Seminar Report

Hall's Theorem

Report

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Graph

Definition

A graph is an ordered triple $G = (V(G), E(G), I_G)$, where:- $V(G)$ is a nonempty set.- $E(G)$ is a set disjoint from $V(G)$. - I_G is an "incidence" relation that associates with each element of $E(G)$ an unordered pair of elements (same or distinct) of $V(G)$.

Elements of $V(G)$ are called the vertices (or nodes or points) of G, and elements of $E(G)$ are called the edges (or lines) of G. $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively.

If, for the edge e of G, $I_G(e) = \{u, v\}$, we write $I_G(e) = uv$.

Example:

If $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, and I_G is given by: - $I_G(e_1) = \{v_1, v_5\}$ - $I_G(e_2) = \{v_2, v_3\}$ - $I_G(e_3) = \{v_2, v_4\}$ - $I_G(e_4) = \{v_2, v_5\}$ - $I_G(e_5) = \{v_2, v_5\}$ - $I_G(e_6) = \{v_3, v_3\}$

Then $(V(G), E(G), I_G)$ is a graph.

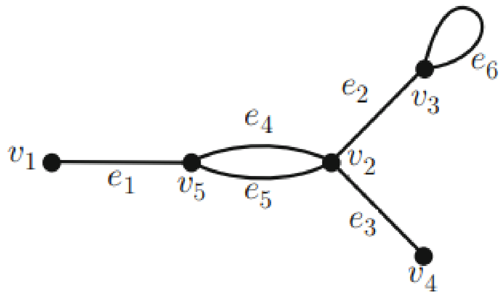

Figure 1: Graph $(V(G), E(G), I_G)$ described in Example

Some Definitions Related to Graph

1. A subset M of the edge set E of a loopless graph G is called independent if no two edges of M are adjacent in G.

2. A matching in G is a set of independent edges.

3. An edge covering of G is a subset L of E such that every vertex of G is incident to some edge of L .Hence, an edge covering of G exists if and only if $\gamma > 0$.

4. A matching M of G is maximum if G has no matching M_0 with $|M_0| > |M|$. M is maximal if G has no matching M_0 strictly containing M. $|M^*(G)|$ is the cardinality of a maximum matching, and $|C^*(G)|$ is the size of a minimum edge covering of G.

5. A set S of vertices of G is said to be saturated by a matching M of G or M-saturated if every vertex of S is incident to some edge of M. A vertex v of G is M-saturated if fv is M-saturated. v is M-unsaturated if it is not M-saturated.

Bipartite Graph

Definition

A graph is bipartite if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a bipartition of the bipartite graph. The bipartite graph G with bipartition (X, Y) is denoted by $G[X, Y]$.

Example:

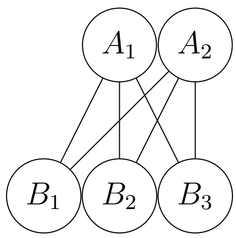

Figure 2: A bipartite graph $K_{2,3}$

Matching in Bipartite Graph

Assignment Problem

Suppose in a factory there are n jobs $j_1, j_2, \ldots, j_n$ and s workers $w_1, w_2, \ldots, w_s$. Also, suppose that each job j_i can be performed by a certain number of workers, and each worker w_j has been trained to do a certain number of jobs. Is it possible to assign each of the n jobs to a worker who can do that job so that no two jobs are assigned to the same worker?

We convert this job assignment problem into a problem in graphs as follows: Form a bipartite graph G with bipartition J, W, where $J = \{j_1, j_2, \ldots, j_n\}$ and $W = \{w_1, w_2, \ldots, w_s\}$, and make j_i adjacent to w_j if and only if worker w_j can do the job j_i. Then our assignment problem translates into the following graph problem: Is it possible to find a matching in G that saturates all the vertices of J?

A solution to the above matching problem in bipartite graphs has been given by Hall.

For a subset $S \subseteq V$ in a graph G, $N(S)$ denotes the neighbor set of S, that is, the set of all vertices, each of which is adjacent to at least one vertex in S.

Definition

Let G be a bipartite graph on the parts X and Y, and let S be a matching of G. If every vertex in X is covered by an edge of S, then we say that S is a perfect matching of X into Y.

For a graph G and a subset T of $V(G)$, we let $N_G(T)$ denote the set of vertices of G that are adjacent to some vertex in T, that is,

$$N_G(T) := \{v \in V(G) \mid vw \in E(G) \text{ for some } w \in T\}.$$

Observe that if G is bipartite on the parts A and B, then $N_G(T) \subseteq B$ for any $T \subseteq A$.

Hall's Theorem

For a bipartite graph G on the parts X and Y, the following conditions are equivalent.

(a) There is a perfect matching of X into Y.

(b) For each $T \subseteq X$, the inequality $|T| \leq |N_G(T)|$ holds.

Proof:

(a) $\Rightarrow$ (b): Let S be a perfect matching of X into Y. As S is a perfect matching, for every $x \in X$, there exists a unique $y_x \in Y$ such that $xy_x \in S$. Define the map $f : X \to Y$ by $f(x) = y_x$. Since S is a matching, the function f is injective. Therefore, for any $T \subseteq X$, we see that $|T| = |f(T)| \leq |N_G(T)|$ because $f(T) \subseteq N_G(T)$.

(b) $\Rightarrow$ (a): Conversely, suppose that $|T| \leq |N_G(T)|$ for each $T \subseteq X$. We will prove that there exists a perfect matching of X into Y by induction on $n := |X|$. If $n = 1$, then the only vertex x in X must be adjacent to some vertex y in Y by condition (b), and, therefore, $\{xy\}$ is a perfect matching of X into Y. Now assume that every bipartite graph on the parts X_0 and Y_0 with $|X_0| < |X|$ and satisfying condition (b) has a perfect matching of X_0 into Y_0. We split the rest of the proof into two cases.

Case 1: For every nonempty proper subset T of X (that is, $T \subset X$), the strict inequality $|T| < |N_G(T)|$ holds. Take $x \in X$ and $y \in$NG($\{x\}$). Let G_0 be the bipartite graph we obtain by removing x and y (and the edges incident to them) from G. Now for every subset A of $X \setminus \{x\}$, we see that

$$|N_{G_0}(A)| \geq |N_G(A)| - 1 \geq |A|,$$

where the last inequality holds because A is a strict subset of X. By induction hypothesis, there exists a perfect matching S_0 in G_0 of $X \setminus \{x\}$ into $Y \setminus \{y\}$. It is clear now that $S_0 \cup \{xy\}$ is a perfect matching in G of X into Y.

Case 2: There exists a nonempty proper subset A of X such that $|A| = |N_G(A)|$. Let G_1 be the subgraph of G induced by the set of vertices $A \cup N_G(A)$, and let G_2 be the subgraph of G we obtain by removing $A \cup N_G(A)$ (and their incident edges) from G. It is clear that $G_1 = (A, N_G(A))$ and $G_2 = (X \setminus A, Y \setminus N_G(A))$ are bipartite graphs.

Let us show that both G_1 and G_2 satisfy condition (b). To show that G_1 satisfies (b), take $T \subseteq A$. It follows by the way G_1 was constructed that $N_{G_1}(T) = N_G(T)$. As a result, $|N_{G_1}(T)| = |N_G(T)| \geq |T|$. Then G_1 satisfies condition (b). In order to argue that G_2 also satisfies condition (b), take $T_0 \subseteq X \setminus A$ and observe that $N_{G_2}(T_0 \cup A) = N_G(A) \cup N_{G_2}(T_0)$, where the union on the right-hand side is disjoint. Since $|N_{G_2}(T_0 \cup A)| \geq |T_0 \cup A|$ and $|N_G(A)| = |A|$, $|N_{G_2}(T_0)| = |N_{G_2}(T_0 \cup A)| - |N_{G_1}(A)| \geq |T_0 \cup A| - |A| = (|T_0| + |A|) - |A| = |T_0|$. Therefore, G_2 also satisfies condition (b).

Since $|A| < |X|$ and $|X \setminus A| < |X|$, our induction hypothesis guarantees the existence of a perfect matching S_1 in G_1 of A into $N_G(A)$ and a perfect matching S_2 in G_2 of $X \setminus A$ into $Y \setminus N_G(A)$. Then it follows from the construction of G_1 and G_2 that $S_1 \cup S_2$ is a perfect matching in G of X into Y, which concludes the proof.

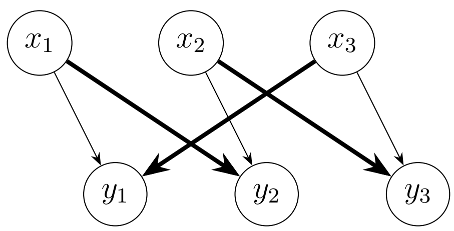

Figure 3: Illustration of a matching with boldfaced edges.

We now give some important consequences of Hall's theorem;

Theorem

A $k(>1)$-regular bipartite graph is 1-factorable

Proof:

Let G be a k-regular bipartite graph with bipartition X and Y. Then, $E(G)$ is the set of edges incident to the vertices of X and is also equal to the set of edges incident to the vertices of Y. Hence, $k|X| = |E(G)| = k|Y|$, and therefore $|X| = |Y|$.

Now, consider any subset $S \subseteq X$. By the definition of a bipartite graph, the neighborhood of S, denoted as $N(S)$, is contained in Y, and $N(N(S))$ contains S.

Let E_1 and E_2 be the sets of edges of G incident to S and $N(S)$, respectively. Then, $E_1 \cup E_2$ contains all the edges of G incident to vertices in S. We have $|E_1| = k|S|$ and $|E_2| = k|N(S)|$. Therefore, since $|E_1| + |E_2| = |E_1 \cup E_2| \leq k|S|$, it follows that $k|N(S)| \leq k|S|$.

As $k > 1$, we can conclude that $|N(S)| \leq |S|$. So, by Hall's theorem (Theorem 5.5.2), G has a matching that saturates all the vertices of X, which means that G has a perfect matching M.

Deletion of the edges in M from G results in a $(k-1)$-regular bipartite graph.

Repeated application of this argument shows that G is 1-factorable.

Bibliography

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