XVII

1:25–2:40PM

19th Jan 2024 (1)

ALGEBRA

(1) (a) $(-3)^4 = 3^4 = 81$

$(-a)^n = -a^n$ if $n \equiv 1, 3, 5, 7 \ldots$

$= a^n$ if $n \equiv 0, 2, 4, 6, 8 \ldots$

(b) $16^{-3/4} = \dfrac{1}{16^{3/4}}$

if $a^{-m} = \dfrac{1}{a^m}$

$= \dfrac{1}{(16^{1/4})^3}$

if $a^{m/n} = (a^{1/n})^m$

$= (a^m)^{1/n}$

$= \dfrac{1}{2^3} \equiv \dfrac{1}{8}$

$16 = 4 \times 4 = 2 \times 2 \times 2 \times 2$

$\equiv 2^4$

$\Rightarrow 16^{1/4} = (2^4)^{1/4} = 2^{4/4} = 2$

$(a+b)^2 = a^2 + b^2 \Longleftrightarrow$ if a or $b = 0$ ✗

$\parallel$

$a^2 + b^2 + 2ab$ ✓

$(a-b)^2 \equiv a^2 + b^2 - 2ab$ ✓

$a^2 - b^2 \equiv (a+b)(a-b)$ ✓

5. d. $\boxed{\dfrac{\frac{y}{x}-\frac{x}{y}}{\frac{1}{y}-\frac{1}{x}}}$

$$\frac{\left(\frac{y}{x}-\frac{x}{y}\right)\left(\frac{y}{x}-\frac{x}{y}\right)^{+}}{\left(\frac{1}{y}-\frac{1}{x}\right)\left(\frac{y}{x}-\frac{x}{y}\right)^{+}}$$

Numerator $N = \dfrac{y}{x} - \dfrac{x}{y} = \dfrac{y^2 - x^2}{xy}$

$$= \frac{\boxed{y^2} - \boxed{x^2}}{xy}$$

Denominator $D = \dfrac{1}{y} - \dfrac{1}{x} = \dfrac{x-y}{xy}$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\cdot\frac{d}{c}$$

$$\frac{N}{D} = \frac{\frac{y^2-x^2}{xy}}{\left\{\frac{x-y}{xy}\right\}} = \frac{y^2-x^2}{\cancel{xy}}\cdot\frac{\cancel{xy}}{x-y}$$

$$= \frac{y^2-x^2}{x-y} = \frac{(y-x)(y+x)}{x-y}$$

$$= \frac{-(x-y)\cdot(y+x)}{x-y} = -(x+y).$$

$x-y = -(y-x)$

$\Rightarrow (y-x) = -(x-y)$

implies

$$\boxed{\frac{1}{\boxed{x}} \quad \frac{1}{\boxed{y}} = \frac{1}{xy}}$$

6 RATIONALIZATION / CONJUGATION

b. $\dfrac{\sqrt{4+h}-2}{h} = y$

$y^2 \not\Downarrow$

$\cancel{\sqrt{4+h} = \sqrt{4}+\sqrt{h}}$

$\sqrt{a+b} = \sqrt{a}+\sqrt{b}$

$\iff a \text{ or } b = 0$

$$= \frac{(\sqrt{4+h}-2)^2}{h(\sqrt{4+h}-2)} = \frac{4+h+4-4\sqrt{4+h}}{\cdots} \not\Downarrow$$

$$\frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$

$\sqrt{4+h}-2$

conjugation $\equiv \sqrt{4+h}+2$

$(a+b)(a-b) = a^2-b^2$

$$= \frac{\overbrace{(\sqrt{4+h}-2)}^{a-b}\cdot\overbrace{(\sqrt{4+h}+2)}^{a+b}}{h(\sqrt{4+h}+2)}$$

$$= \frac{\cancel{4}+\cancel{h}-\cancel{4}}{\cancel{h}(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}.$$

////.

$$\frac{\cancel{4}+h-\cancel{4}}{h(\sqrt{4+h}+2)}$$

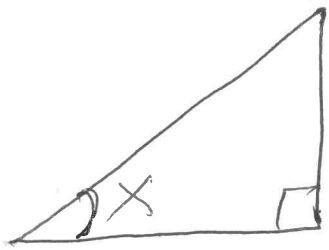
$$= \frac{\cancel{h}}{\cancel{h}}\cdot\frac{1}{(\sqrt{4+h}+2)}$$

$$= \frac{1}{\sqrt{4+h}+2}$$

D TRIG 7. b claim $\frac{2\tan x}{1+\tan^2 x} = \sin 2x$.

left hand side

LHS: $\frac{2\tan x}{1+\tan^2 x} = \frac{2\tan x}{\sec^2 x}$



$= 2\frac{\sin x}{\cos x} \frac{1}{\frac{1}{\cos x}\frac{1}{\cos x}}$

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $1 + \cot^2 x = \operatorname{cosec}^2 x$

$= 2\sin x \cdot \cos x$

$= \sin 2x$.

$=$ RHS

//// QED.

XX | FUNCTIONS

1:25 - 2:40 PM

22nd Jan 2024

Monday (1)

(7) function composition

$f(x) = x$ $\qquad$ $g(x) = 2$

f of x

"english"

$f \circ g = f \text{ of } g \equiv f(g(x))$

$= f(2)$

$= 2$

$g \circ f \equiv g(f(x)) \equiv g(x) = 2$

$f \circ g = g \circ f$

In general $f \circ g \neq g \circ f$

$f(x) \equiv x^2 + 2x - 1 \qquad g(x) = 2x - 3$

(a) $f \circ g$ (?) $= f(g(x)) = f(2x-3)$

$= (2x-3)^2 + 2(2x-3) - 1$

(b) $g \circ f$ (?) $\equiv g(f(x)) = g(x^2+2x-1)$

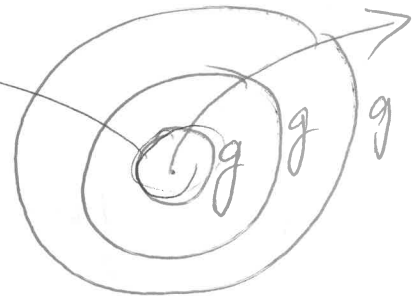
$= 2(x^2+2x-1) - 3$

$= 2x^2 + 4x - 2 - 3$

$= 2x^2 + 4x - 5$ ////

(c) $g_3 \circ g_2 \circ g_1(x)$

Left $\xleftarrow{\text{COMPOSITION}}$ Right



Recall $\equiv g(x) = 2x-3$

$g(g(x)) = g(2x-3)$

$= 2(2x-3) - 3$

$= 4x - 6 - 3$

$g_2 \circ g_1 = 4x - 9$

2 times

$= g \circ g(g(x))$

$= g(4x-9)$

$= 2(4x-9) - 3$

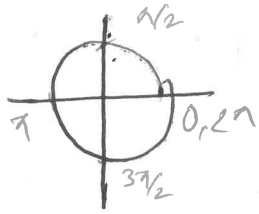
$= 8x - 18 - 3$

$= 8x - 21$ ////

XVII (a) $\sin 2x = \sin x$ $\quad x \in [0, 2\pi]$

Pi / Pie

belongs



$0 \le x \le 2\pi$

$2\cancel{\sin x}\cos x = \cancel{\sin x}$

$2\cos x = 1$

$\cos x = 1/2$

$x = 60^\circ; \pi/3; 300 / 5\pi/3$

[if $x \in (0, 2\pi)$
$\Rightarrow 0 < x < 2\pi$.]

$\sin 2X = \sin X$

$\Rightarrow 2\sin X \cdot \cos X = \sin X$

$\Rightarrow 2\sin X \cos X - \sin X = 0$

$\Rightarrow \sin X \underbrace{(2\cos X - 1)}_{X = \pi/3,\ 5\pi/3} = 0$

$\boxed{} \cdot \boxed{} = 0$

$0 \quad 0$

$0 \le X \le 2\pi$

either $\sin X = 0$ or $2\cos X - 1 = 0$

$\Rightarrow X = \underline{0,\ 2\pi},\ 4\pi$

π

$2\cos X = 1$

$\cos X = 1/2$

$X = \pi/3,\ 5\pi/3$

(c) $\sqrt{a^2+b^2} \neq a+b$

$a = 3$

$b = 4$

LHS $\equiv \sqrt{9+16} = 5$

RHS $a+b \equiv 7$

(d) $\dfrac{1+Pc}{c} = \dfrac{1}{c} + \dfrac{P\cancel{c}}{\cancel{c}} \equiv \dfrac{1}{c} + P \neq \underbrace{1+P}_{RHS}$

true if $c=1$

(c) $\frac{1}{x-y} = \frac{1}{x} - \frac{1}{y}$ FALSE

assume $x = 2$ $y = 2$

LHS $\frac{1}{2-2} = \frac{1}{0}$ | RHS $\frac{1}{2} - \frac{1}{2} = 0$

$\frac{1}{0} \neq 0$

(d) $\frac{1/x}{a/x - b/x} = \frac{1}{\cancel{x}\left[\frac{a}{\cancel{x}} - \frac{b}{\cancel{x}}\right]} = \frac{1}{a-b} = RHS$

LHS.

TRUE

XX | FUNCTIONS (2) given $f(x) = x^3$

$\frac{f(h+2) - f(2)}{h}$

$(a+b)^3$
$= a^3 + b^3 + 3ab(a+b)$

$(h+2)^3$
$= h^3 + 8 + 3 \cdot h \cdot 2(h+2)$
$= h^3 + 8 + 6h(h+2)$
$= h^3 + 8 + 6h^2 + 12h$

$= \frac{(h+2)^3 - 8}{h}$

$= \frac{h^3 + \cancel{8} + 6h^2 + 12h - \cancel{8}}{h} = \frac{h^3 + 6h^2 + 12h}{h} = h^2 + 6h + 12$ ////.

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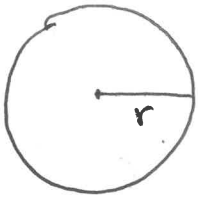
24th Jan 2024
Wed

Chapter-1
FUNCTIONS

$\pi = 3.14\ldots$ → irrational

1



$r > 0$

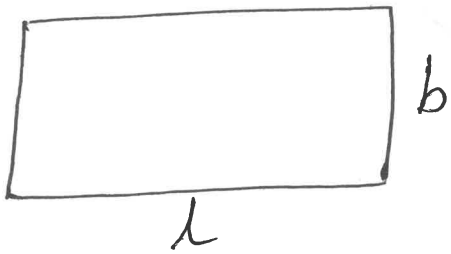
$A(r) \equiv \pi r^2$ ← area

if $r = 1$ $A = \pi$ / $r = 1/2 \Rightarrow A = \frac{\pi}{2^2} = \frac{\pi}{4}$

$A(r)$ is a function of r

#1 independent variable and $A(r)$ → dependent variable

2



#2 independent variable

$p(l, b) = 2l + 2b = 2(l + b)$

if $l = 1$, $b = 2$
↳ $p = 2(1+2) = 6$

$l = 2$, $b = 2$
↳ $p = 2(2+2) = 8$

DOMAIN of function $\equiv$ $f(x) \equiv$ function defined on an independent variable x, then the DOMAIN, $D(f)$ is the collection of x such that $f(x)$ makes sense.

a senseful thing for $f \longrightarrow \sqrt{\ -\ }$ ✗

$\longrightarrow \dfrac{\square}{\boxed{0}}$ ✗

① $f(x) = \sqrt{x-2}$; find $D(f)$

- $x > 2$
- $x \geq 2$

solution: In order to f makes sense we want $x-2 \geq 0$. $\Rightarrow x \geq 2$. Therefore

atleast

the solution is $x \in [2, \infty)$.

belongs to.

② $f(x) = 1/x$; find $D(f)$.

solution: In order to f makes sense, we don't want $x \neq 0$ in $f(x) = 1/x$. Therefore

the $D(f) \equiv$ all Real no. but $x \neq 0$ ✓

$\equiv \mathbb{R} \setminus \{0\}$ set minus.

$= (-\infty, 0) \cup (0, \infty)$

③ $f(x) = \dfrac{1}{x-\pi}$; find $D(f)$. we don't want $x-\pi=0$ in $f(x)=\dfrac{1}{x-\pi}$. Therefore $x \neq \pi$ and hence $D(f) = \mathbb{R} \setminus \{\pi\}$.

④ $f(x) = \dfrac{1}{\sqrt{x-2}}$; find $D(f)$.

$\dfrac{\square}{\square}$; $\sqrt{}$

① Observation: $\dfrac{1}{\sqrt{x-2}}$, in order to make sense we don't want $x-2 \neq 0 \Rightarrow x \neq 2$. (A)

② $\sqrt{x-2} \Rightarrow x-2 \geq 0 \Rightarrow x \geq 2$

$\Rightarrow x \in [2, \infty)$ (A)(A)

Combine (A) and (A)(A), this imply that $x \in (2, \infty)$.

SYMMETRIC:

- EVEN $\rightarrow f(x) = f(-x)$
- ODD $\rightarrow f(x) = -f(-x)$

① $f(x) = 2$ check ~~$f(-x) = -2$~~ why?

idea is $x \longrightarrow -x$

$\Rightarrow f(-x) = 2 = f(x) \Rightarrow f$ is even.

if $f(x) = c$ (constant) then f is an even.

② $f(x) = x$ how $x \longrightarrow -x$

$f(-x) = -x = -f(x)$ hence

f is odd

check: $\boxed{\begin{array}{l} f(x) = \sin x \text{ is odd} \\ f(x) = \cos x \text{ is even} \end{array}}$

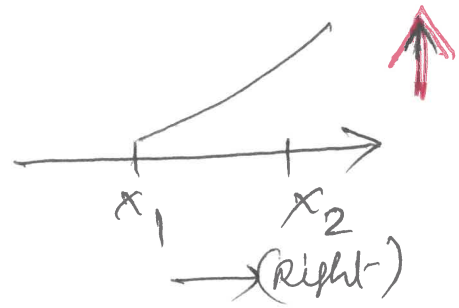
Increasing ($\uparrow$) & Decreasing ($\downarrow$)

① ($\uparrow$) $I = [a, b]$ choose x_1 and x_2

interval from I.

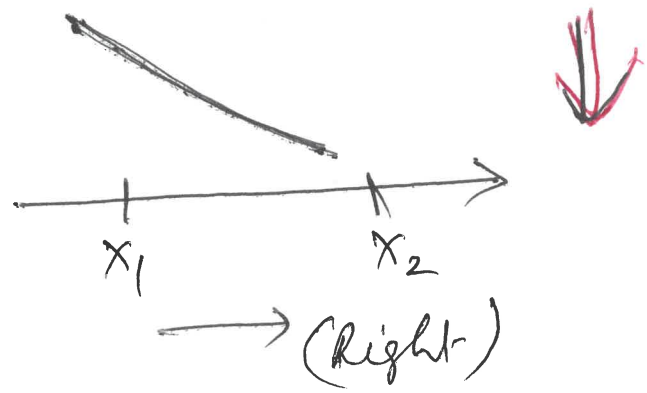
if $f(x_1) \le f(x_2)$ whenever $x_1 \le x_2$

then f is $\uparrow$.

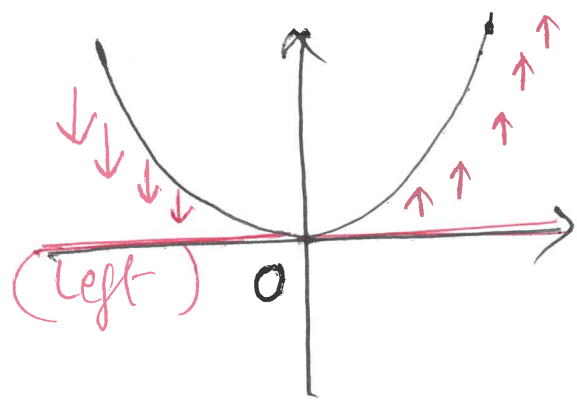


② ($\downarrow$) if $f(x_1) \ge f(x_2)$ whenever $x_1 \le x_2$

then f is $\downarrow$.



check $f(x) = x^2$



HW | due 2nd Feb 2024 Friday

Ex 1.1 page 9

#1, #2, #21–24

#25–29, #30, #31–42

#59–64.

Homework

(B) (b) $2x^2 - 12x + 11$

$(x-b)^2 = x^2 - \boxed{2b}x + b^2$

$2b = 6$

$b = 3$

$b^2 = 9$

$\frac{11}{2} - 9$

$= \frac{11-18}{2}$

$= -7/2$

$= 2\left(x^2 - \boxed{6}x + \frac{11}{2}\right)$

$= 2\left(x^2 - 6x + \square - \square + \frac{11}{2}\right)$

$= 2\left(\underbrace{x^2 - 6x + 9} \overbrace{- 9 + 11/2}\right)$

$= 2\left((x-3)^2 - 7/2\right)$

$= 2(x-3)^2 - 7$

In general $x^2 - Ax + B \equiv (x-T)^2 + C$

(complete the square)

$(x-a)^2 = x^2 - 2ax + a^2$

$A = 2a \Rightarrow a = \frac{A}{2}$

$\hookrightarrow a^2 = \frac{A^2}{4}$ $\quad A = 2a$

$x^2 - 2ax + \cancel{a^2} \cancel{- a^2} + B$

$= (x-a)^2 + (B - a^2)$

$a = \frac{A}{2}$

$2 + 3 \cancel{-3}$

7 (a) $\underbrace{x^2+1x+1}_{(x+a)^2+\square} = x^2+\underline{2xa}+a^2$

$2a=1$

$\Rightarrow a=1/2$

$a^2 \to 1/4$

$\underline{\underline{x^2+x+1}}+1/4-1/4$

$=(x^2+x+1/4)+\underline{1-1/4}$

$=(x+1/2)^2+3/4$

$\left[x^2+2\cdot\frac{1}{2}\cdot x+\left(\frac{1}{2}\right)^2\right]$

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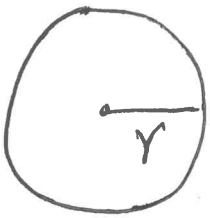
26th Jan 2024 ①

Friday



$A(r) = \cancel{2\pi}\pi r^2 \longrightarrow$ independent variable

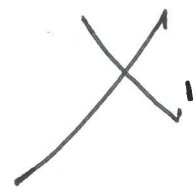
$C(r) = 2\pi r.$

dependent variable.

§1.2 catalogue of some essential fun^n.

① $f(x) = mx + b$ → linear x

$\deg(x) = 1$



↓ AFFINE.

Def $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$

$L \rightarrow$ LINEAR. where α and β are constants and x, y are independents.

(Q1) $f(x) = 9x + 3 \rightarrow$ MATH/2413.004

claim f is affine but not linear.

Sol^n: we will show that f is not linear.

Let α and β be some constants and x and y be two independent variables, then

$f(\alpha x + \beta y) = 9(\alpha x + \beta y) + 3$ ✓ ②

$\downarrow$

$f(x) = 9x + 3$ $\quad = \boxed{9\alpha x + 9\beta y + 3} = LHS$

COMPARE

RHS $\alpha f(x) + \beta f(y)$

$= \alpha(9x+3) + \beta(9y+3)$

$= 9\alpha x + 3\alpha + 9\beta y + 3\beta$

$= \boxed{9\alpha x + 9\beta y + 3\alpha + 3\beta}$

Thus, $f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y)$, Hence f is an affine fun^n.

Q2 $f(x) = 6x$

Claim f is linear.

For any constants α and β, with independent variables x and y

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

LHS: $f(\alpha x + \beta y) = 6(\alpha x + \beta y)$

$f(x) = 6x$

$= \alpha 6x + \beta 6y$

$= \alpha f(x) + \beta f(y) = RHS.$

thus $\boxed{f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)}$

hence $f(x) = 6x$ is linear.

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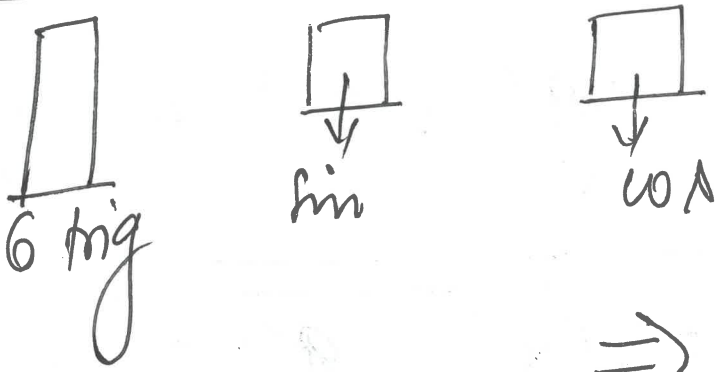
show that $C(r) = 2\pi r$ is a LINEAR function in $r > 0$. (DIY)

TRIG:



$-1 \le \sin x \le 1$

$-1 \le \cos x \le 1$

$\Rightarrow \quad |\sin x| \le 1$

$\Rightarrow \quad |\cos x| \le 1$

Periodic if $f(x + \tau) = f(x)$ and τ is the period.

τ tau

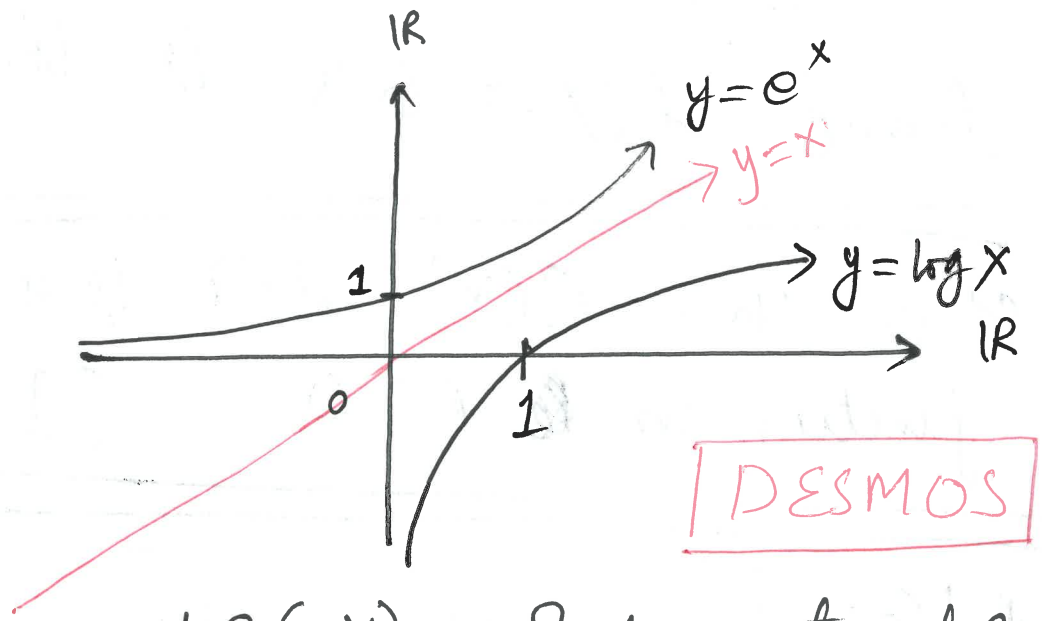
for example $\sin(x + 2\pi) = \sin x \qquad \tau = 2\pi$

Exponential and Logarithms

① $f(x) = e^x$ exponential function if $x = 1$,
$f(1) = e = 2.718\ldots$

② $f(x) = \overbrace{\log x}^{\text{my preference}}$ / $\ln x$ (or) natural logarithm.

$e^x \equiv -ve$



$\log(-x) \equiv ?$ does not make sense.

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page 20. $f(x) = \sqrt{x}$ $\quad g(x) = \sqrt{2-x}$

ⓐ $fog \equiv f(g(x)) \equiv f(\sqrt{2-x})$

$= \sqrt{\sqrt{2-x}}$ ✓

$= (2-x)^{1/4}$ ✓

$f(x) = \sqrt{x}$, ① $f_2 = f \circ f$

$$f_2(x) = x^{1/4} \longrightarrow x^{\frac{1}{2^2}}$$

② $f_3(x) \equiv f \circ f \circ f\,(x) = x^{1/8} \longrightarrow x^{\frac{1}{2^3}}$

③ $f_4(x) = x^{1/16} \longrightarrow x^{1/2^4}$

④ $f_5(x) = x^{1/32} \longrightarrow x^{1/2^5}$

hence $f_n(x) = x^{\frac{1}{2^n}}$ ✓

$\downarrow$

n composition of f with itself.

<u>show</u> $f_n = f_m \Longleftrightarrow n = m$.

① $\Rightarrow$ assume $f_n = f_m$

$$\square = \square$$

$\Rightarrow$ conclude $n = m$

$\Leftarrow$ assume $n = m$

$$f_n = \square = \square = f_m$$

Voila.

expression will involve n

expression with m replace

thus $f_n = f_m \Leftrightarrow n = m$.

$\forall\, n, m$.

for all.

1:25–2:40

Monday (1)

29th Jan 2024

§1.2 FUNCTIONS

page 22 #38 $f = \sqrt{3-x}$ $g = \sqrt{x^2-1}$.

(a) $f+g$, $f-g$, fg, f/g also domains.

HW HW

$f+g = \sqrt{3-x} + \sqrt{x^2-1}$ //// $\rightarrow$ domain of $f+g$.

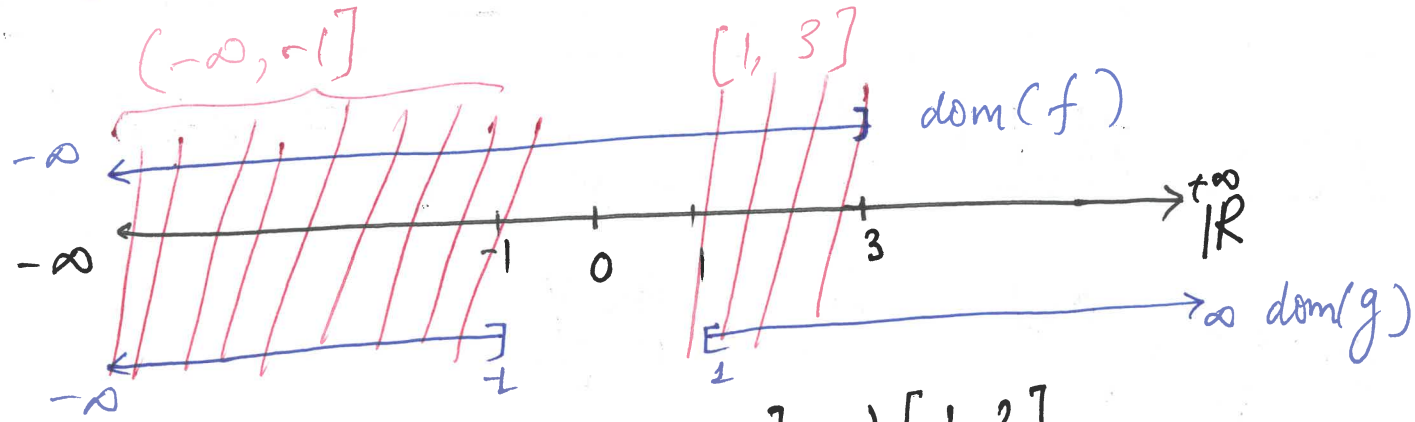
$\text{dom}(f) = (-\infty, +3]$ ✓

$\text{dom}(g) = \text{dom}(\sqrt{x^2-1})$ we want $x^2-1 \geq 0$

$\Rightarrow x^2 \geq 1 \Rightarrow x \in (-\infty, -1] \cup [1, \infty)$ ✓

intersection (common)

(b) $\text{dom}(f+g) = \text{dom}(f) \cap \text{dom}(g)$



ans $\text{dom}(f+g) = (-\infty, -1] \cup [1, 3]$.

$$f/g = \frac{\sqrt{3-x}}{\sqrt{x^2-1}}$$

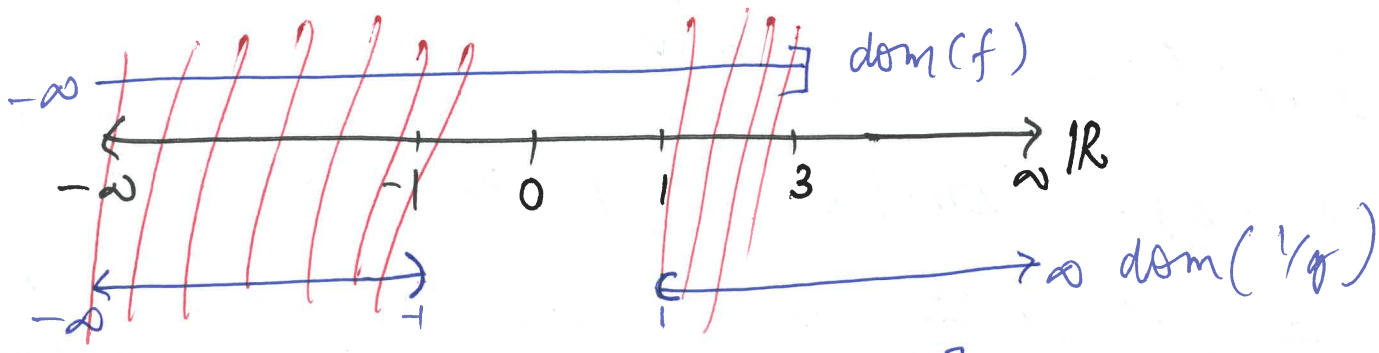
$$\mathrm{dom}\left(\frac{f}{g}\right) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$$

$$= \mathrm{dom}(f) \cap \mathrm{dom}\left(\frac{1}{g}\right)$$

Recall $\mathrm{dom}(g)$

$= (-\infty, -1] \cup [1, \infty)$

$\Rightarrow \mathrm{dom}(1/g) = (-\infty, -1) \cup (1, \infty)$.



$\mathrm{dom}\left(\frac{f}{g}\right) = (-\infty, -1) \cup (1, 3]$. ////.

<u>page 23</u> <u>#44</u> $f = \frac{1}{1+x}$ $g = \sin 2x$

$g(x) = \sin 2x$

ⓐ $f \circ g$, $g \circ f$, $f \circ f$, $\underline{g \circ g}$.

$g \circ g = g(g(x)) = g(\sin 2x) = \sin(2(\sin 2x))$

(43) $f = x + \frac{1}{x}$ $\quad g = \frac{x+1}{x+2}$ $\quad g(t) = \frac{t+1}{t+2}$ (3)

$f \circ f$, $\underline{g \circ f}$, $g \circ g$, $f \circ f$

$$g \circ f = g(f(x)) = g\Big(\overbrace{x + \frac{1}{x}}^{t}\Big)$$

$$= \frac{t+1}{t+2} \quad \text{where } t \neq \cancel{\frac{x+1}{x+2}}$$

$$t = x + \frac{1}{x}$$

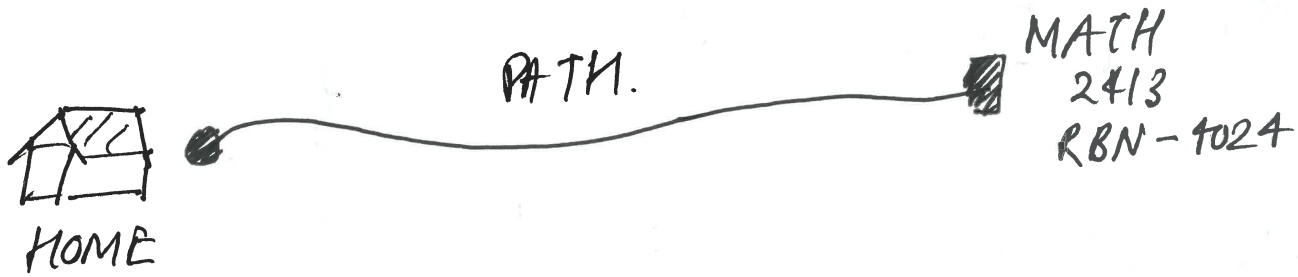
$$= \frac{x + \frac{1}{x} + 1}{x + \frac{1}{x} + 2} = \frac{\left(x + \frac{1}{x} + 1\right)\cancel{x}}{\left(x + \frac{1}{x} + 2\right)\cancel{x}}$$

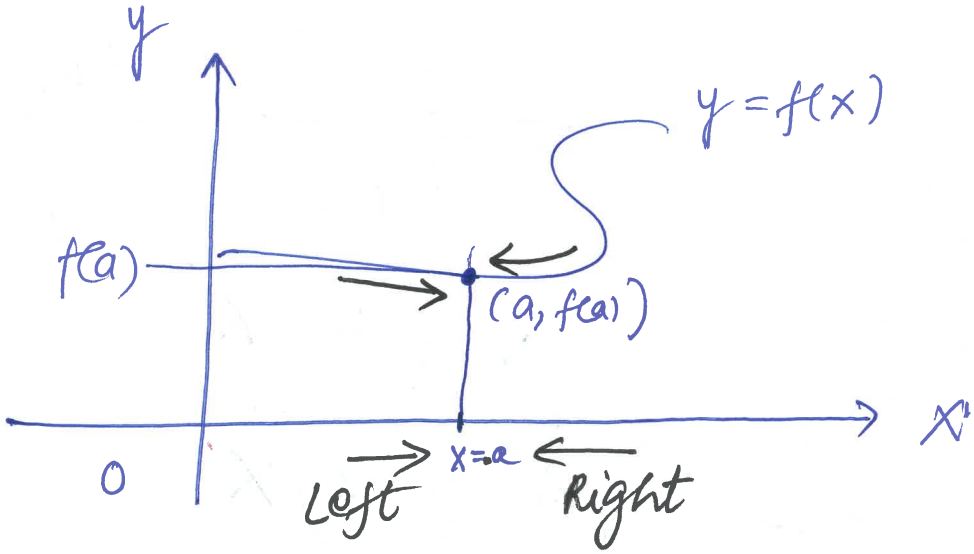
$$= \frac{x^2 + 1 + x}{x^2 + 1 + 2x} = \frac{x^2 + 1 + x}{(x+1)^2} \quad ////$$

$$\cancel{\frac{x^2 + 1 + x}{x^2 + 1}}$$

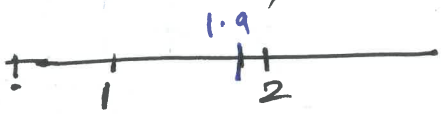
$$\mathrm{dom}(g \circ f) = \mathbb{R} \setminus \{-1\}.$$

§1.3 LIMITS





Example: $f(x) = x^2 - x + 2$, $x = 2$



Left

@ 1.9 → 3.71

@ 1.99 → 3.970100

@ 1.999 → 3.997001

↓ close to 2 ↓ close to 4

Right

@ 2.1 → 4.31

@ 2.01 → 4.03

@ 2.001 → 4.003

↓ close to 2 ↓ close to 4

$\because f(x) = x^2 - x + 2$

$\Rightarrow f(2.01) = (2.01)^2 - (2.01) + 2 = 4.03$

---

① $\lim_{x \to 2^-} f(x) = \boxed{4}$ $\quad f(x) = x^2 - x + 2$

(left hand limit of $f(x)$ @ $x=2$) (LHL)

② $\lim_{x \to 2^+} f(x) = \boxed{4}$ $\quad f(x) = x^2 - x + 2$

(Right hand limit of $f(x)$ @ $x=2$) (RHL)

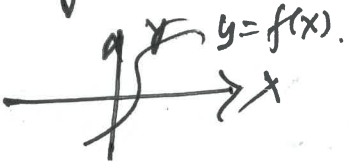
③ $f(2) = (x^2 - x + 2)\Big|_{x=2} = 4 - 2 + 2 = \boxed{4}$

Additionally

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) = 4 \; (< \infty)$

Thus, f is continuous.

Note / Remark / warning

In $\mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R}$  in which for a function $y = f(x)$ where we have only '1' independent variable ('x' here), precisely only 2 paths exist to approach a given limit point i.e. $x = a$.

Reading assignment: Ex → 3 @ page 27
→ 4 @ 28
→ 5 @ 28

Wednesday.

§1.3 ex#3

evaluate $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

if $f(t) = \frac{\sqrt{t^2+9}-3}{t^2}$ then what $f(t)$ looks like as t becomes smaller and smaller!

t	$f(t)$
± 1	$.16228$
$\vdots$	
$.01$	$.166666\ldots$

notation $0.1\bar{6}$. More or less.

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \underbrace{.166667}_{\frac{1}{6}} = .1\bar{6}$$

$\parallel$

$.17$ approximation

---

§#4 $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (theorem)

Taylor's formula

Proof: Recall, the series expansion: Euler-Maclaurin expansion @ origin

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots}$$

Tutorial for $n!$

$n! = n \cdot n-1 \cdot n-2 \cdots 1$

$1! = 1$, $0! = 1$ (HW!), $2! = 2 \cdot 1 = 2$

$3! = 6$, $4! = 24$, $5! = 24 \times 5 = 120$.

$n! = n \cdot (n-1)!$ always.

Recall $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

$$\frac{\sin x}{x} = \overset{1}{\cancel{\frac{x}{x}}} - \overset{x^2}{\cancel{\frac{x^3}{3!\,x}}} + \overset{x^4}{\cancel{\frac{x^5}{5!\cdot x}}} - \overset{6}{\cancel{\frac{x^7}{7!\cdot x}}} + \overset{8}{\cancel{\frac{x^9}{9!\cdot x}}} - \cdots$$

one more line: RHS

$$\left[x - x^3/3! + x^5/5! - x^7/7! + x^9/9! - \cdots\right]/x$$

$$= \frac{x}{x} - \frac{x^3}{3!\cdot x} + \frac{x^5}{5!\,x} - \frac{x^7}{7!\,x} + \frac{x^9}{9!\cdot x} - \cdots$$

$$\frac{\sin x}{x} = 1 + O(x)$$

BIG 'O' notation.

Now, apply the limit $x \longrightarrow 0$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0}\left[1 + O(x)\right]$$

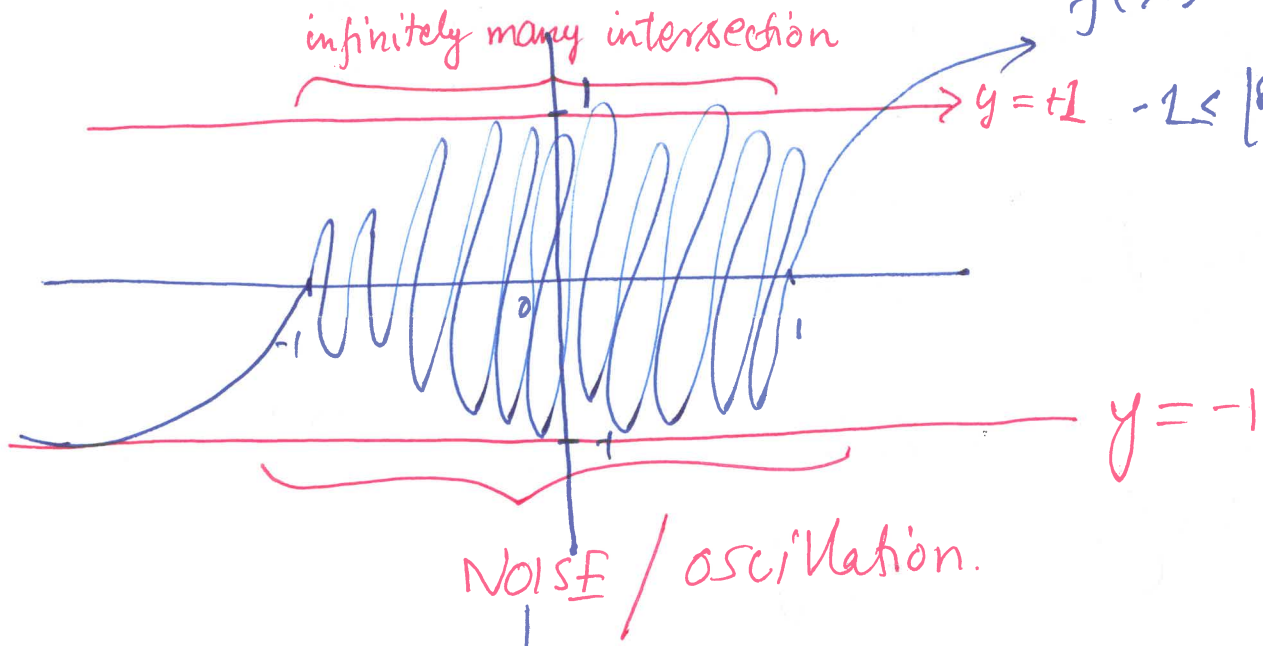
$$= \lim_{x\to 0} 1 + \lim_{x\to 0} O(x)$$

$$= 1 + \left[\lim_{x\to 0} O(x)\right] = 1 + 0 = 1.$$

#5 claim $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = \text{DNE}$

$f(x) = \sin \frac{\pi}{x}$

$-1 \le \left|\sin \frac{\pi}{x}\right| \le 1$



as $x \to 0^{\pm}$, $f(x)$ @ $x \to 0^{\pm}$ changes / oscillates quite frequently between -1 and 1.

Therefore, $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = \text{DNE}$.

however, with the multiplication of x^2 with $f(x)$, yields the value of $\lim_{x \to 0} x^2 \sin \frac{\pi}{x}$. The aforementioned limit is 0. i.e.

$$\lim_{x \to 0} x^2 \sin \frac{\pi}{x} = 0.$$

---

if for some 'β' $\lim_{x \to 0} \sin \frac{1}{x^{\beta}} = \text{DNE}$, then $\lim_{x \to 0} x^{\alpha} \sin \frac{1}{x^{\beta}} = 0$ if $\alpha = 1 + \beta$.

Laws of limit §1.4

① $\lim_{x\to a}[f \pm g] = \lim_{x\to a} f \pm \lim_{x\to a} g$.

② $\lim_{x\to a}(fg) = \lim_{x\to a} f \cdot \lim_{x\to a} g$

③ $\lim_{x\to a}\left(\frac{f}{g}\right) = \dfrac{\lim_{x\to a} f}{\lim_{x\to a} g}$ provided $\lim_{x\to a} g \neq 0$

④ $\lim_{x\to a}(cf) = c \lim_{x\to a} f$.

---

ex 1 (b) @ page 37: $\lim_{x\to -2} \dfrac{x^3+2x^2-1}{5-3x}$

$$= \frac{\lim_{x\to -2} x^3+2x^2-1}{\lim_{x\to -2} 5-3x} = \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} = \frac{\cancel{-8}+\cancel{8}-1}{5+6} = \frac{-1}{11}$$

any function $f(x) \equiv$ polynomial such as
$f(x) = x^3+2x^2-1$

then $\lim_{x\to -2} f(x) = \lim_{x\to -2}(x^3+2x^2-1) = (x^3+2x^2-1)\Big|_{x=-2}$

$= (-2)^3+2(-2)^2-1.$

Ex-4 @ 39 $\lim_{h\to 0}\left[\frac{(3+h)^2-9}{h}\right] \sim \frac{(3+0)^2-9}{0} \equiv$ UNDEFINED

$(3+h)^2 =$

$$= \lim_{h\to 0}\left[\frac{h^2+6h+\cancel{9}-\cancel{9}}{h}\right]$$

$$= \lim_{h\to 0}\left[h+6\right] = 0+6 = 6 \text{ ////.}$$

Ex-5 $\lim_{t\to 0}\left[\frac{\sqrt{t^2+9}-3}{t^2}\right] = \lim_{t\to 0}\left[\frac{\overbrace{\sqrt{t^2+9}}^{a}-\overset{b}{3}}{t^2}\cdot\frac{\overbrace{\sqrt{t^2+9}}^{a}+\overset{b}{3}}{\sqrt{t^2+9}+3}\right]$

$a-b$ $\quad a+b$

$(a-b)(a+b)=a^2-b^2$

$$= \lim_{t\to 0}\left[\frac{\cancel{t^2}+\cancel{9}-\cancel{3^2}}{\cancel{t^2}\sqrt{t^2+9}+3}\right]$$

$$= \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3} = \frac{1}{\sqrt{0+9}+3} = \frac{1}{3+3} = \frac{1}{6} \text{ ////.}$$

---

ex-10 @ 42 $\lim_{x\to 0}\frac{\sin\boxed{7x}}{\boxed{4x}}$.

Recall $\lim_{x\to 0}\frac{\sin\boxed{x}}{\boxed{x}} = 1$

$$= \lim_{x\to 0}\frac{\sin 7x}{4x}\cdot\frac{7}{7} = \underbrace{\left(\lim_{x\to 0}\frac{\sin 7x}{7x}\right)}\left(\frac{7}{4}\right)$$

$$= 1\cdot\frac{7}{4} = \frac{7}{4} \text{ ////.}$$

page 43–44

HW 1–9, 10

11–20, 49–56

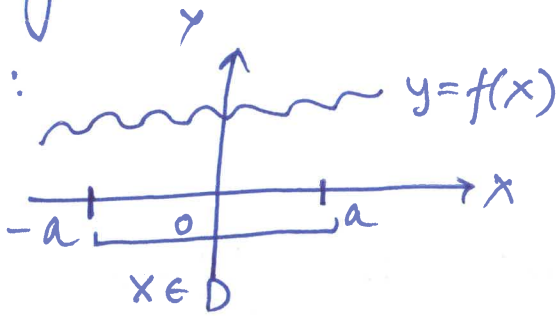
$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ } §1.4 1:25-2:40PM: 2nd feb 2024 ①

Friday

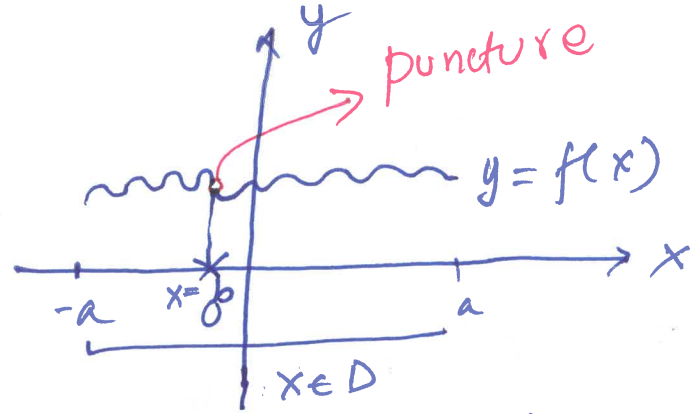
§1.5 continuity: [layman's words] drawing the graph $y = f(x)$ on some domain $D \ni x$, without lifting pen.

$\ni$ belongs to.

for ex:



$y = f(x)$ is $\mathcal{C}(D)$



$y = f(x)$ is dis-continuous @ $x = b$.

example 2 from page 47 $f(x) = \frac{x^2 - x - 2}{x - 2}$, if

we put $x = 2$ in $y = f(x)$,

$$f(2) = \frac{2^2 - 2 - 2}{2 - 2} = \frac{4 - 4}{2 - 2} = \frac{0}{\boxed{0}}$$

UNDEFINED.

$\therefore f$ is not continuous @ $x = 2$.

The function $y = f(x)$ @ $x = a$ is said to be continuous if following holds:

$$\boxed{\underbrace{\overset{①}{\lim_{x \to a^-} f(x)}}_{\text{Left hand Limit}} \overset{②}{=} \underbrace{f(a)}_{\text{functional value } y=f(x) \text{ @ } x=a.} = \overset{③}{\underbrace{\lim_{x \to a^+} f(a)}_{\text{Right-hand Limit}}}.}$$

<u>example 4 @ page 48</u> $f(x) = 1 - \sqrt{1-x^2}$, check the continuity on $[-1, 1]$?

<u>sol^n</u>: In general pick a point in $D \equiv [-1, 1]$, and say that point is 'a'.

② $f(a) = 1 - \sqrt{1-a^2}$ $\forall\, a \in D$.

① <u>compute LHL</u> $\lim_{x \to a^-} f(x)$

$$= \lim_{h \to 0} \left[f(a-h) \right]$$

$$x \to a^- \cong \lim_{h \to 0} (a-h)$$

$$= \lim_{h\to 0}\left[1-\sqrt{1-(a-h)^2}\right]$$

$\because f(x) = 1-\sqrt{1-x^2}$

$$= 1-\sqrt{1-\underbrace{\lim_{h\to 0}(a-h)^2}_{\to a^2}}$$

$\lim_{h\to 0}(a-h)^2 = a^2$

$$= 1-\sqrt{1-a^2} \checkmark$$

With the help of same argument.

<u>RHL</u> $\lim_{x\to a^+} f(x) = \lim_{x\to a^+}\left[1-\sqrt{1-x^2}\right]$

$$= \lim_{h\to 0}\left[1-\sqrt{1-(a+h)^2}\right]$$

$x\to a^+$
$= \lim_{h\to 0}(\cancel{a}+h)$

$$= 1-\sqrt{1-\underbrace{\lim_{h\to 0}(a+h)^2}_{\to a^2}}$$

$$= 1-\sqrt{1-a^2} \checkmark$$

So, for $\forall a \in D$, we have concluded

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x) = 1-\sqrt{1-a^2}. \ (\neq 0)$$

$\therefore f \in \mathcal{C}[-1,1]$.

If f and g are continuous on some domain D.
then, $f \pm g$ is also $\mathcal{C}(D)$
$f \times g$ is also $\mathcal{C}(D)$,
f/g is also $\mathcal{C}(D)$ except @ x for which $g(x)=0$.

cf is also $\mathcal{C}(D)$.
(constant)

<u>example</u>: $y = \tan x$ @ what points of $x \in \mathbb{R}$, $y = \tan x$ is discontinuous?

$y = \tan x = \dfrac{\sin x}{\cos x}$; $\cos x = 0$
@ $x = \frac{\pi}{2} \pm k\pi$, where $k \in \mathbb{N}$

the set
$D_c = \left\{ \frac{\pi}{2} \pm k\pi \,\middle|\, k \in \mathbb{N} \right\}$ or $x = \pm(2n+1)\frac{\pi}{2}$, $n \in \mathbb{N}$.

$= \infty$ because
$k \in \mathbb{N}$ is an ∞ set.

§1.6 Limits with ∞'s $\quad y = \frac{1}{x}$

if $x = .01$, $y = \frac{1}{.01} = 100$

$x = .0001$, $y = \frac{1}{.0001} = 10{,}000$

$x = .00001$, $y = \frac{1}{.00001} = 100{,}000$

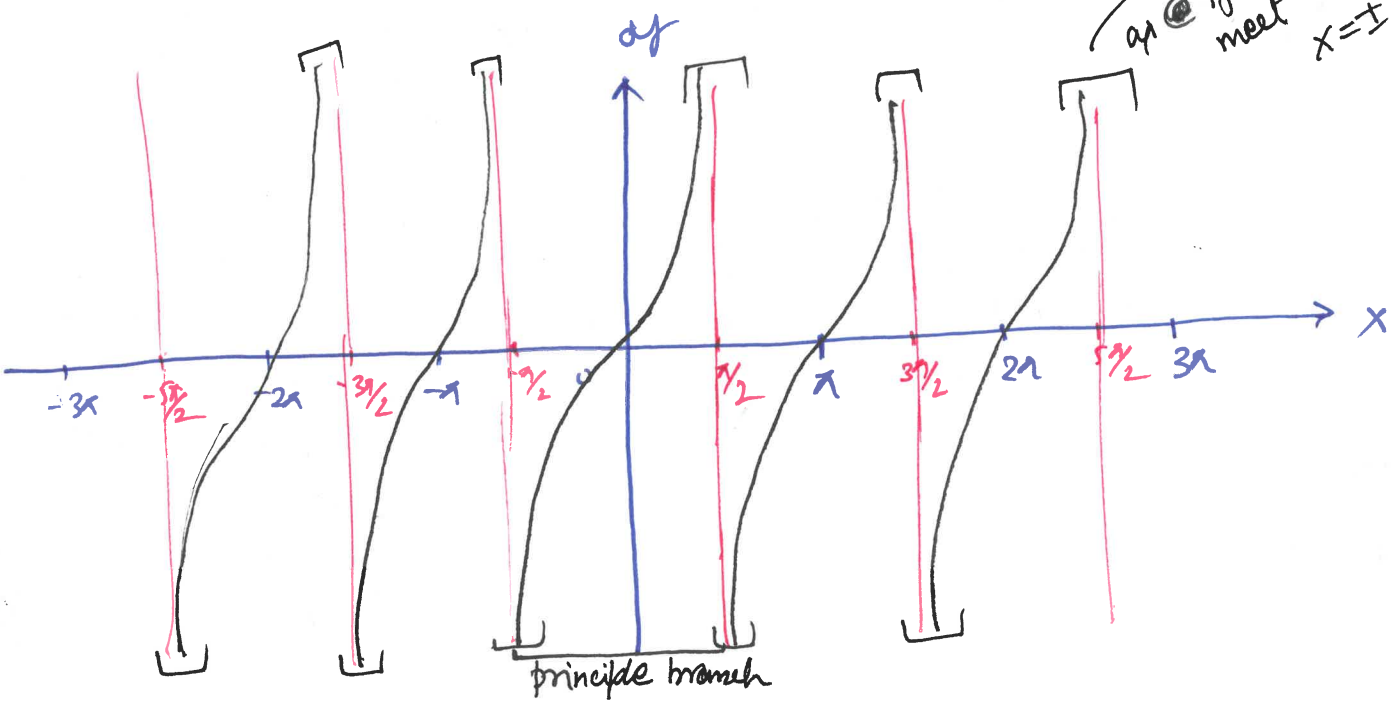
$\uparrow\uparrow\uparrow$ ∞

$$\lim_{x \to 0} \frac{1}{x} = \infty.$$

The concept of "limit with ∞'s" involves the notion of ASYMPTOTES.

For example $y = \tan x$ fails to be continuous @ $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots$

they look like as if they will meet @ $x = \pm\infty$.



VA for $y=f(x)$, the VA is $x=\pm a$ if

$$\underbrace{\lim_{x\to\pm a} f(x)=\pm\infty}_{\text{limit}} \quad \text{or} \quad \underbrace{\lim_{x\to a^{\pm}} f(x)=\pm\infty}_{\text{one-sided limit.}}$$

HA for $y=f(x)$, the HA is $y=L$, if

$$\lim_{x\to\pm\infty} f(x) = L.$$

---

example: find HA of $y=\dfrac{2x^2-1}{3x^2-1}$

$$\lim_{x\to+\infty}\left[\frac{2x^2-1}{3x^2-1}\right] = \lim_{x\to\infty}\left[\frac{\cancel{x^2}\left(2-\frac{1}{x^2}\right)}{\cancel{x^2}\left(3-1/x^2\right)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2-\frac{1}{x^2}}{3-\frac{1}{x^2}}\right] = \frac{2-\lim_{x\to\infty}\frac{1}{x^2}}{3-\lim_{x\to\infty}\frac{1}{x^2}}$$

$$\lim_{x\to\infty}\frac{1}{x^2}=0$$

$$= \frac{2-0}{3-0} = \frac{2}{3}$$

Q2 $f_I : \underbrace{\mathbb{R}}_{\text{domain}} \longrightarrow \underbrace{\mathbb{R}}_{\text{Range}}$ $\quad f_I(x) = x$ ①

1:20-2:40 PM Monday 5th February

$f_c : \mathbb{R} \longrightarrow \mathbb{R}$ $\quad f_c(x) = c$

① <u>claim</u> $f_I \circ f_c = f_c \circ f_I$.

<u>Proof</u>: <u>LHS</u>: $f_I \circ f_c \equiv f_I \circ f_c(x) = f_I(f_c(x))$

$= f_I(c) = c$. —— ①

<u>RHS</u>: $f_c \circ f_I \equiv f_c \circ f_I(x) = f_c(f_I(x)) = f_c(x)$

$= c$. —— ②

By ① and ② LHS = RHS.

② what is the value of $f_I \circ f_c = f_c \circ f_I$.

either $f_I \circ f_c$ or $f_c \circ f_I$ is 'c'.

③ DIY.

<u>Note</u>: f and g $\quad f \circ g \neq g \circ f$ | f is x, g is c then $f \circ g = g \circ f$.

③. ① $f_3(x)$ where $f(x) = 2x - 3$.

① $f_2(x) = f \circ f(x) = f(2x-3)$

$= 2(2x-3) - 3 = 4x - 6 - 3$

$= 4x - 9.$

$f_3(x) = f \circ f \circ f(x) = f \circ \overline{f_2(x)} = f(4x-9)$

$= \cancel{2(4x-9) - 9}$

$= \cancel{8x - 18 - 27}$

$= 2(4x-9) - 3$

$= 8x - 21$

② observe $f_2(x) = f \circ f(x) = f(\alpha x + \beta)$

$= \alpha(\alpha x + \beta) + \beta = \alpha^2 x + \alpha\beta + \beta$ ✓

$f_3(x) = f \circ f_2(x) = f(\alpha^2 x + \alpha\beta + \beta)$

$= \alpha(\alpha^2 x + \alpha\beta + \beta) + \beta$

$= \alpha^3 x + \alpha^2\beta + \alpha\beta + \beta.$ ✓

$f_4(x) = \alpha^4 x + \alpha^3\beta + \alpha^2\beta + \alpha\beta + \beta$ ✓

$$f_n(x) = \alpha^n x + \overbrace{\alpha^{n-1}\beta + \cdots + \alpha\beta + \beta}$$ ③ ~~4~~ ~~5~~

$$= \alpha^n x + \beta \underbrace{\sum_{i=0}^{n-1} \alpha^i}$$

in general

$$\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3$$

$$\alpha^{n-1}\beta + \cdots + \alpha\beta + \beta$$

$$= \beta(\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha + 1)$$

$$= \beta(\underbrace{\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha + \alpha^0}) \checkmark$$

$$\sum_{i=0}^{n-1} \alpha^i \xrightarrow[\alpha \neq 1]{} \frac{\alpha^n - 1}{\alpha - 1}.$$

<u>Ans</u>: $f_n(x) = \alpha^n x + \beta \sum_{i=0}^{n-1} \alpha^i$ ✓

$$= \alpha^n x + \beta[\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha + 1].$$

3. <u>claim</u>: $f_n = f_m \Leftrightarrow n = m$

$\Rightarrow$ Assume $f_n = f_m$. then,

$\Rightarrow f_n(x) = f_m(x)$

$$\Rightarrow \alpha^n x + \beta \sum_{i=0}^{n-1} \alpha^i = \alpha^m x + \beta \sum_{i=0}^{m-1} \alpha^i$$

Compare like expression.

\# comparing algebraic expression

$\alpha^n = \alpha^m$,

$\Rightarrow \boxed{n = m}$. (take log at base a)

$\Leftarrow$ assume $n = m$.

$$f_n(x) = \alpha^n x + \beta \sum_{i=0}^{n-1} \alpha^i$$

$$= \underbrace{\alpha^m x + \beta \sum_{i=0}^{m-1} \alpha^i}$$

$$= f_m(x).$$

$(\because n = m)$

4. $$\underbrace{f_{n+1}(x) = \alpha^{n+1}x + f_n(\beta)}_{\text{Recursion Relationship.}}$$

$f_{2024}(x)$: $n = 2023$ $\quad \because n+1 = 2024$

$\Rightarrow n = 2023$

$$\boxed{f_{2024}(x) = 2^{2024}x + f_{2023}(-3)}$$

§2.1 DERIVATIVES 1:25–2:40PM Wednesday 7th Feb 2024 (1)

$y = f(x)$; small increament in x, say that small increment is simply h, hence, the new x-coordinate is $x+h$.

consider Rate $\dfrac{f(x+h) - f(x)}{x+h-x} \equiv \underbrace{\dfrac{f(x+h) - f(x)}{h}}$

if we allow $h \longrightarrow 0$ in $\dfrac{f(x+h) - f(x)}{h}$, we have

$$\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] := f'(x)$$

derivative of f @ x.

provided that the $\lim\limits_{h \to 0} \left[\dfrac{f(x+h) - f(x)}{h} \right]$ exists.

BTW: $f'(x) = \lim\limits_{h \to 0} \left[\dfrac{f(x+h) - f(x)}{h} \right]$ is called

First principle rule to determine $f'(x)$ @ x.

ex $f(x) = c$ (constant function)

if $f(x) = c \Rightarrow f(x+h) = c$.

now, $$\frac{f(x+h) - f(x)}{h} = \frac{\cancel{c} - \cancel{c}}{h} = 0$$

$$\Rightarrow \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = 0$$

---

ex $f(x) = \boxed{c} \cdot x \longrightarrow$ slope

$\because f(x) = cx \Rightarrow f(x+h) = c(x+h)$

now,

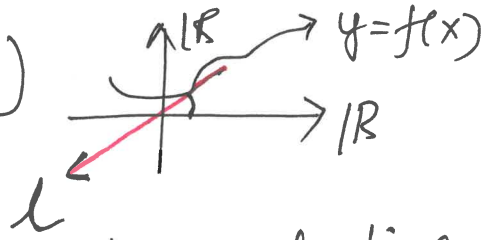
$$\frac{f(x+h) - f(x)}{h} = \frac{c(x+h) - cx}{h}$$

$$= \frac{\cancel{cx} + \cancel{ch} - \cancel{cx}}{\cancel{h}} = c$$

$$\therefore \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = \boxed{c}$$

$f'(x) \equiv$ slope if f is a linear function.

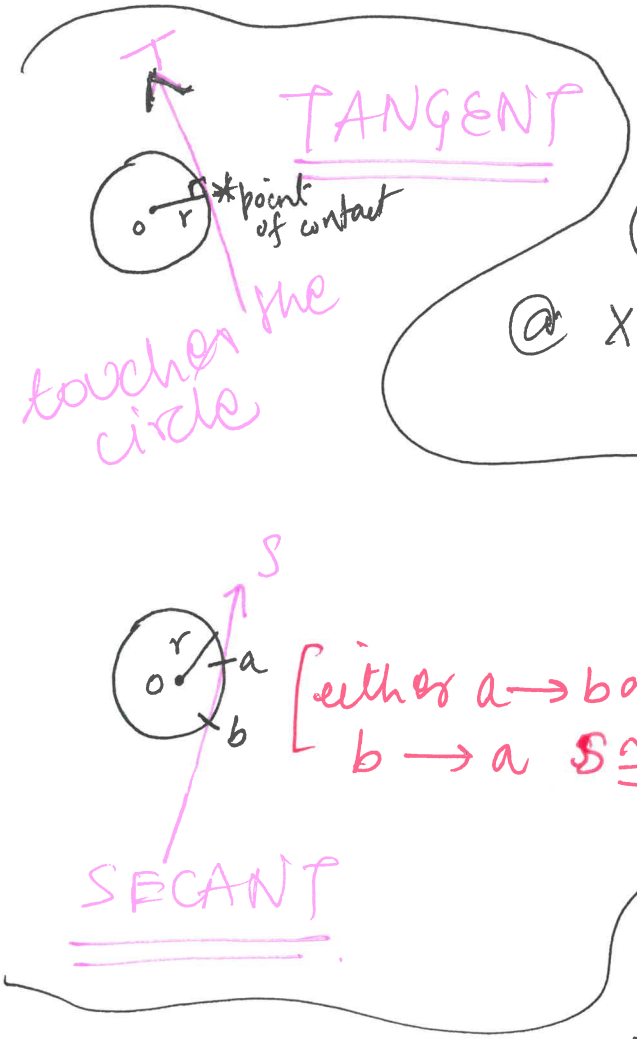
In general: $y = f(x)$



↳ tangent line

$y = (m, c)$

$y = mx + c$



① We need to find the slope @ $x = a$; the slope is given as

$$m\Big|_{x=a} = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right]$$

② ∵ we know $y = f(x)$ and we know $x = a$, it imply that the T passes from $(a, f(a))$

Recall the 1-point slope formula $y - y_0 = m(x - x_0)$

$y_0 \to f(a)$, $x_0 \to a$

Then, the equation of T is

$$\boxed{y = \underbrace{\left[\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right]}_{m} (x - a) + f(a)}$$

ex1 @ 73 determine T of $y = x^2$ @ $(1, 1)$

for $y = \underbrace{x^2}_{f(x)}$; $m\Big|_{x=1} = \lim_{x \to 1} \left[\frac{x^2 - 1}{x - 1} \right]$ $\quad \because a^2 - b^2 = (a+b)(a-b)$ ④

$$= \lim_{x \to 1} \left[\frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} \right]$$

$$= \lim_{x \to 1} \left[x + 1 \right]$$

$$= 2$$

$\therefore$ The equation of T:

$$y - 1 = 2(x - 1) \qquad ////$$

Squeeze theorem: ~~Sp~~ suppose, we have three functions f, g and h.

$$f \le g \le h$$

say $\lim_{x \to a} f = L$ and $\lim_{x \to a} h = L$ and obviously $L \neq \infty$.

then conclusion $\lim_{x \to a} g = L$.

---

find $\lim_{x \to 0} x^2 \sin(1/x)$?

$x^2 \le \boxed{}$

Recall

$$-1 \le \sin(1/x) \le 1$$

$$\Rightarrow \quad \underbrace{-x^2}_{f} \le \underbrace{x^2 \sin(1/x)}_{g} \le \underbrace{x^2}_{h}$$

Additionally $\lim_{x \to 0} (-x^2) = 0$ and $\lim_{x \to 0} x^2 = 0$,

thus by squeeze theorem $\lim_{x \to 0} x^2 \sin(1/x) = 0$.

$$\frac{\alpha^n - 1}{\alpha - 1} = \sum_{i=0}^{n-1} \alpha^i \qquad \alpha \neq 1$$

#45: $f(\tau) = \dfrac{\tau^n - 1}{\tau - 1}$ if $n = 1$ $f(\tau) = 1$

~~f(τ~~ if $n = 2$ $f(\tau) = \dfrac{\tau^2 - 1}{\tau - 1} = \dfrac{(\tau+1)\cancel{(\tau-1)}}{\cancel{\tau-1}}$

$= \tau + 1$

$$\lim_{\tau \to 1} f(\tau) = \lim_{\tau \to 1} (\tau + 1) = 2$$

Similarly: $\lim_{\tau \to 1} f(\tau) \to \dfrac{\tau^3 - 1}{\tau - 1} = 3.$

1:25-2:40PM

Friday 9th feb 2024

①

§ 2.1 $f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$ provided that the $\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$ exists.

Example $f(x) = \sin x$ find $f'(x)$ by using the definition.

solution: For $f(x) = \sin x$,

$$f'(x) = \lim_{h \to 0} \left[\frac{\sin(x+h) - \sin x}{h} \right]$$

$\sin(x+h) = \sin x \cos h + \sin h \cos x$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\sin x (\cos h - 1)}{h} + \frac{\sin h \cos x}{h} \right]$$

$$= \sin x \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{0} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{1}$$

$$= 0 + \cos x = \cos x \text{ ////.}$$

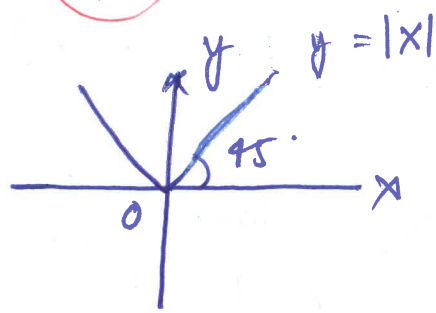
§2.2 What do we mean by "if limit exists" ②

(exist $\exists$)

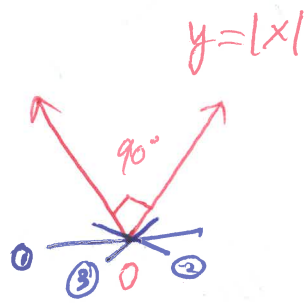
DIFFERENTIABILITY: If $f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$

is the derivate of f @ x, provided that the $\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$ $\exists$, then we say that f is differentiable at x.

For example: $f(x) = |x|$.



Recall: $f(x) = \begin{cases} -x \,;\; x < 0 \\ x \,;\; x \geq 0 \end{cases}$



Left hand side meaning $x < 0 \to -x$.

$$f'(x) = \lim_{h \to 0} \left[\frac{-(x+h) - (-x)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-\cancel{x} - h + \cancel{x}}{h} \right] = -1 \quad (< \infty)$$

<u>Right hand side</u> meaning $x \geq 0 \to +x$

$$f'(x) = \lim_{h \to 0} \left[\frac{(\cancel{x} + h) - \cancel{x}}{\cancel{h}} \right] = \cancel{\lim} \; 1. \quad (< \infty)$$

so, for we have demonstrated the existence of $f'(x)$ (which is not consistent @ $x=0$).

Calculate left hand limit $f(x) = |x|$

$$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{-h}{h} = -1. \quad (< \infty) \; \exists$$

$\downarrow$

left of $x=0$

In the same way, right hand limit for $f(x) = |x|$

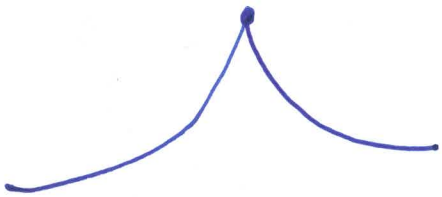
$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1. \quad (< \infty) \; \exists$$

$\therefore$ the limits are not consistent at $x=0$ for $f(x) = |x|$, thus f is not differentiable at $x=0$.

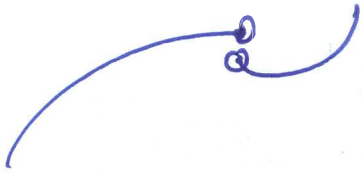
① f is continuous $\not\Longrightarrow$ Differentiable
(eg $f(x) = |x|$)

② f is Differentiable $\Longrightarrow$ f is continuous.

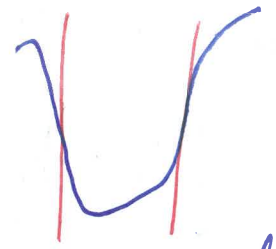
<u>common scenarios for failure of differentiability</u>



<u>corner in f</u>



<u>discontinuity</u>



<u>vertical tangent</u>

$\boxed{\text{Fail to draw a unique tangent to the function} \Rightarrow \text{failure of differentiability.}}$

<u>Notations</u>: ① $f'(x)$ ② $D_x f(x)$

③ $\frac{df}{dx}(x)$ ④ $f^{(1)}(x)$

if $f(x) = c \Rightarrow D_x f(x) = 0$

if $f(x) = \sin x \Rightarrow D_x f(x) = \cos x$.

if $f(x) = e^x \Rightarrow D_x f(x) = e^x$.

Derivative rules

↳ sum rule $[f \pm g]' = f' \pm g'$.

↳ power rule $(x^n)' = n x^{n-1}$.

↳ $(c f(x))' = c f'(x)$

↳ $(\sin x)' = \cos x$; $(\cos x)' = -\sin x$.

---

ex $f(x) = 2x^{1/2} + \frac{1}{3}x^3 - \cos x$

$f'(x) = (2x^{1/2})' + (\frac{1}{3}x^3)' - (\cos x)'$

$= x^{-1/2} + x^2 - (-\sin x)$

$= x^{-1/2} + x^2 + \sin x$. ////.

$$(2x^{1/2})' = 2(x^{1/2})'$$

$$= \cancel{2} \cdot \frac{1}{\cancel{2}} x^{1/2 - 1}$$

$$= x^{-1/2}.$$

(power rule)

$$\left(\tfrac{1}{3} x^3\right)' = \tfrac{1}{3}(x^3)'$$

$$= \tfrac{1}{\cancel{3}} \cdot \cancel{3} x^2$$

$$= x^2.$$

$(x^n)' = n x^{n-1}$

Reading assignment: ex-1 @ 97, ex-3 @ 98, ex-8 @ 102.

Product rule: $(f \times g)' = f'g + g'f$.

Quotient rule: $\left(\frac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$

$g \neq 0$.

<u>example</u>: $y = \underbrace{x}_{f} \underbrace{\sin x}_{g}$ (application of product rule)

$$\Rightarrow \frac{dy}{dx} = (x \sin x)'$$

$$= 1 \cdot \sin x + x \cdot \cos x \quad ////.$$

<u>example</u>: $y = \dfrac{x}{\sin x} \cong \dfrac{f}{g}$

$$\frac{dy}{dx} = \frac{\sin x - x\cos x}{\sin^2 x} \quad ////.$$

<u>example</u> $y = \tan x = \dfrac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{\cos x \cdot \cos x - (-\sin x)\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= 1 + \tan^2 x = \sec^2 x. ////.$$

list of trig- derivatives @ 111.

§ 2.4 ex-6 @ 112.

1:25PM-2:40PM Monday 12th Feb 2024 (1)

$(\tan x)' = \sec^2 x$ by Quotient rule.

recall: $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$ $\tan x = \frac{\sin x}{\cos x}$

$(\cos x)' = -\sin x$

$(\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x$.

csc

$(\cot x)' = -\operatorname{cosec}^2 x$.

$(\sin x)' = \cos x$

$(\tan x)' = \sec^2 x$

$(\sec x)' = \sec x \tan x$.

Chain Rule: $y(x) = f \circ g(x) \equiv f(g(x))$

$\rightarrow y'(x) = \frac{d}{dx} y(x) = [\underbrace{f' \circ g(x)}] \cdot g'(x)$

$$\boxed{\frac{dy}{dx} = f'(g(x)) \cdot g'(x).}$$

1. $y = \sin x$

$\Rightarrow y' = \cos x$

$y = \sin(x^2)$

$y' = 2x \cdot \cos(x^2)$

$\sin(x^2) \neq (\sin x)^2$

$\cancel{\sin(x^2)}$

$\sin(x^2) = \underbrace{\sin x}_{f} \circ \underbrace{x^2}_{g}$

$\Rightarrow (\sin(x^2))' = (\cos x \circ x^2) \cdot (2x)$

$= \cos(x^2) \cdot 2x$

---

2. $y = \sin^3 x$; $y' = 3\sin^2 x \cdot \cos x$

$(x^n)' = n x^{n-1}$

$\sin^3 x = \underbrace{x^3}_{f} \circ \underbrace{\sin x}_{g}$

chain rule

$(\sin^3 x)' = ((x^3)' \circ \sin x) \cdot (\sin x)'$

$= (3x^2 \circ \sin x) \cdot \cos x$

$= 3(\sin^2 x) \cdot \cos x.$

$x^2 \circ \sin x = (\sin x)^2 = \sin^2 x$

3. $y = \sin(x^2 + e^x)$

$y' = \cos(x^2 + e^x) \cdot (2x + e^x)$

<u>Note</u> : $y(x) = \sin u(x)$ $\left(u(x) = x^2 + e^x \right)$

$\Rightarrow y'(x) = \cos u(x) \cdot u'(x)$

---

4. $y = \log_e(x^2 + \tan x)$ $\int$ <u>Recall</u>: $(\log x)' = \frac{1}{x}$.

$\Rightarrow y'(x) = \frac{1}{x^2 + \tan x} \cdot (2x + \sec^2 x)$.

---

5. $y = \log \cos x$; $y' = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$

---

6. $y = \log \sin x$; $y' = \frac{\cos x}{\sin x} = \cot x$.

$\because \int (\log \sin x)' = \int \cot x$

$\Rightarrow \log \sin x = \int \cot x$

7. $y = \sin(\underline{x \tan x})$

$y' = \cos(\tan x) \cdot \sec^2 x.$

---

8. $y = \sqrt{1 - \frac{1}{x^2}} = \underbrace{\sqrt{x}}_{f} \circ \underbrace{\left(1 - \frac{1}{x^2}\right)}_{g}$

if $f = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f' \circ \left(1 - \frac{1}{x^2}\right)$

$= \frac{1}{2\sqrt{1 - \frac{1}{x^2}}}.$

$(x^n)' = n x^{n-1}$

if $n = 1/2$

$(\sqrt{x})' = (x^{1/2})'$

$= \frac{1}{2} x^{1/2 - 1}$

$= \frac{1}{2} \frac{1}{\sqrt{x}}.$

if $g(x) = \cancel{1} - \frac{1}{x^2} \Rightarrow g'(x) = -\frac{(-2)}{x^3} = \frac{2}{x^3}.$

constant

in this case $n = -2$

$\left(\frac{1}{x^2}\right)' = (x^{-2})' \Rightarrow -2x^{-2-1}$

$= \frac{-2}{x^3}$

final result: $y' = \left(\sqrt{1-\frac{1}{x^2}}\right)' = \frac{1}{2\sqrt{1-\frac{1}{x^2}}} \cdot \frac{2}{x^3}$.

---

1. $f(x) = (x+1)(x+2)$ use product rule.

$f'(x) = 1\cdot(x+2) + 1\cdot(x+1) = 2x+3$ ✓

---

2. $f(x) = (x+2)(x+5)$

$f'(x) = 1\cdot(x+5) + 1(x+2) = 2x+7$

---

by product rule: $\underbrace{(x+2)}_{f}\underbrace{(x+5)}_{g}$ $\quad f' = 1$, $g' = 1$

$= 1\cdot(x+5) + 1(x+2)$

$= 2x+7$

---

3. $f(x) = (x+3)(x+4)$

$f'(x) = 2x+7$

---

whenever $f(x) = (x+m)(x+n)$

$f'(x) = 2x+m+n$.

(44)/121 $y(x) = \cos^2 x \equiv \underbrace{x^2}_{f} \circ \underbrace{\cos x}_{g}$ (6)

$y'(x) = 2\cos x(-\sin x)$

$= -\underline{2\sin x \cos x}$

$= -\sin 2x$

$\sin 2x$
$= 2\cos x \sin x$

HW: § 2.3 — 1–26, 27–28, 31–34

§ 2.4 — 1–26

1:25-2:40PM

Wednesday ①

14th Feb 2024

§ chain rule 2.5

composition

$h(x) = f \circ g(x) = f(g(x))$. (assumption)

$h'(x) = f'(g(x)) \cdot g'(x)$.

Ex: $h(x) = (x+1)^{100}$

power rule $(x^n)' = n x^{n-1}$.

$h'(x) = 100(x+1)^{100-1} \cdot \frac{d}{dx}(x+1)$

$= 100(x+1)^{99} \cdot 1$

$= 100(x+1)^{99}$ ////.

Ex: $h(x) = (\underbrace{\sin^2 x + \cos^2 x}_{1})^{10^9}$

Recall: $\sin^2 x + \cos^2 x = 1$

$h'(x) = 0$.

$(x^n)' = nx^{n-1}$

$n=1$

$(x)' = 1x^{1-1} = 1\cdot 1 = 1$

- $y = \tan^8 x = (\underbrace{\tan x}_{x \equiv x^0})^8$

$\Rightarrow y' = 8\tan^{8-1} x \cdot \frac{d}{dx}\tan x$

$= 8\tan^7 x \cdot \sec^2 x.$

- $y = \frac{\sin^2 x}{(x)^2} \Rightarrow y' = \underbrace{2\sin x \cdot \cos x}$ $= \sin 2x$ (double angle formula).

$\hookrightarrow 2x \cdot \frac{d}{dx}x$

- $y = \sin^3(x + e^x + \log x)$

$\begin{cases} (e^x)' = e^x \\ (\log x)' = \frac{1}{x}. \end{cases}$

$y' = 3\sin^2(x + e^x + \log x)(\cos(x + e^x + \log x)) \cdot \left(1 + e^x + \frac{1}{x}\right).$

- $y = \underset{f}{e^{2x}} \cdot \underset{g}{\cos x}.$

$y' =$

$(fg)' = f'g + g'f$

$f = e^{2x} \Rightarrow f' = 2e^{2x}$

$g = \cos x \Rightarrow g' = -\sin x$

$$y' = 2e^{2x}\cdot\cos x - e^{2x}\sin x \quad ////.$$

ex: $y = \underbrace{\left(\underbrace{\sec^2 x - \tan^2 x}_{1} + \underbrace{\cot^2 x - \operatorname{cosec}^2 x}_{-1}\right)^2}_{0^{\square}}$

$= 0$

$\therefore y' = 0.$

#33@121 $\quad y = \left(\dfrac{1-\cos 2x}{1+\cos 2x}\right)^4 \qquad \left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$

$$y' = 4\left(\frac{1-\cos 2x}{1+\cos 2x}\right)^{4-1}\cdot\frac{d}{dx}\left(\frac{1-\cos 2x}{1+\cos 2x}\right) \quad \begin{matrix} \to f \\ \to g \end{matrix}$$

$$\left(\frac{1-\cos 2x}{1+\cos 2x}\right)' = \frac{(1+\cos 2x)(2\sin 2x) - (1-\cos 2x)(-2\sin 2x)}{(1+\cos 2x)^2}$$

$$= \frac{(1+\cos 2x)\,2\sin 2x + (1-\cos 2x)(2\sin 2x)}{(1+\cos 2x)^2}$$

$2\sin 2\,(1+\cancel{\cos 2x}+1-\cancel{\cos 2x})$

$$= \frac{4\sin 2x}{(1+\cos 2x)^2}$$

final answer.

$$y' = 4\left(\frac{1-\cos 2x}{1+\cos 2x}\right)^3 \cdot \frac{4\sin 2x}{(1+\cos 2x)^2}$$

$$y = \left(\frac{1-\cos 2x}{1+\cos 2x}\right)^4$$

$$= \left(\frac{\cancel{2}\sin^2 x}{\cancel{2}\cos^2 x}\right)^4$$

$$= (\tan^2 x)^4$$

$$\boxed{y = \tan^8 x}$$

$$y' = 8\tan^7 x(\sec^2 x)$$

recall double angle formulae

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\because \cos^2 x + \sin^2 x = 1$$

$$\boxed{\cos 2x + 1 = 2\cos^2 x}$$

$$\boxed{1 - \cos 2x = 2\sin^2 x}$$

---

§2.6 Implicit differentiation

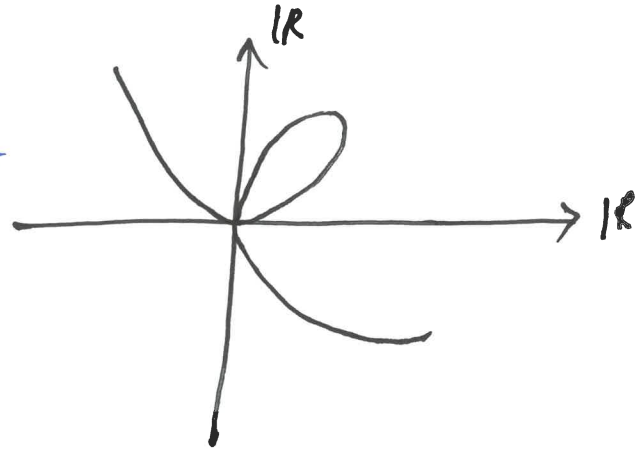
hard to seperate (isolate) independent and dependendent variable.

for example: $y = \sin x$ | $y = \sin(x + y^2 x)$

For example: $y = \sin x \Rightarrow y' = \cos x$. ////.

EXPLICIT

example: $x^3 + y^3 = 6xy$



differentiate both sides.

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$$

chain rule

$(y^3)' = 3y^2 \cdot y'$

$= 3y^2 \frac{dy}{dx}$

$$\Rightarrow \frac{d}{dx}x^3 + \frac{d}{dx}y^3 = 6\frac{d}{dx}(xy).$$

$$\Rightarrow \cancel{3}x^2 + \cancel{3}y^2y' = \cancel{6}^{2}\left[y + xy'\right]$$

$$\Rightarrow x^2 + y^2y' = 2y + 2xy'$$

$$\Rightarrow y^2y' - 2xy' = 2y - x^2$$

$$\Rightarrow y'(y^2 - 2x) = 2y - x^2 \Rightarrow \boxed{y' = \frac{2y - x}{y^2 - 2x}}$$

1:25 - 2:40 PM

Friday

16th Feb 2024

§2.6 Implicit Differentiation

AIM: find y' / (dy/dx) ?

$(fg)' = f'g + g'f$

$(xy)' = y + y' \cdot x$

Recall: $x^3 + y^3 = 6xy$ (what is y'?)

differentiate $x^3 + y^3 = 6xy$ w.r.t. x

$(xy)' =$

$\Rightarrow \cancel{3}x^2 + \cancel{3}y^2 y' = \cancel{6}_2 [y + xy']$

Ex-2 @ 12 s

$\Rightarrow x^2 + y^2 y' = 2y + 2xy'$

$\Rightarrow y^2 y' - 2xy' = 2y - x^2.$

$\Rightarrow y'(y^2 - 2x) = 2y - x^2.$

$$\boxed{y' = \frac{2y - x^2}{y^2 - 2x}}$$

← in general, this is a slope for the tangent line @ (x, y).

Find tangent equation to $x^3+y^3=6xy$ ② ③
(3,3).

@125
ex-2 ⓑ

Recall: $m = \dfrac{2y-x^2}{y^2-2x}$

$$\Rightarrow m\Big|_{(3,3)} = \frac{2y-x^2}{y^2-2x}\Big|_{(3,3)} = \frac{6-9}{9-6} = -1$$

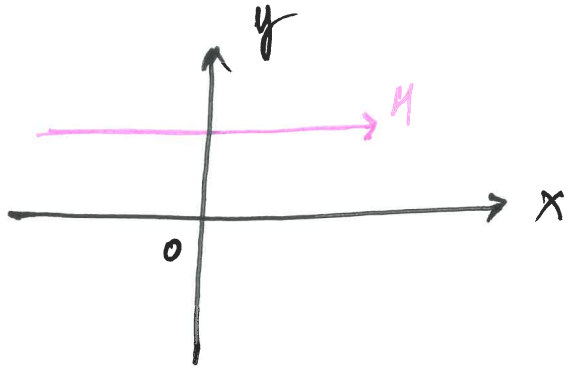
In our situation, we will have

Recall:
$y-y_0=m(x-x_0)$

$$y-3 = -1(x-3)$$

$$\Rightarrow y = -x+3+3 \Rightarrow \boxed{y+x=6}$$ ////.

At what point in 1st quadrant the tangent is horizontal?



Remember: 2 straight line which are //, will share the same slope.

In this case. tangent line 'H' will have slope '0'.

$\because y' = 0 \Rightarrow \dfrac{2y - x^2}{y^2 - 2x} = 0 \Rightarrow 2y - x^2 = 0$

$\Rightarrow \boxed{y = \dfrac{x^2}{2}}$ ① $\longrightarrow$ this will intersect $x^3 + y^3 = 6xy$.

$\Rightarrow x^3 + \left(\dfrac{x^2}{2}\right)^3 = 6x \cdot \dfrac{x^2}{2}$

$\Rightarrow x^3 + \dfrac{x^6}{8} = 3x^3 \Rightarrow \dfrac{x^6}{8} = 2x^3$

$\Rightarrow \boxed{x^6 - 16x^3 = 0}$

$\Rightarrow x^3(x^3 - 16) = 0$

this imply that $x^3 = 0$ or $x^3 - 16 = 0$

$\Rightarrow x^3 = 16$

$\Rightarrow x = 16^{1/3}$

use $x = 16^{1/3}$ in ① to have

$y = \dfrac{(16^{1/3})^2}{2} = \dfrac{16^{2/3}}{2}$. Thus (x, y) - coordinate is $\left(16^{1/3}, \dfrac{16^{2/3}}{2}\right)$ ////.

Review of product Rule:

$$y = (fg) \Rightarrow y' = (fg)' = f'g + fg'.$$

what about $y = fgh$ or $y = fghu$

$(e^x)' = e^x$

for example: $y = \sin x \cdot e^x$

$$y' = \cos x \cdot e^x + e^x \sin x$$

$$y = \sin x \cdot e^x \cdot x^2 \ *$$

For multiplication of multiplie functions, take 'log' for ex

$\log(mn) = \log m + \log n$

$(\log y)' = \frac{y'}{y}$

$$y = fgh$$

$$\log y = \log f + \log g + \log h.$$

$$\Rightarrow \frac{y'}{y} = \frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}$$

ultimately $y' = y\left[\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right]$.

<u>For example</u>:

$$y = e^x \cdot \sin x \cdot (x+2) \cdot \cos x \cdot x^2$$

$$\log y = x + \log \sin x + \log(x+2) + \log \cos x + 2\log x$$

$$\Rightarrow \frac{y'}{y} = 1 + \frac{\cos x}{\sin x} + \frac{1}{x+2} + \frac{(-\sin x)}{\cos x} + \frac{2}{x}$$

multiply by y:

$$y' = y\left[1 + \cot x + \frac{1}{x+2} - \tan x + \frac{2}{x}\right]$$

1:25–2:40 PM

19th Feb 2024 ①

Monday:

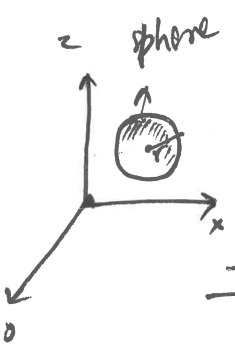
$(cf(x))' = cf'(x)$

§ Related Rates:

Recall: $f'(x) = \frac{df}{dx} \equiv$ rate of change of f w.r.t. 'x'.

In general: we will have "$\frac{d\,f(x)}{dt}$" where 't' is time.

Ex-10/29 Volume of spherical balloon ↑ @ $100\,cm^3/s$.



$$V = \frac{4}{3}\pi r^3 \longrightarrow \text{differentiate w.r.t } t$$

$$\Rightarrow \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4\pi r^3}{3}\right) = \frac{4\pi}{3}\frac{d}{dt}(r^3).$$

$$\frac{dV}{dt} = \frac{4\pi}{3} \cdot 3r^2\frac{dr}{dt}. \qquad \text{(general)}$$

$$\boxed{\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}}$$

rate of change of volume

rate of change of radius.

Recall: $\frac{dV}{dt} = 100$, therefore

$$\Rightarrow 100 = 4\pi r^2 \cdot \frac{dr}{dt}.$$

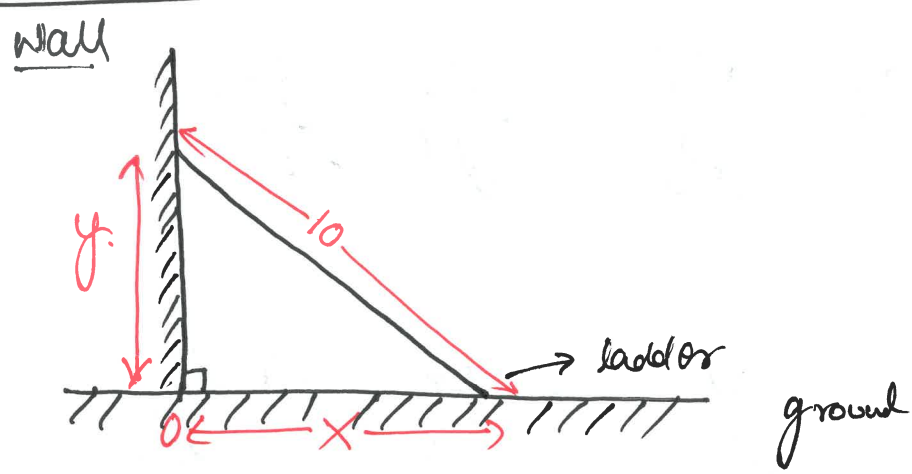
$$\Rightarrow \frac{dr}{dt} = \frac{100}{4\pi r^2}$$

if diameter is 50
$\Rightarrow$ radius is $\frac{50}{2} = 25$

$$\Rightarrow \left.\frac{dr}{dt}\right|_{r=25} = \frac{\cancel{100}}{\cancel{4}\pi \times \cancel{25} \times 25} = \frac{1}{25\pi}.$$

(in units) $\boxed{\left.\frac{dr}{dt}\right|_{r=25} = \left(\frac{1}{25\pi}\right) \text{ cm/s}}$

Ex-2:



by Pythagorean theorem.

$$x^2 + y^2 = 10^2 \Rightarrow x^2 + y^2 = 100$$

differentiate w.r.t t

$$\cancel{2}x \cdot \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}}$$ (in general)

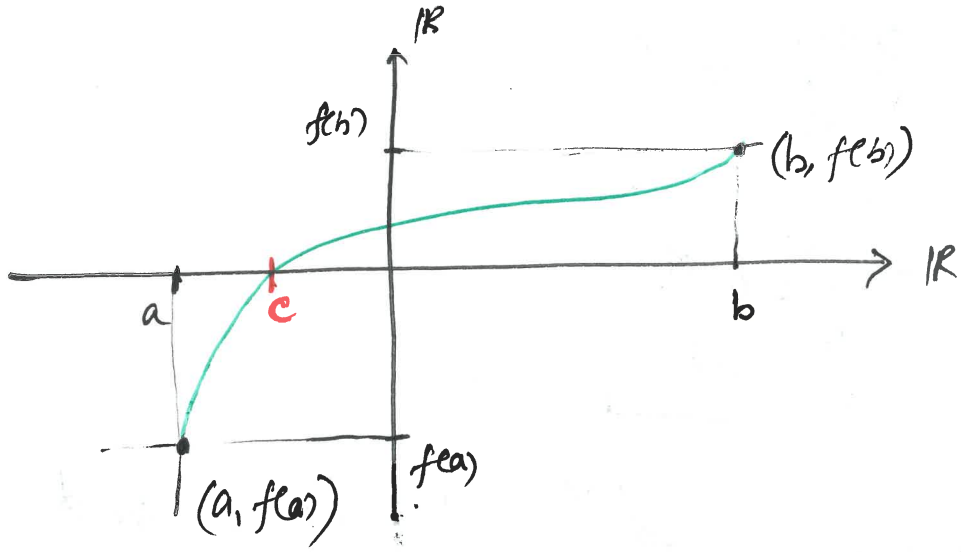
By the question, $\frac{dx}{dt} = 1$ ft/sec. and @ $x = 6$

$y = 8$ $\left(6^2 + y^2 = 100 \Rightarrow y^2 = 100 - 36 \right)$

Thus, $\left.\frac{dy}{dt}\right|_{(x,y)=(6,8)} = \left(-\frac{6}{8}\right)\cdot 1 = -\frac{6}{8} = -\frac{3}{4}$ ft/sec.

$f(c) = 0 \leftarrow$ c is called as root or zeros of f.

IVT (<u>Intermediate Value theorem</u>)



assume

① f is $\mathcal{C}[a,b]$

② $f(a)f(b) < 0$

$\exists$ a c in b/w a and b ∋ $f(c) = 0$.

<u>IVT @ 52</u>: say f is $\mathcal{C}[a,b]$ and let N be any number b/w $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, $\exists$ a number 'c' in (a,b)

∋ $f(c) = N$.

<u>Reading assignment Ex-8 @ 53</u>.

<u>#37 @ 55</u>: <u>claim</u>: $\exists$ a 'c' ∋ $f(c) = 1000$.

here $f(x) = x^2 + 10 \sin x$.

<u>Proof</u>: If $f(c) = 1000 \Rightarrow c^2 + 10\sin c = 1000$

$\Rightarrow c^2 + 10\sin c - 1000 = 0.$

Say $g(x) = x^2 + 10\sin x - 1000$, then $g(c) = 0$ meaning 'c' is the root of g.

Choose $a = 0$, observe $g(a) = g(0) = -1000 (< 0)$

now, choose $b = 100$, observe

$$g(b) = g(100) = 100^2 + 10\sin 100 - 1000$$
$$= 9000 + 10\underbrace{\sin 100}$$
$$\approx > 0$$

Thus, $g(0) \cdot g(100) < 0$, $\therefore$ by application of IVT, $\exists$ 'a' 'c' in b/w 0 and 100

$\therefore$ $g(c) = 0 \Rightarrow \underbrace{c^2 + 10\sin c} = 1000$

$$\boxed{f(c) = 1000}.$$

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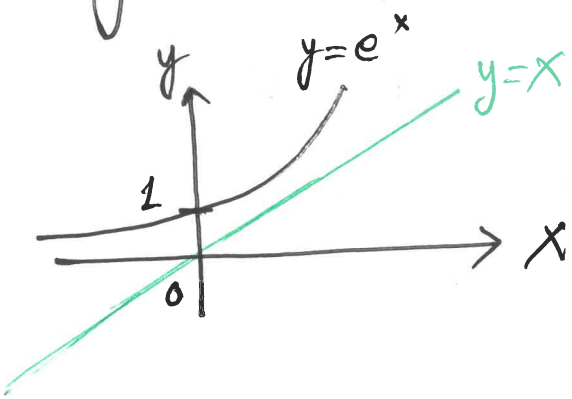
21st feb 2024 ①

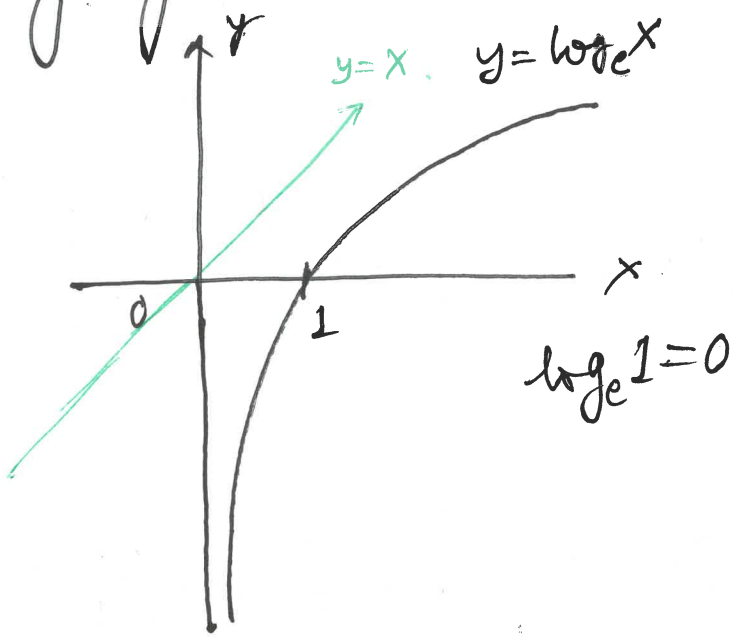
Wednesday

chapter-3 (§1, 2, 3)

inverse function of each other.

$$y = e^{x} \iff \log_e y = x$$





What is e?

$$e = \lim_{x \to 0} \left[(1+x)^{1/x} \right]$$

$$= 2.718\ldots$$

Recall: $$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$n! = n(n-1)(n-2)\cdots 1$

for exa

$3! = 3\times2\times1 = 6$

$5! = 5\times4\times3\times2\times1 = 120$

<u>claim</u>: $\lim_{h \to 0} \left[\frac{e^h - 1}{h} \right] = 1$

<u>Proof</u>: Recall:

$$e^h = 1 + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots$$

$$\Rightarrow e^h - 1 = h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots$$

$$\Rightarrow \frac{e^h - 1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \frac{h^3}{4!} + \cdots$$

$$\Rightarrow \boxed{\lim_{h \to 0} \left[\frac{e^h - 1}{h} \right] = 1}$$ ////.

<u>claim</u> $\frac{d}{dx}(e^x) = (e^x)' = e^x$.

<u>Proof</u>: Recall the first principle definition of $f'(x)$.

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] \quad \text{--- (1)}$$

if $f(x) = e^x$, then $f(x+h) = e^{x+h} = e^x \cdot e^h$

Using (1) will yield

$$(e^x)' = \lim_{h \to 0} \left[\frac{e^x \cdot e^h - e^x}{h} \right]$$

$$= e^x \underbrace{\lim_{h \to 0} \left[\frac{e^h - 1}{h} \right]}_{1}$$

$$= e^x \cdot 1$$

$$\boxed{(e^x)' = e^x}$$

<u>Problem 15 @ 150</u>. we will find domain of f; $f(x) = \dfrac{1 - e^{+x^2}}{1 - e^{1-x^2}}$.

<u>solution</u>: For domain of f, we find x for

which f fails to be defined. That is

$$1 - e^{1-x^2} = 0.$$

$$\Rightarrow e^{1-x^2} = 1. \quad (*)$$ take the $\log_e$

$$\Rightarrow \overbrace{\log_e e^{1-x^2}} = \overbrace{\log_e 1}^{=0}$$

Remember

- $\log_B B = 1$ in general
- $\log_B B^A = A$

$$\Rightarrow (1-x^2) = 0$$

$$\Rightarrow x^2 = 1 \Rightarrow \boxed{x = \pm 1}.$$

Hence, the $\text{dom}(f) = \mathbb{R} \setminus \{\pm 1\}$.

Remember: $\lim_{x\to\infty} e^{x} = \infty$ $\quad \lim_{x\to\infty} e^{x^2} = \infty$

$\lim_{x\to\infty} e^{-x} = 0$ $\quad \lim_{x\to\infty} e^{-x^2} = 0$

#25@150:

$$\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \left(\lim_{x\to\infty} \sigma(3x)\right)$$

$$\sigma(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

SIGMOID FUNCTIONS.

ML / N.N.

Activation function

$$\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \frac{\frac{e^{3x}}{e^{3x}} - \frac{e^{-3x}}{e^{3x}}}{\frac{e^{3x}}{e^{3x}} + \frac{e^{-3x}}{e^{3x}}}$$

DCNN

$$= \lim_{x\to\infty} \left[\frac{1 - e^{-3x}/e^{3x}}{1 + e^{-3x}/e^{3x}}\right]$$

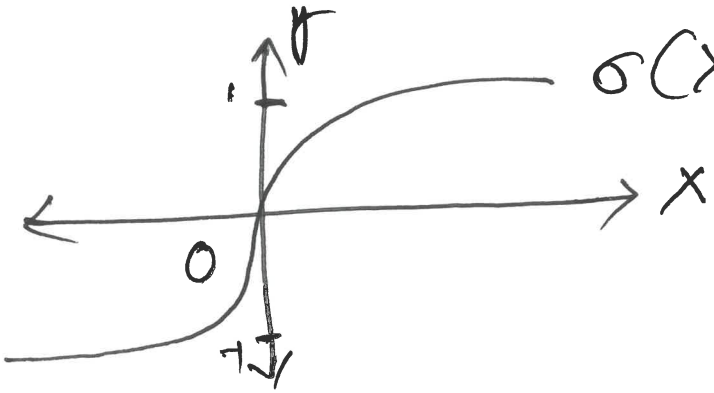
Remember

$$\frac{e^{-3x}}{e^{3x}} = e^{-3x-3x}$$

$$\frac{a^{m}}{a^{n}} = a^{m-n}$$

$$= \lim_{x\to\infty} \left[\frac{1 - e^{-6x}}{1 + e^{-6x}}\right]$$

$$= 1.$$



$\sigma(x)$.

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1st Mar 2024 ①

Friday

Recall:

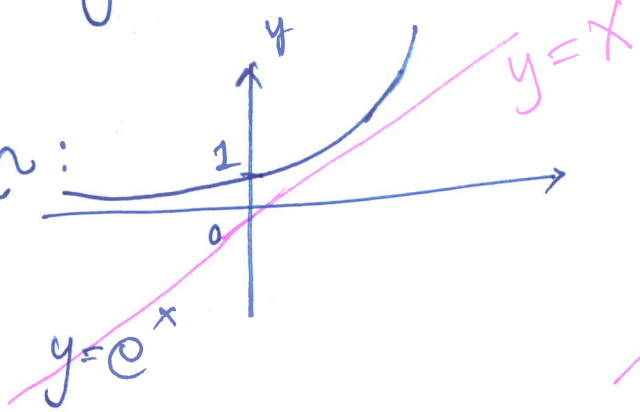
$y = e^x$ (can't be negative) $\quad e = \sum_{n=0}^{\infty} \frac{1}{n!}$

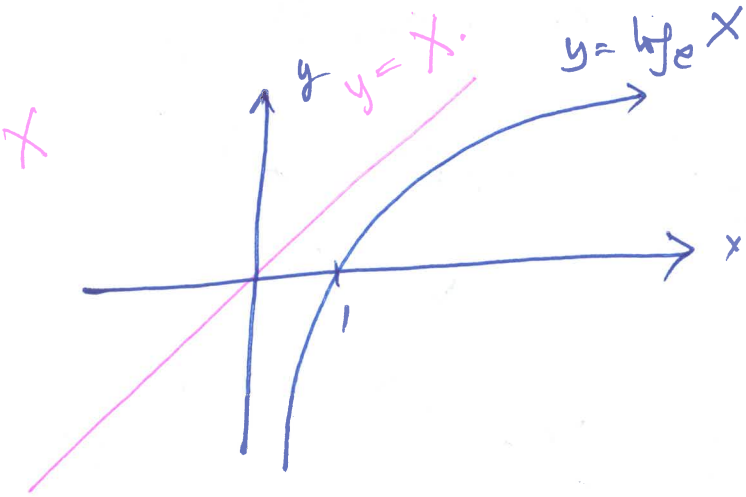
exponential function (simplest fm). $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

inverse of exponential function

$$y = \log_e x = \ln x$$

Graph:





for example: Q159 $\quad e^{5-3x} = 10.$

$$\Rightarrow \log_e e^{5-3x} = \log_e 10$$

$$\Rightarrow 5-3x = \log_e 10 \Rightarrow \boxed{x = \frac{5-\log_e 10}{3}}$$

Remember: ① $\log_A A = 1$

② $\log_A(A^\alpha) = \alpha$ ——— ②

③ $\log_A(mn) = \log_A m + \log_A n$.

④ $\log_A\left(\frac{m}{n}\right) = \log_A m - \log_A n$.

---

page 163 #65 ⓐ $2^{x-5} = 3$. 1st way

sol. $\log_2\left(2^{x-5}\right) = \log_2 3$ (log @ base 2)

$\Rightarrow (x-5) = \log_2 3$ (use ②)

$\Rightarrow \boxed{x = 5 + \log_2 3}$.

$2^{x-5} = 3$ 2nd way

exponent rules to have $2^{x-5} = \frac{2^x}{2^5}$

$$\Rightarrow \frac{2^x}{2^5} = 3$$

$$\Rightarrow 2^x = 2^5 \cdot 3 \qquad (\text{multiply by } 2^5)$$

take the $\log_2$

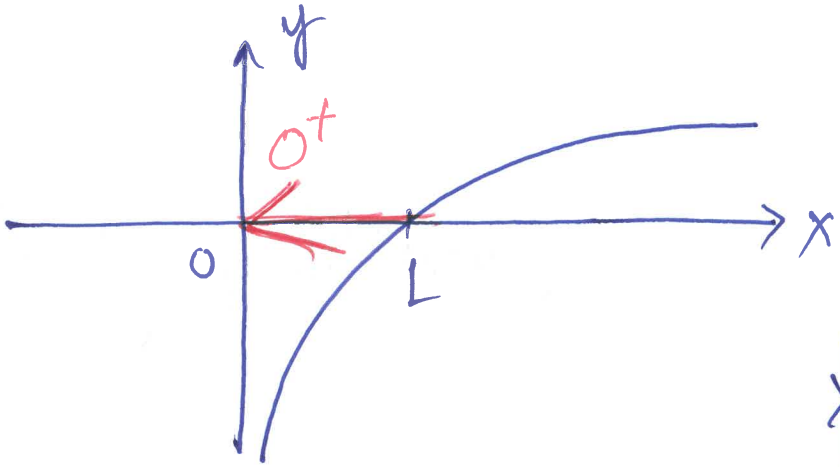
$$\Rightarrow x = \log_2(2^5 \cdot 3)$$

$$= \underbrace{\log_2(2^5)} + \log_2 3 . \quad (\text{rule } 3)$$

$$\boxed{x = 5 + \log_2 3}$$

<u>Recall</u>: $\underbrace{\lim_{x \to \infty} e^x = \infty}_{\text{exponential growth}}$ || $\underbrace{\lim_{x \to \infty} e^{-x} = 0}_{\text{exponential decay}}$.

$\lim_{x \to \infty} \log_{\square} x = \infty$ || $\lim_{x \to 0^+} \log_{\square} x = -\infty$.



$y = \log_{\square} x$

$\lim_{x \to 0^-} \log_{\square} x = \circlearrowleft$

not applicable for negative values of x

<u>Page 162</u>

<u>#26</u> $y = \dfrac{e^x}{1+2e^x}$ (find the inverse of this function.)

first put $e^x = z$. then

$y = \dfrac{z}{1+2z}$ (think of isolating z in terms of y).

$\Rightarrow y(1+2z) = z$ (multiply by $1+2z$).

Recall: $y(1+2z) = z$

$$\Rightarrow y + 2zy = z$$

$$\Rightarrow y = z - 2zy$$

$$\Rightarrow y = z(1-2y)$$

$$\therefore \quad z = \frac{y}{1-2y}$$

$$e^x = \frac{y}{1-2y}$$

(take the natural log)

$$\Rightarrow \boxed{x = \ln\left(\frac{y}{1-2y}\right)}$$

---

1. $y = e^{\cos x} \Rightarrow y' = e^{\cos x} \cdot (-\sin x)$

<u>Rule</u>: $y = e^{f(x)}$

$$\Rightarrow \boxed{y' = f'(x) \cdot e^{f(x)}}$$

application of chain rule.

chain rule.

(2) $y = e^{\cos(x^2)}$

$(\cos x^2)'$ $\quad$ $(x^2)'$

$$y' = e^{\cos x^2}(-\sin x^2) \cdot 2x$$

$$(\log x)' = \frac{1}{x}$$

$$(\log x^2)' = \frac{2}{x} = \frac{1}{x^2} \cdot (x^2)'$$

$$(2\log x)'$$

$$= 2(\log x)' = \frac{2}{x}.$$

<u>general rule</u> $(\log f(x))' = \underbrace{\frac{f'(x)}{f(x)}}_{\text{"control the dynamics"}}$.

$P(x) =$

$$(x+b_1)(x+b_2)\cdots(x+b_n)$$

$$= \prod_{i=1}^{n}(x+b_i)$$

1. $\rightarrow [P'/P] \equiv$ first P' then divide P' by P.
(TEDIOUS)

2. $\log P = \log(x+b_1) + \log(x+b_2) + \cdots + \log(x+b_n)$

<u>differentiate</u>

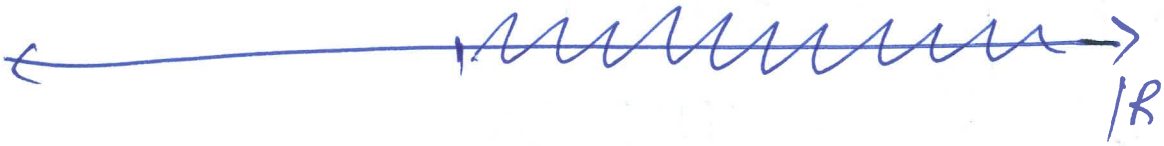
$$\frac{P'}{P} = \frac{1}{x+b_1} + \frac{1}{x+b_2} + \cdots + \frac{1}{x+b_n}$$

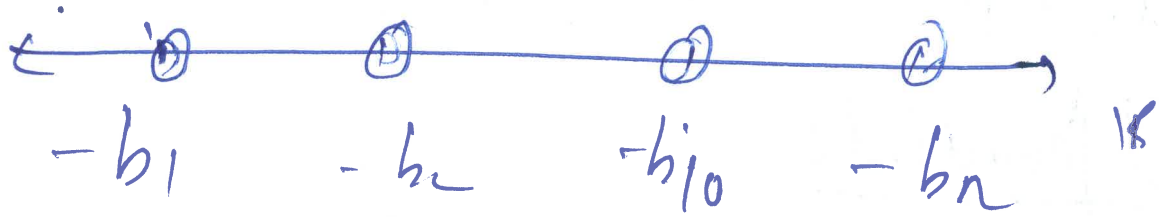
$$\mathrm{dom}(P'/P) = \mathbb{R} \setminus \{-b_1, -b_2, \dots, -b_n\}$$

<u>most of you</u>:

$$(-\infty, -b_n) \cup (-b_n, -b_{n-1}) \cup \dots \cup (-b_1, \infty)$$

WLOG $\underbrace{b_1 < b_2 < b_3 \dots < b_n}_{\text{"chain property"}}$





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Monday ①
4th Mar 2024

#74 @ 170 evaluate $\lim_{x \to \pi} \left[\frac{e^{\sin x} - 1}{x - \pi} \right]$.

solution: note ① by putting $x = \pi$, $\frac{e^{\sin \pi} - 1}{\pi - \pi} = \frac{0}{0}$

② $\lim_{x \to \pi} \left[\frac{e^{\sin x}}{x - \pi} - \frac{1}{x - \pi} \right] = \frac{1}{\pi - \pi} - \frac{1}{\pi - \pi}$

↳ L-Hôpital Rule $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$;

✓ provided the $\lim_{x \to a} \frac{f(x)}{g(x)}$ exist. (check!)

Remember: $f'(a) = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right]$ —— ①

compare ① with $\lim_{x \to \pi} \left[\frac{e^{\sin x} - 1}{x - \pi} \right]$, we get

$a = \pi$ and $f(x) = e^{\sin x}$. Therefore,

$f'(a) = (e^{\sin x})' \Big|_{x = a = \pi} = (\cos x \cdot e^{\sin x}) \Big|_{x = \pi}$

$$= \cos\pi \cdot e^{\sin\pi} = (-1)\cdot e^{0} = (-1)\cdot 1 = -1 \,////.$$

---

§3.4 : exponential growth & decay

$$y = y(t) \; : \quad \frac{dy}{dt} = ky$$

t → time

y → linear function.

(say)

$$\Rightarrow \int \frac{dy}{ky} = \int dt \Rightarrow \boxed{y(t) = y(0)\, e^{kt}}$$

$y(0)$ → initial value of y.

(OVERLOOK as of 4th March 2024).

---

now: $\dfrac{dy}{dt} = ky + b \Rightarrow \boxed{y(t) = y(0)\, e^{kt} - \dfrac{b}{k}}$

---

important : $\dfrac{dy}{dt} = \boxed{\dot{y} = f(y)}$

DYNAMICAL SYSTEM.

ex-1 @ 172:

$t=0$	$t=10$	$t=43$
1950	1960	1993

Given:

population growth rate is proportional to the population, say population is $P \equiv P(t)$. Then,

$$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP.$$

(k is some constant)

we know that $P(t) = P(0)e^{k\cancel{t}\to t}$. As, $P(0) = 2560$, then $P(t) = 2560\, e^{k\cancel{t}\to t}$ ——— (1)

Given, $P(10) = 3040$, then, in (1)

$$3040 = 2560\, e^{k\cdot 10}$$

$$\Rightarrow e^{k\cdot 10} = \frac{3040}{2560} \Rightarrow 10k = \ln\left[\frac{3040}{2560}\right]$$

$$\boxed{k = \frac{1}{10}\ln\left[\frac{3040}{2560}\right]}$$

$k > 0$

then $P(43) = P(0)\, e^{k \cdot 43}$ $\quad k = \frac{1}{10} \ln\left[\frac{3040}{2560}\right]$ ④

$$\boxed{P(43) = 2560\, e^{k \cdot 43}}$$

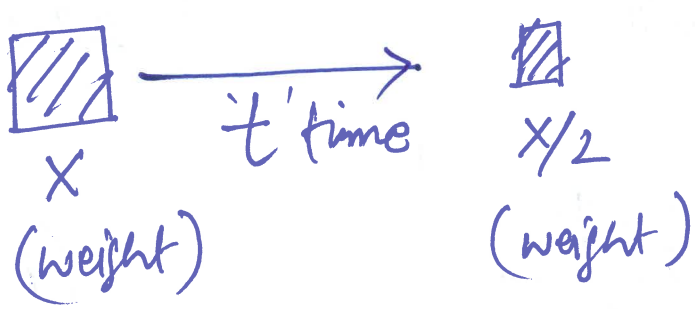
<u>Fact about</u>: $\frac{dy}{dt} = ky$

① if $k > 0$, system is of exponential growth

② if $k < 0$; system is of exponential decay.

③ if $k = 0$; system is constant. $y(t) \equiv C$.
(independent of time).

<u>Ex-2 @ 173</u> (exponential decay).
(half life of radioactive materials).



} <u>half life</u>.

Solution: (a) $m(t) = m(0)e^{kt}$. (AIM) find k? (5)

Incorporate half life. as $m(0) = 100$ and half life of Ra is 1590 yrs, this means that

$$\underbrace{50}_{m(1590)} = \underset{m(0)}{100}\, e^{k \cdot 1590}$$

$$\Rightarrow e^{k \cdot 1590} = \frac{1}{2} \quad \left(= \frac{50}{100}\right)$$

$$\Rightarrow \boxed{k = \frac{1}{1590} \ln \frac{1}{2}} \qquad (k < 0).$$

$$\therefore m(t) = 100 e^{kt} \text{ where } k = \frac{1}{1590} \ln(1/2).$$

(b) $m(1000) = 100 e^{k \cdot 1000}$ where $k = \frac{1}{1590} \ln(1/2)$.

(c) for what t, $m(t) = 30\,mg$?

for this, recall $m(t) = m(0)\,e^{kt}$ where

$k = \frac{1}{1590}\ln(1/2)$.

$$\Rightarrow 30 = 100\,e^{kt}$$

$$\Rightarrow e^{kt} = \frac{3}{10}$$

$$\Rightarrow \boxed{t = \frac{1}{k}\ln\left(\frac{3}{10}\right)}$$

$k = \frac{1}{1590}\ln(1/2)$.

≈ 2762 ms.

1:25–2:40 PM

Wednesday (1)

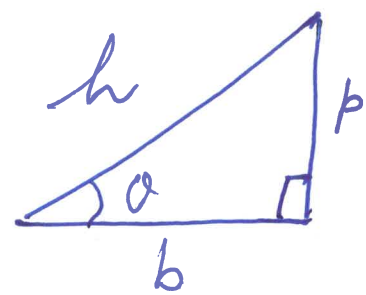
6th Mar 2024

§1–4 of chapter-3 exponential functions and logarithmic functions.

$$\frac{dy}{dt} = ky \longrightarrow \begin{cases} k>0 & \text{"growth" exponential} \\ k<0 & \text{"decay" exponential.} \end{cases}$$

Basic TRIGONOMETRY and Inverse TRIGONOMETRIC FUNCTIONS.

$\sin\theta = \frac{p}{h}$ $\quad \cos\theta = \frac{b}{h}$ $\quad \tan\theta = \frac{p}{b}$



$\downarrow$

$\csc\theta = \frac{h}{p}$; $\sec\theta = \frac{h}{b}$ & $\cot\theta = \frac{b}{p}$.

for example: $\sin\frac{\pi}{6} = \frac{1}{2}$; $\cos\frac{\pi}{3} = \frac{1}{2}$.

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$; $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$; $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

if: $\sin \boxed{y} = x \iff \boxed{\sin^{-1} x} = y$ ①

angle angle

Remember ②: $\sin^{-1} x \neq \frac{1}{\sin x}$

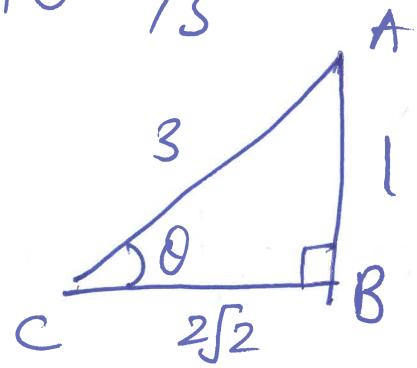
(Page 179 Ex1)

For example ① $\sin^{-1}(1/2) = \theta \Rightarrow \sin\theta = 1/2$, thus

$$\boxed{\theta = \pi/6.}$$

② find $\tan\left(\underbrace{\arcsin \frac{1}{3}}_{\sin^{-1} 1/3}\right) = \tan\left(\sin^{-1}(1/3)\right) = ?$

solution: take $\sin^{-1}(1/3) = \theta \Rightarrow \sin\theta = 1/3$. Lets draw a right ∠d ΔABC,



by Pythagorean theorem, we have

$$BC^2 = 3^2 - 1^2 = 9 - 1 = 8 \Rightarrow \boxed{BC = 2\sqrt{2}}$$

Now, $\tan\left(\sin^{-1} 1/3\right) = \tan\theta = \frac{1}{2\sqrt{2}}$. ////.

~~cos~~ $\cos\theta = \frac{2\sqrt{2}}{3}$.

In the same way, the inverse of "cos" can also be handled, as: ③

if $\boxed{\cos^{-1} x = y \iff \cos y = x.}$ — ②

<u>Remember</u> ① $-\pi/2 \le y \le \pi/2$
② $0 \le y \le \pi$.
} → and sin & cos maintain their 1-1 behaviour in these domain.

<u>Fact</u>: <u>1-1 / Injective function</u>

$f(x) = f(y) \Rightarrow x = y$ —— 1st def.

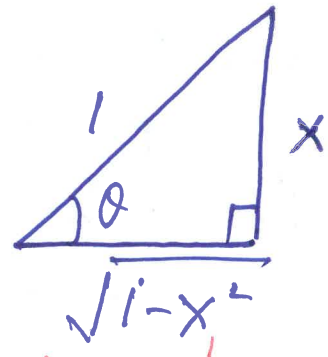
$f(x) \neq f(y) \Rightarrow x \neq y$ —— 2nd def.

<u>For example</u>: if $f(x) = x^2$, then $f(x) = x^2$ is not 1-1. (why)? $f(-1) = (-1)^2 = 1$
$f(1) = 1^2 = 1$ but $-1 \neq 1$.

If $f(x) = x+1$; suppose $f(x) = f(y)$ then
$\Rightarrow x + \cancel{1} = y + \cancel{1} \Rightarrow x = y$.

Exercise 7 @183 | claim $\cos(\sin^{-1}x) = \sqrt{1-x^2}$.

Proof: take $\sin^{-1}x = \theta \Rightarrow \sin\theta = x$



LHS: $\cos(\sin^{-1}x) = \cos\theta$.

RHS: $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta}$

$= \sqrt{\cos^2\theta}$

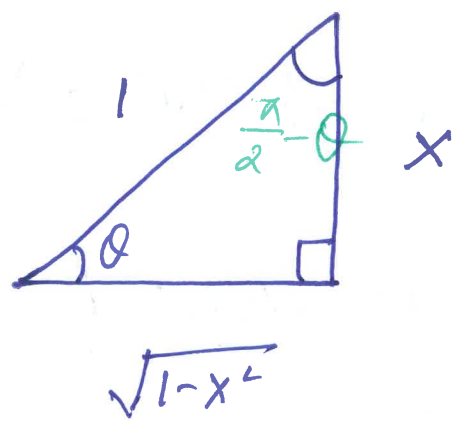
$= \cos\theta$.

Hence proved.

Exercise 12 a claim: $\sin^{-1}x + \cos^{-1}x = \pi/2$.

Proof: take $\boxed{\sin^{-1}x = \theta}$ ① $\Rightarrow \sin\theta = x$.

$\cos\left(\frac{\pi}{2} - \theta\right) = x$.



$\Rightarrow \boxed{\frac{\pi}{2} - \theta = \cos^{-1}x}$ — ②

Add ① & ②.

$\sin^{-1}x + \cos^{-1}x = \cancel{\theta} + \frac{\pi}{2} - \cancel{\theta} = \pi/2$.

① $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$. $\quad -1 < x < 1$

② $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$. $\quad -1 < x < 1$

③ $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

Chain rule

Problem 16@183: find $\frac{d}{dx}(\tan^{-1}(x^2)) = \frac{1}{1+(x^2)^2} \cdot (x^2)'$

$= \frac{2x}{1+x^4}$.

Problem 17@183 find $\frac{d}{dt}(\tan^{-1}t)^2 = 2(\tan^{-1}t) \cdot \frac{1}{1+t^2}$.

$$\boxed{(f(x))^2)' = 2f(x) \cdot f'(x)}$$

Problem 19 @ 183: $\frac{d}{dx}\left(\sin^{-1}(2x+1)\right)$

$$= \frac{1}{\sqrt{1-(2x+1)^2}} \cdot (2x+1)'$$

$$= \frac{2}{\sqrt{1-(2x+1)^2}}.$$

Recall:

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$$

Recall $(\tan^{-1} x)' = \frac{1}{1+x^2}$

Problem 20 @ 183: find $\frac{d}{dx}\left(\tan^{-1}\left(\underbrace{x-\sqrt{1+x^2}}_{w}\right)\right)$

$$= \frac{1}{1+w^2} \cdot \frac{dw}{dx}$$

chain rule

$$\frac{dw}{dx} = \frac{d}{dx}\left(x - \sqrt{1-x^2}\right)$$

$$= 1 - \left(\frac{-x}{\sqrt{1-x^2}}\right)$$

$$= 1 + \frac{x}{\sqrt{1-x^2}}.$$

$$\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$= \frac{d}{dx}\left((1-x^2)^{1/2}\right)$$

$$= \frac{1}{\cancel{2}}(1-x^2)^{1/2-1} \cdot (\cancel{-2}x)$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

Problem 25 @183 $y = \arctan(\cos\theta)$; y'

7

$$y = \tan^{-1}(\cos\theta)$$

$$\frac{dy}{d\theta} = \frac{1}{1+(\cos\theta)^2} \cdot (\cos\theta)'$$

$$= \frac{(\tan^{-1}\theta)'}{} = \frac{1}{1+\theta^2}$$

$$= \frac{-\sin\theta}{1+\cos^2\theta}$$

1:25 - 2:40PM

friday 8th March 202

§3.5 Inverse TRIGONOMETRIC FUNCTIONS

$\sin^{-1} x \leftrightarrow \csc^{-1} x$

$\cos^{-1} x \leftrightarrow \sec^{-1} x$

$\tan^{-1} x \leftrightarrow \cot^{-1} x$.

$\rightarrow \sin(\sin^{-1} x) = x$

$\rightarrow \cos(\cos^{-1} x) = x$

$\rightarrow \tan(\tan^{-1} x) = x$

(check!)

$\rightarrow \sin^{-1}(\sin x) = x$

$\rightarrow \cos^{-1}(\cos x) = x$

$\rightarrow \tan^{-1}(\tan x) = x$

(check!)

definitions.

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}.$$

$$(\tan^{-1} x)' = \frac{1}{1+x^2}.$$

Remember: (@ 183) $(\csc^{-1} x)' = \dfrac{-1}{x\sqrt{x^2-1}}$

$$(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$$

and $$(\cot^{-1} x)' = \frac{-1}{1+x^2}.$$

Problem 37 @ 184 $\lim_{x\to\infty} \tan^{-1}(e^x) = ?$

Remember: $\lim_{x\to\infty} \tan^{-1}(x) = \pi/2$. (1) and

$\lim_{x\to-\infty} \tan^{-1}(x) = -\pi/2$. (2)

$\because \tan(\pi/2) = +\infty$ and $\tan(-\pi/2) = -\infty$.

solution: observe that as $x\to\infty$, $e^x \to \infty$. $\therefore$

by (1), $\lim_{x\to\infty} \tan^{-1}(e^x) = \pi/2$.

Problem 35 #184: $\lim_{x \to 0^+} \tan^{-1}(\ln x) = ?$

Solution: observe that as $x \to 0^+$, $\ln x \to -\infty$.

Thus, 2 $\lim_{x \to 0^+} \tan^{-1}(\ln x) = -\pi/2$.

Problem 36 @184 $\lim_{x \to \infty} \cos^{-1}\left(\frac{1+x^2}{1+2x^2}\right)$.

$$= \lim_{x \to \infty} \cos^{-1}\left(\frac{1/x^2 + \cancel{x^2}/\cancel{x^2}}{1/x^2 + 2\cancel{x^2}/\cancel{x^2}}\right)$$

$$= \lim_{x \to \infty} \cos^{-1}\left(\frac{1/x^2 + 1}{1/x^2 + 2}\right)$$

$$= \cos^{-1}\left(\frac{0+1}{0+2}\right) = \cos^{-1}(1/2) = \pi/3.$$

Problem 28 #184: find y' where

imp

$$y = \tan^{-1}\left[\sqrt{\frac{1-x}{1+x}}\right].$$

Solution 1: Recall $y = \tan^{-1}\left[\underbrace{\boxed{\frac{1-x}{1+x}}}_{X}\right]$.

chain rule

$$y' = \frac{1}{1+X^2} \cdot \frac{d}{dx} X \quad \text{——— } (*)$$

we know $(\tan^{-1} x)' = \frac{1}{1+x^2}$.

we will determine $\frac{d}{dx} X$.

$$\frac{d}{dx} X = \frac{d}{dx}\left[\left(\frac{1-x}{1+x}\right)^{1/2}\right]$$

$$= \frac{1}{2} \cdot \left(\frac{1-x}{1+x}\right)^{1/2 - 1} \cdot \frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

quotient rule.

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2} \cdot \frac{(-1)(1+x) - (1)(1-x)}{(1+x)^2}$$

$$= \frac{1}{\cancel{2}}\left(\frac{1+x}{1-x}\right)^{1/2} \cdot \frac{\cancel{-1} \cancel{-x} - \cancel{1} \cancel{+x}}{(1+x)^2}.$$

$$= -\frac{(1+x)^{1/2}}{(1-x)^{1/2}} \frac{1}{(1+x)^2}.$$

Now, we will simplify ⊛,

$$y' = \underbrace{\frac{1}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2}}_{\frac{1}{1+\cancel{x}^2}} \cdot \underbrace{\left\{ -\frac{(1+x)^{1/2}}{(1-x)^{1/2}} \cdot \frac{1}{(1+x)^2} \right\}}_{\frac{d}{dx}\cancel{x}}$$

$$= \frac{1}{\underset{\frac{1+x}{1+x}}{1} + \frac{1-x}{1+x}} \left\{ -\frac{(1+x)^{1/2}}{(1-x)^{1/2}} \cdot \frac{1}{(1+x)^2} \right\}.$$

$$= \frac{(1+x)^1}{1+\cancel{x}+1-\cancel{x}} \left\{ \frac{-(1+x)^{1/2}}{(1-x)^{1/2}} \cdot \frac{1}{(1+x)^2} \right\}.$$

$$= \frac{-1}{2} \frac{1}{(1-x)^{1/2}} \cdot \frac{1}{(1+x)^{1/2}}$$

$$= \frac{-1}{2\sqrt{(1+x)(1-x)}} = \frac{-1}{2\sqrt{1-x^2}}.$$

solution 2nd way $\quad y = \tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$.

introduce $x = \cos 2\theta$

$\because \sin^2 2\theta + \cos^2 2\theta = 1$

$\sin^2 2\theta \quad x^2 = 1$

$\sin^2 2\theta = \sqrt{1-x^2}$

$$\Rightarrow \frac{dx}{d\theta} = -2(\sin 2\theta) \overset{?}{=} -2\sqrt{1-x^2}.$$

if $x = \cos 2\theta$, $\Rightarrow y = \tan^{-1}\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$

$$\Rightarrow y = \tan^{-1}\sqrt{\frac{\cancel{\cos^2\theta} + \sin^2\theta - \cancel{\cos^2\theta} + \sin^2\theta}{\cos^2\theta + \cancel{\sin^2\theta} + \cos^2\theta - \cancel{\sin^2\theta}}}$$

$1 = \cos^2\theta + \sin^2\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$.

$$= \tan^{-1}\left(\sqrt{\frac{\cancel{2}\sin\theta}{\cancel{2}\cos\theta}}\right)$$

$$= \tan^{-1}(\tan\theta)$$

$$y = \theta \Rightarrow \frac{dy}{dx} = \frac{d\theta}{dx} = \frac{-1}{2\sqrt{1-x^2}}.$$

1:25 – 2:40 PM

18th Mar 2024 (1)

Monday.

§3.7 L-Hospital rule :

Suppose, we want to evaluate

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

Additionally, $\left.\dfrac{f(x)}{g(x)}\right|_{x=a} = \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

If $\lim_{x \to a} \dfrac{f'(x)}{g'(x)}$ exist, then, → very imp!

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

Ex-1 @ 193 find $\lim_{x \to 1} \dfrac{\log x}{x-1}$.

solution: observe that $\left.\dfrac{\log x}{x-1}\right|_{x=1} = \dfrac{\log 1}{1-1}$

$= \dfrac{0}{0}$.

Apply L'Hospital rule to have

$$\lim_{x\to 1} \frac{\log x}{x-1} = \lim_{x\to 1} \frac{(\log x)'}{(x-1)'}$$

$$= \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x} = \frac{1}{1} = 1.$$

<u>Ex-2 @ 193</u> find $\lim_{x\to\infty} \frac{e^x}{x^2}$.

<u>solution</u> as $x \to \infty$, $e^x \longrightarrow \infty$ and also as $x \longrightarrow \infty$, $x^2 \longrightarrow \infty$. Therefore, we have

$$\left.\frac{e^x}{x^2}\right|_{x=\infty} = \frac{\infty}{\infty}.$$ so, apply the L-Hrule

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \frac{1}{2} \underbrace{\lim_{x\to\infty}\left[\frac{e^x}{x}\right]}_{\left(\frac{\infty}{\infty}\right)}.$$

$$= \frac{1}{2} \lim_{x\to\infty}\left[\frac{e^x}{1}\right] = \frac{\infty}{2} = \infty. \quad ///$$

<u>consider</u> $\lim_{x \to \infty} \frac{x^2}{e^x}$ (?)

<u>solution</u>: as $x \to \infty$, $\frac{x^2}{e^x} \to \frac{\infty}{\infty}$. Use L-H rule to have

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \underbrace{\lim_{x \to \infty} \frac{2x}{e^x}}_{\left(\frac{\infty}{\infty}\right)} \overset{\text{L-H rule}}{=} \lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty}.$$

$$= 0.$$

<u>Remember</u>: $\lim_{x \to \infty} x^n e^{-x} = 0$.

<u>Very imp. Ex-5 @ 194</u> $\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$.

<u>solution</u>: let's solve this by L-H rule.

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = \frac{-1}{0} \quad (\text{put } x = \pi^-)$$

$$= -\infty.$$

Here, $\lim_{x \to \pi^-} \frac{(\sin x)'}{(1-\cos x)'}$ does not exist.

However, a straightforward calculation yields

$$\lim_{x \to \pi^-} \frac{\sin x}{1-\cos x} = \frac{\sin \pi}{1-\cos \pi} = \frac{\sin \pi}{1-(-1)} = \frac{0}{2}$$
$$= 0.$$

<u>Ex-6 @ 195</u> evaluate $\lim_{x \to 0^+} x \log x$.

<u>solution</u>: Note that

$$\lim_{x \to 0^+} x \log x = \lim_{x \to 0^+} \frac{\log x}{(1/x)}. \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0^+} \frac{(\log x)'}{(1/x)'}$$

$$= \lim_{x \to 0^+} \frac{1/x}{(-1/x^2)} = \lim_{x \to 0^+} -\frac{x^2}{x}.$$

$$= \lim_{x \to 0} (-x) = 0.$$

Thus, $\boxed{\lim_{x \to 0^+} x \log x = 0.}$

<u>Remember</u> $\frac{0}{0}$, $\frac{\infty}{\infty}$ are forms of indeterminate limits.

Other forms of indeterminate limits involves powers.

$$\lim_{x \to a} [f(x)]^{g(x)}.$$

$$0^0$$
$$\infty^0$$
$$1^\infty.$$

<u>For example</u>: evaluate $\lim_{x \to 0^+} x^x$.

<u>solution</u>: observe that as $x \to 0^+$,

$x^x \to 0^0$ (the top indeterminate form).

consider $y = x^x$, take 'log'

$$\Rightarrow \log y = x \log x.$$

$$\Rightarrow \lim_{x \to 0^+} \log y = \lim_{x \to 0^+} x \log x$$

$$\Rightarrow \log y = 0$$

"DIY"

---

so far in chapter 3

1. Exponential fun
2. Logarithmic functions
3. Logarithmic differentiation
4. $y' = ky \rightarrow$ exponential decay or exponential growth
5. inverse trig functions.
6. Limits with indeterminate p...

Chapter-4 §1 (Absolute max/min)

Application of differentiation.

Note: Absolute max $\Rightarrow$ global max

" min $\Rightarrow$ global min.

Definition: Let D be a domain of f and say $c \in D$. Then

(belongs to)

1. if $\underbrace{f(c)}_{\text{absolute maximum @ } c} \geq f(x) \quad \forall\, x \in D$. then

2. if $\underbrace{f(c)}_{\text{absolute minimum @ } c} \leq f(x) \quad \forall\, x \in D$. then

CRITICAL NUMBER of f' A number 'c' is called as critical number of f', if either $f'(c) = 0$ or $f'(c)$ does not exist.

How to find absolute max/min [page 207]

Note f' as a function with domain $[a, b]$.

1. find 'c' such that $f'(c) = 0$ or $f'(c)$ does not exist. (find out critical numbers).

2. find $f(a)$ and $f(b)$.

3. Largest value in step 1 and step 2 will be Absolute max and

Smallest value in step 1 and step 2 will be Absolute min.

EX-6 @ 207 find absolute max / absolute min $f(x) = x^3 - 3x^2 + 1$ $\quad -\frac{1}{2} \le x \le 4$.

solution : step ① (critical numbers)

$f'(x) = 3x^2 - 6x$, put $f'(x) = 0$

$\Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x-2) = 0 \Rightarrow x = 0$ and $x = 2$

step ② : $f(0) = 1$ $\quad f(2) = -3$

$f(1/2) = 1/8$ $\quad f(4) = 17$.

Absolute max.

Absolute min.

question 33 @ 209 find critical numbers

$f(\theta) = 2\cos\theta + \sin^2\theta$.

<u>solution</u>: $f'(\theta) = -2\sin\theta + \sin 2\theta$.

put $f'(\theta) = 0$.

$\Rightarrow -2\sin\theta + \sin 2\theta = 0$.

<u>Recall</u> $\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow 2\sin\theta - \sin 2\theta = 0$.

$\Rightarrow 2\sin\theta - 2\sin\theta\cos\theta = 0$

$\Rightarrow 2\sin\theta(1-\cos\theta) = 0$

$\Rightarrow \sin\theta = 0$

$1 - \cos\theta = 0$

$\cos\theta = 1$

$\boxed{\theta_s = n\pi}$

$\boxed{\theta_c = 2n\pi}$

$n \in \mathbb{Z} \cup \{0\}$

natural numbers.

0 is also included

$n \in \mathbb{Z} \cup \{0\}$.

0 is also included.

Because every θ_c can be written in terms of θ_s, therefore critical numbers of f are $\theta_s = n\pi$.

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Wednesday
20th Mar 2024

@ 210

§4.2 Mean Value theorems

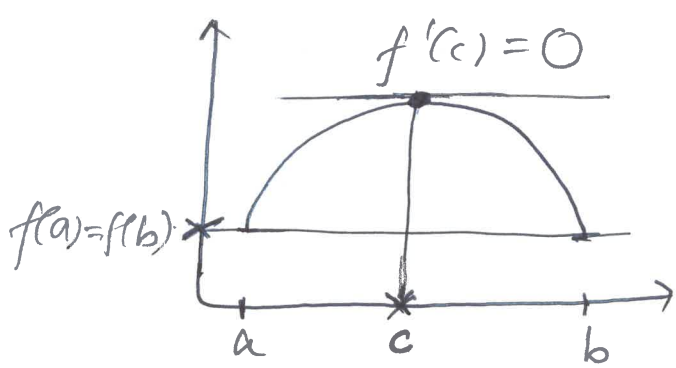
↳ IVT

→ Rolle's theorem → Lagrange - Mean value theorem.

Rolle's theorem: Let f be a function which satisfies following three conditions over $[a,b]$.

1. $f \in \mathcal{C}[a,b]$. 2. $f \in \mathcal{D}(a,b)$.
3. $f(a) = f(b)$ (but $a \neq b$). Then, $\exists$ a 'c' $\in (a,b)$ ni (such that)

$$f'(c) = 0.$$



Problem 1 @215 | verify Rolle's theorem
$f(x) = 5 - 12x + 3x^2$ over $[1,3]$.

solution: check the validity of Rolle's theorem

1. f is a polynomial (deg 2) and thus, $f \in C[1,3]$.

2. f is a polynomial (deg 2), so, it is $D(1,3)$

3. $f(1) = 5 - 12 + 3 = -4$ and
$f(3) = 5 - 36 + 3 \cdot 3^2 = -4$
$\Big\}\ f(1) = f(3)$.

conditions of Rolle's theorem are satisfied.
Then, by Rolle's theorem, $\exists$ 'a' 'c' $\in (1,3)$,
$\therefore$ is $f'(c) = 0$.

$$\begin{cases} f(x) = 5 - 12x + 3x^2 \\ f'(x) = -12 + 6x \end{cases}$$

$\Rightarrow -12 + 6c = 0$

$\Rightarrow \boxed{c = 2}$

EX-2 @ 211 | Prove that $\underbrace{x^3+x-1}_{f(x)}=0$ has only one root (real).

Proof: Show 2 things

① $\exists$ a possibility of having a root of $x^3+x-1=0$.

② only one real root.

① we will use IVT. Observe that

$f(0)=-1$ & $f(1)=1+1-1=1$

Therefore, by IVT, $\exists$ 'c' s.t. $f(c)=0$ where $c\in(0,1)$.

② Assume that we have two real roots of f, that is $f(a)=0$, and $f(b)=0$ where 'a' and 'b' are the roots. obviously

$f(a)=f(b)$.

Then by Rolle's theorem, we have "a" 'c' $\in (0,1)$ is $f'(c) = 0$

$$\Rightarrow \boxed{3c^2 + 1 = 0}$$

≥ 1

$$\begin{cases} f = x^3 + x - 1 \\ f' = 3x^2 + 1 \end{cases}$$

Since $3c^2 + 1 \neq 0$, thus by contradiction, f can't have two real roots.

Lagrange Mean value theorem

Let f be a function satisfying following two conditions

(1) $f \in \mathcal{C}[a,b]$ (2) $f \in \mathcal{D}(a,b)$, then

$\exists$ a 'c' $\in (a,b)$ is (such that)

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$ or

$$f(b) - f(a) = f'(c)(b - a)$$

Problem #9 @ 215 verify ~~Rolle's theorem~~ LMVT

$f(x) = 2x^2 - 3x + 1$ over $[0,2]$

Solution: observe that f is a polynomial (deg 2), therefore it is $C[0,2]$ and $D(0,2)$. Now, by LMVT,

$$f'(c) = \frac{f(2) - f(0)}{2-0} = \frac{f(2) - f(0)}{2}$$

$$\Rightarrow (4c-3) = \frac{(8 - 6 \cancel{+1}) - (\cancel{1})}{2} = 1$$

$$\Rightarrow c = 1 \in (0,2).$$

EX-5 @214| Given $f(0)=-3$, $f'(x)\le 5$ ⑥
(and f is differentiable).
How large $f(2)$ can be?

Solution: Because f is differentiable, it is also continuous over $(0,2)$. Then by LMVT

$$f'(x) = \frac{f(2)-f(0)}{2-0} = \frac{f(2)-f(0)}{2}.$$

$$\Rightarrow \boxed{2f'(x) + f(0) = f(2)}$$

$$\boxed{f(2) = -3 + 2f'(x)} \quad \because f(0)=-3.$$

Since $f'(x)\le 5$

$$\Rightarrow \underbrace{-3+2f'(x)}_{f(2)} \le -3+10 \le 7$$

Thus $\boxed{f(2)\le 7}$.

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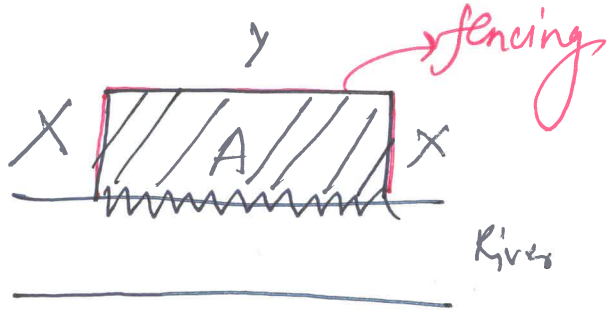
29th Mar 2024 (1)

Friday

§ 4.5 Optimization problems

↳ $f' \to$ objective function (max or min)

EX-1 @232 According to the question,



$2x + y = 2400.$

$\Rightarrow \boxed{y = 2400 - 2x}$ —— (1)

Area of rectangle $A = xy$.

$\Rightarrow A = x(2400 - 2x) = A(x)$ $\quad 2400x - 2x^2$

Now, we find $A'(x)$ as follows.

$A'(x) = 2400 - 4x.$ Then, find critical numbers of $A(x)$, by making $A'(x) = 0$.

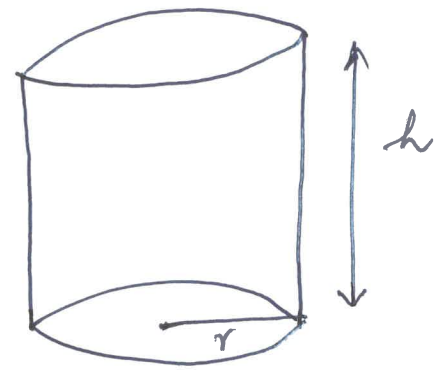
$\Rightarrow 2400 - 4x = 0$

$\Rightarrow \boxed{x = 600}$

whether the objective function is optimized or not.

(4) Draw conclusion.

"can"



Ex-2 @ 233 Volume $V = \pi r^2 h$

and according to the problem

$$\pi r^2 h = 1000 \text{ cm}^3.$$

$$\Rightarrow \boxed{h = \frac{1000}{\pi r^2}} \quad \text{——— (1)}$$

Aim is to find dimensions of can so that manufacturing cost is minimized.

(directly proportional) $\propto$ surface area of can. (S)

$$\boxed{S = 2\pi r^2 + 2\pi r h.}$$

step (1)

Then, $S(r) = 2\pi r^2 + 2\cancel{\pi r} \cdot \frac{1000}{\cancel{\pi} r^{\cancel{2}}}$

$$S(r) = 2\pi r^2 + \frac{2000}{r}$$

$\left(\frac{2000}{r}\right)' = (2000\, r^{-1})'$

$= -2000\, r^{-2}$

if $r = \left(\frac{500}{\pi}\right)^{1/3}$, $h = \frac{1000}{\pi \cdot \left(\frac{500}{\pi}\right)^{2/3}}$

$((1000)^3)^{1/3} = 1000$

$$h = \left(\frac{\overset{2}{\cancel{1000}} \times \overset{2}{\cancel{1000}} \times \overbrace{1000}^{2 \times 500}}{\cancel{500} \times \cancel{500}}\right)^{1/3} \cdot \frac{1}{\pi^{1-2/3}} \rightarrow \pi^{1/3}$$

$$= 2 \cdot \left(\frac{500}{\pi}\right)^{1/3} = 2r^{*}.$$

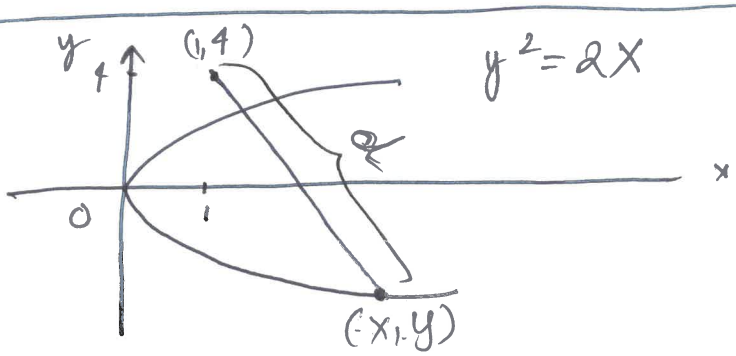
Therefore, the dimension for which the manufacturing cost gets minimized are

$$r = \left(\frac{500}{\pi}\right)^{1/3} \quad \text{and}$$

step ④

$$h = 2\left(\frac{500}{\pi}\right)^{1/3}.$$

Ex-3 @ 234



Let $D'(y) = 0 \Rightarrow y^3 = 8 \Rightarrow y = 2$ Step 2 3

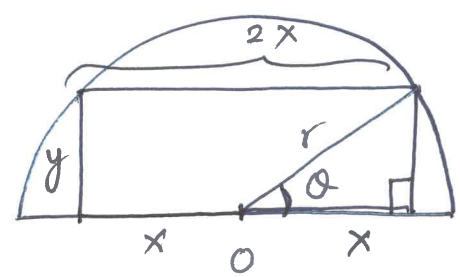
$D''(y) = 3y^2 \Rightarrow D''(y)\Big|_{y=2} = > 0$ Step 3

Thus, $D(y)$ gets minimized @ $y = 2$.

Lastly, if $y = 2$, $x = \frac{y^2}{2} = \frac{2^2}{2} = \frac{4}{2} = 2$

Thus, the point on parabola $y^2 = 2x$ which is closest to $(1, 4)$ is $(2, 2)$. Step 4

EX-5 @ 236



$A(x, y) = 2xy$.

Use polar substitution of x and y,

$x = r\cos\theta$ and $y = r\sin\theta$.

↳ (so that $x^2 + y^2 = r^2$).

1:25-2:40 PM

Monday 1 Apr 2024

①

Homework - A discussion

Remember Q#3 $\tan(a-b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$

plug $a = \pi/4$ and $b = \theta$,

$$\boxed{\tan(\pi/4 - \theta) = \frac{1 - \tan\theta}{1 + \tan\theta}}$$

Integrals (introductory level).

Recall: $(x^n)' = nx^{n-1}$

$\{ (x^3)' = 3x^2$

$$(x^3)' = 3x^2$$

$$\Rightarrow \boxed{\frac{1}{3}(x^3)' = x^2}$$

$$\boxed{\frac{1}{7}x^7}' = x^6$$

4. $\int \frac{1}{x}\,dx = \ln x + c$

5. $\int \sec^2 x\,dx = \tan x + c.$

6. $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c.$

7. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c.$

8. $\int dx(\tan x) = \int \tan x\,dx = -\log\cos x + c$

<u>observe</u> :- $(\log\cos x)' = \frac{\sin x}{\cos x} = \tan x$

9. $\int \cot x\,dx = \log\sin x + c.$

<u>observe</u> $(\log\sin x)' = \frac{\cos x}{\sin x} = \cot x$

Evaluation theorem (@282). If $\int f = F$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

no need of constant of integration in definite integrals

ex-1@283

Find $\int_1^3 e^x\,dx = e^x\Big|_1^3 = e^3 - e$

#22@289

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx$$

Recall

$$\int \frac{dx}{1+x^2} = \tan^{-1}x + c$$

$$= 8\tan^{-1}x\Big|_{1/\sqrt{3}}^{\sqrt{3}} = 8\left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right].$$

$$= 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

(check)

#1@289 $I_1 = \int_{-2}^{3} (x^2 - 3)\,dx.$

antiderivative

$\int x^2 - 3 = \frac{x^3}{3} - 3x + c$

#47 @ 289 $I_1 = \int \frac{\sin x}{1-\sin^2 x}\,dx$

Recall: $1-\sin^2 x = \cos^2 x$ then $I_1 = \int \frac{\sin x}{\cos^2 x}\,dx$

$$= \int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}\,dx = \int \tan x \cdot \sec x\,dx$$

$$= \sec x + C.$$

#25 @ 289 $I_1 = \int_{-1}^{1} e^{u+1}\,du = \int_{-1}^{1} e^{u} e\,du$

$$= e\int_{-1}^{1} e^{u}\,du$$

$$= e\, e^{u}\Big|_{-1}^{1} = e\left[e - e^{-1}\right]$$

$$= e^2 - 1.$$

#21 @ 289 $I_1 = \int_{0}^{\sqrt{3}/2} \frac{dr}{\sqrt{1-r^2}} = \sin^{-1}x\Big|_{0}^{\sqrt{3}/2} = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}0$

$$= \frac{\pi}{3} - 0 = \frac{\pi}{3}.$$

1:25 – 2:40 PM

3rd April 2024 (1)

Wednesday.

Ex-6@286 evaluate

$$I = \int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$$

integrand.

$$\int t^n\,dt = \frac{t^{n+1}}{n+1} + c$$

solution: $I = \int_1^9 \left(2 + \sqrt{t} - \frac{1}{t^2}\right) dt$

$\sqrt{t} \to t^{1/2}$, $\frac{1}{t^2} \to t^{-2}$

$$= 2t\Big|_1^9 + \frac{t^{1/2+1}}{1/2+1}\Big|_1^9 - \frac{t^{-2+1}}{-2+1}\Big|_1^9$$

$$= 2(9-1) + \frac{2}{3}t^{3/2}\Big|_1^9 + \frac{1}{t}\Big|_1^9$$

$$= 16 + \frac{2}{3}\left[9^{3/2} - 1^{3/2}\right] + \left(\frac{1}{9} - \frac{1}{1}\right)$$

$$= 16 + \frac{2\cdot 26}{3} - \frac{8}{9} = 16 + \frac{52}{3} - \frac{8}{9} = 16 + \frac{148}{9}$$

$$= \frac{292}{9} = 32\frac{4}{9}$$

always remember following properties of ③ definite integrals

$$\int_a^b f\,dx = \int_a^b f(a+b-x)\,dx$$

$a = 0$ ↓

$$\int_0^b f\,dx = \int_0^b f(b-x)\,dx.$$

Ex-9 @ 306 $\displaystyle\int_{-1}^{1} \boxed{\frac{\tan x}{1+x^2+x^4}}\,dx = ?$ → $f(x)$

$\tan(-x) = -\tan x.$

Solution: observe that $f(-x) = \dfrac{-\tan x}{1+x^2+x^4}$

therefore by property ⑤ $\displaystyle\int_{-1}^{1} \frac{\tan x\,dx}{1+x^2+x^4} = 0.$

#46 @ 307 $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6}\,dx = 0$ by Property 5

evaluate
<u>ex—5@303</u> $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

observe if $u = \cos x$, then $du = +\sin x \, dx$

then $\int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u} = -\int \frac{du}{u}$

$$= -\log|u| + c$$

$$= \log \frac{1}{|u|} + c$$

$$= \log \frac{1}{|\cos x|} + c = \log|\sec x| + c.$$

<u>#47 @307</u> evaluate $\int_0^1 \frac{e^z + 1}{e^z + z} dz.$

observe $u = e^z + z \Rightarrow du = (e^z + 1) dz$

@ $z = 0$, $u = e^0 + 0 = 1$ and

@ $z = 1$, $u = e^1 + 1 = e + 1.$

then,

e. take $u = e^x \Rightarrow du = e^x dx$ then

$$\int \sqrt{1+e^x}\, e^x dx = \underbrace{\int \sqrt{1+u}\, du}$$

$\hookrightarrow t = 1+u$

$dt = du$

this is computationally expensive

$$= \int \sqrt{t}\, dt$$

<u>#52 @ 307</u> $I = \int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

<u>solution</u>: take $u = \sin^{-1} x \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$

also, @ $x = 0$, $u = \sin^{-1} 0 = 0$ and

@ $x = 1/2$, $u = \sin^{-1} 1/2 = \pi/6$. Then

$$I = \int_0^{\pi/6} u\, du = \frac{u^2}{2}\Big|_0^{\pi/6} = \frac{1}{2}\frac{\pi^2}{36} = \frac{\pi^2}{72}.$$

1:25 PM - 2:40 PM Friday 5th April 2024 (1)

Fundamental theorem of calculus

observe that $\boxed{\int f\,dx = \text{function}}$, for example

$\int \sin x\,dx = -\cos x + C$, on the other hand

$\boxed{\int_a^b f\,dx = \text{number}}$ for example $\int_0^{2\pi} \cos x\,dx = 0$

now, what about $\int_a^x f(t)\,dt$ ~~A~~ this?
$(a < \infty)$

The answer to this question is given by the fundamental theorem of calculus. (page 295)

say $f \in C[a,b]$,

① if $g(x) = \int_a^x f\,dt$ then $g'(x) = f(x)$.

② if $F' = f$ then $\int_a^b f\,dx = F(b) - F(a)$.

Ex-5 @ 295. find $\frac{d}{dx}\int_1^{x^4} \sec t\, dt$.

solution: say $g(x) = \int_1^{x^4} \sec t\, dt$.

observe

$$g'(x) = \frac{d}{dx} g(x) = \frac{d}{dx}\int_1^{x^4} \sec t\, dt.$$

by the application of fundamental theorem of Calculus.

chain rule.

$$g'(x) = (\sec x^4)\, \frac{d}{dx} x^4.$$

$$= (\sec x^4) \cdot 4x^3 \quad ////.$$

#5@298 $g(x) = \int_1^{x} \frac{dt}{t^3+1} = \int_1^{x} \underbrace{\frac{1}{t^3+1}}_{f(t)} \cdot dt$

Again by fundamental theorem of Calculus

$$g'(x) = \frac{1}{x^3+1} \quad ////.$$

#6 @ 298 $g(x) = \int_3^x e^{t^2-t}\,dt$ here $f(t) =$

$e^{t^2}-t$ and hence, by fundamental theorem of Calculus. $g'(x) = e^{x^2-x}$ $(= f(x).)$.

compare $\int_a^x f\,d\theta$

#8 @ 289 $F(x) = \int_x^{10} \tan\theta\,d\theta$

apply the property of definite integral

$$f(x) = \int_{10}^{x} (-\tan\theta)\,d\theta = -\int_{10}^{x} \tan\theta\,d\theta$$

$$\boxed{\text{Recall } \int_a^b f = -\int_b^a f}$$

so by the fundamental theorem of Calculus.

$f'(x) = -\tan x$. ////.

#12 @ 289 find y' if $y = \int_{\sin x}^{1} \sqrt{1+t^2}\, dt$

Solution:

chain rule

$$y' = -\sqrt{1+\sin^2 x} \cdot \frac{d}{dx}(\sin x)$$

$$= -\sqrt{1+\sin^2 x} \cdot \cos x.$$

#14 @ 289 $y(x) = \int_{\sin x}^{\cos x} (1+v^2)^{10}\, dv.$ — (1)

observe.

$$y(-x) = \int_{\sin(-x)}^{\cos(-x)} (1+v^2)^{10}\, dv = \int_{-\sin x}^{\cos x} (1+v^2)^{10}\, dv$$

$$\Rightarrow -y(-x) = \int_{\cos x}^{-\sin x} (1+v^2)^{10}\, dv \quad \text{— (2)}$$

Add 1 and 2

$$y(x) + (-y(-x)) = \int_{\sin x}^{\cos x} (1+v^2)^{10}\,dv + \int_{\cos x}^{-\sin x} (1+v^2)^{10}\,dv$$

Recall $\int_a^c f + \int_c^b f = \int_a^b f$

$$\Rightarrow y(x) - y(-x) = \int_{\sin x}^{-\sin x} (1+v^2)^{10}\,dv.$$

$$\Rightarrow \underbrace{y(-x) - y(x)}_{:\,Y(x)} = \int_{-\sin x}^{\sin x} (1+v^2)^{10}\,dv$$

consider

$$Y(-x) = y(x) - y(-x)$$

$$= -(y(-x) - y(x))$$

$$= -Y(x)$$, hence $Y(x)$ is an odd function.

⑥

$$y'(x) = (1+\cos^2 x)^{10} \cdot \frac{d}{dx}\cos x - (1+\sin^2 x)^{10} \cdot \frac{d}{dx}\sin x$$

$$= (1+\cos^2 x)^{10}(-\sin x) - (1+\sin^2 x)^{10} \cdot \cos x.$$

////.

$$y(x) = \int_{\sin x}^{\cos x} (1+v^2)^{10}\, dv$$

1:25 – 2:40 PM

Wednesday ①

10th April 2024

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx \quad \text{—— ①}$$

Recall : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\, dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} \quad \text{—— ②}$$

① + ②

$$I + I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx$$

$$2I = \int_0^{\pi/2} dx = x \Big|_0^{\pi/2} = \pi/2$$

$$I = \frac{\pi}{2}\left(\frac{1}{2}\right) \Rightarrow \boxed{I = \pi/4}$$

9:05-10:20 AM

Monday

15th April 2024

Revise

1. $\displaystyle\int_1^e \frac{x^2+x+1}{x}\,dx.$

sol: $\displaystyle\int_1^e \left(x+1+\frac{1}{x}\right)dx = \left(\frac{x^2}{2}+x+\ln x\right)\Big|_1^e$

$= \text{D.I.Y.}$

2. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2\sin x}{1+x^6}\,dx$

sol: $f(x) = \dfrac{x^2\sin x}{1+x^6}$ then $\underbrace{f(-x) = -f(x)}_{\text{odd function.}}$

$\therefore \displaystyle\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0.$

8. $I = \int_0^1 \frac{e^z+1}{e^z+z}\,dz$

take $u = e^z + z \Rightarrow du = (e^z+1)\cdot dz$.

change the limits; @ $z=0$, $u=1$
@ $z=1$, $u = e+1$.

then $I = \int_1^{e+1} \frac{du}{u}$ D. I. Y.

4. if $y(x) = \int_{\sin x}^{\cos x} (1+v^2)^{10}\,dv$, (i) find $y'(x)$. ?

sol: $y' = (1+\cos^2 x)^{10}(-\sin x) - (1+\sin^2 x)^{10}\cdot\cos x$

by FTC.

(ii) $Y(x) = y(-x) - y(x)$ then show that Y is an odd function.

D. I. Y.