Microfoundations of Discounting

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July 3, 2019

Abstract

An important question in economics is how people evaluate payoffs in the future. The standard phrasing of the problem is in part psychological. The value we attach to a future payoff is the dollar value of the payoff discounted by a factor whose functional form is determined subjectively and whose (objective) argument is how long we have to wait for the payoff. Here we explore how discounting arises from growth rate optimization, in four specifications of a Riskless Intertemporal Payoff Problem (RIPP). Exponential discounting is recovered as optimizing a multiplicative growth rate, and hyperbolic discounting as optimizing an additive growth rate. Preference reversal occurs in two of the specifications we study.

Keywords: Decision theory, Hyperbolic discounting, Ergodicity economics

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¶We wish to thank Ryan Singer for fruitful discussions and comments.

1 Introduction

This paper studies temporal discounting in a deterministic setting. In our model a decision maker chooses between two known and different payoffs to be received at known and different times by comparing the growth rates of wealth associated with each option. The model is further specified by assumptions about the wealth dynamics. Depending on the specification, the model produces different forms of discounting – including exponential and hyperbolic.

Preference reversal is a behavioral phenomenon documented during the past half a century in many studies in economics and psychology (Lichtenstein and Slovic, 1971; Lindman, 1971; Grether and Plott, 1979; Loomes and Sugden, 1983; Tversky, Slovic and Kahneman, 1990; Ainslie, 1992; Laibson, 1997). It takes various forms in different contexts. In its original psychological context (Tversky, 1969; Lichtenstein and Slovic, 1971) it refers to the intransitivity in decision making under uncertainty. It also refers to the phenomenon in which a decision maker changes his mind between two options as time passes.

In our model preference reversal is a consequence of growth rate optimization. Specifications that lead to standard exponential discounting do not generate preference reversal, but others do. Specifying the wealth dynamic reflects the situation of a decision maker. Thus a single behavioral model, a single optimand, is consistent with a range of observed behaviors if the situation of the decision maker is formally taken into account.

The main contribution of this paper is the prediction of hyperbolic discounting, and hence preference reversal, from considerations that do not violate the standard von Neumann-Morgenstern axioms. Our framework does not assume any form of irrationality or inconsistency on the part of the decision maker. Changes in the type of discounting come from different specifications of wealth dynamics, *i.e.* properties of the situation of the decision maker, not properties of the decision maker himself. Thus, utility functions or temporal preferences are not part of our model specification – such idiosyncratic properties can be derived from a full specification of the situation.

The paper also contributes to the growing branch of ergodicity economics (Peters and Gell-Mann, 2016; Berman, Peters and Adamou, 2017; Peters and Adamou, 2018). It proposes an alternative framework to expected utility theory or prospect theory, suggesting that agents maximize the growth of their resources averaged over time. This joins recent evidence of a strong effect of changes to the wealth dynamic on choices under uncertainty (Meder et al., 2019).

The paper is organized as follows. Section 2 lays out our model and the basic setup of the problem we are addressing. In Section 3 we present different specifications for the problem in question. We describe how a decision maker will discount payoffs in each specification under our model, giving rise to preference reversal. We conclude in Section 4.

1.1 Related literature

Observations of preference reversal have led to various explanations and theories. One such theory is hyperbolic discounting (Ainslie, 1992; Sozou, 1998; Laibson, 1997), suggesting that the valuation of choices falls hyperbolically in time. This is in contrast to the standard assumption of exponential discounting in economic theory, where no such reversal occurs. Hyperbolic discounting has been established as a plausible explanation for preference reversal. Yet, the dynamically inconsistent preferences it induces have challenged standard economic theory (Laibson, 1997; Starmer, 2000; Thaler, 2016).

Rubinstein (2003) suggested that the same experiments supporting hyperbolic discounting, can also be used to reject it under different axioms. In addition, various behavioral explanations for hyperbolic discounting have been given in the economics literature. All require some form of uncertainty in payoffs or time. One approach considers the information available to decision makers. Sozou (1998); Dasgupta and Maskin (2005) suggested that decision makers learn over time, which can lead to preference reversal. This approach implicitly assumes constructivist rationality similar to that of Smith (2003). In the most basic sense, the methodological approach is to posit the cognitive situation of the agent and to deduce a discounting rule.

Sozou (1998), for example, optimizes expected dollar payments (assuming risk-neutrality). The idea is that a future payment may be conditional on the survival of the payer. Modeling death as a Poisson process with unknown hazard rate, a decision maker learns about the rate over time through Bayesian updating. As he imagines payments in a more distant the future, the associated estimate of the hazard rate for such payments decreases. Under the assumption of an exponential prior distribution for the rate, this leads to hyperbolic discounting.

Dasgupta and Maskin (2005) also assume risk neutrality. They keep constant the hazard rate for death of the payer of Sozou (1998), and assume in addition that the payment may not only occur at a fixed time but with a time-dependent rate which is non-analytic with an atom at the fixed time of Sozou (1998).

It is important to note that the approaches to hyperbolic discounting in this literature involve some form of uncertainty and consider as the decision criterion the expectation value of a payout. It is now known that such decision criteria – expectation values of non-ergodic observables – necessarily lead to a-physical decision theories when individual decision makers are concerned. As a consequence, models in this vein that attempt to generate realistic behavior need to make inelegant ad-hoc assumptions, such as the non-analytic form of the payout rate.

As described, this paper discusses temporal discounting, and in particular hyperbolic discounting and preference reversal phenomena, without uncertainty or risk and – crucially – without a-physical expectation values. Our approach optimizes growth rates and is a generalisation of another strand of literature Holling (1959); Kacelnik (1997).

The work by Radner (1998) optimizes survival probabilities in models of wealth dynamics with an absorbing boundary, which is interpreted as bankruptcy or (economic) death. The author notes that the survival criterion leads to behavior which is different from expected-value maximization. Broadly speaking, growth rate optimization cautions against bankruptcy, although if bankruptcy is represented as an absorbing barrier, the mathematics will be extreme, and it may be difficult to define growth rates.

2 Model

We begin by defining a *Riskless Intertemporal Payoff Problem* (RIPP).

Definition 1 *Riskless Intertemporal Payoff Problem*

A Riskless Intertemporal Payoff Problem (RIPP) is a vector $\{t_0, x(t_0), t_a, \Delta x_a, t_b, \Delta x_b\}$ *– a decision maker at time* t_0 *with wealth* $x(t_0)$*, chooses between two future cash payoffs, one earlier than the other, whose amounts and payoff times are known with certainty. The two options are:*

a. an earlier payoff of Δx_a *at time* $t_a > t_0$*; and*

b. a later payoff of Δx_b *at time* $t_b > t_a$*.*

A formal decision theory has to provide a criterion for choosing a or b. Here we explore what happens if that criterion is growth rate maximization. A growth rate, g, is defined as the scale parameter of time for an underlying dynamic of wealth. Wealth dynamics can take different forms. Ignoring payoffs Δx_a and Δx_b, a standard assumption is that wealth grows exponentially in time at rate r. We label this dynamic as multiplicative. It corresponds to investing wealth in income-generating assets, where the income is proportional to the amount invested. In this case wealth follows

$$x(t) = x(0)\, e^{rt}\,, \tag{2.1}$$

and the scale parameter of time is $g = r$.

Another possibility is additive dynamics, where wealth grows linearly in time, at a rate k. It corresponds to something like labor income, namely to situations where investment income is negligible, and wealth instead changes by a net flow that does not depend on wealth itself. In this case wealth follows

$$x(t) = kt + x(0)\ , \tag{2.2}$$

and the scale parameter of time is $g = k$.

The functional form of the growth rate differs between the dynamics. The growth rate between time $t + \Delta t$ and t under multiplicative dynamics it is $r = \frac{\log x(t+\Delta t) - \log x(t)}{\Delta t}$ and under additive dynamics is $k = \frac{x(t+\Delta t) - x(t)}{\Delta t}$. This can be generalized to other possible wealth dynamics (Peters and Gell-Mann, 2016; Peters and Adamou, 2018).

Now, given a specific wealth dynamic, a RIPP implies two growth rates – g_a, associated with choice a; g_b, associated with choice b. This allows formulating a single axiom:

Axiom 1 *The Optimization of Growth*

Given a wealth dynamic, time t_0, an initial wealth $x(t_0)$, and tuples $(t_a, \Delta x_a)$ and $(t_b, \Delta x_b)$, such that the vector $\{t_0, x(t_0), t_a, \Delta x_a, t_b, \Delta x_b\}$ is a RIPP:

1. *$(t_a, \Delta x_a) \succ (t_b, \Delta x_b)$ if and only if $g_a > g_b$*
2. *$(t_a, \Delta x_a) \sim (t_b, \Delta x_b)$ if and only if $g_a = g_b$*
3. *$(t_a, \Delta x_a) \prec (t_b, \Delta x_b)$ if and only if $g_a < g_b$*

In words, Axiom 1 postulates that a decision maker will prefer choice a if her wealth grows faster under this choice than under choice b, and vice versa. Indifference only occurs if the growth rates are equal. Axiom 1 trivially satisfies the von Neumann-Morgenstern axioms – completeness is satisfied by design, while continuity and independence are irrelevant, since in this setup all the payoffs and times are certain. It also satisfies transitivity.

Proposition 1 *Optimization of Growth is Transitive*

Under the notation of Axiom 1, the Transitivity Axiom is satisfied.

2.1 Setup

Our setup is illustrated in Fig. 1. We note again that in this setup there is no uncertainty in the payoffs or in the times in which they are realized. Thus, there is no risk. In addition, we focus on cases in which $\Delta x_b > \Delta x_a$, since a larger and earlier payoff would usually be preferred, and our results would remain unchanged if $\Delta x_b < \Delta x_a$.

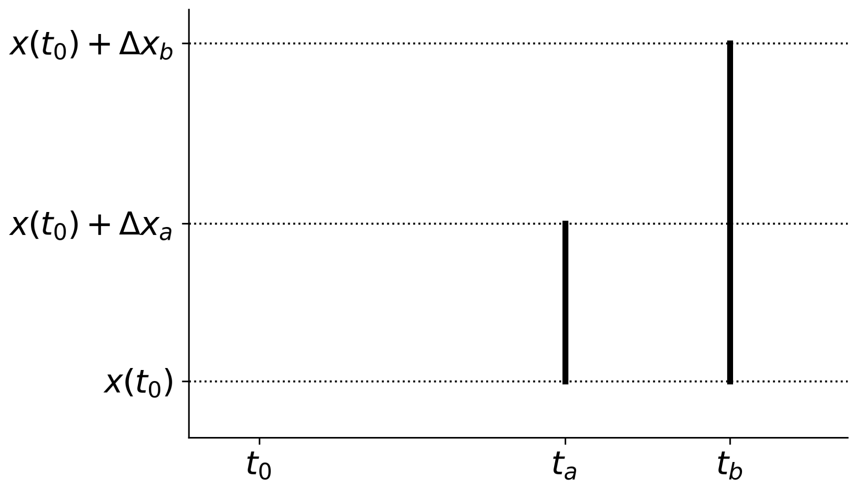

Figure 1: The basic setup of the model. A decision maker faces a choice at time t_0 between option a, which guarantees a payoff of Δx_a at time t_a, and option b, which guarantees a payoff of $\Delta x_b > \Delta x_a$ at time $t_b > t_a$.

This setup corresponds to a standard question that arises in the context of temporal discounting, *e.g.* "would you prefer to receive \$100 tomorrow or \$200 in a month's time?" Despite its apparent simplicity, answering this question requires additional assumptions. Or, put another way, the problem is underspecified. One extra assumption needed concerns the dynamics under which the decision-maker's wealth grows. Often it is assumed that wealth grows exponentially, compounding continuously at a constant riskless rate like funds in a savings account. Another assumption concerns the *time frame* of the decision – specifically whether a decision-maker accepting the earlier payoff at t_a is free immediately to make his next decision, or whether he must wait until the later time t_b (or, indeed, some other time) before the decision can be repeated. Such assumptions are needed to compute decision-maker's maximand – the growth rate of his wealth – so that the options can be compared quantitatively.

We will describe four different specifications of this basic setup. In each we will calculate the growth rates, g_a and g_b, of wealth associated with options a and b. The decision maker prefers the option whose growth rate is larger.

We will also infer the discount factor (DF) from this analysis. This is the multiplicative factor, δ, by which the later payoff, Δx_b, must be multiplied to equal the earlier payoff, Δx_a, when the payoff amounts and times are such that the decision maker is indifferent between the two options $((t_a, \Delta x_a) \sim (t_b, \Delta x_b))$. In symbols,

$$\delta \equiv \left. \frac{\Delta x_a}{\Delta x_b} \right|_{g_a = g_b}, \tag{2.3}$$

i.e. the ratio of payoffs under the constraint that the growth rates of wealth are equal.

As we show below, this setup predicts decisions equivalent to hyperbolic and exponential discounting under different specifications. Some specifications of the model predict preference reversal.

3 Results

3.1 Specification

We begin by describing four different specifications for our basic setup. Each specifies two aspects necessary to quantify the growth rate of wealth: the time frame of the decision; and the dynamics under which wealth evolves.

The time frame is a key aspect, often left unspecified in similar setups in the literature. Consider the following scenarios:

1. Every january Bob the bureacrat is given a budget. Bob must choose a project to fund with his allocated budget. All projects cost the entire allocated budget(there is no question of saving). He is paid upon the completion of each project.

2. Dana, the developer, loves to work and always wants to keep busy with her development projects, she always gets paid at their completion. Dana has a choice between a project that lasts three months and a project that lasts six months, she can only work on one project at time.

In the first scenario, the important element to note is that no matter which choice is made, it will not affect the timing of future choices. Said otherwise, the time frame is independent of the choice, so we say it is *fixed*.

In the second scenario, the time frame depends on the choice made. We call this the *elastic* time frame because Dana is more flexible to pursue other opportunities if she chooses the shorter project. On the other hand, if she chooses the longer project, it locks her in a for a longer time period, which means it also changes when she will have another choice.

In our model, we must choose the time period over which the growth rates of wealth in each option are computed. We can choose it to be the time period associated with each payoff, *i.e.* $t_a - t_0$ for option a and $t_b - t_0$ for option b. This specification corresponds to Dana's situation, the elastic time frame specification. Or we can choose it always to be the longer time period, $t_b - t_0$, resembling Bob's dilemma, the fixed time specification.

It should be noted that in the fixed time period there is also the question of overlapping projects. If the next choice occurs before the previous project has finished there is an implied multi-tasking capability that is not implied in the elastic time frame. If the frequency of the budget is given by t_b and the duration of projects is t_a, then the growth rate will fluctuate between $t_b g_a$ and $(t_b - 1)g_a$ If an agent can only undertake a fixed number of projects, then if the overlap of the projects is frequent enough to exceed his multi-tasking capability then the growth rate $t_b g_a$ is not achieveable.

As described in Section 2, the wealth dynamics can also take different forms, and we will address two specific common cases: additive and multiplicative wealth dynamics. We note that under the multiplicative dynamics it is assumed that the payoff itself is re-invested at the risk-free rate. For additive dynamics there is essentially no re-investment of the payoff – income in this dynamic is independent of wealth.

We will discuss the four specifications, as illustrated in Fig. 2. In each case we will: compute the growth rates g_a and g_b associated with each option; compare them to determine the conditions under which each option is preferred; elicit the form of temporal discounting equivalent to our decision model; and, finally, determine whether preference reversal is predicted.

3.2 Case A – Fixed time frame with additive dynamics

Specification: the period for computing the growth rate is that between the decision (t_0) and the later payoff (t_b); and the wealth dynamics are additive, with growth rate k.

We begin by writing down the final wealth under the two options, evaluated at t_b:

$$x_a(t_b) = x(t_0) + \Delta x_a + k(t_b - t_0)\,; \tag{3.1}$$

$$x_b(t_b) = x(t_0) + \Delta x_b + k(t_b - t_0)\,. \tag{3.2}$$

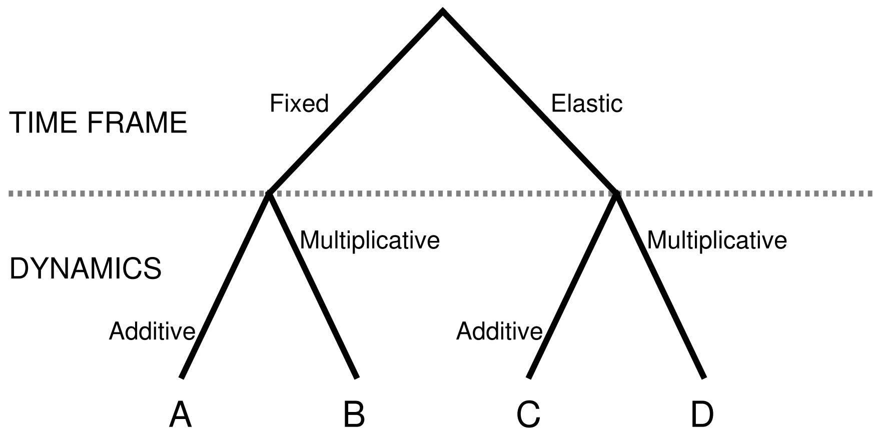

Figure 2: The four model specifications, determined by specifying a time frame and wealth dynamics. The labels A, B, C, and D, are used for the different cases.

The growth rates are:

$$g_a = \frac{x_a(t_b) - x(t_0)}{t_b - t_0} = \frac{\Delta x_a}{t_b - t_0} + k\,; \tag{3.3}$$

$$g_b = \frac{x_b(t_b) - x(t_0)}{t_b - t_0} = \frac{\Delta x_b}{t_b - t_0} + k\,. \tag{3.4}$$

We assume $\Delta x_b > \Delta x_a$, so option b is always preferred to option a. This is a trivial case – if we assume additive wealth dynamics and comparing the growth rates at the same time (or assuming repetition over fixed periods), then the only thing that matters to the decision-maker is payoff size. In this case, the discount factor δ cannot be defined, since the later, larger payoff is always preferred and the indifference condition is never satisfied.

3.3 Case B – Fixed time frame with multiplicative dynamics

Specification: the period for computing the growth rate is that between the decision (t_0) and the later payoff (t_b); and the wealth dynamics are multiplicative, with growth rate r.

This is the specification that corresponds to the standard assumptions usually considered in temporal discounting – that wealth is continuously compounding at the risk-free rate and that payoffs are re-invested at this rate.

We note that in this case the earlier payoff, Δx_a, if chosen, is treated as growing exponentially from t_a to t_b. The wealths evolve from t_0 to t_b as follows:

$$x_a(t_b) = x(t_0)\, e^{r(t_b-t_0)} + \Delta x_a e^{r(t_b-t_a)}\,; \tag{3.5}$$

$$x_b(t_b) = x(t_0)\, e^{r(t_b-t_0)} + \Delta x_b\,. \tag{3.6}$$

The corresponding growth rates are:

$$g_a = \tfrac{1}{t_a-t_0}\log\left(\tfrac{x_a(t_a)}{x(t_0)}\right) = \tfrac{1}{t_b-t_0}\log\left(1+\tfrac{\Delta x_a e^{r(t_b-t_a)}}{x(t_0)e^{r(t_b-t_0)}}\right)+r \tag{3.7}$$

$$g_b = \tfrac{1}{t_b-t_0}\log\left(\tfrac{x_b(t_b)}{x(t_0)}\right) = \tfrac{1}{t_b-t_0}\log\left(1+\tfrac{\Delta x_b}{x(t_0)e^{r(t_b-t_0)}}\right)+r\,. \tag{3.8}$$

The criterion $g_a > g_b$ is actually very simple, since only the second term in the logarithm is different and so only this must be compared. Thus, $g_a > g_b$ if

$$\Delta x_a e^{r(t_b-t_a)} > \Delta x_b\,, \tag{3.9}$$

or, in terms of the delay, if

$$\Delta x_a e^{rD} > \Delta x_b\,. \tag{3.10}$$

The discount factor is similarly expressed by setting the growth rates to be equal. Then we get $\Delta x_a e^{rD} = \Delta x_b$ and

$$\delta = \frac{\Delta x_a}{\Delta x_b} = e^{-rD}\,, \tag{3.11}$$

which is the standard exponential discounting result. The interpretation is straightforward: if it is possible to re-invest the earlier payoff such that, by the time of the later payoff, it will exceed the later payoff amount, then option a is preferable to option b (and *vice versa*). Note that, with this specification, the horizon is irrelevant. All that matters is the payoff amount after possible re-investment.

3.4 Case C – Elastic time frame with additive dynamics

Specification: the period for computing the growth rate is that between the decision (t_0) and the chosen payoff; and the wealth dynamics are additive, with growth rate k.

We can write down the final wealth under the two options, evaluated at t_a and t_b respectively:

$$x_a(t_a) = x(t_0) + \Delta x_a + k(t_a - t_0)\,; \tag{3.12}$$

$$x_b(t_b) = x(t_0) + \Delta x_b + k(t_b - t_0)\,. \tag{3.13}$$

The growth rates are:

$$g_a = \frac{x_a(t_a) - x(t_0)}{t_a - t_0} = \frac{\Delta x_a}{t_a - t_0} + k\,; \tag{3.14}$$

$$g_b = \frac{x_b(t_b) - x(t_0)}{t_b - t_0} = \frac{\Delta x_b}{t_b - t_0} + k\,. \tag{3.15}$$

It follows that the criterion $g_a > g_b$ is

$$\frac{\Delta x_a}{t_a - t_0} > \frac{\Delta x_b}{t_b - t_0}\,. \tag{3.16}$$

This criterion suggests that, under this specification, the only thing that matters to the decision maker is the linear payoff rate of each option.

If we treat the payoff amounts, Δx_a and Δx_b, and payoff times, t_a and t_b, as fixed parameters of the problem, then we can elicit the dependence of the decision on the decision time, t_0. When the payoffs are far ahead in the future, the ratio of the growth rates approaches $\lim_{t_0 \to -\infty} \frac{g_a}{g_b} = \frac{\Delta x_a}{\Delta x_b}$, and $g_a < g_b$ since we have assumed $\Delta x_a < \Delta x_b$. When the earlier payoff is imminent, *i.e.* as $t_0 \to t_a$, g_a grows without bound while g_b remains finite and so $g_a > g_b$. In other words, as time passes, our decision model under this specification predicts preference reversal from the later, larger payoff to the earlier, smaller payoff. This is illustrated in Fig. 3.

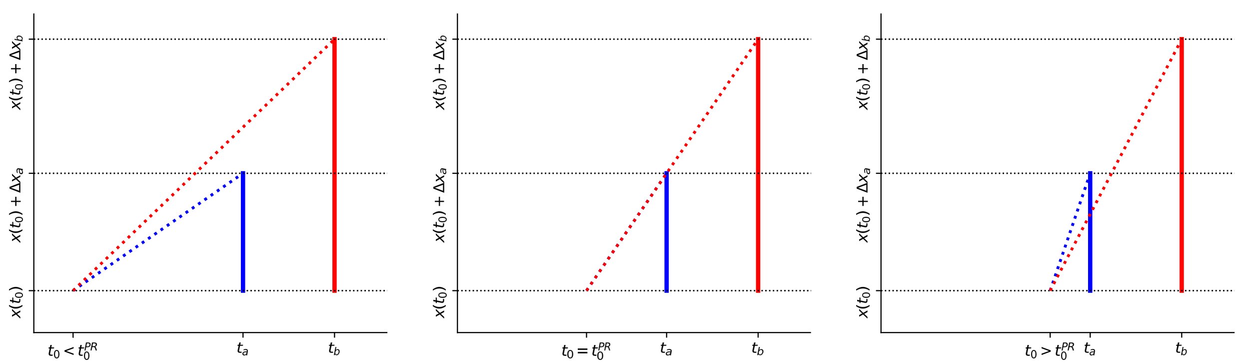

Figure 3: Preference reversal in case C. From left to right panel, t_0 increases, that is, the time of the payoffs approaches, while all other parameters are unchanged. Initially, option b is preferable, having the higher growth rate. At a later time $t_0 = t_0^{PR}$, given by Eq. (3.17), both options imply equal growth, and preference reversal occurs. At later times option a is preferable.

We can compute the decision time, t_0^{PR}, at which preference reversal occurs by setting $g_a = g_b$ to give

$$t_0^{\text{PR}} = \frac{\Delta x_b t_a - \Delta x_a t_b}{\Delta x_b - \Delta x_a}. \tag{3.17}$$

We can also find the effective discount factor under this specification. When $g_a = g_b$, we have

$$\delta = \frac{\Delta x_a}{\Delta x_b} = \frac{t_a - t_0}{t_b - t_0} = \frac{1}{1 + \frac{t_b - t_a}{t_a - t_0}}, \tag{3.18}$$

where we have made the final manipulation to express δ in hyperbolic form. We see that the discount factor depends on two time periods: that between decision and the earlier payoff, $t_a - t_0$, which we will call the *horizon*; and that between the two payoffs, $t_b - t_a$, which we which we will call the *delay*.[1] If we define $H \equiv t_a - t_0$ and $D \equiv t_b - t_a$, we can write the discount factor as

$$\delta = \frac{1}{1 + D/H}, \tag{3.19}$$

which is expressed in the conventional way as a hyperbolic function of the delay, D. The psychological degree of discounting parameter used in mainstream models is replaced here by $1/H$, the reciprocal of the horizon. As the horizon gets shorter, $1/H$ becomes larger, δ gets smaller, and the later payoff becomes less favorable. No knowledge of the decision-maker's psychology is required in this setup – only the postulate that she prefers her wealth to grow faster rather than slower.

Finally, we note that the background growth rate, k, of the decision-maker's wealth does not appear in the decision criterion. This is because wealth growth under additive dynamics is not affected by exogenous cash flows: the gain $k\Delta t$ over period Δt occurs regardless of other payoffs received. This contrasts with multiplicative dynamics, where payoffs can be subjected to the growth process through re-investment.

3.5 Case D – Elastic time frame with multiplicative dynamics

Specification: the time frame for computing the growth rate is time to the chosen payoff; and the wealth dynamics are multiplicative, with growth rate r.

We follow the same steps as in the previous cases. Wealth evolves to:

$$x_a(t_a) = x(t_0)\, e^{r(t_a - t_0)} + \Delta x_a\,; \tag{3.20}$$

$$x_b(t_b) = x(t_0)\, e^{r(t_b - t_0)} + \Delta x_b\,. \tag{3.21}$$

[1] Indeed, the problem is fully specified by these two time periods and the two payoff amounts. The actual times, t_0, t_a, t_b, are not needed to specify the problem because, when computing growth rates, only elapsed times matter. The time origin is arbitrary.

The corresponding growth rates are:

$$g_a = \tfrac{1}{t_a - t_0} \log\left(\tfrac{x_a(t_a)}{x(t_0)}\right) = \tfrac{1}{t_a - t_0} \log\left(1 + \tfrac{\Delta x_a}{x(t_0) e^{r(t_a - t_0)}}\right) + r \tag{3.22}$$

$$g_b = \tfrac{1}{t_b - t_0} \log\left(\tfrac{x_b(t_b)}{x(t_0)}\right) = \tfrac{1}{t_b - t_0} \log\left(1 + \tfrac{\Delta x_b}{x(t_0) e^{r(t_b - t_0)}}\right) + r\,. \tag{3.23}$$

This setting displays preference reversal: $g_a < g_b$ for t_0 sufficiently far away from t_a (long horizon); and $g_a > g_b$ for t_0 sufficiently close to t_a (short horizon). No closed-form expression for the reversal time, t_0^{PR}, is available.

Similarly, the discount factor δ cannot be derived explicitly. However, if we assume small payoffs relative to wealth, *i.e.* $\Delta x_a \ll x(t_0)\, e^{r(t_a - t_0)}$ and $\Delta x_b \ll x(t_0)\, e^{r(t_b - t_0)}$, then, setting $g_a = g_b$ and using the first-order approximation $\log(1+\epsilon) \approx \epsilon$ for $\epsilon \ll 1$, we get

$$\delta = \frac{\Delta x_a}{\Delta x_b} \approx \frac{(t_a - t_0) e^{r(t_a - t_0)}}{(t_b - t_0) e^{r(t_b - t_0)}} = \frac{e^{r(t_a - t_b)}}{1 + \frac{t_b - t_a}{t_a - t_0}}\,. \tag{3.24}$$

Using the previous definitions of H and D, we can write this as

$$\delta \approx \frac{e^{-rD}}{1 + D/H}\,, \tag{3.25}$$

which is a hybrid of hyperbolic and exponential discounting. We note again that only the elapsed times, H and D, appear in the discount factor. However, that the background wealth growth rate, r, no longer cancels out when dynamics are multiplicative, as does k when they are additive.

Case D also displays another type of preference reversal. Varying initial wealth $x(t_0)$ while keeping all other parameters fixed can lead to a switch from $g_a > g_b$ to $g_a < g_b$, as illustrated in Fig. 4.

This type of reversal is further illustrated in the following numerical example: Suppose an agent faces a choice between receiving \$10 after 1 year (choice a) or \$25 after two years (choice b), and we assume the agent has no access to a rate of interest. The model states that the agent will evaluate the growth rate of each choice. If the agent has initially \$100, she will evaluate that the growth rate in choice a is

$$g_a = \frac{1}{1} \log\left(\frac{100 + 10}{100}\right) \approx 0.095 \text{ per annum}\,. \tag{3.26}$$

Similarly, for choice b it is

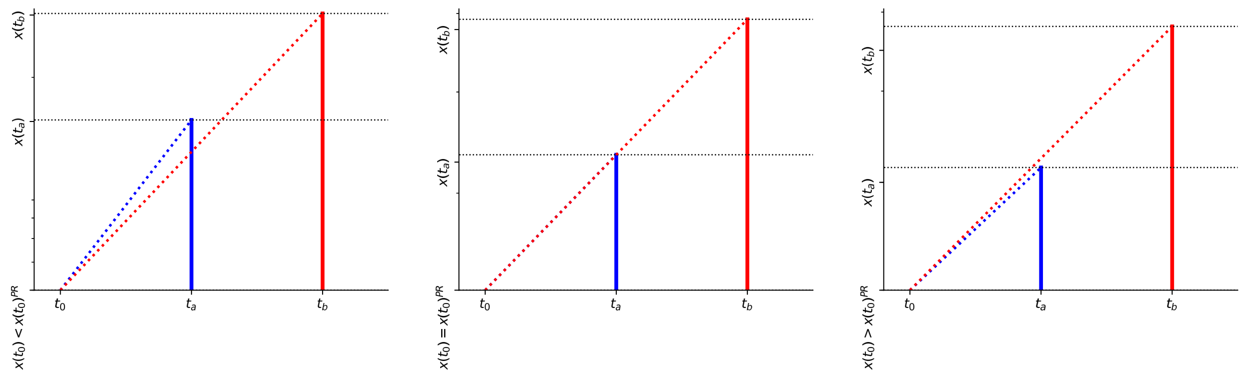

Figure 4: Preference reversal in response to wealth changes, in case D, logarithmic vertical scales. Initial wealth $x(t_0)$ increases from left to right panel (\$500, \$2277, \$5,000), while all other parameters are unchanged (t_0 = today, t_a = 1 year from today, t_b = 2 years from today, Δx_a = \$1000, Δx_b = \$2500, $r = 0.03$ per annum). At low wealth, option a is preferable, having the higher growth rate, according to Eq. (3.22) and Eq. (3.23). At a greater wealth, $x(t_0)^{PR} \approx \$2277$, both options imply equal growth, and preference reversal occurs. At even greater wealth, option b is preferable: the poor behave optimally by choosing the small early payoff.

$$g_b = \frac{1}{2}\log\left(\frac{100+25}{100}\right) \approx 0.112 \text{ per annum}\,. \tag{3.27}$$

Therefore, the agent would prefer the later option because $0.112 > 0.095$. Now suppose that the agent has initially only \$10, a similar calculation yields $g_a \approx 0.69$ and $g_b \approx 0.63$. This means that a poorer individual would prefer the earlier choice.

The difference $g_a - g_b$, from Eq. (3.22) and Eq. (3.23), is shown as a function of $x(t_0)$ in Fig. 5. This type of preference reversal can be expressed as follows: under certain circumstances, it is growth-optimal for people of lower wealth to choose a small early payoff, whereas it is growth-optimal for wealthier individuals to hold out until the later larger payoff. This predicts the findings of Epper et al. (2018), that "individuals with relatively low time discounting are consistently positioned higher in the wealth distribution".

4 Discussion

This paper describes a model in which a decision maker chooses between two certain payoffs realized at different, certain points in time. We assume that a decision is made by comparing the growth rate of wealth associated with each option. We then study temporal discounting in a deterministic setting. In this setting the model is consistent with the standard von

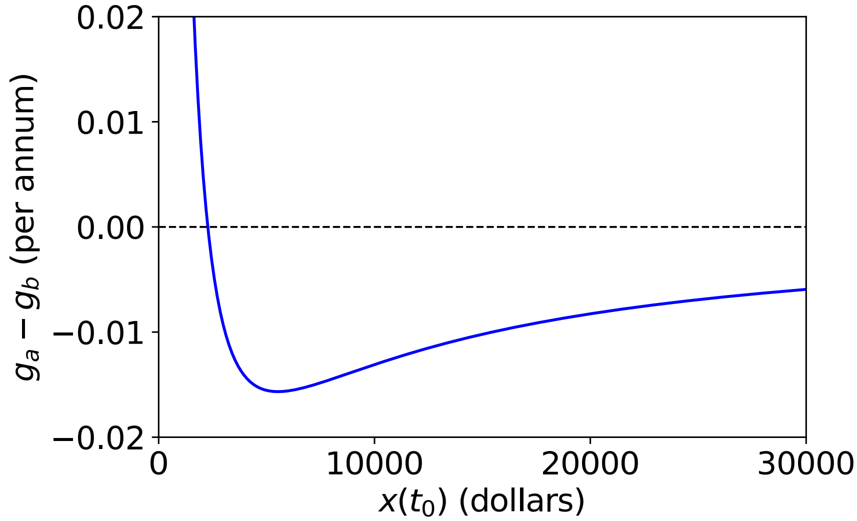

Figure 5: The difference $g_a - g_b$ is positive when the earlier payoff is preferable, and otherwise negative. We see that for small initial wealths $x(t_0)$ the earlier smaller payoff is preferred, whereas for large initial wealth the later larger payoff is preferred (parameters as in Fig. 4).

Neumann-Morgenstern axioms.

The main finding is that discounting can be interpreted as growth rate optimization. We find that depending on the wealth dynamics assumed by the decision maker, growth rate optimization can be equivalent to hyperbolic discounting. It then predicts preference reversal. It can also be equivalent to a mixed case of hyperbolic and exponential discounting, which also implies preference reversal. Under multiplicative dynamics, we find that growth rate optimization reproduces standard exponential discounting. This reveals the standard form of discounting as one of many possible forms of discounting, each of which is optimal under a different type of wealth growth.

The model predicts another type of reversal under multiplicative dynamics. It can be growth-optimal for people of lower wealth to choose a small early payoff, whereas it is growth-optimal for wealthier individuals to hold out until the later larger payoff. This is inline with the experimental and empirical findings of Epper et al. (2018), that "individuals with relatively low time discounting are consistently positioned higher in the wealth distribution".

We emphasize that no knowledge of the decision-maker's psychology is required in our setup – only the postulate that she prefers her wealth to grow faster rather than slower. This postulate is enough for to predicting a decision maker's discount rates used in mainstream models. Yet, we also note that this paper discusses discounting from a theoretical perspective. An important complementary step of this research would be comparing the theoretical predictions of the results to empirical and experimental results. In particular, the predicted discount factors and discount rates can be compared to results from controlled experiments. This is planned for future work.

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A Proofs

A.1 Proof of proposition 1

We assume three tuples $A \equiv (t_a, \Delta x_a)$, $B \equiv (t_b, \Delta x_b)$ and $C \equiv (t_c, \Delta x_c)$, where $t_a < t_b < t_c$. Given time t_0 $(< t_a)$ and an initial wealth $x(t_0)$, the vectors $\{t_0, x(t_0), t_a, \Delta x_a, t_b, \Delta x_b\}$ and $\{t_0, x(t_0), t_b, \Delta x_b, t_c, \Delta x_c\}$ are both RIPPs.

If $A \prec B$ and $B \prec C$ then $g_a < g_b$ and $g_b < g_c$. Also $t_a < t_b < t_c$. Therefore $\{t_0, x(t_0), t_a, \Delta x_a, t_c, \Delta x_c\}$ is a RIPP and $g_a < g_c$, so $A \prec C$.

$\blacksquare$