Chapter 4: Multiple Regression Models

General Form of the Multiple Regression Model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \varepsilon$$

where y is the dependent variable

$x_1, x_2, \ldots, x_k$ are the independent variables

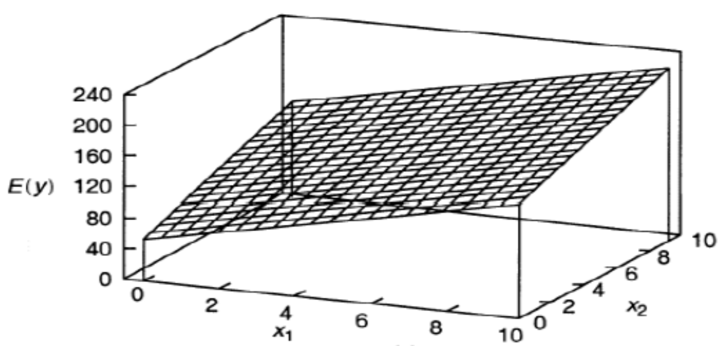
$E(y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k$ is the deterministic portion of the model

β_i determines the contribution of the independent variable x_i

Note: The symbols $x_1, x_2, \ldots, x_k$ may represent higher-order terms for quantitative predictors (e.g., $x_2 = x_1^2$) or terms for qualitative predictors.

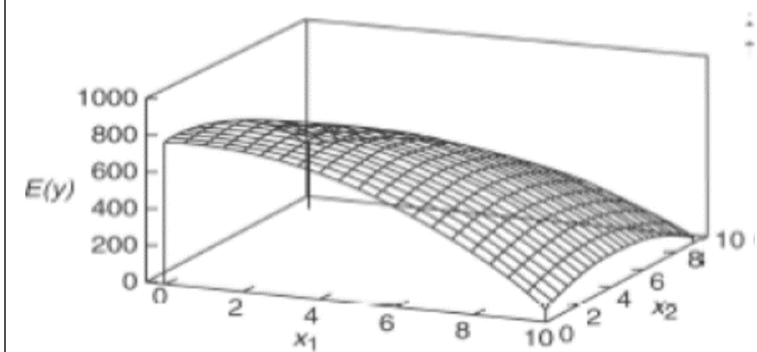
$$E(\varepsilon) = 0,\ \mathrm{Var}(\varepsilon) = \sigma^2$$

$E(Y) = 50 + 10x_1 + 7x_2$



(a)

$E(Y) = 50 + 10x_1 + 7x_2 + 5\,x_1\,x_2$



Example of interaction between two variables.
Time to run a specified obstacle course (min)

Dose of drug (mg)		Low	High
Age	< 65 yo	1.9	1.1
	>=65 yo	2.1	2.0

An Age by Drug Interaction can be observed in the table to the left

Data for Multiple Linear Regression

Observation	y Value	x_1	x_2	$\cdots$	x_k
1	y_1	x_{11}	x_{21}		x_{k1}
2	y_2	x_{12}	x_{22}		x_{k2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
n	y_n	x_{1n}	x_{2n}		x_{kn}

The least squares function is

$$S(\beta_0, \beta_1, \ldots, \beta_k) = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{k} \beta_j x_{ij} \right)^2$$

The function S must be minimized with respect to the coefficients.

σ^2 is estimated by SSE/n-(k+1)

Matrix notation is typically used.

See Appendix B in textbook

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

B.2 Matrices and Matrix Multiplication

Although it is very difficult to give the formulas for the multiple regression least squares estimators and for their estimated standard errors in ordinary algebra, it is easy to do so using **matrix algebra**. Thus, by arranging the data in particular rectangular patterns called **matrices** and by performing various operations with them, we can obtain the least squares estimates and their estimated standard errors. In this section and Sections B.3 and B.4, we define what we mean by a matrix and explain various operations that can be performed with matrices. We explain how to use this information to conduct a regression analysis in Section B.5.

Three matrices, **A**, **B**, and **C**, are shown here. Note that each matrix is a rectangular arrangement of numbers with one number in every row–column position.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -1 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Definition B.1 A **matrix** is a rectangular array of numbers.*

The numbers that appear in a matrix are called **elements** of the matrix. If a matrix contains r rows and c columns, there will be an element in each of the row–column positions of the matrix, and the matrix will have $r \times c$ elements. For example, the matrix **A** shown previously contains $r = 3$ rows, $c = 2$ columns, and $rc = (3)(2) = 6$ elements, one in each of the six row–column positions.

Definition B.2 A number in a particular row–column position is called an **element** of the matrix.

Definition B.3 A matrix containing r rows and c columns is said to be an $\boldsymbol{r \times c}$ **matrix** where r and c are the **dimensions** of the matrix.

Definition B.4 If $r = c$, a matrix is said to be a **square matrix**.

Requirement for Matrix Multiplication

$$\mathbf{AB}$$

$$r \times d \qquad d \times c$$

Find the product **AB**, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & 0 \end{bmatrix}$$

Solution

If we represent the product **AB** as

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

The Data Matrices Y and X and the $\hat{\beta}$ Matrix

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{21} & \cdots & x_{k1} \\ 1 & x_{12} & x_{22} & \cdots & x_{k2} \\ 1 & x_{13} & x_{23} & \cdots & x_{k3} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{1n} & x_{2n} & \cdots & x_{kn} \end{bmatrix} \quad \hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix}$$

Special types of matrices:

- Square matrix
- Identity matrix
- Diagonal matrix