

*Seminar Report*

# Hall's Theorem

*Report*

*Submitted by*

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# Graph

## Definition

A graph is an ordered triple  $G = (V(G), E(G), I_G)$ , where:-  $V(G)$  is a nonempty set.-  $E(G)$  is a set disjoint from  $V(G)$ . -  $I_G$  is an "incidence" relation that associates with each element of  $E(G)$  an unordered pair of elements (same or distinct) of  $V(G)$ .

Elements of  $V(G)$  are called the vertices (or nodes or points) of  $G$ , and elements of  $E(G)$  are called the edges (or lines) of  $G$ .  $V(G)$  and  $E(G)$  are the vertex set and edge set of  $G$ , respectively.

If, for the edge  $e$  of  $G$ ,  $I_G(e) = \{u, v\}$ , we write  $I_G(e) = uv$ .

## Example:

If  $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ ,  $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ , and  $I_G$  is given by: -  $I_G(e_1) = \{v_1, v_5\}$  -  $I_G(e_2) = \{v_2, v_3\}$  -  $I_G(e_3) = \{v_2, v_4\}$  -  $I_G(e_4) = \{v_2, v_5\}$  -  $I_G(e_5) = \{v_2, v_5\}$  -  $I_G(e_6) = \{v_3, v_3\}$

Then  $(V(G), E(G), I_G)$  is a graph.

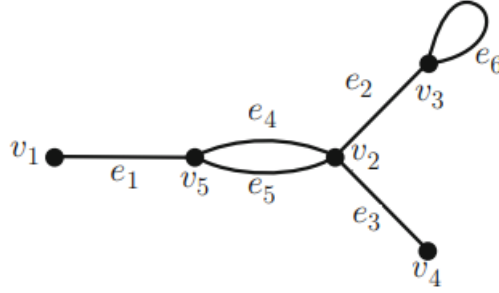


Figure 1: Graph  $(V(G), E(G), I_G)$  described in Example

## Some Definitions Related to Graph

1. A subset  $M$  of the edge set  $E$  of a loopless graph  $G$  is called independent if no two edges of  $M$  are adjacent in  $G$ .
2. A matching in  $G$  is a set of independent edges.
3. An edge covering of  $G$  is a subset  $L$  of  $E$  such that every vertex of  $G$  is incident to some edge of  $L$ . Hence, an edge covering of  $G$  exists if and only if  $\gamma > 0$ .
4. A matching  $M$  of  $G$  is maximum if  $G$  has no matching  $M_0$  with  $|M_0| > |M|$ .  $M$  is maximal if  $G$  has no matching  $M_0$  strictly containing  $M$ .  $|M^*(G)|$  is the cardinality of a maximum matching, and  $|C^*(G)|$  is the size of a minimum edge covering of  $G$ .

5. A set  $S$  of vertices of  $G$  is said to be saturated by a matching  $M$  of  $G$  or  $M$ -saturated if every vertex of  $S$  is incident to some edge of  $M$ . A vertex  $v$  of  $G$  is  $M$ -saturated if  $fv$  is  $M$ -saturated.  $v$  is  $M$ -unsaturated if it is not  $M$ -saturated.

## Bipartite Graph

### Definition

A graph is bipartite if its vertex set can be partitioned into two nonempty subsets  $X$  and  $Y$  such that each edge of  $G$  has one end in  $X$  and the other in  $Y$ . The pair  $(X, Y)$  is called a bipartition of the bipartite graph. The bipartite graph  $G$  with bipartition  $(X, Y)$  is denoted by  $G[X, Y]$ .

### Example:

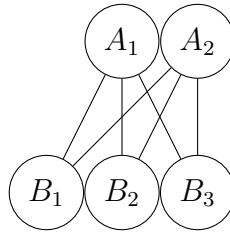


Figure 2: A bipartite graph  $K_{2,3}$

## Matching in Bipartite Graph

### Assignment Problem

Suppose in a factory there are  $n$  jobs  $j_1, j_2, \dots, j_n$  and  $s$  workers  $w_1, w_2, \dots, w_s$ . Also, suppose that each job  $j_i$  can be performed by a certain number of workers, and each worker  $w_j$  has been trained to do a certain number of jobs. Is it possible to assign each of the  $n$  jobs to a worker who can do that job so that no two jobs are assigned to the same worker?

We convert this job assignment problem into a problem in graphs as follows: Form a bipartite graph  $G$  with bipartition  $J, W$ , where  $J = \{j_1, j_2, \dots, j_n\}$  and  $W = \{w_1, w_2, \dots, w_s\}$ , and make  $j_i$  adjacent to  $w_j$  if and only if worker  $w_j$  can do the job  $j_i$ . Then our assignment problem translates into the following graph problem: Is it possible to find a matching in  $G$  that saturates all the vertices of  $J$ ?

A solution to the above matching problem in bipartite graphs has been given by Hall.

For a subset  $S \subseteq V$  in a graph  $G$ ,  $N(S)$  denotes the neighbor set of  $S$ , that is, the set of all vertices, each of which is adjacent to at least one vertex in  $S$ .

## Definition

Let  $G$  be a bipartite graph on the parts  $X$  and  $Y$ , and let  $S$  be a matching of  $G$ . If every vertex in  $X$  is covered by an edge of  $S$ , then we say that  $S$  is a perfect matching of  $X$  into  $Y$ .

For a graph  $G$  and a subset  $T$  of  $V(G)$ , we let  $N_G(T)$  denote the set of vertices of  $G$  that are adjacent to some vertex in  $T$ , that is,

$$N_G(T) := \{v \in V(G) \mid vw \in E(G) \text{ for some } w \in T\}.$$

Observe that if  $G$  is bipartite on the parts  $A$  and  $B$ , then  $N_G(T) \subseteq B$  for any  $T \subseteq A$ .

## Hall's Theorem

*For a bipartite graph  $G$  on the parts  $X$  and  $Y$ , the following conditions are equivalent.*

- (a) *There is a perfect matching of  $X$  into  $Y$ .*
- (b) *For each  $T \subseteq X$ , the inequality  $|T| \leq |N_G(T)|$  holds.*

### Proof:

(a)  $\Rightarrow$  (b): Let  $S$  be a perfect matching of  $X$  into  $Y$ . As  $S$  is a perfect matching, for every  $x \in X$ , there exists a unique  $y_x \in Y$  such that  $xy_x \in S$ . Define the map  $f : X \rightarrow Y$  by  $f(x) = y_x$ . Since  $S$  is a matching, the function  $f$  is injective. Therefore, for any  $T \subseteq X$ , we see that  $|T| = |f(T)| \leq |N_G(T)|$  because  $f(T) \subseteq N_G(T)$ .

(b)  $\Rightarrow$  (a): Conversely, suppose that  $|T| \leq |N_G(T)|$  for each  $T \subseteq X$ . We will prove that there exists a perfect matching of  $X$  into  $Y$  by induction on  $n := |X|$ . If  $n = 1$ , then the only vertex  $x$  in  $X$  must be adjacent to some vertex  $y$  in  $Y$  by condition (b), and, therefore,  $\{xy\}$  is a perfect matching of  $X$  into  $Y$ . Now assume that every bipartite graph on the parts  $X_0$  and  $Y_0$  with  $|X_0| < |X|$  and satisfying condition (b) has a perfect matching of  $X_0$  into  $Y_0$ . We split the rest of the proof into two cases.

**Case 1:** For every nonempty proper subset  $T$  of  $X$  (that is,  $T \subset X$ ), the strict inequality  $|T| < |N_G(T)|$  holds. Take  $x \in X$  and  $y \in N_G(\{x\})$ . Let  $G_0$  be the bipartite graph we obtain by removing  $x$  and  $y$  (and the edges incident to them) from  $G$ . Now for every subset  $A$  of  $X \setminus \{x\}$ , we see that

$$|N_{G_0}(A)| \geq |N_G(A)| - 1 \geq |A|,$$

where the last inequality holds because  $A$  is a strict subset of  $X$ . By induction hypothesis, there exists a perfect matching  $S_0$  in  $G_0$  of  $X \setminus \{x\}$  into  $Y \setminus \{y\}$ . It is clear now that  $S_0 \cup \{xy\}$  is a perfect matching in  $G$  of  $X$  into  $Y$ .

**Case 2:** There exists a nonempty proper subset  $A$  of  $X$  such that  $|A| = |N_G(A)|$ . Let  $G_1$  be the subgraph of  $G$  induced by the set of vertices  $A \cup N_G(A)$ ,

and let  $G_2$  be the subgraph of  $G$  we obtain by removing  $A \cup N_G(A)$  (and their incident edges) from  $G$ . It is clear that  $G_1 = (A, N_G(A))$  and  $G_2 = (X \setminus A, Y \setminus N_G(A))$  are bipartite graphs.

Let us show that both  $G_1$  and  $G_2$  satisfy condition (b). To show that  $G_1$  satisfies (b), take  $T \subseteq A$ . It follows by the way  $G_1$  was constructed that  $N_{G_1}(T) = N_G(T)$ . As a result,  $|N_{G_1}(T)| = |N_G(T)| \geq |T|$ . Then  $G_1$  satisfies condition (b). In order to argue that  $G_2$  also satisfies condition (b), take  $T_0 \subseteq X \setminus A$  and observe that  $N_{G_2}(T_0 \cup A) = N_G(A) \cup N_{G_2}(T_0)$ , where the union on the right-hand side is disjoint. Since  $|N_{G_2}(T_0 \cup A)| \geq |T_0 \cup A|$  and  $|N_G(A)| = |A|$ ,  $|N_{G_2}(T_0)| = |N_{G_2}(T_0 \cup A)| - |N_G(A)| \geq |T_0 \cup A| - |A| = (|T_0| + |A|) - |A| = |T_0|$ . Therefore,  $G_2$  also satisfies condition (b).

Since  $|A| < |X|$  and  $|X \setminus A| < |X|$ , our induction hypothesis guarantees the existence of a perfect matching  $S_1$  in  $G_1$  of  $A$  into  $N_G(A)$  and a perfect matching  $S_2$  in  $G_2$  of  $X \setminus A$  into  $Y \setminus N_G(A)$ . Then it follows from the construction of  $G_1$  and  $G_2$  that  $S_1 \cup S_2$  is a perfect matching in  $G$  of  $X$  into  $Y$ , which concludes the proof.

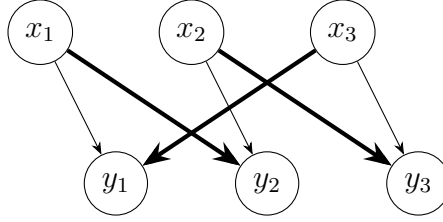


Figure 3: Illustration of a matching with boldfaced edges.

**We now give some important consequences of Hall's theorem;**

## Theorem

*A  $k(> 1)$ -regular bipartite graph is 1-factorable*

## Proof:

Let  $G$  be a  $k$ -regular bipartite graph with bipartition  $X$  and  $Y$ . Then,  $E(G)$  is the set of edges incident to the vertices of  $X$  and is also equal to the set of edges incident to the vertices of  $Y$ . Hence,  $k|X| = |E(G)| = k|Y|$ , and therefore  $|X| = |Y|$ .

Now, consider any subset  $S \subseteq X$ . By the definition of a bipartite graph, the neighborhood of  $S$ , denoted as  $N(S)$ , is contained in  $Y$ , and  $N(N(S))$  contains  $S$ .

Let  $E_1$  and  $E_2$  be the sets of edges of  $G$  incident to  $S$  and  $N(S)$ , respectively. Then,  $E_1 \cup E_2$  contains all the edges of  $G$  incident to vertices in  $S$ . We have  $|E_1| = k|S|$  and  $|E_2| = k|N(S)|$ . Therefore, since  $|E_1| + |E_2| = |E_1 \cup E_2| \leq k|S|$ , it follows that  $k|N(S)| \leq k|S|$ .

As  $k > 1$ , we can conclude that  $|N(S)| \leq |S|$ . So, by Hall's theorem (Theorem 5.5.2),  $G$  has a matching that saturates all the vertices of  $X$ , which means that  $G$  has a perfect matching  $M$ .

Deletion of the edges in  $M$  from  $G$  results in a  $(k - 1)$ -regular bipartite graph.

Repeated application of this argument shows that  $G$  is 1-factorable.

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# Bibliography

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