Lecture 26

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1 NP-Hardness Reductions

Recall from the last lecture that our goal is to prove NP-hardness of some problems using reduction. We continue with the 3-SAT problem first.

3-SAT Problem

Problem 1 (3-SAT Problem). We are given a boolean 3-CNF formula Φ with n variables and m clauses. For any $y \in \{0,1\}^n$, we use $\Phi(y)$ to denote the value of the formula when assigning y to the variables of the formula. The 3-SAT problem asks does there exists at least one assignment y such that $\Phi(y) = 1$ or not? In other words, is this 3-CNF formula *ever* satisfiable or not?

3-SAT is in NP. This is done in the last lecture.

3-SAT is NP-hard. The goal is to show that *any* polynomial time algorithm for 3-SAT also implies a polynomial time algorithm for Circuit-SAT (and since Circuit-SAT is NP-hard, this implies P = NP, which in turn implies 3-SAT is also NP-hard). The reduction is as follows.

Reduction: Given an input C to the Circuit-SAT problem, we turn it into a 3-CNF formula Φ as follows:

1. Define a new variable for every *wire* in the circuit C (including the input wires called variables and the output wire). These are all the variables in the formula Φ.
2. For every gate G in the circuit C we add the following clauses to the formula:
   (a) AND: Let a be the variable for the output wire of this gate and b and c be the variables for input wires. We want to have $a = b \wedge c$ in our formula. To do so, we add the following clauses to Φ:
   $$(a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee b) \wedge (\bar{a} \vee c).$$
   (b) OR: Let a be the variable for the output wire of this gate and b and c be the variables for input wires. We want to have $a = b \vee c$ in our formula. To do so, we add the following clauses to Φ:
   $$(\bar{a} \vee b \vee c) \wedge (a \vee \bar{b}) \wedge (a \vee \bar{c}).$$
   (c) NOT: Let a be the variable for the output wire of this gate and b be the variable for the input wire. We want to have $a = \bar{b}$ in our formula. To do so, we add the following clauses to Φ:
   $$(a \vee b) \wedge (\bar{a} \vee \bar{b}).$$
3. This concludes the description of the formula Φ (we emphasize that for the *output* gate, if z is variable for the output wire, we add the singleton clause z to Φ as well).

We then simply run our (supposed) algorithm for 3-SAT on the formula Φ and if the algorithm outputs Φ is satisfiable, we output that C is satisfiable and otherwise output C is not satisfiable.

Proof of Correctness: Let C be any given circuit and Φ be the resulting 3-CNF formula in the reduction. We prove that C is satisfiable if and only if Φ is satisfiable. This will immediately imply the correctness.

In order to do this, we first prove that Φ and C are *logically equivalent*. To do this, we need to show that the set of clauses introduced for each gate work exactly the same as the gate itself:

- AND-gates – By writing the truth table of each of the two expressions we have:

$a = b \wedge c$

a	b	c	True/False
0	0	0	True
0	0	1	True
0	1	0	True
0	1	1	False
1	0	0	False
1	0	1	False
1	1	0	False
1	1	1	True

$(a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee b) \wedge (\bar{a} \vee c)$

a	b	c	True/False
0	0	0	True
0	0	1	True
0	1	0	True
0	1	1	False
1	0	0	False
1	0	1	False
1	1	0	False
1	1	1	True

  So AND-gates and the resulting clauses in the reductions are equivalent.

- OR-gates – By writing the truth table of each of the two expressions we have:

$a = b \vee c$

a	b	c	True/False
0	0	0	True
0	0	1	False
0	1	0	False
0	1	1	False
1	0	0	False
1	0	1	True
1	1	0	True
1	1	1	True

$(\bar{a} \vee b \vee c) \wedge (a \vee \bar{b}) \wedge (a \vee \bar{c})$

a	b	c	True/False
0	0	0	True
0	0	1	False
0	1	0	False
0	1	1	False
1	0	0	False
1	0	1	True
1	1	0	True
1	1	1	True

  So OR-gates and the resulting clauses in the reductions are equivalent.

- NOT-gates – By writing the truth table of each of the two expressions we have:

$a = \bar{b}$

a	b	True/False
0	0	False
0	1	True
1	0	True
1	1	False

$(a \vee b) \wedge (\bar{a} \vee \bar{b})$

a	b	True/False
0	0	False
0	1	True
1	0	True
1	1	False

  So NOT-gates and the resulting clauses in the reductions are equivalent.

We can now use this to finalize the proof. Recall that we need to prove C is satisfiable if and only if Φ is satisfiable. As any other "if and only if" statement, we need to prove this in two steps:

(i) If C is satisfiable, then Φ is satisfiable also: Pick a satisfying assignment x to C and consider every wire of this circuit. Let y be the assignment of these wires to variables in Φ. By the above part, C and Φ are logically equivalent and thus y satisfies every clause in Φ. Finally, since for the output wire we have a singleton clause z (which is equivalent to $C(x)$ which in turn is 1), the $\Phi(y) = 1$. This means Φ is satisfiable.

(ii) If Φ is satisfiable, then C is satisfiable also. Pick a satisfying assignment y to Φ and consider the variables assigned to the *input* wires. Let x be the assignment of these input wires in C. By the above part, C and Φ are logically equivalent and so every wire of circuit C on the input x will get the same value as the corresponding variable y in Φ. This in particular means that the output wire gets the value 1 on the input x (because the output wire is a singleton clause) and thus $C(x) = 1$. This means C is satisfiable.

Runtime analysis: We also need to show that *if* we have a poly-time algorithm for 3-SAT the above reduction runs in poly-time. This is true because the size of Φ is at most a constant factor larger than size of the circuit (each gate is replaced by at most 3 clauses) and we create Φ in linear-time from C. Thus a poly-time algorithm for 3-SAT on Φ implies a poly-time algorithm for Circuit-SAT on Φ.

This concludes the proof of NP-hardness of the 3-SAT problem.

Maximum Independent Set Problem

Unlike Circuit-SAT which might be a bit cumbersome to work with, 3-SAT is an excellent problem for doing reductions from. We now use this to prove NP-hardness of a fundamental graph problem, namely, the *maximum independent set* problem.

Given an undirected graph $G(V, E)$, a set $S \subseteq V$ of vertices is called an *independent set* if there is *no* edge with both endpoints in S (in other words, no vertices in S are neighbor to each other).

Problem 2 (**Maximum Independent Set Problem**). Given an undirected graph $G(V, E)$ output the size of the largest independent set in G. We abbreviate this problem by ***MaxIndSet***.

Is MaxIndSet in NP? MaxIndSet is *not* a decision problem and hence cannot be in NP.

MaxIndSet is NP-hard. Nevertheless, we prove that MaxIndSet is NP-hard, meaning that a poly-time algorithm for MaxIndSet implies P = NP. We do this by a reduction from 3-SAT (or alternatively, show that 3-SAT can be reduced to MaxIndSet).

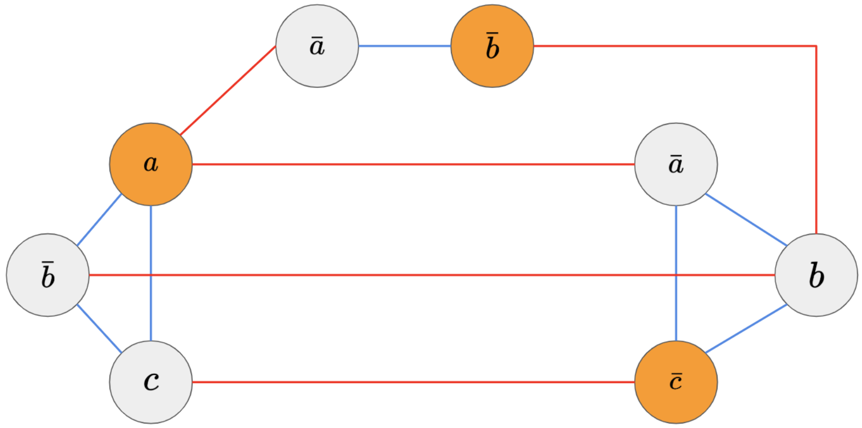
Reduction: Given any instance Φ of the 3-SAT problem, we create an instance $G(V, E)$ of maximum independent set as follows:

1. For any clause in Φ, we add (at most) three vertices corresponding to the literals of the clause to V.
2. Two vertices in G are connected to each other if either (1) they corresponds to the variables in the same clause (and were added to G together in the last step), or (2) they correspond to a variable and its negation. See Figure 1 for an example.

We then simply run our (supposed) algorithm for MaxIndSet on G and let o denote size of maximum independent in G output by the algorithm. If o is equal to to the number of clauses in Φ, we output Φ is satisfiable and otherwise we output it is not.

Proof of Correctness: Let k denote the number of clauses in Φ. We first claim that any independent set in G has size *at most* k. This is because all vertices inside a clause are connected to each other and hence any independent set can pick at most one variable from a clause. Now to prove the correctness of the reduction (similar to any other reduction), we should prove that the size of maximum independent set in G is *exactly* k if and only if Φ is satisfiable.

(i) Maximum independent set size in G is $k \implies \Phi$ is satisfiable: Let $S = \{v_1, \ldots, v_k\}$ be an independent set of size k in G. Note that (1) each vertex belongs to a unique clause (because vertices in each clause are connected to each other and hence cannot be part of an independent set together), and (2) a vertex and its negation do not appear *simultaneously* in S (because each variable-vertex is connected to its



$$(a \vee \bar{b} \vee c) \wedge (\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee b \vee \bar{c})$$

Figure 1: An illustration of the reduction of 3-SAT to MaxIndSet (here blue edges connect vertices in the same clause and red edges connect a vertex to its negation). The orange vertices form an independent set in G corresponding to the assignment of the variables $a = 1$, $b = 0$, and $c = 0$.

negation). We can thus make the *literals* corresponding to $v_1, \ldots, v_k$ in Φ to be true in Φ (and pick an arbitrary assignment of remaining variables) and satisfy all clauses in Φ. This means that Φ is satisfiable.

(ii) Φ is satisfiable $\implies$ maximum independent size in G is k: Let y be a satisfying assignment of Φ. We pick a set S of vertices in G by picking one variable-vertex from each clause corresponding to a true literal in y (since y is a satisfying assignment, there exists always one such literal and if there is more than one we can pick arbitrarily). Since in any assignment of y, we will never have a literal and its negation to be true simultaneously and since we are only picking exactly one literal per each clause, the set S is an independent set of size k. Finally since any independent set has size at most k in G, we get the result.

Runtime analysis: We can create the graph G in polynomial time from Φ and *if* we have a poly-time algorithm for MaxIndSet, we will obtain a poly-time algorithm for 3-SAT as well.

This concludes the proof of NP-hardness of MaxIndSet problem since we proved a poly-time algorithm for MaxIndSet implies a poly-time algorithm for an NP-hard problem, namely, 3-SAT. [1]

Concluding Remarks on NP-Hardness Reductions

The general strategy of proving a problem NP-hard in this course is always the same as the two different reductions we already saw: to show problem A is NP-hard, we will find another NP-hard problem B and show that solving B in polynomial time reduces to solving A in polynomial time; this implies that a polynomial time algorithm for A will give a polynomial time algorithm for B which in turn, by definition, implies that P = NP. [2]

[1] We can trace the chain: poly-time algorithm for MaxIndSet $\underbrace{\implies}_{\textit{this reduction}}$ poly-time algorithm for 3-SAT $\underbrace{\implies}_{\textit{previous reduction}}$ poly-time algorithm for Circuit-SAT $\underbrace{\implies}_{\textit{Cook-Levin Theorem}}$ poly-time algorithm for all NP problems, i.e., P=NP.

[2] Note that this task involves two different steps: (1) finding the appropriate problem B in the first place, and (2) performing the reduction. However, in this course, you will *typically* be told which problem you should reduce from, namely, the choice of B, or at least a small set of candidates for B, will be given to you.

Each reduction also works as follows (more or less): give an instance problem B, we should create another instance of problem A such that the answer to problem A uniquely determines the answer to problem B. Finally, for the proof of correctness, we should use the same strategy as any other reduction: the answer to problem A in this new instance is "something" if and only if the answer to problem B in the given instance is "something". We should also not forget to prove that the reduction itself takes polynomial time and thus a poly-time algorithm for A indeed gives a poly-time algorithm for B[3].

In the following, we list several other "(relatively) easy-to-show NP-hard" problems that appear very frequently, and are worth knowing.

- **Maximum Clique Problem.** A clique in an undirected graph $G(V, E)$ is any set T of vertices such that there is an edge between any pairs of vertices.

  The maximum clique problem asks for finding the size of the largest clique in a given graph. Maximum clique problem is an NP-hard problem and the easiest way to prove this is to do a reduction from the maximum independent set problem (this is something quite easy at this point and you are encouraged to do this on your own for practice).

- **Minimum Vertex Cover Problem.** A vertex cover in an undirected graph $G(V, E)$ is any set U of vertices such that any edge in G has at least one end point in U (namely, every edge is covered by U).

  The minimum vertex cover problem asks for finding the size of the smallest vertex cover in a given graph. Minimum vertex cover problem is also NP-hard and again can be proven so by a reduction from the maximum independent set problem.

- **3-Coloring Problem.** A 3-coloring of an undirected graph $G(V, E)$ is a function $C : V \rightarrow \{1, 2, 3\}$ that assigns one of the colors $\{1, 2, 3\}$ to vertices of the graph so that every edge of the graph has two different colors at its end point, i.e., $C(u) \neq C(v)$ for every edge $\{u, v\} \in E$.

  The 3-coloring problem asks whether a given undirected graph admits a 3-coloring or not (this is a decision problem). One way to prove 3-coloring is NP-hard is by a reduction from the 3-SAT problem; see CLRS book or Chapter 12.10 of Erickson's book (this is a nice reduction and you are strongly encouraged to check it).

- **Hamiltonian Cycle Problem.** A Hamiltonian cycle in an undirected graph $G(V, E)$ is a cycle that passes through *every* vertex.

  The Hamiltonian cycle problem asks whether a given undirected graph has any Hamiltonian cycle or not (this is a decision problem). Hamiltonian cycle can also be shown to be NP-hard by a reduction from the 3-SAT problem; see CLRS book or Chapter 12.11 of Erickson's book. This problem has several other well-known NP-hard variants such as finding a Hamiltonian path instead (a path that goes through every vertex) or the traveling salesman problem (TSP).

- **Minimum Set Cover Problem.** A set cover of a collection sets $S_1, \ldots, S_m$ where each $S_i \subseteq \{1, \ldots, n\}$ is any collection of sets C such that every element in $\{1, \ldots, n\}$ belongs to at least one of the sets in C (namely, every element is covered by C) or in other words $\bigcup_{i \in C} S_i = \{1, \ldots, n\}$.

  The minimum set cover problem asks for finding the size of the smallest set cover of a given collection of sets. Set cover can be proven NP-hard by a reduction from the minimum vertex cover problem (this is a relatively easy reduction and it would be a good idea to think about it on your own – simply try to find the (rather direct) connection between set cover and minimum vertex cover). Set cover problem also has several other well-known NP-hard variants such as the hitting set problem or the dominating set problem.

- **Knapsack Problem.** We have already seen this problem in details when studying dynamic programming algorithms. Knapsack problem is also another NP-hard problem and can be proven so by a reduction from 3-SAT (see CLRS book). It is important to emphasize again that the solutions we designed for this problem using dynamic programming were *not* polynomial time (see Lecture 24 for

[3] If the reduction is *not* poly-time, there is no reason a poly-time algorithm for A give a poly-time algorithm for B, no?

more details)[4]. Knapsack problem also has several other well-known NP-hard variants such as the Partition problem and the Subset Sum problem.

The above list is by no means a comprehensive list of NP-hard problems (not even the most interesting ones). However, it will hopefully give you a general sense of various NP-hard problems.

[4]This means that quite unfortunately we have not proved P=NP...