CS 323

Homework # 2: due Mar 7

Problem 1 a) Let

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{pmatrix}$$

Find a lower triangular matrix L, a unit upper triangular matrix U, and a permutation matrix P , such that $LU = PA$. Use the algorithm given in class, incorporating the partial pivoting strategy. Do it by hand computation.

b) Use Matlab's `lu` command (type help lu to learn about it) to find a unit lower triangular matrix L, an upper triangular matrix U, and a permutation matrix P such that $PA = LU$. To enter the matrix A into Matlab, type the following (the semicolons separate the rows):

```
A = [1 1 3; 3 -1 1; 1 3 0]
```

Problem 2. In this problem we consider the question of whether a small value of the residual $\|Az - b\|$ means that z is a good approximation to the solution x of the linear system $Ax = b$. We showed in class that,

$$\frac{\|x - z\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|Az - b\|}{\|b\|}.$$

which implies that if the condition number $\|A\|\|A^{-1}\|$ of A is small, a small relative residual implies a small relative error in the solution. We now show computationally what can happen if the condition number is large. A standard example of a matrix that is ill-conditioned is the Hilbert matrix H, with entries $(H_{ij}) = 1/(i+j-1)$. For $n = 8, 12, 16$ (where H is of dimension $n \times n$), use Matlab to solve the linear system of equations $Hx = b$, where b is the vector Hy and y is the vector with $y_i = 1/\sqrt{n}, i = 1 \ldots n$. Clearly, the true solution is given by $x = y$, and we let z denote the approximation obtained by Matlab. Then calculate for each value of n the following quantities: (i) the relative error $\|x - z\|/\|x\|$, (ii) the relative residual $\|Hz - b\|/\|b\|$, (iii) the condition number $\|H\|\|H^{-1}\|$, and (iv) the product of the quantities in (ii) and (iii). Arrange all these numbers in a table. The Matlab commands `norm` and `cond` can be used to compute the norm and condition numbers, respectively. When vectors are input, Matlab writes them as row vectors. To convert y to a column vector, write it as y'. To solve the linear system $Hz = b$ in Matlab, type `z = H \ b`. An example of a Matlab loop is given below; the semicolon keeps Matlab from writing unwanted output to the screen. To avoid potential problems, type clear before running a new value of n. Example of a Matlab Loop:

```
for i=1:10
    y(i) = 1/sqrt(10);
end
```

Problem 3. Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$$

a) Find the iteration matrix M for the Jacobi method and Gauss Seidel method.
b) Determine whether Jacobi method converges.

Problem 4. Write a Matlab code for the power method applied to the matrix

$$A = \begin{pmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{pmatrix}$$

with initial vector $z^{(0)} = [1, 0, 0, 0]^T$. Calcuate the successive difference of eigenvalue $\delta\lambda^{(m)} := \lambda_1^{(m)} - \lambda_1^{(m-1)}$ $(m = 2, \ldots 10)$ and successive ratios $\delta\lambda^{(m+1)}/\delta\lambda^{(m)}$. To find the maximum m and max index k of vector z, type `[m,k] = max(z)`. Fill the results in a table in the following format

m	$\lambda_1^{(m)}$	$\lambda_1^{(m)} - \lambda_1^{(m-1)}$	Ratio
1	6.0000	-	-
2	11.5000	5.5000	-
3	13.1304	1.6304	0.2964
...	...	...	...

b) Use the *Matlab* routine `eig` to compute the eigenvalues of the matrix A. Sort the eigenvalues in the descending order such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq |\lambda_4|.$$

Does the ratios in part a) converges to λ_2/λ_1?

Problem 5. The least squares approximation method taught in class is for discrete data. There is a corresponding method for approximating data given as a continuous function. Consider approximating $f(x)$ by a quadratic polynomial

$$\hat{f}(x) = a_1 + a_2 x + a_3 x^2$$

on the interval $0 \leq x \leq 1$. Do so by choosing a_1, a_2 and a_3 to minimize the root-mean-square error

$$E(a_1, a_2, a_3) = \int_0^1 [a_1 + a_2 x + a_3 x^2 - f(x)]^2 dx.$$

Derive the linear system satisfied by the optimum choices of a_1, a_2, a_3. What is its relation to the Hilbert matrix in Problem 2?