Unit-5

Chapter 14

Combinatories

1 Counting Techniques

1.1 Permutation

different arrangement which can be made out of a given number of things by taking some or all at a , are called ***Permutations***.

example, consider arranging the digits 1, 2 and 3. The possible arrangements as follows:

123, 132, 231, 213, 312, 321.

se are 6 permutations. Here the same three digits 1, 2 and 3 have been used but the number changes n the order of digits is changed.

s, forming numbers with given digits means arranging the digits and hence it is the problem of mutation.

Notations

r and n be positive integers such that $1 \le r \le n$.

n, the number of all permutations of n things taking r at a time is denoted by $P(n, r)$ or ^nP_r

n, the number of all permutations of n dissimilar things taken r at a time is given by

eorem 1: *Let $1 \le r \le n$. Then the number of all permutations of n dissimilar things taken r at a time is given by*

$$^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{\lfloor n}{\lfloor n-r}.$$

heorem 2: *The number of all permutations of n different things taken all at a time is given by* $^nP_n = \lfloor n$.

roof: We have, $^nP_r = \frac{\lfloor n}{\lfloor n-r}$ Put $r = n$, we find $^nP_n = \frac{\lfloor n}{\lfloor n-n} = \frac{\lfloor n}{\lfloor 0} = \frac{\lfloor n}{1} = \lfloor n.$

Theorem 3: *Prove that* $\lfloor 0 = 1$.

Proof: We have,

$$^nP_r = \frac{\lfloor n}{\lfloor n-r} \qquad \text{[put } n = r]$$

$$\Rightarrow \quad ^nP_n = \frac{\lfloor n}{\lfloor 0} \qquad (^nP_n = \lfloor n)$$

$$\Rightarrow \quad \lfloor n = \frac{\lfloor n}{\lfloor 0} \quad \Rightarrow \quad \lfloor 0 = \frac{\lfloor n}{\lfloor n} = 1.$$

Example 1: *Evaluate the following:*

(i) ${}^{12}P_4$ (ii) ${}^{6}P_4$ (iii) ${}^{8}P_8$ (iv) ${}^{30}P_2$

Solution: (i) ${}^{12}P_4 = \dfrac{\lfloor\underline{12}}{\lfloor\underline{12-4}} = \dfrac{\lfloor\underline{12}}{\lfloor\underline{8}} = \dfrac{12 \times 11 \times 10 \times 9 \times \lfloor\underline{8}}{\lfloor\underline{8}} = 11{,}880$

(ii) ${}^{6}P_4 = \dfrac{\lfloor\underline{6}}{\lfloor\underline{6-4}} = \dfrac{\lfloor\underline{6}}{\lfloor\underline{2}} = \dfrac{6 \times 5 \times 4 \times 3 \times 2}{2} = 360$ or ${}^{6}P_4 = 6 \times 5 \times 4 \times 3 = 360$

(iii) ${}^{8}P_8 = \lfloor\underline{8} = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$

(iv) ${}^{30}P_2 = \dfrac{\lfloor\underline{30}}{\lfloor\underline{30-2}} = \dfrac{\lfloor\underline{30}}{\lfloor\underline{28}} = \dfrac{30 \times 29 \times \lfloor\underline{28}}{\lfloor\underline{28}} = 870$

Example 2: *If* ${}^{20}P_r = 6{,}840$, *find r.*

Solution: We have, ${}^{20}P_r = 6{,}840$

$$\Rightarrow \quad \frac{\lfloor\underline{20}}{\lfloor\underline{20-r}} = 6{,}840$$

$$\Rightarrow \quad \frac{\lfloor\underline{20}}{\lfloor\underline{20-r}} = \frac{20 \times 19 \times 18 \times \lfloor\underline{17}}{\lfloor\underline{17}}$$

$$\Rightarrow \quad \frac{\lfloor\underline{20}}{\lfloor\underline{20-r}} = \frac{\lfloor\underline{20}}{\lfloor\underline{17}} \quad \Rightarrow \quad 20 - r = 17 \text{ or } r = 3$$

Example 3: *If* ${}^{n}P_4 = 20 \times {}^{n}P_2$, *find n.*

Solution: We have, ${}^{n}P_4 = 20 \times {}^{n}P_2$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 20n(n-1)$$

$$(n-2)(n-3) = 20 \text{ or } n^2 - 5n - 14 = 0 \text{ or } (n-7)(n+2) = 0 \text{ or } n = 7 \quad (\text{as } n \neq -2)$$

Example 4: ${}^{(n+5)}P_{n+1} = \dfrac{11(n-1)}{2} \times {}^{(n+3)}P_n$, *find n.*

Solution: We have,

$${}^{n+5}P_{n+1} = \frac{11(n-1)}{2} \times {}^{(n+3)}P_n$$

$$\Rightarrow \quad \frac{\lfloor\underline{n+5}}{\lfloor\underline{(n+5)-(n+1)}} = \frac{11(n-1)}{2} \cdot \frac{\lfloor\underline{n+3}}{\lfloor\underline{n+3-n}}$$

$$\Rightarrow \quad \frac{(n+5)(n+4)\lfloor\underline{n+3}}{4 \times \lfloor\underline{3}} = \frac{11(n-1)}{2} \times \frac{\lfloor\underline{n+3}}{\lfloor\underline{3}}$$

$$\Rightarrow \quad (n+5)(n+4) = 22(n-1)$$

$$\Rightarrow \quad n^2 + 9n + 20 = 22n - 22$$

$$\Rightarrow \quad n^2 - 13n + 42 = 0 \text{ or } (n-6)(n-7) = 0 \text{ or } n = 6, 7.$$

ample 5: *If* $^nP_4 = 2 \times {}^5P_3$, *find* n.

lution:

$$^nP_4 = 2 \times {}^5P_3$$

$\Rightarrow$ $$n(n-1)(n-2)(n-3) = 2 \times 5 \times 4 \times 3$$

$\Rightarrow$ $$n(n-1)(n-2)(n-3) = 120$$

$\Rightarrow$ $$(n^2 - 3n)(n^2 - 3n + 2) = 120$$

$\Rightarrow$ $m(m+2) = 120$ where $m = n^2 - 3n$

$\Rightarrow$ $$m^2 + 2m - 120 = 0$$

$\Rightarrow$ $$(m+12)(m-10) = 0$$

$\Rightarrow$ $$m = -12, \text{ or } m = 10$$

$\Rightarrow$ $$n^2 - 3n = -12 \quad \text{or} \quad n^2 - 3n = 10$$

$\Rightarrow$ $$n^2 - 3n + 12 = 0 \quad \text{or} \quad n^2 - 3n - 10 = 0$$

$\Rightarrow$ $$n = \frac{3 \pm \sqrt{9 - 48}}{2} \quad \text{or} \quad (n-5)(n+2) = 0$$

$\Rightarrow$ $$n = \frac{3 \pm i\sqrt{39}}{2} \quad \text{or} \quad n = 5 \quad \text{or } n = -2$$

$\Rightarrow$ $$n = 5$$ [neglecting negative and imaginary value]

xample 6: *Find n if* $^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_n$.

olution: We have, $^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_n$

$\Rightarrow$ $$\frac{\lfloor 9}{\lfloor 9-5} + 5 \cdot \frac{\lfloor 9}{\lfloor 9-4} = \frac{\lfloor 10}{\lfloor 10-n}$$

$\Rightarrow$ $$\frac{\lfloor 9}{\lfloor 4} + 5 \cdot \frac{\lfloor 9}{\lfloor 5} = \frac{\lfloor 10}{\lfloor 10-n}$$

$\Rightarrow$ $$\frac{\lfloor 9}{\lfloor 4} + \frac{\lfloor 9}{\lfloor 4} = \frac{\lfloor 10}{\lfloor 10-n}$$

$\Rightarrow$ $$2 \times \frac{\lfloor 9}{\lfloor 4} = \frac{10 \times \lfloor 9}{\lfloor (10-n)}$$

$\Rightarrow$ $$5 \times \lfloor 4 = \lfloor 10-n$$

$\Rightarrow$ $$\lfloor 10-n = \lfloor 5 \quad \Rightarrow \quad 10 - n = 5 \quad \Rightarrow \quad n = 5.$$

xample 7: *Prove that*

$$^nP_r = {}^{n-1}P_r + r \cdot {}^{(n-1)}P_{r-1}$$

Solution: R.H.S. $= {}^{(n-1)}P_r + r\cdot {}^{(n-1)}P_{r-1} = \dfrac{\underline{|n-1}}{\underline{|n-1-r}} + r\cdot\dfrac{\underline{|n-1}}{\underline{|(n-1)-(r-1)}} = \underline{|n-1}\left[\dfrac{1}{\underline{|n-r-1}} + \dfrac{r}{\underline{|n-r}}\right]$

$$= \underline{|n-1}\left[\frac{1}{\underline{|n-r-1}} + \frac{r}{(n-r)\,\underline{|n-r-1}}\right] = \underline{|n-1}\left[\frac{n-r+r}{(n-r)\,\underline{|n-r-1}}\right]$$

$$= \frac{n\,\underline{|n-1}}{(n-r)\,\underline{|n-r-1}} = \frac{\underline{|n}}{\underline{|n-r}} = {}^nP_r = \text{L. H. S.}$$

Example 8: *Prove that*

(i) ${}^nP_n = {}^nP_{(n-1)}$ (ii) ${}^nP_r = n\cdot {}^{n-1}P_{(r-1)}$

Solution: (i) ${}^nP_{n-1} = \dfrac{\underline{|n}}{\underline{|n-(n-1)}} = \dfrac{\underline{|n}}{\underline{|1}} = \underline{|n} = {}^nP_n$ (ii) ${}^nP_r = \dfrac{\underline{|n}}{\underline{|n-r}} = \dfrac{n\,\underline{|n-1}}{\underline{|(n-1)-(r-1)}} = n\cdot {}^{(n-1)}P_{r-1}$

Exercise

1. Prove that ${}^9P_3 + 3\cdot {}^9P_2 = {}^{10}P_3$.
2. Prove that ${}^nP_n = 2\cdot {}^nP_{(n-2)}$.
3. Find all permutations of 7 objects taken 3 at a time.
4. (i) If ${}^nP_4 : {}^nP_5 = 1:2$, find n. (ii) If ${}^{(n-1)}P_3 : {}^{(n+1)}P_3 = 5:12$, find n.
   (iii) If ${}^{2n+1}P_{n-1} : {}^{(2n-1)}P_n = 3:5$, find n.
5. If ${}^{11}P_r = {}^{12}P_{r-1}$, find r.
6. (i) If ${}^{2n}P_3 = 100\times {}^nP_2$, find n. (ii) If $16\,{}^nP_3 = 13\,{}^{n+1}P_3$, find n. (iii) If ${}^nP_5 = 20\times {}^nP_3$, find n.

Answers

3.	210	**4.**	(i) 6	(ii) 8	(iii) 4	**5.**	9	**6.**	(i) 13	(ii) 15	(iii) 8

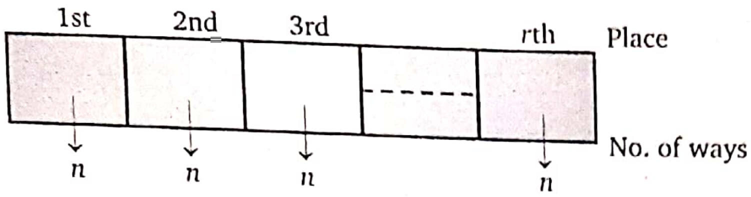
14.1.2 Permutation with Repetition

If out of n objects in a set, p objects are exactly alike of one kind, q objects exactly alike of second kind and r objects exactly alike of third kind and remaining objects are all different then the number of permutation of n objects taken all at a time is

$$= \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}$$

Theorem 4: *The number of permutations of n different objects taken r at a time when each object may be replaced any number of times in each arrangement is n^r.*

Proof: The number of permutations of n objects taken r at a time is the same as the number of ways of filling r places with n different objects.



Number of ways of filling first place $= n$,

Number of ways of filling second place $= n$, (Since the object used in filling the first place can be repeated)

Number of way of filling third place $= n$,

....

....

....

Number of way of filling rth place $= n$

Therefore, by the fundamental principle of counting, the required number of permutations

$$= n \cdot n \cdot n \,.....\, r \text{ (times)} = n^r$$

Simple Practical Problems on Permutation

Example 9: *How many ways are there to arrange the nine letters in the word ALLAHABAD.*

Solution: Since the word ALLAHABAD contains 4A's and 2L's, therefore, the total number of arrangement of nine letters in the word ALLAHABAD $= \frac{\lfloor 9}{\lfloor 4 \lfloor 2} = 7{,}560$ ways.

Example 10: *Find the number of different messages that can be represented by sequences of four dashes and three dots.*

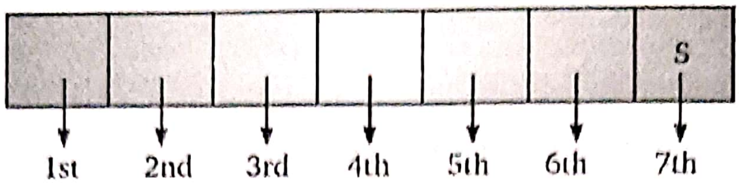
Solution: The number of different message sent by 4 deshes and 3 dots $= \frac{\lfloor 7}{\lfloor 4 \lfloor 3} = 35$

Example 11: *Find the number of ways to point 12 offices so that 3 of them are green, 2 of them pink, 2 of them yellow and the remaining one white.*

Solution: The required number of ways $= \frac{\lfloor 12}{\lfloor 3 \lfloor 2 \lfloor 2 \lfloor 5} = 1{,}66{,}320$

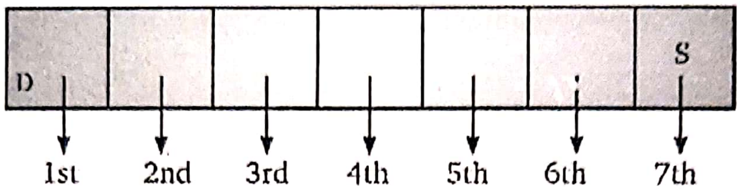
Example 12: *The letters of the word "TUESDAY" are arranged in a line each arrangement ending with letter S. How many different arrangements are possible? How many of them start with letter D?*

Solution: There are seven different letter in the word "TUESDAY"



when last place is filled with letter S then remaining 6 places can be filled by 6 letters in

$${}^6P_6 = 6! \text{ ways} \qquad = 720 \text{ ways}$$



Now, when the letter 'D' is also fixed, then remaining 5 places can be filled by 5 letters in

$${}^5P_5 = 5! \text{ ways} \qquad = 120 \text{ ways.}$$

Example 13: *How many signals can be made by using 6 flags of different colours when any number of them may be hoisted at a time?*

Solution: Number of signals that can be made using 1 flag $= {}^6P_1 = 6$

Number of signals that can be made using 2 flags $= {}^6P_2 = \dfrac{6!}{(6-2)!} = 6 \times 5 = 30$

Number of signals that can be made using 3 flags $= {}^6P_3 = \dfrac{6!}{(6-3)!} = 6 \times 5 \times 4 = 120$

Number of signals that can be made using 4 flags $= {}^6P_4 = \dfrac{6!}{(6-4)!} = 6 \times 5 \times 4 \times 3 = 360$

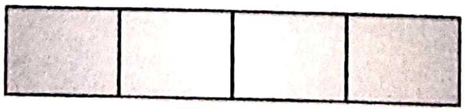
Number of signals that can be made using 5 flags $= {}^6P_5 = \dfrac{6!}{(6-5)!} = 6 \times 5 \times 4 \times 3 \times 2 = 720$

Number of signals that can be made using 6 flags $= {}^6P_6 = 6! = 720$

Therefore the total signals that can be made $= 6 + 30 + 120 + 360 + 720 + 720 = 1{,}956$

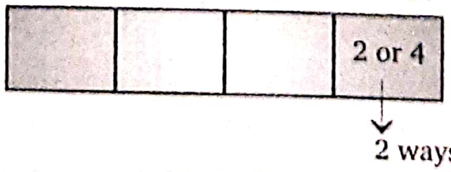
Example 14: *Find the number of 4 digit numbers that can be formed using the digits 1, 2, 3, 4 and 5 if no digit is used more than once in a number. How many of these number will be even?*

Solution: (i)



4 places can be filled by 5-digits in ${}^5P_4$ ways $= \dfrac{5!}{(5-4)!}$ ways $= \dfrac{5!}{1!}$ ways $= 120$ ways

(ii)



For even numbers, unit's place can be filled in 2 ways (either digit 2 or digit 4)

If units place is filled by any one digit, then remaining 3 places can be filled by remaining 4 digit in

$$^4P_3 = \frac{4!}{(4-3)!} \qquad = 4! = 24 \text{ ways}$$

Hence, total number of ways $= 2 \times 24 = 48$ ways.

Example 15: *How many 4 letter words, with or without meaning, can be formed out of the letters of the word 'LOGARITHMS' if repetition of letters is not allowed?*

Solution: The word 'LOGARITHMS' contains different letters

$\therefore$ the number of required words = number of arrangements of 10 letters, taken 4 at a time

$$= {}^{10}P_4 \qquad = 10 \times 9 \times 8 \times 7 = 5{,}040$$

Example 16: *It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangement are possible?*

Solution: In a row of 9 positions, the 2nd, 4th, 6th and 8th are the even places.

These 4 places can be occupied by 4 women in ${}^4P_4 = \lfloor\underline{4} = 24$ ways

The remaining positions can be occupied by 5 men in ${}^5P_5 = \lfloor\underline{5} = 120$ ways

$\therefore$ the total number of seating arrangements $= 24 \times 120 = 2{,}880$.

Example 17: *How many words, with or without meaning, can be founded by using all letters of the word 'DELHI', using each letter exactly once?*

Solution: The word 'DELHI' contains 5 different letters.

$\therefore$ the number of required words = number of arrangements of 5 letters taken all at a time

$$= {}^5P_5 = \lfloor\underline{5} = 5 \times 4 \times 3 \times 2 = 120.$$

Example 18: *In an examination hall, there are four rows of chairs. Each row has 8 chairs, one behind the other. There are two classes appearing for the examination with 16 students in each class. It is desired that in each row all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many way can these 32 students be seated?*

Solution: There are 2 ways of selecting the rows for a class *i.e.*, either I and III or II and IV.

Now, 16 students of one class can be arranged in 16 chairs in ${}^{16}P_{16} = \lfloor\underline{16}$ ways

and 16 students of another class can be arranged in 16 remaining chairs in ${}^{16}P_{16} = \lfloor\underline{16}$ ways

$\therefore$ the required number of ways $= [2 \times \lfloor\underline{16} \times \lfloor\underline{16}\,]$

Example 19: *How many odd numbers greater than 80,000 can be formed using the digits 2, 3, 4, 5 and 8. If each digit is used only once in a number?*

Solution:

Ten Thousand	Thousand	Hundreds	Ten's	Unit's
8				3 or 5
1 way				2 way

Since, it is an odd number greater than 80,000, therefore unit's place can be filled by 2 ways (either by digit 3 or by digit 5) and ten thousand's can be filled by only 1 digit *i.e.*, 8.

Also other 3 places can be filled by remaining 3 digits in ${}^3P_3 = 3!$ ways

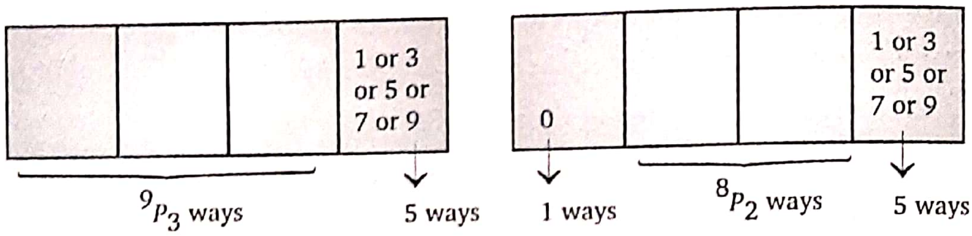
Total number of ways $= 2 \times 1 \times 3! = 12$ (By F.P.C.).

Example 20: *How many four-digit numbers (between 1,000 and 10,000) are there with distinct 2 of these, how many are odd numbers ?*

Solution: (i) Total number of arrangements of ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 taking 4 at a time is ${}^{10}P_4$. But these arrangements, also include those numbers which has 0 at thousand's place. Such numbers are not the four digit numbers. So, we have to exclude those three digit numbers when 0 is in thousand's place, we have to arrange remaining 9 digits by taking 3 digits at a time. The number of such arrangements is ${}^9P_3$.

Hence, the total number of four digit numbers $= {}^{10}P_4 - {}^9P_3 = 5{,}040 - 504 = 4{,}536.$

(ii) Now, since we want the odd number, therefore unit's place can be filled in 5 (1 or 3 or 5 or 7 or 9) ways.



The required number of odd numbers = number of all 4 digited number (including 0) having an odd digit in thousand's place – number of these 4 digited odd numbers which have 0 in thousand's place

$$= 5 \times {}^9P_3 - 1 \times {}^8P_2 \times 5 = 5 \times 9 \times 8 \times 7 - 8 \times 7 \times 5 = 2{,}240.$$

Example 21: *Find the number of permutations of n different things taken r at a time each of which includes 3 particular things ?*

Solution: Given r places, 3 places by particular things can be filled in ${}^rP_3$ ways.

Now, we have to fill up the remaining $(r - 3)$ places with the remaining $(n - 3)$ things in ${}^{n-3}P_{r-3}$ ways.

Total number of permutations $= {}^rP_3 \times {}^{n-3}P_{r-3}$.

Example 22: *Find the number of permutations of n different things taken r at a time from which three particular things are to be excluded.*

Solution: We exclude the 3 particular things from the total number of things and form permutations of the remaining $(n - 3)$ things taken r at a time.

Therefore, the required number of permutations $= {}^{n-3}P_r$.

Example 23: *How many permutation can be made out of the letters of the word 'TRIANGLE'. How many of these will begin with T and end with E ?*

Solution: The word TRIANGLE contains 8 different letters.

The number of all permutation made out of 8 letters $= {}^8P_8 = \lfloor 8 = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320.$

If the word starts with the letter T and ends with letter E then, the number of ways to form the words $= {}^8P_6$

$$= \frac{\lfloor 8}{\lfloor 8-6} = \frac{\lfloor 8}{\lfloor 2} = 20{,}160$$

Example 24: *In how many ways can 10 books be arranged on a shelf so that a particular pair of books shell be (i) always together (ii) never together ?*

Solution: Since a particular pair of books is always together, let us tie these two books together and consider the pair as one book. Then, we shall have to arrange 9 books on the shelf.

This can be done $\quad {}^9P_9 = \lfloor\underline{9}$ ways

Now, in each of these arrangements, the two books comparising the particular pair can be arranged among themselves in $\lfloor\underline{2}$ *i.e.*, 2 ways.

$\therefore \quad$ the required number of ways $= 2 \times (\lfloor\underline{9})$

(ii) We know that 10 books can be arranged on the shelf in ${}^{10}P_{10} = \lfloor\underline{10}$ ways

Also, the number of ways of arranging 10 books so that a particular pair is always together $= 2 \times (\lfloor\underline{9})$

$\therefore$ the number of ways of arranging 10 books so that a particular pair is never together

$$= (\lfloor\underline{10} - 2\lfloor\underline{9}) \quad = (10 \times \lfloor\underline{9} - 2 \times \lfloor\underline{9}) \quad = (8 \times \lfloor\underline{9})$$

Example 25: *There are 6 English, 4 Sanskrit and 5 Hindi books. In how many ways can they be arranged on a shelf so as to keep all the books of the same language together?*

Solution: Let us make one packet for each of the books on the same language.

Now, 3 packets can be arranged in ${}^3P_3 = \lfloor\underline{3} = 6$ ways

6 books on English can be arranged in $\lfloor\underline{6} = 720$ ways

4 books on Sanskrit can be arranged in $\lfloor\underline{4} = 24$ ways

5 books on Hindi can be arranged in $\lfloor\underline{5} = 120$ ways

$\therefore$ the required number of ways $= 6 \times 720 \times 24 \times 120.$

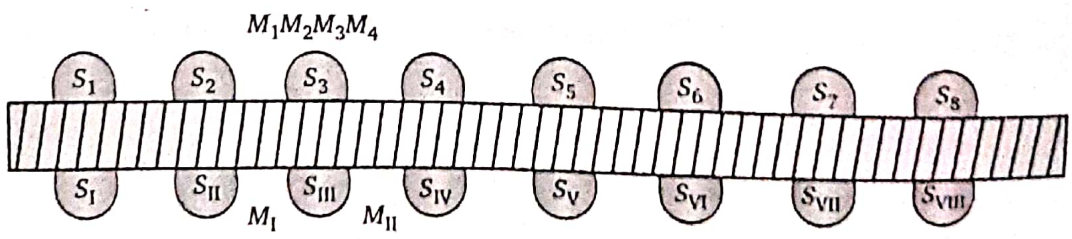
Example 26: *In how many ways 9 pictures can be hung of 6 picture nails on a wall (in a row) ?*

Solution: Here, it is obvious that only 6 pictures out of 9 will be hung and 3 will be left out. The number of ways in which pictures can be hung on 6 picture nails on a wall is the same as the number of arrangements of 9 things, taking 6 at a time.

Hence, the required number $= {}^9P_6 \; = 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 60{,}480.$

Example 27: *A tea party is arranged for 16 people along the two sides of a long table with 8 chairs on each side. Four men wish to sit on one particular side and two on the other side. In how many ways can they be seated ?*

Solution: The order of seating arrangement is as below,



(i) 4 persons (M_1, M_2, M_3, M_4) are to sit one-one side in any one side in any of 8 chairs in ${}^8P_4$ ways.

(ii) 2 persons (M_1 and M_2) wist to sit at other side on any of the 8 chairs. This can be in ${}^8P_2$ ways.

(iii) Rest 10 persons can sit on any of the remaining 10 chairs in 10 chairs in ${}^{10}P_{10}$ ways.

$\therefore$ Total number of ways of sitting $= {}^8P_4 \times {}^8P_2 \times {}^{10}P_{10}$ (by using F.P.C.)

Example 28: *In how many ways can 6 balls of different colours, namely, white, black, blue, red, green and yellow be arranged in a row in such a ways that the white and black balls are never together ?*

Solution: Let x be the number of all arrangements, each one of which contains white and black balls together. And, let y be the number of arrangements, none of which contains white and black balls together.

Then, $$x + y = {}^6P_6 = \lfloor 6 = 720.$$

Now, considering a white and a black ball tied together as one ball.

The number of ways of arranging the 5 balls $= {}^5P_5 = \lfloor 5 = 120$.

Also, these 2 balls may be arranged among themselves in $\lfloor 2 = 2$ ways.

$\because$ $x = 2 \times 120 = 240$, So, $y = 720 - 240 = 480$

Hence, the requisite number of ways = 480.

Example 29: *In how many ways can the word 'PENCIL' be arranged so that N is always next to E ?*

Solution: Keeping E and N together and considering then as one letter, we have to arrange 5 letters at 5 places. This can be done in ${}^5P_5 = \lfloor 5 = 120$ ways.

Example 30: *The principal wants to arrange 5 students on the platform such that the boy Salim occupies the second position and such that the girl Sita is always adjacent to the girl Rita. How many such arrangements are possible?*

Solution: Let the seats be arranged as shown

O	S	O	O	O
I	II	III	IV	V

Keep Salim fixed at the second position. Since, Sita and Rita are to sit together, none of the two can occupy the first seat. This seat can be occupied by any of the remaining two students in 2 ways. Now, two seats, namely, III , IV, V, may be occupied by Sita and Rita in 4 ways. The remaining seat may now be occupied by the 5th student in one way only.

$\therefore$ the number of required arrangement $= 2 \times 4 \times 1 = 8$.

Example 31: *Find the number of ways in which 5 boys and 3 girls can be seated in a row so that two girls are together.*

Solution: The 5 boys may occupy 5 places in ${}^5P_5 = \lfloor 5 = 120$ ways.

Since no. two girls are to sit together, we may arrange the 3 girls at the 6 places. i. e., X B X B X B X B X B

Now, the 3 girls at 6 places may be seated in ${}^6P_3 = 6 \times 5 \times 4 = 120$ ways.

$\therefore$ The required number of ways $= 120 \times 120 = 14{,}400$.

Example 32: *How many permutations of the letters of the word 'APPLE' are there ?*

Solution: Here there are 5 objects, two of which are of same kind and the others are each of its own kind.

$\therefore$ The required number of permutations $= \dfrac{\lfloor 5}{\lfloor 2 \times \lfloor 1 \times \lfloor 1 \times \lfloor 1} = 60.$

Example 33: (i) *Find how many arrangements can be made with the letters of the word 'MATHEMATICS'.*

(ii) *In how many of them are the vowels together?*

Solution: (i) There are 11 letters in the word 'MATHEMATICS'

Out of these letters M occurs twice. A occurs twice, T occurs twice and the rest are all different.

$\therefore$ Required number of arrangement $= \dfrac{\lfloor 11}{\lfloor 2 \times \lfloor 2 \times \lfloor 2} = 49,89,600.$

(ii) The given word consists of 4 vowels namely A, E, A and I. Treating these four vowels as one letter, we have to arrange 8 letters (MTHMTCS) + (AEAI);

out of which M occurs twice, T occurs twice and the rest are all different.

$\therefore$ The number of such arrangements $= \dfrac{\lfloor 8}{(\lfloor 2) \times (\lfloor 2)} = 10,080$

Corresponding to each such arrangement, the four vowels in which A occurs twice and the rest are all different, can be arranged amongest themselves in $\dfrac{\lfloor 4}{\lfloor 2} = 12$ ways.

$\therefore$ Total number of arrangements in which vowels are always together $= 10080 \times 12 = 1,20,960$

Example 34: *In the different permutations of the word 'EXAMINATION' are listed as in a dictionary, how many items are there in the list before the first word starting with E ?*

Solution: Starting with E, we have to find the number of arrangements of the remaining 10 letters, EXAMINATION, out of which there are 2I's, 2N's, 2 A's and the rest of the letters are distinct.

The number of such arrangements is $= \dfrac{\lfloor 10}{\lfloor 2 \times (\lfloor 2) \times (\lfloor 2)} = 4,53,600$

Clearly, we will have to start the next word with E, so the required numbers of items = 4,53,600.

Example 35: *How many numbers greater than a million can be formed with digits 2, 3, 0, 3, 4, 2, 3?*

Solution: Any number greater than one million will contain all the seven digits.

Now, we have to arrange seven digits, out of which 2 occurs twice, 3 occurs thrice and the rest are distinct.

The number of such arrangements $= \dfrac{\lfloor 7}{(\lfloor 2) \times \lfloor 3} = 420$

These arrangements will also include those which contain 0 at the million's place. Keeping 0 fixed at the million's place, the remaining 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct can be arrange in $= \dfrac{\lfloor 6}{(\lfloor 2) \times \lfloor 3} = 60$ ways.

$\therefore$ The number of required numbers $= 420 - 60 = 360.$

Example 36: *In a shipment, there are 40 floppy disks of which 5 are defective, determine:*

(i) *In how many ways can we select five floppy disks?*

(ii) *In how many ways can we select five non-defective floppy disks?*

(iii) *In how many ways can we select five floppy disks containing exactly three defective floppy disks?*

(iv) *In now many ways can we select five disks containing at least 1 defective floppy disk?*

[U.P.T.U. (B.Tech.) 2004]

Solution: (i) The required number of ways

$$= 40C_5 = \frac{\lfloor 40}{\lfloor 5 \lfloor 35} = \frac{40 \times 39 \times 38 \times 37 \times 36 \times \lfloor 35}{5 \times 4 \times 3 \times 2 \times \lfloor 35} = 39 \times 38 \times 37 \times 12 = 6580088$$

(ii) The non-defective floppy are 35

The number of ways to select five non-defective floppy $= 35C_5 = \frac{35}{\lfloor 5 \lfloor 30}$

$$= \frac{35 \times 34 \times 33 \times 32 \times 31 \times \lfloor 30}{5 \times 4 \times 3 \times 2 \times 1 \times \lfloor 30} = 324632$$

(iii) The number of ways to select five floppy contain exactly 3 defective floppy disks $= \frac{35C_2 \times 5C_3}{40C_5}$

(iv) The required selection $= \frac{35C_4 \times 5C_1 + 35C_3 \times 5C_2 + 35C_2 \times 5C_3 + 35C_1 \times 5C_4}{40C_5}$

Example 37: *Given a cube and four identical balls. In how many way can the balls be arranged on the corners of the cube?. Two arrangements are considered the same if by any rotation of the cube they can be transformed into each other.* [U.P.T.U. (B.Tech.) 2005]

Solution: The balls can be arranged on the eight corners of the cube in the number of ways.

$$= 8C_4 = \frac{\lfloor 8}{\lfloor 4 \lfloor 4} = \frac{8 \times 7 \times 6 \times 5 \times \lfloor 4}{\lfloor 4 \times 4 \times 3 \times 2} = 70 \text{ ways}$$

Now, two arrangements are same iff the two corners between the two arrangements become common.

Hence, the total ways $= 70 - 65 = 65$

Example 38: *(a) How many selections any number at a time may be made from three white balls, four green balls, one red ball one black ball, if at least one must be chosen.*

(b) There are twelve students in the class. Find the number of ways that the twelve students take three different tests if four students are to take each test. [U.P.T.U. (B.Tech.) 2008]

Solution: (a) Total number of balls = 9.

The probability that no ball is chosen = 1

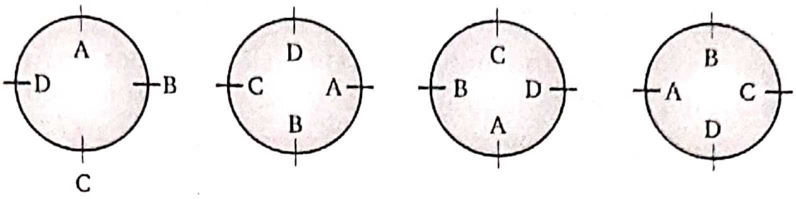
Then, number of selection $= \lfloor 9 - 1 = 362880 - 1 = 362879$

(b) The required number of ways $= \frac{\lfloor 12}{\lfloor 3 \lfloor 3 \lfloor 3 \lfloor 3} = 369600$

14.1.3 Circular Permutations

So far we have been considering the arrangements of objects in a line. Such permutation are known as linear permutations. In circular permutations, what really matters is the position of an object relative to the others. If we consider the linear permutations ABCD, BCDA, CDAB and DABC. Then, clearly they are distinct.

Now, we arrange A, B, C, D along the circumference of a circle as shown



If we consider the position of an object retative to others then we find that the above four arrangements are the same.

Theorem 5: *The number of circular permutations of n different objects, is* $\lfloor\underline{n-1}$.

Proof : Fixing the position of an object can be done in n ways, as the position of anyone of them may be fixed. Thus, each circular permutation corresponding to n linear permutations depending upon where from we start.

Since, there are $\lfloor\underline{n}$ linear permutations, it follows that there are $\frac{\lfloor\underline{n}}{n}$ *i.e.*, $\lfloor\underline{(n-1)}$ circular permutations.

Theorem 6: *The number of ways in which n persons can be seated round a table is* $\lfloor\underline{n-1}$.

Proof: Let us fix the position of one person and then arrange the remaining $(n-1)$ persons in all possible ways. Clearly, this can be done in $\lfloor\underline{(n-1)}$ ways. Hence, the required number of ways $= \lfloor\underline{n-1}$.

Theorem 7: *Show that the number of ways in which n different beads can be arranged to form a necklace is* $\frac{1}{2}\lfloor\underline{(n-1)}$.

Proof : Fixed the position of one bead, the remaining $(n-1)$ beads can be arranged in $\lfloor\underline{n-1}$ ways.

In case of arranging the beads, there is no distinction between the clockwise and anticlockwise arrangements. So, the required number of ways $= \frac{1}{2}\lfloor\underline{(n-1)}$.

Example 39: *A gentleman has 6 friends to invites. In how many ways can he send invitation cards to them, if he has 3 servants to carry the cards ?*

Solution: Two friends can be invited by the same servant. Hence, in this problem, the servant is repeatable (R) and the friend is non-repeatable (NR).

$$\text{Number of ways } = [R]^{NR} = 3^6 = 729.$$

Example 40: *A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. Find the total signals that can be made ?*

Solution: Two arms may have the same position, but same arms cannot have two positions at a time. Hence, position is repeatable (R), but arm is non-repeatable (NR)

$$\text{Number of ways } = (R)^{NR} = 4^5 = 1{,}024$$

But, in one case, when all the 5 arms will be in rest position, no signal will be made.
Hence, the required number of signals $= 1{,}024 - 1 = 1{,}023$

Example 41: *Find the number of way in which 3 friends can stay in 2 hotels ?*

Solution: Two friends can stay in the same hotel but same friend cannot stay in two hotel at a time. Hence, hotel is repeatable (R) and friend is non-repeatable (NR).

$$\text{Number of ways } = (R)^{NR} = (2)^3 = 8.$$

Example 42: *Find the number of ways in which Dr. Bansal can give 2 hotels to his 3 sons?*

Solution: A son can get 2 hotels but a hotel cannot be given again and again.
Hence, son is repeatable (R) but hotel is non-repeatable (NR).

$$\text{Number of ways } = (R)^{NR} = (3)^2 = 9.$$

Example 43: *Find the number of ways in which 6 pictures can be hung on 9 picture nails (in a row)?*

Solution: A picture can not hung on two picture nails at a time, but two pictures can hung on one picture nail. Hence, picture nail is repeatable (R) but picture is non-repeatable (NR).

$$\text{Number of ways } = (R)^{NR} = (9)^6.$$

Example 44: *In how many ways can 8 students be arranged in*

(i) a line *(ii) a circle.*

Solution: (i) The number of ways in which 8 students can be arranged in a line $= \lfloor 8$

$$= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320.$$

(ii) The number of ways in which 8 students can be arranged in a circle $= \lfloor 8 - 1 = \lfloor 7$

$$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040.$$

Example 45: *In how many ways can a party of 4 men and 4 women be seated at a circular table so that no two women are adjacent ?*

Solution: The 4 men can be seated at the circular table such that there is a vacant seat between every pair of men.

The number of ways in which these 4 men can be seated at the circular table $= \lfloor 3 = 6$.

Now, the 4 vacant seats may be occupied by 4 women in ${}^4P_4 = \lfloor 4 = 24$ ways.

$\therefore$ The required number of ways $= 6 \times 24 = 144$.

Example 46: *In how many ways can 7 persons sit around a table so that all shall not have the same neighbours in any arrangements.*

Solution: 7 persons set at a round table in $\lfloor 6$ ways.

But, in clockwise and anticlockwise arrangements, each person will have the same neighbours.

So, the required number of ways $= \dfrac{1}{2} \times \lfloor 6 = 360$.

Answer

1.	60	**2.**	120	**3.**	24	**4.**	5,040
5.	840	**6.**	720	**8.**	720; 120	**9.**	576
10.	5,040; 120	**11.**	480	**12.**	31,10,400	**13.**	3
14.	2,520	**15.**	(a) 60 (b) 2,77,200 (c) 49,89,600			**16.**	30
17.	(a) $\lfloor 6$, (b) 120			**18.**	$\lfloor 4 \times \lfloor 5 = 2{,}880$		

14.2 Combinations

Combinations: Each of the different groups or selections which can be formed by taking some or all of a number of objects, irrespective of their arrangements, is called a combination. The total number of combinations of n distinct objects taking r $(1 \le r \le n)$ at a time is denoted by $C(n, r)$ or ${}^nC_r$ or $\begin{pmatrix} n \\ r \end{pmatrix}$.

where, ${}^nC_r$ is defined only when n and r integers such that $n \ge r$ and $n > 0$, $r \ge 0$.

Illustration: *The combinations of 4 objects a, b, c, d taking 2 at a time are* ab, ac, ad, bc, bd, cd

14.2.1 Difference between a Permutation and a Combination

In combination, only a group is made and the order in which the objects are arranged is immaterial. In permutation, not only a group is formed, but also an arrangement in definite order is considered.

Note: We used the word 'arrangement' for permutation and selections for combinations.

Theorem 8: *The number of all combinations of n distinct objects, taken r at a time, is given by*

$$ {}^nC_r = \frac{\lfloor n}{\lfloor r \, \lfloor n-r} $$

Proof: Let the number of all combinations of n objects, taken r at a time, be x. Then, ${}^nC_r = x$.

Now, each combination contains r objects, which may be arranged amongst themselves is $\lfloor r$ ways.

Thus, each combination gives rise to $\lfloor r$ permutations.

$\therefore$ x combinations will give rise to $x \times \lfloor r$ permutations.

So, the number of permutations of n things, taken r at a time is $x \times \lfloor r$

Hence, $$ {}^nP_r = x \times \lfloor r = {}^nC_r \times \lfloor r $$

$\therefore$ $$ {}^nC_r = \frac{{}^nP_r}{\lfloor r} = \frac{\lfloor n}{\lfloor r \, \lfloor n-r} \qquad \left[\because {}^nP_r = \frac{\lfloor n}{\lfloor n-r}\right] $$

Note: We can write $$ {}^nC_r = \frac{n\,(n-1)\,(n-2) \ldots \text{ n factors}}{\lfloor r}. $$

Theorem 9: *Let $0 \le r \le n$. Prove that* $\quad {}^nC_r = {}^nC_{n-r}$.

Proof: We have $\quad {}^nC_{n-r} = \dfrac{\lfloor n}{\lfloor n-r \lfloor n-(n-r)} = \dfrac{\lfloor n}{\lfloor r \lfloor n-r} = {}^nC_r$.

Theorem 10: *To prove that* ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$.

Proof: We have,

$$ {}^nC_r + {}^nC_{r-1} = \frac{\lfloor n}{\lfloor r \lfloor n-r} + \frac{\lfloor n}{\lfloor r-1 \lfloor n-(r-1)} = \frac{\lfloor n}{\lfloor r \lfloor n-r} + \frac{\lfloor n}{\lfloor r-1 \lfloor n-r+1} = \frac{\lfloor n \cdot (n-r+1)}{\lfloor r \cdot \lfloor (n-r+1)} + \frac{\lfloor n \cdot r}{\lfloor r \lfloor n-r+1} $$

$$ = \left\{\frac{\lfloor n}{\lfloor r \lfloor n-r+1}\right\} \cdot \{n-r+1+r\} = \frac{(n+1)\lfloor n}{\lfloor r \lfloor n-r+1} = \frac{\lfloor n+1}{\lfloor r \lfloor n+1-r} = {}^{n+1}C_r $$

Theorem 11: *If $1 \le r \le n$, prove that $n \times {}^{(n-1)}C_{r-1} = (n-r+1) \times {}^nC_{r-1}$.*

Proof: We have $\quad n \times {}^{(n-1)}C_{r-1} = n \times \dfrac{\lfloor n-1}{\lfloor r-1 \times \lfloor (n-1)-(r-1)} = \dfrac{n \times \lfloor n-1}{\lfloor r-1 \lfloor n-r}$

$$ = \frac{\lfloor n\,(n-r+1)}{\lfloor r-1 \times \lfloor n-r \times (n-r+1)} = (n-r+1) \times \frac{\lfloor n}{\lfloor r-1 \times \lfloor n-r+1} = (n-r+1) \times {}^nC_{r-1} $$

Theorem 12: *If n and r are positive integers such that $1 \le r \le n$ then* $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$.

Proof : We know that

$$ {}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} \quad \text{and} \quad {}^nC_{r-1} = \frac{\lfloor n}{\lfloor r-1 \times \lfloor n-r+1} $$

$$ \therefore \quad \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{\lfloor n}{\lfloor r \lfloor n-r} \times \frac{\lfloor r-1 \lfloor n-r+1}{\lfloor n} = \frac{\lfloor r-1 \times (n-r+1) \lfloor n-r}{r \times \lfloor (r-1) \lfloor n-r} = \left(\frac{n-r+1}{r}\right) $$

Theorem 13: *If $1 \le r \le n$, prove that ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$.*

Proof:

$$ {}^nC_r + {}^nC_{r+1} = \frac{\lfloor n}{\lfloor r \lfloor n-r} + \frac{\lfloor n}{\lfloor (r+1) \lfloor n-r-1} $$

$$ = \frac{\lfloor n}{\lfloor r \times (n-r) \lfloor n-r-1} + \frac{\lfloor n}{(r+1) \lfloor r \times \lfloor n-r-1} $$

$$ = \frac{\lfloor n}{\lfloor r \lfloor n-r-1}\left(\frac{1}{n-r} + \frac{1}{r+1}\right) $$

$$ = \frac{\lfloor n}{\lfloor r \lfloor n-r-1} \times \frac{(n+1)}{(n-r)(r+1)} = \frac{(n+1)\lfloor n}{(r+1)\lfloor r \times (n-r) \times \lfloor n-r-1} $$

$$ = \frac{\lfloor n+1}{\lfloor r+1 \lfloor n-r} = \frac{\lfloor n+1}{\lfloor r+1 \lfloor (n+1)-(r+1)} = {}^{n+1}C_{r+1} $$

Theorem 14: *Prove that* ${}^nC_p = {}^nC_q \Rightarrow p = q$ *or* $p + q = n$.

Proof: We have, ${}^nC_p = {}^nC_q = {}^nC_{n-q}$

$$\Rightarrow \quad p = q \quad \text{or} \quad p = n - q \quad \text{or} \quad p + q = n.$$

Evaluate

Example 47: *If* ${}^nP_r = 720$ *and* ${}^nC_r = 120$, *find* r.

Solution: We know that $${}^nC_r = \frac{{}^nP_r}{\lfloor r}$$

$$\therefore \quad 120 = \frac{720}{\lfloor r} \Rightarrow \lfloor r = \frac{720}{120} = \lfloor 3$$

Hence, $r = 3$.

Example 48: *Prove that* ${}^{2n}C_n = \dfrac{2^n \times [1 \cdot 3 \cdot 5 \ldots . (2n-1)]}{\lfloor n}$.

Solution: $${}^{2n}C_n = \frac{\lfloor 2n}{\lfloor n \lfloor 2n - n} = \frac{\lfloor 2n}{(\lfloor n)^2}$$

$$= \frac{(2n)(2n-1)(2n-2) \ldots \ldots 4 \cdot 3 \cdot 2 \cdot 1}{(\lfloor n)^2}$$

$$= \frac{[2n(2n-2)(2n-4) \ldots . 4 \cdot 2] \times [(2n-1)(2n-3) \ldots . 5 \cdot 3 \cdot 1]}{(\lfloor n)^2}$$

$$= \frac{2^n [n(n-1)(n-2) \ldots . 2 \cdot 1][(2n-1)(2n-3) \ldots . 5 \cdot 3 \cdot 1]}{(\lfloor n)^2}$$

$$= \frac{2^n \lfloor n [1 \cdot 3 \cdot 5 \ldots . (2n-3)(2n-1)]}{(\lfloor n)^2}$$

$$= \frac{2^n \times [1 \cdot 3 \cdot 5 \ldots . (2n-3)(2n-1)]}{\lfloor n}.$$

Example 49: *If* ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} :: 3 : 4 : 5$ *find* n *and* r.

Solution: $$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{4} \text{ and } \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{4}{5} \Rightarrow \frac{r}{n-r+1} = \frac{3}{4} \text{ and } \frac{r+1}{n-r} = \frac{4}{5}$$

$$\Rightarrow \quad 3n - 7r = -3 \text{ and } 4n - 9r = 5$$

$$\Rightarrow \quad r = 27,\ n = 62$$

Exercise

1. Verify that ${}^8C_4 + {}^8C_3 = {}^9C_4$.

2. (i) If ${}^nC_7 = {}^nC_5$, find n. (ii) If ${}^nC_{10} = {}^nC_{14}$, find ${}^nC_{27}$

   (iii) If ${}^{20}C_r = {}^{20}C_{r+6}$, find r. (iv) If ${}^nP_r = 1680$ and ${}^nC_r = 70$, find n and r.

   (v) If ${}^{n+1}C_{r+1} : {}^nC_r = 11 : 6$ and ${}^nC_r : {}^{n-1}C_{r-1} = 6 : 3$, find n and r.

3. If ${}^nC_r : {}^{15}C_{r-1} = 11 : 5$ find r.

4. If ${}^{18}C_r = {}^{18}C_{r+2}$, find ${}^rC_5$.

5. Find ${}^{25}C_{22} - {}^{24}C_{21}$.

Answer

2.	(i) 12 (ii) 2925 (iii) 7 (iv) $n = 8, r = 4$ (v) $n = 10, r = 5$				
5.	276	**3.**	8	**4.**	56

14.3 Practical Problems on Combinations

Example 50: *In how many ways can a committee of 5 members be selected from 6 men and 5 ladies, consisting of 3 men and 2 ladies?*

Solution: 3 men out of 6 and 2 ladies out of 5 can be selected in ${}^6C_3 \times {}^5C_2 = \dfrac{6\times5\times4}{3\times2\times1} \times \dfrac{5\times4}{2\times1} = 200$ ways.

Example 51: *A committee of 5 is to be formed out of 6 men and 4 ladies. In how many ways can this be done, when (a) atleast 2 ladies are included (b) atmost 2 ladies are included.*

Solution: (a) We have to make selection of

(i) (2 ladies out of 4) and (3 men out of 6) or (ii) (3 ladies out of 4) and (2 men out of 6) or

(iii) (4 ladies out of 4) and (1 man out of 6)

The number of ways of these selections are

Case (*i*) ${}^4C_2 \times {}^6C_3 = 6 \times 20 = 120$, **Case** (*ii*) ${}^4C_3 \times {}^6C_2 = 4 \times 15 = 60$,

Case (*iii*) ${}^4C_4 \times {}^6C_1 = 1 \times 6 = 6$.

$\therefore$ Total number of ways to select such a committee $= 120 + 60 + 6 = 186$

(b) We have to make a selection of

(i) (1 lady out of 4) and (4 men out of 6) or (ii) (2 ladies out of 4) and (3 men out of 6)

The number of ways of these selections are

Case (*i*) ${}^4C_1 \times {}^6C_4 = 4 \times 15 = 60$, **Case** (*ii*) ${}^4C_2 \times {}^6C_3 = 6 \times 20 = 120$

Case (iii) *No of lady is selected then*

Number of ways of selecting committee = ${}^{6}C_{5} = 6$

$\therefore$ the required number of ways = 60 + 120 + 6 = 186

Example 52: *An examination paper containing 12 questions consists of two parts, A and B, part A contains 7 questions and part B contains 5 questions. A candidate is required to attempt 8 questions, selecting atleast 3 from each part. In how many ways the candidate select the questions ?*

Solution: We have,

(i) (3 out of 7 from A) and (5 out of 5 from B) or (ii) (4 out of 7 from A) and (4 out of 5 from B) or (iii) (5 out of 7 from A) and (3 out of 5 from B)

The number of ways of these selections are

Case (i) ${}^{7}C_{3} \times {}^{5}C_{5} = 35 \times 1 = 35.$ **Case** (ii) ${}^{7}C_{4} \times {}^{5}C_{4} = 35 \times 5 = 175,$

Case (iii) ${}^{7}C_{5} \times {}^{5}C_{3} = 21 \times 10 = 210$

$\therefore$ the number of ways of selecting the questions = (35 + 175 + 210) = 420.

Example 53: *If 20 lines are drawn in a plane such that no two of them are parallel and no three are concurrent, then in how many points will they intersect each other ?*

Solution: Since, no two lines are parallel and no three pass through the same point, their point of intersection, (i. e., number of ways of selecting two lines from 20 lines) = ${}^{20}C_{2} = \dfrac{20 \times 19}{2 \times 1} = 190.$

Example 54: *How many lines can be drawn through 21 points on a circle ?*

Solution: One line can be drawn by joining any 2 points, from 21 given points.

$\therefore$ Total no. of lines $= {}^{21}C_{2} = \dfrac{21 \times 20}{2 \times 1} = 210.$

Example 55: *There are 15 points in a plane, no three of which are collinear. Find the number of triangles formed by joining them.*

Solution: One triangle is formed by selecting a group of 3 points from 15 given points.

$\therefore$ No. of triangles $= {}^{15}C_{3} = \dfrac{15 \times 14 \times 13}{3 \times 2 \times 1} = 455.$

Example 56: *Everybody in a room shakes hands with everybody else. The total number of handshakes is 66. How many people are there in the room?*

Solution: Let there are n persons in the room.

$\therefore$ Total no. of handshakes is same as the number of ways of selecting 2 persons among n persons, *i. e.*, ${}^{n}C_{2}$.

$$\therefore \quad {}^{n}C_{2} = 66 \qquad \text{[given]}$$

$$\frac{n(n-1)}{2} = 66 \Rightarrow n^2 - n - 132 = 0$$

$$\Rightarrow \quad (n-12)(n+11) = 0 \Rightarrow n = 12 \text{ or } n = -11$$

But, no. of persons cannot be negative then $n = 12$.

Example 57: *Determine the number of 5 cards combination out of a deck of 52 cards, if there is exactly one ace in each combination.*

Solution: One ace can be selected from 4 aces and 4 other cards can be selected from remaining 48 cards.

$\therefore$ No. of ways $= {}^4C_1 \times {}^{48}C_4 = 4 \times \dfrac{48 \times 47 \times 46 \times 45}{4 \times 3 \times 2 \times 1} = 7{,}78{,}320.$

Example 58: *From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen?*

Solution: There are two possibilities:

Case I: *When the 3 particular students join the party. In this case, we have to select 7 more students from the remaining 22 students. This can be done in* ${}^{22}C_7$ ***ways.***

Case II: *When the 3 particular students do not join the party. In this case, we have to select 10 students from the remaining 22 students. This can be done in* ${}^{22}C_{10}$ *ways*

So, total number of ways $= {}^{22}C_7 + {}^{22}C_{10} = 170544 + 646646 = \dfrac{22 \times 19 \times 17 \times 3 \times 8}{10 \times 9 \times 8}(720 + 2{,}730) = 8{,}17{,}190.$

Example 59: *A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done, if atleast 5 women have to be included in a committee? In how many of these committees:*

(a) the women are in majority, (b) the men are in majority?

Solution: The committee can be selected in the following way:

9 Women	5	6	7	8	9	
8 Men	7	6	5	4	3	$\longrightarrow$ (at least 5 women)

$\therefore$ Total, number of ways $= {}^9C_5 \times {}^8C_7 + {}^9C_6 \times {}^8C_6 + {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3$

$= 1{,}008 + 2{,}352 + 2{,}016 + 630 + 56 = 6{,}062$

Number of committees in which women are in majority $= 2{,}016 + 630 + 56 = 2{,}702$

Number of committees in which men are in majority $= 1{,}008.$

Example 60: *If m parallel lines in a plane are intersected by a family of n parallel lines, find the number of parallelograms formed.*

Solution: A parallelogram is formed when two lines from the set of m parallel lines are intersected by the other 2 lines from another set of n parallel lines.

Hence, the number of parallelograms formed in this case

$$= {}^mC_2 \times {}^nC_2 = \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn(m-1)(n-1)}{4}.$$

Example 61: *A student is allowed to select at most n books from a collection of (2n + 1) books. If the total number of ways in which he can select at least one book is 255, find the value of n.*

Solution: The student may select at most n books out of $(2n + 1)$ books. Thus, according to the given condition.

$${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots + {}^{2n+1}C_n = 255$$

$\Rightarrow \quad \frac{1}{2}[2 \cdot {}^{2n+1}C_1 + 2 \cdot {}^{2n+1}C_2 + 2 \cdot {}^{2n+1}C_3 + \dots + 2 \cdot {}^{2n+1}C_n] = 255$

$\Rightarrow \quad \frac{1}{2}[({}^{2n+1}C_1 + {}^{2n+1}C_1) + ({}^{2n+1}C_2 + {}^{2n+1}C_2) + \dots + ({}^{2n+1}C_n + {}^{2n+1}C_n)] = 255$

$\Rightarrow \quad \frac{1}{2}[({}^{2n+1}C_1 + {}^{2n+1}C_{2n}) + ({}^{2n+1}C_2 + {}^{2n+1}C_{2n-1}) + \dots$

$+ ({}^{2n+1}C_n + {}^{2n+1}C_{n+1})] = 255 \quad (\because {}^mC_r = {}^mC_{m-r})$

$\Rightarrow \quad \frac{1}{2}[({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots + {}^{2n+1}C_{2n})] = 255$

$\Rightarrow \quad \frac{1}{2}[({}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \dots + {}^{2n+1}C_{2n+1}) - ({}^{2n+1}C_0 + {}^{2n+1}C_{2n+1})] = 255$

$\Rightarrow \quad \frac{1}{2}[2^{2n+1} - (1+1)] = 255 \qquad (\because {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n)$

$\Rightarrow \quad 2^{2n+1} - 2 = 510$ or $2^{2n+1} = 512$ or $2^{2n+1} = 2^9$ or $2n + 1 = 9$ or $n = 4$.

Example 62: *A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done when*

(i) at least 2 ladies are included, (ii) at most 2 ladies are included.

Solution: Total gents = 6, Total ladies = 4

(i) At least 2 ladies out of 4 ladies are included in the following ways

(1) 3 gents out of 6 gents + 2 ladies out of 4 ladies = ${}^6C_3 \times {}^4C_2$ ways

(2) 2 gents out of 6 gents + 3 ladies out of 4 ladies = ${}^6C_2 \times {}^4C_3$

(3) 1 gent out of 6 gents + 4 ladies out of 4 ladies = ${}^6C_1 \times {}^4C_4$

The total number of ways of forming the committee = ${}^6C_3 \times {}^4C_2 + {}^6C_2 \times {}^4C_3 + {}^6C_1 \times {}^4C_4 = 120 + 60 + 6 = 186$

(ii) At most 2 ladies out of 4 ladies are included in the following ways

(1) No lady and 5 gents = ${}^6C_5$ ways

(2) 1 lady and 4 gents = ${}^4C_1 \times {}^6C_4$ ways

(3) 2 ladies and 3 gents = ${}^4C_2 \times {}^6C_3$ ways

Thus, the required ways of forming committee = ${}^6C_5 + {}^4C_1 \times {}^6C_4 + {}^4C_2 \times {}^6C_3 = 186$.

Example 63: *A polygon has 44 diagonals. Find the numbers of its sides.*

Solution: The number of diagonals of n sided polygon = ${}^nC_2 - n = \frac{n(n-3)}{2}$

Given that: $\frac{n(n-3)}{2} = 44$

$\Rightarrow \quad n^2 - 3n = 88$ or $n^2 - 3n - 88 = 0$ or $(n-11)(n+8) = 0$ or $n = 11 \quad [\because n > 0]$

Example 64: *A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?*

Solution: Total red balls = 5. Total white balls = 6

6 balls can be selected from the box so that there are at least two balls of each colour in the following ways:

(i) 2 red balls and 4 white balls = ${}^5C_2 \times {}^6C_4$ ways

(ii) 3 red balls and 3 white balls = ${}^5C_3 \times {}^6C_3$ ways

(iii) 4 red balls and 2 white balls = ${}^5C_4 \times {}^6C_2$

Hence, the required ways = ${}^5C_2 \times {}^6C_4 + {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2 = 150 + 200 + 75 = 425$

Example 65: *In how many ways can 21 identical books on English and 19 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together?*

Solution: Let E denote the position of an English book and H of Hindi book on a shelf. Since no two books of Hindi must be together on the shelf therefore, we first arrange all books of English in a row. Since all the books of English are identical so they can be arranged in a row in only one way as shown below

$$H\,E\,H\,E\,H\,E\,H \ldots\ldots H\,E\,H\,E$$

As there are 21 books of English so there will be 22 places for Hindi books. Now we have 19 identical books of Hindi, therefore out of 22 places, 19 places.

Total no. of ways $= {}^{22}C_{19} = \dfrac{\lfloor 22}{\lfloor 19 \lfloor 22-19} = \dfrac{22 \times 21 \times 20}{3 \times 2 \times 1} = 1540$

Example 66: *In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers ? Assume that the team of 11 players requires 5 batsman, 3 all-rounders, 2 bowlers and 1 wicket keeper.*

Solution: The selection of team is divided into 4 phases :

(i) 5 batsman out of 10 batsman can be selected in ${}^{10}C_5$ ways.

(ii) 3 all-rounders out of 5 all-rounders can be selected in ${}^5C_3$ ways.

(iii) 2 bowlers out of 8 bowlers can be selected in ${}^8C_2$ ways.

(iv) 1 wicket keeper out of 2 wicket keepers can be selected in ${}^2C_1$ ways.

$\therefore$ Team can be selected in ${}^{10}C_5 \times {}^5C_3 \times {}^8C_2 \times {}^2C_1$ ways = 1,41,120 ways.

Example 67: *In how many ways can a committee of 3 ladies and 4 gentlemen be appointed from a group consisting of 8 ladies and 7 gentlemen ? What will be the number of ways, if Aishwarya of this committee refuses to serve in a committee in which Salman is a member?*

Solution: (i) 3 ladies can be selected from 8 ladies in ${}^8C_3$ ways and 4 gentlemen from 7 gentlemen in ${}^7C_4$ ways.

$\therefore$ No. of ways in which the committee can be appointed $= {}^8C_3 \times {}^7C_4 = 56 \times 35 = 1{,}960$.

(ii) First we find the no. of committees of 3 ladies and 4 gentlemen in which both Aishwarya and Salman are members. In this case, we can select 2 other ladies from the remaining 7 in ${}^7C_2$ ways and 3 other gentlemen from remaining 6 in ${}^6C_3$ ways.

$\therefore$ No. of ways in which both Aishwarya and Salman are always included $= {}^7C_2 \times {}^6C_3 = 21 \times 20 = 420$

Hence, the required no. of committees in which Aishwarya and Salman do not serve together

$$= 1{,}960 - 420 = 1{,}540.$$

Answers

1.	(a) 1,365 (b) 1,001 (c) 364	**2.**	25	**3.**	35,960	**4.**	35
5.	35,960	**6.**	1,464	**7.**	722	**8.**	(i) 57 (ii) 210
9.	11	**10.**	$^{52}C_{18} \times {}^{35}C_2 + {}^{52}C_{19} \times {}^{35}C_1 + {}^{52}C_{20}$				
11.	64	**12.**	20	**13.**	63	**14.**	41
15.	(i) 120 (ii) 344	**16.**	2,200				

14.4 Permutations and Combinations with Unlimited Repetitions

Let $U(n, r)$ denote-permutation of n-objects with unlimited repetitions, and $V(n, r)$ denote the number of r-combinations with unlimited repetitions, then

$U(n, r) = n^r$ and $V(n, r) = (n-1+r, n-1)$ consider the set $\{\infty, a_1, \infty, a_2, \ldots \infty, a_n\}$ where $a_1, a_2, \ldots a_n$ are all distinct. Any r-combination is of the form $\{x_1, a_1, x_2, a_2 \ldots x_n, a_n\}$ where each x_i is non-negative integer and

$$x_1 + x_2 + \ldots + x_n = r$$

The number $x_1 + x_2 + \ldots + x_n$ are called **Repetition Numbers**. Conversely any sequence of non-negative integers $x_1 + x_2 + \ldots + x_n$, where $\sum_{i=1}^{n} x_i = r$

Corresponding to $a-r$ combination $\{x_1, a_1, x_2, a_2, \ldots x_n, a_n\}$. The following results are made.

The number of r-combinations of $\{\infty, a_1, \infty, a_2, \ldots \infty, a_n\}$

= The number of non-negative integers solution of $x_1 + x_2 + x_3 + \ldots + x_n = r$

= The number of ways of placing r-indistinguishable balls in n numbered boxes.

= The number of binary number with $n-1$ one r and r-zeros $= C(n-1+r, r) = C(n-1+r, n-1)$

Example 68: *How may solution are there to* $x_1 + x_2 + x_3 + x_4 + x_5 = 16$. *Where each* $x_i \geq 2$?

Solution: Let $x_i = y_i + 2$ where $y_i \geq 0$

∴

Then $$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

or $$y_1 + 2 + y_2 + 2 + y_3 + 2 + y_4 + 2 + y_5 + 2 = 16$$

$$y_1 + y_2 + y_3 + y_4 + y_5 = 6$$

∴ The number of integer solution of $x_1 + x_2 + x_3 + x_4 + x_5 = 16$

is the same as the number of integral solution as $y_1 + y_2 + y_3 + y_4 + y_5 = 6$

Therefore, there are $C(5-1+6, 6) = C(10, 6)$ such solution.

Example 69: *How many integral solutions are there of* $x_1 + x_2 + x_3 + x_4 + x_5 = 30$?

Where $$x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2, x_5 \geq 0$$

[U.P.T.U. (B.Tech.) 2009]

Solution: Let $x_1 = y_1 + 2, x_2 = y_2 + 2, x_3 = y_3 + 4, x_4 = y_4 + 2, x_5 = y_5 + 2$

Now, $$x_1 + x_2 + x_3 + x_4 + x_5 = 30$$

Then, $y_1+2+y_2+2+y_3+4+y_4+2+y_5+2=30$

or $y_1+y_2+y_3+y_4+y_5=19$

The number of integral solutions of $x_1+x_2+x_3+x_4+x_5=30$

Where $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2, x_5 \geq 0$

is the same as the number of integral solution of $y_1+y_2+y_3+y_4+y_5=19$

Therefore, there are $C(5-1+19, 19)=C(23, 19)$ such solution

Example 70: *Enumerate the number of non-negative integral to the inequality* $x_1+x_2+x_3+x_4+x_5 \leq 19$

[R.G.P.V. (B.E.) Raipur 2005, 2009; Pune (B.E.) 2008]

Solution: We can express the given inequality

$$x_1+x_2+x_3+x_4+x_5 \leq 19$$

As
$$x_1+x_2+x_3+x_4+x_5=0$$
$$x_1+x_2+x_3+x_4+x_5=1$$
$$x_1+x_2+x_3+x_4+x_5=2$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$x_1+x_2+x_3+x_4+x_5=19$$

The number of non-negative integral solution of $x_1+x_2+x_3+x_4+x_5=0$ is $C(5-1+0, 0)$

The number of non-negative integral solution of $x_1+x_2+x_3+x_4+x_5=1$ is $C(5-1+1, 1)$

....

....

The number of non-negative integral solution of $x_1+x_2+x_3+x_4+x_5=19$ is $C(5-1+19, 19)$

$\therefore$ The number of non-negative integral solution of

$$x_1+x_2+x_3+x_4+x_5 \leq 19$$

is $C(5-1+0, 0)+C(5-1+1, 1)+\ldots+C(5-1+19, 19)$

Example 71: *Find the number of 3-combinations of* $\{\infty, a_1, \infty, a_2, \infty, a_3, \infty, a_4\}$ [Osmania (B.E.) 2008]

Solution: We have $n=4, r=3$

$\therefore$ The number of 3-combination of the given set is $C(4-1+3, 3)=C(6, 3)=6C_3=\dfrac{6\times5\times4}{3\times2}=20$

Example 72: *Find the number of non-negative integral solution to*

(i) $n_1+n_2+n_3+n_4=20$ [U.P.T.U. (B.Tech.) 2004]

(ii) $C_1+C_2+C_3+C_4=25$ [U.P.T.U. (B.Tech.) 2005]

Solution: We have $n=4, r=20$

$\therefore$ The number of non-negative integral solution

$$=C(4-1+20, 20) \quad =C(23, 20) \quad =23C_{20} \quad =\frac{\lfloor 23}{\lfloor 20 \lfloor 3}=1,771$$

(ii) The given equation is $C_1+C_2+C_3+C_4=25$

We have $n=4, \ r=25$

The number of solutions $=C(4-1+25, 25) \quad =C(28, 25) \quad =28C_{25}=3276$

mple 73: *Find the number of ways of placing 8 similar balls in 5 numbered bones.*

[M.K.U. (B.E.) 2004]

tion: The number of ways of placing similar balls in 5 numbered boxes

$$= C(5-1+8,8) \quad = C(12,8) \quad = 12C_8 \quad = \frac{\lfloor 12}{\lfloor 8 \lfloor 4} = 495$$

mple 74: *How many outcomes are possible by casting a 6 faced die 10 times.* [Kurukshetra (B.E.) 2007]

ution: The number of ways of costing a 6 faced die 10 times

$$= C(6-1+10,10) \quad = C(15,10) \quad = 15C_{10} = \frac{\lfloor 15}{\lfloor 10 \lfloor 5} = 30,03$$

mple 75: *How many solution does $x_1 + x_2 + x_3 = 13$ have where x_1, x_2, x_3 are non-negative integers with $\le 4, x_2 \le 5$ and $x_3 \le 6$.* [U.P.T.U. (B.Tech.) 2003; P.T.U. (B.E.) Punjab 2006; Nagpur (B.E.) 2004, 2008]

lution: Let S be the set of all integers solutions of the given equation with each $x_i \ge o, i = 1, 2, 3$ and let A_2 and A_3 be the set of integer solutions with $x_1 > 4$, $x_2 > 5$ and $x_3 > 6$ or equivalently $x_1 \ge 5$, $x_2 \ge 6$ and ≥ 7 respectively. Then the number of solutions with $0 \le x_1 \le 4$, $0 \le x_2 \le 5$ and $0 \le x_3 \le 6$ will be $|A'_1 \cap A'_2 \cap A'_3|$.

Now $|S|$ = total number of solutions $= {}^{13+3-1}C_{13} = {}^{15}C_{13} = \dfrac{15\times 14}{2\times 1} = 105$

$|A_1|$ = number of solutions with $x_1 \ge 5$ $= {}^{(13-5)+3-1}C_{13-5} = {}^{10}C_8 = \dfrac{10\times 9}{2\times 1} = 45$

$|A_2|$ = number of solutions with $x_2 \ge 6$ $= {}^{(13-6)+3-1}C_{13-6} = {}^{9}C_7 = \dfrac{9\times 8}{2\times 1} = 36$

$|A_3|$ = number of solutions with $x_3 \ge 7$ $= {}^{(13-7)+3-1}C_{13-7} = {}^{8}C_6 = \dfrac{8\times 7}{2\times 1} = 28$

$|A_1 \cap A_2|$ = number of solutions with $x_1 \ge 5$ and $x_2 \ge 6$ $= {}^{(13-5-6)+3-1}C_{13-5-6} = {}^{4}C_2 = \dfrac{4\times 3}{2\times 1} = 6$

$|A_1 \cap A_3|$ = number of solutions with $x_1 \ge 5$ and $x_3 \ge 7$ $= {}^{(13-5-7)+3-1}C_{13-5-7} = {}^{3}C_1 = 3$

$|A_2 \cap A_3|$ = number of solutions with $x_2 \ge 6$ and $x_3 \ge 7$ $= {}^{(13-6-7)+3-1}C_{13-6-7} = {}^{2}C_0 = 1$

$|A_1 \cap A_2 \cap A_3|$ = number of solutions with $x_1 \ge 5$, $x_2 \ge 6$ and $x_3 \ge 7$

$= 0$, since $x_1 \ge 5$, $x_2 \ge 6$ and $x_3 \ge 7$ and so $x_1 + x_2 + x_3$ would exceed 13. Now by the principle of inclusion-exclusion, we have

$$|A'_1 \cap A'_2 \cap A'_3| = |S| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

$$= 105 - 45 - 36 - 28 + 6 + 1 + 3 - 0 = 6.$$

Thus, the number of solutions with $0 \le x_1 \le 4$, $0 \le x_2 \le 5$ and $0 \le x_3 \le 6$ is 6.

A box contains 3 white 3 black balls and 4 red balls. In how many ways three balls be drawn from the box if atleast one black ball is to be include in the draw?

Find the 3-combinations of $\{3, a, 2, b, 2, c, 1, d\}$

Find the number of 4-combinations of $\{\infty, a_1, \infty, a_2, \infty, a_3, \infty, a_4, \infty, a_5\}$

Find the number of non-negative integral solution to $x_1 + x_2 + x_3 + x_4 + x_5 = 50$

How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 20$, where $x_i \geq 2 \quad \forall_i = 1, 2, 3, 4, 5$

Find the number of integral solution to $x_1 + x_2 + x_3 + x_4 + x_5 = 20$

where
$$x_1 \geq 3,\ x_2 \geq 2,\ x_3 \geq 4,\ x_4 \geq 6,\ x_5 \geq 0$$

12. How many different outcomes are possible form tossing 10 similar dice?

13. Enumerate the number of non-negative integral solutions to the inequality
$$x_1 + x_2 + x_3 + x_4 + x_5 \geq 19$$

14. Find the number of integral solution to $x_1 + x_2 + x_3 + x_4 + x_5 \geq 50$

where
$$x_1 \geq 4,\ x_2 \geq 7,\ x_3 \geq -14,\ x_4 \geq 10$$

Answer

1.	(i) 1260 (ii) 1716	2.	(i) 45 (ii) 21 (iii) 35	3.	20	4.	54054
5.	(i) 720 (ii) 210 (iii) 360	6.	64				
7.	$aaa, aab, aac, aad, bba, bbd, cca, ccb, ccd, abc, abd, acd, bcd$			8.	70	9.	316, 251
10.	$C(14, 10)$	11.	$C(9, 5)$	12.	3003		
13.	$C(5-1+0, 4) + C(5-1+1, 4) + \ldots + C(5-1+19, 4)$			14.	$C(54, 3)$		

14.5 Pigeonhole Principle

This principle is also known as ***Dirilhlet Drawer Principle*** or ***the Shoe Box Principle***. It can be stated as follows.

14.5.1 Theorem of Pigeonhole Principle

[U.P.T.U. (B.Tech.) 2005, 2006, 2007; U.P.T.U. (M.C.A.) 2008]

If n pigeons are assigned to m pigeonholes, and $m<n$, then at least one pigeonhole contains two or more pigeons.

Proof: Let $H_1, H_2 \ldots H_m$ denote m pigeonholes and $P_1, P_2, \ldots P_m, P_{m+1}, \ldots P_n$ denote n pigeons. We consider the assignment of n pigeons to m pigeonholes as follows:

Let the pigeon P_1 is assigned to pigeonhole H_1, pigeon P_2 is assigned to pigeonhole H_2... and pigeon P_m to pigeonhole H_m. This assigns as many as pigeons possible to individuals pigeonholes. Since $m < n$, there are $(n-m)$ pigeons that have not yet been assigned. At least one pigeonhole will be assigned to a second pigeon.

Example 77: *Show that among 13 people, there are at least two people who were born in the same month.*

Solution: We consider 13 peoples as pigeons and 12 months (January, February, ... December) as the pigeonholes. Here the number of pigeons is grater than the number of pigeonholes, therefore by pigeonhole principle there will be atleast two people who were born in the same month.

Example 78: *If eight people are chosen in any way from some group, at least two of them will have been born on the same day of the week.* **[Delhi (B.E.) 2005]**

Solution: Here each person (pigeon) is assigned to the day of the week (pigeonhole) on which he or she was born. Since there are eight people and only seven days of the week, then by pigeon hole principle at least two people must be assigned to the same day of the week.

Example 79: *Show that if we choose 9 single shoes out of 8 distinct pairs of shoes, then we are sure to have a pair.* **[Rohtak (B.E.) 2008]**

Solution: Here 8 distinct pairs correspond to 8 pigeonholes and 9 single shoes correspond to pigeons. It means $m = 8$ and $n = 9$, $m < n$ then by pigeonhole principle, there must be one pigeonhole with 2 shoes which means that we have chosen atleast one pair.

Example 80: *Find the minimum number of students in a class so that three of them are born in the same month.* **[Kurukshetra (B.E.) 2007; Osmania (B.E.) 2009]**

Solution: Here $n = 12$ (months) which represent the number of pigeonholes.

$$m = k + 1 = 3 \text{ or } k = 3, \; k \text{ is positive integer}$$

Hence, the minimum number of pigeons $= kn + 1 = 2 \times 12 + 1 = 25$ students.

Example 81: *How many students must there be in a class to guarantee that at least two students receive the same score in the final examination if the examination is graded on a scale from zero to one hundred points.*

Solution: Here $m = 2$ But $m = k + 1$, k is any integer

then $$k + 1 = 2 \Rightarrow k = 1$$

$$m = 101$$

So, total number of students in class to guarantee that atleast two students receive the same score.

$$= kn + 2 \quad = 1 \times 101 + 1 \quad = 102.$$

Example 82: *Show that if any five integers from 1 to 8 are chosen, then atleast two of them will have a sum 9.*

Solution: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, \}$

The different sets, each containing two numbers whose sum is equal to 9 are

$$A_1 = \{1, 8\},\ A_2 = \{2, 7\},\ A_3 = \{3, 6\},\ A_4 = \{4, 5\}.$$

Each of the five number chosen from 1 to 8 must belong to one of these sets. The four sets are consider as pigeonholes and five chosen numbers are taking as pigeon. Since $m = 4$, $n = 5$ *i.e.* $4 < 5$ Then by pigeonhole principle we can say that the two of the selected numbers must belong to the same set whose sum is 9.

Example 83: *Show that in any room of people who have been doing handshaking there will always be at least two people who have shaken hands the same number of times.*

Solution: Let the pigeonholes be labelled with the different numbers of hands shaken and we put the people (pigeons) into their correct pigeonhole. Suppose there are n people, then since only shake hands with each person at most once, the lables on the pigeonholes will go from o to $(n-1)$ *i.e.* we have n holes and n people. It is not possible for the o^{th} and $(n-1)^{th}$ the holes both to be occupied, because if one person has shaken hands with nobody then there can not be any one person who has shaken hands with every other person. Thus, we have at most $(n-1)$ holes occupied at any one time. Hence, by the pigeonholes principle at least one of the holes has two occupants, which shows that there are at least two people who have shaken hands the same number of time.

Example 84: *What shall be the minimum number of words that must begin with the same alphabet at 27 English words.*

Solution: Here, $m = kn+1 = 27$ (Pigeonholes), $n = 26$ (number of alphabet) = (pigeons) -1, where k is any integer

$$\therefore \qquad k \times 26 + 1 = 27 \Rightarrow k = 1 \qquad \text{and} \qquad m = k+1 = 2$$

Hence at least two words must begin with the same alphabet.

Example 85: *If there are 13 boys in a class, then find minimum number of boys born in the same months.*

Solution: Consider month as pigeons *i.e.* $n = 12$ then find pigeonholes m. But $n = 12$, $kn+1 = 13$

$$\Rightarrow \qquad k \times 12 + 1 = 13 \Rightarrow k = 1, \qquad \text{then} \quad m = k+1 = 2$$

Example 86: *Show that if seven numbers from 1 to 12 are chosen, then two of them will add up to 13.*

Solution: There are six sets each containing two number that add up to 13 as follows:

$$A_1 = \{1, 12\},\ A_2 = \{2, 11\},\ A_3 = \{3, 10\},\ A_4 = \{4, 9\}\ A_5 = \{5, 8\},\ A_6 = \{6, 7\}$$

Each of the seven numbers belongs to one of these six sets. Since there are 6 sets, then by pigeonholes principle two of the numbers chosen belong to the same set. These numbers add upto 13.

Example 87: *If 20 candidates appear in a competitive examination then show that there exists at least two among them whose Roll number differ by a multiple of 19.* **[U.P.T.U. (B.Tech.) 2004]**

Solution: Let X be the set of Roll number of 20 candidates and Y be the set of remainder when any positive integer is divided by 19.

$$i.e., \qquad Y = \{0, 1, 2, \ldots, 18\}.$$

Here we define a function $f : X \to Y$ as $f(x)$ = the remainder obtained when any positive integer is divided by 19. Since $|X| = 20$ and $|Y| = 19$ *i.e.*, $|X| > |Y|$, therefore by the pigeonhole principle function f cannot be one-one. This shows that there exists two distinct Roll numbers x_1 and x_2 such that $f(x_1) = f(x_2)$

x_1, x_2 can be written as $x_1 = 19n_1 + f(x_1)$ and $x_2 = 19n_2 + f(x_2)$

where n_1 and n_2 are positive integers.

This gives $\qquad x_1 - x_2 = 19(n_1 - n_2). \qquad [\because f(x_1) = f(x_2)]$

Now $n_1 - n_2$ is an integer. Therefore $x_1 - x_2$ is a multiple of 19.

Hence there are at least two candidate whose Roll numbers differ by a multiple of 19.

Example 88: *If any 51 integers are chosen from the set {1, 2, 3, ..., 100} then show that among the chosen integers there exist two integers such that one is multiple of the other.* [U.P.T.U. (B.Tech.) 2005]

Solution: We know that every positive integer x can be written as $= 2^p \cdot r$, where r is odd and $p \geq 0$ (for example, $12 = 2^2 \cdot 3$ and $13 = 2^0 \cdot 13$). We shall call this odd integer r the odd part of integer x.

If x is in {1, 2, ...,100} then its odd part r can only be in {1, 3, 5, ..., 99}.

Let S be any set of 51 integers chosen from the set {1, 2, 3, ..., 100}. Define a function

$$f: S \to \{1, 3, 5, 7, \ldots, 99\} \text{ by } f(x) = \text{odd part of } x.$$

Since $|S| = 51$ and $|\{1, 3, 5, 7, \ldots, 99\}| = 50$, *it* follows from the pigeonhole principle that f cannot be one-to-one. Thus there exist two integers x and y in S such that their odd parts $f(x)$ and $f(y)$ are same. That is, $f(x) = f(y)$. Therefore, we have

$$x = 2^p \cdot r \text{ and } y = 2^q \cdot r \text{ for some integers } p \text{ and } q.$$

If $p \geq q$ then $x = 2^{p-q} \cdot y$ and therefore x is a multiple of y. If $p < q$ then y is a multiple of x. Thus either x is multiple of y or is multiple of x.

Example 89: Let any given five points to be placed in the interior of an equilateral triangle of side 1. Show that there exists two points within a distance of at most 1/2.

Solution: Let us divide the given equilateral triangle into four equal triangles (pigeonholes) as shown in the figure. The given five points (pigeons) P_1, P_2, P_3, P_4, P_5, say, will be placed in these four triangles. By the pigeonhole principle at least two of them must belong to one of the four small triangles. Clearly the distance between these points (P_4 and P_5 as shown in the Fig. 14.1) cannot exceed the side of the triangle which is $\frac{1}{2}$.

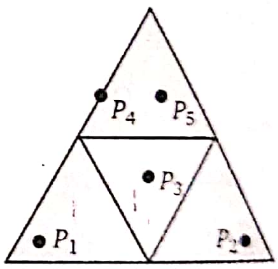


Fig. 14.1

Hence we conclude that there exists two points within a distance of at most $\frac{1}{2}$.

Example 90: *Given any five points inside a square of side 1, show that there exists two points within a distance of at most* $1/\sqrt{2}$.

Solution: Let us divide the given square into four equal squares as shown in the Fig. 14.2 The given five points (pigeons) will be placed in these four squares (pigeonholes). By the pigeonhole principle at least two of them must belong to one of the four small squares. The distance between these two points cannot exceed the length of the diagonal of the small square which is $\frac{1}{2}\sqrt{1^2+1^2}$; *i.e.,* $\frac{1}{\sqrt{2}}$.

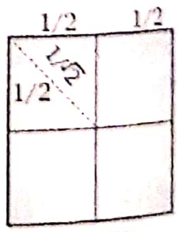


Fig. 14.2

Hence we conclude that there exists two points within a distance of at most $\frac{1}{\sqrt{2}}$.

Example 91: *Show that any sequence of* n^2+1 *distinct integers contains an increasing subsequence of length* $n+1$.

Solution: Suppose $a_1, a_2, a_3, \ldots, a_{n^2+1}$ is a sequence of n^2+1 distinct integers (pigeons). Corresponding to each a_k, we now consider an ordered pair (x_k, y_k), where x_k is the length of a longest increasing subsequence starting at a_k and y_k is the length of a longest decreasing subsequence starting at a_k.

Suppose, if possible, that there is no increasing or decreasing subsequence of length $n+1$ in the sequence a_i of n^2+1 integers. Then it is clear that the set $\{(x_k, y_k)\}$ has at most n^2 distinct ordered pairs (pigeonholes). It follows by the pigeonhole principle that there must exist two elements a_i and a_j in the sequence which correspond to the same ordered pair; i.e.,$(x_i, y_i) = (x_j, y_j)$. But this is not possible because if $a_i < a_j$ then we must have $x_i > x_j$ and if $a_i > a_j$ then we must have $y_i > y_j$. In each case, we have a contradiction. This moves that there is either an increasing subsequence or a decreasing subsequence of length $n+1$.

14.5.2 The Extended Pigeonhole Principle

Theorem: *If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain at least* $\left[\frac{n-1}{m}\right]+1$ or $\left[\frac{n}{m}+1\right]$ *pigeons* $=\left[\frac{p}{q}\right]$ *denotes the largest integer, less than or equal to the rational number* $\left[\frac{p}{q}\right]$.

Proof: *We shall prove this theorem by contradiction. Let each pigeonhole contain no more than* $\left[\frac{n-m}{m}\right]$ *pigeons. Then, there are at most* $m\cdot\left[\frac{n-1}{m}\right]$ *pigeons in all but* $\left[\frac{n-1}{m}\right] < \frac{m(n-1)}{m} \Rightarrow \left[\frac{n-1}{m}\right] = n-1$

i.e. there $(n-1)$ pigeons in all. Which is contradiction our assumption. Hence one of the pigeonholes must contain at least $\left[\frac{n-1}{m}\right]+1$ pigeons.

Example 92: *Show that if any 26 people are selected then we may choose a subset of 4 so that all 4 were born on the same day of the week.*

Solution: We assign each person to the day of the week on which she or he was born. Then number of pigeons (people) are to be assigned to 7 pigeonholes with $n = 26$ and $m = 7$ at least.

$$\left[\frac{n-1}{m}+1\right] \text{ or } \left[\frac{n-1}{m}\right]+1 = \left[\frac{26-1}{7}\right]+1 = 4 \text{ people}$$

must have been on the same day of the week.

Example 93: *There are 38 different time periods during which classes at university can be scheduled if there are 677 different classes, how many different rooms will be needed?*

Solution: Assign each class to a time period then 677 different classes (pigeons) has to be assigned to 38 pigeonholes. Here $n = 677$, $m = 38$.

Then by generalized pigeonhole principle, one of the pigeonholes must contain at least

$$\left[\frac{677}{38}\right]+1 = [17.81]+8 = 17+1 = 18 \text{ pigeons}$$

Example 94: *In a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the groups.*

Solution: Let A be one of the six people of group. There are either three or more among the other five people who are friends of A or enemies of A. Then by generalized pigeonhole principle five objects are divided into two sets, one has at least $\left[\frac{5}{2}+1\right] = 3$ objects. Let B, C and D are friends of A. If any two of these individual are friends then these two and A form a group of three mutual friends. Otherwise B, C and D form a set of three mutual enemies. In the second case suppose B, C and D are enemies of A. If any two of these individual are enemies then these two and A form a group of three mutual enemies otherwise B, C and D form a group of three mutual friends.

Example 95: *How many letters one must choose from the set of 15 $A^{'s}$, $20^{'s}$ B and 25'C, so that 12 identical letters will always be included in the selection.*

Solution: We see that there are three types of letters (pigeonholes) to be chosen from the set having $15A^{'s}$, $20B^{'s}$ and $25C^{'s}$, so that 12 identical letters should always be included in the selection is the smallest integer n such that

$$k+1=12, \quad k=\left[\frac{n-1}{3}\right]$$

or

$$\left[\frac{n-1}{3}\right]+1=12 \Rightarrow [n-1]=11\times 3 \Rightarrow n=34$$

Thus, 34 is the minimum number of letters to be chosen from the set having $15A^{'s}$ $20B^{'s}$ and $25'C$ so that 12 identical letters will always be included in the selection.

Example 96: *What is the minimum number of students required in a class to be that at least five will receive the same grade if there are four possible grades A, B, C and D.*

Solution: *There are four grade A, B, C, D which is the pigeonholes and at least 5 students must receive the same grade. We have to find minimum number of students which is pigeons (n) such that*

$$\left[\frac{n-1}{4}\right]+1=5$$

or

$$\left[\frac{n-1}{4}\right]=4 \quad \text{or} \quad (n-1)=16 \quad \text{or} \quad n=17.$$

Example 97: *Show that if seven colours are used to paint 50 cars, at least eight cars will have the same colour.*

Solution: Let $n=50$ cars $=$ pigeons, $m=7$ colours $=$ pigeonholes

Then, by generalized pigeonhole principle $\left[\frac{n-1}{m}\right]+1=\left[\frac{50-1}{7}\right]+1=8$ Cars will be the same colour.

Example 98: *Show that among 1000 people there are at least 84 people who were born in the same month.*

Solution: Where $n=1000$ people $=$ pigeons, $m=12$ months $=$ pigeonholes

Then, by generalized pigeonhole principle, we have $\left[\frac{n-1}{m}\right]+1=\left[\frac{1000-1}{12}\right]+1=83+1=84$ people. who were born in the same month.

Example 99: *Prove that among 1,00,000 people there are two who born on the same time.*

[R.G..P.V. (B.E.) Bhopal 2001, 2007]

Solution: Let n = pigeons = 1,00,000 people.

A person can be born at any second $=24\times 60\times 60=86,400$ seconds *i. e.*, $m=86,400$ = pigeonholes. By generalized pigeonhole principle

$$\left[\frac{1,00,000-1}{86,400}\right]+1=(1.16)+1=1+1=2 \text{ people are born on same time.}$$