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Slides adopted from Prof. Ninghui Li at Purdue

COMPUTER SECURITY CS 526

TOPIC 3

CRYPTOGRAPHY: OTP, INFORMATION THEORETIC SECURITY, AND STREAM CIPHERS

READINGS FOR THIS LECTURE

- Required reading from wikipedia
  - One-Time Pad
  - Information theoretic security
  - Stream cipher
  - Pseudorandom number generator

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BACKGROUND

RANDOM VARIABLE

Definition

A **discrete random variable, X,** consists of a finite set $\mathcal{X}$, and a probability distribution defined on $\mathcal{X}$. The probability that the random variable **X** takes on the value x is denoted **Pr**[**X** =x]; sometimes, we will abbreviate this to **Pr**[x] if the random variable **X** is fixed. It must be that

$$0 \leq \Pr[x] \quad \text{for all } x \in \mathrm{X}$$

$$\sum_{x \in \mathrm{X}} \Pr[x] = 1$$

EXAMPLE OF RANDOM VARIABLES

- Let random variable $\mathbf{D_1}$ denote the outcome of throwing one die (with numbers 0 to 5 on the 6 sides) randomly, then $\mathcal{D}$={0,1,2,3,4,5} and $\mathbf{Pr}[\mathbf{D_1}=i] = 1/6$ for $0 \le i \le 5$
- Let random variable $\mathbf{D_2}$ denote the outcome of throwing a second such die randomly
- Let random variable $\mathbf{S_1}$ denote the sum of the two dice, then $\mathcal{S}$ ={0,1,2,…,10}, and

$$\mathbf{Pr}[\mathbf{S_1}=0] = \mathbf{Pr}[\mathbf{S_1}=10] = 1/36$$

$$\mathbf{Pr}[\mathbf{S_1}=1] = \mathbf{Pr}[\mathbf{S_1}=9] = 2/36 = 1/18$$

…

- Let random variable $\mathbf{S_2}$ denote the sum of the two dice modulo 6, what is the distribution of $\mathbf{S_2}$?

RELATIONSHIPS BETWEEN TWO RANDOM VARIABLES

Definitions

Assume **X** and **Y** are two random variables,

then we define:

- **joint probability**: **Pr**[x, y] is the probability that **X** takes value x and **Y** takes value y.
- **conditional probability**: **Pr**[x|y] is the probability that **X** takes value x given that **Y** takes value y.
  - **Pr**[x|y] = **Pr**[x, y] / **Pr**[y]
- **independent random variables**: **X** and **Y** are said to be independent if **Pr**[x,y] = **Pr**[x]P[y], for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

EXAMPLES

- Joint probability of $\mathbf{D_1}$ and $\mathbf{D_2}$, for $0 \le i,j \le 5$, $\mathbf{Pr}[\mathbf{D_1}=i, \mathbf{D_2}=j] = ?$
- What is the conditional probability $\mathbf{Pr}[\mathbf{D_1}=i \mid \mathbf{D_2}=j]$ for $0 \le i, j \le 5$?
- Are $\mathbf{D_1}$ and $\mathbf{D_2}$ independent?
- Suppose $\mathbf{D_1}$ is plaintext and $\mathbf{D_2}$ is key, and $\mathbf{S_1}$ and $\mathbf{S_2}$ are ciphertexts of two different ciphers, which cipher would you use?

THINK AFTER CLASS

- What is the joint probability of $\mathbf{D_1}$ and $\mathbf{S_1}$?
- What is the joint probability of $\mathbf{D_2}$ and $\mathbf{S_2}$?

- What is the conditional probability $\mathbf{Pr}[\mathbf{S_1}=s \mid \mathbf{D_1}=i]$ for $0 \le i \le 5$ and $0 \le s \le 10$?
- What is the conditional probability $\mathbf{Pr}[\mathbf{D_1}=i \mid \mathbf{S_2}=s]$ for $0 \le i \le 5$ and $0 \le s \le 5$?

- Are $\mathbf{D_1}$ and $\mathbf{S_1}$ independent?
- Are $\mathbf{D_1}$ and $\mathbf{S_2}$ independent?

BAYES' THEOREM

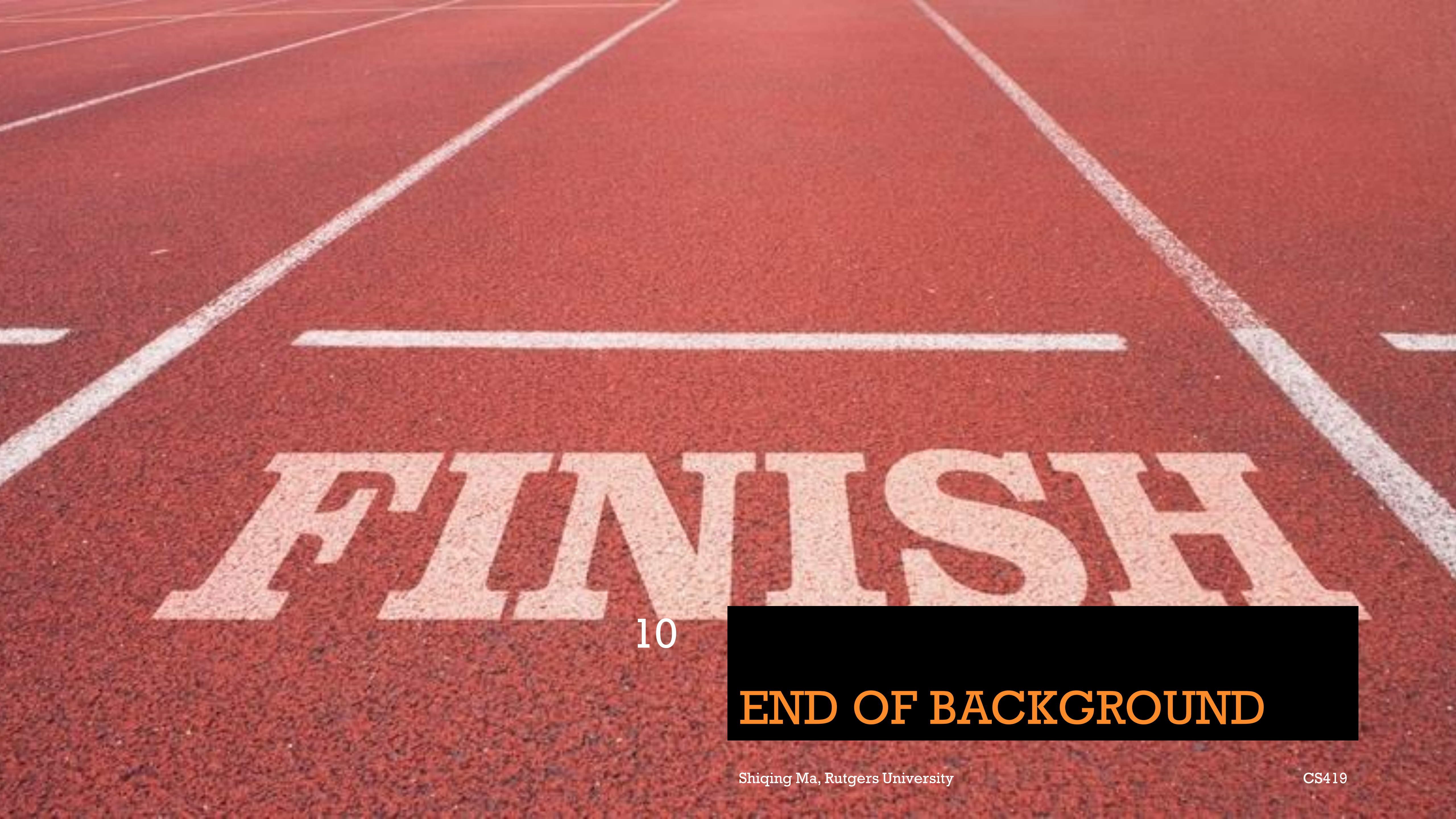
If P[y] > 0 then

$$P[x|y] = \frac{P[x]P[y|x]}{P[y]}$$

$$P[y] = \sum_{x \in X} P[x, y] = \sum_{x \in X} P[x]p[y|x]$$

Corollary

X and Y are independent random variables iff P[x|y] = P[x], for all x ∈ X and all y ∈ Y.



END OF BACKGROUND

ONE-TIME PAD

- Fix the vulnerability of the Vigenere cipher by using very long keys
- Key is a random string that is at least as long as the plaintext
- Encryption is similar to shift cipher
- Invented by Vernam in the 1920s

ONE-TIME PAD

Let Z_m ={0,1,…,m-1} be the alphabet.

Plaintext space = Ciphtertext space = Key space = $(Z_m)^n$

The key is chosen uniformly randomly

Plaintext $X = (x_1\ x_2 \ldots x_n)$

Key $K = (k_1\ k_2 \ldots k_n)$

Ciphertext $Y = (y_1\ y_2 \ldots y_n)$

$e_k(X) = (x_1+k_1\ \ x_2+k_2 \ldots x_n+k_n) \bmod m$

$d_k(Y) = (y_1-k_1\ \ y_2-k_2 \ldots\ y_n-k_n) \bmod m$

THE BINARY VERSION OF ONE-TIME PAD

Plaintext space = Ciphtertext space = Keyspace = $\{0,1\}^n$

Key is chosen randomly

For example:

- Plaintext is 11011011
- Key is 01101001
- Then ciphertext is 10110010

BIT OPERATORS

- Bit AND

  $0 \wedge 0 = 0$ $0 \wedge 1 = 0$ $1 \wedge 0 = 0$ $1 \wedge 1 = 1$

- Bit OR

  $0 \vee 0 = 0$ $0 \vee 1 = 1$ $1 \vee 0 = 1$ $1 \vee 1 = 1$

- Addition mod 2 (also known as Bit XOR)

  $0 \oplus 0 = 0$ $0 \oplus 1 = 1$ $1 \oplus 0 = 1$ $1 \oplus 1 = 0$

- Can we use operators other than Bit XOR for binary version of One-Time Pad?

HOW GOOD IS ONE-TIME PAD?

Intuitively, it is secure …

- The key is random, so the ciphertext is completely random

How to formalize the confidentiality requirement?

- Want to say "certain thing" is not learnable by the adversary (who sees the ciphertext). But what is the "certain thing"?

Which (if any) of the following is the correct answer?

- The key.
- The plaintext.
- Any bit of the plaintext.
- Any information about the plaintext.
  - E.g., the first bit is 1, the parity is 0, or that the plaintext is not "aaaa", and so on

SHANNON (INFORMATION-THEORETIC) SECURITY = PERFECT SECRECY

Basic Idea: Ciphertext should reveal no "information" about plaintext

Definition. An encryption over a message space $\mathcal{M}$ is perfectly secure if

$\forall$ probability distribution over $\mathcal{M}$

$\forall$ message $m \in \mathcal{M}$

$\forall$ ciphertext $c \in \mathcal{C}$ for which $\Pr[C=c] > 0$

We have

$$\Pr[\mathbf{PT}=m \mid \mathbf{CT}=c] = \Pr[\mathbf{PT}=m].$$

EXPLANATION OF THE DEFINITION

- Pr [**PT** = m] is what the adversary believes the probability that the plaintext is m, before seeing the ciphertext
- Pr [**PT** = m | **CT**=c] is what the adversary believes after seeing that the ciphertext is c
- Pr [**PT**=m | **CT**=c] = Pr [**PT** = m] means that after knowing that the ciphertext is C_0, the adversary's belief does not change.

EQUIVALENT DEFINITION

Definition. An encryption scheme over a message space $\mathcal{M}$ is perfectly secure if $\forall$ probability distribution over $\mathcal{M}$, the random variables **PT** and **CT** are independent. That is,

$\forall$ message $m \in \mathcal{M}$

$\forall$ ciphertext $c \in \mathcal{C}$

$$\Pr[\mathbf{PT}=m \wedge \mathbf{CT}=c] = \Pr[\mathbf{PT}=m]\Pr[\mathbf{CT}=c]$$

Note that this is equivalent to: When $\Pr[\mathbf{CT}=c] \neq 0$, we have $\Pr[\mathbf{PT}=m] = \Pr[\mathbf{PT}=m \wedge \mathbf{CT}=c] / \Pr[\mathbf{CT}=c] = \Pr[\mathbf{PT}=m \mid \mathbf{CT}=c]$

This is also equivalent to: When $\Pr[\mathbf{PT}=m] \neq 0$, we have $\Pr[\mathbf{CT}=c] = \Pr[\mathbf{PT}=m \wedge \mathbf{CT}=c] / \Pr[\mathbf{PT}=m] = \Pr[\mathbf{CT}=c \mid \mathbf{PT}=m]$

EXAMPLE FOR INFORMATION THEORETICAL SECURITY

- Consider an example of encrypting the result of a 6-side dice (1 to 6).
  - Method 1: randomly generate K=[0..5], ciphertext is result + K.
    - What is plaintext distribution? After seeing that the ciphertext is 6, what could be the plaintext. After seeing that the ciphertext is 11, what could be the plaintext?
  - Method 2: randomly generate K=[0..5], ciphertext is (result + K) mod 6.
    - Same questions.
    - Can one do a brute-force attack?

PERFECT SECRECY

- Fact: When keys are uniformly chosen in a cipher, the cipher has perfect secrecy iff. the number of keys encrypting M to C is the same for any (M,C)
  - This implies that

$$\forall c \forall m_1 \forall m_2 \ \Pr[\mathbf{CT}=c \mid \mathbf{PT}=m_1] = \Pr[\mathbf{CT}=c \mid \mathbf{PT}=m_2]$$

- One-time pad has perfect secrecy when limited to messages over the same length (Proof?)

KEY RANDOMNESS IN ONE-TIME PAD

- One-Time Pad uses a very long key, what if the key is not chosen randomly, instead, texts from, e.g., a book are used as keys.
  - this is not One-Time Pad anymore
  - this does not have perfect secrecy
  - this can be broken
  - How?
- The key in One-Time Pad should never be reused.
  - If it is reused, it is Two-Time Pad, and is insecure!
  - Why?

USAGE OF ONE-TIME PAD

- To use one-time pad, one must have keys as long as the messages.
- To send messages totaling certain size, sender and receiver must agree on a shared secret key of that size.
  - typically by sending the key over a secure channel
- This is difficult to do in practice.
- Can’t one use the channel for send the key to send the messages instead?
- Why is OTP still useful, even though difficult to use?

USAGE OF ONE-TIME PAD

- The channel for distributing keys may exist at a different time from when one has messages to send.
- The channel for distributing keys may have the property that keys can be leaked, but such leakage will be detected
  - Such as in Quantum cryptography

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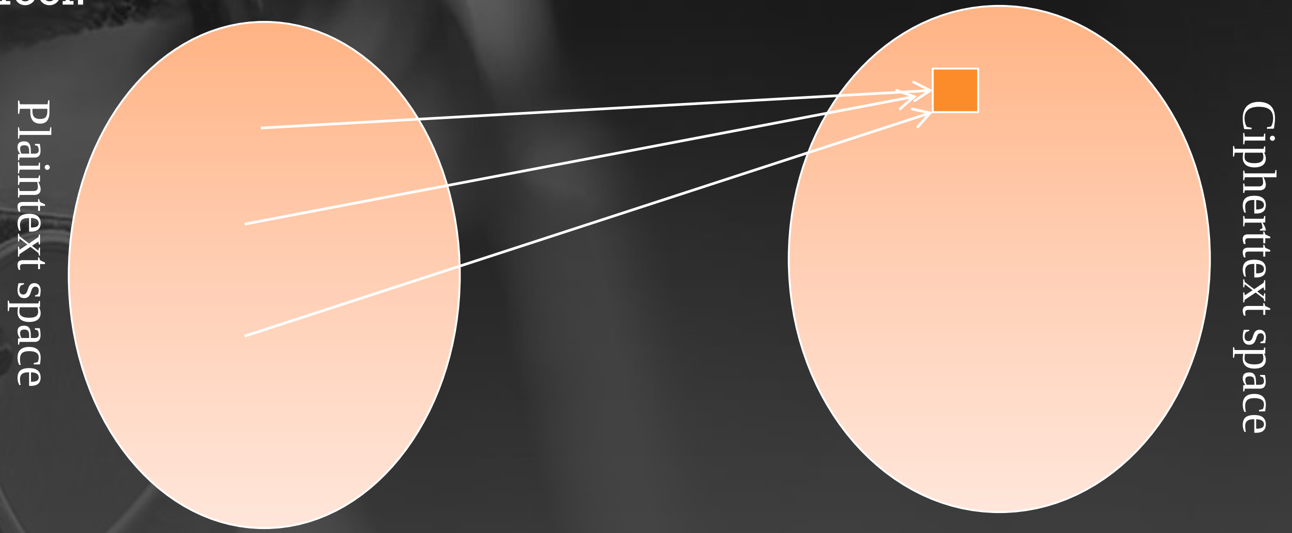
THE "BAD NEWS" THEOREM FOR PERFECT SECRECY

- Question: OTP requires key as long as messages, is this an inherent requirement for achieving perfect secrecy?
- Answer. Yes. Perfect secrecy implies that key-length ≥ msg-length

Proof:



- Implication: Perfect secrecy difficult to achieve in practice

STREAM CIPHERS

- In One-Time Pad, a key is a random string of length at least the same as the message
- Stream ciphers:
  - Idea: replace "rand" by "pseudo rand"
  - Use Pseudo Random Number Generator
  - PRNG: $\{0,1\}^s \rightarrow \{0,1\}^n$
    - expand a short (e.g., 128-bit) random seed into a long (e.g., 10^6 bit) string that "looks random"
  - Secret key is the seed
  - $E_{key}[M] = M \oplus PRNG(key)$

THE RC4 STREAM CIPHER

- A proprietary cipher owned by RSA, designed by Ron Rivest in 1987. Became public in 1994.
- Simple and effective design.
- Variable key size (typical 40 to 256 bits),
- Output unbounded number of bytes.
- Widely used (web SSL/TLS, wireless WEP).
- Extensively studied, not a completely secure PRNG, first part of output biased, when used as stream cipher, should use RC4-Drop[n]
  - Which drops first n bytes before using the output
  - Conservatively, set n=3072

PSEUDO RANDOM NUMBER GENERATOR

- Useful for cryptography, simulation, randomized algorithm, etc.
  - Stream ciphers, generating session keys
- The same seed always gives the same output stream
  - Why is this necessary for stream ciphers?
- Simulation requires uniform distributed sequences
  - E.g., having a number of statistical properties
- **Cryptographically secure pseudo-random number generator** requires unpredictable sequences
  - satisfies the "next-bit test“: given consecutive sequence of bits output (but not seed), next bit must be hard to predict
- Some PRNG’s are weak: knowing output sequence of sufficient length, can recover key.
  - Do not use these for cryptographic purposes

PROPERTIES OF STREAM CIPHERS

- Typical stream ciphers are very fast
- Widely used, often incorrectly
  - Content Scrambling System (uses Linear Feedback Shift Registers incorrectly),
  - Wired Equivalent Privacy (uses RC4 incorrectly)
  - SSL (uses RC4, SSLv3 has no known major flaw)

SECURITY PROPERTIES OF STREAM CIPHERS

- Under known plaintext, chosen plaintext, or chosen ciphertext, the adversary knows the key stream (i.e., PRNG(key))
  - Security depends on PRNG
  - PRNG must be "unpredictable"
- Do stream ciphers have perfect secrecy?
- How to break a stream cipher in a brute-force way?
- If the same key stream is used twice, then easy to break.
  - This is a fundamental weakness of stream ciphers; it exists even if the PRNG used in the ciphers is strong

USING STREAM CIPHERS IN PRACTICE

- If the same key stream is used twice, then easy to break.
  - This is a fundamental weakness of stream ciphers; it exists even if the PRNG used in the ciphers is strong
- In practice, one key is used to encrypt many messages
  - Example: Wireless communication
  - Solution: Use Initial vectors (IV).
  - $E_{key}[M] = [IV, M \oplus PRNG(key \mid\mid IV)]$
    - IV is sent in clear to receiver;
    - IV needs integrity protection, but not confidentiality protection
    - IV ensures that key streams do not repeat, but does not increase cost of brute-force attacks
    - Without key, knowing IV still cannot decrypt
  - Need to ensure that IV never repeats! How?

NEXT CLASS

- Cryptography: Semantic Security, Block ciphers, encryption modes, cryptographic functions