Double Turán Problem

Ray Tsai
Advisor: Prof. Jacques Verstraete

May 8, 2025

Contents

1 Introduction **2**
1.1 Link graphs and hypergraphs 3
1.2 Main results : the induced case 4
1.3 Main results : the non-induced case 5
1.4 Definitions and Notations 7

2 The induced double Turán problem **8**
2.1 Proof of Theorem 2 9
2.2 Proof of Theorem 1 10
2.3 Proof of Theorem 3 11

3 The non-induced double Turán problem **12**
3.1 Proof of Theorem 6 12
3.2 Proof of Theorem 4 14
3.3 Proof of Theorem 5 15
3.4 Proof of Theorem 7 17
3.5 Proof of Theorem 8 19

4 Concluding Remarks **20**

1 Introduction

This thesis focuses on a variation of the *Turán problem* in extremal combinatorics. The fundamental question in extremal hypergraph theory is determining the maximum number of edges in an n-vertex r-uniform graph that does not contain a prescribed r-uniform graph F as a subgraph. These maxima, denoted $\mathrm{ex}(n, F)$, are referred to as the *extremal numbers* or *Turán numbers* for F. One of the cornerstones of extremal graph theory, concerning the case F is a clique, is Turán's Theorem [19]. To state the theorem, we need the *Turán graphs* $T_k(n)$, which denotes a complete multipartite graph with n vertices and k parts of size $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$.

Theorem A (Turán's Theorem). *The maximum number of edges in an n-vertex graph containing no clique of order $r+1$ is $e(T_r(n))$, with equality only for $T_r(n)$.*

Simonovits [5] observed via the Erdős-Stone Theorem [3] that the asymptotic value of $\mathrm{ex}(n, F)$ may be obtained whenever F is non-bipartite:

Theorem B (Erdős-Stone Theorem, Simonovits' Theorem). *Let F be any graph of chromatic number $r+1 \geq 3$. Then $\mathrm{ex}(n, F) = (1+o(1))T_r(n)$ as $n \to \infty$.*

There are a number of proofs of the Erdős-Stone Theorem. A very general framework involves *Szemerédi's Regularity Lemma*, which may be stated as follows. A pair (U, V) of disjoint sets of vertices in a graph G is called *ϵ-regular* if for any $X \subseteq U$ and $Y \subseteq V$ of size at least $\epsilon|U|$ and $\epsilon|V|$ respectively,

$$\left| \frac{e(X,Y)}{|X||Y|} - \frac{e(U,V)}{|U||V|} \right| < \epsilon.$$

The following was proved by Szemerédi [18]:

Theorem C (Szemerédi's Regularity Lemma). *For all $\epsilon > 0$, there exist m and M such that for every graph G, there exists a partition $(V_1, V_2, \dots, V_k)$ of $V(G)$ such that $m \leq k \leq M$ and $|V_1| \leq |V_2| \leq \cdots \leq |V_k| \leq |V_1| + 1$ and all but at most ϵk^2 pairs (V_i, V_j) are ϵ-regular.*

The value of $\mathrm{ex}(n, F)$ for bipartite F is in general wide open, and the order of magnitude of $\mathrm{ex}(n, K_{4,4})$ or $\mathrm{ex}(n, C_8)$ is not known – see Füredi and Simonovits [8] for a history of the bipartite Turán problem. There is also no analog of the above theorems for r-uniform hypergraphs; the asymptotic value of $\mathrm{ex}(n, K_k^r)$ is not known for any $k > r \geq 3$, where K_k^r denotes the complete r-uniform hypergraph on k vertices. The asymptotic value of $\mathrm{ex}(n, K_4^3)$ was conjectured by Turán [19] to be $\frac{5}{9}\binom{n}{3}$, and this remains open despite decades of intensive research.

In this thesis, we investigate closely related problems which we refer to as *double Turán problems*. To describe these problems, let $G_1, G_2, \dots, G_m$ be graphs with the same vertex set $V(G_i) = [n]$ for $i \in [m]$. For a graph F, we say that $G_1, G_2, \dots, G_m$ is *double F-free* if $E(F) \not\subseteq E(G_i) \cap E(G_j)$ for $1 \le i < j \le m$. In other words, F does not appear in the intersection of any two of the graphs G_i. We call a copy of F in the intersection of two of the graphs G_i a *double* F. Let $\phi(m, n, F)$ denote the maximum value of $\sum_{i=1}^m e(G_i)$ such that $G_1, G_2, \dots, G_m$ does not contain a double F. We say that graphs $G_1, G_2, \dots, G_m$ are *induced* to mean that every G_i is an induced subgraph of $\bigcup_{i=1}^m G_i$. In other words, if $\{u, v\} \in E(G_i)$ and $u, v \in V(G_j)$, then $\{u, v\} \in E(G_j)$. Let $\phi^*(m, n, F)$ denote the maximum value of $\sum_{i=1}^m e(G_i)$ such that $G_1, G_2, \dots, G_m$ does not contain a double F and $G_1, G_2, \dots, G_m$ are induced. Clearly, $\phi^*(m, n, F) \le \phi(m, n, F)$, and the study of $\phi^*(m, n, F)$ and $\phi(m, n, F)$ is motivated by certain hypergraph extremal problems.

1.1 Link graphs and hypergraphs

Apart from the intrinsic interest in studying $\phi(m, n, F)$, a motivation is that $\phi(m, n, F)$ is closely connected to pure hypergraph extremal problems via the notion of *link graphs*. Let H be a triple system, that is, a set of three-element subsets of a finite set $[n]$. These three-element subsets form the edge-set $E(H)$ of H, while $V(H) = V$ is the vertex set of H. For $i \in V(H)$, the *link graph* of i, denoted H_i, is the graph with $V(H_i) = V(H) \backslash \{i\}$ and $E(H_i) = \{\{j, k\} : \{i, j, k\} \in E(H)\}$. A handy idea in extremal hypergraph theory is to reduce a hypergraph extremal problem to extremal problems for the link graphs. For instance, a triple system H does not contain a tetrahedron, i.e. four triples on four vertices, if and only if all its link graphs are triangle-free.

In the current context, given a graph F, let F^+ denote the triple system with $V(F^+) = V(F) \cup \{x, y\}$ and $E(F^+) = \{e \cup \{x\}, e \cup \{y\} : e \in E(F)\}$. Then $\phi(n, n, F)$ and $\mathrm{ex}(n, F^+)$ are intimately related: if H is an F^+-free triple system with vertex set $[n]$, then clearly the link graphs $H_1, H_2, \dots, H_n$ are double F-free, which implies $\mathrm{ex}(n, F^+) \le \phi(n, n, F)$. This relates the double Turán problem to hypergraph extremal problems.

Now let G be the graph consisting of all pairs contained in triples in F^+. The *generalized Turán problem* asks for the maximum number $\mathrm{ex}(n, G, K_3)$ of triangles in a graph H with vertex set $[n]$ that does not contain G. This problem was studied by Alon and Shikhelman [1] and Kostochka, Mubayi and Verstraete [10, 12, 14]. Similar to how link graphs relate to hypergraph extremal problems, the generalized Turán problem is related to $\phi^*(n, n, F)$ as follows: define $H_i = \{\{j, k\} : \{i, j\}, \{j, k\}, \{i, k\} \in E(H)\}$. Then $H_1, H_2, \dots, H_n$ are induced and double F-free, so $\phi^*(n, n, F) \ge \mathrm{ex}(n, G, K_3)$. This relates the induced double Turán problem to extremal problems for triangles in graphs.

1.2 Main results : the induced case

The determination of $\phi^*(m, n, F)$ turns out to be fairly straightforward when F is a non-bipartite graph: the extremal objects are simply m copies of the same extremal graph for F.

Theorem 1. *For $r \geq 3$, there exists $n_0(r)$ such that if $n \geq n_0(r)$ and F is a graph of chromatic number r, then for all $m \geq 3$,*

$$\phi^*(m, n, F) = m \cdot \mathrm{ex}(n, F),$$

with equality only for identical extremal n-vertex F-free graphs.

In the case $F = K_r$, we shall see the theorem is true for all $n \geq 3$:

Theorem 2. *Let $m, n, r \geq 3$. Then $\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n))$ with equality for induced K_r-free graphs $G_1, G_2, \dots, G_m$ only if $G_1 = G_2 = \cdots = G_m = T_{r-1}(n)$.*

In the case F is a bipartite graph, even determining the order of magnitude of $\phi^*(m, n, F)$ appears to be difficult. In fact, we do not even know the order of magnitude of $\phi^*(m, n, P)$ when P is a path with two edges. In this thesis, we propose the following very broad conjecture:

Conjecture A. *Let F be any non-empty graph and $m, n \geq 1$. Then*

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

It is clear that a single complete graph K_n does not contain a double F, and neither do identical copies $G_1, G_2, \dots, G_m$ of an extremal n-vertex F-free graph. Thus we have the trivial lower bound

$$\phi^*(m, n, F) \geq \max\left\{\binom{n}{2}, m \cdot \mathrm{ex}(n, F)\right\}.$$

This conjecture is true when F is non-bipartite, by Theorem 1. If F is bipartite, then the upper bounds on $\phi^*(m, n, F)$ are more difficult to come by, especially when m is large. For instance, we know

$$\mathrm{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2}),$$

and so Conjecture A implies that an n-vertex graph not containing the octahedron graph has $O(n^{5/2})$ triangles. In fact, it is also the case that $\mathrm{ex}(2n, K_{2,2,2}, K_3) \geq \phi^*(n, n, K_{2,2})$: if we have double $K_{2,2}$-free induced graphs $G_1, G_2, \ldots, G_n$ with vertex set $[n]$, then let H be the graph with $V(H) = [2n]$ consisting of all triangles with vertex set $\{i, j, k\}$ such that $n < k \leq 2n$ and $\{i, j\} \in E(G_k)$. The graph H is $K_{2,2,2}$-free and $|E(H)| = \sum_{i=1}^{n/2} e(G_i)$. Similarly, we have

$$\mathrm{ex}(n, K_{1,2,2}, K_3) \leq \phi^*(n, n, K_{1,2})$$

and so Conjecture A implies that an n-vertex graph not containing the octahedron graph has $O(n^2)$ triangles, which is conjectured by Mubayi and Verstraete [14]. The conjecture proposes more generally that if F is a tree, then $\phi^*(n, n, F) = O(n^2)$. In fact, it is possible to prove the following theorem using the *removal lemma* as in [12] as well as a construction for $\phi(n, n, P)$ in this work:

Theorem 3. *Let P be a path with two edges. Then $\phi(n, n, P) = \Omega(n^{5/2})$, whereas $\phi^*(n, n, P) = o(n^{5/2})$, as $n \to \infty$. In particular,*

$$\lim_{n \to \infty} \frac{\phi^*(n, n, P)}{\phi(n, n, P)} = 0.$$

If M is a matching with two edges, and M^+ is the graph obtained from two copies of K_4 sharing one edge by removing that edge, then $\mathrm{ex}(n, M^+, K_3) \leq \phi^*(n, n, M)$. If F is the triple system consisting of all four triangles in M^+, then Füredi [7] showed $\mathrm{ex}(n, M^+) = O(n^2)$, answering a conjecture of Erdős [4]. It is possible to adapt Füredi's proof to give $\phi^*(n, n, M) = O(n^2)$, so in this case, $\mathrm{ex}(n, M^+, K_3) = \Theta(\phi^*(n, n, M))$. For improvements of the constant factor, see Mubayi and Verstraete [13] and Pikhurko and Verstraete [15]. We shall see that for some bipartite F, if m is not too large relative to n, then Conjecture A is also true.

1.3 Main results : the non-induced case

Determining $\phi(m, n, F)$ even when F is a complete graph is challenging. The forth theorem we give is well-suited to the case of certain bipartite graphs, and is due to Wilson:

Theorem 4. *Let F be a graph. If there exists an extremal F-free n-vertex graph with maximum degree at most $n^{1/2}/m^2$, then*

$$\phi(m, n, F) = \binom{n}{2} + \binom{m}{2}\mathrm{ex}(n, F).$$

Since $\binom{n}{2} + m - 1 \le \phi^*(m,n,F) \le \phi(m,n,F)$ for any graph F with at least two edges, this theorem shows $\phi^*(m,n,F) = (1+o(1))\binom{n}{2}$ whenever the conditions on m in the theorem are satisfied. In particular, if P is the path with two edges, and $m = o(n^{1/4})$ as $n \to \infty$, then

$$\binom{n}{2} + m - 1 \le \phi^*(m,n,P) \le \phi(m,n,P) = \binom{n}{2} + \binom{m}{2}\left\lfloor \frac{n}{2} \right\rfloor .$$

When F is bipartite, the value of $\phi(m,n,F)$ for larger m appears to be difficult to determine. We investigate the case $F = P$ more closely.

Theorem 5. *Let P be the path with two edges. Then as $n \to \infty$,*

$$\phi(m,n,P) = \begin{cases} \left(\frac{1}{2} + o(1)\right) n^2, & \sqrt{n}/m \to \infty \\ \Theta(n^2), & m = \Theta(\sqrt{n}) \\ \left(\frac{1}{2} + o(1)\right) mn^{3/2}, & \sqrt{n} < m \le n \\ \left(\frac{1}{2} + o(1)\right) \sqrt{m} n^2, & n < m \le n^2 \\ \Theta(n^3), & m = \Theta(n^2) \\ (1 + o(1))\, mn, & m/n^2 \to \infty \end{cases}$$

Interestingly, while Conjecture A proposes $\phi^*(m,n,P) = O(n^2 + mn)$ for all $m, n \ge 1$, the above theorem shows $\phi(m,n,P)$ is much larger, of order at least $mn^{3/2}$ when $m \ge \sqrt{n}$.

Our first theorem on $\phi(m,n,F)$ for non-bipartite graphs F uses the notion of *supersaturation* – see Erdős and Simonovits [6]. We determine the asymptotic value of $\phi(m,n,F)$ as $m \to \infty$ when F is a non-bipartite graph:

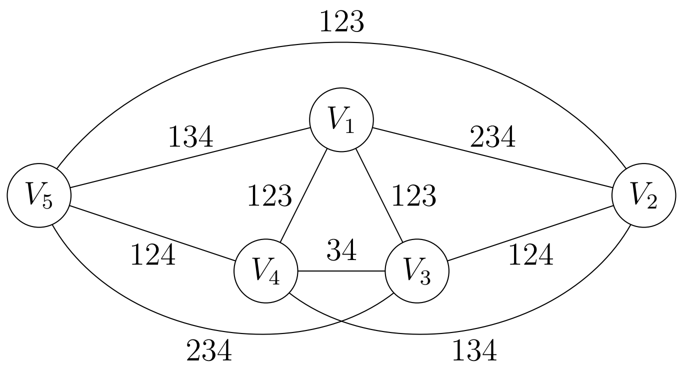
Theorem 6. *Let $n \ge 1$ and let F be a non-bipartite graph. Then as $m \to \infty$,*

$$\phi(m,n,F) = (1 + o(1))m \cdot \mathrm{ex}(n,F).$$

The next result we present concerns non-bipartite graphs. To state the theorem, we require the notion of M-color Ramsey numbers. Define $R_M(r)$ to be the *M-color Ramsey number* for the complete graph K_r, that is, the minimum N such that there exists a monochromatic F in any coloring of $E(K_N)$ with M colors. Suppose we have a monochromatic K_r-free coloring $c : E(K_N) \to 2^{[m]}$. For $i \in [m]$, let $H_i = \{\{u,w\} \in E(K_N) : i \in c(u,w)\}$. Then $H_1, H_2, \ldots, H_m$ are double K_r-free. If we replace the vertices of K_N with disjoint sets $V_1, V_2, \ldots, V_N$ whose sizes add up to n, and then let

$$G_i = \{\{x,y\} : (x,y) \in V_u \times V_w,\, i \in c(u,w),\, 1 \le u < w \le N\}$$

and make each V_i cliques in G_1, then $G_1, G_2, \ldots, G_m$ is also double K_r-free. We call $G_1, G_2, \ldots, G_m$ an (m, n, N)-blowup.



Example of an $(4, n, 5)$-blowup not containing a double K_3.

Let $f(m, n, r)$ denote the maximum of $e(G_1) + e(G_2) + \cdots + e(G_m)$ such that $G_1, G_2, \ldots, G_m$ is a double K_r-free (m, n, N)-blowup for some $N < R_{\binom{m}{2}}(r)$. This turns out to be exactly the construction which determines $\phi(m, n, F)$ when F is a complete graph:

Theorem 7. *Let $r \geq 2$ and $m, n \geq 1$. Then*

$$\phi(m, n, K_r) = f(m, n, r).$$

While computing $f(m, n, r)$ is a finite calculation, the Ramsey number $R_{\binom{m}{2}}(r)$ unfortunately appears to be intractable in general; it is known that $R_2(3) = 6$ and $R_3(3) = 17$ and $R_2(4) = 18$, but no further multicolor Ramsey numbers are known [2, 11]. In the special case $r = m = 3$, the following holds:

Theorem 8. *For all $n \geq 1$,*

$$\phi(3, n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

1.4 Definitions and Notations

Denote the set of first n positive integers as $[n] = \{1, 2, \ldots, n\}$. Given a set X, we denote 2^X as the power set of X. Let $G = (V, E)$ be a graph. Let $V(G)$ denote the vertex set and $E(G)$ denote the edge set of G. Let $e(G) = |E(G)|$ be the number of edges in G. For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v. Given two graphs G_1, G_2, we denote $G_1 \cup G_2$ as the graph on $V(G_1) \cup V(G_2)$ with edge set $E(G_1 \cap G_2) = E(G_1) \cup E(G_2)$. Similarly, we define $G_1 \cap G_2$ as the graph on $V(G_1) \cap V(G_2)$ with edge set $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$. In this thesis, we reserve n to denote the number of vertices in a graph. We call a n-vertex complete graph K_n, and a complete bipartite graph $K_{a,b}$, where a, b are the sizes of its parts. Given graph G, H, define $G + H$ as the graph fully connecting G, H, i.e. $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$. Given graphs G and F, we say that G is F-free if G does not contain F as a subgraph. We denote $\mathrm{ex}(n, F)$ to be the maximum possible number of edges an F-free graph on n vertices, and we call a F-free graph achieving this maximum an extremal graph for F. Let v be a vertex from $G_1, G_2, \ldots, G_m$. Unless otherwise specified, we denote $d(v)$ as the sum of the degree of v over all G_i.

2 The induced double Turán problem

We prove the theorems for $\phi^*(m, n, F)$ in this chapter. In particular, the main theorem we prove is Theorem 1 for general non-bipartite graphs F and in the special case of cliques. We will first introduce two observations that significantly simplify the problem.

The first observation is that the determination of $\phi^*(m, n, F)$ can be reduced down to the case of two graphs, which is stated in the following lemma:

Lemma 9. *Let $n, m, k \geq 2$ with $m \geq k$, F be some graph. Then*

$$\phi^*(m, n, F) \leq \frac{m}{k} \cdot \phi^*(k, n, F).$$

Moreover, let $G_1, \ldots, G_m$ be induced double F-free graphs on $[n]$ and suppose $\sum_{i=1}^{k} e(G_i) = \phi^*(k, n, F)$ only if $G_1 = \cdots = G_k$. Then $\sum_{i=1}^{m} e(G_i) = \phi^*(m, n, F)$ only if $G_1 = \cdots = G_m$.

Proof. Let $G_1, \ldots, G_m$ be induced double F-free graphs on $[n]$. Put $G_{i+m} = G_i$ for all $i \in [m]$. Then

$$\sum_{i=1}^{m} e(G_i) = \frac{1}{k} \sum_{i=1}^{m} [e(G_i) + \cdots + e(G_{i+k-1})] \leq \frac{1}{k} \sum_{i=1}^{m} \phi^*(k, n, F) = \frac{m}{k} \cdot \phi^*(k, n, F),$$

which establishes the upper bound. The lower bound follows from the construction with $G_1 = \cdots = G_m$ to be n-vertex extremal graphs for F.

Now suppose $\sum_{i=1}^{m} e(G_i) = (m/k)\phi^*(k, n, F)$ and $G_1 \neq G_2$. By assumption $\sum_{i=1}^{k} e(G_i) < \phi^*(k, n, F)$. But then $\sum_{i=1}^{k} e(G_{i+j}) > \phi^*(k, n, F)$ for some $j \geq 1$, contradiction. □

Now that we may determine $\phi^*(m, n, F)$ by examining $\phi^*(m, n, F)$, the second observation is that $\phi^*(m, n, F)$ can be further reduced to a finite optimization problem on a single variable. To state the lemma, we introduce the following construction function:

Definition 10. *For $n \geq t \geq 1$ and F some graph, define*

$$\mathcal{C}(n, t, F) := \binom{n-t}{2} + (n-t)t + 2\mathrm{ex}(t, F).$$

The construction described by $\mathcal{C}(n, t, F)$ are graphs G_1, G_2 on $[n]$, such that G_2 is a t-vertex extremal graph for F and $G_1 = G_2 + K_{n-t}$.

Lemma 11. *Let F be some graph. For $n \geq 1$,*

$$\phi^*(2, n, F) = \max_{0 \leq t \leq n} \mathcal{C}(n, t, F).$$

Moreover, the equality holds for graphs G_1, G_2 on $[n]$ only if G_1, G_2 are the construction described by $\mathcal{C}(n, t_{max}, F)$, where $t_{max} \in [n]$ is a maximizer for $\mathcal{C}(n, t, F)$.

Proof. Let G_1, G_2 be induced double F-free graphs on $[n]$. Put $T = V(G_1) \cap V(G_2)$, $t = |T|$, $s = |V(G_1) \backslash T|$, and $n - t - s = |V(G_2) \backslash T|$. Note that $t, s \in \mathbb{Z}_{\geq 0}$. Since G_1, G_2 are induced subgraphs of $G_1 \cup G_2$, we have $G_1[T] = G_2[T] = G_1 \cap G_2$. But then $G_1 \cap G_2$ is F-free, so $e(G_1[T]) = e(G_2[T]) \leq \mathrm{ex}(t, F)$. Notice there can be at most $t(n-t)$ edges between T and $(V(G_1) \cup V(G_2)) \backslash T$. Since $G[V(G_1) \backslash T] \leq \binom{s}{2}$ and $G[V(G_2) \backslash T] \leq \binom{n-t-s}{2}$,

$$e(G_1) + e(G_2) \leq \binom{s}{2} + \binom{n-s-t}{2} + t(n-t) + 2\mathrm{ex}(t, F).$$

But then $\binom{n-t}{2} > \binom{s}{2} + \binom{n-t-s}{2}$ for $0 < s < n - t$, so

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2\mathrm{ex}(t, F) = \mathcal{C}(n, t, F).$$

This establishes the upper bound. From this we also know that $e(G_1) + e(G_2) = \mathcal{C}(n, t, F)$ only if G_1, G_2 are the construction described by $\mathcal{C}(n, t, F)$. The result now follows. $\square$

2.1 Proof of Theorem 2

By Lemma 9, it suffices to prove the theorem for $m = 3$. Let G_1, G_2, G_3 be induced double K_r-free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, K_r)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\phi^*(3, n, K_r) \geq 3\mathrm{ex}(n, K_r)$. Consequently, we must have

$e(G_1) + e(G_2) \geq 2\mathrm{ex}(n, K_r)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\mathrm{ex}(n, K_r)$, it suffices to show that $G_1 = G_2 = T_{r-1}(n)$. In particular, we will use Lemma 11 to show that G_1, G_2 is an extremal configuration without containing a double K_r.

Let $t = |V(G_1 \cap G_2)|$. By Turán's Theorem,

$$\mathrm{ex}(t, K_r) - \mathrm{ex}(t-1, K_r) = e(T_{r-1}(t)) - e(T_{r-1}(t-1)) = t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, K_r) - \mathcal{C}(n, t-1, K_r) = -t + 1 + 2[\mathrm{ex}(t, K_r) - \mathrm{ex}(t-1, K_r)] = t + 1 - 2\left\lceil \frac{t}{r-1} \right\rceil. \quad (1)$$

For $r \geq 4$, $\mathcal{C}(n, t, K_r)$ is strictly increasing on t, so by Lemma 11,

$$\phi^*(2, n, K_r) = \mathcal{C}(n, n, K_r) = 2\mathrm{ex}(n, K_r) = e(G_1) + e(G_2)$$

and $G_1 = G_2 = T_{r-1}(n)$, as desired.

Now suppose $r = 3$. Equation (1) shows that $\mathcal{C}(n, t, K_r)$ is non-decreasing on t and $\mathcal{C}(n, t, K_r) > \mathcal{C}(n, t, K_r)$ for even t. By Lemma 11, we now have

$$\phi^*(2, n, K_r) = \max[\mathcal{C}(n, n, K_r), \mathcal{C}(n, n-1, K_r)] = 2\mathrm{ex}(n, K_r) = e(G_1) + e(G_2),$$

and either $G_1 = G_2 = T_{r-1}(n)$, or $G_2 = T_{r-1}(n-1)$ and $G_1 = G_2 + K_1$. If the latter case is true, then $e(G_3) \geq \mathrm{ex}(n, F) > e(G_2)$, and this contradiction completes the proof. $\square$

2.2 Proof of Theorem 1

If F is a graph of chromatic number $r + 1 \geq 3$, then Theorem B shows $\mathrm{ex}(n, F) = (1 + o(1))\mathrm{ex}(n, K_{r+1})$ as $n \to \infty$. In this section, we prove Theorem 1 following the same line of reasoning as in the proof of Theorem 2.

Proof of Theorem 1. By Lemma 9, it suffices to prove the theorem for $m = 3$. Let G_1, G_2, G_3 be induced double F-free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, F)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\phi^*(3, n, F) \geq 3\mathrm{ex}(n, F)$. Consequently, we must have $e(G_1) + e(G_2) \geq 2\mathrm{ex}(n, F)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\mathrm{ex}(n, F)$, it suffices to show that $G_1 = G_2$ are n-vertex F-free extremal graphs. In particular, we will use Lemma 11 to show that G_1, G_2 is an extremal configuration without containing a double F.

Let $t = |V(G_1 \cap G_2)|$. If $t < \sqrt{n}$, then

$$2\mathrm{ex}(n, F) \geq 2e(T_{r-1}(n)) \geq 2\left\lfloor \frac{n^2}{4} \right\rfloor \geq \binom{n}{2} + \binom{\sqrt{n}}{2} > \mathcal{C}(n, t, F).$$

Thus $t \geq \sqrt{n}$. But then for large enough t, any extremal t-vertex F-free graph contains a spanning complete $(r-1)$-partite subgraph $T_{r-1}(t)$, so we may add $\mathrm{ex}(t-1, F) - e(T_{r-1}(t-1))$ egdes to $T_{r-1}(t)$ and still avoid F as a subgraph. Hence for large enough t, we have $\mathrm{ex}(t, F) \geq \mathrm{ex}(t-1, F) - e(T_{r-1}(t-1)) + e(T_{r-1}(t))$, and so

$$\mathrm{ex}(t, F) - \mathrm{ex}(t-1, F) \geq e(T_{r-1}(t)) - e(T_{r-1}(t-1)) \geq t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, F) - \mathcal{C}(n, t-1, F) = -t + 1 + 2[\mathrm{ex}(t, F) - \mathrm{ex}(t-1, F)] \geq t + 1 - 2\left\lceil \frac{t}{r-1} \right\rceil. \quad (2)$$

For $r \geq 4$, $\mathcal{C}(n, t, F)$ is strictly increasing on t, so by Lemma 11,

$$\phi^*(2, n, F) = \mathcal{C}(n, n, F) = 2\mathrm{ex}(n, F) = e(G_1) + e(G_2),$$

and $G_1 = G_2$ are n-vertex F-free extremal graphs, as desired.

Now suppose $r = 3$. Equation (2) shows that $\mathcal{C}(n, t, F)$ is strictly increasing for even t and $\mathcal{C}(n, t, F) \geq \mathcal{C}(n, t-1, F)$ for odd t. By Lemma 11, we now have

$$\phi^*(2, n, F) = \max[\mathcal{C}(n, n, F), \mathcal{C}(n, n-1, F)] = 2\mathrm{ex}(n, F) = e(G_1) + e(G_2),$$

and either $G_1 = G_2$ are n-vertex extremal F-free graphs, or G_2 is an $(n-1)$-vertex extremal F-free graph and $G_1 = G_2 + K_1$. If the latter case is true, then $e(G_3) \geq \mathrm{ex}(n, F) > e(G_2)$, and this contradiction completes the proof. □

2.3 Proof of Theorem 3

According to Theorem 5, $\phi(n, n, P) = (1/2 + o(1))n^{5/2}$. So to prove Theorem 3, it suffices to show $\phi^*(n, n, P) = o(n^{5/2})$.

Let $G_1, G_2, \ldots, G_n$ be induced and double P-free and let $\epsilon > 0$. Let $d_i(v)$ be the degree of vertex v in the graph G_i. Let I be the set of pairs (i, v) such that $d_i(v) \geq \sqrt{n}/\epsilon + 1$. Since

$G_1, G_2, \dots, G_n$ do not contain a double P,

$$\sum_{(i,v)\in I} \binom{d_i(v)}{2} \leq n^3.$$

The maximum possible value of $\sum_{(i,v)\in I} d_i(v)$ subject to this constraint is when $d_i(v) = \sqrt{n}/\epsilon + 1$ for all (i,v), in which case $|I| \leq 2\epsilon^2 n^2$ and so

$$\sum_{(i,v)\in I} d_i(v) \leq (2\epsilon^2 n^2) \cdot \left(\frac{\sqrt{n}}{\epsilon} + 1\right) = 3\epsilon n^{5/2}$$

for large enough n. Remove all edges of G_i on vertex v such that $(i,v) \in I$. The total number of edges removed is at most $3\epsilon n^{5/2}$. Let $G_1', G_2', \dots, G_n'$ be the remaining subgraphs of $G_1, G_2, \dots, G_n$. If $e(G_i') \leq \epsilon n^{3/2}$, then remove all edges of G_i'. The number of edges removed in this process is at most $\epsilon n^{5/2}$. The remaining graphs $G_1'', G_2'', \dots, G_m''$ have each at least $\epsilon n^{3/2}$ edges and maximum degree at most $\sqrt{n}/\epsilon$. In particular, each G_i'' contains a matching M_i of size at least $e(G_i'')/2\Delta(G_i'') = \epsilon^2 n/2$. If $m \leq \epsilon n$, then

$$\sum_{i=1}^{n} e(G_i) \leq 4\epsilon n^{5/2} + \sum_{i=1}^{m} e(G_i'') \leq 4\epsilon n^{5/2} + \phi(m,n,P) \leq 5\epsilon n^{5/2}$$

by Theorem 5. If $m > \epsilon n$, then we apply Szemerédi's Regularity Lemma to find, for some $\delta > 0$ depending only on ϵ, a matching say M_1 in G_1'' such that for some pair of set $X, Y \subseteq V(M_1)$ of size at least δn each, there is a set E of at least $\delta^3 n^2$ edges $\{x,y\}$ of $G_1'' \cup G_2'' \cup \dots \cup G_m''$ such that $x \in X$ and $y \in Y$. Since G_1'' is induced, $E \subseteq E(G_1)$. In particular, there are at least $\delta^5 n^3/4$ copies of P in G_1. We can repeat the argument in the remaining graphs $G_i'' : i \in [2,m]$ to get say M_2 in G_2'' as above, which gives $\delta^5 n^3/4$ copies of P in G_2. If we do this $4\delta^{-5}$ times, then we have found n^3 copies of P in the first $4\delta^{-5}$ graphs, and two of them have the same edge-set. We conclude $\sum_{i=1}^{n} e(G_i) \leq 5\epsilon n^{5/2}$ if n is large enough. Since ϵ is arbitrary, we are done. $\square$

3 The non-induced double Turán problem

In this section, we prove our main theorems on $\phi(m,n,F)$.

3.1 Proof of Theorem 6

We need the following *saturation theorem*, which may be found in [6].

Proposition 12. *Let F be any non-empty graph with k vertices. For all $\epsilon > 0$, there exists $\delta > 0$ such that if G is any n-vertex graph with $\mathrm{ex}(n, F) + \epsilon n^2$ edges, then G contains δn^k copies of F.*

Proof of Theorem 6. Let $k = |V(F)|$ and let $\epsilon > 0$. Let $G_1, G_2, \ldots, G_m$ be double F-free. Reorder $G_1, G_2 \ldots, G_m$ so that $e(G_i) \geq \mathrm{ex}(n, F)+\epsilon n^2$ for $1 \leq i \leq \ell$ and $e(G_i) < \mathrm{ex}(n, F)+\epsilon n^2$ for $\ell < i \leq m$. Then each $G_i : 1 \leq i \leq \ell$ contains at least δn^k copies of F, by Proposition 12. On the other hand, there are at most n^k copies of F such that $F \subseteq G_i$ for some $i \in [m]$. Therefore $\ell \leq 1/\delta$ and

$$\begin{aligned}\sum_{i=1}^{m} e(G_i) &= \sum_{i=1}^{\ell} e(G_i) + \sum_{i=\ell+1}^{m} e(G_i) \\ &\leq \frac{1}{\delta}\binom{n}{2} + (m-\ell)\mathrm{ex}(n, F) + (m-\ell)\epsilon n^2 \\ &\leq m \cdot \mathrm{ex}(n, F) + \epsilon m n^2 + \frac{1}{\delta}\binom{n}{2}.\end{aligned}$$

Since F is not bipartite, $\mathrm{ex}(n, F) = \Theta(n^2)$ and so $\phi(m, n, F) \leq m \cdot \mathrm{ex}(n, F) + (\epsilon + 1/\delta m)mn^2$. Since ϵ was arbitrary and δ is a constant depending only on ϵ, we conclude $\phi(m, n, F) \leq (1 + o(1))m \cdot \mathrm{ex}(n, F)$ as $m \to \infty$. $\square$

Let F be a bipartite graph with $k \geq 2$ vertices and $j \geq 1$ edges. A strong version of a conjecture of Simonovits [16, 17] would suggest that for all $\epsilon > 0$, there exists $\delta > 0$ such that every n-vertex graph G with at least $p\binom{n}{2}(1+\epsilon)\mathrm{ex}(n, F)$ edges contains at least $\delta p^j n^k$ copies of F. For instance, this is known to be true whenever the asymptotic behavior of $\mathrm{ex}(n, F)$ is known, which includes the case $F = K_{2,t}$. If F is bipartite and $m \cdot \mathrm{ex}(n, F)/n^2 \to \infty$ as $m, n \to \infty$, then this conjecture with the same proof as above shows $\phi(m, n, F) = (1+o(1))m\cdot \mathrm{ex}(n, F)$. When F contains a cycle, then there exists $\alpha > 0$ such that $\mathrm{ex}(n, F) \geq n^{1+\alpha}$ for large enough n. Thus, we conclude that if F contains a cycle and the Simonovits conjecture is true for F, then $\phi(m, n, F) = (1 + o(1))m \cdot \mathrm{ex}(n, F)$ for $m \geq n$ and $n \to \infty$. In particular, this shows $\phi(m, n, K_{2,t}) = (1 + o(1))m \cdot \mathrm{ex}(n, F)$ for $m \geq n$ as $n \to \infty$.

We also present a weaker version of the above theorem that holds for all graphs F, which adopts a similar but simpler proof:

Proposition 13. *Let $n, k \geq 1$ and let F be a graph with k vertices. If $m \cdot \mathrm{ex}(n, F)/n^k \to \infty$, then*

$$\phi(m, n, F) = (1 + o(1))m \cdot \mathrm{ex}(n, F),$$

as $m \to \infty$.

Proof. Let $G_1, G_2, \ldots, G_m$ be double F-free. Write $e(G_i) = \mathrm{ex}(n,F) + t_i$ for each $i \in [m]$. Reorder $G_1, \ldots, G_m$ so that $e(G_i) > \mathrm{ex}(n,F)$ for $1 \leq i \leq \ell$ and $e(G_i) \leq \mathrm{ex}(n,F)$ for $\ell < i \leq m$. Then each $G_i : 1 < i \leq \ell$ contains at least t_i copies of F, and so there are $T = \sum_{i=1}^{\ell} t_i$ copies of F over all G_i. But then there are at most n^k copies of F such that $F \subseteq G_i$ for some $i \in [m]$, so $T \leq n^k = o(m) \cdot \mathrm{ex}(n,F)$. It now follows that

$$\sum_{i=1}^{m} e(G_i) \leq T + m \cdot \mathrm{ex}(n,F) = (1+o(1))m \cdot \mathrm{ex}(n,F).$$

□

3.2 Proof of Theorem 4

We first show that for all $m, n \geq 1$ and graph F,

$$\phi(m,n,F) \leq \binom{n}{2} + \mathrm{ex}(n,F)\binom{m}{2}.$$

Thereafter, we show that if there is an extremal F-free graph with maximum degree at most $n^{1/2}/m^2$, then the above bound is tight.

Proof of the upper bound. For $S \subseteq [m]$, let E_S denote the set of edges that are contained in exactly $\{G_i\}_{i \in S}$. Then

$$\sum_{i=1}^{m} e(G_i) = \sum_{S \subseteq [m]} |S||E_S| \leq \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq 2} (|S|-1)|E_S|.$$

Let $A_S = \bigcup_{T \supseteq S} E_T$, i.e., the set of edges that are contained in all G_i with $i \in S$. When $|S| \geq 2$, the edge set A_S is F-free and thus

$$|A_S| = \sum_{T \supseteq S} |E_T| \leq \mathrm{ex}(n,F).$$

Hence,

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S|-1)|E_S| = \sum_{\substack{S \subseteq [m], \\ |S|=2}} \sum_{T \supseteq S} \frac{(|T|-1)|E_T|}{\binom{|T|}{2}} \leq \sum_{\substack{S \subseteq [m], \\ |S|=2}} \sum_{T \supseteq S} |E_T| \leq \binom{m}{2}\mathrm{ex}(n,F),$$

as each $T \in [m]$ with $|T| \geq 2$ is counted $\binom{|T|}{2}$ times in total and $|T| - 1 \leq \binom{|T|}{2}$. This proves the upper bound.

Proof of the lower bound. We need to show there exists a construction such that the graph with edge set E_S is an extremal F-free graph, for all $S \subseteq [m]$ of size 2. Let $M = \binom{m}{2}$ and $H_1, \ldots, H_M$ be copies of an extremal F-free graph on n vertices such that H_i with maximum degree $\triangle \leq n^{1/2}/m^2$ for all $i \in [m]$. It suffices to show that we can embed each H_i onto $[n]$ such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each H_i and iteratively decrease the number of intersecting edges. Define a (u, v, i)-*swap* by swapping the embedding of vertex u and v of H_i, i.e. replacing each edge $\{u, w\} \in E(H_i)$ with the edge $\{u, w\}$ and each edge $\{v, w\} \in E(H_i)$ with the edge $\{v, w\}$. This preserves the type of isomorphism of H_i. Given a vertex v, let $N(v) = N_{H_1}(v) \cup \cdots \cup N_{H_M}(v)$. Suppose there exists an intersecting edge $\{u, w\} \in E(H_i) \cap E(H_j)$. Since $|N(u)| \leq M \cdot \triangle \leq n^{1/2}/2$, $|N(u) \cup N(N(u))| \leq \triangle + \triangle(\triangle - 1) \leq n/4$, so there exists a vertex $v \notin N(u) \cup N(N(u))$. Since $N(u) \cap N(v) = \emptyset$, performing a (u, v, i)-swap reduces the number of intersecting edges. The result now follows from iterating this process. □

3.3 Proof of Theorem 5

Let $G_1, \ldots, G_m$ be graphs on $[n]$ not containing a double P. We first show the following claims:

Claim 1. $\phi(m, n, P) \leq mn(1 + \sqrt{4n^2/m + 1})/4$.

Proof. Since there is no double P in $G_1, G_2, \ldots, G_m$,

$$\sum_{i=1}^{m} \#\{P \subseteq G_i\} \leq \#\{P \subseteq K_n\}.$$

For all G_i, each vertex v in G_i along with two of its neighbors form one unique P, so

$$\#\{P \subseteq G_i\} = \sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2}.$$

By Jensen's inequality,

$$\sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2} \geq n \binom{\sum_{v \in V(G_i)} d_{G_i}(v)/n}{2} = n \binom{2e(G_i)/n}{2} \geq \frac{2(e(G_i))^2}{n} - e(G_i).$$

On the other hand, since each three vertices in G can form at most three P's,

$$\#\{P \subseteq K_n\} \leq 3\binom{n}{3} \leq \frac{n^3}{2}.$$

Combining the above inequalities yields and using Jensen's inequality once more yields

$$\frac{2m}{n}\left(\frac{1}{m}\sum_{i=1}^{m}e(G_i)\right)^2-\sum_{i=1}^{m}e(G_i)\overset{\text{Jensen}}{\leq}\sum_{i=1}^{m}\frac{2(e(G_i))^2}{n}-e(G_i)\leq\frac{n^3}{2}.$$

Solving the quadratic equation gives

$$\sum_{i=1}^{m}e(G_i)\leq mn\cdot\frac{1+\sqrt{4n^2/m+1}}{4}.$$

This proves the claim. □

Claim 2. *$\phi(m,n,P)\leq(mn^{3/2}+n^2)/2$.*

Proof. For each vertex $u\in[n]$, define H_u as the $m\times n$ bipartite graph with edge set $E(H_u):=\{\{v,i\}:\{u,v\}\in E(G_i)\}$. If H_u contains a quadrilateral $\{v,i\},\{v,j\},\{w,i\},\{w,j\}$, then $\{u,v\},\{u,w\}$ form a double P in $G_i\cap G_j$, contradiction. Thus we conclude that H_u is quadrilateral-free, and therefore $e(H_u)\leq m\sqrt{n}+n$, by the Kővari-Sós-Turán Theorem [9]. It now follows that

$$\sum_{i=1}^{m}e(G_i)=\frac{1}{2}\sum_{u\in V(G)}e(H_u)\leq\frac{1}{2}(mn^{3/2}+n^2).$$

This proves the claim. □

Claim 2 along with the construction of one complete graph now yield the desired bounds for $m\leq\sqrt{n}$. On the other hand, Claim 1 along with the construction of m extremal graphs for P yield the desired bounds for $m=\Theta(n^2)$. The bound for the case for $m/n^2\to\infty$ follows from Proposition 13.

Thus it remains to show that $\phi(m,n,P)\geq(1/2+o(1))mn^{3/2}$ for $\sqrt{n}<m\leq n$ and $\phi(m,n,P)\geq(1/2+o(1))\sqrt{m}n^2$ for $n<m\leq n^2$.

We first prove the case $\sqrt{n}\leq m\leq n$. Suppose $G_1,G_2,\dots,G_n$ are graphs on $[n]$ containing no double P and $\sum_{i=1}^{n}e(G_i)\geq(1/2+o(1))n^{5/2}$, with $e(G_1)\geq e(G_2)\geq\cdots\geq e(G_n)$. Then $G_1,G_2,\dots,G_m$ are graphs with no double P and $\sum_{i=1}^{m}e(G_i)\geq(1/2+o(1))mn^{3/2}$. Hence, it suffices to prove the case for $m=n$.

Consider a finite projective plane with n points and n lines, with prime q chosen so that $n=(1+o(1))(q^2+q+1)$ as $q\to\infty$. Let $S_1,\dots,S_n\subseteq[n]$ be the n lines of the projective plane. Note that each line S_i contains $q+1$ points, and the intersection of any two distinct lines S_i,S_j contains $|S_i\cap S_j|=1$ point.

Define $G_1, \dots, G_n$ to be graphs on $[n]$, each with edge set

$$E(G_i) \coloneqq \{\{j,k\} \subseteq [n] : j \neq k,\, j+k \in S_i \mod n\}.$$

Note that the intersection of distinct G_i, G_j is P free: since $|S_i \cap S_j| = 1$, if $\{a,b\}, \{a,c\} \in E(G_i) \cap E(G_j)$, then $a+b = a+c$ so $b = c$.

We now count the number of edges in $G_1, \dots, G_n$. Since $|S_i| = q+1$, for each point $j \in [n]$, there are $q+1$ choices for $k \in [n]$ such that $j+k \in S_i$. But then we have to avoid counting the same edge twice and loops, so the number of edges in G_i is

$$e(G_i) = \frac{n(q+1) - \#\text{loops counted for } G_i}{2}.$$

If $j \in [n]$ is even, then $k = j/2$ is the unique number in $[n]$ such that $k+k = j \mod n$. If $j \in [n]$ is odd, then $k = (n+j)/2$ is the unique number in $[n]$ such that $k+k = j \mod n$, as n is even. Hence, for each $j \in S_i$, there exists a unique $k \in [n]$ such that $k+k = j \mod n$, and thus

$$\#\text{loops counted for } G_i = |S_i| = q+1.$$

Since $q+1 = (1+o(1))n^{1/2}$, the number of edges in $G_1, \dots, G_n$ is

$$\sum_{i=1}^{n} e(G_i) = n \cdot \frac{n(q+1) - (q+1)}{2} = \left(\frac{1}{2} + o(1)\right) n^{5/2},$$

as $n \to \infty$.

The case for $n < m \leq n^2$ is similar. Consider the finite projective plane P with n points defined above. Since $|S_i| = q+1 > \sqrt{n} \geq n^2/m$, we may further place a smaller projective plane P_i with n^2/m points inside each line S_i. Since each line of P_i has size roughly $n/\sqrt{m}$, each S_i contains roughly m/n lines, and thus we now have m small lines in total. Define G_i' on each small line the same way we defined G_i on S_i. Following the same line of calculations above, the construction of $G_1', \dots, G_m'$ now gives $\sum_{i=1}^{m} e(G_i) = (1/2 + o(1))\sqrt{m}n^2$, provided $m \leq n^2$. This completes the proof. □

3.4 Proof of Theorem 7

We now prove Theorem 7. Notice that we trivially have $f(m,n,r) \leq \phi(m,n,K_r)$, so it suffices to show the reverse inequality. That is, we need to show that there exists a blowup construction meeting the desired bound.

Let $G_1, G_2, \dots, G_m$ be graphs on $[n]$ with no double K_r and $\sum_{i=1}^{m} e(G_i) = \phi(m,n,K_r)$.

Observe that any pair $\{i, j\} \subseteq [n]$ must be in some G_i, otherwise, we may add it to G_1 without creating a double K_r.

We call vertices v, v' *clones* if for all $u \in [n] \backslash \{v, v'\}$ and $i \in [m]$, the edge $\{u, v\} \in E(G_i)$ if and only if $\{u, v'\} \in E(G_i)$. Furthermore, we call $\{v, v'\}$ a *light edge* if $\{v, v'\}$ is in exactly one graph G_i.

We now apply Algorithm 1 to $G_1, G_2, \ldots, G_m$.

Algorithm 1 symmetrization algorithm

```
while ∃ a light edge whose endpoints are not clones do
    among all vertices incident to such an edge, select a vertex v with maximum degree
    B_v ← collection of vertices sending a light edge to v that are not clones of v
    while B_v ≠ ∅ do
        pick u ∈ B_v
        j ← colour of the light edge from u to v
        for 1 ≤ i ≤ m do
            if i ≠ j then;
                N_{G_i}(u) ← N_{G_i}(v)
            else if i = j then
                N_{G_i}(u) ← (N_{G_i}(v) \ {u}) ∪ {v}
            end if
        end for
    end while
end while
```

Claim 3. *Algorithm 1 terminates.*

Proof. Notice that at the end of the 'while $B_v \neq \emptyset$' loop, every vertex sending a light edge to v is a clone of v. This implies v along with the set L_v of vertices receiving light edges from v induce a clique of size at least two in some G_i, and an empty graph in every other graph G_j with $j \neq i$. Moreover, any vertex $w \notin L_v$ sends edges to either all or none of the vertices in L_v, and if w is incident to L_v, then w sends edges to L_v in at least two graphs. It now follows that no light edge incident with a vertex in L_v will be picked again in an iteration of the out most while loop. Thus the algorithm can run through at most $n/2$ such iterations, and so it terminates. $\square$

Claim 4. *The resulting graphs $G'_1, G'_2, \ldots, G'_m$ do not contain a double K_r and $\sum_{i=1}^{m} e(G'_i) = \phi(m, n, K_r)$.*

Proof. Note that we replace u by a clone of v in the for loop of Algorithm 1. Since $\{u, v\}$ remains a light edge in this step, u and v cannot both belong to a double K_r in the modified graphs. Furthermore, any double K_r containing u after the for loop arises from a double K_r containing v prior to the for loop. But then $G_1, G_2, \ldots, G_m$ contained no double K_r to begin with, so $G'_1, G'_2, \ldots, G'_m$ do not contain a double K_r.

We now show that the algorithm does not reduce the number of edges. By our choice of v, we know $d(v) \geq d(u)$ for all $u \in B_v$ prior to the for loop. Hence, replacing u with a clone of v does not decrease the number of edge over a complete iteration of the inner while loop. Therefore, $\sum_{i=1}^{m} e(G'_i) = \phi(m, n, K_r)$. □

Hence, the algorithm results in graphs $G'_1, G'_2, \ldots, G'_m$ with $\phi(m, n, K_r)$ edges and the additional property that light edges come in 'clone cliques.' We may thus partition the vertex set $[n]$ into k disjoint sets $V_1, V_2, \ldots, V_k$, such that each V_i induces a clique of light edges from the same graph. Moreover, for distinct $i, j \in [k]$, define S_{ij} to be the set of all edges between V_i and V_j, and note that any edge in S_{ij} appears in at least two modified graphs. The sets S_{ij} now yield a k-blowup. Notice that if the pattern of the k-blowup contains a double K_r, then the original graphs $G_1, G_2, \ldots, G_m$ must have contained a double K_r as well, contradiction. Thus the k-blowup is double K_r-free.

It remains to show that $k < R_M(K_r)$. For each edge $\{i, j\} \subseteq [k]$ in the pattern of the k-blowup, we assign an arbitrary distinct pair $\{a, b\} \subseteq L_{ij} \subseteq [m]$ to $\{i, j\}$. If $k \geq R_M(K_r)$, then there exists K_r in the pattern of the k-blowup colored by some distinct pair $\{a, b\} \subseteq [m]$. But then this implies the pattern of the k-blowup contains a double K_r, contradiction. This completes the proof. □

3.5 Proof of Theorem 8

It is not hard to see that $\phi(2, n, K_3) = \binom{n}{2} + \lfloor n^2/4 \rfloor$: if G_1, G_2 is double triangle-free, then we have

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_1 \cap G_2) \leq \binom{n}{2} + \mathrm{ex}(n, K_3)$$

and so $\phi(2, n, K_3) \leq \binom{n}{2} + \lfloor n^2/4 \rfloor$. Taking $G_1 = K_n$ and $G_2 = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ meets this bounds. The main result of this section is to show for all $n \geq 1$,

$$\phi(3, n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Let G_1, G_2, G_3 be double triangle-free. Define $H_k \subseteq G$ to be the graph with edges contained in at least k of the G_i's and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \le n^2/2$. Notice H_2 must not contain any triangles with two edges in H_3, so

$$e(H_2) + e(H_3) \le \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \ne v, N_{H_3}(u) \cap N_{H_3}(v) \ne \emptyset\}|.$$

Let H_3' be the graph with the same vertex set as H_3 and edge set $\{\{u, v\} : u \ne v, N_{H_3}(u) \cap N_{H_3}(v) \ne \emptyset\}$. It suffices to show that $n/2 \ge e(H_3) - e(H_3')$.

Let $d_1 \ge d_2 \ge \cdots \ge d_n$ and $f_1 \ge f_2 \ge \cdots \ge f_n$ each be the degree sequence of H_3 and H_3', respectively. We show that $f_i \ge d_i - 1$ for all i. Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i. Let $S_i = |N_{H_3}(v_1) \cup \cdots \cup N_{H_3}(v_i)|$. Since

$$\sum_{u \in S_i} d_{H_3}(u) \ge d_1 + \cdots + d_i,$$

we have that $|S_i| \ge i$. But then $S_i \backslash \{u_1, \ldots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H_3'}(u) \ge d_i - 1$. Hence, $f_i \ge d_i - 1$ for all i, which yields

$$e(H_3') = \frac{1}{2} \sum_{i=1}^{n} f_i \ge \frac{1}{2} \sum_{i=1}^{n} (d_i - 1) = e(H_3) - \frac{n}{2}.$$

This proves Theorem 8. □

4 Concluding Remarks

- For Theorem 1, we may not be able to achieve the same result with smaller n. For example, consider F to be the bowtie graph, i.e. the 5-vertex graph with two triangles sharing a vertex. The n-vertex extremal graph for F is given by $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ plus an edge when $n \ge 5$, otherwise it is the complete graph. For $n = 5$, the construction $G_1 = K_4, G_2 = K_5$ then shows that $\phi^*(2, 5, F) > 2 \cdot \mathrm{ex}(5, F)$. Fortunately, for non-bipartite F with $|V(F)| = k$, it is not hard to show $n \ge k^2$ is sufficient to avoid this issue.

- We note that Theorem 7 may be generalized to any family of non-bipartite graphs up to asymptotic error via Szemerédi's Regularity lemma

- One could ask for the analogous results for hypergraphs. That is, if F is an r-uniform hypergraph, let $\phi(m, n, F)$ be the maximum number of edges over m double F-free r-uniform hypergraphs on $[n]$. Again, we have $\phi(m, n, F) \ge \binom{n}{r} + (m - 1) \cdot \mathrm{ex}(n, F)$.

Another direction of generalization is to relax the constraint to no copies of F contained in the intersection of k of the graphs $G_1, G_2, \ldots, G_m$. Many of the theorems and proofs also hold in this case. For instance, the proof of Theorem 4 applies for this generalization by merely changing the numbers.

References

[1] N. Alon and C. Shikhelman. Many T copies in H-free graphs. *J. Combin. Theory Ser. B*, 121:146–172, 2016. ISSN 0095-8956,1096-0902. doi: 10.1016/j.jctb.2016.03.004. URL https://doi.org/10.1016/j.jctb.2016.03.004.

[2] D. Conlon and A. Ferber. Lower bounds for multicolor Ramsey numbers. *Adv. Math.*, 378:Paper No. 107528, 5, 2021. ISSN 0001-8708,1090-2082. doi: 10.1016/j.aim.2020.107528. URL https://doi.org/10.1016/j.aim.2020.107528.

[3] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946. ISSN 0002-9904. doi: 10.1090/S0002-9904-1946-08715-7. URL https://doi.org/10.1090/S0002-9904-1946-08715-7.

[4] P. Erdős. Problems and results in combinatorial analysis. In *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977)*, volume No. XIX of *Congress. Numer.*, pages 3–12. Utilitas Math., Winnipeg, MB, 1977. ISBN 0-919628-19-2.

[5] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar*, 1:51–57, 1966. ISSN 0081-6906.

[6] P. Erdős and M. Simonovits. Supersaturated graphs and hypergraphs. *Combinatorica*, 3(2):181–192, 1983. ISSN 0209-9683. doi: 10.1007/BF02579292. URL https://doi.org/10.1007/BF02579292.

[7] Z. Füredi. Hypergraphs in which all disjoint pairs have distinct unions. *Combinatorica*, 4(2-3):161–168, 1984. ISSN 0209-9683. doi: 10.1007/BF02579216. URL https://doi.org/10.1007/BF02579216.

[8] Z. Füredi and M. Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdős centennial*, volume 25 of *Bolyai Soc. Math. Stud.*, pages 169–264. János Bolyai Math. Soc., Budapest, 2013. doi: 10.1007/978-3-642-39286-3_7. URL https://doi.org/10.1007/978-3-642-39286-3_7.

[9] T. Kővari, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954. doi: 10.4064/cm-3-1-50-57. URL https://doi.org/10.4064/cm-3-1-50-57.

[10] A. Kostochka, D. Mubayi, and J. Verstraëte. Turán problems and shadows III: expansions of graphs. *SIAM J. Discrete Math.*, 29(2):868–876, 2015. ISSN 0895-4801,1095-7146. doi: 10.1137/140977138. URL https://doi.org/10.1137/140977138.

[11] H. Lefmann. A note on Ramsey numbers. *Studia Sci. Math. Hungar.*, 22(1-4):445–446, 1987. ISSN 0081-6906,1588-2896.

[12] D. Mubayi and S. Mukherjee. Triangles in graphs without bipartite suspensions. *Discrete Math.*, 346(6):Paper No. 113355, 19, 2023. ISSN 0012-365X,1872-681X. doi: 10.1016/j.disc.2023.113355. URL https://doi.org/10.1016/j.disc.2023.113355.

[13] D. Mubayi and J. Verstraëte. A hypergraph extension of the bipartite Turán problem. *J. Combin. Theory Ser. A*, 106(2):237–253, 2004. ISSN 0097-3165,1096-0899. doi: 10.1016/j.jcta.2004.02.002. URL https://doi.org/10.1016/j.jcta.2004.02.002.

[14] D. Mubayi and J. Verstraëte. A survey of Turán problems for expansions. In *Recent trends in combinatorics*, volume 159 of *IMA Vol. Math. Appl.*, pages 117–143. Springer, [Cham], 2016. ISBN 978-3-319-24296-5; 978-3-319-24298-9. doi: 10.1007/978-3-319-24298-9_5. URL https://doi.org/10.1007/978-3-319-24298-9_5.

[15] O. Pikhurko and J. Verstraëte. The maximum size of hypergraphs without generalized 4-cycles. *J. Combin. Theory Ser. A*, 116(3):637–649, 2009. ISSN 0097-3165,1096-0899. doi: 10.1016/j.jcta.2008.09.002. URL https://doi.org/10.1016/j.jcta.2008.09.002.

[16] A. Sidorenko. A correlation inequality for bipartite graphs. *Graphs Combin.*, 9(2):201–204, 1993. ISSN 0911-0119. doi: 10.1007/BF02988307. URL https://doi.org/10.1007/BF02988307.

[17] M. Simonovits. Extremal graph problems, degenerate extremal problems, and supersaturated graphs. In *Progress in graph theory (Waterloo, Ont., 1982)*, pages 419–437. Academic Press, Toronto, ON, 1984.

[18] E. Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978.

[19] P. Turán. Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941.