distinguished

Example 2.1.3. In our example, suppose that $SD(A)$ can be divided into a proper subchain that is not λ-consistent and a second subchain with exactly one distinguished node, denoted as node 7, which serves as the end node (but not the attachment node). Let $\tilde{A}$ and x be specified as described for the proper subchain, which includes nodes $\{1,2,3,4,5\}$, and let the other entries of $\tilde{A}$ be arbitrary in magnitude except for $a_{q+1q}^{\tilde{}} = a_{64}^{\tilde{}}$.

Based on the given signed digraph $SD(A)$, consider the matrix given by:

$$\tilde{A} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & \tilde{a_{64}} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } x = \begin{bmatrix} -2 \\ 1 \\ -2 \\ -2 \\ 1 \\ x_6 \\ x_7 \end{bmatrix}$$

Solving by the algorithm outlined in the theorem's proof:

1. Set $x_7 = 1$ and solve the equation $\tilde{A}x = 0$ at node 7, resulting in $x_6 = -1$.

2. Specify the value of $\tilde{a_{64}}$ by solving the equation at node 6, yielding $\tilde{a_{64}} = 1/2$.

3. Modify the values of $\tilde{A}$ at the attachment node, i.e., node 4, to satisfy the equation at node 4 given by:

$$\tilde{a_{43}}x_3 + \tilde{a_{45}}x_5 + \tilde{a_{46}}x_6 = 0$$

Set $\tilde{a_{43}} = 3/2$, resulting in the final $\tilde{A}$ and x:

$$\tilde{A} = \begin{bmatrix} -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } x = \begin{bmatrix} -2 \\ 1 \\ -2 \\ -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Chapter 3

Strictly Sinusoidal Trajectories

3.1 Strictly Sinusoidal Trajectories

Definition 3.1.1. A strictly sinusoidal trajectory for the equation $\dot{x}(t) = \tilde{A}x$ which satisfies $\ddot{x}_i = -x_i$ and $x_i \neq 0$ for all i.

Lemma 3.1.2. *Suppose A is irreducible, all 2-cycles in $SD(A)$ are positive, and $SD(A)$ contains no k-cycle for $k > 2$. Then there exist $\tilde{A} \in Q(A)$ and sinusoidal x solving $\dot{x} = \tilde{A}x$.*

Proof. When all 2-cycles in $SD(A)$ are positive, it implies that for every pair of nodes i and j in the signed digraph with $i \neq j$, the product of the corresponding entries a_{ij} and a_{ji} in A is positive, i.e., $\lambda_i a_{ij} = \lambda_j a_{ji}$ for some positive constants λ_i and λ_j.

As a result, the matrix formed by scaling the entries of A by $\lambda_i^{1/2}$ and $\lambda_j^{-1/2}$ is symmetric and the symmetric matrix has only real eigenvalues. Since A is diagonally similar to this symmetric matrix, it also has only real eigenvalues.

Furthermore, because A has only real eigenvalues, it cannot support a sinusoidal trajectory since sinusoidal functions have purely imaginary eigenvalues.

Therefore, under the given conditions on $SD(A)$, the existence of $\tilde{A}$ and a sinusoidal trajectory x satisfying $\dot{x} = \tilde{A}x$ is not possible.

$\square$

Lemma 3.1.3. *Suppose A is irreducible and $SD(A)$ has no 1-cycle or k-cycle, $k > 2$, but at least one negative 2-cycle. Then there exist $\tilde{A} \in Q(A)$ and strictly sinusoidal x solving $\dot{x} = \tilde{A}x$.*

Proof. We will prove above lemma in a constructive approach.Therefore, suppose signed digraph $SD(A)$ contains a negative 2-cycle. The question of existence is resolved by showing that it is always possible to attach straight chains to any subsystem with strictly sinusoidal nodes. In fact, only $\pm \sin t$ and $\pm \cos t$ are required as node values and the entries of $\tilde{A}$ are specified and modified as needed. We will illustrate this with an example, attaching a straight chain to a subsystem with node set $\{2, 3, \ldots, p\}$ with strictly sinusoidal node at node 1. Let's say node 1 has the value $\sin t$. The idea is illustrated in Figure 3.1. We tentatively assign node p the value $\cos t$ if p is even, or $\sin t$ if p is odd. Then we'll set $|\tilde{a}_{p,p-1}| = 1$, so that the sign of $\tilde{a}_{p,p-1}$ determines whether node $p-1$ in $\dot{x} = \tilde{A}x$ is $\pm\dot{x_p}$, that is $\pm \sin t$ if p is even, $\pm \cos t$ if p is odd.

Consider the equation at row $p-1$. Specifying either $|\tilde{a}_{p-1,p-2}| = |\tilde{a}_{p-1,p-2}| = 1/2$ or $|\tilde{a}_{p-1,p}| = 1$, $|\tilde{a}_{p-1,p-2}| = 2$, as needed according to edge signs, allows us to keep $x_{p-1} = \pm\dot{x_p}$. This procedure extends down to row 1 of $\dot{x} = Ax$. If there's a sign inconsistency at row 1, we start over with the opposite sign for node p to correct it. If node 1 is $\cos t$, we adjust our strategy accordingly, assigning node p the value $\sin t$ for even p or $\cos t$ for odd p, and finally, we adjust the magnitudes of $\tilde{a}_{1j}$ as necessary. $\square$

Example 3.1.4. Consider an example with matrix A of order 5 that satisfies the property of the lemma. Our goal is to find $\tilde{A} \in Q(A)$ and a strictly sinu-

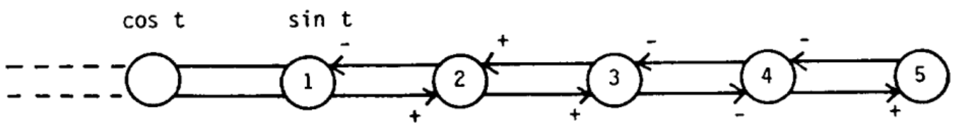


Figure 3.1: An illustration of the idea of the proof of Lemma 2. Attachment of a staright chain with node set $\{2, 3, 4, 5\}$ to subsystem at node 1.

soidal vector x satisfying $\dot{x} = \tilde{A}x$. To achieve this, we employ the algorithm outlined in the proof.

Let's start by attaching a straight chain with node set $\{2, 3, \ldots, 5\}$ to a subsystem with strictly sinusoidal node values at node 1, where node 1 has the value $\sin t$. We represent $\tilde{A}$ as:

$$\tilde{A} = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 & 0 \\ 0 & a_{32} & 0 & a_{34} & 0 \\ 0 & 0 & a_{43} & 0 & a_{45} \\ 0 & 0 & 0 & a_{54} & 0 \end{bmatrix}$$

Solving the system $\dot{x} = \tilde{A}x$, we have:

$$\begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 & 0 \\ 0 & a_{32} & 0 & a_{34} & 0 \\ 0 & 0 & a_{43} & 0 & a_{45} \\ 0 & 0 & 0 & a_{54} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \sin t \end{bmatrix} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \cos t \end{bmatrix}$$

Equating the fifth row equation, we find $a_{54}x_4 = \cos t$. Setting $a_{54} = 1$, we obtain $x_4 = \cos t$.

Solving the row 4 equation yields $a_{43}x_3 + a_{45} \sin t = -\sin t$. Choosing the node value of node 3 is $\sin t$, we choose $a_{43} = -1$ and $a_{45} = -2$ to satisfy the equation.

Similarly, we solve the row 3 equation by equating it to $a_{32}x_2 + a_{34}\cos t = -\cos t$. Choosing the node value of node 2 to be $-\cos t$, we set $a_{32} = 1$ and $a_{34} = 2$ to satisfy the equation.

Continuing, we solve the row 2 equation by setting it equal to $a_{21}x_1 + a_{23}\sin t = \sin t$. We choose the node value of node 1 as $\sin t$, thus we set $a_{21} = 1/2$ and $a_{23} = 1/2$ to satisfy the equation.

Finally, solving the row 1 equation, we have $a_{12}(-\cos t) = \cos t$. To satisfy this equation, we set $a_{12} = -1$.

Hence, after completing all the steps, we obtain:

$$\tilde{A} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } x = \begin{bmatrix} \sin t \\ -\cos t \\ \sin t \\ \cos t \\ \sin t \end{bmatrix}$$

This $\tilde{A}$ and x combination satisfies the equation;

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sin t \\ -\cos t \\ \sin t \\ \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \\ \cos t \\ -\sin t \\ \cos t \end{bmatrix}$$

Theorem 3.1.5. *Suppose A is irreducible and $SD(A)$ has no k-cycle, $k > 2$. If there exists a strictly sinusoidal trajectory x solving $\dot{x} = \tilde{A}x$ for some $\tilde{A} \in Q(A)$ then $SD(A)$ has at least one negative 2-cycle and $SD(A)$ is not λ-consistent.*

Proof. Suppose we have a matrix $\tilde{A}$ that belongs to the qualitative class of A, denoted as $Q(A)$, along with a strictly sinusoidal trajectory x characterized by certain constants λ_j. Let's assume there exists a 1-cycle at node i within the system. Now, consider the quantity Λ, defined as $\Lambda = \sum_{i=1}^{n} \lambda_i x_i^2/2$. for all nodes i.

If the signed digraph $SD(A)$ were to be λ-consistent, then along the trajectory x, the derivative of Λ would be calculated as $\dot{\Lambda} = \sum_{i=1}^{n} \lambda_i \tilde{a_{ii}} x_i^2$, which must be greater than zero. However, this contradicts the periodicity property, where $\Lambda(x(t))$ is equal to $\Lambda(x(t + 2\pi))$. Lemma 1 concludes that $SD(A)$ must indeed contain a negative 2-cycle.

□

Corollary 3.1.6. *If A is irreducible and $SD(A)$ has no k-cycle, $k > 2$, and exactly one 1-cycle, then no $\tilde{A} \in Q(A)$ admits a strictly sinusoidal trajectory.*

Proof. Given these conditions, $SD(A)$ is λ-consistent, as per Theorem 3.1.5, which effectively rules out the existence of a strictly sinusoidal trajectory. □

Chapter 4

Constant Trajectories

4.1 Introduction

In our investigation of solutions to $Ax = 0$, where x is non-zero but some $x_i = 0$, we focus on cases where $n > 2$, excluding the trivial scenario of $n = 1$ where A reduces to the zero matrix. Assuming the irreducibility of A and the absence of cycles longer than 2 in $SD(A)$, if $Ax = 0$ and $x \neq 0$, we can divide the signed digraph $SD(A)$ and the graph $G(A)$ into distinct subgraphs.

To achieve this partitioning, we introduce the concept of a white block, which represents a maximal connected subgraph of nodes in $SD(A)$ corresponding to non-zero components of x. Nodes not included in white blocks are labeled as black and form part of black blocks.

This partitioning scheme can be understood through what we term as a "color test".

4.2 Constant Trajectories and 0-coloring

Definition 4.2.1. A constant trajectory $x(t) \in \mathbb{R}^n$ for $\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j$ satisfies $\dot{x}_i = 0$ and $x_i \neq 0$ for some i.

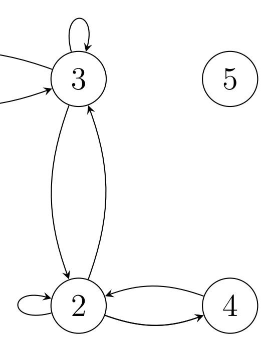
Definition 4.2.2. A 0-coloring is a scheme for coloring all nodes of $SD(A)$ which has no k-cycle, $k > 2$, black or white, so that:

1. no black node is a neighbor of exactly one white node;
2. each maximal white block as a subgraph is either: a single undistinguished node; or a digraph which has at least two nodes, which has each end node distinguished, and which is not λ-consistent.

Example 4.2.3. Consider the matrix

$$\tilde{A} = \begin{bmatrix} 3 & 0 & -3 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ -2 & 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and let } x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

be a solution corresponding to the equation $Ax = 0$. Thus, we color nodes $\{1, 3, 4\}$ white since the corresponding x_i values for nodes $\{1, 3, 4\}$ are non-zero. Conversely, nodes $\{2, 5\}$ are colored black, indicating that the corresponding x_i values for these nodes are zero. Additionally, this coloring satisfies the criteria of 0-coloring, as node 2 is a neighbor of two white nodes, i.e., nodes 3 and 4.



Theorem 4.2.4. *Suppose A is an irreducible matrix of order ≥ 2 and $SD(A)$ contains no k-cycle, $k > 2$. Then there exists $\tilde{A} \in Q(A)$ and a vector $x \neq 0$*

satisfying $\tilde{A}x = 0$ if and only if $SD(A)$ admits a 0-coloring with at least one white node.

Proof. First, we are going to prove in a forward direction i.e, $SD(A)$ admits a 0-coloring with at least one white node. Suppose, $n \geq 2$, and we have a non-zero solution x to the equation $Ax = 0$, with the constraint $x \neq 0$. We proceed by coloring all nodes corresponding to non-zero entries in x as white, while the remaining nodes are colored black.

When all $x_i \neq 0$, Theorem 1 implies condition (ii) is satisfied, as all nodes are white. However, in case where both black and white nodes exist, we need to ensure condition (i) holds true. This condition becomes evident when we examine the equation corresponding to the jth row in $Ax = 0$ for some $x_j = 0$. we see condition (i) must be fulfilled.

Each white block represents a subsystem of equations $\bar{A}\bar{x} = 0$ within $Ax = 0$, adhering to the conditions outlined in Theorem 1. Consequently, every white block must fulfill condition (ii) to maintain consistency with the theorem's requirements.

Now, let's consider the **converse** scenario: assume $SD(A)$ admits a 0-coloring with at least one white node. Here's how we construct a non-zero vector x that satisfies $\tilde{A}x = 0$.

We begin by examining the subsystem of equations $\tilde{A}x = 0$ associated with a white block. According to Theorem 1, such a subsystem has a solution where each component is non-zero. We define the components of the complete vector x accordingly.

Suppose node j is black and connected to a node within this white block. By condition (i), node j must be linked to other white nodes $k_1, k_2, \ldots, k_q$, where $q \geq 1$. We assign arbitrary positive values to $|\tilde{a}_{jk_i}|$ and choose non-zero x_{k_i} values that satisfy the jth row equation in $\tilde{A}x = 0$. Using the idea of Theorem 2.1.2, we extend these x-values throughout their respective white blocks. Given that $G(A)$ forms a tree, we can systematically extend these values to all white nodes and black nodes connected to white nodes.

We assign arbitrary magnitudes to all other entries in $\tilde{A}$ corresponding to edges in $SD(A)$ originating from black nodes, while setting the components of x corresponding to black nodes to zero. This process completes the construction of $\tilde{A}$ and x satisfying $\tilde{A}x = 0$ with $x \neq 0$.

□

Example 4.2.5. Let's consider the scenario where $SD(A)$ allows a 0-coloring with at least one white node. We aim to find a non-zero solution x satisfying $Ax = 0$. We begin by examining the subsystem of equations $Ax = 0$ associated with a white block, comprising nodes $\{1, 3\}$, assuming they form the maximal white block. Additionally, nodes 4 and 6 are also white but are undistinguished nodes. According to Theorem 2.1.2, such a subsystem has a non-zero solution for each component. Let nodes $\{2, 5\}$ be black, and node 2 is linked to white nodes $\{1, 3, 4, 6\}$.

We construct the matrix $\tilde{A}$ as follows:

$$\tilde{A} = \begin{bmatrix} 3 & 0 & -3 & 0 & 0 & 0 \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} & \tilde{a}_{24} & 0 & \tilde{a}_{26} \\ -2 & \tilde{a}_{32} & 2 & 0 & 0 & 0 \\ 0 & \tilde{a}_{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{a}_{62} & 0 & 0 & 0 & 0 \end{bmatrix}$$

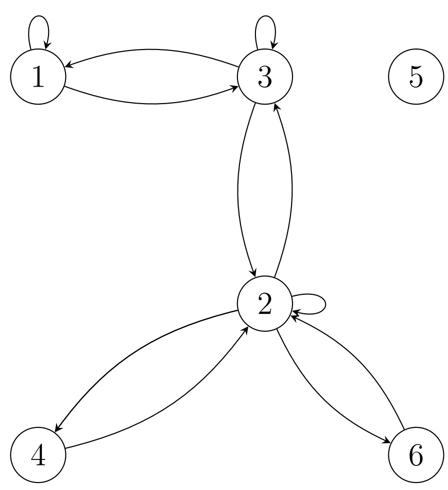
And let x be:

$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ x_4 \\ 0 \\ x_6 \end{bmatrix}$$

Node 4 is connected to other white nodes 4 and 6. We choose arbitrary positive values for $|\tilde{a}_{24}|$ and $|\tilde{a}_{24}|$ and determine x_4 and x_6 so that they satisfy the 2nd row equation in $\tilde{A}x = 0$. We assign arbitrary magnitudes to all other entries in $\tilde{A}$ corresponding to edges originating from black nodes, while setting the components of x corresponding to black nodes to zero. This process completes the construction of $\tilde{A}$ and x, ensuring $\tilde{A}x = 0$ with $x \not\equiv 0$. Hence, $\tilde{A}$ and x using the algorithm of the proof are given by:

$$\tilde{A} = \begin{bmatrix} 3 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 & 1 \\ -2 & 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

Its associated signed digraph is given by;



Undirected Block graph B(A)

To explore the presence of multiple eigenvalues in matrix A, we introduce the concept of an undirected block graph denoted as $B(A)$. We begin by assuming the existence of a nontrivial 0-coloring scheme for $SD(A)$. In this scheme, we remove from $SD(A)$ all black nodes that are not connected to any white nodes, along with any edges connected to or from these isolated black nodes.

The nodes of $B(A)$ are then composed of the remaining black nodes denoted as $b_1, b_2, \ldots$, along with the maximal white blocks denoted as $w_1, w_2, \ldots$. An edge (b_i, w_j) belongs to $B(A)$ if and only if some node in b_i is connected by a 2-cycle to some node in w_j. Now, we define the concept of branching within $B(A)$.

Definition 4.2.6. $B(A)$ is branched at a black node if that black node in $B(A)$ is connected to more than two white nodes

Theorem 4.2.7. *Suppose A is irreducible and $SD(A)$ contains no k-cycle, $k > 2$. Then 0 is an eigenvalue in at least two Jordan blocks of some $\tilde{A} \in Q(A)$ if and only if $SD(A)$ admits a 0-coloring for which $B(A)$ is branched at a black node.*

Proof. Suppose we have a matrix $\tilde{A}$ belonging to the set $Q(A)$, and 0 appears as an eigenvalue in two or more Jordan blocks of A. In such a case, there must exist two linearly independent solutions x and y to the equations $\tilde{A}x = \tilde{A}y = 0$.This is because the Jordan Canonical Form (JCF) of a matrix $\tilde{A}$ is a block-diagonal matrix consisting of Jordan blocks, where each Jordan block corresponds to an eigenvalue of A. If 0 is an eigenvalue with more than one Jordan block, it implies that there are multiple linearly independent eigenvectors associated with the eigenvalue 0, each corresponding to a different Jordan block.

We select x in such a way that the number of components with $x_i = 0$ is maximal. If the 0-colorings associated with x and y are identical, then it follows that $x_i = 0$ if and only if $y_i = 0$. By rearranging and renumbering the components, we can arrange for x_i and y_i to be equal and non-zero.

Consequently, the 0-coloring associated with $x - y$ would have more zero components than x, leading to a contradiction. Therefore, we can assume, without loss of generality, that the 0-coloring associated with x contains the minimal number of white nodes, and that the 0-colorings associated with x and y are distinct.

Consider a scenario where we have a white block in the 0-coloring for vector x, denoted as $\bar{x}$, with associated submatrix $\bar{A}$. Let's suppose that this white block is attached at node i to exactly one black node, denoted as node j. Such a configuration must always exist.

Now, let's introduce another vector y with its corresponding 0-coloring. If node j is white in the 0-coloring for y, then we can express $\bar{A}\bar{y} + \xi = 0$, where ξ is a vector with only one non-zero entry corresponding to the attachment of node i to the white node j.

Since any vector in the kernel of $\bar{A}$ must be proportional to $\bar{x}$, we can infer that $\bar{y_k} = \alpha \bar{x_k}$ for all nodes in the white block, including node i. Upon considering the ith row equation, we find $\xi_i = 0$, leading to a contradiction. This contradiction indicates that node j must be black in the 0-coloring for y.

By leveraging the tree structure of $G(A)$, this reasoning extends to all nodes that are black in the 0-coloring for x and are attached to white nodes. It demonstrates that no white block of the 0-coloring for y can entirely contain a white block of the 0-coloring for x.

Consequently, if the two colorings differ, it implies that some white block and its adjacent black nodes from the coloring for x lie completely within a black block of the coloring for y.

By coloring a node of $SD(A)$ white if it's white in either the x or the y 0-coloring, and black otherwise, we effectively create a 0-coloring for $SD(A)$. The associated block graph $B(A)$ resulting from this coloring is branched.

For the converse part, let's consider a 0-coloring for $SD(A)$ exists, and the associated block graph $B(A)$ is branched. Using the reasoning from the first part of Theorem 3, we can construct a vector y and a modified matrix $\tilde{A}$ such that $\tilde{A}y = 0$, ensuring that $y_i \neq 0$ if and only if node i is white in the 0-coloring.

This implies that the components of y corresponding to black nodes are zero, while the edge values from black nodes can take arbitrary values. Within a branched component of $B(A)$, there exists a straight path with white block end nodes. By recoloring all nodes of $SD(A)$ not in this straight path to black, we achieve a distinct 0-coloring.

Using Theorem 4.2.3 once more, we can construct a new vector x (with the same modified matrix $\tilde{A}$) that is not proportional to y, but still satisfies $\tilde{A}x = 0$. This process demonstrates how, starting from a branched block graph $B(A)$, we can iteratively generate distinct solutions to the system of equations $\tilde{A}x = 0$ with unique 0-colorings.

$\square$

Chapter 5

Sinusoidal Trajectories

Definition 5.0.1. A sinusoidal trajectory for our equation satisfies $\ddot{x}_i = -x_i$ and $x_i \not\equiv 0$ ("not the constant function with the value zero") for some i.

Definition 5.0.2. A Im-coloring is a scheme for coloring all nodes of $SD(A)$ which has no k-cycle, $k > 2$, black or white, so that:

1. no black node is a neighbor of exactly one white node;
2. each maximal white block as a subgraph contains at least one negative 2-cycle and is not λ-consistent.

5.1 Sinusoidal Trajectories and Im-coloring

Theorem 5.1.1. *Suppose A is an irreducible matrix of order > 2, and $SD(A)$ has no k-cycles, $k > 2$. If there exists a sinusoidal trajectory for $\dot{x} = \tilde{A}x$, $x \neq 0$, for some $A \in Q(A)$ then $SD(A)$ admits an Im-coloring with at least one white node.*

Proof. If $\dot{x} = \tilde{A}x$ is a sinusoidal trajectory (x≡0), color node i white if $x_i \equiv 0$; otherwise color node i black. Theorem 3.1.5 together with a line of reasoning parallel to that in the first part of the proof of theorem 4.2.3; it becomes evident that such a coloring constitutes a nontrivial Im-coloring. $\square$

Theorem 5.1.2. *Suppose A is irreducible and $SD(A)$ contains no k-cycle, $k > 2$. Then ι is an eigenvalue in at least two Jordan blocks of some $A \in Q(A)$ if and only if $SD(A)$ admits an Im-coloring for which $B(A)$ is branched at a black node.*

Proof. The proof is completely similar to the proof of Theorem 4.2.6(using Theorem 5.1.1)and is omitted. $\square$

Bibliography

[1] Clark Jeffries, Victor Klee, and Pauline van den Driessche. Qualitative stability of linear systems. *Linear Algebra and its Applications*, 87:1–48, 1987.

[2] Clark Jeffries and P. van den Driessche. Eigenvalues of matrices with tree graphs. *Linear Algebra and its Applications*, 101:109–120, 1988.