Double Turán Problem

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1 Introduction

1.1 Definitions and Notation

Let $G = (V, E)$ be a graph. Let $V(G) = V$ denote the vertex set and $E(G) = E$ denote the edge set of G. We note by $v(G) = |V|$ the number of vertices and $e(G) = |E|$ the number of edges in G. For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v.

Given $G_1, \ldots, G_m$ subgraphs of G, we denote $G_{i_1,\ldots,i_k}$ as the subgraph of G with edge set $E(G_{i_1,\ldots,i_k}) = \bigcap_{\alpha=1}^{k} E(G_\alpha)$.

In this thesis, we reserve n to denote the number of vertices in a graph. Given a graph F, we denote $\mathrm{ex}(n, F)$ to be the extremal number for F on a graph with n vertices, i.e. the maximum number of edges in a n-vertex graph that does not contain F as a subgraph. We define the Turán density of F as

$$\pi(F) := \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{2}}.$$

1.2 Problem Statement

Given graph G with n vertices, let $G_1, \ldots, G_m$ be subgraphs of G. Let F be a graph with at least one edge. Our goal is to determine the maximum sum of the number of edges over all G_i's, i.e. $\sum_{i=1}^{m} e(G_i)$, with the constraint of $E(G_i) \cap E(G_j)$ not including some graph F for all distinct i, j.

In this thesis, we mainly put our attention on the case where F is non-bipartite. We will first consider the case where $G_1, \ldots, G_m$ are induced subgraphs, and then shift our focus to the general case. At the end, we will discuss the case where F is bipartite.

2 Induced Case

In this section, we assume that $G_1, \ldots, G_m$ are induced subgraphs of G. Given graph H, let $\mathcal{T}(H)$ be the graph with an additional vertex connecting to all vertices in H.

We first show a simplier case where F is a triangle.

2.1 Triangle F

Theorem 2.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j. Then*

$$\sum_{i=1}^{n} e(G_i) \leq n \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = \cdots = G_n = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

Lemma 2.1.1. *Suppose $E(G_1) \cap E(G_2)$ does not include K_3. Then*

$$e(G_1) + e(G_2) \leq 2 \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$, unless n is odd and $G_1 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and $G_2 = \mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$.

Proof. Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both G_1 and G_2. Let $A = V(G_1) \backslash C$, and let $B = V(G_2) \backslash C$. For simplicity, put $a = |A|$, $b = |B|$, and $c = |C|$. We may assume that $a + b + c = n$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to a, b, c. Since G_1, G_2 are induced graphs, we have $\{u, v\} \in E(G_1)$ if and only if $\{u, v\} \in E(G_2)$, for $u, v \in C$. This implies the subgraph of G_1 induced by C is identical to the subgraph of G_2 induced by C. In other words, $E(G_1[C]) = E(G_2[C]) = E(G_i) \cap E(G_j)$, which is triangle-free. By Mantel's Theorem, $e(G_1[C]) \leq \left\lfloor \frac{c^2}{4} \right\rfloor$, with equality if and only if $G_1[C] = K_{\lceil \frac{c}{2} \rceil, \lfloor \frac{c}{2} \rfloor}$. Hence, we may write

$$\begin{aligned} e(G_1) + e(G_2) &\leq \binom{|V(G_1)|}{2} + \binom{|V(G_2)|}{2} - 2\left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor\right] \\ &= \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor\right]. \end{aligned} \tag{2.1}$$

Define $f(a,b,c)$ as the function on the right-hand-side of (1). We show that $f(a,b,c)$ attains its maximum at $a=b=0$ and $c=n$. Note that

$$\begin{aligned}
f(a,b-2,c+2)-f(a,b,c) &= \binom{a+c+2}{2}-\binom{a+c}{2} \\
&\quad -2\left[\binom{c+2}{2}-\binom{c}{2}-\left\lfloor\frac{(c+2)^2}{4}\right\rfloor+\left\lfloor\frac{c^2}{4}\right\rfloor\right] \\
&= 2(a+c)+1-2[2c+1-(c+1)] \\
&= 2a+1>0.
\end{aligned}$$

By symmetry, $f(a-2,b,c+2)>f(a,b,c)$, and thus f attains its maximum when c is $n-1$ or n, that is, $a+b\le 1$. Equation (1) now yields,

$$e(G_1)+e(G_2)\le f(a,b,c)\le 2\left\lfloor\frac{n^2}{4}\right\rfloor.$$

Assume that $a=0$. When $c=n$, the equality holds only if $G_1=G_2=K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$. If $c=n-1$, then the equality holds only if n is odd and $G_1=G[C]=K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor}$ and G_2 is G_1 with all vertices connected with the only remaining vertex, that is, $G_2=\mathcal{T}(K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor})$. □

We now prove Theorem 2.1.

Proof of Theorem 2.1. We may assume that $n>1$. Put $G_{n+i}=G_i$. By Lemma 3.2,

$$\sum_{i=1}^{n}e(G_i)=\frac{1}{2}\sum_{i=1}^{n}(e(G_i)+e(G_{i+1}))\le\frac{1}{2}\sum_{i=1}^{n}2\left\lfloor\frac{n^2}{4}\right\rfloor=n\left\lfloor\frac{n^2}{4}\right\rfloor.$$

Suppose the equality holds. By Lemma 3.2, we are done if n is even. Suppose n is odd and $G_i=\mathcal{T}(K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor})$ for some i. By Lemma 3.2, one of G_i and G_{i+1} is $K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor}$ and the other is $\mathcal{T}(K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor})$, for all i. Hence,

$$G_{i+1}=K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor},G_{i+2}=\mathcal{T}(K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor}),\dots$$

and the alternation proceeds. But then $G_{n+i}=G_i=K_{\lceil\frac{n-1}{2}\rceil,\lfloor\frac{n-1}{2}\rfloor}$ as n is odd, and this contradiction completes the proof. □

2.2 Non-bipartite F

Theorem 2.2. *Let F be $(r+1)$-colorable, with $r\ge 2$. Suppose that $E(G_i)\cap E(G_j)$ is F-free for distinct i,j. For large enough n,*

$$\sum_{i=1}^{n}e(G_i)\le n\cdot\mathrm{ex}(n,F),$$

with equality if and only if $G_1=G_2=\cdots=G_n$ are n-vertex extremal graphs for F.

By the same argument as in Theorem 2.1, it suffices to prove the following lemma:

Lemma 2.2.1. *Let F be $(r+1)$-colorable, with $r \geq 2$. Suppose $E(G_1) \cap E(G_2)$ does not include F. For large enough n,*

$$e(G_1) + e(G_2) \leq 2 \cdot \text{ex}(n, F),$$

with equality if and only if $G_1 = G_2$ are n-vertex extremal graphs for F, unless n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2 = \mathcal{T}(G_1)$.

Proof. TODO: fix this proof. Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both G_1 and G_2. Let $A = V(G_1) \backslash C$, and let $B = V(G_2) \backslash C$. For simplicity, put $a = |A|$, $b = |B|$, $c = |C|$, and $r = \chi(F)$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to a, b, c. Since G_1, G_2 are induced graphs, we have $E(G_1[C]) = E(G_2[C]) = E(G[C]) = E(G_i) \cap E(G_j)$, which is F-free. Hence, we may write

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \text{ex}(c, F)\right]. \tag{2.2}$$

Define $f(a, b, c)$ as the function on the right-hand-side. We show that $f(a, b, c)$ attains its maximum at $a = b = 0$ and $c = n$. By a theorem of Simonovits, for large enough c, $\text{ex}(c, F) = \text{ex}(c, K_{r+1}) + \text{ex}(c, \tilde{F})$, where $\tilde{F}$ is the family of residue subgraphs of F after F is embedded into $T_r(c)$. Hence, we may write

$$\begin{aligned} f(a, b-2, c+2) - f(a, b, c) &= \binom{a+c+2}{2} - \binom{a+c}{2} \\ &\quad - 2\left[\binom{c+2}{2} - \binom{c}{2} - \text{ex}(c+2, F) + \text{ex}(c, F)\right] \\ &\geq 2a - 2c - 1 + 2[\text{ex}(c+2, K_{r+1}) - \text{ex}(c, K_{r+1})] > 0, \end{aligned}$$

as shown in the proof of Lemma 3.4. By symmetry, we also have $f(a-2, b, c+2) > f(a, b, c)$. Thus, f attains its maximum when c is $n-1$ or n. Equation (5) now yields,

$$e(G_1) + e(G_2) \leq \max\left[2 \cdot \text{ex}(n, F), 2 \cdot \text{ex}(n-1, F) + n - 1\right].$$

Assume that $a = 0$. Since

$$\begin{aligned} 2 \cdot \text{ex}(n, F) - [2 \cdot \text{ex}(n-1, F) + n - 1] &\geq 2[\text{ex}(n, K_{r+1}) - \text{ex}(n-1, K_{r+1})] \qquad (2.3)\\ &\quad - n + 1 \qquad (2.4)\\ &= 2\left(n - \left\lceil \frac{n}{r} \right\rceil\right) - n + 1 \qquad (2.5)\\ &\geq n + 1 - 2\left\lceil \frac{n}{2} \right\rceil \geq 0, \end{aligned}$$

we have

$$e(G_1) + e(G_2) \leq 2 \cdot \text{ex}(n, F). \tag{2.6}$$

If $c = n$, the equality for (9) holds only if $G_1 = G_2$ are n-vertex extramal graphs for F. Suppose $c = n-1$ and the equality holds. Observe that equation (6) is equal to zero only when $r = 2$ and n is odd. Hence, if $c = n-1$, the equality for (9) could only be achieved when $r = 2$, n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2 = \mathcal{T}(G_1)$. $\square$

3 General Case

We now relax the assumption that $G_1, \ldots, G_m$ are induced subgraphs. The trivial construction of putting $G_1 = K_n$ and $G_2, \ldots, G_m$ to be extremal graphs for F yields the lower bound

$$\sum_{i=1}^{m} e(G_i) = \binom{n}{2} + (m-1)\mathrm{ex}(n, F). \tag{3.1}$$

In this section we examine whether this bound is tight. The following is an asymptotic result on the number of G_i's:

Theorem 3.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include r-vertex graph F for distinct i, j. Then for large enough n,*

$$\sum_{i=1}^{m} e(G_i) \leq m(1 + o_m(1))\mathrm{ex}(n, F),$$

as $m \to \infty$.

Proof. Fix $\epsilon > 0$. Reorder the G_i's such that $G_1, \ldots, G_{m'}$ are all the G_i's containing at least $(\pi_n(F) + \epsilon)\binom{n}{2}$ edges. A theorem of Simonovits states that G contains at least δn^r copies of F for some $\delta = \delta(\epsilon)$. Since there can be at most $\binom{n}{r}$ copies of F across all G_i's, we have

$$m'\delta n^r \leq \binom{n}{r} \leq n^r \implies m' \leq \frac{1}{\delta}.$$

It now follows that

$$\begin{aligned}
\sum_{i=1}^{m} e(G_i) &\leq \sum_{i=1}^{m'} e(G_i) + \sum_{i=m'+1}^{m} e(G_i) \\
&\leq m'\binom{n}{2} + (m - m')(\pi_n(F) + \epsilon)\binom{n}{2} \\
&\leq m\left[\pi_n(F) + \epsilon - \frac{m'}{m}(1 - \pi_n(F) - \epsilon)\right]\binom{n}{2} \\
&= m(1 + o_m(1))\mathrm{ex}(n, F),
\end{aligned}$$

as $m \to \infty$. □

3.1 Triangle F

Consider F to be a triangle. Simply counting the number of triangles in each G_i shows the following:

Theorem 3.2. *For any $\epsilon > 0$, if $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j, then*

$$\sum_{i=1}^{m} e(G_i) < m(1+\epsilon)\frac{n^2}{4} + (1-\epsilon)\frac{n^2}{2\epsilon}.$$

Claim 3.2.1. *There are less than $\frac{2}{\epsilon}$ number of G_i's with $e(G_i) \geq (1+\epsilon)\frac{n^2}{4}$.*

Proof. Suppose $e(G_i) \geq (1+\epsilon)\frac{n^2}{4}$ for $1 \leq i \leq k$. Let $K_3(G)$ denote the number of triangles in graph G. By the Moon-Moser inequality,

$$K_3(G_i) \geq \epsilon(1+\epsilon)\frac{n^3}{12}.$$

Since there are no overlapping traingles from different G_i's,

$$\binom{n}{3} \geq \sum_{i=1}^{k} K_3(G_i) \geq \frac{\epsilon(1+\epsilon)}{12} kn^3.$$

Rearranging yields $k < \frac{2}{\epsilon}$. $\square$

By the claim,

$$\sum_{i=1}^{m} e(G_i) < \frac{2}{\epsilon}\binom{n}{2} + \left(m - \frac{2}{\epsilon}\right)(1+\epsilon)\frac{n^2}{4} \leq m(1+\epsilon)\frac{n^2}{4} + (1-\epsilon)\frac{n^2}{2\epsilon},$$

which proves Theorem 3.2.

It can be easily shown that the bound in Theorem 3.2 is tight when $m = 2$, as

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_{1,2}) \leq \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor.$$

This result is also true for $m = 3$:

Proposition 3.3. *Let G_1, G_2, G_3 be subgraphs of some G such that no triangle is contained in any two graphs, then*

$$e(G_1) + e(G_2) + e(G_3) \leq \binom{n}{2} + \frac{n^2}{2}.$$

Proof. Define $H_k \subseteq G$ be the graph with edges contained in at least k number of G_i's and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \leq \frac{n^2}{2}$. Notice H_2 must not contain any triangles with two edges in H_3, so

$$e(H_2) + e(H_3) \leq \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}|.$$

Let H'_3 be the graph with the same vertex set as H_3 and edge set $\{\{u,v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}$. It suffices to show that $\frac{n}{2} \geq e(H_3) - e(H'_3)$.

Let $d_1 \geq d_2 \geq \cdots \geq d_n$ and $f_1 \geq f_2 \geq \cdots \geq f_n$ each be the degree sequence of H_3 and H'_3, respectively. We show that $f_i \geq d_i - 1$ for all i. Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i. Let $S_i = |N_{H_3}(v_1) \cup \cdots \cup N_{H_3}(v_i)|$. Since

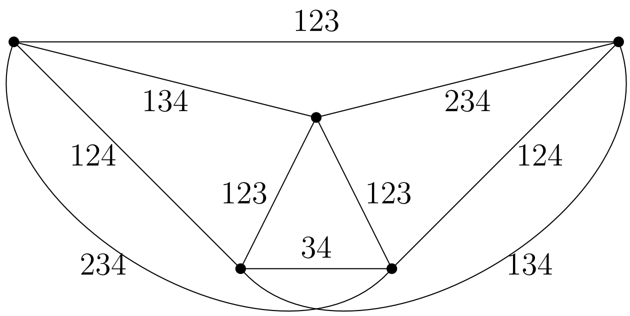
$$\sum_{u \in S_i} d_{H_3}(u) \geq d_1 + \cdots + d_i,$$

we have that $|S_i| \geq i$. But then $S_i \backslash \{u_1, \ldots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H'_3}(u) \geq d_i - 1$. Hence, $f_i \geq d_i - 1$ for all i, which yields

$$e(H'_3) = \frac{1}{2}\sum_{i=1}^{n} f_i \geq \frac{1}{2}\sum_{i=1}^{n}(d_i - 1) = e(H_3) - \frac{n}{2}.$$

□

However, the bound in Proposition 3.1 is not tight for $m \geq 4$, as shown in the following graph:



The number on each edge denotes the set of G_i's that contain the edge.

The above graph contains 29 edges, which exceeds the bound $\binom{5}{2} + 3\lfloor \frac{5^2}{4} \rfloor = 28$ by 1. By blowing up the above graph, we can construct a graph with $n \in 10\mathbb{Z}$ vertices that contains

$$5\binom{n/5}{2} + 29 \cdot \frac{(n/5)^2}{4}$$

edges, which exceeds the bound $\binom{n}{2} + 3\lfloor \frac{n^2}{4} \rfloor$ by $\frac{n^2}{100}$.

3.2 Bipartite F