CS 323

Homework # 2: due Mar 7

Problem 1 a) Let

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{pmatrix}$$

Find a lower triangular matrix L, a unit upper triangular matrix U, and a permutation matrix P , such that $LU = PA$. Use the algorithm given in class, incorporating the partial pivoting strategy. Do it by hand computation.

b) Use Matlab's `lu` command (type help lu to learn about it) to find a unit lower triangular matrix L, an upper triangular matrix U, and a permutation matrix P such that $PA = LU$. To enter the matrix A into Matlab, type the following (the semicolons separate the rows):

```
A = [1 1 3; 3 -1 1; 1 3 0]
```

Solution: Step 1: Since 3 is the largest element in the 1st column, we do not need to do pivoting. So the permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 2: Perform the Gauss Elimination, we get

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 2/5 & 1 \end{bmatrix}. \quad U = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 10/3 & -1/3 \\ 0 & 0 & 14/5 \end{bmatrix}.$$

Problem 2. In this problem we consider the question of whether a small value of the residual $\|Az - b\|$ means that z is a good approximation to the solution x of the linear system $Ax = b$. We showed in class that,

$$\frac{\|x - z\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|Az - b\|}{\|b\|}.$$

which implies that if the condition number $\|A\|\|A^{-1}\|$ of A is small, a small relative residual implies a small relative error in the solution. We now show computationally what can happen if the condition number is large. A standard example of a matrix that is ill-conditioned is the Hilbert matrix H, with entries $(H_{ij}) = 1/(i+j-1)$. For $n = 8, 12, 16$ (where H is of dimension $n \times n$), use Matlab to solve the linear system of equations $Hx = b$, where b is the vector Hy and y is the vector with $y_i = 1/\sqrt{n}, i = 1 \dots n$. Clearly, the true solution is given by $x = y$, and we let z denote the approximation obtained by Matlab. Then calculate for each value of n the following quantities: (i) the relative error $\|x - z\|/\|x\|$, (ii) the relative residual $\|Hz - b\|/\|b\|$, (iii) the condition number $\|H\|\|H^{-1}\|$, and (iv) the product of the quantities in (ii) and (iii). Arrange all these numbers in a table. The Matlab commands `norm` and `cond` can be used to compute the norm and condition numbers, respectively. When vectors are input, Matlab writes them as row vectors. To convert y to a column vector, write it as y'. To solve the linear system $Hz = b$ in Matlab, type `z = H \ b`. An example of a Matlab loop is given below; the semicolon keeps Matlab from writing unwanted output to the screen. To avoid potential problems, type clear before running a new value of n. Example of a Matlab Loop:

```
for i=1:10
    y(i) = 1/sqrt(10);
end
```

Solution:

n	$\|x - z\|/\|x\|$	$\|Hz - b\|/\|b\|$	$\|H\|\|H^{-1}\|$	$cond(H)\|Hz - b\|/\|b\|$
8	1.9237e-07	1.1554e-16	3.3873e+10	3.9136e-06
12	0.2369	6.1967e-17	3.9202e+16	2.4292
16	60.3329	6.5680e-16	1.6715e+18	1.0978e+03

Problem 3. Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$$

a) Find the iteration matrix M for the Jacobi method and Gauss Seidel method.
b) Determine whether Jacobi method converges.

Solution: Jacobi method: Take N be the diagonal elements of A and $P = N - A$, we have

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let $[x_1^k, x_2^k]$ be the k-th Jacobi iteration, then

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}}_{N^{-1}} \left[\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \begin{bmatrix} x_1^k \\ x_2^k \end{bmatrix} \right]$$

So we have

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} = \begin{bmatrix} b_1 + x_2^k \\ (b_2 + x_1^k)/4 \end{bmatrix}$$

Gauss Seidel method: Take N be the lower matrix of A and $P = N - A$, we have

$$N = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Let $[x_1^k, x_2^k]$ be the k-th Gauss Seidel iteration, then

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1/4 & 1/4 \end{bmatrix}}_{N^{-1}} \left[\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{P} \begin{bmatrix} x_1^k \\ x_2^k \end{bmatrix} \right]$$

So we have

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} b_1 + x_2^k \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 + x_2^k \\ (b_1 + x_2^k + b_2)/4 \end{bmatrix}.$$

b) Since the matrix A is diagonal dominant, the Jacobi method converges.

Problem 4. Write a Matlab code for the power method applied to the matrix

$$A = \begin{pmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{pmatrix}$$

with initial vector $z^{(0)} = [1, 0, 0, 0]^T$. Calcuate the successive difference of eigenvalue $\delta\lambda^{(m)} := \lambda_1^{(m)} - \lambda_1^{(m-1)}$ $(m = 2, \dots 10)$ and successive ratios $\delta\lambda^{(m+1)}/\delta\lambda^{(m)}$. To find the maximum m and max index k of vector z, type `[m,k] = max(z)`. Fill the results in a table in the following format

m	$\lambda_1^{(m)}$	$\lambda_1^{(m)} - \lambda_1^{(m-1)}$	Ratio
1	6.0000	-	-
2	11.5000	5.5000	-
3	13.1304	1.6304	0.2964
...	...	...	...

b) Use the *Matlab* routine `eig` to compute the eigenvalues of the matrix A. Sort the eigenvalues in the descending order such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq |\lambda_4|.$$

Does the ratios in part a) converges to λ_2/λ_1?

Solution: a)

m	$\lambda_1^{(m)}$	$\lambda_1^{(m)} - \lambda_1^{(m-1)}$	Ratio
1	6.0000	-	-
2	11.5000	5.5000	-
3	13.1304	1.6304	0.2964
4	14.3146	1.1841	0.7263
5	14.7587	0.4442	0.3751
6	14.9184	0.1597	0.3595
7	14.9726	0.0543	0.3398
8	14.9909	0.0182	0.3359
9	14.9970	0.0061	0.3341
10	14.9990	0.0020	0.3336

b) The eigenvalues are $\{15, 5, -1\}$. The ratio in part a) does converge to $|\lambda_2/\lambda_1| = 5/15 = 1/3$.

Problem 5. The least squares approximation method taught in class is for discrete data. There is a corresponding method for approximating data given as a continuous function. Consider approximating $f(x)$ by a quadratic polynomial

$$\hat{f}(x) = a_1 + a_2 x + a_3 x^2$$

on the interval $0 \le x \le 1$. Do so by choosing a_1, a_2 and a_3 to minimize the root-mean-square error

$$E(a_1, a_2, a_3) = \int_0^1 [a_1 + a_2 x + a_3 x^2 - f(x)]^2 dx.$$

Derive the linear system satisfied by the optimum choices of a_1, a_2, a_3. What is its relation to the Hilbert matrix in Problem 2?

Solution: The values (a_1, a_2, a_3) minimizing the function $E(a_1, a_2, a_3)$ satisfy

$$\frac{\partial E}{\partial a_1}(a_1, a_2, a_3) = 0 \quad \frac{\partial E}{\partial a_2}(a_1, a_2, a_3) = 0 \quad \frac{\partial E}{\partial a_3}(a_1, a_2, a_3) = 0$$

We calculate the partial derivatives and get

$$\begin{aligned}
\frac{\partial E}{\partial a_1} &= \int_0^1 [a_1 + a_2 x + a_3 x^2 - f(x)] dx = 0 \\
\frac{\partial E}{\partial a_2} &= \int_0^1 [a_1 + a_2 x + a_3 x^2 - f(x)] x dx = 0 \\
\frac{\partial E}{\partial a_3} &= \int_0^1 [a_1 + a_2 x + a_3 x^2 - f(x)] x^2 dx = 0
\end{aligned}$$

Rewrite these three equations into a linear system,

$$\begin{aligned}
a_1 \int_0^1 dx + a_2 \int_0^1 x dx + a_3 \int_0^1 x^2 dx &= \int_0^1 f(x) dx \\
a_1 \int_0^1 x dx + a_2 \int_0^1 x^2 dx + a_3 \int_0^1 x^3 dx &= \int_0^1 f(x) x dx \\
a_1 \int_0^1 x^2 dx + a_2 \int_0^1 x^3 dx + a_3 \int_0^1 x^4 dx &= \int_0^1 f(x) x^2 dx
\end{aligned}$$

Since $\int_0^1 x^n dx = \frac{1}{n+1}$, we obtain

$$\begin{aligned}
a_1 + a_2/2 + a_3/3 &= \int_0^1 f(x) dx \\
a_1/2 + a_2/3 + a_3/4 &= \int_0^1 f(x) x dx \\
a_1/3 + a_2/4 + a_3/5 &= \int_0^1 f(x) x^2 dx
\end{aligned}$$

Finally, we end up with the system of linear equations

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \int_0^1 f(x)dx \\ \int_0^1 f(x)xdx \\ \int_0^1 f(x)x^2dx \end{bmatrix}$$