Midterm II Practice Problems

CS 323 , Spring 2019

Problem 1 [10+10 points]
Given the data points $(1,2),(2,0),(3,1),(4,-1)$,
a) Find the cubic interpolation polynomial in the Lagrange form. DO NOT SIMPLIFY.

Solution

$$p(x) = 2\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} - \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}.$$

b) Find the cubic interpolation polynomial in the Newton form. DO NOT SIMPLIFY.
Solution Caculate the divided difference using the divided difference table,

x_i	y_i	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
1	2	$\frac{2-0}{1-2} = -2$	$\frac{-2-1}{1-3} = 1.5$	$\frac{1.5-(-1.5)}{1-4} = -1$
2	0	$\frac{0-1}{2-3} = 1$	$\frac{1-(-2)}{2-4} = -1.5$	–
3	1	$\frac{1-(-1)}{3-4} = -2$	–	–
4	-1	–	–	–

The interpolation polynomial in the Newton's form is

$$p(x) = 2 - 2(x-1) + 1.5(x-1)(x-2) - (x-1)(x-2)(x-3).$$

Problem 2

Construct a piecewise linear interpolating polynomials for the function

$$f(x) = \sin x, \text{ at } x_0 = 0, x_1 = \pi/2, x_2 = \pi,$$

and find a bound for the absolute error on the interval $[0, \pi]$.

Solution: The piecewise linear function

$$p_1(x) = \begin{cases} \frac{2}{\pi}x & 0 \le x \le \frac{\pi}{2} \\ 1 - \frac{2}{\pi}(x - \frac{\pi}{2}) & \frac{\pi}{2} \le x\pi. \end{cases}$$

By error formula,

$$|f(x) - p_1(x)| \le \frac{M_2}{2}|(x-0)(x-\frac{\pi}{2})|$$

where $M_2 = \max_{0\le x\le\frac{\pi}{2}} |f''(x)|$. Since $f'(x) = \cos x$ and $f''(x) = -\sin x$, we get

$$M_2 = \max_{0\le x\le\frac{\pi}{2}} \sin x = 1$$

So

$$|f(x) - p_1(x)| \le \frac{1}{2}\left|(x-0)(x-\frac{\pi}{2})\right|.$$

To estimate the error on the interval $[0.\frac{\pi}{2}]$, we take maximum and get

$$|f(x) - p_1(x)| \le \frac{1}{2}\max_{0\le x\le\frac{\pi}{2}} |(x-0)(x-\frac{\pi}{2})| = \frac{1}{2}\left(\frac{\pi}{4}\right)^2$$

Similarly (by symmetry), on the interval $[\frac{\pi}{2}, \pi]$

$$|f(x) - p_1(x)| \le \frac{1}{2}\max_{\frac{\pi}{2}\le x\le\pi} |(x-\frac{\pi}{2})(x-\pi)| = \frac{1}{2}\left(\frac{\pi}{4}\right)^2$$

Problem 3 Let $f(x) = \frac{1}{1+x^2}$ defined on the interval $[-2, 2]$.
a) Approximate the integral $\int_{-2}^{2} f(x)dx$ by Trapezoid method T_4 with 4 equally spaced subintervals.
Solution

$$T_4 := \frac{1}{2}[\frac{1}{5} + 2 * \frac{1}{2} + 2 * 1 + 2 * \frac{1}{2} + \frac{1}{5}] = \frac{11}{5}.$$

b) Approximate the integral $\int_{-2}^{2} f(x)dx$ by Simpson method S_2 with 2 equally spaced subintervals.

Solution

$$T_2 := \frac{2}{6}[\frac{1}{5} + 4 * \frac{1}{2} + 2 * 1 + 4 * \frac{1}{2} + \frac{1}{5}] = \frac{32}{15}.$$

Problem 4 Let $f(x)$ be a cubic polynomial defined on interval $[-1, 1]$. Determine a Gaussian intergration formula with minimal number of nodes such that the integral formula

$$\sum_{i=0}^{n} f(x_i)w_i$$

is exact for cubic polynomials.

Solution: With $n+1$ nodes and weights, the Gaussian integral is exact for polynomials of deg $2n+1$. For cubic polynomials $(2n+1) = 3$, we have $n = 1$. So we need two nodes and weights of Gaussian formula exact for cubic polynomials. The formula for $n = 1$ gives

$$f(-\sqrt{\frac{1}{3}}) + f(\sqrt{\frac{1}{3}})$$