Unit - 3

Algebraic structure :- A set 'S' along with a binary operation '$\star$' is known as an algebraic structure, if it follows the following property:

- Closure property : $\forall\ a, b \in S$, $a \star b \in S$

eg. $(N, +)$, $(Z, +)$, $(Z, -)$, $(Z, \times)$, $(N, \times)$, $(Q-0, \div)$

$a, b \in N$

$a+b \in N$

$a+b = \bigcirc \in N$

Semi Group :- A set 'S' along with a binary operation '$\star$' is known as an semi-group if it follows the following properties :

- Closure property : $\forall\ a, b \in S$, $a \star b \in S$
- Associative property : $\forall\ a, b, c \in S$

  $a+(b+c) = (a+b)+c$

eg. $(N, +)$, $(N, \times)$

$(Z, +)$, $(Z, -)$, $(Z, \star)$

$(\text{Matrix}, \times)$ $(\text{Matrix}, +)$

Monoid :- A set 's' along with a binary operation '$\star$' is known as an monoid if it follows the following properties :

- Closure property : $\forall\ a, b \in S$, $a \star b \in S$
- Associative property : $\forall\ a, b, c \in S$

  $a+(b+c) = (a+b)+c$

- Existence of Identity : $\forall a \in S$, $\exists e \in S$ s.t. $a \star e = a = e \star a$

eg. $2 + \boxed{0} = 2$

$2 \times \boxed{1} = 2$

Group : A set 'S' along with a binary operation '$\star$' is known as an group if it follows the following properties :

- Closure property : $\forall a, b \in S$, $a \star b \in S$
- Associative property : $\forall a, b, c \in S$
  $a + (b + c) = (a + b) + c$
- Existence of Identity : $\forall a \in S$, $\exists e \in S$
  s.t. $a \star e = a = e \star a$
- Existence of Inverse : For $a \in S$, $\exists a^{-1} \in S$ s.t. $a \star a^{-1}$
  $= e$ (identity element) $= a^{-1} \star a$

eg. $2 + \boxed{(-2)} = 0$

$2 \times \boxed{\frac{1}{2}} = 1$

Abelian group : A set 'S' along with a binary operation '$\star$' is known as an albelian group if it follows the following properties :

- Closure property : $\forall a, b \in S$, $a \star b \in S$
- Associative property : $\forall a, b, c \in S$
  $(a + (b + c)) = (a + b) + c$
- Existence of Identity : $\forall a \in S$, $\exists e \in S$
  s.t. $a \star e = a = e \star a$
- Existence of Inverse : For $a \in S$, $\exists a^{-1} \in S$
  s.t. $a \star a^{-1} = e$ (identity element) $= a^{-1} \star a$
- Commutative : $\forall a, b \in S$
  $a \star b = b \star a$

Set with binary composition

A.S.	Properties
1. Alegebraic Structure	Closure
2. Semi group	Closure, Associative
3. Monoid	Closure, Associative, Identity
4. Group	Closure, Associative, Identity, Inverse
5. Abelian Group	Closure, Associative, Identity, Inverse, Commu

Q. Show that cube root of unity is a group.

$G = \{1, \omega, \omega^2\}$ under $*$

Soln:

$*$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	$\omega^3 = 1$
ω^2	ω^2	1	ω

- Closure : Entries in the table belongs to G
  $\therefore$ G is closed under $*$.
- Associative : Cube roots of unity follow associative property.
- Identity : Rows and column corresponding to $1 \in G$ is unchanged $\therefore$ $1 \in G$ is identity element.
- Inverse : Corresponding to entry '1' (which is identity element). we observe inverse of
  - 1 is 1
  - ω is ω^2
  - ω^2 is ω

Q. $Z_m = \{0, 1, 2, \ldots (m-1)\}$ is an abelian group w.r.t addition modulo 'm'.

Addition Modulom ($+_m$)

Soln: Closure : $\forall\, a, b \in Z_m$

$0 \le a +_m b \le (m-1)$ by def^n of addition module m

$\Rightarrow a +_m b \in Z_m$. It is closed

Addition Modulo m ($+_m$)

$(a+b) \bmod m' = a +_m b$ = remainder when $(a+b)$ is divided by 'm'

Multiplication Modulo 'm'

$(a \times b) \bmod m' = a \times_m b$ = remainder when $(a \times b)$ is divided by 'm'

- Associative : $(a +_m b) +_m c = a +_m (b +_m c)$

  Addition module 'm' is associative.

- Identity : $\forall a \in Z_m$ $\quad 0 +_m a = a$

  $\because 0 \le a \le (m-1)$

- Inverse : $\forall a \in Z_m$

For $a \ne 0 \Rightarrow a +_m \underline{(m-a)}$ = Identity element $= 0$ $\quad a +_m m-a = m \Rightarrow m/m$ rem $= 0$

For $a = 0 \Rightarrow$ Inverse $\to$ '0'

- Commutative : $\forall a, b \in Z_m$

  $a +_m b = b +_m a$

  So, it is commutative.

$(R, +_m)$ ✓

$\Rightarrow Z_m$ is abelian group.

$(R, \times_m)$ ✗

Inverse of 0 ✗

Q. If $G = Q - \{-1\}$. P.T. it forms a group under $*$ defined as

$a * b = a + b + ab$

Soln: Closure : $\forall a, b \in G$

$a * b = a + b + ab \in G$

$a + b + ab = -1$

$a(1+b) = -1(1+b)$

$\boxed{a = -1}$

or

$\boxed{b = -1}$ $\quad$ but $a \ne -1 \ne b$

$-1 \notin G$

- Associative : $a, b, c \in G$

$\because$ a, b, c are rational no.

$\Rightarrow$ rational no. are associative.

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- Identity : $\forall\ a \in G$ let $e \in G$ be identified.
by defⁿ :

$$a * e = a$$
$$\cancel{a} + e + ae = \cancel{a}$$
$$e(1+a) = 0$$

$\Rightarrow$ $e = 0$ or $1+a = 0 \longrightarrow$ This is not possible

$\therefore$ $e = 0 \in G$ is identity element.

- Inverse : let $a \in G$ $\quad a^{-1} \in G$

$a * a^{-1} = e = 0 \Rightarrow a + a^{-1} + aa^{-1} = 0$

$\Rightarrow$ $a^{-1} = \dfrac{a}{1+a} \in G$

$1 + a \neq 0$

$\therefore$ G is a group.

Order of a group : No. of elements in a group.
Finite or Infinite group : order finite $\rightarrow$ Finite group
Order infinite $\rightarrow$ infinite group

Cyclic Monoid / Group
$N = \{1, 2, 3, \ldots\ldots\}$

$1 = 1$
$2 = 1+1$
$3 = 1+1+1$
$\vdots$
$n = 1+1+1+1+\ldots\ldots + n$ times

- A ~~cyclic~~ Monoid / Group $(G, *)$ is said to be cyclic if it can be generated by a ~~simple~~ single element of G.

$G = \{a^n \mid a \in G\}$
$a^n = a * a * a * \ldots\ldots n$ times

Eg. $(N, +)$ is a cyclic monoid

Th-1: Every cyclic group is Abelian.

Sol^n: Let $x, y \in G$

$\because$ G is cyclic, so it has a generator $a \in G$

$x = a^m \qquad y = a^n \qquad m, n \in Z$

$x.y = a^m . a^n = a^{m+n}$

$= a^{n+m}$ [Integers are commutative

$= a^n . a^m$

$= y.x$

$\Rightarrow$ $xy = yx$

$\Rightarrow$ Commutativity is verified, so G is abelian.

Th-2: If $a \in G$ is generator of cyclic group G then a^{-1} is also its generator.

Let $a \in G$ is generator

then every element of G can be written in terms of G.

Let $x \in G \Rightarrow x = a^n \qquad n \in Z$

$\Rightarrow \quad = (a^{-1})^{-n}$

$= (a^{-1})^p \qquad p = -n \in Z$

$\Rightarrow$ a^{-1} is also a generator of G.

Eg. $Z^+_4 = \{0, 1, 2, 3\}$ Addition Modulo

1

$1 +_4 1 = 2$

$1 +_4 1 +_4 1 = 3$

$1 +_4 1 +_4 1 +_4 1 = 0$

$\therefore$ Cyclic group

$1 +_4 \underline{3} = 0$

$\therefore$ Inverse of '1' $= 3$

3

$3 +_4 3 = 2$

$3 +_4 3 +_4 3 = 1$
$3 +_4 3 +_4 3 +_4 3 = 0$

Order of cyclic group : If a is generator of group then the least +ve integer m is called order of cyclic group if $a^m = e$.

Eg. 1. $Z_4^+ = \{0, 1, 2, 3\}$
Order = 4

2. $\{1, \omega, \omega^2\}$ X
Order = 3.

Order of an element in a group : The least +ve integer 'm' such that $a^m = e$ then m is called order of 'a'.
If no such integer exists, then element is said to be infinite order.

Eg. Let $x = \{a, b\}$
a b
aa, ab, ba, bb
aaa, aab, aba, baa, bba ------
$X^* = \{\underbrace{\ }_{\text{empty string}}, a, b, aa, ab, ba, bb, aaa, aab \ldots\}$
Operation — concatenation.
$a(bc) = (ab)c$ Associative
$aab + \omega = aab$ Identity

Free Monoid : Let X be the set of generators. Then set X^* of all the possible combinations of elements of X (including empty string, repetition allowed) is called a free monoid under the ~~string~~ composition usually taken as concatenation.

Eg. Let $X = \{a, b\}$ → Set of generator

Define set $X^{-1} = \{a^{-1}, b^{-1}\}$, contains inverses.

$$X \cup X^{-1} = \{a, b, a^{-1}, b^{-1}\}$$

Now, set of generators

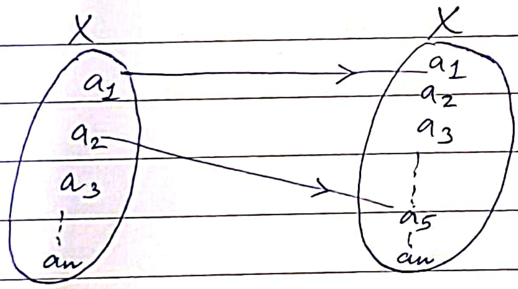
$a \quad b \quad a^{-1} \quad b^{-1}$

$aa^{-1} \quad a^{-1}b \quad ab^{-1} \quad ab$ - - - - - - - -

Free Group : Let X be the given set. Define X^{-1} which contains inverses of elements of X. Generate $X \cup X^{-1}$. Free group $F(X)$ is the set of all sequences of elements of $X \cup X^{-1}$: such that cancellation is allowed like $aa^{-1} = e$ $a^{-1}a$.

Permutation

Let $X = \{a_1, a_2 \ldots\ldots a_n\}$ then permutation $f : X \to X$ is a bijection or one-one onto map.



$$f = \begin{pmatrix} a_1 & a_2 & \cdots\cdots & a_n \\ f(a_1) & f(a_2) & \cdots\cdots & f(a_n) \end{pmatrix}$$

Permutation Group

Set of all possible permutations is called permutation group / Symmetric group (P_n or S_n) under composition of mappings.

$\boxed{\text{defined as } f * g = g \circ f}$

Degree of Permutation group : No. of elements in the set that is to be permutated.

Q. Set of all permutations under composition of mapping is group.

Proof: Let $X = \{1, 2, ---- n\}$

$f : X \to X$

$S_n = \{ f : \forall f : X \to X \text{ is one-one} \}$

Define $f * g = g \circ f$

① Closure - $\forall f, g \in S_n$

$$f * g = g \circ f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & n \\ g(1) & g(2) & \cdots & g(n) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \cdots & n \\ g(f(1)) & g(f(2)) & \cdots & g(f(n)) \end{pmatrix} \in S_n$$

② Associative : $\forall f, g, h \in S_n$

$$f \circ (g \circ h) = (f \circ g) \circ h$$

Composition of mapping is associative.

③ Identity : $e \in S_n$

$e \circ f = f \circ e = f \quad \forall f \in S_n$

$$e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

$$e \circ f = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix} = f$$

④ Inverse : $f \in S_n \cdots f^{-1} \in S_n$

$$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix} \qquad f^{-1} = ? = \begin{pmatrix} f(1) & f(2) & \cdots & f(n) \\ 1 & 2 & \cdots & n \end{pmatrix}$$

$f \circ f^{-1} = f^{-1} \circ f = e$

OR f is a bijection $\therefore f^{-1}$ exists.

$f\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} & \\ & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

Inverse : $\begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

* Abelian – (1) S_n is not abelian for $n \geq 3$

(2) $n = 1$ $\{1\}$

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UNIT-4 COMBINATONES [Counting Techniques]

1. Principle of Inclusion & Exclusion

① $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

2. Pigeonhole Principle

If n objects (pigeons) are to be distributed in m ways (holes) where $n > m$, then $\exists$ atleast one hole which contains more than one object (pigeon).

Proof : $f : X \to Y$ is onto mapping s.t. $n(X) > n(Y)$

$X \to$ set of objects

$Y \to$ set of holes

then f being a function, every element of 'X' should be mapped onto some element of Y, which is possible only if atleast two elements of 'X' are mapped to same element of 'Y'.

$\Rightarrow$ f is many-one.

Q. A bag contains 10 Red, 10 white and 10 blue marbles. What is the minimum no. of marbles that we choose randomly from the bag to ensure that we get 4 marbles of same colour.

↓ Worst case scenario

10 R 10 B 10 W (Objects)

$\overset{R}{\boxed{3}} + \overset{B}{\boxed{3}} + \overset{W}{\boxed{3}} + 1$ (Worst Case Scenario)

Let us first distribute these objects equally and postpone the given task (4 marbles of same colour) until possible. After this add '1' more to the collection.

$\therefore$ minimum no. of attempts needed to complete the task.

$= 3+3+3+1 = 10$

Q. Minimum no. of students in class, so that atleast 3 of them share same birth month.

No. of holes = 12

$2 \times 12 = 24$ (Equally first)

$24 + 1 = 25$ (3 students same birth month)

Q. In cse dept, a club can be formed with either 10 students of 1^{st} year or 8 of 2^{nd} year or 6 from 3^{rd} year or 4 from final year, what is minimum no of students we may choose randomly from dept, to ensure that club is formed.

$$\underset{1^{st}\ yr}{\overset{10-1}{\boxed{9}}} + \underset{2^{nd}\ yr}{\overset{8-1}{\boxed{7}}} + \underset{3^{rd}\ yr}{\overset{6-1}{\boxed{5}}} + \underset{4^{th}\ yr}{\overset{4-1}{\boxed{3}}} + 1 = 25$$

Strong form of pigeon hole principle

Let $q_1, q_2, \ldots, q_n$ be +ve integers.

If $q_1 + q_2 + q_3 + \ldots + q_n - n + 1$ objects (or $1 + \Sigma(q_i - 1)$) are put into ~~the 1st box contains at~~ 'n' boxes then either the 1^{st} box contains atleast q_1 objects or 2nd box contains atleast q_2 objects or ----- n^{th} box contain atleast q_n objects.

Q. If there are 35 different time period, during which class can be scheduled. Find the no. of rooms required, if 679 different classes are to be scheduled.

Soln: $\left\lceil \frac{679}{35} \right\rceil = \lceil 19.4 \rceil = 20$ rooms

Permutation & Combination

- Arrangement | Selection

a, b, c

- ab, ba, ac, ca, bc, cb | ab, bc, ac.

- ^nP_r
arranging r objects of out n
$r \le n$

^nC_r
selecting r objects out of n

- $^nP_r = \dfrac{\lfloor n}{\lfloor n-r}$

$^nC_r = \dfrac{\lfloor n}{\lfloor r \, \lfloor n-r}$

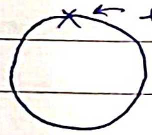
- Repitition allowed : n^r
$n \rightarrow$ no. of objects to be arranged
$r \rightarrow$ no. of ways / place at which they are to be arranged.

eg. 6 6 6 6 6 $= 6^5$

- Circular permutations (n) :

→ Repetition not allowed $= \lfloor n-1$

x ← fixed (1)



→ no. of ways of necklace (identical objects) $= \dfrac{1}{2} \lfloor n-1$

- Identical objects
Total objects $= n$
$K_1 \rightarrow$ identical
$K_2 \rightarrow$ identical
no. of ways $= \dfrac{\lfloor n}{\lfloor K_1 \, \lfloor K_2}$

Q. Distribute (no. of ways) 6 identical candies to 4 children.

Soln:

$$\boxed{x_1} + \boxed{x_2} + \boxed{x_3} + \boxed{x_4} = 6$$

$x_1, x_2, x_3, x_4 \geq 0$

★ |
Star – bars Technique.
(objects) (b)
(O)

★ | ★ ★ | ★ ★ | ★

0 | ★ ★ | ★ ★ | ★ ★

$$\boxed{\text{No. of ways} = \binom{O+b-1}{O} \text{ or } \binom{O+b-1}{b-1}}$$

combination combination

Ans. No. of ways $= \binom{6+4-1}{6} = {}^9C_6$

Q. How many integer solⁿ are possible for $x_i \geq 0$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

Solⁿ: Objects = 16

$b = 5$

No. of ways $= \binom{O+b-1}{O} = \binom{16+5-1}{16} = {}^{20}C_{16}$

Q. $x_1 + x_2 + x_3 + x_4 + x_5 = 16$ s.t. $x_1 \geq 2$ $x_2 \geq 3$ $x_5 \geq 2$

Let $x_1 = 2 + y_1$ $y_1 \geq 0$

$x_2 = 3 + y_2$ $y_2 \geq 0$

$x_3 = y_3$ $y_3 \geq 0$

$x_4 = y_4$ $y_4 \geq 0$

$x_5 = 2 + y_5$ $y_5 \geq 0$

$y_1 + y_2 + y_3 + y_4 + y_5 = 16 - 7 = 9$ $y_i \geq 0$

No. of ways = $\binom{9+5-1}{9} = {}^{13}C_9 = {}^{13}C_4$ $\quad {}^nC_r$ or ${}^nC_{n-r}$

Generating function

$(1+x)^n = 1 + {}^nC_1 x^1 + {}^nC_2 x^2 + \dots\dots + {}^nC_r x^r + \dots\dots {}^nC_n x^n$ $\quad$ (${}^nC_0 x^0$)

$+ 0 + 0 + 0 \dots\dots$

$a_{n+2} x^{n+1}$ $= 0$

$$\boxed{A(z) = \sum_{r=0}^{\infty} a_r z^r}$$

a_r → constants

$A(z) = a_0 + a_1 z + a_2 z^2 + \dots\dots + a_n z^n + \dots\dots \infty$

Q1. Find Generating function

$\underset{a_0}{1}, \underset{a_1}{1}, \underset{a_2}{1}, \underset{a_3}{1}, \underset{a_4}{1}, \underset{a_5}{1}, \underset{a_6}{1}, 0, 0, \dots\dots$

Let G.F. be $A(z) = \sum_{r=0}^{\infty} a_r z^r$

$A(z) = 1 + z + z^2 + z^3 + z^4 + z^5 + z^6$

$A(z) = 1\left(\frac{1-z^7}{1-z}\right) = 1\left(\frac{z^7-1}{z-1}\right)$

Q2. $1, \frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \dots\dots \left(\frac{r+1}{3^r}\right), \dots\dots$

Let G.F. be $A(z) = \sum_{r=0}^{\infty} a_r z^r$

$A(z) = 1 + \frac{2}{3}z + \frac{3}{9}z^2 + \frac{4}{27}z^3 + \dots\dots + \left(\frac{r+1}{3^r}\right)z^r + \dots\dots$

$= 1 + 2\left(\frac{z}{3}\right) + 3\left(\frac{z}{3}\right)^2 + 4\left(\frac{z}{3}\right)^3 + \dots\dots$

$+ (r+1)\left(\frac{z}{3}\right)^r + \dots\dots$

$= (1-x)^{-2} = \left(1-\frac{z}{3}\right)^{-2}$

1. $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^n$ (Finite)

2. $(1+x)^{-1} = 1-x+x^2-x^3+\ldots\ldots$

3. $(1-x)^{-1} = 1+x+x^2+x^3+\ldots\ldots$

4. $(1+x)^{-n} = 1-nx+\frac{n(n+1)}{\lfloor 2}x^2 - \frac{n(n+1)(n+2)}{\lfloor 3}x^3+\ldots\ldots$

5. $(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\ldots\ldots$

Q. $a_r = 5^r + (-1)^r 3^r + 8^r + {}^3C_r$

Soln: $A(z) = \sum_{r=0}^{\infty} 5^r z^r + \sum_{r=0}^{\infty} (-3)^r z^r + \sum_{r=0}^{\infty} (8)^r z^r + \boxed{\sum_{r=0}^{\infty} {}^3C_r z^r}$

$= (1+5z+(5z)^2+\ldots\ldots) +$

$= \frac{1}{1-5z} + \frac{1}{1+3z} + \frac{1}{1-8z} + (1+z)^3$

Recurrence Relation / Difference Equation

$\boxed{F_n = F_{n-1} + F_{n-2}}$ Fibonacci Series

$F_0 = 0$
$F_1 = 1$] Initial Condition

Solution of Recurrence relation :-

Q1. $y_{n+2} - y_{n+1} + y_n = 0$, $y_0 = 1$ $y_1 = \frac{1+\sqrt{3}}{2}$

Soln: $y_n = \alpha^n$ is $\lambda\alpha^n$

Characteristic eqn:

$$\alpha^{n+2} - \alpha^{n+1} + \alpha^{n} = 0$$

$$\alpha^{n}[\alpha^2 - \alpha + 1] = 0$$

No use

Trivial Soln : $\alpha = 0$ (n times) or $\alpha^2 - \alpha + 1 = 0$

$$\alpha^2 - \alpha + 1 = 0$$

$$\alpha = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2} \text{ (complex roots)}$$

Method : Find characteristic roots by putting $y_n = \alpha^n$ in RHS of RR.

→ If roots are real and distinct

$$GS = C_1(\alpha_1)^r + C_2(\alpha_2)^r + \ldots\ldots + C_n(\alpha_n)^r$$

→ Real and equal [α(2 times)]

$$GS = (C_1 + C_2 r)\alpha^r$$

→ Complex Roots $\alpha \pm i\beta$

$$GS = \rho^r [A\cos(r\theta) + B\sin(r\theta)]$$

$$\rho = \sqrt{\alpha^2 + \beta^2} \qquad \theta = \left(\tan^{-1}\frac{\beta}{\alpha}\right)$$

Now, $\rho = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

$$\theta = \tan^{-1}\frac{(\sqrt{3}/2)}{(1/2)} = \frac{\pi}{3}$$

$$\boxed{a_n = (1)^n \left[C_1 \cos\frac{\pi n}{3} + C_2 \sin\frac{\pi n}{3} \right]}$$

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Q1. $a_{r+2} - 2a_{r+1} + a_r = 2^r$ — ① $a_0 = 2,\ a_1 = 1$

Soln: $a_r = \alpha^r$ is the soln.

Characteristic eqn:

$$\alpha^{r+2} - 2\alpha^{r+1} + \alpha^r = \cancel{2^r}\ 0$$

$$\alpha^r [\alpha^2 - 2\alpha + 1] = \cancel{2^r}\ 0$$

$\alpha^r = \cancel{2^r}\ 0$ $\qquad$ $\alpha^2 - 2\alpha + 1 = \cancel{2^r}\ 0$

$\alpha = 1, 1$ $\qquad$ 1 is the root with multiplicity 2.

$$GS = (C_1 + C_2 r)\alpha^r$$
$$= (C_1 + C_2 r)(1)^r$$

Finding Particular Solution :-

1. $f(r) = b^r$

$P.S. = A \cdot b^r$

2. $f(r) =$ (Polynomial of degree k) $\cdot b^r$

$(a_0 + a_1 r + a_2 r^2 + \cdots a_k r^k)\, b^r$

$P.S. = (a_0 + a_1 r + \cdots a_k r^k)\, b^r$

3. $f(r) = \sin(br) / \cos(br)$

$P.S. = A \sin br + B \cos br$

4. $f(r) = b^r \sin br / b^r \cos br$

$P.S. = (A \sin br + B \cos br) \cdot b^r$

Here, $f(r) = 2^r$

$P.S. = C_3 (2^r)$

Put in ①

$C_3(2)^{r+2} - 2C_3(2)^{r+1} + C_3(2)^r = 2^r$

$C_3[\cancel{4} - \cancel{4} + 1] = 1$

$\boxed{C_3 = 1}$

$P.S. = 2^r$

Final Solution $= G.S. + P.S. = (C_1 + C_2 r)(2)^r + (2)^r$

$a_r = (C_1 + C_2 r)(1)^r + (2)^r$ —— ①

Put $a_0 = 2,\ a_1 = 1$ in ②

$a_0 = 2 = C_1 + 1 \Rightarrow \boxed{C_1 = 1}$

$a_1 = 1 = (C_1 + C_2) + 2 \Rightarrow \boxed{C_2 = -2}$

$$\boxed{a_r = (1 - 2r)(1)^r + (2)^r}$$

Case II: $a_{r+2} - 2a_{r+1} + a_r = r^2 \cdot 2^r$

$\therefore\ P.S. = (C_3 + C_4 r + C_5 r^2) \cdot (2^r)$

Put in ① replacing r by $r+2$ replacing r by $r+1$

$\cancel{2^r}[[(C_3 - C_4(r+2) + C_5(r+2)^2)4] - 4[C_3 + C_4(r+1) + C_5(r+1)^2]$
$+ (C_3 + C_4 r + C_5 r^2)] = r^2 \cdot \cancel{2^r}$

Putting $r = 0$

$4[C_3 - 2C_4 + 4C_5] - 4[C_3 + C_4 + C_5] + C_3 = 0$

$\cancel{4C_3} - 8C_4 + 16C_5 - \cancel{4C_3} + 4C_4 + 4C_5 + C_3 = 0$

$C_3 - 4C_4 + 20C_5 = 0$

Putting $r = -1$

$4[C_3 - C_4 + C_5] - 4[C_3 +] + [C_3 - C_4 + C_5] = 1$

$\cancel{4C_3} - 4C_4 + 4C_5 - \cancel{4C_3} + C_3 - C_4 + C_5 = 1$

$C_3 - 5C_4 + 5C_5 = 1$

Putting $r = -2$

$$4[C_3] - 4[C_3 \cancel{+} - C_4] + [C_3 - 2C_4 + 4C_5] = 4$$

$$\cancel{4C_3} - \cancel{4C_3} + 4C_4 + C_3 - 2C_4 + 4C_5 = 4$$

$$C_3 - 2C_4 + 4C_5 = 4$$