

An example showing the matrix version of the simple linear model

```
> library(faraway)
> #Matrix approach to Simple Linear Model
> #Experiment: 4 rats were injected with a dose(mcg) of a drug.
> #The time (minute) to run a maze was recorded for each rat.
> # x=dose y=time
```

Model:  $time_i = \beta_0 \times 1 + \beta_1 \times dose_i + error_i$

```
> dose=c(0, 2, 4, 6) ← vector of doses
> y=c(7, 5, 4, 4) ← vector of responses(time)
```

$n=4$

```
> X=as.matrix(cbind(1,dose)) ← creates X-matrix
> X #this is what the X-matrix looks like
```

	dose
[1,]	0
[2,]	2
[3,]	4
[4,]	6

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} \quad Y = \begin{bmatrix} 7 \\ 5 \\ 4 \\ 4 \end{bmatrix}$$

```
> y #this is what the vector of y responses looks like
```

```
[1] 7 5 4 4
```

```
> #create a data frame named "example" for later use
> example <- data.frame(dose,y)
> example
```

the four observations had the format  $(x_i, y_i) i=1,2,3,4$

	dose	y
1	0	7
2	2	5
3	4	4
4	6	4

$$y_1 = \beta_0 \times 1 + \beta_1 \times dose_1 + Error_1$$

$$\vdots$$

$$y_4 = \beta_0 \times 1 + \beta_1 \times dose_4 + Error_4$$

$$\text{or } \underline{Y} = \underline{X} \underline{\beta} + \underline{Error}$$

```
> XtX <- t(X) %*% X
> XtX #show XtX - should be a 2x2 symmetric matrix about the
```

diagonal

	dose
dose	4 12
	12 56

← the (1,2) element = the (2,1) element

$$X'X = \begin{bmatrix} 4 & 12 \\ 12 & 56 \end{bmatrix}$$

and is  $p \times p$

p=#parameters

$X'$  is  $p \times n$   $X$  is  $n \times p$

$X'X$  is  $p \times p$

$$(X'X) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 12 & 56 \end{bmatrix} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \quad \text{sample size} \quad \text{Page 2}$$

```
> XtXi <- solve(t(X) %*% X) #compute XtX inverse
> XtXi #show XtX inverse
```

$$\begin{array}{cc} & \text{dose} \\ & 0.70 \quad -0.15 \\ \text{dose} & -0.15 \quad 0.05 \end{array} \leftarrow (X'X)^{-1}$$

```
> I2 <- XtX %*% XtXi
> I2 #should get back the Identity (2x2) matrix
```

$$\begin{array}{cc} & \text{dose} \\ & 1 \quad 0 \\ \text{dose} & 0 \quad 1 \end{array} (X'X)(X'X)^{-1} = \begin{bmatrix} 4 & 12 \\ 12 & 56 \end{bmatrix} \begin{bmatrix} 0.70 & -0.15 \\ -0.15 & 0.05 \end{bmatrix} = \begin{bmatrix} 2.8-1.8 & -.6+.6 \\ 8.4-8.4 & -1.8+2.8 \end{bmatrix}$$

$$\begin{array}{cc} & \text{dose} \\ & 1 \quad 0 \\ \text{dose} & 0 \quad 1 \end{array} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$$

```
> Xty <- t(X) %*% y
> Xty
```

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 50 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$X'Y \xrightarrow{[1,1]} \begin{array}{cc} & [1,1] \\ & 20 \\ \text{dose} & 50 \end{array}$

```
> Bhat <- XtXi %*% t(X) %*% y #Beta hat is equal to
(X'X)inverse * X'y
```

$$\begin{array}{cc} & [1,1] \\ & 6.5 \\ \text{dose} & -0.5 \end{array} \quad \hat{\beta} = (X'X)^{-1} X'Y = \begin{bmatrix} 0.70 & -0.15 \\ -0.15 & 0.05 \end{bmatrix} \begin{bmatrix} 20 \\ 50 \end{bmatrix} = \begin{bmatrix} 14-7.5 \\ -3+2.5 \end{bmatrix} = \begin{bmatrix} 6.5 \\ -0.5 \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} 6.5 \\ -0.5 \end{bmatrix}$$

```
> #compute the standard errors of Beta hat which can be found by
> #taking the square root of the diagonal elements of the
matrix XtXi * sigma_hat
```

```
> mod <- lm(y ~ dose, data=example)
> modsum <- summary(mod)
> modsum #check to be sure the matrix computations agree with
the output from R
```

Call:

```
lm(formula = y ~ dose, data = example)
```

Residuals:

1	2	3	4
0.5	-0.5	-0.5	0.5

$SSE = (0.5)^2 + (-0.5)^2 + (-0.5)^2 + (0.5)^2 = 1.0$   
 $SSE = [e_1, e_2, e_3, e_4] \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \sum e_i^2 = 1.0$

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	6.5000	0.5916	10.987	0.00818 **
dose	-0.5000	0.1581	-3.162	0.08713 .

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.7071 on 2 degrees of freedom  
Multiple R-squared: 0.8333, Adjusted R-squared: 0.75  
F-statistic: 10 on 1 and 2 DF, p-value: 0.08713

```
> sigma_hat=sqrt(deviance(mod)/df.residual(mod)) #compute the
estimate of sigma
```

```
> sigma_hat
[1] 0.7071068
```

$\sqrt{MSE} = \sqrt{\frac{1}{2}} = \sqrt{.5} = 0.7071$

```
> sigma_hat_of_Beta_hat=sqrt(diag(XtXi))*sigma_hat
> sigma_hat_of_Beta_hat
```

```
dose
0.5916080 0.1581139
```

known as the variance-covariance matrix of  $\hat{\beta}$

$\hat{\sigma}^2 \times (X'X)^{-1} = \begin{bmatrix} (.5)(.70) & (.5)(-.015) \\ (.5)(-.015) & (.5)(0.05) \end{bmatrix}$

$= \begin{bmatrix} \sqrt{0.35} & \sim \\ \sim & \sqrt{0.025} \end{bmatrix}$

$= \begin{bmatrix} 0.35 & -0.075 \\ -0.075 & 0.025 \end{bmatrix}$

std error of  $\hat{\beta}_0 = \sqrt{.35} = .5916$  and std error of  $\hat{\beta}_1 = \sqrt{0.025} = .1581$