ECE 271A: Quiz #2

Due on October 30, 2023 at 11:59pm

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Part A

Using the training data in `TrainingSampleDCT_8.mat` compute the histogram estimate of the prior $P_Y(i), i \in \{cheetah, grass\}$. Using the results of problem 2 compute the maximum likelihood estimate for the prior probabilities. Compare the result with the estimates that you obtained last week. If they are the same, interpret what you did last week. If they are different, explain the differences.

Solution

Let n be the total sample size, C_j be the count of samples for class j, and $\pi_j = P_Y(j)$. To derive the ML estamator for $P_{C_1,\dots,C_N}(c_1,\dots,c_N) = \frac{n!}{\prod_{k=1}^N c_k!}\prod_{j=1}^N \pi_j^{c_j}$, given the constraint $\sum_j^N \pi_j = 1$, we first take the logarithm of $P_{C_1,\dots,C_N}(c_1,\dots,c_N)$ and get

$$\ln P_{C_1,\dots,C_N}(c_1,\dots,c_N) = \ln \frac{n!}{\prod_{k=1}^N c_k!} + \sum_{j=1}^N c_j \ln \pi_j.$$

Let $\theta = (\pi_1,\dots,\pi_N)^T$, and let $L(\theta,\lambda) = \ln P_{C_1,\dots,C_N}(c_1,\dots,c_N) + \lambda\left(\sum_j^N \pi_j - 1\right)$. Then,

$$\nabla_\theta L = \left(\frac{c_1}{\pi_1},\dots,\frac{c_N}{\pi_N}\right)^T + (\lambda,\dots,\lambda)^T = 0$$
$$\frac{\partial}{\partial\lambda} L = \sum_{j=1}^N \pi_j - 1 = 0.$$

For $1 \le j \le N$, we get $\frac{c_j}{\pi_j} + \lambda = 0$, and so $c_j + \pi_j\lambda = 0$. Then, $\sum_{j=1}^N (c_j + \pi_j\lambda) = n + \lambda = 0$, and thus $\lambda = -n$. We also take the Hessian of L. Notice that

$$\frac{\partial^2 L}{\partial\pi_i\partial\pi_j} = \begin{cases} -\frac{c_j}{\pi_j^2}, & i = j \\ 0, & i \ne j \end{cases},$$

and so $\nabla_\theta^2 L = \operatorname{diag}\left(-\frac{c_i}{\pi_i^2},\dots,-\frac{c_N}{\pi_N^2}\right)$. Since the Hessian of L is obviously negative definite, the critical point we are looking for is indeed a maximum point. Therefore, we obtain the ML estamator $\theta^* = \left(\frac{c_1}{n},\dots,\frac{c_N}{n}\right)$.

In the case of `TrainingSampleDCT_8.mat`, there are two classes in total, with 250 samples of class *cheetah* and 1053 samples of class *grass*. By our ML estamator, we get

$$P_Y(cheetah) = \frac{c_{cheetah}}{n} = \frac{250}{1053+250} \approx 0.192 \qquad P_Y(grass) = \frac{c_{grass}}{n} = \frac{1053}{1053+250} \approx 0.808.$$

This result surprisingly coincides with our estimation last week, implying that our intuitive approach of designating the ratio between the sample sizes of each class as the prior probability is in fact somewhat optimal.

Part B

Using the training data in `TrainingSampleDCT_8.mat`, compute the maximum likelihood estimates for the parameters of the class conditional densities $P_{X|Y}(x|cheetah)$ and $P_{X|Y}(x|grass)$ under the Gaussian assumption. Denoting by $X = \{X_1, \ldots, X_{64}\}$ the vector of DCT coefficients, create 64 plots with the marginal densities for the two classes - $P_{X_k|Y}(x_k|cheetah)$ and $P_{X_k|Y}(x_k|grass)$, $k = 1, \ldots, 64$ - on each. Use different line styles for each marginal. Select, by visual inspection, what you think are the best 8 features for classification purposes and what you think are the worst 8 features (you can use the subplot command to compare several plots at a time). Hand in the plots of the marginal densities for the best-8 and worst-8 features (once again you can use subplot, this should not require more than two sheets of paper). In each subplot indicate the feature that it refers to.

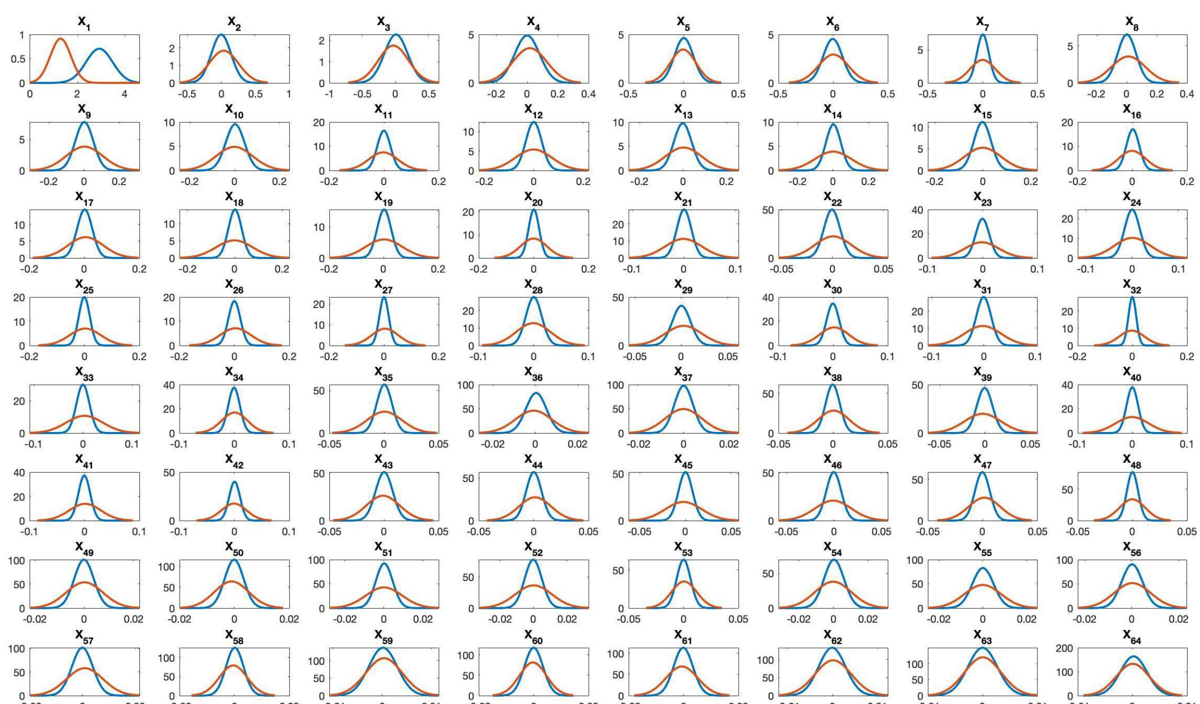
Solution

We know that the ML estamates for a multivariate Gaussian distribution of class i are

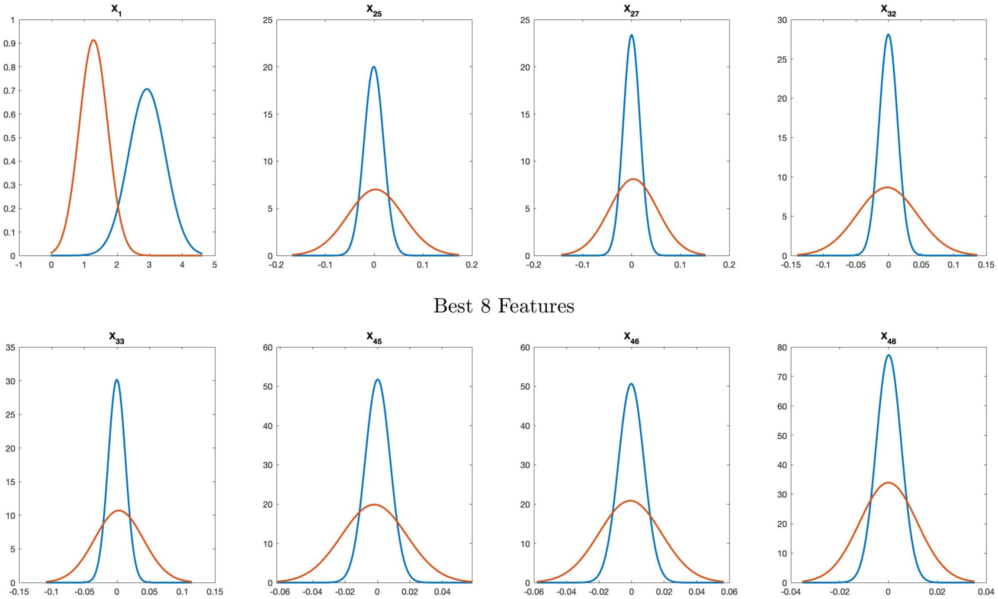
$$\mu_i = \frac{1}{n}\sum_j x_j^{(i)} \qquad\qquad \Sigma_i = \frac{1}{n}\sum_j (x_j^{(i)} - \mu_i)(x_j^{(i)} - \mu_i)^T.$$

Suppose that $X \sim N(\mu_i, \Sigma_i)$. By applying the projection matrix, we can transform the 64-dimension Gaussian distribution onto any desired dimension. Let $A_k = \frac{a_k a_k^T}{a_k^T a_k}$ be the projection matrix to the k-th dimension unit vector $a_k = (0\ldots1\ldots0)$. Then, we know $A_k x \sim N(A_k\mu_i, A_k\Sigma_i A_k^T)$, and $N(A_k\mu_i, A_k\Sigma_i A_k^T)$ is the projection of the 64-dimension Gaussian distribution onto the k-th unit vector. Notice that $A_k\mu_i$ simply extracts the k-th entry of μ_i, namely μ_{i_k}, which becomes our one dimension mean on the k-th dimention. Similarly, $A_k\Sigma_i A_k^T$ extracts the entry on the k-th row k-th column, namely $\Sigma_{i_{kk}}$, which becomes our variance on the k-th dimension. Thus, we obtain the marginal density $P_{X_k|Y}(x_k|i) \sim N(\mu_{i_k}, \Sigma_{i_{kk}})$.

The following are the plots for each of the 64 features for each class:

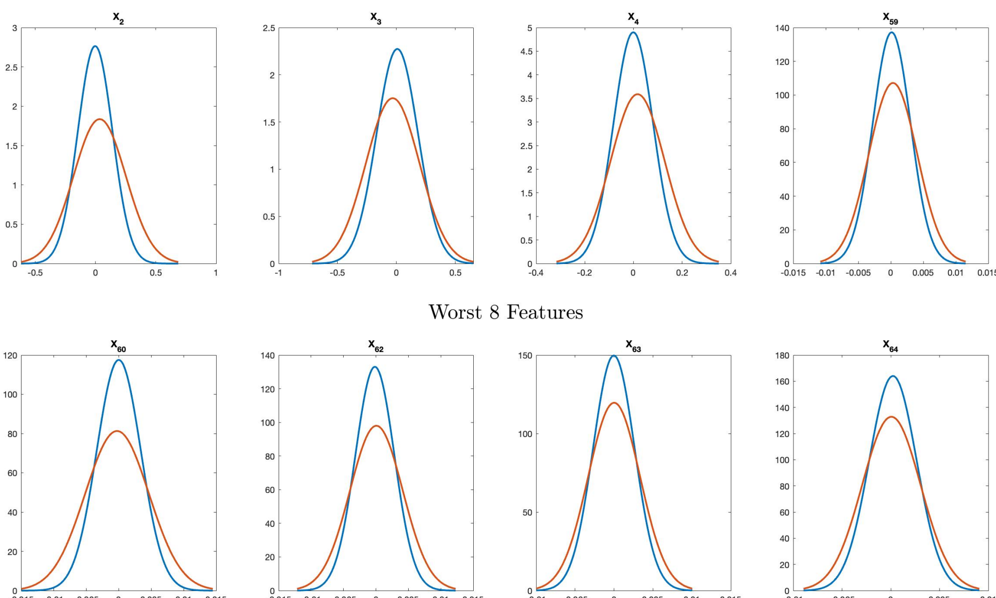


By visual inspection, we pick $\{1, 25, 27, 32, 33, 45, 46, 48\}$ to be the best 8 features because the two distributions appear to have less ambibuous decision ranges.



Best 8 Features

Conversely, we pick $\{2, 3, 4, 59, 60, 62, 63, 64\}$ to be the worst 8 features, as the two distributions are highly overlapped.



Worst 8 Features

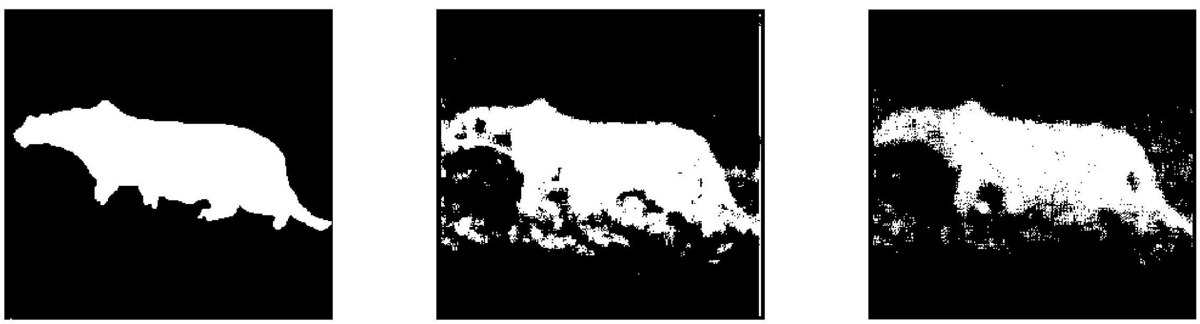
Part C

Compute the Bayesian decision rule and classify the locations of the cheetah image using (i) the 64-dimensional Gaussians, and (ii) the 8-dimensional Gaussians associated with the best 8 features. For the two cases, plot the classification masks and compute the probability of error by comparing with cheetah mask.bmp. Can you explain the results?

Solution

The following are the results of (i) and (ii).

Error Rate: 8.34% Error Rate: 4.29%



Truth All 64 Features Best 8 Features

Overall, (ii) exhibits a better performance comparing to (i). The reason might be that we have removed ambiguous features that have highly overlapping distributions between the two classes, e.g. the worst 8 features listed in part B, which could cause confusion to our decision. In other words, we have filtered out the features that possibly possess high bayes risks, and thus we obtain an enhanced result.

MATLAB Code

main.m

```
load('../dataset/TrainingSamplesDCT_8.mat');
zigzag = load('../dataset/Zig-Zag Pattern.txt');
cheetah = imread('../dataset/cheetah.bmp');
cheetah_mask = imread('../dataset/cheetah_mask.bmp');
target = im2double(cheetah);
mask = im2double(cheetah_mask);

training_BG = TrainsampleDCT_BG;
training_FG = TrainsampleDCT_FG;

zigzag = zigzag + 1;

[row_BG, col_BG] = size(training_BG);
[row_FG, col_FG] = size(training_FG);
[row_TG, col_TG] = size(target);

prior_BG = row_BG / (row_BG + row_FG);
prior_FG = row_FG / (row_BG + row_FG);

% pick cheetah if (p(x | grass) / p(x | cheetah)) < threshold
threshold = prior_FG / prior_BG;

mean_FG = zeros(1, 64);
mean_BG = zeros(1, 64);
cov_FG = zeros(64, 64);
cov_BG = zeros(64, 64);

for r = 1:row_FG
    cov_FG = cov_FG + training_FG(r,:)' * training_FG(r,:);
    mean_FG = mean_FG + training_FG(r,:);
end

for r = 1:row_BG
    cov_BG = cov_BG + training_BG(r,:)' * training_BG(r,:);
    mean_BG = mean_BG + training_BG(r,:);
end

mean_FG = mean_FG / row_FG;
mean_BG = mean_BG / row_BG;
cov_FG = (cov_FG / row_FG) - mean_FG' * mean_FG;
cov_BG = (cov_BG / row_BG) - mean_BG' * mean_BG;

figure;

for k = 1:64
    subplot(8, 8, k);
```

```
    x = linspace(min(mean_FG(k) - 3 * sqrt(cov_FG(k, k)), ...
        mean_BG(k) - 3 * sqrt(cov_BG(k, k))), ...
        max(mean_FG(k) + 3 * sqrt(cov_FG(k, k)), ...
        mean_BG(k) + 3 * sqrt(cov_BG(k, k))), 1000);
    p_BG = (1/(sqrt(2*pi * cov_BG(k, k)))) * exp(-0.5 * ...
        ((x - mean_BG(1,k)).^2/cov_BG(k, k)));
    p_FG = (1/(sqrt(2*pi * cov_FG(k, k)))) * exp(-0.5 * ...
        ((x - mean_FG(1,k)).^2/cov_FG(k, k)));
    plot(x, p_BG, 'LineWidth', 2);
    hold on;
    plot(x, p_FG, 'LineWidth', 2);
    title("X_{" + k + "}");
end

figure;
for i = 1:8
    subplot(2, 4, i);
    k = best8(i);
    x = linspace(min(mean_FG(k) - 3 * sqrt(cov_FG(k, k)), ...
        mean_BG(k) - 3 * sqrt(cov_BG(k, k))), ...
        max(mean_FG(k) + 3 * sqrt(cov_FG(k, k)), ...
        mean_BG(k) + 3 * sqrt(cov_BG(k, k))), 1000);
    p_BG = (1/(sqrt(2*pi * cov_BG(k, k)))) * exp(-0.5 * ...
        ((x - mean_BG(1,k)).^2/cov_BG(k, k)));
    p_FG = (1/(sqrt(2*pi * cov_FG(k, k)))) * exp(-0.5 * ...
        ((x - mean_FG(1,k)).^2/cov_FG(k, k)));
    plot(x, p_BG, 'LineWidth', 2);
    hold on;
    plot(x, p_FG, 'LineWidth', 2);
    title("X_{" + k + "}");
end

figure;
for i = 1:8
    subplot(2, 4, i);
    k = worst8(i);
    x = linspace(min(mean_FG(k) - 3 * sqrt(cov_FG(k, k)), ...
        mean_BG(k) - 3 * sqrt(cov_BG(k, k))), ...
        max(mean_FG(k) + 3 * sqrt(cov_FG(k, k)), ...
        mean_BG(k) + 3 * sqrt(cov_BG(k, k))), 1000);
    p_BG = (1/(sqrt(2*pi * cov_BG(k, k)))) * exp(-0.5 * ...
        ((x - mean_BG(1,k)).^2/cov_BG(k, k)));
    p_FG = (1/(sqrt(2*pi * cov_FG(k, k)))) * exp(-0.5 * ...
        ((x - mean_FG(1,k)).^2/cov_FG(k, k)));
    plot(x, p_BG, 'LineWidth', 2);
    hold on;
    plot(x, p_FG, 'LineWidth', 2);
    title("X_{" + k + "}");
end
```

```
best8 = [1 25 27 32 33 45 46 48];
worst8 = [2 3 4 59 60 62 63 64];

E = zeros(8, 64);

for i = 1:8
    E(i, best8(i)) = 1;
end

mean8_FG = zeros(1, 8);
mean8_BG = zeros(1, 8);
cov8_FG = zeros(8, 8);
cov8_BG = zeros(8, 8);

for r = 1:row_FG
    v = E * training_FG(r,:)';
    cov8_FG = cov8_FG + v * v';
    mean8_FG = mean8_FG + v';
end

for r = 1:row_BG
    v = E * training_BG(r,:)';
    cov8_BG = cov8_BG + v * v';
    mean8_BG = mean8_BG + v';
end

mean8_FG = mean8_FG / row_FG;
mean8_BG = mean8_BG / row_BG;
cov8_FG = (cov8_FG / row_FG) - mean8_FG' * mean8_FG;
cov8_BG = (cov8_BG / row_BG) - mean8_BG' * mean8_BG;

A_64 = zeros(row_TG, col_TG);
A_8 = zeros(row_TG, col_TG);

for r = 5:row_TG-3
    for c = 5:col_TG-3
        block = target(r - 4:r + 3, c - 4:c + 3);
        dctBlock = dct2(block);
        X = zeros(1, 64);
        for i = 1:8
            for j = 1:8
                X(zigzag(i, j)) = dctBlock(i, j);
            end
        end
        A_64(r, c) = int8(mvn(X, mean_BG, cov_BG)/ ...
            mvn(X, mean_FG, cov_FG) <= threshold);
        A_8(r, c) = int8(mvn(X * E', mean8_BG, cov8_BG)/ ...
            mvn(X * E', mean8_FG, cov8_FG) <= threshold);
```

```
    end
end

figure;

subplot(1, 3, 1);
imagesc(mask);
axis off
colormap(gray(255));
axis equal tight;

subplot(1, 3, 2);
imagesc(A_64);
axis off
colormap(gray(255));
axis equal tight;

subplot(1, 3, 3);
imagesc(A_8);
axis off
colormap(gray(255));
axis equal tight;

error64 = 0;
error8 = 0;
for r = 1:row_TG
    for c = 1:col_TG
        if (A_64(r, c) ~= mask(r, c))
            error64 = error64 + 1;
        end

        if (A_8(r, c) ~= mask(r, c))
            error8 = error8 + 1;
        end
    end
end

error_rate64 = error64 / (row_TG * col_TG);
error_rate8 = error8 / (row_TG * col_TG);
disp(error_rate64);
disp(error_rate8);
```

mvn.m

```
function result = mvn(x, mean, cov)
  [~, dim] = size(mean);
  d = (x - mean) * inv(cov) * (x - mean)';
  c = 1/sqrt((2 * pi)^dim * det(cov));
  result = c * exp(-0.5 * d);
end
```