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Form: *Best of Luck!*

Math 2413 Exam 1

Exam Information

- This exam has 10 questions for a total of 100 points.
- Partial credit will be given for partially correct work.
- You must show all work.
- Anyone caught cheating will receive an automatic zero for this exam, and there may be more severe consequences.
- No calculators, phones, or other electronic devices may be used during the exam.
- You have 75 minutes to complete the exam.

I certify that I have read, understand, and agree to abide by the above rules.

Signature:____

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR.

1. (10 points) 1. Simplify $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$.

2. Establish $\dfrac{2\tan x}{1+\tan^2 x} = \sin 2x$.

Solution

① $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2} = \dfrac{1}{9}\left(\dfrac{x^2y^{-1/2}}{x^{3/2}y^3}\right)^2 = \dfrac{x}{9y^7}$

② LHS $\dfrac{2\tan X}{1+\tan^2 X} = \dfrac{2\tan X}{\sec^2 X} = \dfrac{2\sin X}{\cos X\cdot\sec^2 X}$

$= \underbrace{2\sin X\cdot\cos X}_{\text{by definition of double angle formule}}$

$= \sin 2X$

$=$ RHS.

Hence established

2. (10 points) Find the limit of following:

1. $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta+\tan\theta}$.
2. $\lim_{t\to -3} \dfrac{t^2-9}{2t^2+7t+3}$.

Sol:

(1.) $\displaystyle\lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta+\tan\theta}\right] = \lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta}\,\frac{1}{1+\frac{\tan\theta}{\theta}}\right]$

$\displaystyle = \left[\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right]\cdot\left[\lim_{\theta\to 0}\frac{1}{1+\frac{\tan\theta}{\theta}}\right] = 1\cdot\frac{1}{1+\lim_{\theta\to 0}\frac{\tan\theta}{\theta}}$

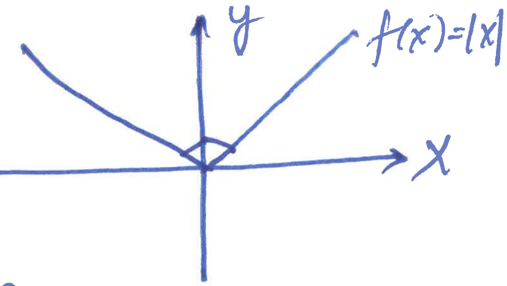
$\displaystyle = \frac{1}{1+1} = \frac{1}{2}.\quad \left(\text{as } \lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1\right).$

(2.) $\displaystyle\lim_{t\to -3}\left[\frac{t^2-9}{2t^2+7t+3}\right] = \lim_{t\to -3}\left[\frac{(t+3)(t-3)}{(t+3)(2t+1)}\right]$

$\displaystyle = \lim_{t\to -3}\frac{t-3}{2t+1} = \frac{6}{5}.$

3. (10 points) The desirable properties of any function $f : \mathbb{R} \to \mathbb{R}$ defined over reals are usually *continuity* and *differentiability*. Often times, function which are continuous need not to be differentiable but the converse of this is certainly true.

   Demonstrate the former statement that is, *continuity of a function f do not imply that f is differentiable* with an example with proper justification.



solution consider $f(x) = |x|$

$$= \begin{cases} -x, & x \le 0 \\ x, & x > 0 \end{cases}$$

compute the limit (left and right) at $x=0$ for $f(x) = |x|$.

$$\lim_{h \to 0^+} \left[\frac{|0+h| - |0|}{h} \right] = \lim_{h \to 0^+} \frac{h}{h} = 1 \text{ --- (RHL)}$$

$$\lim_{h \to 0^-} \left[\frac{|0+h| - |0|}{h} \right] = \lim_{h \to 0^-} \left[\frac{-h}{h} \right] = -1 \text{ --- (LHL)}.$$

therefore, $f'(0)$ does not exist. Hence f is not differentiable; but obviously continuous.

4. (10 points) What is the limiting behavior of function $f(x) = x^2 \sin^2\left(\frac{1}{x}\right)$ as x tends to 0.

<u>Solution</u>: Since $-1 \leq \sin(1/x) \leq 1$, then

$\Rightarrow 0 \leq \sin^2(1/x) \leq 1.$

$\Rightarrow x^2 \cdot 0 \leq x^2 \cdot \sin^2(1/x) \leq 1 \cdot x^2$

$\Rightarrow 0 \leq x^2 \sin^2(1/x) \leq x^2$

As, $x^2 \sin^2(1/x^2)$ is squeezed by 0 and x^2 and both of them tends to 0 as $x \to 0$. Thus $\lim_{x \to 0} \left[x^2 \sin^2(1/x)\right] = 0.$

5. (10 points) Discuss the continuity of the function $f(x) = 3x^4 - 5x$ at $x = 2$.

Solution: (1^{st} way). Observe that $f(x) = 3x^4 - 5x$ is a polynomial. As polynomials are continuous function in their domain. Thus f is continuous at $x=2$. (its a polynomial of deg 4).

(2nd way) $f(x)\Big|_{x=2} = 3\cdot 2^4 - 5\cdot 2 = 38$. ——— 1

$\lim_{h\to 0} f(2-h) = \lim_{h\to 0} 3\cdot(2-h)^4 - 5(2-h) = 38$ ——— 2

$\lim_{h\to 0} f(2+h) = \lim_{h\to 0} 3(2+h)^4 - 5(2+h) = 38$ ——— 3

Since, the functional value in 1, left hand limit in 2 & right hand limit in 3 are all equal to each other, i.e. 38, thus f is continuous.

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6. (10 points) Use the definition of derivative to find the derivative of $f(x) = \cos x$.

Solution: Recall the derivative definition as

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right].$$

$$\therefore (\cos x)' = \lim_{h \to 0} \left[\frac{\cos(x+h) - \cos x}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right]$$

$$= \lim_{h \to 0} \left[\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right]$$

$$= 0 - \sin x \cdot 1 = -\sin x.$$

7. (15 points) Consider the function in x and y variables as $x^3+y^3 = 6xy$. Find the y'. Determine the equation of tangent for the given curve at $(3,3)$.

Solution: Differentiate $x^3+y^3=6xy$ as

$\Rightarrow x^2+y^2y' = 2y+2xy'$

$\Rightarrow (y^2-2x)y' = 2y-x^2$,

$\therefore \boxed{y' = \dfrac{2y-x^2}{y^2-2x}}$ —— ①

Now, we determine slope @ $(x,y)=(3,3)$ from ①; $y'\Big|_{(3,3)} = -1$. Then equation of tangent is given as

$y-3 = -1(x-3) \Rightarrow \boxed{y+x=6}$

8. (10 points) Define $y = \left(\frac{1-\cos 2x}{1+\cos 2x}\right)^4$.

1. Find y'.
2. Show that $y' = 8\left(y^{1/8-1} + y^{1/8+1}\right)$.

Solution ① (1st way) $y' = 4\left(\frac{1-\cos 2x}{1+\cos 2x}\right)^3 \frac{d}{dx}\left(\frac{1-\cos 2x}{1+\cos 2x}\right)$

$$= 4\left(\frac{1-\cos 2x}{1+\cos 2x}\right)^3 \left[\frac{(1+\cos 2x)(+2\sin 2x) - (1-\cos 2x)(-2\sin 2x)}{(1+\cos 2x)^2}\right]$$

$$= 4\left(\frac{-\cos 2x+1}{1+\cos 2x}\right)^3 \left[\frac{2\sin 2x + \cancel{2\sin 2x\cos 2x} + 2\sin 2x - \cancel{2\sin 2x\cos 2x}}{(1+\cos 2x)^2}\right]$$

$$= 16\left(\frac{1-\cos 2x}{1+\cos 2x}\right)^3 \frac{\sin 2x}{(1+\cos 2x)^2}.$$

2nd way use double angle formula to have

$$y' = (\tan^8 x)' = 8\tan^7 x \sec^2 x.$$

Note: $y = \left(\frac{1-\cos 2x}{1+\cos 2x}\right)^4 = \left(\frac{2\sin^2 x}{2\cos^2 x}\right)^4 = \tan^8 x.$

9. (10 points) For two functions f and g defined over some domain where g fails to vanish. This simply means that $g \neq 0$ over that domain of interest.

   1. Supply $(fg)'$.
   2. Supply $\left(\frac{f}{g}\right)'$. The division by function g is possible as $g \neq 0$ according to our assumption.
   3. Extract the double of $f'g$ in terms of previous two results.

solution ① $(fg)' = f'g + g'f$

② $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

③ $(fg)' = f'g + g'f$ ——— ①

$g^2(f/g)' = f'g - g'f$ ——— ②

$$\boxed{(fg)' + g^2(f/g)' = 2f'g}.$$

10. (5 points) Define what it means to be a function which is either an *odd* or an *even* function. Show that the derivative of an *even* function is an odd function.

solution: ODD : $f(-x) = -f(x)$ —— (1)

EVEN : $f(-x) = f(x)$. —— (2)

consider an even function from (2);

$f(-x) = f(x)$

$\Rightarrow \underbrace{f'(-x)(-x)' = f'(x)}_{\text{by chain rule.}}$

$\therefore \boxed{-f'(-x) = f'(x)}$

↳ odd function.