Double Turán Problem

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Contents

1 Introduction

TODO: Add introduction and motivation.

1.1 Definitions and Notation

Denote the set of first n positive integers as $[n] = \{1, 2, \ldots, n\}$. Given a set X, we denote 2^X as the power set of X.

Let $G = (V, E)$ be a graph. Let $V(G) = V$ denote the vertex set and $E(G) = E$ denote the edge set of G. We note by $v(G) = |V|$ the number of vertices and $e(G) = |E|$ the number of edges in G. For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v.

Given graphs $G_1, \ldots, G_m$ on some vertex set V, we denote $G_{i_1,\ldots,i_k}$ as graph on V with edge set $E(G_{i_1,\ldots,i_k}) = \bigcap_{\alpha=1}^{k} E(G_{i_\alpha})$. Given two graphs G_1, G_2, we denote $G_1 \cup G_2$ as the graph on $V(G_1) \cup V(G_2)$ with edge set $E(G_1 \cap G_2) = E(G_1) \cup E(G_2)$. Let s

In this thesis, we reserve n to denote the number of vertices in a graph. We call a n-vertex complete graph K_n, and a complete bipartite graph $K_{a,b}$, where a, b are the size of its parts. We denote P_n as a path with n edges, and C_n as a cycle with n edges. Given graph G, H, define $G + H$ as the graph fully connecting G, H, i.e. $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$.

Given graphs G and F, we say that G is F-free if G does not contain F as a subgraph. We denote $\mathrm{ex}(n, F)$ to be the maximum possible number of edges an F-free graph on n vertices, and we call a F-free graph achieving this maximum an extremal graph for F. Given graphs $G_1, \ldots, G_m$ on the same set of vertices and F, we say that $G_1, \ldots, G_m$ are pairwise F-free if $E(G_i) \cap E(G_j)$ does not contain F for $i \neq j$.

1.2 Problem Statement

Let $\mathrm{ex}_2(n, m, F)$ be the maximum possible number of edges that m pairwise F-free graphs on n vertices can have. Our goal is to determine $\mathrm{ex}_2(n, m, F)$ for different forbidden graphs F. A trivial construction with $G_1 = K_n$ and $G_2, \ldots, G_m$ to be extremal graphs for F yields the lower bound $\mathrm{ex}_2(n, m, F) \geq \binom{n}{2} + (m - 1)\mathrm{ex}(n, F)$. In this work, we use this bound as a benchmark to either show its tightness or to improve it.

Additionally, we are also interested in a more restrictive version where $G_1, \ldots, G_m$ are induced subgraphs of $G_1 \cup \cdots \cup G_m$. Let $\mathrm{ex}_2^*(n, m, F)$ as the maximum possible number of edges that m pairwise F-free graphs on n vertices can have, with the constraint that each graph is an induced subgraph of their union. A trivial construction with $G_1 = \cdots = G_m$ to be extremal graphs for F yields the lower bound $\mathrm{ex}_2^*(n, m, F) \geq m \cdot \mathrm{ex}(n, F)$. This is the benchmark we use to determine $\mathrm{ex}_2^*(n, m, F)$. Similar to the non-induced case, we will use this bound as a benchmark and base our work on it.

2 Induced Version

In this section, we investigate the case where $G_1, \dots, G_m$ are induced subgraphs of $G_1 \cup \dots \cup G_m$ and are pairwise F-free, for some specified F. Unless otherwise specified, when we say $G_1, \dots, G_m$ are induced subgraph, we mean that they are induced subgraphs of $G_1 \cup \dots \cup G_m$.

The following lemma shows that the problem can be reduceed to only two graphs.

Lemma 2.1. *Let $n, m, k \geq 1$ such that $2 \leq k \leq m$, F be some graph, and $G_1, \dots, G_m$ be pairwise F-free induced subgraphs on n vertices. Then*

$$\mathrm{ex}_2^*(n, m, F) \leq \frac{m}{k} \cdot \mathrm{ex}_2^*(n, k, F).$$

Moreover, if $\sum_{i=1}^k e(G_i) = \mathrm{ex}_2^*(n, k, F)$ only if $G_1 = \dots = G_k$, then $\sum_{i=1}^m e(G_i) = \mathrm{ex}_2^*(n, m, F)$ only if $G_1 = \dots = G_m$ and $\mathrm{ex}_2^*(n, m, F) = \frac{m}{k} \cdot \mathrm{ex}_2^*(n, k, F)$.

Not putting equality because I'm unsure if a construction for k subgraphs can always generalize to m subgraphs. For example, if $F = K_3$ and n is odd, the $G_1 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and $G_2 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$ construction cannot be generalized to $m = n+1$ subgraphs.

Proof. Let $G_1, \dots, G_m$ be induced subgraphs of $G_1 \cup \dots \cup G_m$ with $E(G_i) \cap E(G_j)$ not containing F for $i \neq j$. Put $G_{i+m} = G_i$ for all $i \in [m]$. Then

$$\sum_{i=1}^m e(G_i) = \frac{1}{k} \sum_{i=1}^m [e(G_i) + \dots + e(G_{i+k-1})] \leq \frac{1}{k} \sum_{i=1}^m \mathrm{ex}_2^*(n, k, F) = \frac{m}{k} \cdot \mathrm{ex}_2^*(n, k, F),$$

which establishes the upper bound.

Suppose $\sum_{i=1}^k e(G_i) = \mathrm{ex}_2^*(n, k, F)$. By assumption $G_1 = \dots = G_k$, so $e(G_i) = \mathrm{ex}_2^*(n, k, F)/k$ for $1 \leq i \leq k$. Hence, the construction $G_1 = \dots = G_m$ meets the upperbound. On the other hand, if $G_1 \neq G_2$ then $\sum_{i=1}^k e(G_i) < \mathrm{ex}_2^*(n, k, F)$. Since $\sum_{i=1}^k e(G_{i+j}) \leq \mathrm{ex}_2^*(n, k, F)$ for all $j \geq 1$, we have $\sum_{i=1}^m e(G_i) < \frac{m}{k} \cdot \mathrm{ex}_2^*(n, k, F)$. Thus the extremal condition is met only when $G_1 = \dots = G_m$. $\square$

Lemma 2.1 allows us to reduce the problem to the case for two subgraphs G_1, G_2. Let $C = V(G_1) \cap V(G_2)$, $c = |C|$, $d = |V(G_1) \backslash A|$, and $n - c - d = |V(G_2) \backslash C|$. Note that $c, d \in \mathbb{Z}_{\geq 0}$. Since G_1, G_2 are induced subgraphs of $G_1 \cup G_2$, $G_1[C] = G_2[C] = G_{1,2}$. But then $G_{1,2}$ is F-free, so $e(G_1[C]) = e(G_2[C]) \leq \mathrm{ex}(c, F)$. Thus, given c, d, the optimal construction to maximize the number of edges over G_1, G_2 is to put $G_{1,2}$ as an extremal graph for F on c vertices and connect all edges that are not induced in A. This yields the inequality

$$e(G_1) + e(G_2) \leq \binom{d}{2} + \binom{n-c-d}{2} + (n-c)c + 2\mathrm{ex}(c, F).$$

But then notice that $\binom{n-c}{2} > \binom{d}{2} + \binom{n-c-d}{2}$ for $0 < d < n-c$. This implies our construction is optimized when $d = 0$ or $d = n - c$, that is, to let G_2 contain G_1 or the other way around. Hence, we may assume $d = 0$ and define the construction function as

$$\mathcal{C}(n, c, F) := \binom{n-c}{2} + (n-c)c + 2\mathrm{ex}(c, F),$$

i.e. the number of edges over two induced graphs in the above construction. Since $e(G_1) + e(G_2) \leq \mathcal{C}(n, c, F)$, we have the following

Lemma 2.2. *Let F be some graph. For $n \geq 1$,*

$$\mathrm{ex}_2^*(n, 2, F) = \max_{0 \leq c \leq n} \mathcal{C}(n, c, F).$$

Moreover, let G_1, G_2 be induced pairwise F-free subgraphs and c_{max} be some maximizer of $\mathcal{C}(n, c, F)$. Then $e(G_1) + e(G_2) = \mathrm{ex}_2^*(n, 2, F)$ only if G_1, G_2 are the construction described by $\mathcal{C}(n, c_{max}, F)$.

Thus the problem is reduced to maximizing $\mathcal{C}$ over c. In particular, $\mathcal{C}(n, n, F)$ gives our benchmark construction of $G_1 = G_2$ being the extremal graphs for F on n vertices. For $0 \leq k \leq c \leq n$, define

$$\Delta_k\mathcal{C}(n, c, F) := \mathcal{C}(n, c, F) - \mathcal{C}(n, c-k, F) = \frac{1}{2}k(k - 2c + 1) + 2[\mathrm{ex}(c, F) - \mathrm{ex}(c-k, F)]$$

and denote $\Delta\mathcal{C} = \Delta_1\mathcal{C}$. Most of the work in this section will show that the maximum of $\mathcal{C}$ happens when $c \geq n - k$ by proving that $\Delta_k\mathcal{C}(n, c, F) > 0$ for all $c \leq n - k$.

Lemma 2.3. *Let $n, c_0 \geq 1$, $m \geq 2$, and F be some graph. If $\mathcal{C}(n, c, F) < 2 \cdot \mathrm{ex}(n, F)$ for $0 \leq c < c_0$ and $\mathrm{ex}(c, F) - \mathrm{ex}(c-1, F) > \frac{c-1}{2}$ for $c_0 \leq c \leq n$, then*

$$\mathrm{ex}_2^*(n, m, F) = m \cdot \mathrm{ex}(n, F)$$

and the extremal condition is met if and only if all m induced pairwise F-free subgraphs are equal and extremal graphs for F.

This should be if and only if and I will strengthen it shortly.

Proof. By Lemma 2.1 and Lemma 2.2, it suffices to show $\mathcal{C}(n, c, F)$ has a unique maximum of $2\mathrm{ex}(n, F)$ at $c = n$. We may assume $c \geq c_0$ by assumption. Suppose $c < n$. Since $\mathrm{ex}(c, F) - \mathrm{ex}(c-1, F) > \frac{c-1}{2}$,

$$\Delta\mathcal{C}(n, c, F) = -c + 1 + 2[\mathrm{ex}(c, F) - \mathrm{ex}(c-1, F)] > 0.$$

Thus, $\mathcal{C}$ is strictly increasing with respect to c for $c \geq c_0$, so $\mathcal{C}$ has a unique maximum of $2 \cdot \mathrm{ex}(n, F)$ at $c = n$, which yields the unique extremal construction of $G_1 = G_2$ being extremal graphs for F on n vertices. $\square$

2.1 Complete Graph F

Lemma 2.4. *For $n \geq 1$ and $r \geq 2$,*

$$\mathrm{ex}(n, K_{r+1}) - \mathrm{ex}(n-1, K_{r+1}) \geq \frac{n-1}{2},$$

with equality if and only if n is odd and $r = 2$.

Proof. By Turán's Theorem,

$$\mathrm{ex}(n, K_{r+1}) - \mathrm{ex}(n-1, K_{r+1}) = \delta(T_r(n)) = n - \left\lceil \frac{n}{r} \right\rceil \geq n - \left\lceil \frac{n}{2} \right\rceil.$$

The result now follows. □

The following theorem for complete graphs with more than 3 vertices now follows directly from Lemma 2.3 and Lemma 2.4:

Theorem 2.5. *Let $n \geq 1$, $m \geq 2$, and $r \geq 3$, and let $G_1, \dots, G_m$ be pairwise K_{r+1}-free induced subgraphs on n vertices. Then*

$$\mathrm{ex}_2^*(n, m, K_{r+1}) = m \cdot e(T_r(n)),$$

and $\sum_i e(G_i) = \mathrm{ex}_2^*(n, m, K_{r+1})$ if and only if $G_1 = \cdots = G_m = T_r(n)$.

Surprisingly, the triangle case is more complicated than the case for larger complete graphs. As shown in Lemma 2.4, the condition given by Lemma 2.3 is not satisfied for all n in the triangle case, and there are indeed constructions of induced subgraphs G_1, G_2 that meet the extremal condition but are neither equal nor both complete bipartite graphs. For odd n, consider $G_1 = K_{\frac{n-1}{2}, \frac{n-1}{2}}$ and $G_2 = K_{\frac{n-1}{2}, \frac{n-1}{2}} + K_1$. The number of edges over G_1, G_2 is $\frac{(n-1)^2}{2} + n - 1 = \frac{n^2-1}{2} = 2\left\lfloor \frac{n^2}{4} \right\rfloor$, which meets the benchmark construction of two complete bipartite graphs. We will show that this is the only deviant construction for the triangle case.

Theorem 2.6. *Let $n \geq 1$, $m \geq 2$, and let $G_1, \dots, G_m$ be pairwise K_3-free induced subgraphs on n vertices. Then*

$$\mathrm{ex}_2^*(n, m, K_3) = m \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Moreover, $\sum_i e(G_i) = \mathrm{ex}_2^*(n, m, K_3)$ if and only if $G_1 = \cdots = G_m = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, unless n is odd and $m = 2$, in which case $e(G_1) + e(G_2) = \mathrm{ex}_2^*(n, 2, K_3)$ if and only if either $G_1 = G_2 = K_{\frac{n+1}{2}, \frac{n-1}{2}}$ or $G_1 = K_{\frac{n-1}{2}, \frac{n-1}{2}}$ and $G_2 = G_1 + K_1$.

Proof. We first show the following claim.

Claim 2.6.1. $\mathrm{ex}_2^*(n, 2, K_3) = 2\left\lfloor \frac{n^2}{4} \right\rfloor$*, and the extremal condition is met only when $G_1 = G_2 = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, unless n is odd, $G_1 = K_{\frac{n-1}{2}, \frac{n-1}{2}}$, and $G_2 = K_{\frac{n-1}{2}, \frac{n-1}{2}} + K_1$.*

Proof. Consider $\Delta_2\mathcal{C}(n,c,K_3)$. Since

$$\Delta_2\mathcal{C}(n,c,K_3) = -2c+3+2\left[\left\lfloor\frac{c^2}{4}\right\rfloor - \left\lfloor\frac{(c-2)^2}{4}\right\rfloor\right] = -2c+3+2(c-1) = 1 > 0,$$

$\mathcal{C}(n,c,K_3)$ has a maximum of $2\left\lfloor\frac{n^2}{4}\right\rfloor$ when $c \geq n-1$, so $\mathrm{ex}_2^*(n,2,K_3) = 2\left\lfloor\frac{n^2}{4}\right\rfloor$ by Lemma 2.2. We are done if $c = n$, so assume that $c = n-1$. Then in the extremal condition, $G_1 = G_{1,2} = K_{\lfloor\frac{n-1}{2}\rfloor,\lceil\frac{n-1}{2}\rceil}$ and

$$e(G_1) + e(G_2) = 2\left\lfloor\frac{(n-1)^2}{4}\right\rfloor + \deg(v),$$

where v is the only vertex not in $G_{1,2}$. But then to meet the extremal condition,

$$\deg(v) = 2\left\lfloor\frac{n^2}{4}\right\rfloor - 2\left\lfloor\frac{(n-1)^2}{4}\right\rfloor = \begin{cases} n & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.}\end{cases}$$

Hence, n must be odd and G_2 must be a copy of G_1 with all vertices adjacent to the only remaining vertex, i.e. $G_2 = G_1 + K_1$. □

By Lemma 2.1 and the above claim, it remains to show that for odd n and $m = 3$, $G_1 = \cdots = G_3 = K_{\frac{n+1}{2},\frac{n-1}{2}}$ if the extremal condition is met. Suppose not. The above claim then guarantees one of the subgraphs, say G_1, is $K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor} + K_1$. But then by the claim $G_2 = G_3 = G_1 + K_1$, which contradicts that G_2, G_3 are pairwise K_3-free. This completes the proof. □

Combining Theorem 2.5 and Theorem 2.6, we have the following theorem for complete graphs:

Theorem 2.7. *For $n \geq 1$ and $m, r \geq 2$, let $G_1, \ldots, G_m$ be pairwise K_{r+1}-free induced subgraphs on n vertices. Then*

$$\mathrm{ex}_2^*(n,m,K_{r+1}) = m \cdot e(T_r(n)).$$

Moreover, $\sum_i e(G_i) = \mathrm{ex}_2^*(n,m,K_{r+1})$ if and only if $G_1 = \cdots = G_m = T_r(n)$, unless $r = 2$, n is odd, and $m = 2$, in which case the $e(G_1) + e(G_2) = \mathrm{ex}_2^*(n,2,K_3)$ if and only if either $G_1 = G_2 = K_{\frac{n+1}{2},\frac{n-1}{2}}$ or $G_1 = K_{\frac{n-1}{2},\frac{n-1}{2}}$ and $G_2 = G_1 + K_1$.

Since n cannot be both even and odd, we also have the following corollary:

Corollary 2.8. *For $n \geq 2$, let $G_1, \ldots, G_n$ be pairwise K_{r+1}-free induced subgraphs on n vertices. Then*

$$\mathrm{ex}_2^*(n,n,K_{r+1}) = n \cdot e(T_r(n)).$$

and $\sum_i e(G_i) = \mathrm{ex}_2^*(n,n,K_3)$ if and only if $G_1 = \cdots = G_n = T_r(n)$.

2.2 Non-bipartite F

For non-bipartite F, it is hard to determine the extremal graphs for F in general, but their structures becomes more apparent when n is large.

More specifically, the Erdős-Stone Theorem tells us that for large n, the extremal graph for F mimics the structure of the Turán graph. With this idea in mind, the following theorem is a generalization of Theorem 2.7 for large n.

Theorem 2.9. *Let $n \geq 1$, $m, r \geq 2$, F be a $(r+1)$-colorable graph, and $G_1, \dots, G_m$ be pairwise F-free induced subgraphs on n vertices. Then for large enough n,*

$$\mathrm{ex}_2^*(n, m, F) = m \cdot \mathrm{ex}(n, F),$$

and $\sum_i e(G_i) = \mathrm{ex}_2^*(n, m, F)$ if and only if $G_1 = G_2 = \cdots = G_n$ are n-vertex extremal graphs for F, unless $r = 2$, n is odd, and $m = 2$, in which case $e(G_1)+e(G_2) = \mathrm{ex}_2^*(n, 2, F)$ if and only if when either $G_1 = G_2$ are n-vertex extremal graph for F, or G_1 is the $(n-1)$-vertex extremal graph for F and $G_2 = G_1 + K_1$.

This proof only works for $r \geq 3$. Ignore the case $r = 2$ for now.

Proof. It suffices to show for $m = 2$ by Lemma 2.1. We first show that $\mathcal{C}(n, c, F)$ fails to meet the desired bound for small c.

Claim 2.9.1. *If $c \leq \frac{n}{2}$, then $\mathcal{C}(n, c, F) < 2\mathrm{ex}(n, F)$.*

Proof. Write $c = kn$ for some $k \in [0, 1/2]$. Since

$$\mathcal{C}(n, kn, F) = \binom{(1-k)n}{2} + k(1-k)n^2 + 2\mathrm{ex}(kn, F),$$

it suffices to show that

$$\mathrm{ex}(n, F) - \mathrm{ex}(kn, F) > \frac{1}{2}\binom{(1-k)n}{2} + \frac{k(1-k)}{2}n^2$$

for all $k \in [0, 1/2]$. By the Erdős-Stone theorem, $\mathrm{ex}(n, F) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} + o(n^2)$ and so the left-hand-side is at least

$$\mathrm{ex}(n, F) - \mathrm{ex}(kn, F) \geq \mathrm{ex}(n, F) - \mathrm{ex}(n/2, F) \geq \left(1 - \frac{1}{r}\right)\left(\frac{n^2}{2} - \frac{n^2}{8}\right) - o(n^2) \geq \frac{3n^2}{16} - o(n^2).$$

On the right-hand-side,

$$\frac{1}{2}\binom{(1-k)n}{2} + \frac{k(1-k)}{2}n^2 = \frac{1-k^2}{4}n^2 + o(n^2) \leq \frac{n^2}{4} + o(n^2)$$

The problem is here. If $r = 2$, there does not exist $\alpha \in (0, 1]$ such that for $c \leq \alpha n$ the claim works: Erdős-Stone gives us $\mathrm{ex}(n, F) - \mathrm{ex}(kn, F) \geq \frac{1}{4}\left(1 - \alpha^2\right)n^2 + o(n^2)$, which exceeds the bound $\frac{n^2}{4} + o(n^2)$ for the right-hand-side for any $\alpha > 0$. □

Thus we may assume that $c > \frac{n}{2}$. A theorem of Simonovits states that for large enough c, $\mathrm{ex}(c, F) = \mathrm{ex}(c, K_{r+1}) + \mathrm{ex}(c, \tilde{F})$, where $\tilde{F}$ is the family of residue subgraphs of F after F is embedded into $T_r(c)$. This implies

$$\mathrm{ex}(c, F) - \mathrm{ex}(c-1, F) \geq \mathrm{ex}(c, K_{r+1}) - \mathrm{ex}(c-1, K_{r+1}),$$

as we assume n is sufficiently large. Thus by Lemma 2.3 and Lemma 2.4, we are done if $r > 3$.

The remaining proof is for $r = 3$.

The above inequality also implies that for $r = 2$,

$$\Delta_2 \mathcal{C}(n, c, F) \geq \Delta_2 \mathcal{C}(n, c, K_3),$$

which is positive by the proof of Claim 2.6.1. Thus when c is $n-1$ or n, $\mathcal{C}(n, c, F)$ attains its maximum, and plugging in $c = n$ and $c = n-1$ yields

$$\mathcal{C}(n, c, F) \leq \max\left[2 \cdot \mathrm{ex}(n, F), 2 \cdot \mathrm{ex}(n-1, F) + n - 1\right].$$

By Lemma 2.4,

$$2 \cdot \mathrm{ex}(n, F) - [2 \cdot \mathrm{ex}(n-1, F) + n - 1] \geq 2[\mathrm{ex}(n, K_3) - \mathrm{ex}(n-1, K_3)] - n + 1 \geq 0,$$

with equality only if n is odd. Hence, $\mathcal{C}(n, c, F) \leq 2 \cdot \mathrm{ex}(n, F)$. We may assume that n is odd and $c = n-1$, otherwise we are done by Lemma 2.1. Then in the extremal condition, $G_1 = G_{1,2}$ is the extremal graph for F on $n-1$ vertices, and G_2 must be $G_1 + K_1$. It remains to show that for $m \geq 3$, $G_1 = \cdots = G_m$ are extremal graphs for F when the extremal condition is met, and this follows from the argument in the proof of Theorem 2.6. $\square$

For small n, we may not be able to achieve the same result. Consider the case when F is the bowtie graph, i.e. the 5-vertex graph with two triangles sharing a vertex. For $n \leq 4$, the n-vertex extremal graph for F is the complete graph K_n. For $n \geq 5$, the n-vertex extremal graph for F is then $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ plus an edge, and so $\mathrm{ex}(n, F) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$. But then in this case when $n = 5$,

$$\mathcal{C}(5, 4, F) = 2e(K_4) + 4 = 16 > \mathcal{C}(5, 5, F) = 2\left(\left\lfloor \frac{5^2}{4} \right\rfloor + 1\right) = 14.$$

This yields an instance where the construction $G_1 = K_{v(F)-1}$ and $G_2 = K_n$ beats our benchmark construction. Thus the following lemma gives a lower bound for n to avoid losing to this construction.

Lemma 2.10. *Let $n \geq 1$, $r \geq 2$, and F be $(r+1)$-colorable with $|V(F)| = t \geq 3$. If $n > t^2 - 3t + 2$ and r divides n, then*

$$\mathcal{C}(n, n, F) > \mathcal{C}(n, t-1, F).$$

Proof. We need to show that

$$2\mathrm{ex}(n, F) - \binom{n}{2} > \binom{t-1}{2}.$$

Since $\mathrm{ex}(n, F) \geq e(T_r(n)) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} \geq \frac{n^2}{4}$,

$$2\mathrm{ex}(n, F) - \binom{n}{2} \geq \frac{n^2}{2} - \binom{n}{2} = \frac{n}{2} > \frac{t^2 - 3t + 2}{2} = \binom{t-1}{2}.$$

$\square$

2.3 Bipartite F

2.4 Hypergraph F

3 General Version

TODO: add introduction.

Theorem 3.1. *For all n and graph F,*

$$\mathrm{ex}_2(n, m, F) = m(1 + o(1))\mathrm{ex}(n, F)$$

as $m \to \infty$.

Proof. Let $r = v(F)$. Pick $\epsilon > 0$. Reorder $G_1, \dots, G_m$ so that $G_1, \dots, G_{m'}$ are all the G_i's containing at least $(1 + \epsilon)\mathrm{ex}(n, F)$ edges. A theorem of Simonovits states that G contains at least δn^r copies of F for some $\delta = \delta(\epsilon)$. Since there can be at most $\binom{n}{r}$ copies of F across all G_i's,

$$m'\delta n^r \leq \binom{n}{r} \leq n^r \implies m' \leq \frac{1}{\delta}.$$

It now follows that

$$\begin{aligned}
\sum_{i=1}^{m} e(G_i) &= \sum_{i=1}^{m'} e(G_i) + \sum_{i=m'+1}^{m} e(G_i) \\
&\leq \frac{1}{\delta}\binom{n}{2} + \left(m - \frac{1}{\delta}\right)(1 + \epsilon)\mathrm{ex}(n, F) \\
&= m\left[1 + \epsilon + \frac{1}{m\delta}\left(\frac{\binom{n}{2}}{\mathrm{ex}(n, F)} - (1 + \epsilon)\right)\right]\mathrm{ex}(n, F).
\end{aligned}$$

Since ϵ is arbitrary, the result follows. □

Theorem 3.2. *For large enough n, suppose that $G_1, \dots, G_m$ are graphs on common vertex set $[n]$ with no copy of F contained in any k of the G_i's. If there exists extremal F-free subgraph H on n vertices such that $\binom{m}{k}\Delta(H) = o(n^{1/2})$, then*

$$\mathrm{ex}_2(n, m, F) = (k - 1)\binom{n}{2} + \mathrm{ex}(n, F)\binom{m}{k}.$$

Proof. For $S \subseteq [m]$, let E_S denote the set of edges that are contained in exactly $\{G_i\}_{i \in S}$. Then

$$\sum_{i=1}^{m} e(G_i) = \sum_{S \subseteq [m]} |S||E_S| \leq (k - 1)\binom{n}{2} + \sum_{S \subseteq [m], |S| \geq k} (|S| - k + 1)|E_S|.$$

Let $A_S = \bigcup_{T \supseteq S} E_T$, i.e. the set of edges that are contained in all G_i with $i \in S$. When $|S| \geq k$, the edge set A_S is F-free and thus

$$\sum_{T \supseteq S} |E_T| \leq \mathrm{ex}(n, F).$$

Hence,

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq k}} (|S| - k + 1)|E_S| = \sum_{\substack{S \subseteq [m], \\ |S| = k}} \sum_{T \subseteq S} \frac{(|T| - k + 1)|E_T|}{\binom{|T|}{k}} \leq \sum_{\substack{S \subseteq [m], \\ |S| = k}} \sum_{T \subseteq S} |E_T| \leq \binom{m}{k} \mathrm{ex}(n, F),$$

as each $T \in [m]$ with $|T| \geq k$ is counted $\binom{|T|}{k}$ times in total and $|T| - k + 1 \leq \binom{|T|}{k}$. This proves the upper bound.

Now we show the bound is tight. In particular, we need to show there exists a construction such that the graph with edge set E_S is an extremal F-free graph, for all $S \subseteq [m]$ of size k. Let $M = \binom{m}{k}$ and $H_1, \ldots, H_M$ be copies of an extremal F-free graph on n vertices with $\Delta(H_i) = o(n^{1/2})$ for all i. It suffices to show that we can embed each H_i onto $[n]$ such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each H_i and iteratively decrease the number of intersecting edges. Define a (u, v, i)-swap by swapping the embedding of vertex u and v of H_i, i.e. replacing each edge $\{u, w\} \in E(H_i)$ with the edge $\{u, w\}$ and each edge $\{v, w\} \in E(H_i)$ with the edge $\{v, w\}$. This perserves the type of isomorphism of H_i. Given a vertex v, let $N(v) = N_{H_1}(v) \cup \cdots \cup N_{H_M}(v)$. Suppose there exists an intersecting edge $\{u, w\} \in E(H_i) \cap E(H_j)$. Since $|N(u)| \leq M \cdot \Delta(H_i) = o(n^{1/2})$, $|N(u) \cup N(N(u))| = o(n)$ so there exists a vertex $v \notin N(u) \cup N(N(u))$. Since $N(u) \cap N(v) = \emptyset$, performing a (u, v, i)-swap reduces the number of intersecting edges. The result now follows from iterating this process. $\square$

3.1 Triangle F

Consider F to be a triangle. Simply counting the number of triangles in each G_i shows the following:

Theorem 3.3. *For all n, m and $\epsilon > 0$,*

$$\mathrm{ex}_2(n, m, K_3) < \left(m \cdot \frac{1+\epsilon}{4} + \frac{1}{2\epsilon} - \frac{1}{2}\right) n^2.$$

Claim 3.3.1. *There are less than $\frac{2}{\epsilon}$ number of G_i's with at least $(1+\epsilon)\frac{n^2}{4}$ edges.*

Proof. Suppose $e(G_i) \geq (1+\epsilon)\frac{n^2}{4}$ for $1 \leq i \leq k$. Let $K_3(G)$ denote the number of triangles in graph G. By the Moon-Moser inequality,

$$K_3(G_i) \geq \epsilon(1+\epsilon)\frac{n^3}{12}.$$

Since there are no overlapping traingles from different G_i's,

$$\binom{n}{3} \geq \sum_{i=1}^{k} K_3(G_i) \geq \frac{\epsilon(1+\epsilon)}{12} k n^3.$$

Rearranging yields $k < \frac{2}{\epsilon}$. $\square$

By the claim,

$$\sum_{i=1}^{m} e(G_i) < \frac{2}{\epsilon}\binom{n}{2} + \left(m - \frac{2}{\epsilon}\right)(1+\epsilon)\frac{n^2}{4} \le m(1+\epsilon)\frac{n^2}{4} + (1-\epsilon)\frac{n^2}{2\epsilon},$$

which proves Theorem 3.3.

When $m = 2$,

$$e(G_1) + e(G_2) \le \binom{n}{2} + e(G_{1,2}) \le \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor,$$

which meets the benchmark bound and so $\mathrm{ex}_2(n, 2, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor$.

This result is also true for $m = 3$:

Proposition 3.4. *For all n,*

$$\mathrm{ex}_2(n, 3, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Proof. Define $H_k \subseteq G$ be the graph with edges contained in at least k number of G_i's and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \le \frac{n^2}{2}$. Notice H_2 must not contain any triangles with two edges in H_3, so

$$e(H_2) + e(H_3) \le \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \ne v, N_{H_3}(u) \cap N_{H_3}(v) \ne \emptyset\}|.$$

Let H_3' be the graph with the same vertex set as H_3 and edge set $\{\{u, v\} : u \ne v, N_{H_3}(u) \cap N_{H_3}(v) \ne \emptyset\}$. It suffices to show that $\frac{n}{2} \ge e(H_3) - e(H_3')$.

Let $d_1 \ge d_2 \ge \cdots \ge d_n$ and $f_1 \ge f_2 \ge \cdots \ge f_n$ each be the degree sequence of H_3 and H_3', respectively. We show that $f_i \ge d_i - 1$ for all i. Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i. Let $S_i = |N_{H_3}(v_1) \cup \cdots \cup N_{H_3}(v_i)|$. Since

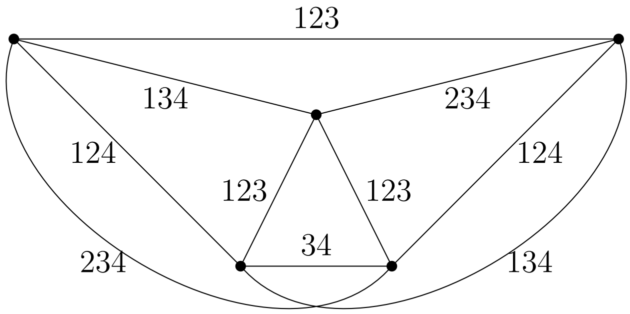
$$\sum_{u \in S_i} d_{H_3}(u) \ge d_1 + \cdots + d_i,$$

we have that $|S_i| \ge i$. But then $S_i \backslash \{u_1, \ldots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H_3'}(u) \ge d_i - 1$. Hence, $f_i \ge d_i - 1$ for all i, which yields

$$e(H_3') = \frac{1}{2}\sum_{i=1}^{n} f_i \ge \frac{1}{2}\sum_{i=1}^{n}(d_i - 1) = e(H_3) - \frac{n}{2}.$$

□

However, the bound in Proposition 3.1 is not tight for $m \ge 4$, as shown in the following graph:



The number on each edge denotes the set of G_i's that contain the edge.

The above graph contains 29 edges, which exceeds the bound $\binom{5}{2} + 3\lfloor \frac{5^2}{4} \rfloor = 28$ by 1. By blowing up the above graph, we can construct a graph with $n \in 10\mathbb{Z}$ vertices that contains

$$5\binom{n/5}{2} + 29 \cdot \frac{(n/5)^2}{4}$$

edges, which exceeds the bound $\binom{n}{2} + 3\lfloor \frac{n^2}{4} \rfloor$ by $n^2/100$.

3.2 Bipartite F

In this section, we discuss the case where F is bipartite. In particular, we focus on the cases where $F \subseteq K_{2,2}$ is P_2, a path of length 2, or M_2, a matching with two edges.

Theorem 3.5.

$$\mathrm{ex}_2(n, m, P_2) \leq \left(\frac{1}{2} + o(1)\right) n^2\sqrt{m}$$

as $n \to \infty$ or $m \to \infty$.

Proof. Since there are no overlapping P_2's in different G_i's,

$$\sum_{i=1}^{m} \#\{P_2 \subseteq G_i\} \leq \#\{P_2 \subseteq G\}$$

For each G_i, each vertex v in G_i and two of its neighbors form one unique P_2, so

$$\#\{P_2 \subseteq G_i\} = \sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2}.$$

And by Jensen's inequality,

$$\sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2} \geq n\binom{d_{G_i}(v)/n}{2} = n\binom{2e(G_i)/n}{2} \geq \frac{2(e(G_i))^2}{n} - e(G_i).$$

On the other hand, since each three vertices in G can form at most three P_2's,

$$\#\{P_2 \subseteq G\} \leq 3\binom{n}{3} \leq \frac{n^3}{2}.$$

Combining the above inequalities yields

$$\frac{2m}{n}\left(\frac{1}{m}\sum_{i=1}^{m} e(G_i)\right)^2 - \sum_{i=1}^{m} e(G_i) \overset{Jensen's}{\leq} \sum_{i=1}^{m} \frac{2(e(G_i))^2}{n} - e(G_i) \leq \frac{n^3}{2},$$

and solving the quadratic equation gives

$$\sum_{i=1}^{m} e(G_i) \leq mn \cdot \frac{1+\sqrt{4n^2/m+1}}{4} = \left(\frac{1}{2}+o(1)\right) n^2\sqrt{m},$$

as $n \to \infty$ or $m \to \infty$. $\square$

When $m = n$, the following projective plane construction shows the above bound is tight asymptotically:

Theorem 3.6.

$$\mathrm{ex}_2(n, n, P_2) = \left(\frac{1}{2}+o(1)\right) n^{5/2},$$

as $n \to \infty$.

Proof. It suffices to show the tightness of the bound in Theorem 3.5. Consider a finite projective plane of order q. The projective plane has $n = q^2 + q + 1$ points and n lines. Let $S_1, \ldots, S_n \subseteq [n]$ be the n lines of the projective plane. Note that each line S_i contains $q + 1$ points, and the intersection of any two distinct lines S_i, S_j contains $|S_i \cap S_j| = 1$ point. Define $G_1, \ldots, G_n$ to be graphs on $[n]$, each with edge set $E(G_i) = \{\{j, k\} \subseteq [n] : j \neq k,\ j + k \in S_i \mod n\}$. Note that the intersection of distinct G_i, G_j is P_2 free: since $|S_i \cap S_j| = 1$, if $\{a, b\}, \{a, c\} \in E(G_i) \cap E(G_j)$, then $a + b = a + c$ so $b = c$. We now count the number of edges in $G_1, \ldots, G_n$. Since $|S_i| = q + 1$, for each point $j \in [n]$, there are $q + 1$ choices for $k \in [n]$ such that $j + k \in S_i$. But then we have to avoid counting the same edge twice and loops, so the number of edges in G_i is

$$e(G_i) = \frac{n(q+1) - \#\text{loops counted for } G_i}{2}.$$

If $j \in [n]$ is even, then $k = j/2$ is the unique number in $[n]$ such that $k + k = j \mod n$. If $j \in [n]$ is odd, then $k = (n + j)/2$ is the unique number in $[n]$ such that $k + k = j \mod n$, as n is even. Hence, for each $j \in S_i$, there exists a unique $k \in [n]$ such that $k + k = j \mod n$, and thus

$$\#\text{loops counted for } G_i = |S_i| = q + 1.$$

Since $q + 1 = (1 + o(1))n^{1/2}$, the number of edges in $G_1, \ldots, G_n$ is

$$\sum_{i=1}^{n} e(G_i) = n \cdot \frac{n(q+1) - (q+1)}{2} = \left(\frac{1}{2}+o(1)\right) n^{5/2},$$

as $n \to \infty$. $\square$

Theorem 3.7. *For all* n, m,

$$\mathrm{ex}_2(n, m, M_2) \leq n^{5/2}.$$

Proof. Notice that $\#\{M_2 \subseteq G\} = \binom{e(G_i)}{2}$. On the other hand, each four vertices in G can form at most three M_2's, so $\#\{M_2 \subseteq G\} \leq 3\binom{n}{4} \leq \frac{n^4}{8}$. By the same argument as in Theorem 3.4, we have

$$\sum_{i=1}^{n} \binom{e(G_i)}{2} = \sum_{i=1}^{n} \#\{M_2 \subseteq G_i\} \leq \#\{M_2 \subseteq G\} \leq \frac{n^4}{8}.$$

By Jensen's inequality,

$$\sum_{i=1}^{n} \binom{e(G_i)}{2} \geq n\binom{\sum_{i=1}^{n} e(G_i)/n}{2} = \frac{1}{2n}\left[\left(\sum_{i=1}^{n} e(G_i)\right)^2 - n\sum_{i=1}^{n} e(G_i)\right].$$

Combining the above inequalities yields

$$\left(\sum_{i=1}^{n} e(G_i)\right)^2 - n\sum_{i=1}^{n} e(G_i) \leq \frac{n^5}{4},$$

and solving the quadratic inequality gives

$$\sum_{i=1}^{n} e(G_i) \leq n^{5/2}.$$

□

We may obtain an exact result if we forbid both P_2 and M_2 at the same time:

Theorem 3.8. *For all n, m,*

$$\mathrm{ex}_2(n, m, \{P_2, M_2\}) = n^2 - n.$$

Proof. Denote the set of G_i's as $\{G_i\} = \{G_1, \ldots, G_n\}$, and the set of distinct pairs of G_i's as $\{G_i\}^2 = \{\{G_j, G_k\} : j \neq k\}$. Consider the bipartite graph H with vertex set $V(H) = \{G_i\} \sqcup E(K_n)$ and edge set $E(H) = \{\{G_j, e\} \in \{G_i\} \times E(K_n) : e \in G_j\}$. Define $\phi : \{G_i\}^2 \to 2^{E(K_n)}$ by sending each $\{G_j, G_k\}$ to their common edge set $E(G_j) \cap E(G_k)$. Notice that each distinct G_j, G_k have at most one edge in common, so $|\phi(G_j, G_k)| \leq 1$. On the other hand, each edge $e \in E(G)$ can be obtained via ϕ by $\binom{d_H(e)}{2}$ possible distinct pairs (G_j, G_k), and thus $|\phi^{-1}(e)| = \binom{d_H(e)}{2}$. But then

$$\binom{n}{2} \geq \sum_{(G_j, G_k) \in \{G_i\}^2} |\phi(G_j, G_k)| = \sum_{e \in E(K_n)} |\phi^{-1}(e)| = \sum_{e \in E(K_n)} \binom{d_H(e)}{2}.$$

By Jensen's inequality,

$$\sum_{e \in E(K_n)} \binom{d_H(e)}{2} \geq \binom{n}{2}\binom{\sum_{e \in E(K_n)} d_H(e)/\binom{n}{2}}{2} = \binom{n}{2}\binom{\sum_{i=1}^{n} e(G_i)/\binom{n}{2}}{2}.$$

Combining the above inequalities yields

$$2\binom{n}{2}^2 \geq \left(\sum_{i=1}^{n} e(G_i)\right)^2 - \binom{n}{2}\sum_{i=1}^{n} e(G_i),$$

and the result now follows from solving the quadratic inequality.

To see that this bound is tight, consider the construction such that for each distinct $i, j \in [n]$, $E(G_i) \cap E(G_j)$ contains exactly one unique edge $e \in K_n$. The number of edges in this construction is $2\binom{n}{2} = n^2 - n$. $\square$