XVII ALGEBRA 9:05–10:20 19th Jan 2024

FRIDAY 1

1. (a) $(-3)^4 = (-3)(-3)(-3)(-3)$
$= +81$

$(-a)^n = -a^n$ if $n =$ odd

$(-a)^n = a^n$ $n =$ even

$(|a| < \infty)$.

(f.) $16^{-3/4} = \frac{1}{16^{3/4}} = \frac{1}{8}$.

$a^{-n} = \frac{1}{a^n}$

$\boxed{16} = \boxed{2}^4 \Rightarrow 16^{1/4} = (2^4)^{1/4} = 2^1 = 2$

↓ cube

$\Rightarrow 16^{3/4} \equiv 2^3 = 8$

5. RATIONAL EXPRESSIONS

(d.) $\dfrac{\frac{y}{x} - \frac{x}{y}}{\frac{1}{y} - \frac{1}{x}}$

$\left(\frac{y}{x} - \frac{x}{y}\right)\left(\frac{1}{y} - \frac{1}{x}\right)$

$\underbrace{\left(\frac{1}{y} - \frac{1}{x}\right) \times \left(\frac{y}{1} - \frac{x}{1}\right)}$

$\Rightarrow \left(\frac{1}{y} - \frac{1}{x}\right)(y - x)$

$= \frac{y-x}{y} - \frac{y-x}{x}$

2

Recall $\dfrac{N\left[\dfrac{y}{x} - \dfrac{x}{y}\right]}{D\left[\dfrac{1}{y} - \dfrac{1}{x}\right]}$

$y = 2$
$x = 4$

$$\left\{\frac{\cancel{\frac{2}{4}} - \frac{4}{2}}{\frac{1}{\cancel{2}} - \frac{1}{4}}\right\}$$

↓ individual cancellations.

N → numerator

$$\frac{y}{x} - \frac{x}{y} = \frac{y^2 - x^2}{xy}$$

D → denominator

$$\frac{1}{y} - \frac{1}{x} = \frac{x - y}{xy}$$

$a^2 - b^2 = (a+b)(a-b)$

$$\Rightarrow \frac{N}{D} = \frac{\frac{y^2 - x^2}{xy}}{\left(\frac{x-y}{xy}\right)} = \frac{y^2 - x^2}{\cancel{xy}} \cdot \frac{\cancel{xy}}{x-y} = \frac{y^2 - x^2}{x - y}$$

$$= \frac{(y-x)(x+y)}{x-y}$$

$$\left\{\frac{y-x}{x-y} = \frac{-\cancel{(x-y)}}{\cancel{x-y}} = -1\right\}$$

$$= (-1)(x+y). \quad ////$$

6 b RATIONALIZATION

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

$$\frac{\sqrt{4+h} - 2}{h} = y \text{ (say)}$$

$$y = \left(\frac{\sqrt{4+h} - 2}{h}\right) \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$$

conjugation $\equiv \sqrt{4+h} + 2$

$y^2 = [\quad\quad]$

$$y = \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \longrightarrow \text{conjugate}$$

$$(a+b)(a-b) = a^2 - b^2$$

$$= \frac{\cancel{(4+h)} - \cancel{4}}{\cancel{h}\,\sqrt{4+h} + 2}$$

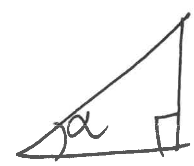
$$= \frac{1}{\sqrt{4+h} + 2} \checkmark$$

$a = \sqrt{4+h} \qquad b = 2$

$$a^2 - b^2 = (\sqrt{4+h})^2 - (2)^2$$
$$= ((4+h)^{1/2})^2 - 4$$
$$= 4+h - 4$$

(D) TRIG (7.) (6.) claim

$$\frac{2\tan X}{1+\tan^2 X} = \sin 2X$$



(1) $\sin^2\alpha + \cos^2\alpha = 1$

(2) $\tan^2\alpha + 1 = \sec^2\alpha$

(3) $1 + \cot^2\alpha = \operatorname{cosec}^2\alpha$

LHS: $$\frac{2\tan X}{1+\tan^2 X}$$

$$= \frac{2\tan X}{\sec^2 X}$$

$$= \frac{\dfrac{2\sin X}{\cancel{\cos X}}}{\dfrac{1}{\cancel{\cos X}\cdot\cos X}} \longrightarrow \frac{2\sin X}{\cos X} \cdot \frac{1}{\dfrac{1}{\cos X \cdot \left[\cos X\right]}}$$

$= 2\sin X \cdot \cos X$

$= \sin 2X$

$= \text{RHS}$ ////

$= 2\sin X \cdot \cos X$

$= \sin 2X$

$= \text{RHS}$ ////

XX | FUNCTIONS 9:05–10:20 AM 22nd Jan 2024

Mon (1)

$f(x) = x$ ✓ $g(x) = 2.$ ✓

(1) $f \circ g = f(g(x)) = f(2) = 2$

(2) $g \circ f = g(f(x)) = g(x) = 2$

$f \circ g = g \circ f$

---

book problems:

(7) $f(x) = x^2 + 2x - 1$ $g(x) = 2x - 3$

(a) $f \circ g \rightarrow\ ? = f(g(x)) = f(2x-3)$

$= (2x-3)^2 + 2(2x-3) - 1$ ////

$(2x-3)^2 = 4x^2 - 2 \cdot 2x \cdot 3 + 9 \leftarrow (a-b)^2 = a^2 - 2ab + b^2$

$= 4x^2 - 12x + 9 + 4x - 6 - 1 = 4x^2 - 8x - 6$

(b) $g \circ f = g(f(x)) = g(x^2 + 2x - 1)$

$= 2(x^2 + 2x - 1) - 3$

$= 2x^2 + 4x - 2 - 3$

$= 2x^2 + 4x - 5$

actually : $2\sin x\cos x = \sin x$

$\Rightarrow \underbrace{2\sin x\cos x - \sin x = 0}$ $\Rightarrow \boxed{\sin x}\boxed{(2\cos x - 1)} = 0$

$\boxed{}\cdot\boxed{} = 0$

either $\boxed{} = 0$

or $\boxed{} = 0$

$\sin x = 0$ $\underline{2\cos x - 1 = 0}$

$x = 0, 2\pi$ $x = \pi/3, 5\pi/3$

Fact

if $\sin x = 0$ $x = 0\pi, \pi, 2\pi, 3\pi \ldots (n\pi)$ $n \in \mathbb{Z}$.

---

(1) /XVII

LHS $\dfrac{y/x}{a/x - b/x} = \dfrac{1}{x\left[\frac{a}{x} - \frac{b}{x}\right]}$

$= \dfrac{1}{a-b}$

$=$ RHS ✓

TRUE.

XX | (2) $f(x) = x^3$ $\underbrace{\left[\dfrac{f(h+2) - f(2)}{h}\right]}$ $= ?$

→ ?

$f(x) = x^3$

$f(2) = 2^3 = 8$

- $f(h+2)$ ✓
- $f(2)$ ✓
- $\dfrac{}{h}$ ..

$f(x) = x^3$ (given)

$(a+b)^3$
$= a^3 + b^3 + 3ab(a+b)$

4

$\Rightarrow \quad x \longrightarrow h+2$

$$f(h+2) = (h+2)^3 = h^3 + 8 + 3 \cdot h \cdot 2(h+2)$$

$$= h^3 + 8 + 6h(h+2)$$

$$\boxed{f(h+2) = h^3 + 8 + 6h^2 + 12h}$$

$$f(h+2) - f(2) = f(h+2) - 8$$

$$\frac{f(h+2) - f(2)}{h} = \frac{h^3 + 6h^2 + 12h}{h}$$

$$= h^2 + 6h + 12 \quad ////.$$

9:05–10:20 AM

24th Jan 2024

wednesday (1)

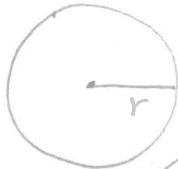
chapter – 1

FUNCTIONS & LIMITS

irrational ← $\pi = 3.14\ldots$

$= 22/7$ → rational

for example: (1)



$r > 0$

$C(r) = 2\pi r$ → circumference

$r = 2$

$C = 2\pi \cdot 2 = 4\pi$

$r = 1, C = 2\pi \cdot 1 = 2\pi$

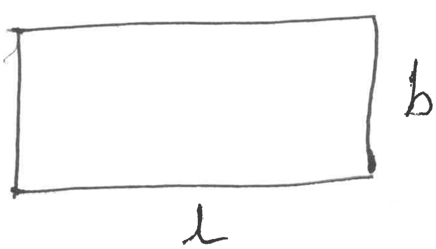
$\underbrace{C(r)}_{\text{dependent variable}} = 2\pi r$ function on 'r'

independent variable

Also, in $C(r) = 2\pi r$ → only 1 presence of 'r' (independent variable)

1 variable dependent function.

(2)



$P(l,b) = 2(l + b)$

l, b → independent variable.

→ Domain of $f(X)$ → usual notation for a function 'f' whose independent variable is x.

↓

all those values of x for which f ~~is defi~~ makes sense. (avoid $\sqrt{-}$, or $\frac{}{0}$)

→ say $f(x) = \frac{1}{x}$ imagine if $x = 1$ ②

$f(1) = \frac{1}{1} = 1$ ✓

if $x = -1$ $f(-1) = \frac{1}{-1} = -1$

however, if $x = 0$

$f(0) = \frac{1}{0} = \text{DNE}$

Domain of $f(x) = \frac{1}{x} \equiv$ all Real no. but not $x = 0$.

$= \mathbb{R} \setminus \{0\}$.

→ $f(x) = \frac{1}{x-1}$ $\notin \mathbb{R}$ $D(f) = \mathbb{R} \setminus \{1\}$

→ $f(x) = \sqrt{x-1}$ $\xrightarrow{\text{we want}}$ $x - 1 \geqslant 0$

$\Rightarrow x \geqslant 1$

$D(f) = \underbrace{\text{all Real no. that are } \geqslant 1}$

$= [1, \infty)$

$\mapsto f(x) = \dfrac{1}{\sqrt{x-4}}$ $\begin{cases} \sqrt{\ } \\ \dfrac{\square}{\square} \end{cases}$

① Observation : $\sqrt{x-4} \Rightarrow x-4 \geq 0 \Rightarrow x \geq 4$

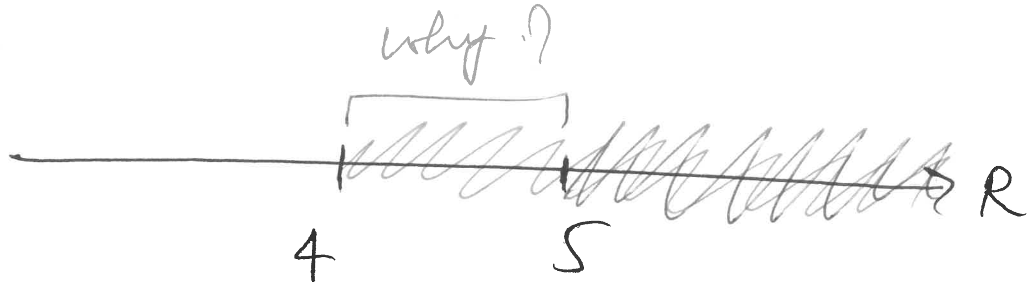
② observation $\dfrac{1}{\sqrt{x-4}}$; $\sqrt{x-4} = 0$ (check)

$\Rightarrow x-4=0$

$\Rightarrow x = 4$

this implies that $f(4) = \infty$ / DNE.

Therefore, the domain of f is all Real no. that are greater than 4, i.e. $(4, \infty)$



Symmetric nature of function

① if $f(x) = f(-x) \rightarrow$ EVEN FUNCTION

② if $f(x) = -f(-x) \rightarrow$ ODD FUNCTION.

example: $f(x) = 2$ (check its oddeness or evenness).

if $f(-x) = 2 = f(x)$ hence its even.

In particular, all functions defined $f(x) = c$ for some ~~som~~ constant c are even function.

example: $f(x) = 2x$ ✓ plug $-x$ in $f(x)$

$f(-x) = 2(-x) = \underbrace{-2x} = -f(x)$; its an odd.

example: $f(x) = \cos x \longrightarrow$ even function

$f(x) = \sin x \longrightarrow$ odd function.

Increasing and Decreasing function (page 7 / 2nd edition)

say interval $I = [a, b]$ (example $[0, 2]$, $(-4, 3)$)

choose x_1 and $x_2 \in I$

$\in$: belongs to

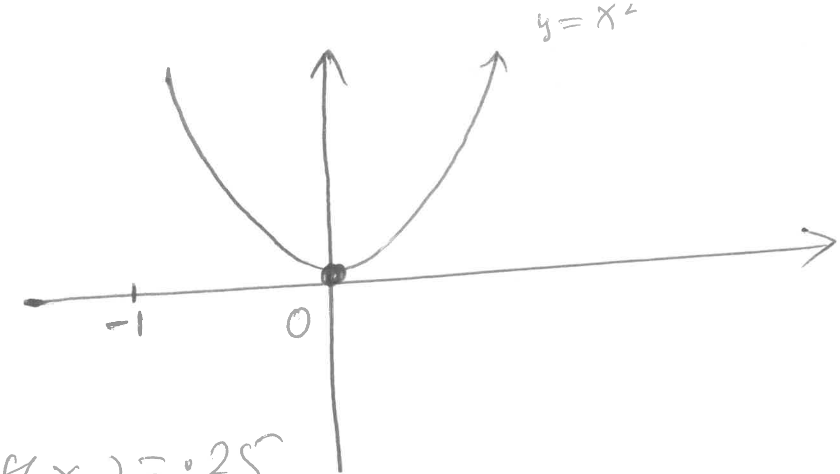
ⓐ if ~~f(x)~~ $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$

then f is increasing function

ⓑ if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$ then

f is ~~an~~ decreasing function

$y = x^2$



from $[-1, 0]$

$x_1 \to -.5 \Rightarrow f(x_1) = .25$

$x_2 \to 0 \Rightarrow f(x_2) = 0$

$f(x_1) > f(x_2)$ whenever $x_1 < x_2 \Rightarrow f$ is $\downarrow$ on $[-1, 0]$

from $[0, 1]$ $\quad x_1 \to 0 \quad x_2 \to 1$ clearly $x_1 < x_2$

$f(x_1) = 0$ and $f(x_2) = 1 \Rightarrow f(x_1) < f(x_2)$

thus f is $\uparrow$ on $[0, 1]$.

HW / due 2nd Feb 2024 Friday

Exercise 1.1 page 9 #1, #2, #21–24, #25–29, #30, #31–42, #59–64.

Homework name

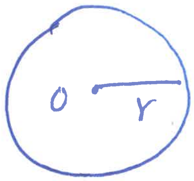
$f(x)=x$ and $g(x)=2 \Rightarrow f \circ g = g \circ f$ 6

9:05-10:20 AM

26th Jan 2024 Friday (1)

A CATALOG of Essential functions

Recall:



$A(r) \equiv \pi r^2$ (quadratic)

$C(r) \equiv 2\pi r$

functions of r → 1 independent r.

$C(r) = 2\pi r \equiv \underbrace{2\pi r^1}_{\text{algebraic expression}}$

$C(r)$ → circumference

algebraic expression $\deg(r) = 1$

$\deg(r) = 1$

Affine / Linear	quadratic
1	2

cubic

3

$f(X) \equiv mX + C$ → Affine (linear)

$$\underbrace{L(\alpha X + \beta y) = \alpha L(X) + \beta L(y)}$$

LINEAR.

constants, variable

(1) $f(X) = X + 1$. observe for some constants α and β and an independent variable y,

$$f(\alpha X + \beta y) = (\alpha X + \beta y) + 1 \neq \underline{\alpha f(X) + \beta f(y)}$$

$$\alpha f(x) + \beta f(y)$$
$$= \alpha(x+1) + \beta(y+1)$$
$$= \alpha x + \alpha + \beta y + \beta$$
$$= \alpha x + \beta y + \underbrace{\alpha + \beta}$$
$$\alpha x + \beta y + 1$$

$f(x) = x + 1$ ②

$\downarrow$

$f(y) = y + 1$

$\alpha + \beta \neq 1$

$\boxed{f(x) = x+1 \text{ is not Linear but Affine.}}$

② $C(r) = 2\pi r$

$$C(\alpha r + \beta s) = 2\pi(\alpha r + \beta s)$$
$$= 2\pi \alpha r + 2\pi \beta s$$
$$= \alpha(2\pi r) + \beta(2\pi s)$$
$$= \alpha C(r) + \beta C(s)$$

$C(r)$ is LINEAR ////.

Q1) $f(x) = 3x + 2$ ③

for some constants α and β and an independent variable y,

$$f(\alpha x + \beta y) = 3(\alpha x + \beta y) + 2$$
$$= \boxed{3\alpha x + 3\beta y + 2}$$

check $\alpha f(x) + \beta f(y)$
$$= \alpha(3x+2) + \beta(3y+2)$$
$$= \alpha 3\cancel{x} + 2\alpha + 3y\beta + 2\beta$$
$$= \boxed{3\alpha x + 3y\beta + 2\alpha + 2\beta}$$

COMPARE

thus, $\boxed{f(\alpha x + \beta y) \neq \alpha f(x) + \beta f(y)}$

$\therefore$ f is not linear ////.

Q2) $f(x) = 4x \rightarrow f(y) = 4y$

$4x + \beta$
$4(x+0)$

$$f(\alpha x + \beta y) = 4(\alpha x + \beta y) = \underline{4\alpha x} + \underline{4\beta y}$$
$$= \alpha f(x) + \beta f(y) = \underline{\text{Linear}}$$

④

Polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

example: $p(x) = x^2 - 1$ $\quad$ $p(x) = x^3 - \frac{4}{3}x + 1$

In TRIG: $\boxed{-1 \le \sin x \le 1} \Longrightarrow |\sin x| \le 1$

$\boxed{-1 \le \cos x \le 1} \Longrightarrow |\cos x| \le 1$

finit range

Periodic $f(x + \underbrace{T}_{\text{period of } f}) = f(x)$

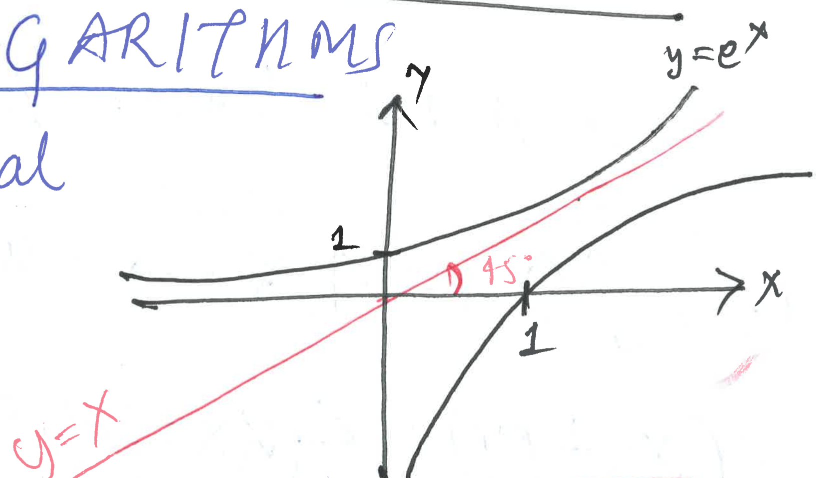
for example: $\sin(x + 2\pi) = \sin x$

$\cos(x + 2\pi) = \cos x$.

EXPONENTIAL & LOGARITHMS

$y = e^x \longrightarrow$ exponential

$y = \log x / \underbrace{\ln x}_{\text{natural logarithm.}}$



desmos

page -20 $f(x)=\sqrt{x}$ and $g(x)=\sqrt{2-x}$ (5)

(a) $f\circ g = f\circ g\,(x) = f(g(x))$

$= f(\sqrt{2-x})$

$= \sqrt{\sqrt{2-x}} = (2-x)^{1/4}$

(b) $g\circ f = g\circ f(x) = g(f(x)) = g(\sqrt{x})$

$= \sqrt{2-\sqrt{x}}.$

(c) $f\circ f = f\circ f(x) = f(\sqrt{x}) = \sqrt{\sqrt{x}}$

find $\underbrace{f\circ f\circ \cdots \circ f}_{n^{th}\text{ composition}} = f_n(x)$

$f = x^{1/2}$

$f_2 = f(f) = f(x^{1/2})$

$= (x^{1/2})^{1/2}$

$= x^{1/4}$

$= x^{(\frac{1}{2})^2}$

$f_3(x) = f(f_2(x))$

$= f(x^{1/4}) = (x^{1/4})^{1/2} = x^{1/4\cdot 1/2} = x^{1/8} = x^{(\frac{1}{2})^3}$

one more time: $f(x) = x^{1/2}$? $f_n(x)$ 6

(TRY to understand the fashion)

$n=2, \; f_2 \equiv f \circ f(x) = f(f(x)) = f(x^{1/2}) = (x^{1/2})^{1/2}$

$= x^{1/4}$

$= \cancel{x^{(1/2)^2}}$

$n=3 \to f_3 \equiv f(f_2(x)) = x^{1/8} = x^{\frac{1}{2^3}}$

$n=4 \quad f_4 \to x^{1/16} \to x^{\frac{1}{2^4}}$

thus $\quad f_n(x) = x^{\frac{1}{2^n}}$

Show $\quad f_n = f_m \Longleftrightarrow n = m$

§ 1.2 functions

9:05 – 10:20 AM

Mon ①

19th Jan 2024

Page 22 #38 $f(x) = \sqrt{3-x}$ $\quad g(x) = \sqrt{x^2-1}$

? $f+g$; $f-g$; fg ; f/g ? domain

① $f+g = f(x) + g(x) = \sqrt{3-x} + \sqrt{x^2-1}$ ///

$$dom(f+g) = dom(f) \underbrace{\cap}_{\text{intersection}} dom(g)$$

for example: $D_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$D_2 = \{2, 4, 6, 8, 10\}$

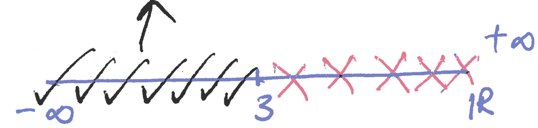
then $D_1 \cap D_2 = \{2, 4, 6, 8, 10\} = D_2$

$D_3 = \{5, 10, 11\} \to D_1 \cap D_3 = \{5, 10\}$.

$D_2 \cap D_3 = \{10\}$

$*$ $dom(f) = dom(\sqrt{3-x}) = (-\infty, 3]$

we want $3 - x \geq 0 \Rightarrow 3 \geq x$

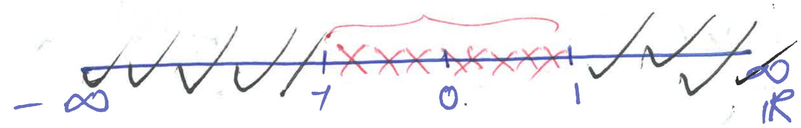


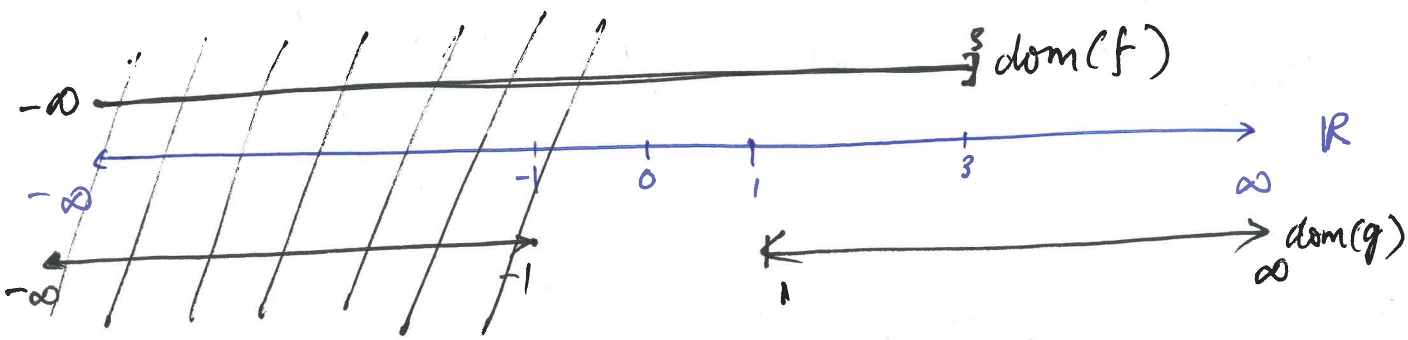
$\mathrm{dom}(g) = \mathrm{dom}(\sqrt{x^2-1})$; we want $x^2-1 \geq 0$ ②

$\Rightarrow x^2 \geq 1 \longrightarrow (-1 — 1)$

$\therefore \mathrm{dom}(\sqrt{x^2-1})$

$= (-\infty, -1] \cup [1, \infty)$





certainly, the common portion is

$(-\infty, -1] \underset{\downarrow}{\cup} [1, 3]$

union

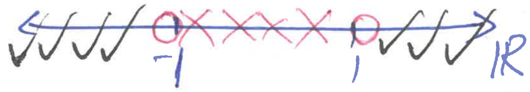
④ $f/g = \cancel{\sqrt{3-x}} \; \dfrac{\sqrt{3-x}}{\sqrt{x^2-1}}$; determine dom(f/g)

we don't want $\sqrt{x^2-1} = 0$. Therefore, if $x^2-1=0$

$\Rightarrow x^2=1 \Rightarrow x = \pm 1$. This imply $x \neq \pm 1$

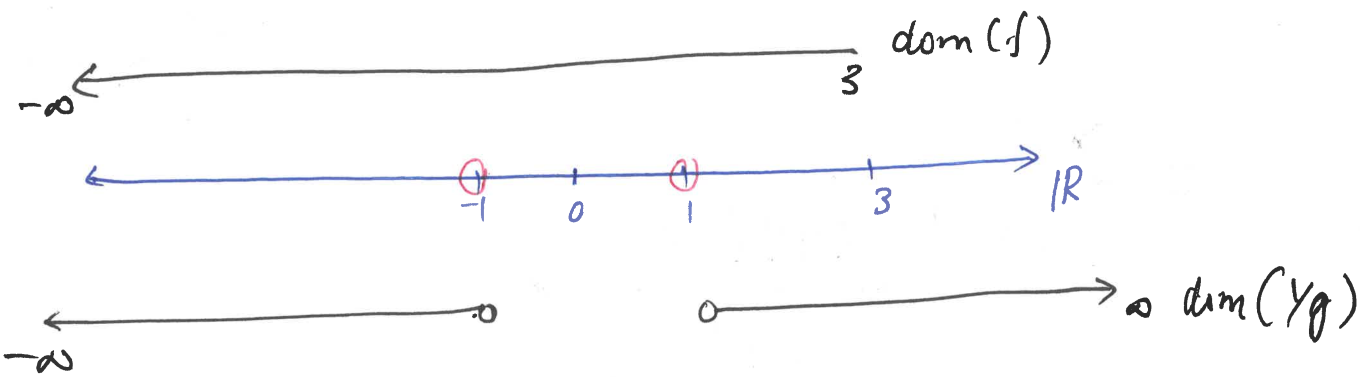
also, we don't want $x^2-1<0 \Rightarrow x^2-1>0$ (3)

$\Rightarrow x^2>1$



$\mathrm{dom}(1/g) = \mathrm{dom}\left(\frac{1}{g}\right) = (-\infty,-1) \cup (1,\infty)$.

and $\mathrm{dom}(f) = \mathrm{dom}(f) = (-\infty, 3]$.



$$\therefore\ \mathrm{dom}(f/g) = \mathrm{dom}(f) \cap \mathrm{dom}(1/g)$$
$$= (-\infty,3] \cap \big((-\infty,-1)\cup(1,\infty)\big)$$
$$= (-\infty,-1) \cup (1,3].$$

exclude ± 1.

---

page 23 (44) $f = \dfrac{x}{1+x}$ $\quad g = \sin 2x$.

determine $f\circ f,\ g\circ g,\ f\circ g,\ g\circ f$.

$$f \circ f = f\left(\frac{x}{1+x}\right) \qquad \text{Replacement}: f(x) = \frac{\boxed{x}}{1+\boxed{x}} \quad (4)$$

$$= \frac{\boxed{\dfrac{x}{1+\cancel{x}}}}{1+\boxed{\dfrac{x}{1+x}}}$$

$$1 + \frac{x}{1+x} = \frac{x+1+x}{x+1}$$

$$= \frac{x}{\cancel{1+x}\left[\dfrac{x+1+x}{\cancel{x+1}}\right]}$$

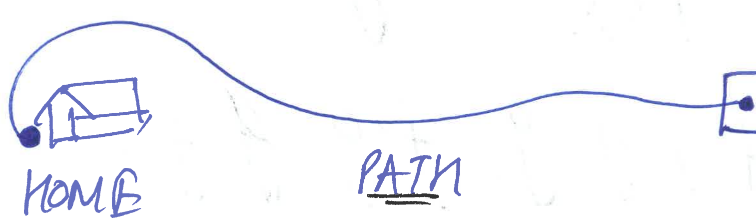
$$= \frac{x}{2x+1}$$

---

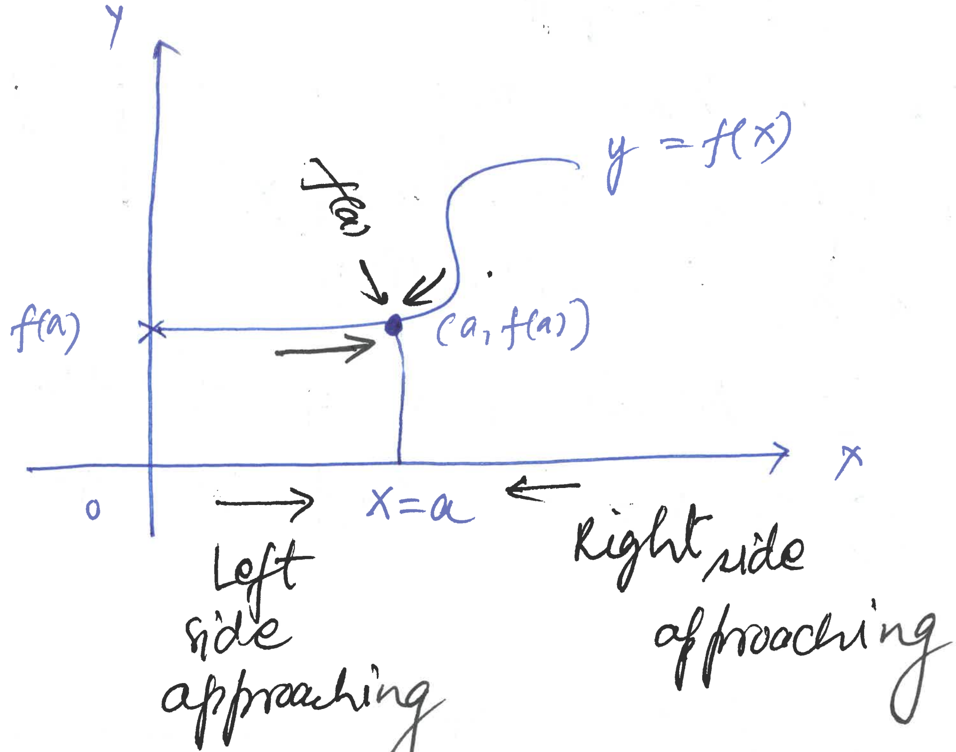
§1.3 LIMIT(S) of a function at a point 'a'

$\Updownarrow$

~~gradual~~ collection of information for $y=f(x)$ when you approached $x \longrightarrow a$ gradually



MATH 2413
RBN4024



$f(x) = x^2 - x + 2$, lets study $f(x)$ near $x=2$

Left of 2 | Right of 2

$1 \quad 1.9 \quad 1.99 \quad 2$

@ $1 \longrightarrow f(1) = 2$

@ $1.9 \rightarrow 3.71$

@ $1.99 \rightarrow 3.97$

@ $1.999 \rightarrow 3.997001 \Rightarrow 4 \Leftarrow$; obviously

@ $2.1 \rightarrow 4.31$

@ $2.01 \rightarrow 4.030100$

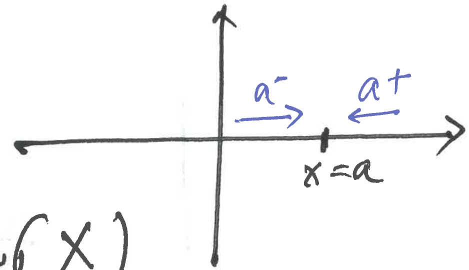
@ $2.001 \rightarrow 4.003001$

if $f(2) = 4$. this means that

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x) = f(2) = 4; \text{ (Left)}$$

$$\lim_{x \to 2^+} = \lim_{x \to 2} f(x) = f(2) = 4. \quad \text{(Right)}$$

*In 2D / $\mathbb{R}\times\mathbb{R}$ / $\mathbb{R}^2$ in which we have only one independent variable (x) i.e. $y=f(x)$ is the function, then, $\exists$, precisely only '2' directions of limit for $f(x)$ @ $x=a$. That is $x\to a^-$ and $x\to a^+$



Reading Assignment

Ex-3 @ 27 page
Ex-4 @ 28
Ex-5 @ 28.

9:05 – 10:20 AM

31st Jan 2024

Wed ①

§1.3 ex-3 $\lim\limits_{t \to 0} \underbrace{\dfrac{\sqrt{t^2+9}-3}{t^2}}_{f(t)}$

t	$f(t)$
± 1	$.16228$
$\pm .01$	$.166667$

$.166667 \to .1\overline{6}$ recurring notation

$\to .1666 \ldots\ldots$

$\pm .0001 \to .166666 \ldots$

$\Rightarrow .166666$

$\lim\limits_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2} \to .16666$

numerical result

§ ~~#4~~ example claim $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ (theorem)

Proof: Recall $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$

divide by x

Fact $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$

for example $3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 24$, $5! = 120$

$\dfrac{\sin x}{x} = \cancel{\dfrac{x}{x}} - \dfrac{x^3}{3! \cdot x} + \dfrac{x^5}{5! \cdot x} - \dfrac{x^7}{7! \cdot x} + \ldots$

$\dfrac{\sin x}{x} = 1 - \underbrace{\dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \ldots}_{O(x)}$

big O

$\Rightarrow \lim\limits_{x \to 0} \dfrac{\sin x}{x} = \lim\limits_{x \to 0} \left[1 - \cancel{x} \ldots O(x) \right] = \lim\limits_{x \to 0} 1 - \lim\limits_{x \to 0} O(x)$

$= 1 - 0 = 1.$

ex-5? $\lim\limits_{x \to 0} \sin \frac{\pi}{x}$ $x = 1/n$ $n \in \mathbb{Z}$ integer

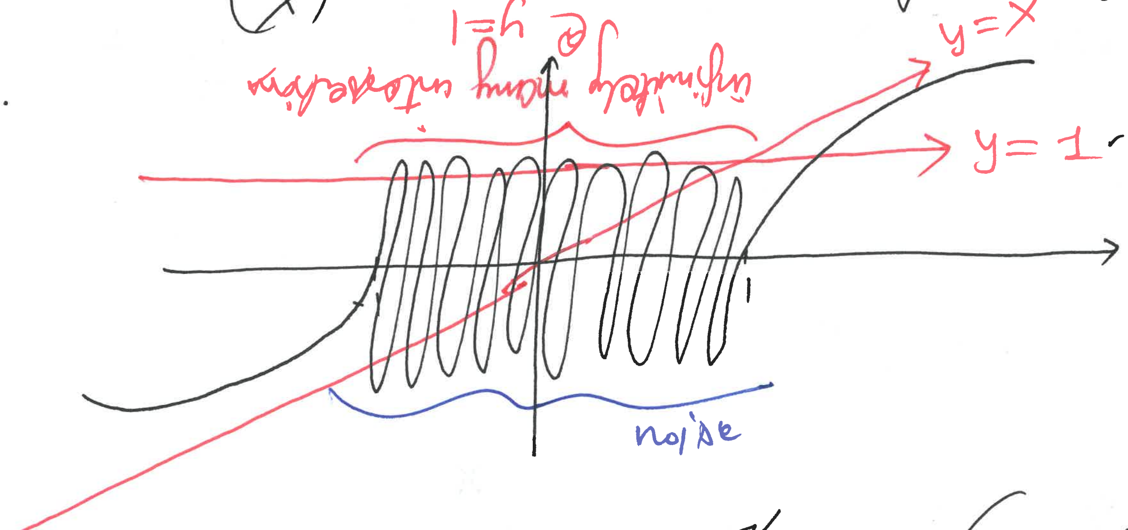
$\Rightarrow 1/x = n$

observe if $x \to 0$ then $n \to \infty$ ✓

Recall: $x = \frac{1}{n} \Rightarrow n = \frac{1}{x}$ if $x \to 0 \Rightarrow 1/x \to 1/0$

then $\lim\limits_{x \to 0} \sin \frac{\pi}{x} = \lim\limits_{n \to \infty} (\sin n\pi) = \lim\limits_{n \to \infty} (0) = 0$

Fact $\sin n\pi = 0$, however it is also true that $\sin\left(\frac{\pi}{x}\right) = 1$ for infinitely many values of x.



$|\sin x| \le 1$

Therefore, $\lim\limits_{x \to 0} \sin \frac{\pi}{x} \nexists$. (very important)

---

change $f(x) = \sin \frac{\pi}{x} \longrightarrow x^2 \sin \frac{\pi}{x}$ then, everything becomes easy. In general if $f(x) = \sin \frac{A}{x^\alpha}$ then

$F(x) = x^{\alpha+1} f(x)$ limit exist ~~and~~ @ $x=0$ ③

and in particular $\lim_{x\to 0} f(x) = 0$. $(\alpha>1)$

1.4 § limit calculation Acknowledge LIMIT LAWS @ page 35

Rough sketch $\lim_{x\to a}[f \pm g] = \lim_{x\to a} f \pm \lim_{x\to a} g$.

$\lim_{x\to a} (f(x)\, g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

$\lim_{x\to a} \frac{f}{g} = \frac{\lim_{x\to a} f}{\lim_{x\to a} g} = \square$ but ~~g~~ $\lim_{x\to a} g(x) \neq 0$

$\lim_{x\to a} (cf) = c \lim_{x\to a} f(x)$.

ex-1 @ 37

$$\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$$

$$= \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)} = \frac{\lim_{x\to -2} x^3 + \lim_{x\to -2} 2x^2 - \lim_{x\to -2} -1}{\lim_{x\to -2} 5 - \lim_{x\to -2} 3x}$$

constants are not affected!

$$= \frac{\cancel{-8} + \cancel{8} - 1}{5+6} = \frac{-1}{11} \text{ ////.}$$

$2(-2)^2 = 2\cdot 4 = 8$

ex-4 @ 39 $\lim_{h \to 0} \left[\frac{(3+h)^2 - 9}{h} \right] =$

limit law do not help in the beginning here!

- we need to simplify

$(a+b)^2 = a^2 + 2ab + b^2$

$$= \lim_{h \to 0} \left[\frac{\cancel{9} + 6\cancel{h} + h^2 - \cancel{9}}{\cancel{h}} \right]$$

$$= \lim_{h \to 0} (6+h) = \lim_{h \to 0} 6 + \cancel{\lim_{h \to 0} h}^{\,0}$$

$$= 6 + 0 = 6 ////.$$

RATIONALIZATION

ex:5 $\lim_{t \to 0} \left[\frac{\sqrt{t^2+9} - 3}{t^2} \right] = \lim_{t \to 0} \left[\frac{\overset{a}{\sqrt{t^2+9}} \overset{-}{-} \overset{b}{3}}{t^2} \cdot \frac{\overset{a+b}{\sqrt{t^2+9} + 3}}{\sqrt{t^2+9} + 3} \right]$

$(a-b)(a+b) = a^2 - b^2$

$$= \lim_{t \to 0} \left[\frac{\cancel{t^2} + \cancel{9} - \cancel{9}}{\cancel{t^2} \sqrt{t^2+9} + 3} \right]$$

$$= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9} + 3} = \frac{1}{\sqrt{\underset{\text{nine}}{9}} + 3} = \frac{1}{6} = .16666---$$

$f(x) = \sin \frac{\pi}{x}$

then @ $x \to 0$

$\lim_{x \to 0} f(x) \not\exists$

$f(x) = x^2 \sin \frac{\pi}{x}$

then @ $x \to 0$ $\lim_{x \to 0} f(x) = 0.$

very important ex-9 @ 41

(ex – ~~10~~ @ pg 42)

$$\lim_{x\to 0}\left[\frac{\sin 7x}{4x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin 7x}{4x}\cdot\frac{\cancel{7}}{\cancel{7}}\right]$$

$$= \frac{7}{4}\lim_{x\to 0}\underbrace{\left[\frac{\sin 7x}{7x}\right]}_{1}$$

$$= \frac{7}{4}\cdot 1 = \frac{7}{4} \;////.$$

Recall:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1}$$

$$\overbrace{\square\cdot\frac{*}{*} = \square}$$

page 43–44
Hw: 1 – 9, 10
11–28, 49–56.

$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ DNE

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

9:05-10:20 AM

§ 1.3

2nd Feb 2024 (1)

Friday

§ 1.4 calculating limits / limit laws.

↳ $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t} = \frac{1}{6}$ and $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

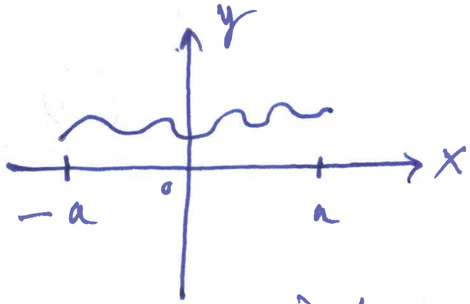
---

§ 1.5 CONTINUITY

technically if for $y=f(x)$ and @ $x=a$, we say that $f(x)$ is continuous at $x=a$, if following holds

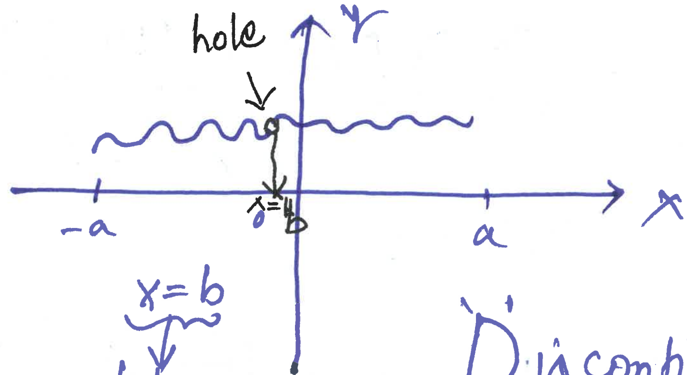
$$\underbrace{\lim_{x\to a^-} f(x)}_{\boxed{LHL}\ (1)} = \underbrace{f(x)\Big|_{x=a} = f(a)}_{\text{functional value @ } x=a\ (2)} = \underbrace{\boxed{RHL}}_{(3)};\ \lim_{x\to a^+} f(x)$$

In layman words: drawing the graph of $y=f(x)$ without pen-up.



$y = f(x)$

Continuous



x=b
poles where discontinuity is observed

Discontinuous

Ex-~~24~~ @ 47 $f(x) = \frac{x^2 - x - 2}{x - 2}$ if $x = 2$, then $f(2) =$

$= \frac{4 - 2 - 2}{2 - 2} = \frac{0}{\boxed{0}}$ $\boxed{\text{UNDEFINED}}$,

thus f is discontinuous @ $x = 2$

TRIVIAL example: $y = \frac{1}{x}$ @ $x = 0$, $y = 1/x$ is discontinuous ✓

Ex- 4 @ page 48: $f(x) = 1 - \sqrt{1 - x^2}$ and dom(f) $= [-1, 1]$.

claim $f \in \mathcal{C}([-1, 1])$.

$\in$ → belongs to.

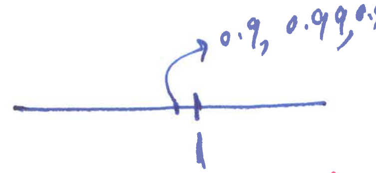
$[-1, 1]$ → set

$\mathcal{C}$ → continuity.

$f(x) = 1-\sqrt{1-x^2}$. For continuity @ $x = a \in [-1,-1]$ ③

$\boxed{f(a) = 1-\sqrt{1-a^2}}$. Now, we will find LHL & RHL

<u>LHL</u> : $\lim_{x\to a^-} f(x) = \lim_{x\to a^-}\left[1-\sqrt{1-x^2}\right]$



$= 1 - \lim_{x\to a^-}\left[\sqrt{1-x^2}\right]$

$= 1-\sqrt{1-\left(\lim_{x\to a^-} x^2\right)}$

$\boxed{\lim_{x\to a^-} f(x) = 1-\sqrt{1-a^2}}$

we need to find $\lim_{x\to a^-} x^2$

denote $x \to a^-$ as $a^- = a-h$ with $h \to 0$. Thus,

$\lim_{h\to 0} (a-\underbrace{h}_{\text{(small) decrement}})^2$

$= a^2$

<u>RHL</u>: $\lim_{x\to a^+} f(x) = \lim_{x\to a^+}\left[1-\sqrt{1-x^2}\right]$

$= 1 - \lim_{x\to a^+}\sqrt{1-x^2}$

$= 1-\sqrt{1-\lim_{x\to a^+} x^2}$

$\lim_{x\to a^+} x^2$

$= \lim_{h\to 0}(a+h)^2$

$= a^2$

$\boxed{\lim_{x\to a^+} f(x) = 1-\sqrt{1-a^2}}$, Thus, f is $\mathcal{C}[-1,1]$.

if f and g are $e \Rightarrow f \pm g$ is so,
$f \times g$ is so
f/g is so but $g \neq 0$
cf is so.

for example $f(x) = \sin x$ and $g(x) = \cos x$
then $f \pm g = \sin x \pm \cos x$ continuous.
$f \times g = \sin x \cos x = \frac{\sin 2x}{2}$ continuous.

(double angle)
($\sin 2x = 2\sin x \cos x$)

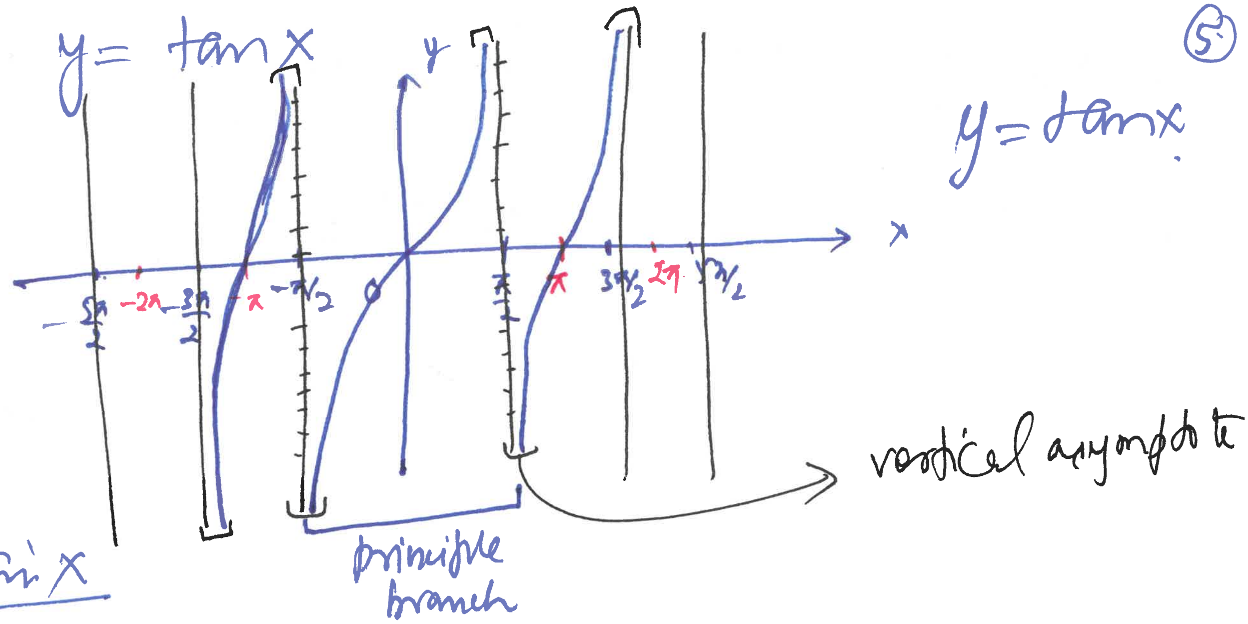
$f/g = \frac{\sin x}{\cos x} = \tan x$ but not at $\cos x = 0$
$\Rightarrow x = (2n+1)\frac{\pi}{2}$ $n \in \mathbb{Z}$.

Fact $\sin x = 0$
@ $x = n\pi$
$n \in \mathbb{Z}$

so far we have discussed limits at finite point value.
Now, we will discuss limits @ $\pm\infty$.

§ 1.5 limits @ $\pm \infty \longleftrightarrow$ ASYMPTOTES

For example $y = \tan x$



$\because \tan x = \frac{\sin x}{\cos x}$

$\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$

@ $x = n\pi \equiv 0, \pm\pi, \pm 2\pi \ldots$ $\tan x = 0 \rightarrow$ cross @ x axis.

def. vertical asymptote: $y = f(x)$ the VA. 'equation of VA.

for $y = f(x)$ is $\boxed{x = a}$ if

$$\lim_{x \to a} f(x) = \pm\infty, \qquad \text{(limit value)}$$

$$\lim_{x \to a^\pm} f(x) = \pm\infty. \qquad \text{(one sided limit value).}$$

?. VA of $y = \frac{1}{x}$ is $x = 0$.

horizontal asymptote: for $y = f(x)$, $y = L$ @ is HA if $\lim_{x\to\pm\infty} f(x) = L$.

calculate this limit to find the HA of $y=f(x)$.

For ex $y = \underbrace{\frac{x^2-1}{x^2+1}}$; find HA of $y=f(x)$

In order to find the HA of $y = \frac{x^2-1}{x^2+1}$ we will compute $\lim_{x\to\infty} \frac{x^2-1}{x^2+1}$.

$$\lim_{x\to\infty}\left[\frac{x^2-1}{x^2+1}\right] = \lim_{x\to\infty}\left[\frac{\cancel{x^2}(1-1/x^2)}{\cancel{x^2}(1+1/x^2)}\right]$$

$$= \left[\frac{1-\lim_{x\to\infty} 1/x^2}{1+\lim_{x\to\infty} 1/x^2}\right] = \frac{1-0}{1+0} = 1$$

Note: $\lim_{x\to\infty}\frac{1}{x^n} = 0$ if $n \in \mathbb{Z}_+$

so, HA $\equiv y=1$ for $y = \frac{x^2-1}{x^2+1}$.

Ex-1 @ page 73 find eq. of tangent to $y = x^2$ @ $\underbrace{(1,1)}_{(x_0, y_0)}$.

Recall

$y - y_0 = m(x - x_0)$

m ↓ slope and (x_0, y_0)

Solution : Recall :

$$m_x = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right].$$

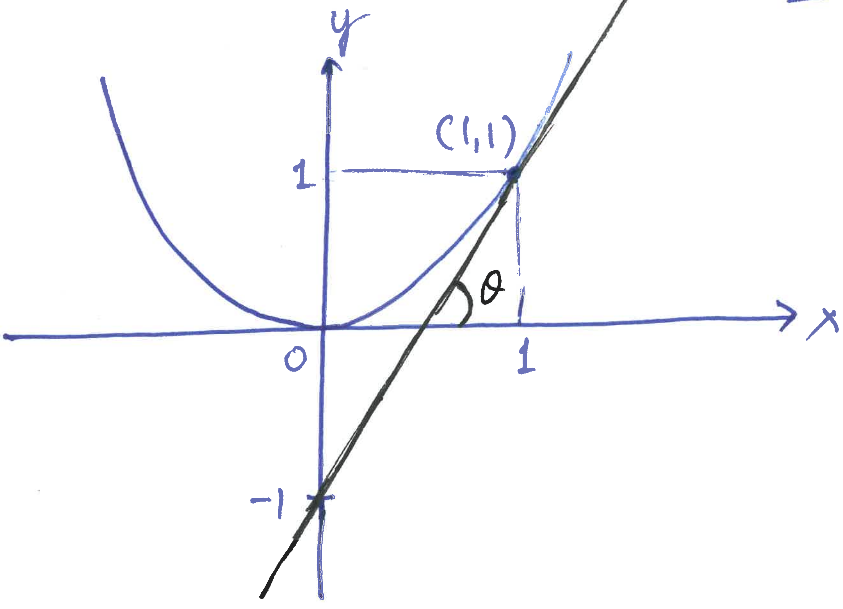
Here $f(x) = x^2 \Rightarrow \underbrace{m_x = 2x}_{\text{in general.}}$ (Refer pag 2)

Therefore $m_x \Big|_{x=1} = 2x \Big|_{x=1} = 2 \cdot 1 = 2.$

By the point-slope formula we have

$$y - 1 = 2(x-1) \Rightarrow y - 1 = 2x - 2$$

$$\Rightarrow \boxed{y = 2x - 1}$$



$\tan\theta = 2 = m_x$

(2) $f(x) = x^2$, $f(x+h) = (x+h)^2$ (2)

$= x^2 + 2xh + h^2$

Recall $f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$

$= \lim_{h \to 0} \left[\frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \right]$

$= \lim_{h \to 0} \left[\frac{2xh + h^2}{h} \right] = \lim_{h \to 0} \left[2x + h \right]$

$= 2x + \underbrace{\lim_{h \to 0} h}_{0}$

$= 2x.$

Thus, $f'(x) = 2x$ for $f(x) = x^2$.

In general if $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$.

DERIVATIVES help in determining the equation of tangents for $y = f(x)$ in $\mathbb{R} \times \mathbb{R}$. (cartesian plane).

Essentially, by the first principle of derivative the slope of tangent line for $y = f(x)$, i.e.

$m_x = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] = f'(x)$

$y = f(x)$ (general) 9:05-10:20AM

5th feb 2024 ①

Monday

$$\downarrow$$

$$\frac{[f(x+h) - f(x)]}{\underbrace{x+h-x}} \rightarrow \frac{\Delta f}{\Delta x} = \left.\frac{\text{change in } f}{\text{change in } x}\right\}$$

$$\boxed{\text{rate of change of } f \text{ w.r.t } x}$$

$$\frac{f(x+h)-f(x)}{h} \xrightarrow[h\to 0]{\text{limiting process}} \underbrace{\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]}$$

derivative by FIRST PRINCIPLE

Def:

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] \equiv \text{derivative of } f \text{ w.r.t. } x.$$

provided that the $\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$ exist.

① $f(x) = c$, then $f(x+h) = c$, Now,

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h\to 0}\left[\frac{c-c}{h}\right] = 0$$

if, $f(x) = \text{constant} \Rightarrow f'(x) = 0$.

$$\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta + \tan \theta} \right)$$

Recall $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ ✓

$$= \lim_{\theta \to 0} \frac{\sin \theta}{\theta \left[1 + \frac{\tan \theta}{\theta} \right]}$$

$$= \underbrace{\left(\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \right)}_{1} \left(\lim_{\theta \to 0} \frac{1}{1 + \frac{\tan \theta}{\theta}} \right)$$

$$= 1 \cdot \lim_{\theta \to 0} \frac{1}{1 + \frac{\tan \theta}{\theta}}$$

$$= 1 \cdot \frac{1}{1 + \underbrace{\lim_{\theta \to 0} \frac{\tan \theta}{\theta}}_{1}}$$

$$= 1 \cdot \frac{1}{1+1}$$

$$= 1/2 \; ////.$$

$$\lim_{\theta \to 0} \frac{\tan \theta}{\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin \theta}{\theta \cdot \cos \theta}$$

$$= \underbrace{\left(\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \right)}_{} \cdot \lim_{\theta \to 0} \frac{1}{\cos \theta}$$

$$= 1 \cdot \left(\frac{1}{\lim_{\theta \to 0} \cos \theta} \right)$$

$$= 1 \cdot 1$$

$$= 1$$

#18) $\lim_{h\to 0}\left[\frac{(2+h)^3-8}{h}\right]$ $\quad (2+h)^3$

$= 8+h^3+12h+6h^2$

$=\lim_{h\to 0}\left[\frac{\cancel{8}+h^3+12h+6h^2-\cancel{8}}{h}\right]$

$=\lim_{h\to 0}\left[h^2+12+6h\right]=12$

---

53) $\lim_{x\to 0}\frac{\sin 3x}{5x^3-4x}$

$\lim_{x\to 0}\frac{\sin x}{x}=1$

$=\lim_{x\to 0}\left[\frac{3\sin 3x}{3x(5x^2-4)}\right]$

$=\lim_{x\to 0}\frac{3\sin 3x}{3x(5x^2-4)}$

$=3\left[\lim_{x\to 0}\frac{\sin 3x}{3x(5x^2-4)}\right]=3\left[\lim\frac{\sin 3x}{3x}\lim\frac{1}{5x^2-4}\right]$

$1 \quad -\frac{1}{4}$

$g(x) = x^2 \sin\left(\frac{1}{x}\right)$, now, find f and h?

however,

$\boxed{?} \leq x^2 \leq \boxed{?}$

no definitive way

$-1 \leq \sin(1/x) \leq 1$

multiply by x^2

Remember

$-1 \leq \sin x \leq 1$

$-1 \leq \cos x \leq 1$

$\Rightarrow \underbrace{-x^2}_{f} \leq g(x) \leq \underbrace{\cancel{1}x^2}_{h}$

$\lim_{x \to 0} (-x^2) = 0$; $\lim_{x \to 0} x^2 = 0$

and $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$.

$\therefore$ by squeeze theorem $\lim_{x \to 0} x^2 \sin(1/x) = 0$.

$$\lim_{\tau\to 0} \frac{\sin \tau}{\tau} = 1$$ fact

$$\lim_{\tau\to 0} \frac{\sin^2 \tau}{\tau} = ?$$

$$\lim_{\tau\to 0} \frac{1-\cos^2 \tau}{\tau}$$ $\because \sin^2\tau + \cos^2\tau = 1$

$\frac{0}{0}$

$$= \lim_{\tau\to 0} \frac{1-\cos^2 \tau}{\tau}$$

hint: $\boxed{\lim_{\tau\to 0} \frac{\sin^2 \tau}{\tau} = \lim_{\tau\to 0} \frac{1-\cos^2 \tau}{\tau} \rightarrow ?}$

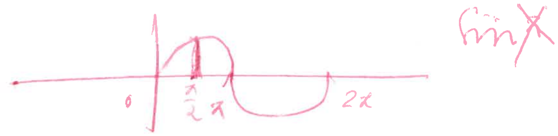
$\rightarrow ?$

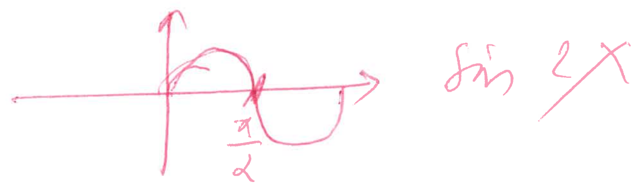
Example-11 @ page 43

$\sin(x+2\pi) = \sin x \longrightarrow$ periodicity

$$S_2(x+2\pi) = \sin(\underline{\sin(x+2\pi)})$$
$$= \sin(\sin(x))$$
$$= S_2(x)$$

$\sin(2x) \rightarrow$





Q3 Application of squeeze theorem

let's say we have three functions f, g, h

say $f \le g \le h$

@ $x=a$ if $\lim_{x\to a} f = L$ and

$\lim_{x \to a} h = L$, then

your conclusion is $\boxed{\lim_{x\to a} g = L}$

§2 #1 $S(x) = \sin x \to S_n(\tau)$

$S_2(\tau) = \sin(\sin\tau)$ $\quad S_3(\tau) = \sin(\sin(\sin(\tau)))$

$\therefore S_n(\tau) = \sin(\underbrace{\dots}_{n-2}(\sin(\tau)))$

$= \sin(S_{n-1}(\tau)) \equiv S_{n-1}(\sin\tau)$

(2) $\lim_{\tau\to 0} \frac{S_n(\tau)}{1(\tau)} = 1 = \lim_{\tau\to 0} \frac{\sin(S_{n-1}(\tau))}{1(\tau)} = 1$

$\because \lim_{\tau\to 0} \frac{\sin\boxed{\tau}}{\boxed{\tau}} = 1$ comparison

---

$x = .5 \in [0,1] = I$

$x^3 = .5^3 = .125$

$f_{[\#]}(d) = d^{\#}$

$f(x) = x^2$

$f: \underset{\text{domain}}{I} \longrightarrow \underset{\text{range}}{I}$

Next-week's HW

9:05-10:20 AM

Wednesday ①

7th Feb 2024

§1.5 12-13, 15-18, 29-30

§1.6 13-33

§2.1 3-6.

chapter-2 §1 9:05-10:20 AM 9th feb 2024 ①

FRIDAY

determining $f'(x)$ for $f(x) = \sin x \equiv \cancel{S(x)}$

Recall : $f'(x) = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$

Solution : $f'(x) = \lim_{h\to 0}\left[\frac{\sin(x+h) - \sin x}{h}\right]$

$f(x) = \sin x$

$$= \lim_{h\to 0}\left[\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}\right]$$

Recall
$\sin(x+h)$
$= \sin x \cosh h +$
$\cos x \sin h.$

$$= \lim_{h\to 0}\left[\frac{\sin x(\cos h - 1) + \cos x \sin h}{h}\right]$$

$$= \lim_{h\to 0}\left[\sin x \frac{(\cos h - 1)}{h} + \cos x \frac{\sin h}{h}\right]$$

$$= \lim_{h\to 0} \sin x \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \frac{\sin h}{h}$$

key limit law.

@ p.43 1

$$= \sin x \boxed{\lim_{h\to 0} \frac{\cos h - 1}{h}} + \cos x \boxed{\lim_{h\to 0} \frac{\sin h}{h}} = \sin x \cdot 0 + \cos x \cdot 1$$

$= \cos x.$

§ chapter 2 section 2

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$; provided that $\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$ exist.

Note: If $f'(x)$ exists meaning that the $\lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$ exist, then we say that f is differentiable @ x.

For example: if $f(x) = \sin x$, then $f'(x) = \cos x$ and we say (simply) that $\sin x$ is differentiable at x.

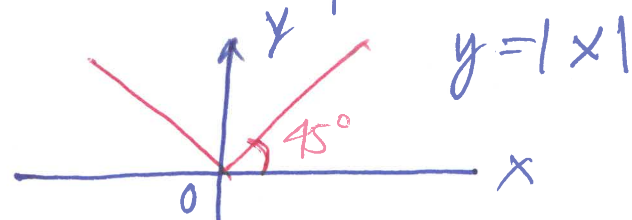
Notation for derivatives: $'$ (prime); $\frac{d}{dx}$, D_x.

For example: say $f(x) = c \Rightarrow f'(x) = 0$, or

$\frac{d}{dx} f(x) = 0$; $D_x f(x) = 0$.

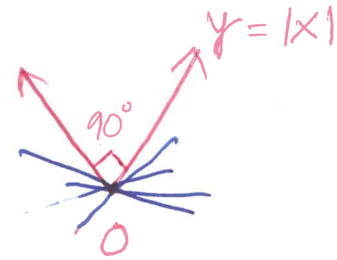
$\boxed{\text{EXIST}}$ → value is not ∞ / DNE.

Lets discuss "EXIST" ? (in terms of derivative) ③



consider $f(x) = |x|$.

def of $f(x) = \begin{cases} -x & \text{if } x < 0 \\ +x & \text{if } x \geq 0 \end{cases}$



LHS (left hand side) meaning $x < 0$

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-(x+h) - (-x)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\cancel{-x} - \cancel{h} + \cancel{x}}{\cancel{h}} \right] = -1$$

RHS (right hand side) meaning $x \geq 0$

$$f'(x) = \lim_{h \to 0} \left[\frac{(x+h) - (x)}{h} \right] = 1.$$

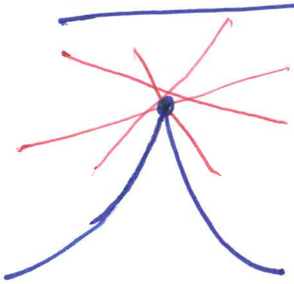
This means that, $\underbrace{f'(x)}_{=-1}\Big|_{x<0} \neq \underbrace{f'(x)}_{=1}\Big|_{x \geq 0}$

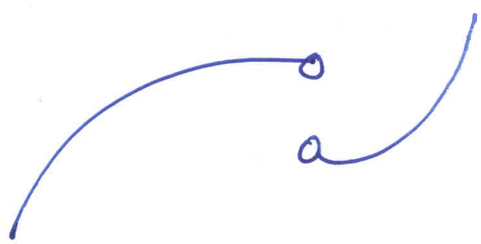
Thus, f is ~~not~~ differentiable (it has to be consistent) at any $x < 0$ and $x \geq 0$, but @ $x = 0$, the derivatives are not consistent.)

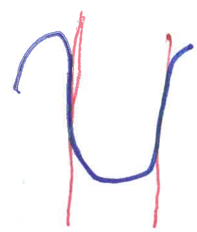
Thus, $f(x)$ is not differentiable @ $x = 0$.

<u>Note</u>: continuity $\not\Rightarrow$ Differentiability;
differentiability $\Longrightarrow$ continuity.

<u>common scenarios for failure of differentiability</u>.







<u>corner</u>
cusp

<u>discontinuity</u>

<u>vertical tangent</u>.

simplest explanation for failure of differentiability is $\Longrightarrow$ unable to draw a unique tangent @ point of differentiation.

- $f(x) = c \Rightarrow f'(x) = 0 \Rightarrow f''(x) = 0$

1st derivative   2nd derivative

- $f(x) = cx \Rightarrow f'(x) = c \Rightarrow f''(x) = 0$

- $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow f''(x) = -\sin x$

- $f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow f''(x) = e^x.$

---

§2.3: <u>sum rule</u> $[f \pm g]' = f' \pm g'$

<u>power rule</u>: $(x^n)' = n x^{n-1}$

$$(c)' = 0$$

$$(c f(x))' = c f'(x)$$

$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x.$$

say
$$f(x) = 2x^{1/2} + \frac{1}{3}x^3 - \cos x$$ ⓒ

$$\Rightarrow f'(x) = (2x^{1/2} + 1/3\, x^3 - \cos x)'$$
$$= (2x^{1/2})' + \left(\frac{1}{3}x^3\right)' - (\cos x)'$$

$$= x^{-1/2} + x^2 - (-\sin x) \quad ////$$
$$= x^{-1/2} + x^2 + \sin x \quad ////.$$

$$(2x^{1/2})' = 2(x^{1/2})' = \cancel{2}\cdot\frac{1}{\cancel{2}}x^{1/2-1} = x^{-1/2}$$
$$(c\,f(x))'$$
$$= c\,f'(x)$$

$$\left(\frac{1}{3}x^3\right)' = \frac{1}{3}(x^3)' = \frac{1}{3}\,3x^{3-1} = x^2$$

<u>Reading problems</u>: ex-1 @ 97, ex-3 @ 98, ex-8 @ 102

§2.4: Product Rule:

$$D_x(fg) = \frac{d}{dx}(fg) = (fg)' = f'g + g'f$$

quotient rule $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

$g \neq 0$

for example: $y = \underbrace{x}_{f}\,\underbrace{\sin x}_{g}$

$$y' = 1 \cdot \sin x + x \cdot \cos x = \sin x + x\cos x. ////$$

for example $y = \frac{\sin x}{x} \equiv \frac{f}{g}$

$$y' = \frac{x \cdot \cos x - \sin x \cdot 1}{x^2} \quad ////.$$

Reading assignment §2.4 → EX-6 @ 112,

list of TRIG's derivatives @ 111.

§ 2.4 9:05-10:20 AM Monday ①

12th Feb 2024

$f(x) = \tan x \quad f'(x) = ?$

$= \dfrac{\sin x}{\cos x} = \dfrac{F}{G}$ (quotient)

$$\left(\frac{F}{G}\right)' = \frac{F'G - FG'}{G^2}$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' =$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$(\cot x)' = -\operatorname{cosec}^2 x$

$(\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x$

$(\cos x)' = -\sin x.$

---

chain rule :→ differentiation of functions which are in composition formate.

$h(x) = f \circ g(x) = f(g(x))$

example: $\sin(x) \xrightarrow{n\text{ó}} \sin(\underbrace{\sin(\dots \sin x)}_{n-1})$

$\sin x \xrightarrow{2\text{ö}} \sin(\sin(x))$

Rule of chain: $h'(x) = f'(g(x)) \cdot g'(x)$

for example: $h(x) = \sin x$

$h'(x) = \cos x$

$\rightsquigarrow h(x) = \sin(x^2) = \underbrace{\sin(x)}_{f} \circ \underbrace{x^2}_{g}$

$h'(x) = (\cos x \circ x^2) \cdot 2x$

$= \cos x^2 \cdot 2x.$

---

for example $h(x) = \tan(\sin x)$

$h'(x) = \sec^2(\sin x) \cdot \cos x.$

---

for ex: $h(x) = \tan(\sin x + e^x)$

$h'(x) = \sec^2(\sin x + e^x) \cdot (\cos x + e^x)$. ////.

ex: $y = \sin\left(e^x + \frac{1}{x}\right)$

$$y' = \cos\left(e^x + \frac{1}{x}\right) \cdot \left(e^x - \frac{1}{x^2}\right)$$

$$\left(\frac{1}{x}\right)' = (x^{-1})' = (-1)x^{-1-1} = -x^{-2}$$

$$x^{-n} = \frac{1}{x^n}$$

power rule

$$\left(\frac{3}{x}\right)' = 3\left(\frac{1}{x}\right)' = \frac{-3}{x^2}$$

$$(cf(x))' = cf'(x)$$

$$(3x^{-1})' = -3x^{-2}$$

---

ex $y = x - \log(\sin x)$

$$(\log x)' = \frac{1}{x}$$

$$y' = 1 - \frac{1}{\sin x} \cdot \cos x$$

$$y' = 1 - \cot x.$$

$$y = \underbrace{(x+1)}_{f}\underbrace{(x+2)}_{g} \qquad (fg)' = f'g + g'f$$

$$y' = 1\cdot(x+2) + (x+1)\cdot 1 = (x+2) + (x+1)$$
$$= 2x+3$$

$$y = (x+3)(x+1)$$
$$y' = 2x+4$$

ex $y = x+3$

$$\frac{dy}{dx} = \frac{d}{dx}(x+3)$$
$$= \overbrace{\frac{d}{dx}(x)}^{1} + \overbrace{\frac{d}{dx}(3)}^{0}$$
$$= 1$$

$$y = (x+1)(x+4)$$
$$y' = 2x+5$$

$$y = (x+m)(x+n) \to y' = 2x + (m+n).$$

HW : §2.3 1–26, 27–28, 31–34

§2.4 1–26

§2.5 Chain Rule

$h(x) = f \circ g(x) = f(g(x))$, then,

$h'(x) = f'(g(x)) \cdot g'(x)$.

$(x^n)' = nx^{n-1}$

ex: $h(x) = (x+1)^{100}$

$h'(x) = \underbrace{100(x+1)^{100-1}}_{f' \circ g} \cdot \underbrace{(x+1)'}_{g'}$

$(x+1)' = (x)' + 1' = 1 + 0 = 1$

$= 100(x+1)^{99} \cdot 1$

$= 100(x+1)^{99}$ ////.

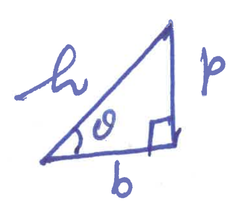
ex: $h(x) = (\underbrace{\sin^2 x + \cos^2 x})^{10^{10^9}}$

$h'(x) = \frac{d}{dx}(1) = 0$.

$\sin^2 x + \cos^2 x = 1$

$\downarrow$

Pythagorean identity



$b^2 + p^2 = h^2$

$\Rightarrow \underbrace{\frac{b^2}{h^2}}_{\cos^2\theta} + \underbrace{\frac{p^2}{h^2}}_{\sin^2\theta} = 1$

$(x^n)' = nx^{n-1}$ 2

ex: $y = \tan^8 x$

$$y' = 8\tan^{8-1} x \cdot (\tan x)'$$
$$= 8\tan^7 x \cdot \sec^2 x. \quad ////.$$

ex: $y = \sin^2 x$

$$y' = 2\sin x \cdot \cos x = \sin 2x$$

$(e^x)' = e^x$

$(\log x)' = \frac{1}{x}$

ex: $y = \sin^3\overbrace{(x + e^x + \log x)}$

$$y' = 3\sin^2(x + e^x + \log x)\cos(x + e^x + \log x)\cdot$$
$$(1 + e^x + \tfrac{1}{x}) ////.$$

ex: $y = \underbrace{e^{2x}}_{f}\ \underbrace{\cos x}_{g}$

$(fg)' = f'g + g'f$

$$y' = 2e^{2x}\cos x - \cancel{e}\sin x\, e^{2x}$$
$$= e^{2x}(2\cos x - \sin x). \quad ////.$$

$(e^{2x})'$
$= 2e^{2x}$

ex $y = (\underbrace{\sec^2 x - \tan^2 x}_{1} + \underbrace{\cot^2 - \operatorname{cosec}^2 x}_{-1})^{30^2}$ (3)

$\Rightarrow y = (1-1)^{D} = 0 \Rightarrow y' = 0.$ ////.

$$\left\{\begin{array}{l} s^2 + c^2 = 1 \\ \sec^2 - \tan^2 = 1 \\ \operatorname{cosec}^2 - \cot^2 = 1 \end{array}\right\}$$

121. / 33.

$$y = \left(\frac{1-\cos 2x}{1+\cos 2x}\right)^4$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

$$y' = 4\left(\frac{1-\cos 2x}{1+\cos 2x}\right)^3 \cdot \frac{4\sin 2x}{(1+\cos 2x)^2}$$

$$\left(\frac{f}{g}\right)' = \left[\frac{1-\cos 2x}{1+\cos 2x}\right]'$$

$$= \frac{(1+\cos 2x)(2\sin 2x) - (1-\cos 2x)(-2\sin 2x)}{(1+\cos 2x)^2}$$

$\sin 2x = 2\sin x \cos x$

$$= \frac{2\sin 2x + \cancel{\sin 4x} + 2\sin 2x - \cancel{\sin 4x}}{(1+\cos 2x)^2}$$

$$= \frac{4\sin 2x}{(1+\cos 2x)^2}$$

$$y = \left(\frac{1-\cos 2x}{1+\cos 2x}\right)^4$$

$$= \left(\frac{\cancel{2}\sin^2 x}{\cancel{2}\cos^2 x}\right)^4$$

$$= \left(\tan^2 x\right)^4$$

$$\boxed{y = \tan^8 x}$$

use double angle formula (1)

$\cos 2x = \cos^2 x - \sin^2 x$

$1-\cos 2x = \underline{2\sin^2 x}$

$1+\cos 2x = \underline{2\cos^2 x}$

how→. $\sin^2 x + \cos^2 x = 1$ in

$\cos 2x = 1 - \sin^2 x - \sin^2 x.$

and

$\cos 2x = \cos^2 x - (1-\cos^2 x)$

---

so far : we have met with <u>nice functions</u>.

↓ they were not computational expensive.

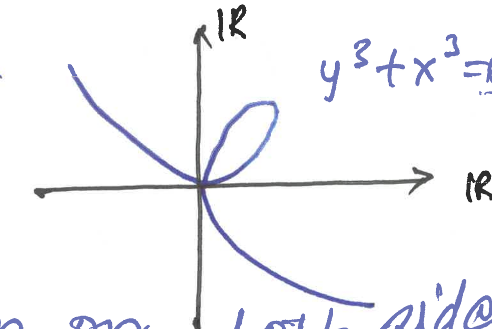
But, Now, we will get involved with <u>implicit functions</u>.

↓ hard to separate (isolate) independent & dependent variables.

<u>For example</u> : $y = x$; however. $y = \sin(xy)$

$y = x^2 + x^3 + 2$; however $y^3 = x^3 + 6xy + x^2$

example: $x^3+y^3=6xy$



$y^3+x^3=6xy$ 5

lets differentiate:

$\Rightarrow$ do the differentiation on both sides of implicit equation.

$(y^3)' = 3y^2 \cdot y'$

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$$

$$\Rightarrow \underbrace{\frac{d}{dx}x^3}_{\text{power rule}} + \frac{d}{dx}y^3 = 6\frac{d}{dx}(xy)$$

$$\Rightarrow \cancel{3}x^2 + \cancel{3}y^2y' = \cancel{6}^{\,2}[y\cdot 1 + xy']$$

$$\Rightarrow x^2 + y^2y' = 2y + 2xy'$$

$$\Rightarrow y^2y' - 2xy' = 2y - x^2$$

$$\Rightarrow y'(y^2-2x) = 2y - x^2$$

$$\Rightarrow \boxed{y' = \frac{2y-x^2}{y^2-2x}}$$

page 54 : #12 $f(x) = 3x^4 - 5x + (x^2+4)^{1/3}$

check f continuity @ $x = 2$

solution: $f(2)$ (functional value)

$= 3\cdot 16 - 10 + 2 = 40.$

LHL: $x \to 2^- \Rightarrow x = 2-h$ with $h \to 0$.

$$\lim_{x\to 2^-}[f(x)] = \lim_{x\to 2^-}\left[3x^4 - 5x + (x^2+4)^{1/3}\right]$$

$$= \lim_{h\to 0}\left[3(2-h)^4 + 5(2-h) + ((2-h)^2+4)^{1/3}\right]$$

$$= \lim_{h\to 0} 3(2-h)^4 + \lim_{h\to 0} 5(2-h) + \lim_{h\to 0}\left((2-h)^2+4\right)^{1/3}$$

$$= 3\cdot 2^4 + 5\cdot 2 + (2^2+4)^{1/3} = 40.$$

§ 2.6 Implicit differentiation

9:05 – 10:20AM

16th feb 2024 (1)

friday.

Folium of descartes $\boxed{x^3 + y^3 = 6xy}$ — (1)

Find dy/dx ; y' (aim try to isolate y')

product rule

$(xy)' = [(x)' \cdot y + x(y)']$
$= [y + xy']$

differentiate (1).

$$\cancel{3}x^2 + \cancel{3}y^2 \underbrace{\frac{dy}{dx}}_{y'} = \cancel{6}^{\,2}\underbrace{[y + xy']}_{2y + 2xy'}$$

$$\Rightarrow x^2 + y^2 y' = 2[y + xy']$$

$$\Rightarrow \underbrace{y^2 y' - 2xy'}_{y'(y^2 - 2x)} = 2y - x^2$$

$$\Rightarrow \boxed{y' = \frac{2y - x^2}{y^2 - 2x}}$$

@ 12:5'

VEx-2

so far, we have got y'. Now, we will determine equation of tangent for $x^3 + y^3 = 6xy$ @ $(3,3)$.

slope @ (3,3) is $m\Big|_{(3,3)} = \frac{2y-x^2}{y^2-2x}\Big|_{(3,3)}$

(particular slope)

$$= \frac{6-9}{9-6} = \frac{-3}{3} = -1.$$

∴ we know slope, that is -1 @ (3,3),

$$y-3 = (-1)(x-3)$$

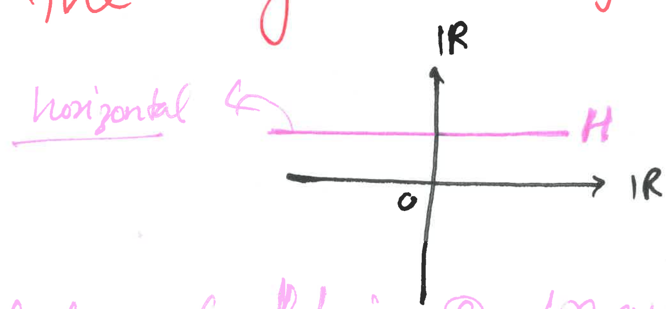
$$\Rightarrow y = -x+3+3 \Rightarrow \boxed{y=-x+6}$$

L passes at (x_0, y_0) with slope m

$$\boxed{y-y_0 = m(x-x_0)}$$

part (b) of #2 @ 125

Q: find x-coordinate in first quadrant is the tangent to $x^3+y^3=6xy$ would be horizontal.



H is // to x-axis.
→ parallel.

$\Rightarrow$ slope of H is 0 because slope of x-axis is 0.

Recall: $y' = \dfrac{2y - x^2}{y^2 - 2x}$. set $y' = 0 \Rightarrow$

$$\frac{2y - x^2}{y^2 - 2x} = 0 \Rightarrow 2y - x^2 = 0 \Rightarrow \boxed{y = \frac{x^2}{2}}$$

Use $x^3 + y^3 = 6xy$ to find x-value by putting $y = x^2/2$.

$$\Rightarrow x^3 + \left(\frac{x^2}{2}\right)^3 = \cancel{6}^{\,3} x \cdot \frac{x^2}{\cancel{2}}$$

$$\Rightarrow x^3 + \frac{x^6}{8} = 3x^3$$

$$\Rightarrow \frac{x^6}{8} = 2x^3 \Rightarrow \boxed{x^6 = 16x^3}$$

$$\Rightarrow x^6 - 16x^3 = 0 \Rightarrow \underbrace{\boxed{x^3}}_{=0}\underbrace{\boxed{(x^3 - 16)}}_{=0} = 0$$

$x^3 = 0 \Rightarrow x = 0$ or $x^3 - 16 = 0 \Rightarrow \boxed{x = 16^{1/3}}$

$y = 0$

④

if $x = 16^{1/3}$ then, ~~from~~ $\cancel{x^3+y^3=6xy}$

$\cancel{\Rightarrow 16 + y^3 = 6\,16^{1/3} \cdot y}$ $y = \frac{x^2}{2}$ will yield

$y = \frac{(16^{1/3})^2}{2}$. Finally the point where tangent is // to a x axis is $\left(16^{1/3}, \frac{(16^{1/3})^2}{2}\right)$.

<u>note</u>: slope x axis = 0

slope y axis = ∞.

$$y = \sin x \cdot e^{x} \cdot (x+2)$$

$y = \underset{f}{\sin x} \cdot \underset{g}{(x+2)}$ $\qquad (fg)' = f'g + g'f$

$y' = \cos x\,(x+2) + 1 \cdot \sin x$

5

If product happens with more than 2 functions then take 'log' and differentiate further.

$(\sin x)' = \cos x$
$(e^x)' = e^x$
$(x+2)' = 1$
$(\log x)' = \frac{1}{x}$
$(\log f(x))' = \frac{f'(x)}{f(x)}$
$\downarrow$
chain rule

for example:

$$y = \sin x \cdot e^x \cdot (x+2)$$

$$\Rightarrow \log y = \log(\sin x \cdot e^x \cdot (x+2))$$

$$\boxed{\log y = \log \sin x + \underbrace{\log e^x}_{x} + \log(x+2)}.$$

Now, differentiate:

$$\frac{y'}{y} = \frac{\cos x}{\sin x} + 1 + \frac{1}{x+2}$$

$$\Rightarrow \frac{y'}{y} = \cot x + 1 + \frac{1}{x+2}$$

$$\Rightarrow \boxed{y' = y\left[\cot x + 1 + \frac{1}{x+2}\right]}$$

$y = (x+1)(x+2)(x+3)$, ⑥

$\log mn = \log m + \log n$

take the log

$\Rightarrow \log y = \log(x+1) + \log(x+2) + \log(x+3)$

$$\Rightarrow \frac{1}{y} \cdot y' = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3}$$

$$\boxed{y' = y\left[\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3}\right]}$$

$y = (x^2+x)(x^2-x)(\sin x^2)x$

$= (x^4 - x^2)(\sin x^2) \cdot x$, take the log

$\log y = \log(x^4 - x^2) + \log(\sin x^2) + \log x$

$$\Rightarrow \frac{y'}{y} = \frac{(x^4 - x^2)'}{x^4 - x^2} + \frac{(\sin x^2)'}{(\sin x^2)} + \frac{1}{x}.$$

$$\frac{y'}{y} = \frac{(4x^3 - 2x)}{x^4 - x^2} + \frac{\cos x^2 \cdot (x^2)'}{\sin x^2} + \frac{1}{x}$$

chain rule. ////.

§2.7 Related Rates

Recall: For a given function $f \rightarrow \frac{df}{dx}$ or $f'(x)$ is the rate of change of f w.r.t. x.

In our example 1 @ 129 $f \equiv$ volume of a sphere

$$V(r) = V = \frac{4}{3}\pi r^3$$

Given: volume ↑ @ rate 100 cm³/sec.

$$\therefore \frac{dV}{dt} = 100$$

$\because V = \frac{4}{3}\pi r^3$, differentiating w.r.t 't'

$$\Rightarrow \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{4}{3}\pi \frac{d}{dt}(r^3) = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$\downarrow$ given

$$\Rightarrow 100 = 4\pi r^2 \frac{dr}{dt}$$

if diameter = 50

$\Rightarrow r = 25$

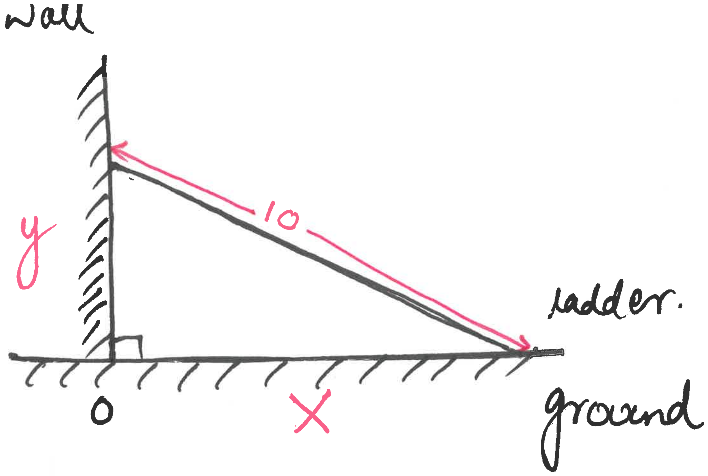
$$\Rightarrow 100 = 4\pi \cdot 25^2 \left(\frac{dr}{dt}\right) \rightarrow ?$$

$4 \times 25 = 100$

$$\Rightarrow 1 = \pi \cdot 25 \frac{dr}{dt} \Rightarrow \boxed{\frac{dr}{dt} = \frac{1}{25\pi}}$$

$$\Rightarrow \boxed{\frac{dr}{dt} = \frac{1}{25\pi} \text{ cm/s}}$$

$\rightarrow$ rate of change of ~~volume~~ radius w.r.t t when radius = 25cm.



$$36 + y^2 = 100 \Rightarrow y^2 = 64 \Rightarrow y = 8$$

$$x^2 + y^2 = 10^2$$

$$\boxed{x^2 + y^2 = 100} \quad (1)$$

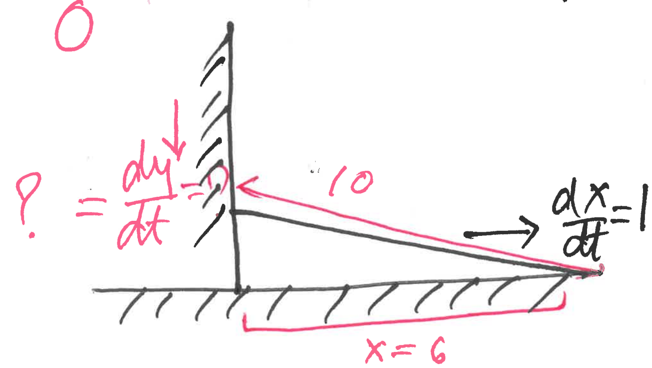
from the problem, $\frac{dx}{dt} = 1\ ft/sec.$

differentiate (1) w.r.t time 't'

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(\cancel{100}) \quad 0$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

(slipped)



$$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}}$$

→ This is your general rate of change of y w.r.t 't' in ~~terms~~ of x, y and $\frac{dx}{dt}$.

Recall $\dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt}$

plug $x = 6$, $y = 8$, $\dfrac{dx}{dt} = 1$

$\Rightarrow \dfrac{dy}{dt} = -\left[\dfrac{6}{8}\right]\cdot 1 = -\dfrac{3}{4}$ ft/sec.

Application of continuity

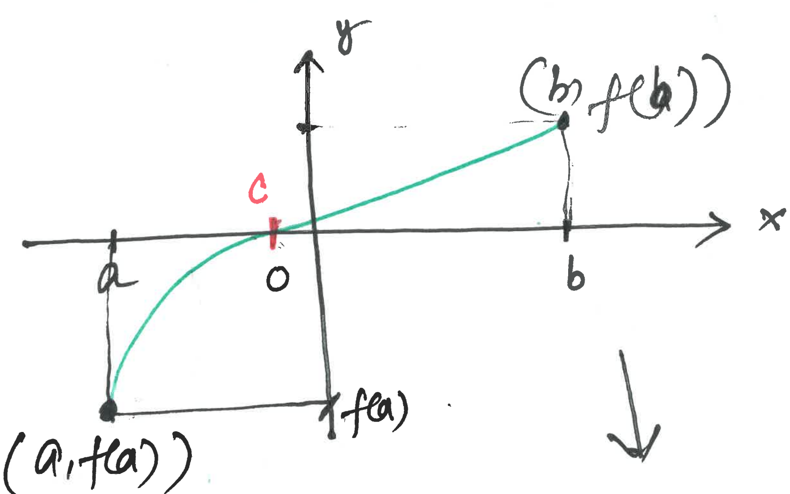
my assumption

① f is cont. on $[a, b]$

② $f(a) < 0$

③ $f(b) > 0$

$\hookrightarrow f(a) f(b) < 0$



application of intermediate value theorem.

$\boxed{f(c) = 0.}$

c is the root or zero of f.

IVT / @ 52: suppose that $f \in C[a,b]$ (5)
and let $N \in [f(a), f(b)]$. (N is in b/w $f(a)$ and $f(b)$) where $f(a) \neq f(b)$. Then $\exists$ a number 'c' $\in [a,b]$ $\ni$

$$f(c) = N.$$

Reading assignment: Ex-8 @ 53

#37 @ 55: $f(x) = x^2 + 10 \sin x$

claim: $\exists$ a 'c' $\ni$ $f(c) = 1000$.

Proof: if $f(c) = 1000 \Rightarrow c^2 + 10 \sin c = 1000$.

$$\Leftrightarrow c^2 + 10 \sin c - 1000 = 0.$$

Say $g(x) = x^2 + 10 \sin x - 1000$.

AIM Simply find a and b $\ni$ such that $g(a)\,g(b) < 0$.

Think of $a = 0 \Rightarrow g(a) = g(0) = 0 + 0 - 1000$
$= < 0$.

think of $b = 100$, (a)

$$g(b) = g(100) = 100^2 + 10\sin 100 - 1000$$

$$= > (+ve)$$

$\because g(a)\,g(b) = g(0)\,g(100) < 0.$

$\exists$ 'c' is $g(c) = 0$

$\Rightarrow c^2 + 10\sin c - 1000 = 0$

$\Rightarrow \boxed{c^2 + 10\sin c = 1000}$ ////.

9:05 - 10:20 AM

21st Feb 2024 ①

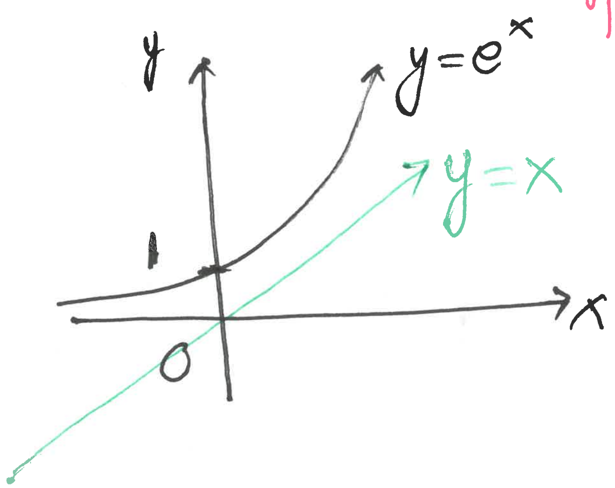
Wednesday

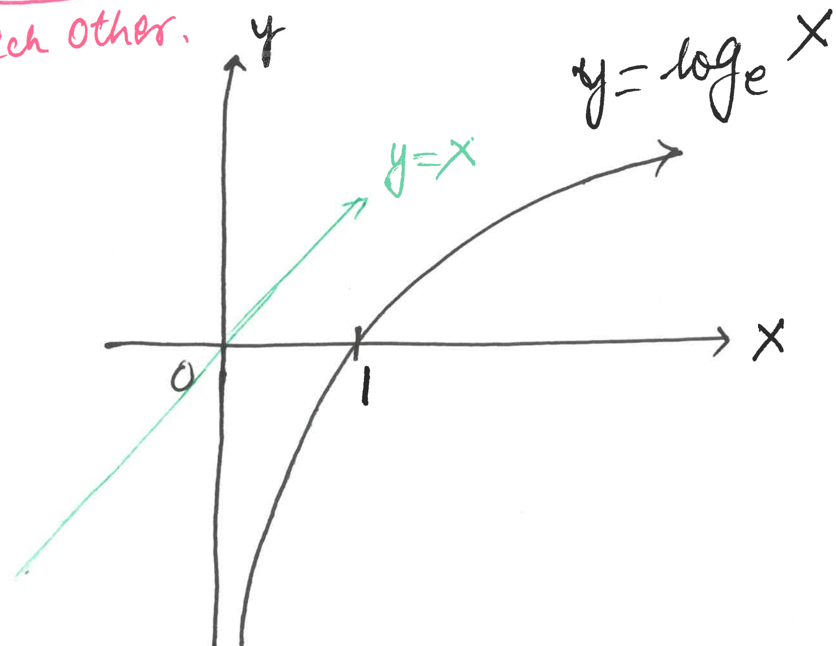
chapter-3 (§ 1, 2, 3)

Recall: Exponential functions:

say: $\boxed{y = e^x} \iff \boxed{\log_e y = x}$

inverse of each other.





what is e? $\boxed{e = \lim_{x \to 0}\left[(1+x)^{1/x}\right]}$

and $e = 2.718\ldots$

one should know that

$$e^x = \underbrace{1 + \frac{x}{1!}}_{\text{are 2 @ } x=1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Claim : $\lim_{h \to 0} \left[\frac{e^h - 1}{h} \right] = 1$

Proof : $e^h = 1 + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots$

$$\Rightarrow e^h - 1 = h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots$$

$$\Rightarrow \frac{e^h - 1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots$$

Thus, $\boxed{\lim_{h \to 0} \left[\frac{e^h - 1}{h} \right] = 1}$ ——— (1)

we will use (1) to demonstrate $\frac{d}{dx}(e^x) = e^x$.

This is done as follows:

Recall :

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right].$$

say $f(x) = e^x$, then

$$(e^x)' = \lim_{h\to 0}\left[\frac{e^{x+h} - e^x}{h}\right]$$

note $e^{x+h} = e^x e^h$

$$= \lim_{h\to 0}\left[\frac{e^x e^h - e^x}{h}\right] = e^x \underbrace{\lim_{h\to 0}\left[\frac{e^h - 1}{h}\right]}_{1}$$

$\therefore\ (e^x)' = e^x \cdot 1 = e^x.$

Problem 15 @ 150 find domain of $f(x) = \dfrac{1-e^{x^2}}{1-e^{1-x^2}}$.

Sol: For domain of f, put $1-e^{1-x^2} = 0$.

$\Rightarrow e^{1-x^2} = 1. \Rightarrow (1-x^2)\log_e e = \overbrace{\log_e 1}^{0}$

Remember $\log_B B = 1$ $\Rightarrow (1-x^2)\cdot 1 = 0$

thus, $x^2 = 1 \Rightarrow x = \pm 1.$

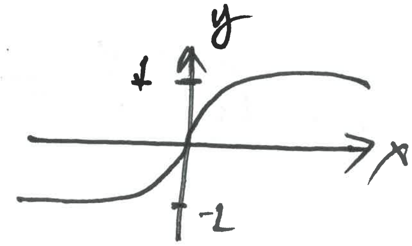
Therefore, $\mathrm{dom}(f) = \mathbb{R}\setminus\{\pm 1\}.$

Remember: $\left.\begin{aligned} \lim_{x\to\infty} e^{-x^2} &= 0 \text{ and} \\ \lim_{x\to\infty} e^{x^2} &= \infty \end{aligned}\right\}$.

Note: Above result are also valid for x instead of x^2 in the argument of $e^{\square}$.

$\boxed{\text{@ 149 in book}}$

#25@150: $\displaystyle \lim_{x\to\infty}\left[\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}\right]$



$$\sigma(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \qquad \left[\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \sigma(3x)\right]$$

SIGMOID FUNCTIONS

↳ used in NN as an AF

↳ DCNN.

$$\lim_{x\to\infty}\left[\frac{e^{3x}/e^{3x} - e^{-3x}/e^{3x}}{e^{3x}/e^{3x} + e^{-3x}/e^{3x}}\right] = \lim_{x\to\infty}\left[\frac{1-e^{-6x}}{1+e^{-6x}}\right]$$

$$= 1.$$

$$\boxed{e^{-3x}/e^{3x} = e^{-3x-3x} = e^{-6x}}$$

Recall: $y = e^x$

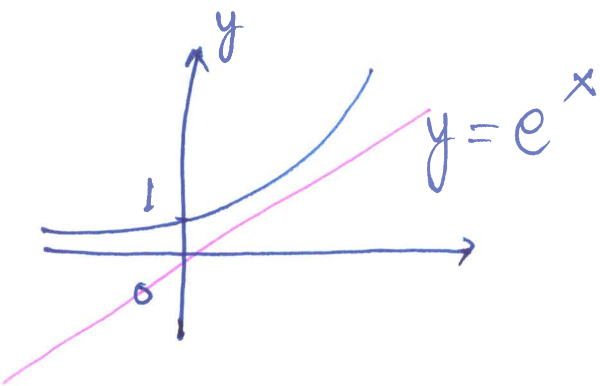
"exponential function"

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

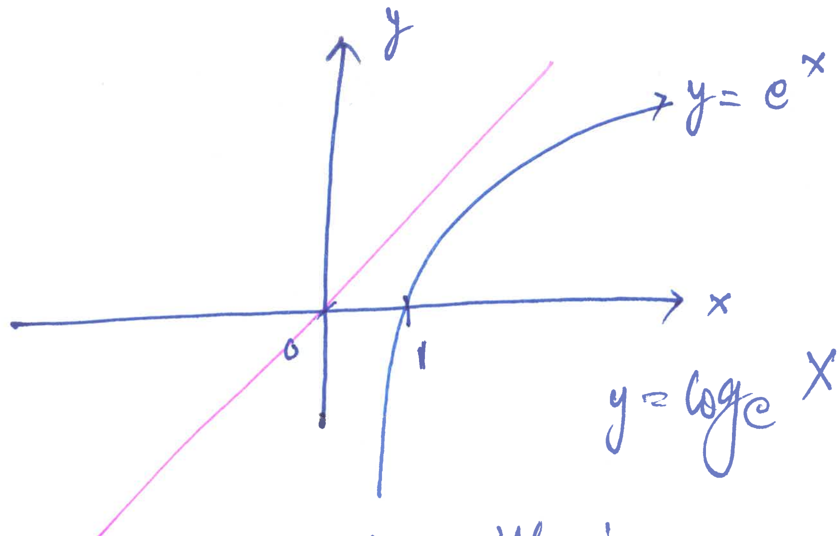
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

inverse of an exponential function

is $y = \log_e X = \log X \equiv \ln X$.



exponential



logarithmic

for example: ex 159 solve the eqn $e^{5-3x} = 10$.

solution: take the "log"

$\Rightarrow \log_e[e^{5-3x}] = \log_e 10$.

Remember $\log_A(A^{\alpha}) = \alpha$; ——— ①

$\log_A A = 1.$

$\log(mn) = \log m + \log n$

$\log\left(\frac{x}{y}\right) = \log x - \log y$

---

by ① $(5-3x)\log_e e = \log 10$

$\Rightarrow \quad 5-3x = \log_e 10$

$\Rightarrow \quad \boxed{\frac{5-\log 10}{3} = x}$

---

page 163 (65) ⓐ $2^{x-5} = 3$

① $\log_2 2^{x-5} = \log_2 3$ (log at base 2)

$\Rightarrow (x-5) = \log_2 3 \Rightarrow \boxed{x = 5 + \log_2 3}$

2 observe $2^{x-5} = 3 \Rightarrow \dfrac{2^x}{2^5} = 3$

$\Rightarrow 2^x = 2^5 \cdot 3$ (take the $\log_2$)

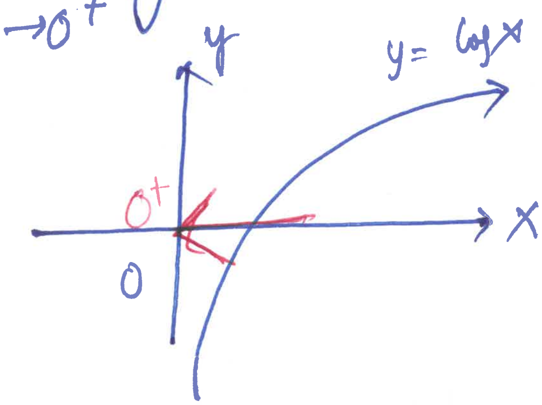
$\Rightarrow \log_2 2^x = \log_2(2^5 \cdot 3)$

$\Rightarrow x = \underbrace{\log_2 2^5} + \log_2 3$

$\boxed{x = 5 + \log_2 3}$ ✓

Recall: $\lim\limits_{x\to\infty} e^x = \infty \quad \| \quad \lim\limits_{x\to\infty} e^{-x} = 0.$

$\lim\limits_{x\to\infty} \log_{\square} x = \infty \quad \| \quad \lim\limits_{x\to 0^+} \log_{\square} x = -\infty.$



Page 162

26 $y = \dfrac{e^x}{1+2e^x}$ ← (find the inverse of this function)

put $e^x = R$, then

$\Rightarrow \quad y = \dfrac{R}{1+2R}$ (isolate R)

$\Rightarrow (1+2R)\, y = R$ (think of keeping R terms separate from y).

$\Rightarrow (1+2R) = \dfrac{R}{y}$

$\Rightarrow \dfrac{1+2R}{R} = \dfrac{1}{y}$ (divide by R).

$\Rightarrow \dfrac{1}{R} + 2 = \dfrac{1}{y}$

$\Rightarrow \dfrac{1}{R} = \dfrac{1}{y} - 2 = \dfrac{1-2y}{y}$

$\Rightarrow$ (if) $\dfrac{1}{R} = \dfrac{1-2y}{y} \Rightarrow R = \dfrac{y}{1-2y}$

flip it!

Recall: $R = \frac{y}{1-2y}$ and $\because R = e^x$ ⑤

$\Rightarrow \quad e^x = \frac{y}{1-2y}$

$\Rightarrow \quad \log_e e^x = \log\left(\frac{y}{1-2y}\right)$

$$\boxed{x = \log_e\left(\frac{y}{1-2y}\right)}$$

---

§3.3: Recall HW 4

$$P(x) = (x+b_1)(x+b_2)\cdots(x+b_n)$$

$$= \prod_{i=1}^{n}(x+b_i)$$

find $\frac{P'}{P}$.

solution : Recall : $P(x)=(x+b_1)(x+b_2)\cdots(x+b_n)$. 6

First way : differentiate P to get $P'(x)$ and then divide $P'(x)$ by $P(x)$ to have $P'(x)/P(x)$.
(! TEDIOUS WORK (AVOID))

second way : take "log" of $P(x)$ and then differentiate

$$\log P(x) = \log(x+b_1) + \log(x+b_2) + \cdots + \log(x+b_n)$$

Remember $(\log x)' = \frac{1}{x}$

$$(\log f(x))' = \frac{f'(x)}{f(x)}.$$

therefore

$$\boxed{\frac{P'(x)}{P(x)} = \frac{1}{x+b_1} + \frac{1}{x+b_2} + \cdots + \frac{1}{x+b_n}}$$

⑦

$\mathrm{dom}(P'/P) = (-\infty, -b_n) \cup (-b_n, -b_{n-1}) \cup \dots \cup (-b_1, \infty)$

$b_1 < b_2 < b_3 < \dots < b_n$ ← make an assumption.

AVOID

simply say $\mathrm{dom}(P'/P) = \mathbb{R} \setminus \{-b_j\}_{j=1}^{n}$.

$= \mathbb{R} \setminus \{-b_1, -b_2, \dots, -b_n\}$.

9:05 - 10:20 AM

Monday (1)

4th March 2024

#7 4@ 70: evaluate $\lim_{x \to \pi} \left[\frac{e^{\sin x} - 1}{x - \pi}\right]$? $(= -1)$.

Solution (try 1) substitute: $\frac{e^{\sin \pi} - 1}{\pi - \pi} = \frac{1-1}{\pi - \pi} = \frac{0}{0}$.

Remember: $\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$ — (1)

If we compare (1) with $\lim_{x \to \pi} \left[\frac{e^{\sin x} - 1}{x - \pi}\right]$

then, we have $a = \pi$ & $f(x) = e^{\sin x}$.

Thus, $f'(x) = \underbrace{\cos x}_{\frac{d}{dx}\sin x} \cdot \left(e^{\sin x}\right)$.

Now, $f'(a) = f'(\pi) = \cos \pi \cdot e^{\sin \pi}$

$= -1 \cdot 1 = -1$.

§ 3.4 Exponential Growth & decay

$k \in \mathbb{R}$. $t \to$ time

$$\frac{dy}{dt} = \underbrace{ky}_{\text{linear function.}} \quad \text{——— } ①$$

$$\Rightarrow \underbrace{\int \frac{dy}{ky} = \int dt}_{} \Rightarrow \boxed{y = y(0)e^{kt}}$$ or more

OVERLOOK @1 of 4th March 2024.

clearly $\boxed{y(t) = y(0)e^{kt}}$

$y(0) \to$ initial value of y

lets change the scenario in ① by including an affine function

$$\Rightarrow \frac{dy}{dt} = \overbrace{ky + b}^{\text{affine function}}$$

$$\boxed{y(t) = y(0)e^{kt} - \frac{b}{k}}$$

Important: $\frac{dy}{dt} = \boxed{\dot{y} = f(y)} \longrightarrow$ DYNAMICAL SYSTEM.

Newton's 2nd law of motion: $F \propto a$ ($F = ma$) (3)

but, however we know that, a displacement denoted by x, and its derivative w.r.t t, will yield 'v', i.e.

$$\frac{dx}{dt} = v \text{ --- } (1)$$

$$\frac{dv}{dt} = a \text{ --- } (2)$$

$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= a \end{aligned}}$$

Ex: 1 @ 172

$t=0$	$t=10$	$t=43$
1950	1960	1993

Given Population growth is directly proportional ($\propto$) to the the population. Therefore

$$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP \quad (k \text{ is some constant})$$

$$\therefore \boxed{P(t) = P(0)e^{kt}} \text{ --- } (A)$$

→ this will help in modelling the population problem.

Also, given in 1960, P = 3040 & in 1950, P = 2560. Therefore with this and

(*.)

$$\underbrace{P(\underset{\downarrow}{10})}_{t=10} = 3040 \quad \text{and} \quad P(t) = \underbrace{2560}_{P(0)} e^{kt}.$$

AIM → solve for k:

For this; observe that $\boxed{3040 = 2560\, e^{k \cdot 10}}$.

$$\Rightarrow e^{10k} = \left(\frac{3040}{2560}\right) \quad ;$$

$$\Rightarrow 10k = \ln\left(\frac{3040}{2560}\right) \qquad \text{(take logs)}$$

$$\Rightarrow \boxed{k = \frac{1}{10}\ln\left(\frac{3040}{2560}\right)} \quad \{\approx 0.017185\}.$$

∴ This simply means that

$$P(t) = 2560\, e^{kt} \quad \text{where } k = \frac{1}{10}\ln\left(\frac{3040}{2560}\right).$$

for 1993, $t = 43$. therefore

$$\boxed{P(43) = 2560\, e^{k \cdot 43}} \quad \text{where } k = \frac{1}{10}\ln\left(\frac{3040}{2560}\right).$$

≈ 5360 million.

Fact about $\dfrac{dy}{dt} = ky$

$y(t)$

① if $k > 0$, then the system is of exponential growth. $\cancel{(k>0)}$

② if $k < 0$ (negative), then the system is of exponential decay. $\cancel{(k<0)}$

③ if $k = 0$, (0), then the system is (identically) constant; meaning $y(t) \equiv c$ (c is constant).

example: population growth corresponds to $k > 0$. and radioactive decay corresponds to $k < 0$.

ex-2 : start with $\underbrace{m(t)}_{\text{mass of } {}^{226}_{88}Ra \text{ w.r.t } t} = m(0)\, e^{kt}$

@ 173 (6)

Through half life data of ${}^{226}_{88}Ra$, we have $t = 1590$ and.

$$\underbrace{100}_{\substack{\text{initial mass} \\ m(0)}} e^{k\cancel{t}590} = \underbrace{50}_{\text{final mass due to half-life } (\equiv 1590)}$$

$$\Rightarrow \quad e^{k \cdot 1590} = \frac{1}{2}$$

$$\Rightarrow \quad k \cdot 1590 = \ln(1/2) \qquad (-ve)$$

$$\Rightarrow \quad \boxed{k = \frac{\cancel{1590}\ 1}{1590} \ln\left(\frac{1}{2}\right)}$$

$\therefore$ In general $m(t) = 100\, e^{kt}$ where $k = \frac{1}{1590} \ln(1/2)$.

⑦

Recall: $m(t) = 100\, e^{kt}$ where $k = \frac{1}{1590}\ln(1/2)$.

$(k < 0)$

(b) $m(1000) = 100\, e^{k \cdot 1000}$ where $k = \frac{1}{1590}\ln(1/2)$

(c) when $(t?)$ mass be 30 mg. from the model eqn

$\Rightarrow \quad 30 = 100\, e^{k \cdot t} \quad \left(k = \frac{1}{1590}\ln(1/2)\right)$

$\Rightarrow \quad e^{kt} = \frac{3}{10}$

$\Rightarrow \quad \boxed{t = \frac{1}{k} \log_e\left(\frac{3}{10}\right)}$ "ln."

we know, $k = \frac{1}{1590}\ln(1/2)$.

≈ 2762.0 yrs.

9:05–10:20 AM

6th Mar 2024 (1)
Wednesday

§ Section 1-7 of chapter-3

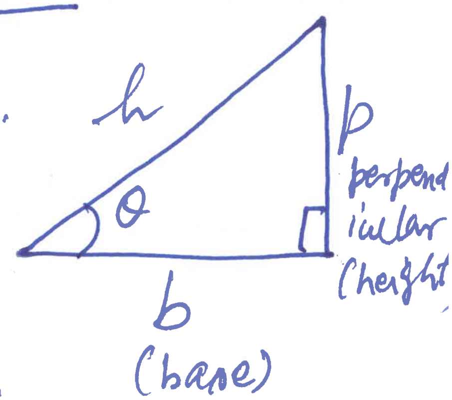
$$\frac{dy}{dt} = ky \Rightarrow y(t) = y(0)\,e^{kt}$$

$\left\{ \text{if } \begin{array}{l} k > 0 \rightarrow \text{exponential growth} \\ k < 0 \rightarrow \text{exponential decay.} \end{array} \right.$

---

§ 3.5: Inverse trigonometric functions

$\sin\theta = \dfrac{p}{h}$ $\qquad \cos\theta = \dfrac{b}{h}$ $\qquad \tan\theta = \dfrac{p}{b}$.



$\csc\theta = \dfrac{h}{p}$ $\qquad$ ~~sec~~ $\sec\theta = \dfrac{h}{b}$ $\qquad \cot\theta = \dfrac{b}{p}$.

Inverse trigonometric functions, we have

$\sin^{-1}$; $\cos^{-1}$; $\tan^{-1}$; $\csc^{-1}$; $\sec^{-1}$;

$\cot^{-1}$; (older times → arc sin; arc cos, ...)

if: $\sin \boxed{y} = x \Longrightarrow \boxed{\sin^{-1} x} = \boxed{y}$ → angle

angle. $\qquad$ angle.

Note: $\sin^{-1} x = y \iff \sin y = x$ and $-\pi/2 \le y \le \pi/2$.

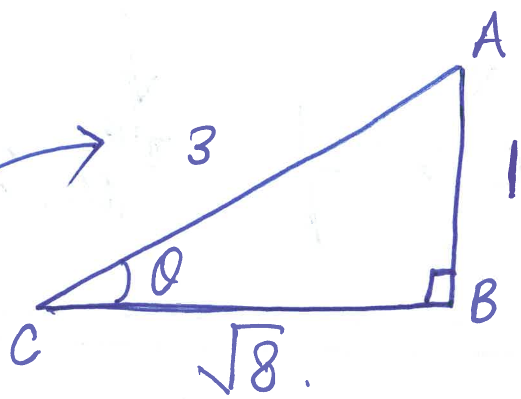
$\therefore$ (because) $-1 \le x \le 1$.

For example: if we have $\sin^{-1}(1/2)$ then $\sin^{-1}(1/2) = \pi/6$ as $\sin \pi/6 = \sin 30^\circ = 1/2$.

? $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta \Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$; $\boxed{\theta = \frac{\pi}{4}}$.

? $\tan(\sin^{-1} 1/3) = ?$

(book @ 179 ; Ex-1 (b)).



$\sin\theta = \frac{1}{3}$.

say $\sin^{-1} 1/3 = \theta \Rightarrow \sin\theta = \frac{1}{3}$

Now, by Pythagorean theorem, $BC^2 = AC^2 - AB^2$
$= 9 - 1$
$= 8$,

therefore $BC = \sqrt{8}$. Then,

$\tan(\sin^{-1} 1/3) = \tan(\theta) = \frac{1}{\sqrt{8}}$.

In the same way, the inverse of 'cos' will also be handled. That is: (3)

$$\boxed{\cos^{-1} X = y \iff \cos y = X \text{ and } 0 \le y \le \pi.}$$

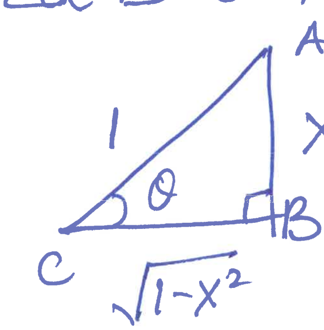
Because $-1 \le X \le 1$.

Exercise-7 @ 183 $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.

Solution: take $\sin^{-1} x = \theta \Rightarrow x = \sin\theta$.

with $\sin\theta = x = \frac{x}{1}$, we have right $\angle$d $\triangle ABC$,

with $AB = x$; $\underbrace{BC = \sqrt{1-x^2}}_{\text{Pythagorean theorem.}}$ and $AC = 1$.



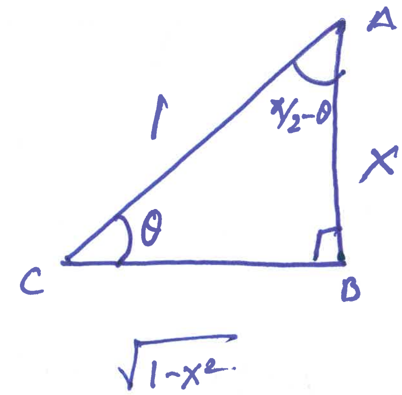
LHS: $\cos(\sin^{-1} x) = \cos\theta$.

RHS: $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$.

Exercise 12 (a) : $\boxed{\sin^{-1} x + \cos^{-1} x = \pi/2}$.

183

Solution : take $\boxed{\sin^{-1} x = \theta} \Rightarrow x = \sin\theta$. (1)



consider $\cos\left(\frac{\pi}{2} - \theta\right) = \frac{x}{1}$.

$\Rightarrow \boxed{\frac{\pi}{2} - \theta = \cos^{-1} x}$ (2)

add (1) and (2)

$\sin^{-1} x + \cos^{-1} x = \theta + \left(\frac{\pi}{2} - \theta\right)$

$= \pi/2$.

add θ and $\frac{\pi}{2} - \theta$

$\Rightarrow \cancel{\theta} + \frac{\pi}{2} - \cancel{\theta} = \frac{\pi}{2}$.

Derivative of $\sin^{-1} x$. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

and $\operatorname{dom}\left(\frac{d}{dx}\sin^{-1} x\right) = \{-1 < x < 1\}$.

Derivative of $\cos^{-1} x$ $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$.

and $\operatorname{dom}\left(\frac{d}{dx}(\cos^{-1} x)\right) = -1 < x < 1$.

Ex-2 @180 | we have $f(x) = \sin^{-1}(x^2-1)$.

(a) dom f. ?

Solution : we know that domain of $\sin^{-1}x$ is $[-1, 1]$. Therefore.

$$\checkmark\ -1 \le x^2-1 \le 1 .$$

$\left\{ \begin{array}{l} \text{put } \sin^{-1}(x^2-1)=\theta \\ \sin\theta = x^2-1. \end{array} \right.$

$$\Rightarrow \quad 0 \le x^2 \le 2$$

$$\Rightarrow \quad \boxed{x \in [-\sqrt{2}, \sqrt{2}]} \equiv \text{dom}\left(\sin^{-1}(x^2-1)\right)$$

(b) find f'(?)

$$f(x) = \sin^{-1}(\underline{x^2-1}).$$

$$\left\{ \frac{d}{dx} \sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \right\}$$

$$f'(x) = \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot (x^2-1)'$$

$$= \frac{1}{\sqrt{1-(x^4+1-2x^2)}} \cdot 2x = \frac{2x}{\sqrt{\cancel{1}-x^4-\cancel{1}+2x^2}}$$

$$f'(x) = \frac{2x}{\sqrt{2x^2 - x^4}}.$$ 6

why this step does not make sense!

$$f'(x) = \frac{2x}{\sqrt{x^2(2-x^2)}} = \frac{2\cancel{x}}{\cancel{x}\sqrt{2-x^2}} = \frac{2}{\sqrt{2-x^2}}$$

$\operatorname{dom}(f') = 2 - x^2 = 0 \Rightarrow x = \pm\sqrt{2}$

$x \in (-\sqrt{2}, \sqrt{2}) \approx (-\sqrt{2}, 0] \cup [0, \sqrt{2})$.

<u>Reason</u>: $f = \sin^{-1}(x^2-1)$ is not differentiable at $x = 0$. Therefore, we should not include $x = 0$ in the domain of f'. Hence

$$\operatorname{dom}(f') = (-\sqrt{2}, 0) \cup (0, \sqrt{2}).$$

9:05–10:20 AM

8th March 2024 (1)

friday

§3.5 Inverse TRIGONOMETRIC FUNCTION.

✓ $\sin^{-1}x$ ✓ $\cos^{-1}x$ ✓ $\tan^{-1}x$

✓ $\csc^{-1}x$ ✓ $\sec^{-1}x$ ✓ $\cot^{-1}x$.

- definitions ✓
- Derivatives

$$\left.\begin{aligned}
(\sin^{-1}x)' &= \frac{1}{\sqrt{1-x^2}} \\
(\cos^{-1}x)' &= \frac{-1}{\sqrt{1-x^2}} \\
(\tan^{-1}x)' &= \frac{1}{1+x^2} \\
(\csc^{-1}x)' &= \frac{-1}{x\sqrt{x^2-1}} \\
(\sec^{-1}x)' &= \frac{1}{x\sqrt{x^2-1}} \\
(\cot^{-1}x)' &= -\frac{1}{1+x^2}
\end{aligned}\right\}\ \text{page @ 183}$$

Problem 16 @ 184 find y' if $y = \tan^{-1}(x^2)$.

solution: $y' = \frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x^2)$.

$(\tan^{-1} x_1)' = \frac{1}{1+x_1^2}$

chain rule

$= \frac{1}{1+(x^2)^2} \cdot (x^2)'$

$= \frac{2x}{1+x^4}$.

Problem 17 @ 184: find y' if $y = (\tan^{-1}x)^2$

solution: $y' = 2(\tan^{-1}x)^{2-1} \cdot (\tan^{-1}x)'$

$= \frac{2\tan^{-1}x}{1+x^2}$.

Problem 25 @ 184: find y' if $y = \tan^{-1}(\cos\theta)$

solution: $y' = \frac{1}{1+\cos^2\theta} \cdot (\cos\theta)' = \frac{-\sin\theta}{1+\cos^2\theta}$.

Remember : $\lim_{x \to \pm\infty} \tan^{-1} x = \pm \frac{\pi}{2}$. $\left\{ \begin{array}{l} \because \tan(+\pi/2) = +\infty \\ \tan(-\pi/2) = -\infty . \end{array} \right.$ (3.)

#37 @184 $\lim_{x \to \infty} \tan^{-1}(e^x) = ?$

solution : Observe that as $x \to \infty$, $e^x \to \infty$.

therefore, $\lim_{x \to \infty} \tan^{-1}(e^x) = \pi/2$.

#36 @184 $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

solution : Observe that as $x \to 0^+$, $\ln x \to -\infty$.

$\therefore \lim_{x \to 0^+} \tan^{-1}(\ln x) = -\pi/2$.

#28 @184 : find y', if $y = \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)$.

solution (1.) $y = \tan^{-1}\left[\left(\frac{1-x}{1+x}\right)^{1/2}\right]$.

Recall: $y = \tan^{-1}\left[\underbrace{\left(\frac{1-x}{1+x}\right)^{1/2}}_{X}\right]$

$\Rightarrow y' = \frac{1}{1+X^2} \cdot (X')$; Now, we will determine X'.

chain rule.

$$X' = \frac{d}{dx}\left[\left(\frac{1-x}{1+x}\right)^{1/2}\right].$$

chain rule.

$$= \frac{1}{2}\left[\left(\frac{1-x}{1+x}\right)^{1/2-1}\right] \cdot \left(\frac{1-x}{1+x}\right)'.$$

quotient rule

$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

$$= \frac{1}{2}\left[\left(\frac{1+x}{1-x}\right)^{1/2}\right] \cdot \frac{(1+x)(-1) - 1(1-x)}{(1+x)^2}$$

$$= \frac{1}{\cancel{2}}\left[\frac{1+x}{1-x}\right]^{1/2}\left(\frac{-\cancel{1}-\cancel{x}-\cancel{1}+\cancel{x}}{(1+x)^2}\right)$$

$$= -\frac{(1+x)^{1/2}}{(1-x)^{1/2}} \cdot \frac{1}{(1+x)^2} \left(\equiv X'\right).$$

and $\dfrac{1}{1+X^2} = \dfrac{1}{1+\frac{1-X}{1+X}} = \dfrac{1}{\frac{1+X+1-X}{1+X}}$

$\because X = \sqrt{\dfrac{1-X}{1+X}}$

$= \dfrac{1+X}{2}.$

Therefore, $\dfrac{dy}{dx} = y' = \dfrac{(1+X)}{2} \cdot \left\{ \dfrac{-(1+X)^{1/2}}{(1-X)^{1/2}} \cdot \dfrac{1}{(1+X)^2} \right\}$

$= -\dfrac{1}{2(1-X)^{1/2}} \cdot \dfrac{1}{(1+X)^{1/2}}$

$= \dfrac{-1}{2\sqrt{(1-X)(1+X)}} = \dfrac{-1}{2\sqrt{1-X^2}}.$

Solution ②: put $X = \cos 2\theta$ (via trigonometric substitution)

$\dfrac{dx}{d\theta} = -2\sin 2\theta = -2\sqrt{1-X^2}.$

also, $\dfrac{d\theta}{dX} = \dfrac{-1}{2\sqrt{1-x^2}}.$

Now, $y = \tan^{-1}\sqrt{\dfrac{1-X}{1+X}} = \sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$ $\left(\because X = \cos 2\theta\right.$

(a)

$$y = \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan^{-1}\sqrt{\frac{\cancel{\cos^2\theta} + \sin^2\theta - \cancel{\cos^2\theta} + \sin^2\theta}{\cos^2\theta + \cancel{\sin^2\theta} + \cos^2\theta - \cancel{\sin^2\theta}}}$$

Remember

$\cos 2\theta = \cos^2\theta - \sin^2\theta$.

$$= \tan^{-1}\sqrt{\frac{\cancel{2}\sin^2\theta}{\cancel{2}\cos^2\theta}}$$

$$= \tan^{-1}\sqrt{\tan^2\theta}$$

$\sin^{-1}(\sin x) = x$

$\cos^{-1}(\cos x) = x$

$\tan^{-1}\tan x = x$

$$= \tan^{-1}\tan\theta$$

$$y = \theta$$

$$\frac{dy}{dx} = \frac{d\theta}{dx} = \frac{-1}{2\sqrt{1-x^2}} \quad ////.$$

9:05 – 10:20 AM

18th March 2024 (1)

Monday.

§ 3.7 ~~XXXX~~ L-Hospital Rule

suppose, $\lim_{x \to a} \dfrac{f(x)}{g(x)} \longrightarrow ?$

$g(x) \neq 0$ @ $x = a$.

however, we also observe that

@ $x = a$ either $\dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ or $\dfrac{f(x)}{g(x)} = \dfrac{\infty}{\infty}$.

if $\lim_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists, then → very important!

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

Ex-1 @ 193 find $\lim_{x \to 1} \dfrac{\log x}{x - 1}$.

sol. clearly, $\left.\dfrac{\log x}{x-1}\right|_{x=1} = \dfrac{\log 1}{1-1} = \dfrac{0}{0}$ form.

Employ the L-Hospital rule,

$$\lim_{x \to 1} \frac{\log x}{x-1} = \lim_{x \to 1} \frac{(\log x)'}{(x-1)'}$$

$$= \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x}$$

$$= 1.$$

<u>EX-2</u> $\lim_{x \to \infty} \frac{e^x}{x^2} \equiv \frac{\infty}{\infty}$

<u>Note</u> as $x \to \infty$, $e^x \to \infty$ and as $x \to \infty$, $x^2 \to \infty$.

Now, apply the L-hospital rule

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \underbrace{\lim_{x \to \infty} \frac{e^x}{2x}}_{\left(\frac{\infty}{\infty}\right)} \overset{\text{again L-Hospital rule}}{=} \lim_{x \to \infty} \frac{e^x}{2} = \frac{\infty}{2}$$

$$= \infty.$$

consider, $\lim_{x \to \infty} \frac{x^2}{e^x} = \left(\frac{\infty}{\infty}\right)$! (important) ③

$$= \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0.$$

⊗ CAUTION EX-5 @194 $\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x}$

check by L-Hospital rule

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi} \frac{\cos x}{-(-\sin x)} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x}$$

$$= \frac{-1}{0} = -\infty.$$

limit do not exist.

However, simple substitution of $x = \pi$,

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \cancel{\frac{\sin \pi}{1 - (+1)}} \quad \frac{\sin \pi}{1 - (\cos \pi)}$$

$$= \frac{0}{1 - (-1)} = \frac{0}{2} = 0.$$

Ex-195 #6 | find $\lim_{x \to 0^+} x \log x$.

Solution: $\lim_{x \to 0^+} x \log x = \lim_{x \to 0^+} \frac{\log x}{\left(1/x\right)} = \left(\frac{-\infty}{\infty}\right)$

$\boxed{\left(\frac{1}{x}\right)' = ?}$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0^+} -\frac{x^2}{x} = \lim_{x \to 0^+} (-x)$$

$$= 0.$$

Indeterminate powers $\lim_{x \to a} [f(x)]^{g(x)}$.

- 0^0
- ∞^0
- 1^∞.

for example $y = x^x$ with $x \to 0^+$.

Refer ex-9 @ 196 :

Recall ~~...~~ evaluate $\lim_{x\to 0^+} x^x$.

(Ex-9 @ 196).

Solution: put $y = x^x$. (with $\lim_{x\to 0^+}$).

$\Rightarrow \log y = x \log x$

"apply the limit"

$\Rightarrow \underbrace{\lim_{x\to 0^+} \log y}_{} = \lim_{x\to 0^+} x \log x$.

$\log y = 0$ (previous example).

$\Rightarrow$ $y = 1$.

In chapter-3
1. Exponential functions
2. Inverse of exp. $\rightsquigarrow$ log
3. $(\log x)'$; (logarithmic diff.)
4. exponential growth and exponential decay.

(5) Inverse trig functions (Q)

(6) L'Hospital rule and indeterminate forms.

Chapter-4 Application of Differentiation.

§4.1: max/min problem.

Goal: find the max/min of f.

You should know "how to find critical numbers of f".

↳ Those values of x @ $f'(x)=0$.

Let's discuss in detail:

(1) Absolute max / absolute min

(also called as global max / global min)

'c' is called Absolute max value of f, if

$f(c) \geq f(x) \quad \forall \; x \in D$ → domain of f.

(similarly) absolute min is given by 'c' ⑦
if $f(c) \leq f(x) \quad \forall x \in D$.

Critical numbers: A critical number 'c' for a function f satisfies either $f'(c) = 0$
or $f'(c)$ does not exist.

Algorithm to find absolute max / absolute min for some f in $[a, b]$.

step ① find 'c' s.t. $f'(c) = 0$ or $f'(c)$ does not exist. (critical numbers).

step ② find $f(a)$ and $f(b)$.

step ③: Largest of step ① and step ② is Absolute max. and smallest of step ① and step ② is absolute min.

Ex 6 @ 207 find absolute max/min for (8.)

$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \le x \le 4.$

solution: find $f'(x) = 3x^2 - 6x$.

Put $f'(x) = 0$ for critical numbers of f.

$\Rightarrow 3x^2 - 6x = 0.$

$\Rightarrow 3x(x-2) = 0. \Rightarrow$ either $x = 0, x = 2.$

Now, evaluate f @ critical numbers.

$f(0) = 1$ and $f(2) = 8 - 12 + 1 = -3.$

also $f(-1/2) = 1/8$ and $f(4) = 17.$

$\therefore$ absolute max is $f(4) = 17$ @ $x = 4$

and absolute min is $f(2) = -3$ @ $x = 2.$

Question # 33 @ 209 find critical numbers of (a)

$f(\theta) = 2\cos\theta + \sin^2\theta$.

solution : $f'(\theta) = -2\sin\theta + \sin 2\theta$

$\Rightarrow f'(\theta) = 0$ (for critical numbers)

$\Rightarrow -2\sin\theta + \sin 2\theta = 0$

$\hookrightarrow 2\sin\theta - \sin 2\theta = 0$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow 2\sin\theta - 2\sin\theta \cdot \cos\theta = 0$

$\Rightarrow 2\sin\theta(1-\cos\theta) = 0$.

either $\sin\theta = 0$ or $1-\cos\theta = 0$.

$\downarrow$

$\cos\theta = 1$

$\Rightarrow \theta_s = n\pi$

$\theta_c = 2n\pi$

$\therefore \theta_c$ is one of the smallest collection of θ_s. Therefore critical numbers of f are $\theta_s = n\pi$

9:05–10:20 AM

Wednesday ①

20th Mar 2024

§4.2 Mean value theorem

↳ IVT (intermediate value theorem)

* Rolle's theorem

* Lagrange's ~~intermediate~~ mean value theorem

Rolle's theorem: Let f be a function and it satisfies following three conditions

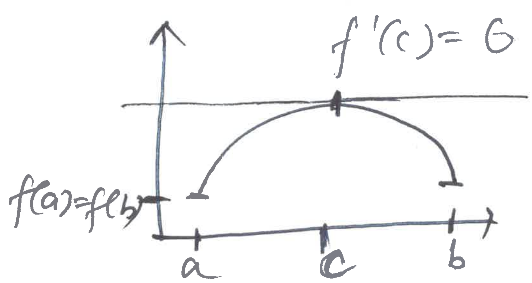
1. $f \in \boxed{\mathcal{C}[a,b]}$
2. $f \in \boxed{\mathcal{D}(a,b)}$

focus!

3. $f(a) = f(b)$.

Then, $\exists$ a number $c \in (a,b)$ s.t. (such that)

$$f'(c) = 0.$$



Problem ① @215 verify Rolle's theorem

$$f(x) = 5 - 12x + 3x^2, \quad [1,3]$$

solution : ⓐ f is a polynomial with deg 2,

$$f \in C[1,3].$$

ⓑ $\because f$ is a polynomial, its differentiable on $(1,3)$.

ⓒ $f(1) = 5 - 12 + 3 = -4$
$f(3) = 5 - 36 + 27 = -4$ $\Rightarrow f(1) = f(3)$.

Then, $\exists$ 'c' $\in (1,3)$ $\ni$ $f'(c) = 0$.

Now, $f'(x) = 6x - 12. \Rightarrow f'(c) = 6c - 12$.

Then, by Rolle's theorem, $6c - 12 = 0$

$$\Rightarrow c = 2.$$

Ex-2 @211 | Show that $x^3+x-1=0$ has only one root. (real)

Proof: Show 2 things $\to$ $\exists$ a root — IVT
$\to$ $\exists$ "only" 1 root. (real). — Rolles theorem

Part ① we will show that $\underbrace{x^3+x-1}_{f(x)}$ has some root.

observe that

$$f(0) = 0^3+0-1 = -1 \ (<0) \text{ and}$$

$$f(1) = 1+1-1 = 1 \ (>0)$$

Then, immediately by IVT, $\exists$ 'a' number 'c' s $c^3+c-1=0$.

Part ② we will show that there is only 'one' root by using Rolle's theorem and contradiction.

so, assume f has two roots and they are 'a' and 'b'. Meaning

$f(a) = 0$ and $f(b) = 0 \Rightarrow f(a) = f(b)$

Observe that f is a polynomial, deg = 3.
Therefore f is $\mathcal{C}[0,1]$ and $\mathcal{D}(0,1)$.

Then by Rolle's theorem there should be a $c \in (0,1)$ s.t. $f'(c) = 0$

$$\begin{cases} f = x^3 + x - 1 \\ f' = 3x^2 + 1 \end{cases}$$

$$\Rightarrow 3c^2 + 1 = 0$$

However, $3c^2 + 1 \geq 1$, thus, by contradiction, $f = x^3 + x - 1$ has only one root!

Lagrange mean value theorem (page 212).
(! the first two conditions will be same as of Rolle's theorem).

(1) $f \in \mathcal{C}[a,b]$ (2) $f \in \mathcal{D}(a,b)$.

Then, $$f'(c) = \frac{f(b)-f(a)}{b-a}$$

where $c \in (a,b)$.

Problem #9 @ 218 verify Rolle's theorem for $f(x) = 2x^2 - 3x + 1$ and $x \in [0,2]$.

Solution: Since f is a polynomial of deg 2, thus f is $\mathcal{C}[0,2]$ and $\mathcal{D}(0,2)$.

Then by LMVT, $\exists$ a 'c' i

$$\underset{4x-3}{f'(c)} = \frac{f(2)-f(0)}{2}$$

$$\Rightarrow (4c-3) = \frac{(8-6+1)-1}{2} = \frac{2}{2} = 1$$

$$\Rightarrow 4c-3 = 1 \Rightarrow c = ? \to 1$$ DIY.

EX-3 @214 Given $f(0)=-3$, $f'(x)\le 5$, (e)

How large $f(2)$ can be? also given that f is differentiable

Solution: (Again), $\because$ f is differentiable on $(0,2)$, it implies that f is $C[0,2]$.

by LMVT,

$$f'(x) = \frac{f(2)-f(0)}{2-0} = \frac{f(2)-f(0)}{2}$$

$$\Rightarrow 2f'(x) = f(2)-f(0)$$

$$\Rightarrow f(2) = f(0) + 2f'(x).$$

$$= -3 + 2f'(x)$$

$\because$ $f'(x)\le 5 \Rightarrow -3+2f'(x) \le -3+10=7$

$\therefore$ $f(2) \le 7$.

9:05 – 10:20 AM Wednesday, 27th Mar 2024 ~~Monday~~ (1)

§4.3 : what f' can provide to us

(1.) Increasing and decreasing functions

(a) if $f'(x) > 0$ on an interval, f is increasing on that interval.

(b) if $f'(x) < 0$ on an interval, f is decreasing on that interval.

EX 1 @ 217 find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and decreasing.

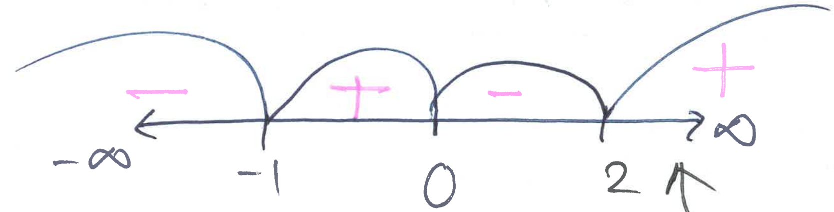
solution : $f'(x) = 12x^3 - 12x^2 - 24x$

$$= 12x(x^2 - x - 2)$$

$$= 12x(x-2)(x+1).$$

Next, put $f'(x) = 0$, to find critical numbers.

$\Rightarrow 12x(x-2)(x+1) = 0 \Rightarrow x = 0, 2, -1.$



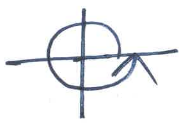
put critical number

So, increasing $[-1, 0] \cup [2, \infty)$

decreasing $(-\infty, -1) \cup (0, 2)$

<u>Ex-3 @ 218</u> find local max / local min of

$$g(x) = x + 2\sin x \qquad 0 \le x \le 2\pi$$



<u>solution</u>: $g'(x) = 1 + 2\cos x.$

put $g'(x) = 0 \Rightarrow 1 + 2\cos x = 0$

$$\Rightarrow \boxed{\cos x = -1/2}$$

$$x = 2\pi/3, \ \frac{4\pi}{3}.$$

determine $g''(x) = -2\sin x.$

$$g''(x)\Big|_{x=\frac{2\pi}{3}} = -2\sin\left(\frac{2\pi}{3}\right) = -2 \cdot \frac{\sqrt{3}}{2} = -ve \quad (-\sqrt{3}).$$

$$g''(x)\Big|_{x=\frac{4\pi}{3}} = -2\sin\frac{4\pi}{3} = -2\cdot\left(\frac{-\sqrt{3}}{2}\right) = \sqrt{3} > 0$$ (3)

+ve

by double derivative test

$x = \frac{4\pi}{3}$ is where local min happens.

$$g(x)\Big|_{x=\frac{4\pi}{3}} = \frac{4\pi}{3} + 2\cdot\sin\frac{4\pi}{3}$$

$$= \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3}.$$

(similarly) $x = \frac{2\pi}{3}$ is where local max happens

$$g(x)\Big|_{x=\frac{2\pi}{3}} = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3}$$

$$= \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3}.$$

Now, we shall define second derivative test.

suppose f'' is continuous near c;

if ① $f'(c)=0$ & $f''(c)>0$ then f has a local minima at $x=c$.

② if $f'(c)=0$ & $f''(c)<0$ then f has a local maxima at $x=c$.

<u>Reading assignment</u>

① First derivative test @ 218

② def. concavity test. @ 220.

9:05 – 10:20 AM

29th Mar 2024 (1)

Friday.

§4.5 Optimization Problems

↪ you have some function & your aim is either max or minimize $\underbrace{\text{that function}}_{f \text{ Objective function}}$.

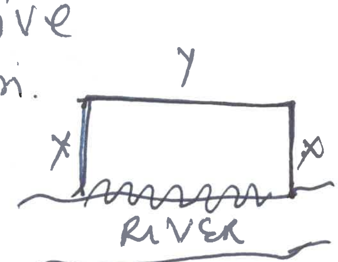
(1) Understand the problem

(2) Put problem into mathematical statements.

(3) Use derivatives to find critical numbers and check the sign of f''.

(4) Conclude your result by providing max/min value.

Ex-1 @ 232 : $A = xy$ → objective function.
area



$P = 2x + y = 2400$. (given).

$\Rightarrow \boxed{y = 2400 - 2x}$

$$A = X(2400 - \cancel{2x}) \equiv A(x, y) \quad \therefore y = 2400 - 2x$$

(convert into 1 variable).

$$\boxed{A = 2400X - 2X^2}$$

$$A'(x) = 2400 - 4.X$$

Now, critical numbers are given as $A'(x) = 0$

$$2400 - 4X = 0$$

$$\Rightarrow \boxed{X = 600}.$$

Also $A''(X) = -4 \Rightarrow A''(x)\Big|_{X=600} = -4 < 0$

$\therefore$ by double derivative test, $A(x)$ gets maximized @ $X = 600.$, $y = 1200.$

$$A_{max} = 600 \times 1200 = 72000 \text{ units}.$$

step involved in optimization

1. construct the objective function (in the previous ex, its Area)

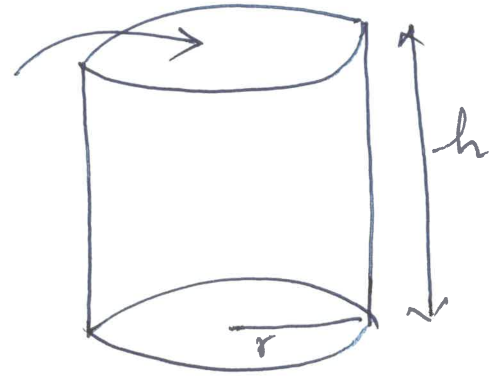
2. find critical numbers of the objective function

3. Use double derivative test to check whether objective function is optimized or not

4. draw conclusion.

Ex-2 @ 233, given,

$V = \pi r^2 h = 1000 \ (cm^3)$



$$\boxed{h = \frac{1000}{\pi r^2}} \text{ —— } (1)$$

In order to min. the cost, we should minimize the surface area of can,

$$S = 2\pi r^2 + 2\pi r h \equiv S(r, h)$$

Then $S(r) = 2\pi r^2 + 2\cancel{\pi r} \cdot \frac{1000}{\cancel{\pi r^2}}$ (4)

$S(r) = 2\pi r^2 + \frac{2000}{r}$ (step 1)

Now, $S'(r) = 4\pi r - \frac{2000}{r^2}$

set $S'(r) = 0$ to have critical numbers
(step 2)

$\Rightarrow 4\pi r = \frac{2000}{r^2} \Rightarrow r^3 = \frac{\cancel{2000}^{500}}{\cancel{4}\pi}$

$\boxed{r^* = \left(\frac{500}{\pi}\right)^{1/3}}$ (>0)

Now, $S''(r) = 4\pi - \frac{2000(-2)r^{-3}}{}$

$= 4\pi + \frac{4000}{r^3}$

$S''(r)\Big|_{r^* = \left(\frac{500}{\pi}\right)^{1/3}} = 4\pi + \frac{4000}{r^3} > 0.$ step 3

Thus, $S(r)$ achieves min. @ $r^* = \left(\frac{500}{\pi}\right)^{1/3}$.

$$\therefore r^* = \left(\frac{500}{\pi}\right)^{1/3}, \quad h = \frac{1000}{\pi \cdot \left(\frac{500}{\pi}\right)^{2/3}} \quad \text{(see 1)}$$

$$= \left[\frac{\cancel{1000}^{\,2\times 500} \times \cancel{1000}^{\,2} \times \cancel{1000}^{\,2}}{\cancel{500} \times \cancel{500}}\right]^{1/3} \frac{1}{\underbrace{\pi^{1-2/3}}_{=\pi^{1/3}}}$$

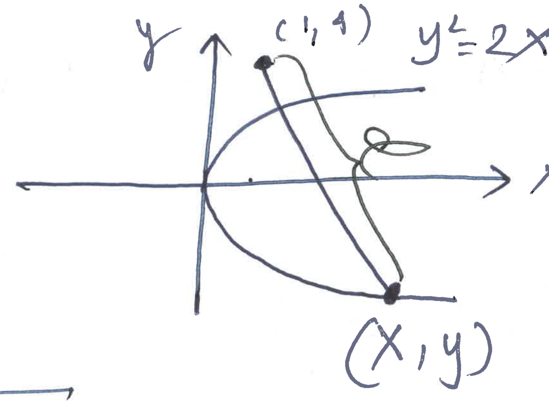
$$= 2\left(\frac{500}{\pi}\right)^{1/3} = 2r^*.$$

Conclusion The desired dimension of can should be $r = \left(\frac{500}{\pi}\right)^{1/3}$ and

$$h = 2\left(\frac{500}{\pi}\right)^{1/3}$$

step 4

EX-3 @ 234 Now, by distance formula (in coordinate geometry) ⑥



$$d(x,y)=\sqrt{(x-1)^2+(y-4)^2}$$

$$\approx d^2=D=(x-1)^2+(y-4)^2$$

Recall (x, y) satisfies $y^2=2x$ and also isolating x is easy $\boxed{x=\frac{y^2}{2}}$ and.

hence

$$D=\left(\frac{y^2}{2}-1\right)^2+(y-4)^2.$$

step ①

differentiate D w.r.t y, to have

$$D'(y)=2\left(\frac{y^2}{2}-1\right)\cdot\frac{\cancel{2}y}{\cancel{2}}+2(y-4).$$

$$=\frac{\cancel{2}y^3}{\cancel{2}}-2y+2y-\cancel{8}=\cancel{2}y^3-8.$$

put $y^3 - 8 = 0 \Rightarrow y = 2$ step 2

Now, $D'' = 3y^2 \Rightarrow D''\big|_{y=2} = 3 \cdot 4 = 12 > 0$ step 3

Thus, $D(y)$ gets minimized @ $y = 2$.

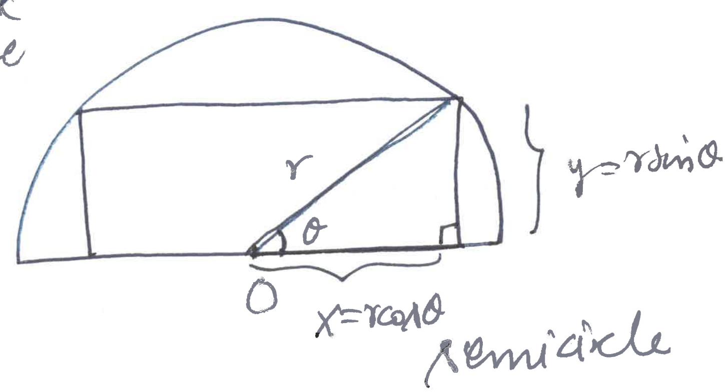
then $X = \frac{y^2}{2} = \frac{4}{2} = 2$. The point is $(2,2)$ which is closest to $(1,4)$ on parabola

$$y^2 = 2X.$$

step 4

Ex 5 @ 236: largest ~~[illegible]~~ rectangle in semicircle.

Why?



$$A = 2Xy$$

$$A \equiv A(\theta) = 2(r\cos\theta)(r\sin\theta)$$

$$\boxed{A(\theta) = r^2 \sin 2\theta}$$

step 1

(8)

$A'(\theta) = 0 \Rightarrow 2r^2\cos 2\theta = 0$

$\Rightarrow \cos 2\theta = 0$

$\Rightarrow 2\theta = \pi/2 \Rightarrow \boxed{\theta = \pi/4}$ step ②

$A''(\theta) = -4r^2 \sin 2\theta$

$A''(\theta)\Big|_{\theta=\pi/4} = -4r^2 < 0$ (for all $r > 0$)

step ③

Therefore max area happens @ $\theta = \pi/4$.

<u>Conclusion</u>: The largest rectangle in the semicircle that can be inscribed with radius r is

$X = r\cos\theta = r\cos \pi/4 = r/\sqrt{2}$

$y = r\sin\theta = r\sin \pi/4 = r/\sqrt{2}$.

step ④

9:05–10:20AM

1st April 2024 (1)

Monday

Homework-L discussion

#3: $T_1(\theta) = \dfrac{1-\tan\theta}{1+\tan\theta}$

$T_{\pi/4}(\theta) = \tan\left(\dfrac{\pi}{4}-\theta\right)$

they are same graphs.

Remember: $\tan(a-b) = \dfrac{\tan a - \tan b}{1+\tan a \tan b}$ —(1)

which means that if $a = \pi/4$ & $b = \theta$

$$\underbrace{\tan\left(\frac{\pi}{4}-\theta\right)}_{T_{\pi/4}(\theta)} = \frac{\tan\pi/4 - \tan\theta}{1+\tan\frac{\pi}{4}\tan\theta} = \underbrace{\frac{1-\tan\theta}{1+\tan\theta}}_{T_1(\theta)}$$

#6| find $\lim_{x \to \infty} \dfrac{\sin^{-1}(\sqrt{\sin x^2}) + \cos^{-1}(\sqrt{\sin x^2})}{e^x}$. (3)

solution: Recall $\boxed{\sin^{-1} x + \cos^{-1} x = \pi/2.}$

if $x \longmapsto \sqrt{\sin x^2}$ then

$$\lim_{x \to \infty} \left[\quad \right] = \lim_{x \to \infty} \left(\frac{\pi}{2}\right) \frac{1}{e^x} = 0. ////.$$

Here, we would like to introduce (general) integral theory to make antiderivative concept robust.

A few integrals

$$(x^n)' = n x^{n-1}$$

Aim:

$$\boxed{???} \xrightarrow{\frac{d}{dx}} x^n$$

$$\frac{x^{n+1}}{n+1} \rightarrow \text{why } \left(\frac{x^{n+1}}{n+1}\right)' = \frac{n+1}{n+1} x^n.$$

Recall $(e^x)' = e^x$

Aim:

$$\boxed{???} \xrightarrow{d/dx} e^x$$

e^x

~~Recall~~ $\cancel{(\ln x)' = 1/x}$

$\cancel{\boxed{???} \xrightarrow{d/dx} (\ln x)}$ ← leave this

⑦ $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$ ⑦

⑧ $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$

Ex 3 @ 284 $\int \underbrace{(10x^4 - 2\sec^2 x)}\,dx$

$= \int 10\cancel{x}x^4\,dx - \int 2\sec^2 x\,dx$

$= 10\int x^4\,dx - 2\int \sec^2 x\,dx$

$= 10\cdot\frac{1}{5}x^5 - 2\tan x + c$

$= 2x^5 - 2\tan x + c.$ ////.

$\frac{d}{dx}$.

Evaluation theorem if $\int f = F$, then

$\int_a^b f\,dx = F(b) - F(a).$ @ 282.

#22 @ 289 $I_1 = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx$

solution: observe $\int \frac{8}{1+x^2}\,dx = 8\tan^{-1}x + C.$

then $I_1 = 8\tan^{-1}\sqrt{3} - 8\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$ check.

#1 @ 289 $I_1 = \int_{-2}^{3} (x^2-3)\,dx$. observe $\int (x^2-3)\,dx = \frac{x^3}{3} - 3x + c$

then $I_1 = \left(\frac{x^3}{3} - 3x\right)\Big|_{-2}^{3} = (3^2 - 3^2) - \left(\frac{(-2)^3}{3} - 3(-2)\right)$

$= -\left(\frac{-8}{3} + 6\right) = -\left(\frac{-8+18}{3}\right)$

$= -10/3.$

#23 @ 289 $I_1 = \int_1^e \frac{x^2+x+1}{x}\,dx$, note that

$\int_1^e \frac{x^2+x+1}{x}\,dx = \int_1^e \left(x + 1 + \frac{1}{x}\right)dx.$ observe that

$\int \left(x + 1 + \frac{1}{x}\right)dx = \frac{x^2}{2} + x + \ln x + c.$

9:05 - 10:20 AM

3rd April 2024 (1.)

Wednesday.

EX-6@286 Evaluate

$$I = \int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt.$$

solution: observe that $I =$

$$\int_1^9 \left(2 + \sqrt{t} - \frac{1}{t^2}\right) dt \quad \text{on simplification.}$$

Recall

$$\int t^n dt = \frac{t^{n+1}}{n+1} + c$$

$$= \int_1^9 (2 + \underset{t^{1/2}}{\sqrt{t}} - t^{-2}) dt.$$

$2 \cdot t^0$

$$\int t^0 dt = t + c$$

$$= \int_1^9 2dt + \int_1^9 \sqrt{t}\, dt - \int_1^9 t^{-2} dt$$

$$\int \sqrt{t} = \int t^{1/2} dt$$

$$= \frac{t^{1/2+1}}{1/2 + 1} + c$$

$$= \frac{2t^{3/2}}{3} + c$$

$$= 2t\Big|_1^9 + \frac{2}{3}t^{3/2}\Big|_1^9 - \frac{t^{-1}}{-1}\Big|_1^9$$

$$\int t^{-2} dt = \frac{t^{-2+1}}{-2+1} + c$$

$$= \underbrace{2t\Big|_1^9}_{①} + \underbrace{\frac{2}{3}t^{3/2}\Big|_1^9}_{②} + \underbrace{\frac{1}{t}\Big|_1^9}_{③}.$$

Properties of integrals

1. $$\int_a^b f\,dx = -\int_b^a f\,dx$$

2. $$\int_a^{\boxed{c}} f\,dx + \int_{\boxed{c}}^b f\,dx = \int_a^b f\,dx$$

3. $$\int_a^a f\,dx = 0$$

4. if f is an odd function then $\int_{-a}^{a} f\,dx = 0$

   Recall: $f(-x) = -f(x)$

   eg: x^3; $\sin x$; $\tan x$.

5. if f is an even function then $\int_{-a}^{a} f\,dx = 2\int_0^a f\,dx$

   Recall: $f(-x) = f(x)$

   eg: $\cos x$; $\sec x$, x^2.

evaluate: $I = \int_{-\pi}^{\pi} \cos x \, dx = 2\int_0^{\pi} \cos x \, dx$

$= 2 \sin x \Big|_0^{\pi} = 0.$

~~the other way is~~ $\cancel{I = \int_{-\pi}^{\pi} \cos(x - \pi) \ldots}$

#46 @ 307 evaluate $I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6} \, dx$

Observe that $\frac{x^2 \sin x}{1+x^6}$ is an odd function.

Therefore, $I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6} \, dx = 0.$

---

evaluate the general integral as follows

$I = \int \sqrt{\sin x} \cdot \cos x \, dx.$

we can easily handle

$\int \sqrt{t} \, dt$

evaluate $I = \int \frac{1}{x}\sqrt{1+\log x}\, dx.$

say $t = 1+\log x \Rightarrow dt = \frac{dx}{x}$ and hence

$I = \int \frac{1}{x}\sqrt{\log x + 1}\, dx = \int dt\, \sqrt{t} \equiv$ DIY.

evaluate $\int x^3 \cos(x^4+2)\, dx$
Ex-1@301

Observe that if $t = x^4+2 \Rightarrow \frac{dt}{4} = x^3 dx$

then $\int \cos(x^4+2)\, x^3 dx = \int \cos t \,\frac{dt}{4}$

$= \frac{\sin t}{4} + C$

$= \frac{\sin(x^4+2)}{4} + C.$

evaluate $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$
ex-8@303

#44 @ 307 $\int_0^{\pi/2} \cos x \sin(\sin x)\,dx.$

solution take $t = \sin x \Rightarrow dt = \cos x\,dx$. hence

@ $x = 0$, $t = \sin 0 = 0$ and @ $x = \pi/2$, $t = \sin\frac{\pi}{2} = 1$

Thus $\int_0^{\pi/2} \cos x \sin(\sin x)\,dx = \int_0^1 dt \sin(t)$

$= -\cos t \Big|_0^1 = -\cos(1) + 1.$

#52 @ 307 $I = \int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx.$

take $t = \sin^{-1} x$ for substitution then

$dt = \frac{dx}{\sqrt{1-x^2}}$. @ $x = 0$, $t = \sin^{-1} 0 = 0$

@ $x = 1/2$, $t = \sin^{-1} 1/2 = \pi/6$.

thus $I = \int_0^{\pi/6} t\,dt = \frac{t^2}{2}\Big|_0^{\pi/6} = \frac{1}{2}\frac{\pi^2}{36} = \frac{1}{12}\frac{\pi^2}{6}$

$= \frac{1}{12}\,3(2)\ldots$

9:05–10:20AM 5th April 2024

Friday.

Fundamental theorem of Calculus

observe that $\boxed{\int f\,dx \equiv \text{function}}$

for example $\int \sin x\,dx = -\underbrace{\cos x}_{\text{function}} + c$,

and $\boxed{\int_a^b f(x)\,dx = \text{number}}$ for example

$$\int_0^{\pi} \cos x\,dx = 0$$

At an introductory level what is $\int_a^x f(t)\,dt$?

The answer to this question is given by the fundamental theorem of calculus. (page 295)

say $f \in C[a,b]$

① If $g(x) = \int_a^x f(t)\,dt$ then $g'(x) = f(x)$.

② if $F' = f$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.

exo — 5 @ 295 find $\frac{d}{dX}\left\{\int_1^{X^4} \sec t \, dt\right\}$ (20)

solution: say $g(X) = \int_1^{X^4} \sec t \, dt$. Then, we want to know $g'(X)$? as

$$g'(X) = \frac{d}{dX} g(X) = \frac{d}{dX} \int_p^{X^4} \sec t \, dt.$$

Apply the fundamental theorem of calculus, to have
by the chain rule.

$$g'(X) = \sec(X^4) \cdot \frac{d}{dX}(X^4)$$

$$= \sec X^4 \cdot 4X^3 \quad ////.$$

#5 @ 298 ~~find~~ $g(X) = \int_1^{X} \frac{dt}{t^3+1}$, observe that

$f(t) = \frac{1}{t^3+1}$, therefore by the application of

fundamental theorem of calculus, we (
have

$$g'(x) = f(x) = \frac{1}{x^3+1}.$$

#6 @ 289 $g(x) = \int_3^x e^{t^2-t}\,dt$, again

by the fundamental theorem of calculus,

$$g'(x) = e^{x^2-x}.$$

#8@289: $f(x) = \int_x^{10} \tan\theta\, d\theta.$

observe that $f(x) = -\int_{10}^{x} \tan\theta\, d\theta$ by the

properties of definite integral. Now, apply the
fundamental theorem of calculus to have

$$f'(x) = -[\tan\theta x] = -\tan x.$$

#12 @289 find y' if

$$y = \int_{\sin x}^{1} \sqrt{1+t^2}\, dt$$

Solution: $y = -\int_{1}^{\sin x} \sqrt{1+t^2}\, dt$ (flip it).

then by the application of fundamental theorem of calculus, we have

$$y' = -\sqrt{1+\sin^2 x} \cdot \frac{d}{dx} \sin x = -\sqrt{1+\sin^2 x} \cdot \cos x.$$

14 @289: $y(x) = \int_{\sin x}^{\cos x} (1+v^2)^{10}\, dv$ — 1

also $y(-x) = \int_{\sin(-x)}^{\cos(-x)} (1+v^2)^{10}\, dv = \int_{-\sin x}^{\cos x} (1+v^2)^{10}\, dv.$

Also
$$-y(-x) = \int_{\cos x}^{-\sin x} (1+v^2)^{10}\, dv. \quad \text{——— (2)}$$

Recall $\int_a^c f + \int_c^b = \int_a^b f$

Add (1) and (2),
$$y(x) + (-y(-x)) = \int_{\sin x}^{\cos x} (1+v^2)^{10}\, dv + \int_{+\cos x}^{-\sin x} (1+v^2)^{10}\, dv$$

$$\boxed{y(x) - y(-x) = \int_{\sin x}^{-\sin x} (1+v^2)^{10}\, dv.}$$

$$\Rightarrow \underbrace{y(-x) - y(x)}_{:Y(x)} = \int_{-\sin x}^{\sin x} (1+v^2)^{10}\, dV.$$

observe that $Y(x) = y(-x) - y(x)$, then

$$Y(-x) = y(x) - y(-x) = -(y(-x) - y(x))$$
$$= -Y(x)$$

Hence $Y(x)$ is an odd function. 6

we still need to find $y'(x)$. therefore fundamental theorem of calculus yields

$$\boxed{y'(x) = (1+\cos^2 x)^{10} \cdot (-\sin x) - (1+\sin^2 x)^{10} \cdot \cos x}$$

9:05-10:20 AM Wednesday 10th April 2024 ①

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx. \quad \left(= \pi/4\right). \qquad \text{——— } ①$$

Recall $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx.$

$$\therefore I = \int_0^{\pi/2} \frac{\sin\left(\pi/2 - x\right)}{\sin\left(\pi/2 - x\right) + \cos\left(\pi/2 - x\right)}\, dx$$

$$= \int_0^{\pi} \frac{\cos x}{\cos x + \sin x}\, dx \qquad \text{——— } ②$$

add ① and ②

$$2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx$$

$$= \int_0^{\pi/2} dx$$

$$= \frac{\pi}{2} \Rightarrow \boxed{I = \left(\frac{\pi}{2}\right) \cdot \frac{1}{2} = \frac{\pi}{4}}.$$

③

$$I = \int_0^{\pi/2} \frac{\sin^{2024} x}{\cos^{2024} x + \sin^{2024} x}\, dx$$

Recall that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

then $I = \int_0^{\pi/2} \frac{\cos^{2024} x}{\sin^{2024} x + \cos^{2024} x}$

$\left\{ \begin{array}{l} \because \sin\left(\frac{\pi}{2} - x\right) = \cos x \\ \cos\left(\frac{\pi}{2} - x\right) = \sin x \end{array} \right.$

add, $2I = \int_0^{\pi/2} dx \Rightarrow \boxed{I = \pi/4}$

///.

9:05-10:20 AM

Monday ①

15th April 2024

Revise

① $\int_1^e \frac{x^2+x+1}{x}\,dx$.

sol: $\int_1^e \left(x+1+\frac{1}{x}\right)dx = \left(\frac{x^2}{2}+x+\ln x\right)\Big|_1^e$

$= D.I.Y.$

② $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6}\,dx$

sol: $f(x) = \frac{x^2 \sin x}{1+x^6}$ then $\underbrace{f(-x) = -f(x)}_{\text{odd function.}}$

$\therefore \int_{-\pi/2}^{\pi/2} f(x)\,dx = 0.$

(3) $I = \int_0^1 \frac{e^z+1}{e^z+z} dz$

take $u = e^z + z \Rightarrow du = (e^z+1)\cdot dz$.

change the limits; @ $z=0$, $u=1$
@ $z=1$, $u = e+1$.

then $I = \int_1^{e+1} \frac{du}{u}$ D. I. Y.

(4) if $y(x) = \int_{\sin x}^{\cos x} (1+v^2)^{10} dv$, (i) find $y'(x)$. ?

sol: $y' = (1+\cos^2 x)^{10}(-\sin x) - (1+\sin^2 x)^{10} \cdot \cos x$

by FTC.

(ii) $Y(x) = y(-x) - y(x)$ then show that
Y is an odd function.

D. I. Y.