In M. Majumdar, ed., Equilibrium and Dynamics, Macmillan, London, 1992, pp. 179-200.

8 Survival under Production Uncertainty*

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1 INTRODUCTION

In this study we explore some questions related to the 'survival' and 'failure' of an economic agent. Our theme is unquestionably removed from the mainstream of analysis in economics, where the emphasis is on optimisation – more on 'achieving the best' than on 'averting failure'. Given some institutional constraints, however, the behaviour of some economic agent (a consumer, a manager of a firm, a small farmer) can perhaps be modelled more realistically as a sequence of decisions meeting a minimal level of some performance index (a nutritional requirement, a minimal rate of dividend, a minimal level of debt service). In what follows, a number of issues will be pursued in simple partial equilibrium frameworks, where the outcome resulting from an action is subjected to exogenous random shocks.

Questions of 'survival' have not entirely escaped the attention of economic theorists. There is by now a large literature on bankruptcy of firms in a world of uncertainty and incomplete markets. Following his assessment of the developments of Walrasian equilibrium theory, Koopmans (1957, pp. 62–3) observed that sufficient conditions ensuring the survival of consumers in an equilibrium posed a 'considerable challenge to further research'. Some years later, in her critique of equilibrium and steady state analysis, Joan Robinson (1962, p. 3) complained that 'the prices that rule in an equilibrium at a particular moment may well be such that some of the individuals in question are in the process of being starved out of existence'; and that an equilibrium that involves 'accumulation of means of production, consumption of exhaustible resources or starvation for some group' is in the course of 'upsetting itself from within and chance events may upset it from without'. Research in the area of Walrasian equilibrium in the 1960s and 1970s shed little light on these issues (not much can be concluded

*The research reported in this paper was started during Majumdar's visit to the Bell Laboratories in 1981. Majumdar wishes to acknowledge the support from the National Science Foundation. Thanks are due to Professors William Brock, Rabi Bhattacharya, Debraj Ray, Tapan Mitra, Shubhashis Gangopadhyay, David Easley and Daniel Spulber for helpful conversations and comments. The views presented here are those of the authors, and not necessarily those of AT&T Bell Laboratories.

about the nature of an equilibrium income distribution!), but the long-run difficulties created by the presence of exhaustible resources in maintaining a positive steady level of consumption was recognised in the elegant paper of Solow (1974), and subsequently by others.[1]

Our approach is closely related to the classical 'gambler's ruin' problem. An economic agent starts with an *initial fortune*. From this fortune is subtracted a positive number, say c. The parameter c is a datum of the problem, and represents the agent's minimum level of 'consumption', or a minimum level of debt service, etc. If the remainder is zero or negative, then the agent is declared 'ruined'. If the remainder is strictly positive, then that quantity becomes an *input* into some productive activity (i.e., an 'investment'). The *output* of this productive activity is a random variable whose probability distribution depends on the size of the corresponding input (in a known way). This output is then the agent's fortune at the beginning of the next period, and so on. The agent is ruined at the first date T, if any, at which his fortune reaches or falls below c. Of course, the agent may never be ruined, in which case we shall set $T = \infty$ and say that the agent *survives* (forever). We shall study conditions on the technology, initial fortune, and c, that determine whether the probability of survival is 1, 0, or strictly between 0 and 1.

Let $P(y, c)$ denote the probability of survival if the initial fortune is y and the required consumption per period is c. (In the classical 'gambler's ruin' problem, $c = 0$).

We give particular attention to two cases. In the first case the decreasing returns to scale are sufficiently strong so that the sustainable stock would be bounded even if c were zero; likewise, the maximum sustainable consumption is finite. In this case, if c does not exceed the maximum sustainable consumption then there are two critical values of y, say $\eta_1(c)$ and $\eta_2(c)$, such that

$$\begin{aligned} P(y, c) &= 0, & y &\leq \eta_1(c), \\ 0 < P(y, c) &< 1, & \eta_1(c) < y &< \eta_2(c), \\ P(y, c) &= 1, & y &\geq \eta_2(c); \end{aligned} \tag{8.1.1}$$

we also show how to calculate the critical initial fortunes, $\eta_1(c)$ and $\eta_2(c)$. In addition, we show that $P(y, c)$ is nondecreasing in y and nonincreasing in c. (Of course, in order to obtain these conclusions we must make a number of assumptions, which we hope the reader will find economically meaningful.) If, on the other hand, c exceeds the maximum sustainable stock, then $P(y, c) = 0$ for all y and $c > 0$.

In the second case we assume 'constant returns to scale', so that in each period the output is proportional to the input. The coefficient of proportionality varies from period to period in a stochastic way; let r_t denote this coefficient in period t. (Note that $r_t - 1$ is the corresponding 'rate of return'.) In the simplest case, we assume that the successive random variables r_t are independent and identically distributed. In this case we can obtain a more precise characterisation of $P(y, c)$. In particular, denoting the expected value of a random variable x by Ex, we show that $P(y, c) = 0$ for all y and c if $E\ln r_t \leqslant 0$, and that $P(y, c)$ depends monotonically on the ratio of y to c if $E\ln r_t > 0$, approaching unity as (y/c) increases without limit (or in some cases reaching unity for sufficiently large but finite (y/c).)

In section 2 we prepare the way by analysing the case of certainty. Sections 3 and 4 are devoted to the cases of 'bounded growth' and 'constant returns', respectively. Section 5 deals with extensions of the model to the case in which the environments in successive periods are Markovian rather than independent and identically distributed.

In the models we discuss here the agent is actually passive, being restricted to an exogenously determined (although stochastic) technology. In Majumdar and Radner (1991) we discuss the case of an 'active' agent who can sequentially choose from some set of available technologies, or even devote resources to expanding that set.

2 SURVIVAL IN A DETERMINISTIC ENVIRONMENT

An economic agent starts with an initial stock y of money or some other reproducible asset (the metaphorical 'corn' of one-sector growth theory). In each period t the agent is required to 'consume' a positive amount c of the beginning-of-period stock, y_t. (See section 1 for alternative interpretations of c.) The remaining stock in that period, x_t, is 'invested', and the corresponding output (principal plus return) is the next begining-of-period stock, y_{t+1}. The outputs are related to the corresponding inputs by a 'production function' g:

$$y_{t+1} = g(x_t)$$

Given the initial stock, y, the required consumption per period, c, and the production function, g, the complete system evolves according to the equations:

$$y_0 = y \tag{8.2.1}$$

$$x_t = y_t - c, \; t \geqslant 0 \tag{8.2.2}$$

$$y_{t+1} = g(x_t), \; t \geqslant 0 \tag{8.2.3}$$

Let T be the first t, if any such that $x_t < 0$; if there is no such t, then $T \equiv \infty$.

If T is finite we shall say that the agent *survives up to* (but not including) *period T*. We shall say that the agent *survives* (forever) if T is infinite (i.e., if $x_t \geqslant 0$ for all t). In this section, we shall investigate conditions on y, c, and g that are sufficient for the agent's survival.

We start with an example.

Example 2.1 Suppose that

$$g(x) = \left.\begin{matrix} 0 \\ 2x \\ \frac{x}{2} + 6 \end{matrix}\right\} , \begin{cases} x \leqslant 0 \\ 0 \leqslant x \leqslant 4 \\ x \geqslant 4 \end{cases} \tag{8.2.4}$$

The graph of g is shown in Figure 8.1. Note that, for nonnegative x,

$$g(x) - x \quad \left.\begin{matrix} > \\ = \\ < \end{matrix}\right\} \; 0 \quad \text{as} \quad \begin{cases} 0 < x < 12 \\ x = 0, 12 \\ x > 12 \end{cases} \tag{8.2.5}$$

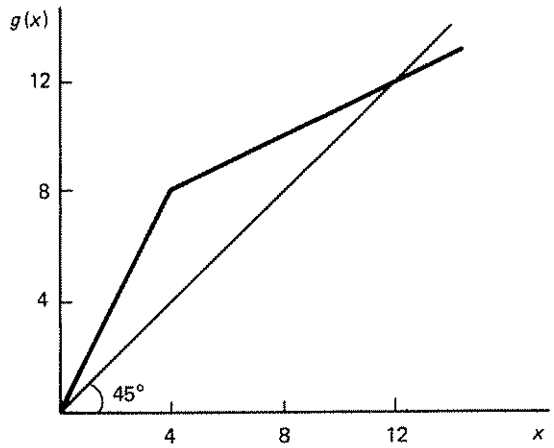

Figure 8.1

Furthermore, $g(x) - x$ attains a maximum of 4 at $x = 4$. Hence 4 is the maximum sustainable consumption per period. However, the agent cannot sustain a per-period consumption equal to 4 unless the starting stock y is at least 8, for if $c = 4$ and $y < 8$, then

$$y_1 = 2x_0 = 2(y_0 - 4) = y_0 + (y_0 - 8) < y_0$$

Continuing in this way, one sees that

$$y_{t+1} = 2x_t = 2(y_t - 4) = y_t + (y_t - 8) < y_t$$

so that not only is y_t decreasing over time, but in larger and larger steps; eventually a time will come when the stock falls below c, and the agent is ruined ($x_t < 0$).

What if $c < 4$? For example, suppose $c = 2$. Figure 8.2 shows the graph of $y = g(x) - 2$. We see that

$$g(x) - 2 - x \quad \left.\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0 \quad \text{as} \quad \begin{cases} 2 < x < 8 \\ x = 2, 8 \\ x < 2, x > 8 \end{cases} \tag{8.2.6}$$

The law of motion for x is

$$\begin{aligned} x_0 &= y - 2 \\ x_{t+1} &= g(x_t) - 2 \end{aligned} \tag{8.2.7}$$

Hence:

1. x_t decreases and becomes negative if $x_0 < 2$ ($y < 4$).
2. $x_t = 2$ for all t if $x_0 = 2$.
3. x_t converges monotonically to 8 if $x_0 > 2$.

Note that the agent survives only in cases 2 and 3 ($y \geqslant 4$). Note, too, that if c were decreased further, then the minimum initial stock needed for survival would be smaller.

The general method is now clear.

The law of motion of (x_t) is:

$$\begin{aligned} x_0 &= y - c \\ x_{t+1} &= g(x_t) - c, \; t \geqslant 0 \end{aligned} \tag{8.2.8}$$

We shall make the following assumptions about the function g:

Assumption 1 (*Productive Technology*) g is continuous and strictly increasing;
$g(x) > x$ for some $x > 0$; $g(x) = 0$ for $x \leqslant 0$.

Assumption 2 (*Bounded Growth*) For some $\bar{x} > 0$, $g(x) < x$ for $x > \bar{x}$.

Assumption 3 (*Concavity*) g is concave for $x \geqslant 0$.

(As we shall indicate below, not all of these assumptions are needed for some of our results.)

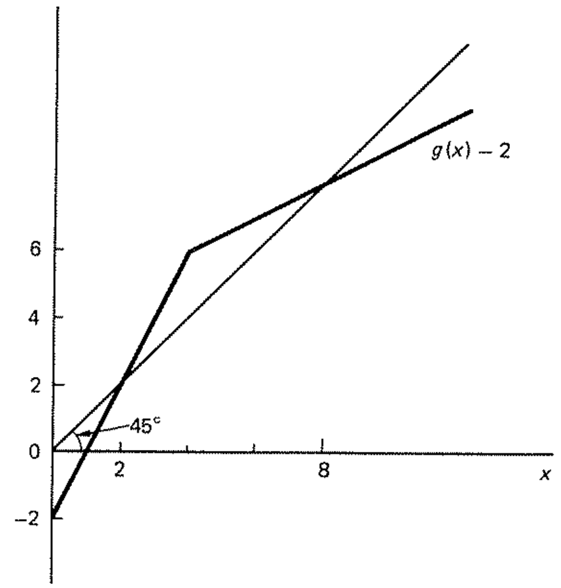

Figure 8.2

Define the *net return function* h by

$$h(x) \equiv g(x) - x \tag{8.2.9}$$

It follows from our assumptions that there is a unique $\hat{x} > 0$ such that $g(\hat{x}) = \hat{x}$, and h satisfies

$$h(x) \quad \left.\begin{matrix} > \\ = \\ < \end{matrix}\right\} \; 0 \quad \text{as} \quad \begin{cases} 0 < x < \hat{x} \\ x = 0, x \\ x > \hat{x} \end{cases} \tag{8.2.10}$$

Since $g(x) = 0$ for $x \leqslant 0$, all statements about g *and* h will be understood to be for nonnegative arguments, unless something *explicit* is said to the contrary. (In any case, we are interested in following the system only up to the 'failure time' T.)

The maximum sustainable consumption is

$$H \equiv \max_x h(x) \tag{8.2.11}$$

From the law of motion, (8.2.8), and the definition of h, (8.2.9),

$$x_{t+1} = x_t + h(x_t) - c \tag{8.2.12}$$

If $c > H$,

$$x_{t+1} - x_t \leqslant H - c < 0 \tag{8.2.13}$$

Hence, if $c > H$, x_t will eventually fall to or below zero. On the other hand, if $0 < c < H$, there will be two roots, say ξ' and ξ'' of the equation

$$x = g(x) - c \tag{8.2.14}$$

which have the properties:

$$0 < \xi' < \xi'' < \hat{x}$$

$$h(x) - c \left.\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0 \quad \text{as} \quad \begin{cases} \xi' < x < \xi'' \\ x = \xi', \xi'' \\ x < \xi', x > \xi'' \end{cases} \tag{8.2.15}$$

An inspection of (8.2.12) and (8.2.15) reveals that:

1. If $x_0 < \xi'$, then x_t reaches or falls *below* 0 in finite time.
2. If $x_0 = \xi'$, then $x_t = \xi'$ for all t.
3. If $x_0 > \xi'$, then x_t converges monotonically to ξ''.

(In Figure 8.2, ξ', $= 2$ and $\xi'' = 8$.)

Note that if $c = H$, there are two possibilities: either $\xi' = \xi''$ (i.e., $h(x)$ attains its maximum H at a unique point ξ') or, for all x in a nondegenerate interval $[\xi', \xi'']$, $h(x)$ attains its maximum (i.e., for all x in $[\xi', \xi'']$, the equation $h(x) = c$ is satisfied). Finally, if $c = 0$. $\xi' = 0$ and $\xi'' = \hat{x}$.

The implications of the foregoing discussion for survival are summarised in the following proposition:

Proposition 1

(1) If $c > H$, then there is no y from which survival (forever) is possible.
(2) If $0 < c \leqslant H$, then there is a ξ', with

$$h(\xi') = c, \xi' > 0$$

such that survival is possible if and only if the initial stock

$$y \geqslant \xi' + c$$

(3) $c = H$ implies $h(\xi') = H$, and ξ' tends to 0 as c tends to 0.

We now give a brief account of the case in which Assumption 2 (Bounded Growth) is replaced by the assumption that g is linear:

$$g(x) = rx,\ r > 1 \tag{8.2.16}$$

(The condition $r > 1$ is implied by Assumption 1. Equation (8.2.16) holds, of course, only for $x \geqslant 0$.) With this specification of g, the law of motion (8.2.8) becomes the simple linear difference equation:

$$x_{t+1} = rx_t - c \tag{8.2.17}$$

It is easily verified that equation (8.2.17) has the solution:

$$x_t = r^t(x_0 - \xi) + \xi$$

$$\xi = \frac{c}{r-1} \tag{8.2.18}$$

Hence (1) x_t reaches 0 in finite time if $x_0 < \xi$; (2) $x_t = \xi$ for all t if $x_0 = \xi$; (3) x_t diverges to (plus) infinity if $x_0 > \xi$. The implications for survival are:

Proposition 2 In the linear case (8.2.16), survival is possible if and only if the initial stock $y \geqslant \xi + c$, or equivalently

$$y \geqslant \left(\frac{r}{r-1}\right) c \tag{8.2.19}$$

Finally, we note that if $r \leqslant 1$ then, for any $c > 0$, there is no initial stock from which survival is possible (recall the 'cake-eating' example of Gale, 1967).

3 UNCERTAIN ENVIRONMENT: THE CASE OF BOUNDED GROWTH

Suppose now that the returns from investment are uncertain rather than deterministic. We model this by supposing that, given the previous input x_{t-1}, the corresponding beginning-of-period stock y_t depends not only on x_{t-1}, but on a random event, E_t. To minimise measure-theoretic difficulties we assume that the set of alternative events at each date is the same for all dates, namely a finite set $\underset{\sim}{E}$.

We shall begin by assuming that the sequence of events (E_t) are independent and identically distributed. (More general cases will be discussed briefly below.)

The law of motion of stocks Y_t and inputs X_t is, given the initial stock y,

$$Y_0 = y \tag{8.3.1}$$

$$X_t = Y_t - c,\ t \geqslant 0$$

$$Y_{t+1} = g(X_t, E_{t+1}),\ t \geqslant 0$$

For each event e in $\underset{\sim}{E}$, the function $g(\cdot, e)$ satisfies Assumptions 1–3. To avoid trivial cases, we assume that $\underset{\sim}{E}$ has at least two elements, and that every event has strictly positive probability.

We denote the stocks and inputs by capital letters to remind the reader that, expect for Y_0 and X_0, they are random variables.[2] As before, let T denote the first t, if any such that $X_t \leqslant 0$; otherwise $T = \infty$. We shall say that the agent *survives with probability* P if prob $(T = \infty) = P$.

We now introduce a further assumption which expresses the idea that the event in $\underset{\sim}{E}$ can be ranked in order of how favourable they are to production. This assumption is substantive, and not made merely for technical convenience. We believe it is plausible in many situations; it has strong implications.

Assumption 4 (Monotonicity) The event set $\underset{\sim}{E}$ is (strictly) linearly ordered by a relation more favourable than; if e is more favourable than e', then for every $x > 0$,

$$g(x, e) > g(x, e') \tag{8.3.2}$$

Since $\underset{\sim}{E}$ is finite, there is a *most favourable event*, $\bar{e}$, and a *least favourable event*, $\underset{\sim}{e}$; these will play an important role in the analysis.

We start with two examples: the first is an admittedly degenerate one, but is still illuminating. The second example will be adapted from the example of section 2, and will pave the way for a general qualitative analysis.

Example 3.1 There are only two elements $(\bar{e}, \underset{\sim}{e})$ in $\underset{\sim}{E}$. The 'good' technology $g(x, \bar{e})$ is

$$\begin{aligned} g(x, \bar{e}) &= 2x && \text{for } x \in [0, 4] \\ &= x/2 + 6 && \text{for } x \geqslant 4 \end{aligned} \tag{8.3.1a}$$

The 'bad' technology $g(x, \underline{e})$ is

$$g(x, \underline{e}) = 3x/2 \qquad \text{for } x \in [0, 4]$$

$$= x/2 + 4 \qquad \text{for } x \geq 4 \tag{8.3.1b}$$

Let $c = 2$ and $Y(0) = 4$. If $\bar{e}$ occurs in *every* period (and this plainly has zero probability) the (stationary) programme $x(t) = 2$, $c = 2$ guarantees survival. However, if $\underline{e}$ occurs *only once* (i.e., in some period $t' \geq 1$, and $\bar{e}$ occurs in all $t \neq t'$), note that with $x(t' - 1) \leq 2$

$$y(t') = g(x(t' - 1), \underline{e}) \leq 3$$

Hence, $x(t') \leq 1$, and

$$y(t' + 1) = g(x(t'), \bar{e}) \leq 2$$

But this means that $x(t' + 1) = 0$ and the agent is ruined. Clearly, the probability of survival is zero. Interestingly, this conclusion does *not* depend on the magnitude of the probability of occurrence of the bad technology.[3] The extreme conclusion that 'all those good years' cannot compensate for *even one* 'bad' year is *not* typical. However, as we shall see shortly, the persistence of bad luck does have serious implications for the survival problem.

Example 3.2 Suppose that $\underline{E}$ has only two states, $\underline{e}$ and $\bar{e}$, with

$$g(x, \underline{e}) = \begin{cases} \dfrac{3x}{2} & , 0 \leq x \leq 5 \\ \dfrac{x}{2} + 5 & , x \geq 5 \end{cases} \tag{8.3.2a}$$

$$g(x, \bar{e}) = \begin{cases} 2x & , 0 \leq x \leq 4 \\ \dfrac{x}{2} + 6 & , x \geq 4 \end{cases} \tag{8.3.2b}$$

(Note that equation (8.3.2b) is the same as equation (8.2.4).) Let $c = 2$. Figure 8.3 shows the graphs of $g(x, \underline{e}) - 2$ and $g(x, \bar{e}) - 2$.

For each event e in $\underline{E}$, define

$$h(x, e) \equiv g(x, e) - x \tag{8.3.3}$$

(recall (8.2.9)). Correspondingly, define $H(e)$ as in (8.2.11), and $\hat{x}(e)$ by

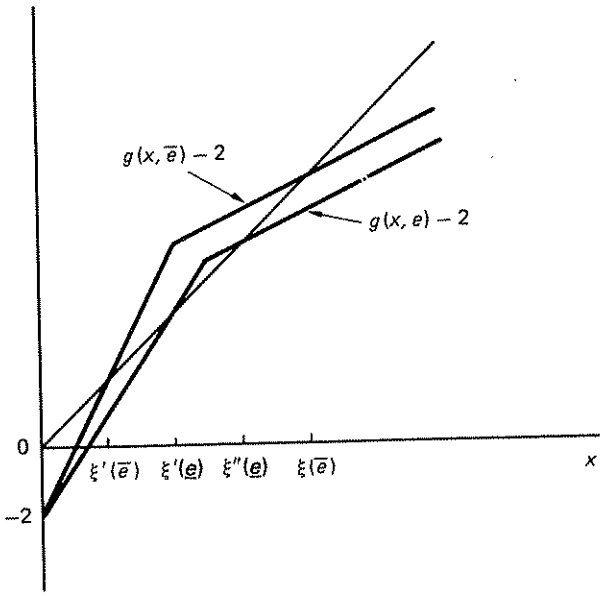

Figure 8.3

$$h[\hat{x}(e), e] = 0, \hat{x}(e) > 0 \tag{8.3.4}$$

Going back to equations (8.3.2a–b), check that:

$$H(\underline{e}) = 7.5, \hat{x}(\underline{e}) = 10$$

$$H(\bar{e}) = 8, \hat{x}(\bar{e}) = 12 \tag{8.3.5}$$

Finally, for each $c > 0$, and each event e for which $H(e) \geqslant c$, define $\xi'(e)$ $\xi''(e)$ as in (8.2.14) and (8.2.15); these roots of $g(x, e) - c = x$ will be distinct if $c < H(e)$. In example 3.2, $c = 2$ and

$$\xi'(\underline{e}) = 4, \xi''(\underline{e}) = 6$$

$$\xi'(\bar{e}) = 2, \xi''(\bar{e}) = 8 \tag{8.3.6}$$

It is no accident (see below) that

$$\xi'(\bar{e}) < \xi'(\underline{e}) < \xi''(\underline{e}) < \xi''(\bar{e}) \tag{8.3.7}$$

We can now analyse the dynamics of the example. If $X_0 < \xi'(\bar{e}) = 2$, then $X_0 < \xi'(\underline{e}) = 4$ as well, and the argument leading up to Proposition 1 in section 2 shows that, whatever the sequence of events, X_t will reach or fall below 0 in finite time ($T < \infty$), and so survival has probability 0. On the other hand, if $X_0 \geqslant \xi'(\underline{e}) > \xi'(\bar{e})$, then the argument leading to Proposition 1 shows that, whatever the sequence of events, X_t cannot fall below X_0, and hence the probability of survival is 1. Now suppose that X_0 is in the *open* interval

$$I \equiv (\xi'(\bar{e}), \xi'(\underline{e})) \tag{8.3.8}$$

Since I is bounded, a sufficiently long *initial* run of *favourable* events ($\bar{e}$) will carry X_t above $\xi'(\underline{e})$, from which point – as we have just seen – survival is assured. Since such an initial run has strictly positive probability, the probability of survival is strictly positive. On the other hand, a sufficiently long initial run of *unfavourable* events ($\underline{e}$) will carry X_t below $\xi'(\bar{e})$, from which point survival has 0 probability. Such an initial run also has positive probability, so that the probability of survival is strictly less than 1. Finally, we must deal with the case in which $X_0 = \xi'(\bar{e})$. As long as $e_t = \bar{e}$, the inputs X_t will remain equal to $\xi'(\bar{e})$. The first time that $e_t = \underline{e}$, the input X_t will fall below $\xi'(\bar{e})$; from that point the probability of survival is 0. Since the unfavourable event will eventually occur, it follows that the probability of survival from X_0 is 0.

To summarise the results for the example 3.2, the probability of survival is 0, strictly between 0 and 1, or 1, according as X_0 is less than or equal to $\xi'(\bar{e})$, strictly between $\xi'(\bar{e})$ and $\xi'(\underline{e})$, or, greater than or equal to $\xi'(\underline{e})$. We shall see that these results carry over directly to the general analysis.

We shall distinguish two cases.

Case 1 $\quad c \leqslant H(\underline{e})$.

It is easy to verify that, as a consequence of the four assumptions, if e is more favourable than e', then

$$\xi'(e) < \xi'(e') \leqslant \xi''(e') < \xi''(e) \tag{8.3.9}$$

In particular, for every event e distinct from $\underline{e}$ and $\bar{e}$,

$$\xi'(\bar{e}) < \xi'(e) < \xi'(\underline{e}) \leqslant \xi''(\underline{e}) < \xi''(e) < \xi''(\bar{e}) \tag{8.3.10}$$

Therefore, an argument almost identical to that used for the example 3.2 leads to the same conclusion.

Case 2 $\quad c > H(\underline{e})$

If $c > H(\bar{e})$, then, using (8.2.13) one can show that, for any X_0, the failure time T is bounded above uniformly over all sequences (E_t). Suppose then that $c \leqslant H(\bar{e})$. For any X_0 and any positive number ε there will be some τ such that, for all sequences (E_t),

$$X_t \leqslant \xi''(\bar{e}) + \varepsilon, \text{ all } t \geqslant \tau \tag{8.3.11}$$

One can show that there is an integer N such that, if $X_t \leqslant \xi''(\bar{e}) + \varepsilon$ and there is a run of N consecutive events $\underline{e}$ following period t (i.e., $e_{t+1} = \ldots = e_{t+N} = \underline{e}$), then

$$X_{t+N} < \xi'(\bar{e}) \tag{8.3.12}$$

As in the example, conditional on (8.3.12), the probability of survival is 0. Hence, to show that the unconditional probability of survival is 0, it suffices to show that, almost surely, there will be a run of N consecutive events $\underline{e}$ some time after period T; for the sake of completeness, this is shown in the Appendix.

Before summarising the results for this section, we note a monotonicity property of the probability of survival. For any given sequence of events and any t, consider X_t as a function of X_0 and c. For each event e, the function $g(\cdot, e)$ is monotonic nondecreasing. Hence X_t is a nondecreasing function of X_0 and a nonincreasing function of c. It follows that the probability of survival has these same properties.

For any initial stock y and consumption rate c, let $P(y, c)$ denote the corresponding probability of survival. If $c \leqslant H(\underline{e})$, define $\eta_1(c)$ and $\eta_2(c)$ by

$$\begin{aligned} \eta_1(c) &= \xi'(\bar{e}) + c \\ \eta_2(c) &= \xi'(\underline{e}) + c \end{aligned} \tag{8.3.13}$$

The following proposition summarises the results of this section:

Proposition 3

(1) The probability of survival, $P(y, c)$ is nondecreasing in y and nonincreasing in c.
(2) If $c > H(\underline{e})$ then, for all y, $P(y, c) = 0$.
(3) If $c \leqslant H(\underline{e})$ then

$$\begin{aligned} &P(y, c) = 0 &&, \text{ for } y \leqslant \eta_1(c) \\ &0 < P(y, c) < 1 &&, \text{ for } \eta_1(c) < y < \eta_2(c) \\ &P(y, c) = 1 &&, \text{ for } y > \eta_2(c) \end{aligned}$$

4 UNCERTAIN ENVIRONMENT: THE CASE OF LINEAR RETURNS

We turn now to the analogue for uncertainty of the linear production function (8.2.16). Using the setup at the beginning of Section 3 we define

$$g(x, e) = r(e)x \tag{8.4.1}$$

where for each event e in $\underset{\sim}{E}$, $r(e)$ is a positive number (not necessarily greater than 1).

We retain the assumption that $\underset{\sim}{E}$ has at least 2 elements, and that every event in $\underset{\sim}{E}$ has strictly positive probability.

Define

$$R_t \equiv 1n\ r(e_t); \tag{8.4.2}$$

Then (R_t) is a sequence of independent, identically distributed random variables. Since $\underset{\sim}{E}$ is finite, the distribution of R_t has finite support (i.e., R_t can take on only finitely many values). The beginning-of-period stocks, Y_t, evolve according to the questions

$$Y_0 = y$$

$$Y_t = e^{R_t}(Y_{t+1} - c),\ t \geqslant 1 \tag{8.4.3}$$

Recall that $P(y, c)$ denotes the probability of survival, given y and c. Define

$$S_0 \equiv 0$$

$$S_t \equiv R_1 + \ldots + R_t \tag{8.4.4}$$

$$Z_t \equiv e^{-S_0} + \ldots + e^{-S_t} \tag{8.4.5}$$

Since (Z_t) is an increasing sequence, it either converges to a finite limit or diverges to $+\infty$. Hence we can define

$$Z \equiv \lim_t Z_t \tag{8.4.6}$$

with the understanding that it may be infinite. Of course, Z is a random variable.

Lemma

$$P(y, c) = \text{prob}\{Z \leqslant \frac{y}{c}\} \tag{8.4.7}$$

Proof It is straightforward to verify that the solution of the difference equation (8.4.3) is

$$\begin{aligned} Y_t &= Y_0 \exp(S_t) - c \sum_{n=1}^{t} \exp\left(\sum_{m=n}^{t} R_m \right) \\ &= e^{S_t} (y - cZ_{t-1}) \end{aligned} \tag{8.4.8}$$

Hence $Y_t > c$ if and only if

$$\begin{aligned} & y - cZ_{t-1} > ce^{-S_t} \\ & y > c\,(Z_{t-1} + e^{-S_t}) = c\, Z_t, \end{aligned} \tag{8.4.9}$$

and (8.4.9) is true for every t if and only if $y \geqslant cZ$, which completes the proof of the lemma.

We shall show that, if $\varrho \equiv ER_t \leqslant 0$, then $P(y, c) = 0$ for all y and c. On the other hand, as we shall now show, if $\varrho > 0$ then $P(y, c)$ is positive provided y/c is large enough, and $P(y, c)$ approaches 1 as y/c increases without limit. The condition here that $\varrho > 0$ corresponds to the condition that $r > 1$ in the deterministic case (8.2.16).

Suppose that $\varrho > 0$. By the Strong Law of Large Numbers,

$$\lim_{t\to\infty} \frac{S_t}{t} = \varrho \quad a.s. \tag{8.4.8'}$$

Hence,

$$\lim_{t\to\infty} (e^{-S_t})^{1/t} = e^{-\varrho}, \; a.s.$$

and so, by the Cauchy Criterion, Z is finite *a.s.*. Let F denote its cumulative distribution functions, i.e.,

$$F(z) \doteqdot \text{prob}(Z \leqslant z) \tag{8.4.10}$$

From the lemma we have

$$P(y, c) = F\left(\frac{y}{c}\right) \tag{8.4.11}$$

so that P is nondecreasing in (y/c), and

$$\lim_{\frac{y}{c} \to \infty} P(y, c) = 1 \tag{8.4.12}$$

To get more information about F, we need some elementary facts about Z. First, if $r > 1$ and $\text{prob}(R_t = 1n\, r) = 1$, then $Z = r/(r-1)$ *a.s.*, as we see from (8.2.19), or directly from the definition (8.4.4)–(8.4.6). For $r > 1$ this is decreasing in r. Now define

$$\bar{r} \equiv \max_{e} r(e)$$

$$\underline{r} = \min_{e} r(e) \tag{8.4.13}$$

and assume $\bar{r} > 1$. Then

$$Z \geqslant \frac{\bar{r}}{\bar{r}-1} \; a.s. \tag{8.4.14}$$

Furthermore,

$$\text{if } \underline{r} > 1 \text{ then } Z \leqslant \frac{\underline{r}}{\underline{r}-1} \; a.s. \tag{8.4.15}$$

On the other hand, if $\underline{r} < 1$, and $\text{prob}(R_t = 1n\, \underline{r}) > 0$, then for any $z > 0$, $\text{prob}(Z \geqslant z) > 0$.

It remains to deal with the case in which $ER_t \leqslant 0$. In this case

$$\underline{\lim}_{t\to\infty} S_t = -\infty$$

(see, e.g., Chung, 1974, pp. 264–70). Hence

$$\overline{\lim}_{t\to\infty} e^{-S_t} = +\infty$$

and so, by equation (8.4.5)

$$\lim_{t\to\infty} Z_t = +\infty$$

Hence the probability of survival is 0, whatever y and c are.

The results for all cases are summarised in the following proposition.

Proposition 4 Let $R_t \equiv 1n\, r(e_t)$, and $\varrho \equiv ER_t$.

(1) If $\varrho \leqslant 0$, then $P(y, c) = 0$ for all y and c.
(2) If $\varrho > 0$, then $P(y, c) = \text{prob}(Z \leqslant \frac{y}{c})$, where Z is defined by equations (8.4.4–8.4.6).

(2a) $P(y, c) = 0$ if $y/c \leqslant \bar{r}/(\bar{r} - 1)$, where $\bar{r} \equiv \max_e r(e)$.

(2b) If $\underline{r} \equiv \min_e r(e) > 1$ then $P(y, c) = 1$ for $y/c \geqslant \underline{r}/(\underline{r} - 1)$.

(2c) If $\underline{r} < 1$ then $P(y, c) < 1$ for all y and c.

5 SOME POSSIBLE EXTENSIONS AND RELATED RESULTS

Let us now indicate briefly some possible extensions of the results derived above. The concavity assumption (Assuption 3) does simplify the exposition; however, it is not really central to our approach. The reader is invited to examine Figure 8.4, in which corresponding to two elements $\underline{e}$ and $\bar{e}$ in $\underline{E}$, the return functions $g(x, \underline{e})$ and $g(x, \bar{e})$ exhibit an initial phase of increasing returns. The arguments leading to Proposition 3 can be obviously adapted to analyse the behaviour of the stochastic process (x_t) and to characterise survival probabilities.

The assumption that the sequence (E_t) is independent and identically distributed can be relaxed in two ways. First, one may consider the situation where (E_t) is a sequence of independent random variables, but are *not* necessarily identically distributed. To pursue this direction one assumes that the families of finite dimensional distributions satisfy the relevant *consistency conditions* (see, e.g., Billingsley, 1979, Cl. 7). Furthermore, assume that the distribution μ_t of E_t assigns positive mass to each

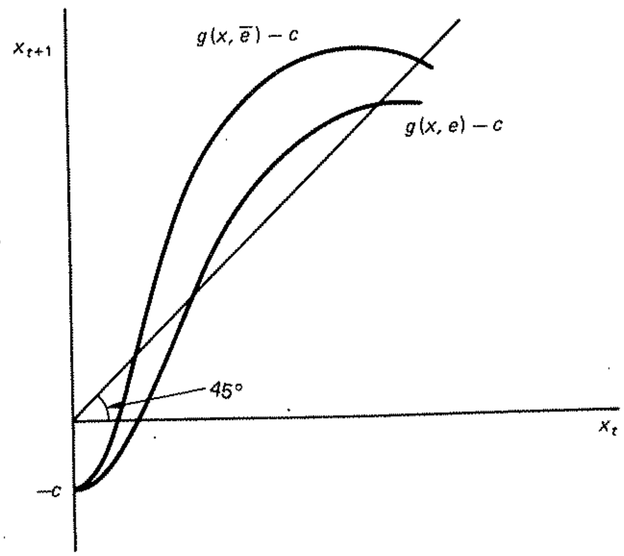

Figure 8.4

element $\underset{\sim}{E}$, (and maintain the Assumptions 1–5). The reader is invited to verify Proposition 3. The result on the almost sure occurrence of a run of N consecutive $\underline{e}$ can be proved by using the Borel–Cantelli lemma (see, e.g., Breiman, 1968, Proposition 3.16, p. 47).

A second possible extension is the relaxation of the independence assumption. Maintain the assumption that the set $\underset{\sim}{E}$ is finite; one can study the case where the sequence (E_t) is a *Markov chain*. Let $F = (F_{ij})$ be the matrix of (stationary) transition probabilities: F_{ij} being the probability that the return function in period $t + 1$ is $g(\cdot\ , e_j)$ given that it is $g(\cdot\ , e_i)$ in period t. Instead of a taxonomic analysis of the long-run behaviour of the process (8.3.1), we indicate some heuristic arguments. Recall that a particular state of a Markov chain is absorbing if it is impossible to leave it. A Markov chain is *absorbing* if (i) it has at least one absorbing state and (ii) from any state it is possible to go to an absorbing state. If a Markov chain *is* absorbing, then with probability 1 the process gets absorbed; the expected number of steps before absorption depends on the initial state and can be calculated (see, e.g., Kemeny *et al.*, 1958, pp. 404–5). If the process starts in an absorbing state, the agent's survival problem reduces to that discussed in section 2. However, an agent may be ruined with positive probability 'on his way' to an absorbing state or even after reaching an absorbing state. One can use Example 3.1 to illustrate some possibilities. Suppose that the event $\bar{e}$ (giving rise to the 'good' technology $g(x,\bar{e})$ in equation (8.3.1a)) is absorbing, and F_{ee}, the one-step transitions probability of moving into $\bar{e}$ from $\underline{e}$ (see equation (8.3.1b)) is positive. If the initial distribution assigns positive probability to $\underline{e}$, then the agent is ruined with positive probability even through $\bar{e}$ is reached with probability 1 (and, if the process happens to state in $\bar{e}$, the agent can survive with probability 1!).

One can also analyse the case where F is strictly positive or, at least, the least favourable event $\underline{e}$ is *recurrent* (see, e.g., Karlin and Taylor, 1975, Ch. 2) and $F_{ee} = p > 0$. By the strong Markov property, a run of N consecutive $\underline{e}$ occurs with probability 1 (see the Appendix for details). Hence, a result analogous to Proposition 3 can be derived. Finally, the case of a linear return function can also be studied without the assumption that (R_t) in (8.4.2) is a sequence of independent, identically distributed random variables. Due to technical complexity, we plan to report the extensions in a forthcoming note.

As mentioned in the Introduction, the model we have developed has affiliations to some earlier studies. Billingsley (1979, pp. 77–9) provides a succinct account of the classical problem of a gambler's ruin (in which the gambler is not required to consume a *positive* amount in every period. Of general interest is the well-known monograph of Dubins and Savage (1976).

The approach we follow is more in line with that of Ray (1984), who examined the prospect of averting failure through borrowing. Sufficient conditions for the expected amount of debt to go to infinity (occurrence of failure) were given. The motivation behind this ingenious analysis came from the literature (cited by him) on rural indebtedness in developing countries.

Yet another class of allied models focuses on optimal policies of an insurance company. Suppose that the claims are made in an insurance company in accordance with a renewal process with inter-arrival times T_1, T_2, and the values of the claims are i.i.d. and independent of the renewal process of when they occurred. If $N(t)$ is the number of claims by time t, and Y_i is the value of the ith claim, then the total value of claims is

$$\sum_{i=1}^{N(t)} Y_i;$$

if α is the rate at which premiums are accumulated, the 'survival' probability for the insurance company is

$$\text{prob}\left[\sum_{i=1}^{N(t)} Y_i > ct + x(0)\right]$$

where $x(0)$ is the initial fortune.

In this formulation (see, e.g., Ross, 1983, p. 238, and the references cited there), the insurance company faces no uncertainty on the income side (and does not engage in any investment activity).

Finally, we should mention that as an example of a nonlinear dynamic system, stability properties of a deterministic process of the type discussed in section 2 were sketched in Dorfmann, Samuelson and Solow (1958, Ch. 11, eq. 11). Some results on the independent, identically distributed case with a strictly concave g have appeared in Mirman and Spulber (1984).

Appendix

LONG RUNS WILL OCCUR

Independent and Identically Distributed Events

Let e in $\underline{E}$ be a given event, $p > 0$ be its probability, and N be a given positive integer. We shall say that a *run of events e of length N occurs* starting at time M if $E_M = E_{M+1} = \ldots = E_{M+N} = e$; here M is a random time. We wish to show that, with probability 1, a run of length N will occur. Divide the sequence of events E_t, $t \geqslant 1$, into successive blocks of length N, and call the kth block B_k. It will suffice to show that, with probability 1, *some* block B_k will eventually be a run of length N. In fact, let us say that block k is a *success* if it is a run of length N, and a *failure* otherwise. The successive blocks are independent, and the probability that B_k is a success is $P^N > 0$. Hence the number K of the *first* successful block has a geometric distribution

$$\text{prob}(K = k) = p^N(1 - p^N)^{k-1}$$

Hence prob $(K < \infty) = 1$.

Markovian Events

Suppose now that the sequence (E_t) of underlying events constitutes a Markov chain; as before, the set $\underline{E}$ of possible events at each date is finite. We make two assumptions about the Markov chain:

(1) The 'least favourable' event $\underline{e}$ is recurrent.
(2) prob $(E_{t+1} = \underline{e} \mid E_t = \underline{e}) \equiv p > 0$.

Let τ and N be as in (8.3.11) and (8.3.12). Define

$$\sigma_1 \equiv \min\{t\colon t \geqslant \tau,\ E = \underline{e}\}$$

Since $\underline{e}$ is recurrent, σ_1 is finite. If $p = 1$, then $E_t = \underline{e}$ for all $t \geqslant \tau$. Therefore, suppose that $p < 1$; then the chain will eventually leave the 'state' $\underline{e}$ after date τ. Let σ_2, σ_3, etc. denote the dates of successive returns to state $\underline{e}$ i.e., for $k \geqslant 2$, σ_k is the first t (if any) such that

$$t > \sigma_{k-1}$$
$$E_{t-1} \neq \underline{e}$$
$$E_t = \underline{e}$$

Again since $\underline{e}$ is recurrent, the sequence $(\sigma_k,\ k \geqslant 1)$ is infinite.
Define the random variable (A_k) by

$$A_k = \begin{cases} 1, \text{ if } E_{\sigma_k} = \ldots = E_{\sigma_{k+N-1}} = \underline{e} \\ 0, \text{ otherwise} \end{cases}$$

(i.e., $A_k = 1$ if and only if there is a run of N events $\underline{e}$ starting at date σ_k). By the Strong Markov Property, the variables (A_k) are independent and identically distributed, with

$$\text{prob } (A_k = 1) = p^{N-1}$$

Let K be the first k such that $A_k = 1$; then K has a geometric distribution, and

$$\text{prob } (K = k) = p^{N-1} \, (1-p^{N-1})^{k-1}$$

Hence

$$\text{prob } (K < \infty) = 1$$

Note that, although the assumptions of this section of the Appendix include the assumptions of the previous section on p. 198 as a special case, the argument is slightly different. In particular, the random variable K defined here is different from the K defined there. Here we look for the *first* run of length N, whereas on p. 198 we were satisfied to identify the first such run that exactly matched one of the predetermined blocks of length N.

Notes

1. A definitive analysis of the substitution condition between capital ('produced means of production') and natural resource that guarantees the attainment of a minimal level of positive consumption is given in Cass and Mitra (1979).
2. Formally, let μ be the probability measure on $\underline{E}$ *and* μ^∞ the corresponding product measure on $\underline{E}^\infty$; then we can take $(\underline{E}^\infty, \mu^\infty)$ to be our basic probability space, and X_t and Y_t will be random variables (real-valued functions) on it. All probability statements (and 'almost surely' statements) are with reference to $(\underline{E}^\infty, \mu^\infty)$.
3. Suppose that $\bar{e}$ (resp. $\underline{e}$) occurs with probability $\bar{\mu}$ (resp. $(1-\bar{\mu})$). Given input x, the expected or average output is $\bar{\mu}\, g(x, \bar{e}) + (1-\bar{\mu})\, g(x, \underline{e})$. For $\bar{\mu}$ sufficiently close to 1, the expected output is close to that given by the 'good technology'. However, this average output does not provide any clue to survival possibilities.

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