Final Practice Problems

CS 323 , Spring 2019

Problem 1 [10+10 points]
a) Give Newton's method for finding $\sqrt[3]{2}$ by solving $x^3 - 2 = 0$.
Solution Given initial guess x_0, the Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad k = 0, 1, 2, \ldots.$$

Here, $f(x_k) = x_k^3 - 2$ and $f'(x_k) = 3x_k^2$, we have

$$x_{k+1} = x_k - \frac{x_k^3 - 2}{3x_k^2} = \frac{2}{3}x_k + \frac{2}{3x_k^2} \quad k = 0, 1, 2, \ldots.$$

b) Find the *first* Newton's iteration to solve for the nonlinear system

$$\begin{aligned} f_1(x, y) &= xy + 1.089 = 0 \\ f_2(x, y) &= x^2 + y^2 - 3.23 = 0 \end{aligned}$$

with initial value $x_0 = 1, y_0 = 0$.
Solution The Newton's method

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

where

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = - \begin{bmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{bmatrix}.$$

Here

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} y_0 & x_0 \\ 2x_0 & 2y_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{bmatrix} = \begin{bmatrix} 1.089 \\ -2.23 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1.115 \\ -1.089 \end{bmatrix}$$

Problem 2
a) Let

$$A = \begin{bmatrix} 19 & 20 \\ 20 & 21 \end{bmatrix}$$

Find the conditional number of A.
Solution: We first compute A^{-1}

$$A^{-1} = \begin{bmatrix} -21 & 20 \\ 20 & -19 \end{bmatrix}$$

By definition, $\|A\| = \max\{19+20, 20+21\} = 41$ and $\|A^{-1}\| = \max\{21+20, 20+19\} = 41$ and The conditional number of A is

$$\|A\|\|A^{-1}\| = 41 * 41 = 1681.$$

b) We consider the error of the solution x of the linear system $Ax = b$. We showed in class that,

$$\frac{\|x - z\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|Az - b\|}{\|b\|}.$$

If $b = [1, 1/2]$ and $Az = [1.001, 0.499]$, estimate the relative error $\frac{\|x-z\|}{\|x\|}$.
Solution
Since $\|Az - b\| = 0.001$ and $\|b\| = 1$, by the error estimate of linear system,

$$\frac{\|x - z\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|b - Az\|}{\|b\|},$$

we have the relative error

$$\frac{\|x - z\|}{\|x\|} \leq 1681 * 0.001 = 1.681.$$

Problem 3 a) Find the least square fitting by linear polynomial for the given data $(0,1)$, $(2,1)$ and $(3,2)$.
Solution Let $y = c_1 x + c_0$. We minimize the error

$$E = \frac{1}{2}|y(0)-1|^2 + \frac{1}{2}|y(2)-1|^2 + \frac{1}{2}|y(3)-2|^2.$$

The pair (c_1, c_0) minimizing the error satisfies

$$\frac{\partial E}{\partial c_0} = (c_0 - 1) + (c_1 2 + c_0 - 1) + (c_1 3 + c_0 - 2) = 0$$
$$\frac{\partial E}{\partial c_1} = (c_0 - 1) * 0 + (c_1 2 + c_0 - 1) * 2 + (c_1 3 + c_0 - 2) * 3 = 0$$

That is,

$$5c_1 + 3c_0 = 4$$
$$13c_1 + 5c_0 = 8$$

Solving for c_0, c_1, we get

$$c_1 = \frac{2}{7} \quad c_0 = \frac{6}{7}$$

So

$$y = \frac{2}{7}x + \frac{6}{7}.$$

b) Find the interpolating polynomial for the given data $(0,1)$, $(2,1)$ and $(3,2)$ in the Newton's form.
Solution:

$$p(x) = 1 + \frac{1}{3}x(x-2).$$

Problem 4

a) Let $f(x)$ be a function defined on the interval $[-1, 1]$ with

$$f(-1) = f(1) = 0, \quad f(-1/2) = f(1/2) = 1, \quad f(0) = 2/3.$$

Find an approximation of $\int_{-1}^{1} f(x)dx$ with Simpson's rule.

Solution

$$S_4 = \frac{1}{6}(f(-1) + 4f(-1/2) + 2f(0) + 4f(1/2) + f(1)) = \frac{14}{9}.$$

b) If $|f^{(4)}(x)| < 5$ in $[-1, 1]$, find an error estimate of the approximation in part a).

Solution

$$|I - S_4| \leq \frac{M_4}{180}(b-a)h^4 = \frac{5}{180} * 2 * (\frac{1}{2})^4 = \frac{1}{18 * 16}.$$

Problem 5 Use Euler's method to find the an approximation to the solution $y(1/2)$ of the problem

$$y'(x) = -x^2 y, \quad y(0) = 1$$

with stepsize $h = 0.5$.

Solution

$$y_1 = y_0 + h * f(x_0, y_0) = 1 + 0.5 * 0^2 * 1 = 1.$$

Problem 6 Trapezoid method is a second order implicit method for soloving the problem

$$y'(x) = f(x, y), \quad y(0) = y_0.$$

The method reads

$$y_{n+1} = y_n + h[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]/2.$$

Use Trapezoid method to find the approximation to the solution $y(1/2)$ of the problem

$$y'(x) = -x^2 y, \quad y(0) = 1$$

with stepsize 0.5.

Solution

$$y_1 = y_0 + h(f(x_0, y_0) + f(x_1, y_1))/2 = 1 + 0.5 * (0 - 0.5^2 y_1)/2.$$

Solving for y_1, we get

$$y_1 = \frac{16}{17}.$$