Homework – D

Calc

Q#2 (1) for sine — follow notes

for cosine

$$(\cos x)' = \lim_{h\to 0}\left[\frac{\cos(x+h)-\cos x}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\cos x\cosh h - \sin x \sin h - \cos x}{h}\right]$$

$$= \lim_{h\to 0}\left[\cos x\,\frac{\cosh h-1}{h} - \sin x\,\frac{\sin h}{h}\right]$$

0 under $h\to 0$ | 1 under $h\to 0$

$$= -\sin x.$$

(2) For $\sin^2(v)$

$$(\sin^2 v)' = \lim_{h\to 0}\left[\overbrace{\frac{\sin^2(v+h) - \sin^2 v}{h}}^{\text{apply the given formulae}}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin(2v+h)\sin h}{h}\right]$$

1 under $h\to 0$

$$= \lim_{h\to 0} \sin(2v+h) = \sin 2v$$ ////.

For $\cos^2 v$

$$(\cos^2 v)' = \lim_{h\to 0}\left[\frac{\cos^2(v+h) - \cos^2 v}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-\sin h\,\sin(2v+h)}{h}\right]$$

1 under $h\to 0$

$$= -\sin 2v.$$

Note that $(\sin^2 v)' + (\cos^2 v)' = \boxed{0}$ ③

↑ constant.

we expect that constant (which is 0 here)

$\therefore \sin^2 v + \cos^2 v = \boxed{1.}$

→ constant.

and differentiation of constant is always 0.

Q#3 Follow class notes and books.

Q#4 if $f(x) = x^n \; \forall n \in \mathbb{N}$. This imply that $f(1) = 1^n = 1$.

From previous HW (HW-V Q#5) we have learned that the slope of tangent line to curve $f(x) = x^n$ is given as

4

$$m = \lim_{x\to 1}\left[\frac{f(x)-1}{x-1}\right] = \lim_{x\to 1}\frac{x^n-1}{x-1} = n.$$

Thus, now we ~~ke~~ know the slope $m=n$ and point $(1,1)$ from where the tangent line passes, therefore

$$y - \underset{\downarrow}{1} = \underbrace{n}_{\text{slope}}(x-\underset{\downarrow}{1})$$

y-coordinate x-coordinate

$(\underbrace{y-y_0=m(x-x_0)}_{\text{point slope formulla}})$.

$$\Rightarrow \boxed{y = nx - n + 1}$$

Solution of HW - T / due by 2nd Feb 2024

Q#2 Given $f_I : \mathbb{R} \to \mathbb{R}$ by $f_I(x) = x$ and $f_c(x) = c$ where $f_c : \mathbb{R} \to \mathbb{R}$.

① claim $f_I \circ f_c = f_c \circ f_I$.

Proof: LHS $f_I \circ f_c \equiv f_I \circ f_c(x) = f_I(f_c(x))$

$= f_I(c) = c$. —— ①

RHS $f_c \circ f_I \equiv f_c \circ f_I(x) = f_c(f_I(x)) = f_c(x)$

$= c$ —— ②.

comparing ① and ②, we immediately have $f_I \circ f_c = f_c \circ f_I$.

② $\because$ (because/since) from previous part $f_I \circ f_c = f_c \circ f_I = c \;\; \forall x \in \mathbb{R}$, thus the desired value is simply 'c', where $c \in \mathbb{R}$.

(3) DIY.

Q#3 is not tough at-all. Infact the first part is directly from our book. Ref. @ page XX problem 7(c).

Ques#3 we have $f:\mathbb{R}\longrightarrow\mathbb{R}$ via $f(x)=\alpha x+\beta$ where α and $\beta\in\mathbb{R}$. Define the n-th composition of 'f' with itself by f_n.

(1) DIY

(2) To find the explicit expression of f_n, we need to observe the pattern for n=2,3 and 4. Therefore, we determine f_2, f_3 and f_4 as follows:

$$f_2 \equiv f_2(x) = f\circ f(x) = f(f(x)) = f(\alpha x+\beta)$$

$$= \alpha(\alpha x+\beta)+\beta = \alpha^2 x+\alpha\beta+\beta.$$

since $\alpha\beta+\beta = \underbrace{f(\beta)}_{n=1}$, thus $f_2(x)=\alpha^2 x+\underbrace{f(\beta)}_{n=1}$

⑧

$$f_3 \equiv f_3(x) = f(f_2(x)) = f(\alpha^2 x + \alpha\beta + \beta)$$

$$= \alpha(\alpha^2 x + \alpha\beta + \beta) + \beta$$

$$= \alpha^3 x + \underbrace{\alpha^2\beta + \alpha\beta + \beta}$$

↳ the presence of 2^{nd} degree in α directs us to evaluate $f_2(x)$ at $x=\beta$ thus $f_2(\beta) = \alpha^2\beta + \alpha\beta + \beta$.

thus $\boxed{f_3(x) = \alpha^3 x + f_2(\beta)}$ or

$$f_3(x) = \alpha^3 x + \beta \sum_{i=0}^{2} \alpha^i$$

1 less than 3. (=2)

Similarly one can have $f_4(x) =$

$$\alpha^4 x + \alpha^3\beta + \alpha^2\beta + \alpha\beta + \beta$$

OR

$$= \alpha^4 x + \beta(\alpha^3 + \alpha^2 + \alpha + 1)$$

1 less than 4 (=3).

$$= \alpha^4 x + \beta \sum_{i=0}^{3} \alpha^i.$$

Therefore, from this observation,

$$f_n(x) = \alpha^n x + \beta \sum_{i=0}^{n-1} \alpha^i \quad \forall\, n \in \mathbb{Z}_+.$$

(3) claim $f_n = f_m \iff n = m$

($\Rightarrow$)

Proof: Assume that $f_n = f_m \ \forall\, x \in \mathbb{R}$,

$$\alpha^n x + \beta \sum_{i=0}^{n-1} \alpha^i = \alpha^m x + \beta \sum_{i=0}^{m-1} \alpha^i .$$

compare like terms. (either)

After comparing like terms, we have $\alpha^n = \alpha^m$, therefore taking the 'log' at base a, yields $n = m$ as desired.

($\Leftarrow$) Assume now that $n = m$. Then

$$f_n(x) = \alpha^n x + \beta \sum_{i=0}^{n-1} \alpha^i$$

$$= \alpha^m x + \beta \sum_{i=0}^{m-1} \alpha^i \quad (\text{as } n = m)$$

$$= f_m(x), \text{ thus } f_n = f_m.$$

Thus, combining both directions, we have desired result.

(4). Upon the observation we have (5)

$f_{n+1}(x) = \alpha^{n+1}x + f_n(\beta)$. As, we need to determine $f_{2024}(x)$, this means that $n+1 = 2024 \Rightarrow n = 2023$.

Therefore, with $\alpha = 2$ & $\beta = -3$, we have finally

$$\boxed{f_{2024}(x) = \alpha^{2024}x + f_{2023}(-3)}$$

////.

Q#2 Recall $\lim_{\tau\to 0} \frac{\sin\tau}{\tau} = 1.$

① $\underbrace{S_n(\tau) = \sin(S_{n-1}(\tau)) = S_{n-1}(\sin(\tau))}$

Recursion relationship.

② Recall $\lim_{\tau\to 0} \frac{\sin \tau}{\tau} = 1$

Same $\Longrightarrow$

$\lim_{\tau\to 0} \frac{\sin(S_{n-1}(\tau))}{\Lambda(\tau)} = 1$

has to be SAME

$\therefore \Lambda(\tau) = S_{n-1}(\tau)$ (one of the simple ready-to-use answers.)

③ Periodicity with 2π means that

$$\sin(\tau + 2\pi) = \sin\tau. \quad \text{——} ①$$

Now apply "sin" above in eq ① we have

$$\sin(\sin(\tau+2\pi)) = \sin(\sin(\tau)) \quad \text{——} ②$$

Observe that by definition for 'n'-times ②
composition, result in eq ② simply
means that

$$S_2(\tau + 2\pi) = S_2(\tau).$$

In similar ways, we can have
$S_3(\tau + 2\pi) = S_3(\tau)$. Thus, in general
we have
$$S_n(\tau + 2\pi) = S_n(\tau).$$

(Repeat the application of "sin" on
eq ① n times and result will be
delivered!).

④ Note that "sin"(·) is an odd function.
meaning
$$\sin(-x) = -\sin x \quad \text{——— ①}$$

Apply "sin" to above, we have
$$\underbrace{\sin(\sin(-x))}_{:= S_2(-x)} = \sin(-\sin x)$$

$$\Rightarrow S_2(-x) = -\sin(\sin x)$$
$$= -S_2(x).$$

$\because \sin(-\alpha) = -\sin\alpha$ (8)

Meaning at 2-times composition we have
$$S_2(-x) = -S_2(x) \equiv \text{ODD}.$$

Now, repeat the above process upto n-times and hence the desired result will be delivered, that is
$$\boxed{S_n(-x) = -S_n(x)}$$

Therefore, $S_n(x)$ is an odd.

Q#3 — DIY

Q#4 — DIY ; learn the specific values of all sin and cos functions

Q#5 $f(\tau) = \dfrac{\tau^{n}-1}{\tau-1}$; note that for $n=1$ (7)

then $f(\tau) = 1$.

for $n=2$ $\quad \dfrac{\tau^2-1}{\tau-1} = \dfrac{\tau-1 \cdot \tau+1}{\tau-1} = \tau+1$

and $\lim\limits_{\tau\to 1} \dfrac{\tau^2-1}{\tau-1} = \lim\limits_{\tau\to 1}(\tau+1) = 2$.

for $n=3$ $\quad \dfrac{\tau^3-1}{\tau-1} = \tau^2+\tau+1$ and

$\lim\limits_{\tau\to 1} \dfrac{\tau^3-1}{\tau-1} = \lim\limits_{\tau\to 1} (\tau^2+\tau+1) = 3$.

Similarly $\dfrac{\tau^4-1}{\tau-1} = \tau^3+\tau^2+\tau+1$

thus $\lim\limits_{\tau\to 1} \dfrac{\tau^4-1}{\tau-1} = 4$.

Hence, in general $\lim\limits_{\tau\to 1} \dfrac{\tau^{\boxed{n}}-1}{\tau-1} = \boxed{n}$

Q#6 (again) Recall $\lim_{\tau \to 0} \frac{\sin \tau}{\tau} = 1$.

Now $\lim_{\tau \to 0} \frac{s^2(\tau)}{\tau} = \lim_{\tau \to 0} \frac{\sin \tau \sin \tau}{\tau}$

by limit laws

$= \left(\lim_{\tau \to 0} \frac{\sin \tau}{\tau}\right)\left(\lim_{\tau \to 0} \sin \tau\right)$

$= 1 \cdot 0$

$= 0$

Q#7 (1.) $f_{[\#]}(\lambda) = \lambda^{\#}$ and $\lambda \in I = [0,1]$.

Hence domain of $f_{[\#]} = I$. Thus the range of $f \equiv I$, since $\lambda \in [0,1]$.