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COMP6651 **40091993**
Prof. Tiberiu Popa Due Date: 22/Feb/2018

1 Prove or Disprove

1. $O(2^n) = O(4^n)$

   Let $O(2^n)$ is member of the set $0(4^n)$
   Using $O - notation$, $0 \leq f(n) \leq$ cg(n)
   $0 \leq 2^n \leq$ c4^n
   $0 \leq \frac{1}{2^n} \leq$ c
   Since, for all value of $n_o \geq 1$ and n $\geq n_o$, C is constant.

   Let $O(4^n)$ is member of the set $0(2^n)$
   Using $O - notation$, $0 \leq f(n) \leq cg(n)$
   $0 \leq 4^n \leq$ c2^n
   $0 \leq 2^n \leq$ c
   Since, It is impossible to exist for all value of $n_o \geq 1$ and n $\geq n_o$. Hence, it is contradicted.
   Hence, $O(2^n) \neq O(4^n)$

2. $O(log(2^n)) = O(log(4^n))$

   Let $O(log(2^n))$ is member of the set $O(log(4^n))$
   Using $O - notation$, $0 \leq f(n) \leq cg(n)$
   $0 \leq log(2^n) \leq clog(4^n)$
   $0 \leq \frac{1}{n} \leq$ c
   Since, for all value of $n_o \geq 1$ and n $\geq n_o$, C is constant. There must exist $f(\bar{n}) \leq \bar{c}g(n)$

   Let $O(log(4^n))$ is member of the set $O(log(2^n))$
   Using $O - notation$, $0 \leq f(n) \leq cg(n)$
   $0 \leq log(4^n) \leq clog(2^n)$
   $0 \leq 2 \leq$ c
   Since, for all value of $n_o \geq 1$ and n $\geq n_o$, C $\geq$ 2.

   Hence, $O(log(2^n)) = O(log(4^n))$.

3. $O(n!) = O((n+1)!)$

   Let $O(n!)$ is member of the set $O((n+1)!)$
   Using $O - notation$, $0 \leq f(n) \leq cg(n)$
   $0 \leq n! \leq$ c$(n+1)!$
   $0 \leq \frac{1}{n+1} \leq$ c
   Since, for all value of $n_o \geq 1$ and n $\geq n_o$, C is constant.

   Let $O((n+1)!)$ is member of the set $O(n!)$
   Using $O - notation$, $0 \leq f(n) \leq cg(n)$
   $0 \leq (n+1)! \leq cn!$

$0 \leq$ n+1 $\leq$ c
Since, It is impossible to exist for all value of $n_o \geq 1$ and n $\geq n_o$. Hence, it is contradicted.

Hence, $n! \neq (n+1)!)$.

2 Problems from the textbook:

1. Exercise 1.1-1

   $= \max(f(n), g(n)) \leq \max(f(n), g(n)) + \min(f(n), g(n))$
   $= f(n) + g(n)$

2. Exercise 3.1-2

   When |a| $\leq$ n, n + a $\leq$ 2n.
   Let $c_1 = 0.5^b$ and $c_2 = 2^b$.
   $0 \leq c_1 n^b \leq (n+a)^b \leq c_2 n^b$
   By putting value of c_1 and c_2, we will get $0 \leq (\frac{n}{2})^b \leq (n+a)^b \leq (2n)^b$
   Therefore, we can say that there exist c_1, c_2 and n_0 for all n $\geq n_0$. Hence, $(n+a)^b = \theta(n^b)$

3. Exercise 3.2-4

   We can say that function is polynomially bounded when $f(n) =$ O(n^k), when k is some constant.

   For $\lceil lgn \rceil$!, It is not polynomially bound because there must exist a, d and n_0 for all n $\geq n_0$.
   It may hold true if n = 2^k
   k = log n
   k! $\leq 2^k$
   Here, n = 2^k is contradicted as k! $\leq 2^k$ condition is not satisfied for k $\geq$ 1 and also factorial function are not exponentially bounded.
   Therefore, $\lceil lgn \rceil$! is not polynomially bounded

   For $\lceil lglgn \rceil$!, Suppose n = 2^{2^k}
   Here, k! $\leq 2^{2^k}$ for k $\geq$ 1.
   Therefore, $\lceil lglgn \rceil$! is polynomially bounded.

4. Exercise 3-4 (Except h)

   (a) $f(n) = O(g(n))$ implies $g(n) = O(f(n))$

   $f(n) = O(g(n))$ implies $g(n) = O(f(n))$ conjecture is not true.
   for eg. n = $O(n^2)$ but $n^2 \neq O(n)$

   (b) $f(n) + g(n) = \Theta(min(f(n), g(n)))$

Let $f(n)$ = n and $g(n) = n^2$.
n + $n^2 \neq \Theta(n)$ Therefore, $f(n) + g(n) = \Theta(min(f(n), g(n)))$ conjecture is not true.

(c) $f(n) = O(g(n))$ implies $lg(f(n)) = O(lg(g(n)))$, where $lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n.

For $f(n) = O(g(n))$, there exist c_1, c_2 and n_0 for all n $\geq n_0$
f(n) $\leq$ cg(n) and f(n) $\geq$ 1
Applying log on both side
$logf(n) \leq logc + logg(n)$
Now we have to prove that $f(n) \leq$ d$logg(n)$

$logc + logg(n) \leq$ d$logg(n)$
$\frac{logc+logg(n)}{logg(n)} \leq$ d
From above, we can say that $logg(n) \geq 1$

(d) $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{f(n)})$

This Conjecture is not true.
Suppose $f(n) = 4^n$, $g(n) = 2^n$ and $f(n) \in O(g(n))$
Using $O - notation$, 0 $\leq$ f(n) $\leq$ cg(n)
$0 \leq 4^n \leq$ c2^n
$0 \leq 2^n \leq$ c
Since, It is impossible to exist for all value of $n_o \geq 1$ and n $\geq n_o$. Hence, it is contradicted.

(e) $f(n) = O((f(n))^2)$

This Conjecture is not true.
Let $f(n) = \frac{1}{n}$
$0 \leq \frac{1}{n} \leq$ c$\frac{1}{n^2}$
$0 \leq n \leq$ c
Since, c must be constant. $0 \leq n \leq$ c implies contradiction.

(f) $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$

This Conjecture is true.
Since, $f(n) = O(g(n))$ implies that there exist a c and n_0 for all n $\geq n_0$
O $\leq f(n) \leq$ c$g(n)$
O $\leq \frac{f(n)}{c} \leq g(n)$

(g) $f(n) = \Theta(f(n/2))$

Let $f(n) = 2^{2n}$
Since, $f(n) = O(g(n))$ implies that there exist a c_1, c_2 and n_0 for all n $\geq n_0$
O $\leq c_1 g(n) \leq f(n) \leq c_2 g(n)$
O $\leq c_1 2^n \leq 2^{2n} \leq c_2 2^n$
O $\leq c_1 \leq 2^n \leq c_2$
Since, c must be constant. O $\leq c_1 \leq 2^n \leq c_2$ implies contradiction.
Therefore, Conjecture is false.

5. Exercise 4.5-1

   (a) $T(n) = 2T(n/4)$ +1

      a = 2, b = 4 and $f(n) = 1$
      $n^{log_4 2} = n^{\frac{1}{2}} = \Theta(n^{\frac{1}{2}})$
      Since, $f(n) = n^{log_2 4-k}$, where k =1
      Using case 1, we can conclude $T(n) = \Theta(\sqrt{n})$

   (b) $T(n) = 2T(n/4) + \sqrt{n}$

      a = 2, b = 4 and $f(n) = \sqrt{n}$
      $n^{log_4 2} = \sqrt{n} = \Theta(\sqrt{n})$
      Since, $f(n) = \sqrt{n}$
      Using case 2, we can conclude $T(n) = \Theta(\sqrt{n} logn)$

   (c) $T(n) = 2T(n/4)$ +n

      a = 2, b = 4 and $f(n) =$ n
      $n^{log_4 2} = \sqrt{n} = \Theta(\sqrt{n})$
      Since, $f(n) =$ n
      $af(n/b) \leq cf(n)$
      $2\frac{n}{4} \leq$ cn
      $\frac{1}{2} \leq$ c
      Using case 3, we can conclude that $T(n) = \Theta(n)$ as it holds regularity condition

   (d) $T(n) = 2T(n/4) + n^2$

      a = 2, b = 4 and $f(n) = n^2$
      $n^{log_4 2} = \sqrt{n} = \Theta(\sqrt{n})$
      Since, $f(n) = n^2$
      $af(n/b) \leq cf(n)$
      $2\frac{n^2}{16} \leq$ cn^2
      $\frac{1}{8} \leq$ cn^2
      Using case 3, we can conclude that $T(n) = \Theta(n^2)$ as it holds regularity condition

6. Exercise 4.5-5

   Let a =1, b = 3, $\epsilon = 1$ and $f(n) = 3n + 2^{3n}$
   $n^{log_3 1} = n^0 = 1 = \Theta(1)$
   Since, $f(n) = 3n + 2^{3n}$
   $af(n/b) \leq cf(n)$
   $3n + 2^{3n} \leq$ c$(3n + 2^{3n})$
   $(n + 2^n) \leq$ c$(3n + 2^{3n})$
   since, $3n < n + 2^n$, It fails regularity condition, even it satisfies $f(n) = \Omega(n^{log_b a+\epsilon})$

7. Exercise 4.1

   (a) $T(n) = 2T(n/2) + n^4$

a = 2, b = 2 and $f(n) = n^4$
$n^{log_2 2}$ = n = $\Theta(n)$
Since, $f(n) = n^4$
a$f(n/b) \leq cf(n)$
2 $\frac{n^4}{16} \leq$ cn^4
$\frac{1}{8} \leq$ c
Using case 3, we can conclude that $T(n) = \Theta(n^4)$ as it holds regularity condition

(b) $T(n) = T(7n/10)$ + n

a = 1, b = $\frac{10}{7}$ and $f(n) = n$
$n^{log_{\frac{10}{7}} 1}$ = 1 = $\Theta(1)$
Since, $f(n) = n$
a$f(n/b) \leq cf(n)$
$\frac{7n}{10} \leq$ cn
0.7 $\leq$ c
Using case 3, we can conclude that $T(n) = \Theta(n)$ as it satisifes regularity condition.

(c) $T(n) = 16T(n/4) + n^2$

a = 16, b = 4 and $f(n) = n^2$
$n^{log_4 16} = n^2 = \Theta(n^2)$
Since, $f(n) = n^2$
Using case 2, we can conclude that $T(n) = \Theta(n^2 logn)$

(d) $T(n) = 7T(n/3) + n^2$

a = 7, b = 3 and $f(n) = n^2$
$n^{log_3 7} = n^{1.77}$ =$\Theta(n^{1.77})$
Since, $f(n) = n^2$
a$f(n/b) \leq cf(n)$
7$\frac{n^2}{9} \leq$ cn^2
$\frac{7}{9} \leq$ c
Using case 3, we can conclude that $T(n) = \Theta(n^2)$ as it holds regularity condition

(e) $T(n) = 7T(n/2) + n^2$

a = 7, b = 2 and $f(n) = n^2$
$n^{log_2 7} = n^{2.80} = \Theta(n^{2.80})$
Since, $f(n) = n^2$
Using case 1, we can conclude that $T(n) = \Theta(n^{log_2 7})$

(f) $T(n) = 2T(n/4) + \sqrt{n}$

a = 2, b = 4 and $f(n) = \sqrt{n}$
$n^{log_4 2} = n^{1.77} = \Theta(n^{1.77})$
Since, $f(n) = n^2$
a$f(n/b) \leq cf(n)$
2$\frac{\sqrt{n}}{4} \leq$ c$\sqrt{n}$

$\frac{1}{2} \leq$ c
Using case 3, we can conclude that $T(n) = \Theta(\sqrt{n})$ as it holds regularity condition

(g) $T(n) = T(n-2) + n^2$

The Master method is not apply to this recurrence.

8. Exercise 4.3

(a) $T(n) = 4T(n/3) + nlogn$

a = 4, b = 3 and $f(n) = nlogn$
$n^{log_3 4} = n^{1.26} = \Theta(n^{1.26})$
Since, $f(n) = nlogn = \Theta(nlogn)$
Using case 1, we can conclude that $T(n) = \Theta(n^{log_3 4})$

(b) $T(n) = 3T(n/3) + n/logn$

a = 3, b = 3 and $f(n) = n/logn$
$n^{log_3 3} = n = \Theta(n)$
Since, $f(n) = n/logn = \Theta(n/logn)$
The Master method is not apply to this recurrence as it falls under case 1 and case 2.

(c) $T(n) = 4T(n/2) + n^2\sqrt{n}$

a = 4, b = 2 and $f(n) = n^2\sqrt{n}$
$n^{log_2 4} = n^2 = \Theta(n^2)$
Since, $f(n) = n^2\sqrt{n} = \Theta(n^2\sqrt{n})$
a$f(n/b) \leq cf(n)$
$4\frac{n^2\sqrt{n}}{4\sqrt{2}} \leq$ c$n^2\sqrt{n}$
$\frac{1}{\sqrt{2}} \leq$ c
Using case 3, we can conclude that $T(n) = \Theta(n^2\sqrt{n})$ as it holds regularity condition

(d) $T(n) = 3T(n/3 - 2) + n/2$

The Master method is not apply to this recurrence.

(e) $T(n) = 2T(n/2) + n/logn$

a = 2, b = 2 and $f(n) = n/logn$
$n^{log_2 2} = n = \Theta(n)$
Since, $f(n) = n/logn = \Theta(n/logn)$
The Master method is not apply to this recurrence as it falls under case 1 and case 2.

(f) $T(n) = T(n/2) + T(n/4) + T(n/8) + n$

The Master method is not apply to this recurrence.

(g) $T(n) = T(n-1) + 1/n$

The Master method is not apply to this recurrence.

(h) $T(n) = T(n-1) + logn$

The Master method is not apply to this recurrence.

(i) $T(n) = T(n-1) + 1/logn$

The Master method is not apply to this recurrence.

(j) $T(n) = \sqrt{n}T(\sqrt{n}) + n$

The Master method is not apply to this recurrence.