CS 323 Quiz 1 February 22, 2019

Name:____

Show all your work

(10) Consider the following two iteration schemes for finding the root $x = 2$ of $F(x) = x^2 - x - 2$: (1) $x_{n+1} = \sqrt{2 + x_n}$, (2) $x_{n+1} = 1 + 2/x_n$, where the starting guess x_0 is chosen in the interval $[5/4, 7]$.

Fixed Point Iteration Theorem says that if (i) f and f' are continuous on a finite interval $[a, b]$, (ii) $a \leq f(x) \leq b$ for all x in $[a, b]$, and (iii) $|f'(x)| \leq L$ for all x in $[a, b]$ and some constant L satisfying $0 < L < 1$, then the iteration scheme $x_{n+1} = f(x_n)$ converges to the unique fixed point s of f for all starting guesses x_0 in $[a, b]$.

For each of these iteration schemes, it is easily seen that both f and f' are continuous on $[5/4, 7]$. Determine which of the remaining two hypotheses above are valid and which are not for each of the two iteration schemes. Each hypothesis must be accompanied by either a demonstration of its validly or a demonstration of why it fails.

Solution:

1) Let $G(x) = \sqrt{2 + x}$. Then $G'(x) = \frac{1}{2}(2 + x)^{-1/2}$. Since $G'(x)$ is a decreasing function,

$$|G'(x)| \leq |G'(5/4)| = 13^{-1/2} < 1 \quad \text{for all } x \in [5/4, 7].$$

To check the last condition, we have

$$\sqrt{13}/2 = G(5/4) \leq G(x) \leq G(7) = 3$$

because $G(x)$ is an increasing function in $[5/4, 7]$. Since $5/4 < \sqrt{13}/2$ and $3 < 7$, we have

$$5/4 < G(x) < 7 \quad \text{for all } x \in [5/4, 7].$$

Thus, the first iterative method is convergent.

2) Let $G(x) = 1 + 2/x$. Then $G'(x) = -2x^{-2}$ and

$$|G'(5/4)| = 32/25 > 1.$$

So the second condition does not hold. Thus, the second iterative method may not converge.