CS 323

Homework # 3: due Apr 9

Problem 1. Use appropriate Lagrange interpolating polynomials of degrees one (at 8.3 and 8.6) and degree two (at 8.1, 8.3 and 8.6) to approximate $f(8.4)$ if

$$f(8.1) = 16.94410, f(8.3) = 17.56492, f(8.6) = 18.50515, f(8.7) = 18.82091$$

Solution:
(degree one) $L_1(x) := 18.50515\frac{x-8.3}{8.6-8.3} + 17.56492\frac{x-8.6}{8.3-8.6}$. The degree one approximation gives $L_1(8.4) = 17.8783$.
(degree two) $L_2(x) = 16.94410\frac{(x-8.3)(x-8.6)}{(8.1-8.3)(8.1-8.6)} + 17.56492\frac{(x-8.1)(x-8.6)}{(8.3-8.1)(8.3-8.6)} + 18.50515\frac{(x-8.1)(x-8.3)}{(8.6-8.1)(8.6-8.3)}$. The degree two approximation gives $L_2(8.4) = 17.8771$.

Problem 2. The data above in problem 1 is generated by the function $f(x) = x\ln x$. Use the error formula to find a bound for the error, and compare the bound to the actual error for the linear and quadratic cases.

Solution:
(degree one): By error estimate, $|f(0.84) - L_1(0.84)| \le \frac{M_2}{2!}|(8.4-8.3)(8.4-8.6)|$ where $M_2 = \max_{8.3\le x 8.6}|f''(x)|$. To estimate M_2, we calculate

$$f'(x) = \ln x + 1, \quad f''(x) = \frac{1}{x}.$$

Since $f''(x)$ is decreasing,

$$M_2 = \frac{1}{8.3}.$$

So the error

$$|f(0.84) - L_1(0.84)| \le \frac{4}{830} \approx 0.0012.$$

(degree two): By error estimate, $|f(0.84)-L_2(0.84)| \le \frac{M_3}{3!}|(8.4-8.1)(8.4-8.3)(8.4-8.6)|$ where $M_3 = \max_{8.3\le x 8.6}|f'''(x)|$. To estimate M_3, we calculate

$$|f'''(x)| = \frac{1}{x^2}.$$

Since $f''(x)$ is decreasing,

$$M_3 = \frac{1}{8.3^2}.$$

So the error

$$|f(0.84) - L_1(0.84)| \le \frac{0.1}{83^2} \approx 0.0000145.$$

Problem 3. Using the divided difference table to construct the interpolating polynomial of degree 4 for the points

$$(-0.1, 5.3), (0, 2), (0.2, 3.19), (0.3, 1)$$

Solution

x_i	$f(x_i)$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
-0.1	5.3	-33	$\frac{779}{6}$	$-\frac{1670}{3}$
0	2	0.595	$-\frac{27.85}{3}$	
0.2	3.19	-2.19		
0.3	1			

$$p(x) = 5.3 - 33(x+0.1) + \frac{779}{6}(x+0.1)x - \frac{1670}{3}(x+0.1)x(x-0.2).$$

Problem 4.

Computer Problem: Before beginning this problem, copy the file `interppoly.m` file in the assignment folder to the directory you will be using when you start up Matlab.

a): Let $[a,b] = [-5,5]$. For each degree $n = 4, 8, 16$, and 32, plot over the interval $[a,b]$ the error function $e(x) = f(x) - P_n(x)$, where $f(x) = 1/(1+x^2)$ and $P_n(x)$ is the polynomial of degree $\le n$ which interpolates f at the equally spaced interpolation points $x_i = a + i(b-a)/n, i = 0...n$. Record the approximate maximum of $|e(x)|$ (as seen from the graph) for each n and approximately where it occurs. To get you started, the case $n = 4$ can be done by typing (or cutting and pasting) the following commands into Matlab

```
% Problem 3a
a=-5;  b=5;
n=4;
xin=linspace(a,b,n+1);
yin= 1./(1+xin.^2);
xout=linspace(a,b,100);
[yout, cof] = interppoly(xin,yin,xout);
ytrue= 1./(1+xout.^2);
plot(xout,ytrue-yout)
```

The other values of n can be done by by entering a new value of n and then re-entering the other commands.

b): Repeat part (a), this time using the polynomial $Q_n(x)$ of degree $\leq n$ which interpolates f at the (Chebyshev) points

$$x_i = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{(2i+1)\pi}{2n+2}\right), i = 0, ..., n.$$

To get you started, the case $n = 4$ can be done by typing (or cutting and pasting) the following commands into Matlab.

```
% Problem 3b
close all  % closes plotting windows
a=-5;  b=5;
n=4;
x=linspace(1,2*n+1,n+1);
xin = (a+b)/2 + (b-a)/2 * cos(x*pi/(2*n+2));
yin= 1./(1+xin.^2);
xout=linspace(a,b,100);
[yout, cof] = interppoly(xin,yin,xout);
ytrue= 1./(1+xout.^2);
plot(xout,ytrue-yout)
```

c. Based on your plots, does the choice of interpolation points make a difference in the error in the approximation? Which choice is better in this case?

Solution:

a. $n = 4$: max of $|e_4(x)| \approx .44$ and occurs at $x \approx \pm 4$.
$n = 8$: max of $|e_8(x)| \approx 1$ and occurs at $x \approx \pm 4.5$.
$n = 16$: max of $|e_16(x)| \approx 14$ and occurs at $x \approx \pm 4.75$.
$n = 32$: max of $|e_16(x)| \approx 4600$ and occurs at $x \approx \pm 4.9$.

b. $n = 4$: max of $|e_4(x)| \approx .4$ and occurs at $x \approx \pm 1.5$.
$n = 8$: max of $|e_8(x)| \approx .17$ and occurs at $x \approx \pm 1$.
$n = 16$: max of $|e_16(x)| \approx .032$ and occurs at $x \approx \pm 1.4$.
$n = 32$: max of $|e_16(x)| \approx .00014$ and occurs at $x \approx \pm 1.1$

c. Clearly, the choice of interpolation points does make a difference and the Chebyshev points give a much smaller error for this problem than the choice of equally spaced points.