JEE-ADVANCED 2021 P1 Q4

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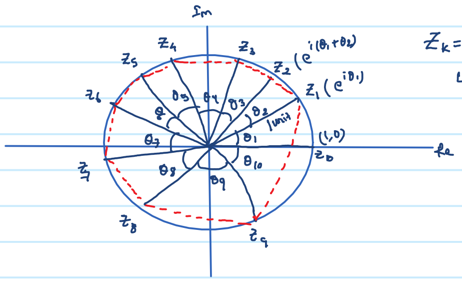
Sol :- Given;

P: $|z_2-z_1| + |z_3-z_2| + |z_4-z_3| + \cdots + |z_{10}-z_9| + |z_1-z_{10}| \le 2\pi$

Q: $|z_2^2-z_1^2| + |z_3^2-z_2^2| + |z_4^2-z_3^2| + \cdots + |z_{10}^2-z_9^2| + |z_1^2-z_{10}^2| \le 4\pi$



$Z_k = Z_{k-1}e^{i\theta_k}$

$\hookrightarrow z_2 = z_1 e^{i\theta_1}$

$= e^{i(\theta_1+\theta_2)}$

Clearly, $|z_k - z_{k-1}|$ is the length of the chord (---)

So, P: $\Sigma|z_k - z_{k-1}|$ is the sum of the lengths of all chords.

which is less than the circumference of the circle. (i.e, $2\pi(1) = 2\pi$.)

So, P: $|z_1-z_2| + |z_3-z_2| + \cdots + |z_{10}-z_9| + |z_1-z_{10}| \le 2\pi$.

But, observe that,

$$\lim_{n\to\infty}\sum_{r=1}^{n}|z_r - z_{r+1}| = 2\pi$$

Now,

Q: $|z_2^2-z_1^2| + |z_3^2-z_2^2| + \cdots + |z_{10}^2-z_9^2| + |z_1^2-z_{10}^2| \le 4\pi$.

$\Rightarrow$ So, insted of proving or disproving the statement Q, we show that the maximum value of $\Sigma|z_r^2 - z_{r-1}^2| = 4\pi$. (for $n\to\infty$) and, statement Q would be deduced from it, automatically, (for finite terms).

We have, $P_{max}: \lim_{n\to\infty}\sum_{r=2}^{n}|z_r^2 - z_{r-1}^2|$

$\Rightarrow P_{max}: \lim_{n\to\infty}\sum_{r=2}^{n}|(z_r - z_{r-1})(z_r + z_{r-1})|$

$\Rightarrow P_{max}: \lim_{n\to\infty}\sum_{r=2}^{n}|z_r - z_{r-1}||z_r + z_{r-1}|$

Now, as $n\to\infty$ $\theta_r\to 0$. So, $z_r \cong z_{r-1}$

$\therefore z_r + z_{r-1} \approx 2z_r$

$\Rightarrow |z_r + z_{r-1}| \approx 2$ (radius=1)

$P_{max}: 2\lim_{n\to\infty}\sum_{r=2}^{n}|z_r - z_{r-1}|$

$P_{max}: 2(2\pi) = 4\pi$

So, $P \le P_{max}$.

Hence, (c) is Correct.