Computer Vision : Assignment 1

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December 29, 2020

1

Marginal pmf function of x_3, x_4, x_5

$$\begin{aligned} p(x_3, x_4, x_5) = & p(1, 1, x_3, x_4, x_5) + p(-1, -1, x_3, x_4, x_5) + \\ & + p(1, -1, x_3, x_4, x_5) + p(-1, 1, x_3, x_4, x_5) \end{aligned}$$

2

Mean of discrete random vairable :

$$E(X) = \frac{1}{6} * \sum_{\{1,2,3,4,5,6\}} x = 3.5$$

3

$E(X)$ mean of discrete or continuous random vector . $E(X_i)$ mean of discrete or continuous random variable .We use equation (25 and 26)

$$(E(X))^T = \begin{pmatrix} E(X_1) & E(X_2) & E(X_3) \end{pmatrix}^T = \begin{pmatrix} E(X_1) \\ E(X_2) \\ E(X_3) \end{pmatrix} = E(X^T) \tag{1}$$

4

Let $x = [x_1 \dots x_n] \in \mathbb{R}^n$ be scalar vector . According to the difinition of outer product , $x \otimes x \in \mathbb{R}^{n \times n}$, when ith row and jth column element $x_{i,j}$ is $x_i * x_j$ .Thus $x_{i,j} = x_i * x_j = x_j * x_i = x_{j,i}$ From the definition of T operator $x_{i,j}$ element in $M \in \mathbb{R}^a \times b$ becomes $x_{j,i}$ element in M^T matrix . So we cam conclude that in case of outer product of vector with itself the result matrix will be symmetric with respect to the diagonal : $xx^T = (xx^T)^T$.

5

I will show that $(R_x)_{i,j} = E(X_i X_j)$ for continuous random vector . The case of discrete RV is identical .
According to definition of correlation matrix
$R_x = \int_{\mathbb{R}^n} xx^T * p(x)dx = R_x = \int_{\mathbb{R}^n} x \otimes x * p(x)dx$.
Let's consider element $(R_x)_{i,j}$:

$$\begin{aligned}(R_x)_{i,j} &= \int_{\mathbb{R}^n} x_i * x_j \; p(x) \; dx = \\ &= \int_{\mathbb{R}^2} x_i * x_j \left(\int_{\mathbb{R}^{n-2}} p(x) dx_1 \dots dx_{i-1} \; dx_{x+1} \dots dx_{j-1} dx_{j+1} \dots dx_n \right) dx_i \; dx_j \\ &\overset{\text{margin pdf}}{=} \int_{\mathbb{R}^2} x_i * x_j \;\; * p(x_i, x_j) \; dx_i dx_j = E(X_i X_j)\end{aligned}$$

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The fact that $R_x = R_x^T$ follows from the proof of **4** that
$xx^T = (xx^T)^T$,
and proof from **5** that
$(R_x)_{i,j} = E(X_i X_j) = E(X_j X_i) = (R_x)_{j,i}$

7

I will show that $(\Sigma_X)_{i,j} = E((X_i - E(X_i) * (X_j - E(X_j))$ for continuous random vector . The case of discrete RV is identical .
Column i in $\mu = E(X)$ is $E(X_i)$ according to RV mean definition .
Thus we can conclude that

$$\begin{aligned}(\Sigma_X)_{i,j} &= \int_{\mathbb{R}^n} (x_i - E(X_i)) * (x_j - E(X_j)) \; p(x) \; dx \\ &= \int_{\mathbb{R}^2} (x_i - E(X_i)) * (x_j - E(X_j)) * \\ &\quad * \left(\int_{\mathbb{R}^{n-2}} p(x) dx_1 \dots dx_{i-1} \; dx_{x+1} \dots dx_{j-1} dx_{j+1} \dots dx_n \right) dx_i \; dx_j \\ &\overset{\text{margin pdf}}{=} \int_{\mathbb{R}^2} (x_i - E(X_i)) * (x_j - E(X_j)) \;\; * p(x_i, x_j) \; dx_i dx_j \\ &= E((x_i - E(X_i)) * (x_j - E(X_j))\end{aligned}$$

8

In **7** we proved that $(\Sigma_X)_{i,j} = E((x_i - E(X_i)) * (x_j - E(X_j))$, thus

$$(\Sigma_X)_{i,i} = E((x_i - E(X_i)) * (x_i - E(X_i)) = E((x_i - E(X_i))^2) = Var(X_i)$$

9

In **7** we proved that $(\Sigma_X)_{i,j} = E((X_i - E(X_i)) * (X_j - E(X_j)))$. We can write it in another way $E(X_i X_j) - \mu_i * \mu_j$. Random variables product is commutative, thus $E(X_i X_j) - \mu_i * \mu_j = E(X_j X_i) - \mu_j * \mu_i = (\Sigma_X)_{j,i}$.
In addition Σ_X $n \times n$ matrix , then $\Sigma_X = (\Sigma_X)^T$

10

In **7** we proved that $(\Sigma_X)_{i,j} = E((X_i - E(X_i)) * (X_j - E(X_j)))$. We will show the equations correctness by means of one element consideration

10.1

$$\begin{aligned} \Sigma_X = E((X - E(X)) * (X - E(X))^T)(\Sigma_X)_{i,j} &= E((X_i - E(X_i)) * (X_j - E(X_j)) \\ &= E(X_i X_j - X_j E(X_i) - X_i E(X_j) + E(X_i)E(X_j)) = \end{aligned}$$

According to mean properties (mean of sum, mean multiplied
by constant and mean of consant

$$= E(X_i X_j) - E(X_j)E(X_i) - E(X_j)E(X_i) + E(X_i)E(X_j) = E(X_i X_j) - E(X_i)E(X_j)$$
$$\Rightarrow \Sigma_X = E(XX^T) - \mu\mu^T$$

10.2

$$\begin{aligned} (\Sigma_X)_{i,j} &= E((X_i - E(X_i)) * (X_j - E(X_j)) \\ &= E(X_i X_j - X_j E(X_i) - X_i E(X_j) + E(X_i)E(X_j)) \\ &= E(X_j(X_i - E(X_i)) - X_i E(X_j) + E(X_i)E(X_j)) \\ &= E(X_j(X_i - E(X_i)) - E(X_j)E(X_i) + E(X_i)E(X_j) = E(X_j(X_i - E(X_i)) \\ &\Rightarrow \Sigma_X = E(X(X - \mu)^T) \end{aligned}$$

10.3

From Σ_X matrix symmetry follows that
$\Sigma_X = E(X(X - \mu)^T) \Rightarrow \Sigma_X = E((X - \mu)X^T)$

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In **9** we proved that $\Sigma_X = (\Sigma_X)^T$. Lets consider RV $Z = [XY]$, when X and Y are RV of length n and m. We also know that

$$\Sigma_Z = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix}$$

From the matrix Σ_Z symmetry we can conclude that $\Sigma_{XY} = \Sigma_{YX}^T$

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12.1

A is scalar matrix and b is scalar vector
Let

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \dots \\ A_m \end{pmatrix}$$

When $A_i \in \mathbb{R}^n$ is row vector

Then

$$AE(X) + b = \begin{pmatrix} A_1 E(X) + b_1 \\ \dots \\ A_m E(X) + b_m \end{pmatrix} = \begin{pmatrix} E(A_1 X + b_1) \\ \dots \\ E(A_n X + b_n) \end{pmatrix} = E(AX + b) = E(Y)$$

12.2

$$\begin{aligned} \Sigma_Y &= E((Y - E(Y)) * (Y - E(Y))^T) = E((AX + b - AE(X) - b) * (\dots)^T) \\ &= E((AX - AE(X)) * (AX - AE(X))^T = E((AX - AE(X)) * (X^T A^T - E(X^T) A^T)) \\ &= E(A(X - E(X))(X^T - E^T(X))A^T) == E(A(X - E(X))(X - E(X))^T A^T) \\ &= AE((X - E(X))(X - E(X))^T) A^T = A \Sigma_X A^T \end{aligned}$$

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We can consider random variable as random vector with length one . Then the variance of random variable will be equivalent to the covariance of corresponding random vector . So the expression $\sum_i X_i = 1^T X$ can be interpreted as affine Transformation of RV X , where $A = 1^T$ and b is zero vector . Thus according to affine transformation formula we can state that $\Sigma_{\sum_i X_i} = \Sigma_1^T X = 1^T \Sigma_X 1 = 1^T (\theta^2) I 1 = \theta^2 * n$

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We can express any subvector of X (call it $Y \in \mathbb{R}^m$) as affine transformation with matrix A

$$A = \begin{pmatrix} A_1 \\ \dots \\ A_m \end{pmatrix}$$

When A_i is a basic vector with 1 in the place j if X_j is included in Y.There is a row A_k for each particular vector X_k. According to the fact : affine transformation of Gausian Random Vector is a Gausian random vector of type $N = (A\mu, A\Sigma A^T)$.

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$E(1_A) = p(A) * 1 + (1 - p(A) * 0) = p(A)$

16

16.1

Correlation matrix of RV V is $E(VV^T)$. So correlation matrix of Z is

$$E(ZZ^T) = E([X^TY^T]^T[X^TY^T] = E\begin{pmatrix} XX^T & XY^T \\ YX^T & YY^T \end{pmatrix} =$$

According to equasion (39) , We can write it:

$$= \begin{pmatrix} E(XX^T) & E(XY^T) \\ E(YX^T) & E(YY^T) \end{pmatrix} \overset{X \perp Y}{=} \begin{pmatrix} E(XX^T) & 0_{n\times n} \\ 0_{m\times m} & E(YY^T) \end{pmatrix} = \begin{pmatrix} R_X & 0_{n\times n} \\ 0_{m\times m} & R_Y \end{pmatrix}$$

16.2

We can move through the same process as in previous subsection
and use equation (40) "$E((Z - E(Z))^T(Z - E(Z))) = \begin{pmatrix} \Sigma_X & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_Y \end{pmatrix}$" , and the fact that if X and Y are uncorrelated , then $\Sigma_{XY} = 0_{n\times m} \quad and \quad \Sigma_{YX} = 0_{m\times n}$

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$$E(XY) = \int_{\mathbb{R}^{n+m}} x * y * p(x,y)dxdy \overset{\text{X Y independent}}{=} \int_{\mathbb{R}^{n+m}} x * y * p(x)p(y)dxdy$$
$$= \int_{\mathbb{R}^n} x * p(x)dx \int_{\mathbb{R}^m} y * p(y)dy = \mu_X\mu_Y$$

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18.1

$$P(X_1) = P([X_1, 0]) + P([X_1, 1])$$
$$\text{Same to } X_2$$

18.2

$$E(X) = 0.5 * [0, 0] + 0.1 * [0, 1] + 0.3 * [1, 0] + 0.1 * [1, 1] = [0.4, 0.2]$$

18.3

$$E(XX^T) = 0.5 * \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + 0.1 * \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 0.3 * \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0.5 * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} 0.8 & 0.1 \\ 0.1 & 0.6 \end{pmatrix}$$

18.4

$$\Sigma_X = E(XX^T) - \mu_X \mu^T = \begin{pmatrix} 0.8 & 0.1 \\ 0.1 & 0.6 \end{pmatrix} - \begin{pmatrix} 0.16 & 0.08 \\ 0.08 & 0.16 \end{pmatrix} = \begin{pmatrix} 0.62 & 0.02 \\ 0.02 & 0.42 \end{pmatrix}$$

18.5

The vectors are not independent since for example :
$p(X = [0, 0]) = 0.5 \neq 0.48 = 0.6 * 0.8 = p(X_1 = 0)p(X_2 = 0)$

18.6

X_1 and X_2 are correlated , since Σ_X in non-zero .