Double Turán Problem

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1 Introduction

1.1 Definitions and Notation

Let $G = (V, E)$ be a graph. Let $V(G) = V$ denote the vertex set and $E(G) = E$ denote the edge set of G. We note by $v(G) = |V|$ the number of vertices and $e(G) = |E|$ the number of edges in G. For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v.

Given $G_1, \ldots, G_m$ subgraphs of G, we denote $G_{i_1,\ldots,i_k}$ as the subgraph of G with edge set $E(G_{i_1,\ldots,i_k}) = \bigcap_{\alpha=1}^{k} E(G_\alpha)$.

In this thesis, we reserve n to denote the number of vertices in a graph. Given a graph F, we denote $\mathrm{ex}(n, F)$ to be the extremal number for F on a graph with n vertices, i.e. the maximum number of edges in a n-vertex graph that does not contain F as a subgraph.

1.2 Problem Statement

Given graph G with n vertices, let $G_1, \ldots, G_m$ be subgraphs of G. Let F be a graph with at least one edge. Our goal is to determine the maximum sum of the number of edges over all G_i's, i.e. $\sum_{i=1}^{m} e(G_i)$, with the constraint of $E(G_i) \cap E(G_j)$ not including some graph F for all distinct i, j.

In this thesis, we mainly put our attention on the case where F is non-bipartite. We will first consider the case where $G_1, \ldots, G_m$ are induced subgraphs, and then shift our focus to the general case. At the end, we will discuss the case where F is bipartite.

2 Induced Case

In this section, we assume that $G_1, \ldots, G_m$ are induced subgraphs of G. Given graph H, let $\mathcal{T}(H)$ be the graph with an additional vertex connecting to all vertices in H.

We first show a simplier case where F is a triangle.

2.1 Triangle F

Theorem 2.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j. Then*

$$\sum_{i=1}^{n} e(G_i) \leq n \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = \cdots = G_n = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

Lemma 2.1.1. *Suppose $E(G_1) \cap E(G_2)$ does not include K_3. Then*

$$e(G_1) + e(G_2) \leq 2 \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$, unless n is odd and $G_1 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and $G_2 = \mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$.

Proof. Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both G_1 and G_2. Let $A = V(G_1) \backslash C$, and let $B = V(G_2) \backslash C$. For simplicity, put $a = |A|$, $b = |B|$, and $c = |C|$. We may assume that $a + b + c = n$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to a, b, c. Since G_1, G_2 are induced graphs, we have $\{u, v\} \in E(G_1)$ if and only if $\{u, v\} \in E(G_2)$, for $u, v \in C$. This implies the subgraph of G_1 induced by C is identical to the subgraph of G_2 induced by C. In other words, $E(G_1[C]) = E(G_2[C]) = E(G_i) \cap E(G_j)$, which is triangle-free. By Mantel's Theorem, $e(G_1[C]) \leq \left\lfloor \frac{c^2}{4} \right\rfloor$, with equality if and only if $G_1[C] = K_{\lceil \frac{c}{2} \rceil, \lfloor \frac{c}{2} \rfloor}$. Hence, we may write

$$\begin{aligned} e(G_1) + e(G_2) &\leq \binom{|V(G_1)|}{2} + \binom{|V(G_2)|}{2} - 2 \left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor \right] \\ &= \binom{a+c}{2} + \binom{b+c}{2} - 2 \left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor \right]. \end{aligned} \tag{2.1}$$

Define $f(a,b,c)$ as the function on the right-hand-side of (1). We show that $f(a,b,c)$ attains its maximum at $a=b=0$ and $c=n$. Note that

$$\begin{aligned}
f(a,b-2,c+2)-f(a,b,c) &= \binom{a+c+2}{2}-\binom{a+c}{2} \\
&\quad -2\left[\binom{c+2}{2}-\binom{c}{2}-\left\lfloor\frac{(c+2)^2}{4}\right\rfloor+\left\lfloor\frac{c^2}{4}\right\rfloor\right] \\
&= 2(a+c)+1-2[2c+1-(c+1)] \\
&= 2a+1>0.
\end{aligned}$$

By symmetry, $f(a-2,b,c+2)>f(a,b,c)$, and thus f attains its maximum when c is $n-1$ or n, that is, $a+b\le 1$. Equation (1) now yields,

$$e(G_1)+e(G_2)\le f(a,b,c)\le 2\left\lfloor\frac{n^2}{4}\right\rfloor.$$

Assume that $a=0$. When $c=n$, the equality holds only if $G_1=G_2=K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$. If $c=n-1$, then the equality holds only if n is odd and $G_1=G[C]=K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor}$ and G_2 is G_1 with all vertices connected with the only remaining vertex, that is, $G_2=\mathcal{T}(K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor})$. □

We now prove Theorem 2.1.

Proof of Theorem 3.1. We may assume that $n>1$. Put $G_{n+i}=G_i$. By Lemma 3.2,

$$\sum_{i=1}^{n}e(G_i)=\frac{1}{2}\sum_{i=1}^{n}(e(G_i)+e(G_{i+1}))\le\frac{1}{2}\sum_{i=1}^{n}2\left\lfloor\frac{n^2}{4}\right\rfloor=n\left\lfloor\frac{n^2}{4}\right\rfloor.$$

Suppose the equality holds. By Lemma 3.2, we are done if n is even. Suppose n is odd and $G_i=\mathcal{T}(K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor})$ for some i. By Lemma 3.2, one of G_i and G_{i+1} is $K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor}$ and the other is $\mathcal{T}(K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor})$, for all i. Hence, $G_{i+1}=K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor}$, $G_{i+2}=\mathcal{T}(K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor}),\ldots$ and the alternation proceeds. But then $G_{n+i}=G_i=K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor}$ as n is odd, and this contradiction completes the proof. □

2.2 Non-bipartite F

Theorem 2.2. *Let F be $(r+1)$-colorable, with $r\ge 2$. Suppose that $E(G_i)\cap E(G_j)$ is F-free for distinct i,j. For large enough n,*

$$\sum_{i=1}^{n}e(G_i)\le n\cdot\mathrm{ex}(n,F),$$

with equality if and only if $G_1=G_2=\cdots=G_n$ are n-vertex extremal graphs for F.

By the same argument as in Theorem 2.1, it suffices to prove the following lemma:

Lemma 2.2.1. *Let F be $(r+1)$-colorable, with $r \geq 2$. Suppose $E(G_1) \cap E(G_2)$ does not include F. For large enough n,*

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n, F),$$

with equality if and only if $G_1 = G_2$ are n-vertex extremal graphs for F, unless n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2 = \mathcal{T}(G_1)$.

Proof. TODO: fix this proof. Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both G_1 and G_2. Let $A = V(G_1) \backslash C$, and let $B = V(G_2) \backslash C$. For simplicity, put $a = |A|$, $b = |B|$, $c = |C|$, and $r = \chi(F)$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to a, b, c. Since G_1, G_2 are induced graphs, we have $E(G_1[C]) = E(G_2[C]) = E(G[C]) = E(G_i) \cap E(G_j)$, which is F-free. Hence, we may write

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \mathrm{ex}(c, F)\right]. \tag{2.2}$$

Define $f(a, b, c)$ as the function on the right-hand-side. We show that $f(a, b, c)$ attains its maximum at $a = b = 0$ and $c = n$. By a theorem of Simonovits, for large enough c, $\mathrm{ex}(c, F) = \mathrm{ex}(c, K_{r+1}) + \mathrm{ex}(c, \tilde{F})$, where $\tilde{F}$ is the family of residue subgraphs of F after F is embedded into $T_r(c)$. Hence, we may write

$$\begin{aligned} f(a, b-2, c+2) - f(a, b, c) &= \binom{a+c+2}{2} - \binom{a+c}{2} \\ &\quad - 2\left[\binom{c+2}{2} - \binom{c}{2} - \mathrm{ex}(c+2, F) + \mathrm{ex}(c, F)\right] \\ &\geq 2a - 2c - 1 + 2[\mathrm{ex}(c+2, K_{r+1}) - \mathrm{ex}(c, K_{r+1})] > 0, \end{aligned}$$

as shown in the proof of Lemma 3.4. By symmetry, we also have $f(a-2, b, c+2) > f(a, b, c)$. Thus, f attains its maximum when c is $n-1$ or n. Equation (5) now yields,

$$e(G_1) + e(G_2) \leq \max\left[2 \cdot \mathrm{ex}(n, F), 2 \cdot \mathrm{ex}(n-1, F) + n - 1\right].$$

Assume that $a = 0$. Since

$$2 \cdot \mathrm{ex}(n, F) - [2 \cdot \mathrm{ex}(n-1, F) + n - 1] \geq 2[\mathrm{ex}(n, K_{r+1}) - \mathrm{ex}(n-1, K_{r+1})] \tag{2.3}$$
$$- n + 1 \tag{2.4}$$
$$= 2\left(n - \left\lceil \frac{n}{r} \right\rceil\right) - n + 1 \tag{2.5}$$
$$\geq n + 1 - 2\left\lceil \frac{n}{2} \right\rceil \geq 0,$$

we have

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n, F). \tag{2.6}$$

If $c = n$, the equality for (9) holds only if $G_1 = G_2$ are n-vertex extramal graphs for F. Suppose $c = n-1$ and the equality holds. Observe that equation (6) is equal to zero only when $r = 2$ and n is odd. Hence, if $c = n-1$, the equality for (9) could only be achieved when $r = 2$, n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2 = \mathcal{T}(G_1)$. $\square$

3 General Case

We now relax the assumption that $G_1, \ldots, G_m$ are induced subgraphs. The trivial construction of putting $G_1 = K_n$ and $G_2, \ldots, G_m$ to be extremal graphs for F yields the lower bound

$$\sum_{i=1}^{m} e(G_i) = \binom{n}{2} + (m-1) \cdot \mathrm{ex}(n, F). \tag{3.1}$$

Thus in this section we examine whether this bound is tight. An asymptotic result on m is the following:

Theorem 3.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include F for distinct i, j. Then for large enough n,*

$$\sum_{i=1}^{m} e(G_i) \le m(1 + o_m(1))\mathrm{ex}(n, F),$$

as $m \to \infty$.

Proof. □

3.1 Triangle F

Consider F to be a triangle. Simply counting the number of triangles in each G_i shows the following:

Theorem 3.2. *For any $\epsilon > 0$, if $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j, then*

$$\sum_{i=1}^{m} e(G_i) < m(1 + \epsilon)\frac{n^2}{4} + O(n^2)$$

Claim 3.2.1. *There are less than $\frac{2}{\epsilon}$ number of G_i's with $e(G_i) \ge (1 + \epsilon)\frac{n^2}{4}$.*

Proof. Suppose $e(G_i) \ge (1 + \epsilon)\frac{n^2}{4}$ for $1 \le i \le k$. Let $K_3(G)$ denote the number of triangles in graph G. By the Moon-Moser inequality,

$$K_3(G_i) \ge \epsilon(1 + \epsilon)\frac{n^3}{12}.$$

Since there are no overlapping traingles from different G_i's,

$$\binom{n}{3} \ge \sum_{i=1}^{k} K_3(G_i) \ge \frac{\epsilon(1 + \epsilon)}{12} kn^3.$$

Rearranging yields $k < \frac{2}{\epsilon}$. □

Proof of Theorem 3.2. By the claim,

$$\sum_{i=1}^{m} e(G_i) < \frac{2}{\epsilon}\binom{n}{2} + \left(m - \frac{2}{\epsilon}\right)(1+\epsilon)\frac{n^2}{4} \leq m(1+\epsilon)\frac{n^2}{4} + O(n^2).$$

□

It can be easily shown that the bound in Theorem 3.2 is tight when $m = 2$, as

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_{1,2}) \leq \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor.$$

This is also true for $m = 3$:

Proposition 3.3. *For any $\epsilon > 0$, if $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j, then*

$$\sum_{i=1}^{3} e(G_i) \leq \binom{n}{2} + \frac{n^2}{2}.$$

Proof. □

3.2 Bipartite F