Midterm I Practice Problems

CS 323 , Spring 2019

Problem 1 [10+10 points]
a) Give Newton's method for finding $\sqrt[3]{2}$ by solving $x^3 - 2 = 0$.
Solution Given initial guess x_0, the Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad k = 0, 1, 2, \ldots.$$

Here, $f(x_k) = x_k^3 - 2$ and $f'(x_k) = 3x_k^2$, we have

$$x_{k+1} = x_k - \frac{x_k^3 - 2}{3x_k^2} = \frac{2}{3}x_k + \frac{2}{3x_k^2} \quad k = 0, 1, 2, \ldots.$$

b) Show that $\sqrt[3]{2}$ is in the interval $[1, 2]$ by mean value theorem. Give a bisection method for finding $\sqrt[3]{2}$.
Solution Since $f(1) = -1 < 0$ and $f(2) = 6 > 0$, by mean value theorem, there is a root in the interal $[1, 2]$. Since $f(x)$ is an increasing function because $f'(x) = 3x^2 \geq 0$, there is only one root $\sqrt[3]{2}$ of $f(x)$. So $\sqrt[3]{2}$ is in $[1, 2]$.
Let $a_0 = 1$ and $b_0 = 2$ and $x_0 = (a + b)/2$. For $k = 0, 1, 2 \ldots,$

$$\begin{cases} a_{k+1} = a_k, b_{k+1} = x_k, x_{k+1} = (a_{k+1} + b_{k+1})/2, & \text{if } f(a)f(x_k) < 0 \\ a_{k+1} = x_k, b_{k+1} = b_k, x_{k+1} = (a_{k+1} + b_{k+1})/2, & \text{if } f(a)f(x_k) > 0 \end{cases}$$

Problem 2

a) Let

$$A = \begin{bmatrix} 19 & 20 \\ 20 & 21 \end{bmatrix}$$

Find the conditional number of A.

Solution: We first compute A^{-1}

$$A^{-1} = \begin{bmatrix} -21 & 20 \\ 20 & -19 \end{bmatrix}$$

By definition, $\|A\| = \max\{19+20, 20+21\} = 41$ and $\|A^{-1}\| = \max\{21+20, 20+19\} = 41$ and The conditional number of A is

$$\|A\|\|A^{-1}\| = 41 * 41 = 1681.$$

b) We consider the error of the solution x of the linear system $Ax = b$. We showed in class that,

$$\frac{\|x-z\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|Az-b\|}{\|b\|}.$$

If $b = [1, 1/2]$ and $Az = [1.001, 0.499]$, estimate the relative error $\frac{\|x-z\|}{\|x\|}$.

Solution

Since $\|Az - b\| = 0.001$ and $\|b\| = 1$, by the error estimate of linear system,

$$\frac{\|x-z\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|b-Az\|}{\|b\|},$$

we have the relative error

$$\frac{\|x-z\|}{\|x\|} \leq 1681 * 0.001 = 1.681.$$

Problem 3 a) Give a Jacobi method for solving $Ax = b$ where

$$A = \begin{bmatrix} 5 & 2 & 3 \\ 0 & -4 & 2 \\ -1 & 1 & 5 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Solution: Let

$$N = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & -2 & -3 \\ 0 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix}$$

Jacobi method:

$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \left[\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -3 \\ 0 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} \right]$$

b) Find the iterate matrix M of Jacobi method in part a) and find the norm $\|M\|$.
Solution:

$$M = \begin{bmatrix} 0 & -2/5 & -3/5 \\ 0 & 0 & -1/2 \\ 1/5 & -1/5 & 0 \end{bmatrix} \quad \text{and} \quad \|M\| = 1$$

c) Does the Jacobi method converge?
Solution: Since $\|M\| \geq 1$, the Jacobi method may not converge.

Problem 4 a) Give a power method to approximate the eigenvector v of the matrix A associated with the largest eigenvalue where

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and initial guess} \quad z^0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solution: Given initial guess $z^{(0)}$, for $k = 0, 1, 2, \dots$

$$\begin{aligned} w^{(k+1)} &= Az^{(k)}, \\ z^{(k+1)} &= w^{(k+1)}/\|w^{(k+1)}\|. \end{aligned}$$

b) Show that the eigenvalues of A in part a) are $\{4, 3, 5\}$. And find the eigenvector v associated with the largest eigenvalue.

Solution: To find the eigenvalues, we use the characteristic polynomial

$$p(\lambda) = det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & -1 \\ 0 & 3-\lambda & -2 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda)(5-\lambda)$$

Solve for $p(\lambda) = 0$ and arrange the eigevalues in descending order, we have

$$\lambda_1 = 5, \lambda_2 = 4, \lambda_3 = 3$$

To find the eigenvector associated with the largest eigenvalue $\lambda_1 = 5$, we solve for $(A - \lambda_1 I)v = 0$ and have

$$\begin{bmatrix} 4-5 & 2 & -1 \\ 0 & 3-5 & -2 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finally, the eigenvector

$$\begin{bmatrix} 1 \\ 1/3 \\ -1/3 \end{bmatrix}$$

c) The error estimate of the power method shows that

$$\|z^m - v\| \approx \|z^0 - v\| \left|\frac{\lambda_2}{\lambda_1}\right|^m \qquad \lambda_i \text{ eigenvalues of } A \text{ and } |\lambda_1| > |\lambda_2| > |\lambda_3| \ldots$$

Estimate the number of iterations of the power method needed such that

$$\|z^m - v\| \leq 10^{-4}.$$

Solution: Since

$$\|z^m - v\| \approx \|z^0 - v\| \left|\frac{\lambda_2}{\lambda_1}\right|^m$$

we set

$$\|z^0 - v\| \left|\frac{\lambda_2}{\lambda_1}\right|^m \leq 10^{-4}.$$

Here, $\lambda_1 = 5$ and $\lambda_2 = 4$ and

$$\|v - z^{(0)}\| = \|[0, 2/3, 1/3]^T\| = 2/3.$$

Hence, we solve for

$$2/3 \left|\frac{4}{5}\right|^m \leq 10^{-4}.$$

and have

$$m \geq \frac{\ln(3/2 * 10^{-4})}{\ln(4/5)} \approx 40.$$

So we need roughly 40 iterations for the power method to converge within the error bound 10^{-4}.