Double Turán Problem

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Contents

1 Introduction

Let F be a graph with at least one edge. A graph G is *F-free* if G does not contain F as a subgraph. The fundamental question in extremal graph theory is to determine the maximum number of edges in an n-vertex F-free graph. These maxima, denoted $\mathrm{ex}(n, F)$, are referred to as the *extremal numbers* or *Turán numbers* for F.

In this thesis, we investigate a closely related problem which we refer to as the *double Turán problem.* Let $G_1, G_2, \ldots, G_m$ be graphs on the same vertex set of size n. We are interested in determining the maximum sum of edges over m graphs $G_1, G_2, \ldots, G_m$ whose pairwise intersection is F-free. We denote this quantity as $\mathrm{ex}_2(m, n, F)$ and refer to an F in the intersection of two of the graphs G_i as a *double F*.

1.1 Definitions and Notation

Denote the set of first n positive integers as $[n] = \{1, 2, \ldots, n\}$. Given a set X, we denote 2^X as the power set of X.

Let $G = (V, E)$ be a graph. Let $V(G) = V$ denote the vertex set and $E(G) = E$ denote the edge set of G. We note by $v(G) = |V|$ the number of vertices and $e(G) = |E|$ the number of edges in G. For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v.

Given graphs $G_1, \ldots, G_m$ on some vertex set V, we denote $G_{i_1,\ldots,i_k}$ as graph on V with edge set $E(G_{i_1,\ldots,i_k}) = \bigcap_{\alpha=1}^{k} E(G_{i_\alpha})$. Given two graphs G_1, G_2, we denote $G_1 \cup G_2$ as the graph on $V(G_1) \cup V(G_2)$ with edge set $E(G_1 \cap G_2) = E(G_1) \cup E(G_2)$. Let s

In this thesis, we reserve n to denote the number of vertices in a graph. We call a n-vertex complete graph K_n, and a complete bipartite graph $K_{a,b}$, where a, b are the size of its parts. We denote P_n as a path with n edges, and C_n as a cycle with n edges. Given graph G, H, define $G + H$ as the graph fully connecting G, H, i.e. $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$.

Given graphs G and F, we say that G is F-free if G does not contain F as a subgraph. We denote $\mathrm{ex}(n, F)$ to be the maximum possible number of edges an F-free graph on n vertices, and we call a F-free graph achieving this maximum an extremal graph for F. Given graphs $G_1, \ldots, G_m$ on the same set of vertices and F, we say that $G_1, \ldots, G_m$ are pairwise F-free if $E(G_i) \cap E(G_j)$ does not contain F for $i \neq j$. Let v be a vertex from $G_1, G_2, \ldots, G_m$. Unless otherwise specified, we denote $d(v)$ as the sum of degree of v over all G_i.

1.2 Problem Statement

Let $\mathrm{ex}_2(m, n, F)$ be the maximum possible number of edges that m pairwise F-free graphs on n vertices can have. Our goal is to determine $\mathrm{ex}_2(m, n, F)$ for different forbidden graphs F. A trivial construction with $G_1 =$

K_n and $G_2, \ldots, G_m$ to be extremal graphs for F yields the lower bound $\mathrm{ex}_2(m,n,F) \geq \binom{n}{2} + (m-1)\mathrm{ex}(n,F)$. In this work, we use this bound as a benchmark to either show its tightness or to improve it.

Additionally, we are also interested in a more restrictive version where $G_1, \ldots, G_m$ are induced subgraphs of $G_1 \cup \cdots \cup G_m$. Let $\mathrm{ex}_2^*(m,n,F)$ as the maximum possible number of edges that m pairwise F-free graphs on n vertices can have, with the constraint that each graph is an induced subgraph of their union. A trivial construction with $G_1 = \cdots = G_m$ to be extremal graphs for F yields the lower bound $\mathrm{ex}_2^*(m,n,F) \geq m \cdot \mathrm{ex}(n,F)$. This is the benchmark we use to determine $\mathrm{ex}_2^*(m,n,F)$. Similar to the non-induced case, we will use this bound as a benchmark and base our work on it.

2 Induced Version

In this chapter, we investigate the case where $G_1, \dots, G_m$ are induced subgraphs of $G_1 \cup \cdots \cup G_m$ and are pairwise F-free, for some specified F. Unless otherwise specified, when we say $G_1, \dots, G_m$ are induced subgraph, we mean that they are induced subgraphs of $G_1 \cup \cdots \cup G_m$.

The following lemma shows that the problem can be reduced to only two graphs.

Lemma 2.1. *Let $n, m, k \geq 2$ with $m \geq k$, F be some graph. Then*

$$\mathrm{ex}_2^*(m, n, F) \leq \frac{m}{k} \cdot \mathrm{ex}_2^*(k, n, F).$$

Moreover, let $G_1, \dots, G_m$ be induced doubly F-free graphs on $[n]$ and suppose that $\sum_{i=1}^{k} e(G_i) = \mathrm{ex}_2^*(k, n, F)$ only if $G_1 = \cdots = G_k$. Then $\sum_{i=1}^{m} e(G_i) = \mathrm{ex}_2^*(m, n, F)$ only if $G_1 = \cdots = G_m$.

Proof. Let $G_1, \dots, G_m$ be induced subgraphs of $G_1 \cup \cdots \cup G_m$ with $E(G_i) \cap E(G_j)$ not containing F for $i \neq j$. Put $G_{i+m} = G_i$ for all $i \in [m]$. Then

$$\sum_{i=1}^{m} e(G_i) = \frac{1}{k} \sum_{i=1}^{m} [e(G_i) + \cdots + e(G_{i+k-1})] \leq \frac{1}{k} \sum_{i=1}^{m} \mathrm{ex}_2^*(k, n, F) = \frac{m}{k} \cdot \mathrm{ex}_2^*(k, n, F),$$

which establishes the upper bound.

Suppose $\sum_{i=1}^{k} e(G_i) = \mathrm{ex}_2^*(k, n, F)$. By assumption $G_1 = \cdots = G_k$, so $e(G_i) = \mathrm{ex}_2^*(k, n, F)/k$ for $1 \leq i \leq k$. Hence, the construction $G_1 = \cdots = G_m$ meets the upperbound. On the other hand, if $G_1 \neq G_2$ then $\sum_{i=1}^{k} e(G_i) < \mathrm{ex}_2^*(k, n, F)$. Since $\sum_{i=1}^{k} e(G_{i+j}) \leq \mathrm{ex}_2^*(k, n, F)$ for all $j \geq 1$, we have $\sum_{i=1}^{m} e(G_i) < \frac{m}{k} \cdot \mathrm{ex}_2^*(k, n, F)$. Thus the extremal condition is met only when $G_1 = \cdots = G_m$. $\square$

Lemma 2.1 allows us to reduce the problem to the case for two subgraphs G_1, G_2. Let $S = V(G_1) \cap V(G_2)$, $t = |T|$, $s = |V(G_1) \backslash T|$, and $n - t - s = |V(G_2) \backslash T|$. Note that $t, s \in \mathbb{Z}_{\geq 0}$. Since G_1, G_2 are induced subgraphs of $G_1 \cup G_2$, we have $G_1[T] = G_2[T] = G_1 \cap G_2$. But then $G_1 \cap G_2$ is F-free, so $e(G_1[T]) = e(G_2[T]) \leq \mathrm{ex}(t, F)$. Fixing s and t, the optimal construction to maximize the number of edges over G_1, G_2 is thus putting $G_1 \cap G_2$ as an extremal graph for F on c vertices and connect all edges that are not induced in A. This yields the inequality

$$e(G_1) + e(G_2) \leq \binom{s}{2} + \binom{n-s-t}{2} + (n-t)t + 2\mathrm{ex}(t, F).$$

But then notice $\binom{n-t}{2} > \binom{s}{2} + \binom{n-t-s}{2}$ for $0 < s < n - t$. This implies our construction is optimized when $s = 0$ or $s = n - t$, that is, to let G_2 contain G_1 or the other way around. Hence, we may assume $s = 0$ and define the following construction function:

Definition 2.2. *For $n \geq t \geq 1$ and F some graph, define*

$$\mathcal{C}(n,t,F) := \binom{n-t}{2} + (n-t)t + 2\mathrm{ex}(t,F).$$

The construction described by $\mathcal{C}(n,t,F)$ are graphs G_1, G_2 on $[n]$, such that G_2 is the t-vertex extremal graph for F and $G_1 = G_2 + K_{n-t}$.

The above discussion also yields the following lemma:

Lemma 2.3. *Let F be some graph. For $n \geq 1$,*

$$\mathrm{ex}_2^*(2,n,F) = \max_{0\leq t\leq n} \mathcal{C}(n,t,F).$$

Moreover, the equality holds for n-vertex graphs G_1, G_2 only if G_1, G_2 are the construction described by $\mathcal{C}(n, t_{max}, F)$, where $t_{max} = \arg\max_{0\leq t\leq n} \mathcal{C}(n,t,F)$.

The problem is now reduced to maximizing $\mathcal{C}$ over t. In particular, $\mathcal{C}(n,n,F)$ gives our benchmark construction of $G_1 = G_2$ being the extremal graphs for F on n vertices. For $0 \leq k \leq t \leq n$, define

$$\Delta_k\mathcal{C}(n,t,F) := \mathcal{C}(n,t,F) - \mathcal{C}(n,t-k,F) = \frac{1}{2}k(k-2t+1) + 2[\mathrm{ex}(t,F) - \mathrm{ex}(t-k,F)]$$

and denote $\Delta\mathcal{C} = \Delta_1\mathcal{C}$. Most of the work in this section will show that the maximum of $\mathcal{C}$ happens when $t \geq n-k$ by proving that $\Delta_k\mathcal{C}(n,t,F) > 0$ for all $c = t \leq n-k$.

Lemma 2.4. *Let $n, t_0 \geq 1$, $m \geq 2$, and F be some graph. If $\mathcal{C}(n,t,F) < 2 \cdot \mathrm{ex}(n,F)$ for $0 \leq t < t_0$ and $\mathrm{ex}(t,F) - \mathrm{ex}(t-1,F) > \frac{t-1}{2}$ for $t_0 \leq t \leq n$, then*

$$\mathrm{ex}_2^*(m,n,F) = m \cdot \mathrm{ex}(n,F).$$

Moreover, the equality holds for n-vertex graphs $G_1, G_2, \ldots, G_m$ if and only if all $G_1 = G_2 = \cdots = G_m$ are extremal graphs for F.

Proof. By Lemma 2.1 and Lemma 2.3, it suffices to show $\mathcal{C}(n,t,F)$ has a unique maximum of $2\mathrm{ex}(n,F)$ at $t = n$. We may assume $t \geq t_0$ by assumption. Suppose $t < n$. Since $\mathrm{ex}(t,F) - \mathrm{ex}(t-1,F) > \frac{t-1}{2}$,

$$\Delta\mathcal{C}(n,t,F) = -t + 1 + 2[\mathrm{ex}(t,F) - \mathrm{ex}(t-1,F)] > 0.$$

Thus, $\mathcal{C}$ is strictly increasing with respect to t for $t \geq t_0$, so $\mathcal{C}$ has a unique maximum of $2 \cdot \mathrm{ex}(n,F)$ at $t = n$, which yields the unique extremal construction of $G_1 = G_2$ being extremal graphs for F on n vertices. □

2.1 Complete Graph F

We will show the following result in this section:

Theorem 2.5. *For $n, m, r \geq 3$,*

$$\mathrm{ex}_2^*(m,n,K_r) = m \cdot \mathrm{ex}(n,K_r),$$

with equality for n-vertex graphs $G_1, G_2, \ldots, G_m$ if and only if $G_1 = \cdots = G_m = T_{r-1}(n)$.

Surprisingly, the proof for the triangle case ($r = 3$) is more complicated than the cases for larger r. Hence, we will first prove the case for $r \geq 4$ and then prove the triangle case separately. In particular, the case for $r \geq 4$ is a direct consequence of the following lemma:

Lemma 2.6. *For $n \geq 2$ and $r \geq 3$,*

$$\mathrm{ex}(n, K_r) - \mathrm{ex}(n-1, K_r) \geq \frac{n-1}{2},$$

with equality if and only if n is odd and $r = 3$.

Proof. By Turán's Theorem,

$$\mathrm{ex}(n, K_r) - \mathrm{ex}(n-1, K_r) = \delta(T_{r-1}(n)) = n - \left\lceil \frac{n}{r-1} \right\rceil \geq n - \left\lceil \frac{n}{2} \right\rceil.$$

The result now follows. $\square$

Theorem 2.7. *For $n, m \geq 2$ and $r \geq 4$,*

$$\mathrm{ex}_2^*(m, n, K_r) = m \cdot \mathrm{ex}(n, K_r),$$

with equality for n-vertex graphs $G_1, G_2, \ldots, G_m$ if and only if $G_1 = \cdots = G_m = T_{r-1}(n)$.

Proof. The result follows from Lemma 2.4 and Lemma 2.6. $\square$

As shown in Lemma 2.6, the condition given by Lemma 2.4 is not satisfied for all n in the triangle case, and there are indeed constructions of induced subgraphs G_1, G_2 that meet the extremal condition but are neither equal nor both complete bipartite graphs. For odd n, consider $G_1 = K_{\frac{n-1}{2}, \frac{n-1}{2}}$ and $G_2 = K_{\frac{n-1}{2}, \frac{n-1}{2}} + K_1$. The number of edges over G_1, G_2 is $\frac{(n-1)^2}{2} + n - 1 = \frac{n^2-1}{2} = 2\left\lfloor \frac{n^2}{4} \right\rfloor$, which meets the benchmark construction of two complete bipartite graphs. We will show that this is the only deviant construction for the triangle case.

Theorem 2.8. *Let $n, m \geq 2$, and let $G_1, \ldots, G_m$ be pairwise K_3-free induced subgraphs on n vertices. Then*

$$\mathrm{ex}_2^*(m, n, K_3) = m \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Moreover, $\sum_i e(G_i) = \mathrm{ex}_2^*(m, n, K_3)$ if and only if $G_1 = \cdots = G_m = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, unless n is odd and $m = 2$, in which case $e(G_1) + e(G_2) = \mathrm{ex}_2^*(2, n, K_3)$ if and only if either $G_1 = G_2 = K_{\frac{n+1}{2}, \frac{n-1}{2}}$ or $G_1 = K_{\frac{n-1}{2}, \frac{n-1}{2}}$ and $G_2 = G_1 + K_1$.

Proof. We claim that $\mathrm{ex}_2^*(2, n, K_3) = 2\left\lfloor \frac{n^2}{4} \right\rfloor$, and the equality holds for G_1, G_2 on $[n]$ only when $G_1 = G_2 = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ unless n is odd, $G_1 = K_{\frac{n-1}{2}, \frac{n-1}{2}}$, and $G_2 = K_{\frac{n-1}{2}, \frac{n-1}{2}} + K_1$.
Consider $\Delta_2 \mathcal{C}(n, t, K_3)$. Since

$$\Delta_2 \mathcal{C}(n, t, K_3) = -2t + 3 + 2\left[\left\lfloor \frac{t^2}{4} \right\rfloor - \left\lfloor \frac{(t-2)^2}{4} \right\rfloor\right] = -2t + 3 + 2(t-1) = 1 > 0,$$

$\mathcal{C}(n, t, K_3)$ has a maximum of $2\left\lfloor \frac{n^2}{4} \right\rfloor$ when $t \geq n-1$, so $\mathrm{ex}_2^*(2, n, K_3) = 2\left\lfloor \frac{n^2}{4} \right\rfloor$ by Lemma 2.3. We are done if $t = n$, so assume that $t = n-1$. Then in the extremal condition, $G_1 = G_{1,2} = K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ and

$$e(G_1) + e(G_2) = 2\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \deg(v),$$

where v is the only vertex not in $G_1 \cap G_2$. But then to meet the extremal condition,

$$\deg(v) = 2\left\lfloor \frac{n^2}{4} \right\rfloor - 2\left\lfloor \frac{(n-1)^2}{4} \right\rfloor = \begin{cases} n & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Hence, n must be odd and G_2 must be a copy of G_1 with all vertices adjacent to the only remaining vertex, i.e. $G_2 = G_1 + K_1$. This shows the claim.

By Lemma 2.1 and our claim, it remains to show that for odd n and $m = 3$, $G_1 = G_2 = G_3 = K_{\frac{n+1}{2}, \frac{n-1}{2}}$ if the extremal condition is met. Suppose not. Our claim then guarantees one of the subgraphs, say G_1, is $K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$. But then by our claim $G_2 = G_3 = G_1 + K_1$, which contradicts that G_2, G_3 are pairwise K_3-free. This completes the proof. □

Theorem 2.5 now follows directly from Theorem 2.7 and Theorem 2.8.

2.2 Non-bipartite F

For non-bipartite F, it is hard to determine the extremal graphs for F in general, but their structures becomes more apparent when n is large.

More specifically, the Erdős-Stone Theorem tells us that for large n, the extremal graph for F mimics the structure of the Turán graph. With this idea in mind, the following theorem is a generalization of Theorem 2.5 for large n.

Theorem 2.9. *Let $m, r \geq 3$, and F be a r-colorable graph. Then for large enough n,*

$$\mathrm{ex}_2^*(m, n, F) = m \cdot \mathrm{ex}(n, F),$$

with equality for n-vertex graphs $G_1, G_2, \ldots, G_m$ if and only if $G_1 = \cdots = G_m$ are extremal F-free graphs.

Proof. By Lemma 2.1, it suffices to show the case for $m = 3$. Let G_1, G_2, G_3 be induced doubly F-free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \mathrm{ex}_2^*(3, n, F)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\mathrm{ex}_2^*(3, n, F) \geq 3\mathrm{ex}(n, F)$. Consequently, we must have $e(G_1) + e(G_2) \geq 2\mathrm{ex}(n, F)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\mathrm{ex}(n, F)$, it suffices to show that $G_1 = G_2$ are n-vertex F-free extremal graphs.

Let $t = |V(G_1 \cap G_2)|$. If $t < \sqrt{n}$, then

$$2\mathrm{ex}(n, F) \geq 2e(T_{r-1}(n)) \geq 2\left\lfloor \frac{n^2}{4} \right\rfloor \geq \binom{n}{2} + \binom{\sqrt{n}}{2} > \mathcal{C}(n, t, F).$$

Thus $t \geq \sqrt{n}$. But then for large enough t, any extremal t-vertex F-free graph contains a spanning complete $(r-1)$-partite subgraph $T_{r-1}(t)$, so we may add $\mathrm{ex}(t-1, F) - e(T_{r-1}(t-1))$ egdes to $T_{r-1}(t)$ and still avoid F as a subgraph. Hence for large enough t, we have $\mathrm{ex}(t, F) \geq \mathrm{ex}(t-1, F) - e(T_{r-1}(t-1)) + e(T_{r-1}(t))$, and so

$$\mathrm{ex}(t, F) - \mathrm{ex}(t-1, F) \geq e(T_{r-1}(t)) - e(T_{r-1}(t-1)) \geq t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, F) - \mathcal{C}(n, t-1, F) = -t + 1 + 2[\mathrm{ex}(t, F) - \mathrm{ex}(t-1, F)] = t + 1 - 2\left\lceil \frac{t}{r-1} \right\rceil. \tag{2.1}$$

For $r \geq 4$, $\mathcal{C}(n,t,F)$ is strictly increasing on t, so by Lemma 2.3,

$$\mathrm{ex}_2^*(2,n,F) = \mathcal{C}(n,n,F) = 2\mathrm{ex}(n,F) = e(G_1) + e(G_2),$$

and $G_1 = G_2$ are n-vertex F-free extremal graphs, as desired.

Now suppose $r = 3$. Equation (2.1) shows that $\mathcal{C}(n,t,F)$ is strictly increasing for even t and $\mathcal{C}(n,t,F) \geq \mathcal{C}(n,t-1,F)$ for odd t. By Lemma 2.3, we now have

$$\mathrm{ex}_2^*(2,n,F) = \max[\mathcal{C}(n,n,F), \mathcal{C}(n,n-1,F)] = 2\mathrm{ex}(n,F) = e(G_1) + e(G_2),$$

and either $G_1 = G_2$ are n-vertex extremal F-free graphs, or G_2 is an $(n-1)$-vertex extremal F-free graph and $G_1 = G_2 + K_1$. If the latter case is true, then $e(G_3) \geq \mathrm{ex}(n,F) > e(G_2)$, and this contradiction completes the proof. $\square$

For small n, we may not be able to achieve the same result. Consider the case when F is the bowtie graph, i.e. the 5-vertex graph with two triangles sharing a vertex. For $n \leq 4$, the n-vertex extremal graph for F is the complete graph K_n. For $n \geq 5$, the n-vertex extremal graph for F is then $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ plus an edge, and so $\mathrm{ex}(n,F) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$. But then in this case when $n = 5$,

$$\mathcal{C}(5,4,F) = 2e(K_4) + 4 = 16 > \mathcal{C}(5,5,F) = 2\left(\left\lfloor \frac{5^2}{4} \right\rfloor + 1\right) = 14.$$

This yields an instance where the construction $G_1 = K_{v(F)-1}$ and $G_2 = K_n$ beats our benchmark construction. Thus the following lemma gives a lower bound for n to avoid losing to this construction.

Lemma 2.10. *Let $n, r, k \geq 3$, and F be r-colorable with $|V(F)| = k$. If $n > k^2 - 3k + 2$ and $r - 1$ divides n, then*

$$\mathcal{C}(n,n,F) > \mathcal{C}(n,k-1,F).$$

Proof. We need to show that

$$2\mathrm{ex}(n,F) - \binom{n}{2} > \binom{k-1}{2}.$$

Since $\mathrm{ex}(n,F) \geq e(T_{r-1}(n)) = \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2} \geq \frac{n^2}{4}$,

$$2\mathrm{ex}(n,F) - \binom{n}{2} \geq \frac{n^2}{2} - \binom{n}{2} = \frac{n}{2} > \frac{k^2 - 3k + 2}{2} = \binom{k-1}{2}.$$

$\square$

3 General Version

TODO: add introduction.

Theorem 3.1. *For all n and graph F,*

$$\mathrm{ex}_2(m,n,F) = m(1+o(1))\mathrm{ex}(n,F)$$

as $m \to \infty$.

Proof. Let $r = v(F)$. Pick $\epsilon > 0$. Reorder $G_1, \ldots, G_m$ so that $G_1, \ldots, G_{m'}$ are all the G_i's containing at least $(1+\epsilon)\mathrm{ex}(n,F)$ edges. A theorem of Simonovits states that G contains at least δn^r copies of F for some $\delta = \delta(\epsilon)$. Since there can be at most $\binom{n}{r}$ copies of F across all G_i's,

$$m'\delta n^r \leq \binom{n}{r} \leq n^r \implies m' \leq \frac{1}{\delta}.$$

It now follows that

$$\begin{aligned}
\sum_{i=1}^{m} e(G_i) &= \sum_{i=1}^{m'} e(G_i) + \sum_{i=m'+1}^{m} e(G_i) \\
&\leq \frac{1}{\delta}\binom{n}{2} + \left(m - \frac{1}{\delta}\right)(1+\epsilon)\mathrm{ex}(n,F) \\
&= m\left[1+\epsilon+\frac{1}{m\delta}\left(\frac{\binom{n}{2}}{\mathrm{ex}(n,F)} - (1+\epsilon)\right)\right]\mathrm{ex}(n,F).
\end{aligned}$$

Since ϵ is arbitrary, the result follows. $\square$

Theorem 3.2. *For large enough n, suppose that $G_1, \ldots, G_m$ are graphs on common vertex set $[n]$ with no copy of F contained in any k of the G_i's. If there exists extremal F-free subgraph H on n vertices such that $\binom{m}{k}\Delta(H) = o(n^{1/2})$, then*

$$\mathrm{ex}_2(m,n,F) = (k-1)\binom{n}{2} + \mathrm{ex}(n,F)\binom{m}{k}.$$

Proof. For $S \subseteq [m]$, let E_S denote the set of edges that are contained in exactly $\{G_i\}_{i\in S}$. Then

$$\sum_{i=1}^{m} e(G_i) = \sum_{S\subseteq[m]} |S||E_S| \leq (k-1)\binom{n}{2} + \sum_{S\subseteq[m], |S|\geq k} (|S|-k+1)|E_S|.$$

Let $A_S = \bigcup_{T\supseteq S} E_T$, i.e. the set of edges that are contained in all G_i with $i \in S$. When $|S| \geq k$, the edge set A_S is F-free and thus

$$\sum_{T \supseteq S} |E_T| \leq \text{ex}(n, F).$$

Hence,

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq k}} (|S| - k + 1)|E_S| = \sum_{\substack{S \subseteq [m], \\ |S|=k}} \sum_{T \subseteq S} \frac{(|T| - k + 1)|E_T|}{\binom{|T|}{k}} \leq \sum_{\substack{S \subseteq [m], \\ |S|=k}} \sum_{T \subseteq S} |E_T| \leq \binom{m}{k} \text{ex}(n, F),$$

as each $T \in [m]$ with $|T| \geq k$ is counted $\binom{|T|}{k}$ times in total and $|T| - k + 1 \leq \binom{|T|}{k}$. This proves the upper bound.

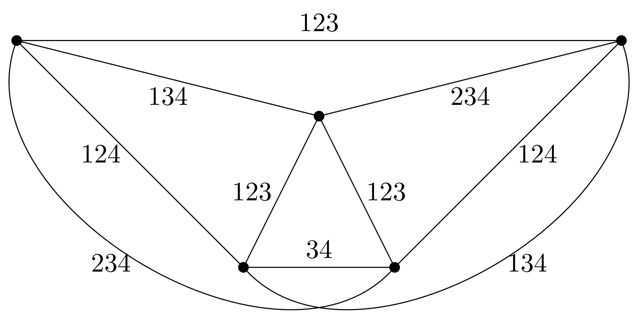
Now we show the bound is tight. In particular, we need to show there exists a construction such that the graph with edge set E_S is an extremal F-free graph, for all $S \subseteq [m]$ of size k. Let $M = \binom{m}{k}$ and $H_1, \ldots, H_M$ be copies of an extremal F-free graph on n vertices with $\Delta(H_i) = o(n^{1/2})$ for all i. It suffices to show that we can embed each H_i onto $[n]$ such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each H_i and iteratively decrease the number of intersecting edges. Define a (u, v, i)-swap by swapping the embedding of vertex u and v of H_i, i.e. replacing each edge $\{u, w\} \in E(H_i)$ with the edge $\{u, w\}$ and each edge $\{v, w\} \in E(H_i)$ with the edge $\{v, w\}$. This perserves the type of isomorphism of H_i. Given a vertex v, let $N(v) = N_{H_1}(v) \cup \cdots \cup N_{H_M}(v)$. Suppose there exists an intersecting edge $\{u, w\} \in E(H_i) \cap E(H_j)$. Since $|N(u)| \leq M \cdot \Delta(H_i) = o(n^{1/2})$, $|N(u) \cup N(N(u))| = o(n)$ so there exists a vertex $v \notin N(u) \cup N(N(u))$. Since $N(u) \cap N(v) = \emptyset$, performing a (u, v, i)-swap reduces the number of intersecting edges. The result now follows from iterating this process. $\square$

3.1 Complete F

It turns out that $\text{ex}_2(m, n, K_r)$ can be determined exactly, but the statement of the theorem requires some definitions. Let $k \geq 2$ and sets $S_{ij} \subseteq [m]$, for $1 \leq i < j \leq k$. We call the following type of construction a *k-blowup*:

Define $G_1, G_2, \ldots, G_m$ by partitioning $[n]$ into k sets $V_1, V_2, \ldots, V_k$ and letting $\{u, v\} \in E(G_h)$ whenever $u \in V_i, v \in V_j, h \in S_{ij}$. Additionally, for each $i \in [k]$ and $\{u, v\} \subseteq V_i$, we place $\{u, v\}$ in exactly one of G_1.

Given a k-blowup, we may define m graphs $H_1, H_2, \ldots, H_m$ on $[k]$ with edge set $E(H_h) \coloneqq \{\{i, j\} : h \in S_{ij}\}$ for $h \in [m]$. We call $H_1, H_2, \ldots, H_m$ the *pattern* of the k-blowup, and we say that a k-blowup is doubly F-free if its pattern is doubly F-free.



Example of a 5-blowup. A label $i \in \{1, 2, 3, 4\}$ at an edge indicates that the edge is in G_i.

For $M \geq 3$, let $R_M(F)$ denote the M-color Ramsey number for F. That is, the minimum number N such that there exists a monochromatic F in any M-coloring of the edges of K_N.

Let $m \geq 3$ and $M = \binom{m}{2}$. Define $B(m, n, F)$ as the maximum of $\sum_{i=1}^{m} e(G_i)$ such that $G_1, G_2, \ldots, G_m$ form a doubly F-free k-blowup, for some $k \leq R_M(F)$.

We are new ready to state the theorem.

Theorem 3.3. *For $n, m \geq 1$ and $r \geq 2$,*

$$\mathrm{ex}_2(m, n, K_r) = B(m, n, K_r).$$

Proof of Theorem 3.3. Notice that we trivially have $B(m, n, K_r) \leq \mathrm{ex}_2(m, n, K_r)$, so it suffices to show the reverse inequality. That is, we need to show that there exist $k \leq R_M(K_r)$ and S_{ij} for $1 \leq i < j \leq k$ such that the k-blowup construction on meets the desired bound.

Let $G_1, G_2, \ldots, G_m$ be graphs on $[n]$ with no double K_r and $\sum_{i=1}^{m} e(G_i) = \mathrm{ex}_2(m, n, K_r)$. Observe that any pair $\{i, j\} \subseteq [n]$ must be in some G_i, otherwise we may add it to G_1 without creating a double K_r.

We call vertices v, v' *clones* if for all $u \in [n] \setminus \{v, v'\}$ and $i \in [m]$, the edge $\{u, v\} \in E(G_i)$ if and only if $\{u, v'\} \in E(G_i)$. Furthermore, we call $\{v, v'\}$ a *light edge* if $\{v, v'\}$ is in exactly one graph G_i.

We now apply Algorithm 1 to $G_1, G_2, \ldots, G_m$.

Algorithm 1 symmetrization algorithm

> **while** $\exists$ a light edge whose endpoints are not clones **do**
>   among all vertices incident to such an edge, select a vertex v with maximum degree
> $B_v \leftarrow$ collection of vertices sending a light edge to v that are not clones of v
>   **while** $B_v \neq \emptyset$ **do**
>   pick $u \in B_v$
> $j \leftarrow$ colour of the light edge from u to v
>   **for** $1 \leq i \leq m$ **do**
>   **if** $i \neq j$ **then**;
> $N_{G_i}(u) \leftarrow N_{G_i}(v)$
>   **else if** $i = j$ **then**
> $N_{G_i}(u) \leftarrow (N_{G_i}(v) \setminus \{u\}) \cup \{v\}$
>   **end if**
>   **end for**
>   **end while**
> **end while**

Claim 3.3.1. *Algorithm 1 terminates.*

Proof. Notice that at the end of the 'while $B_v \neq \emptyset$' loop, every vertex sending a light edge to v is a clone of v. This implies v along with the set L_v of vertices receiving light edges from v induce a clique of size at least two in some G_i, and an empty graph in every other graph G_j with $j \neq i$. Moreover, any vertex $w \notin L_v$ sends edges to either all or none of the vertices in L_v, and if w is incident to L_v, then w sends edges to L_v in at least two graphs. It now follows that no light edge incident with a vertex in L_v will be picked again in an iteration of the out most while loop. Thus the algorithm can run through at most $n/2$ such iterations, and so it terminates. $\square$

Claim 3.3.2. *$G'_1, G'_2, \ldots, G'_m$ do not contain a double K_r and $\sum_{i=1}^{m} e(G'_i) = \mathrm{ex}_2(m, n, K_r)$.*

Proof. Note that we replace u by a clone of v in the for loop of Algorithm 1. Since $\{u, v\}$ remains to be an light edge in this step, u and v cannot both belong to a double K_r in the modified graphs. Furthermore, any double K_r containing u after the for loop arises from a double K_r containing v prior to the for loop. But then $G_1, G_2, \ldots, G_m$ contained no double K_r to begin with, so $G'_1, G'_2, \ldots, G'_m$ do not contain a double K_r.

We now show that the algorithm does not reduce the number of edges. By our choice of v, we know $d(v) \geq d(u)$ for all $u \in B_v$ prior to the for loop. Hence, replacing u with a clone of v does not decrease the number of edge over a complete iteration of the inner while loop. Therefore, $\sum_{i=1}^{m} e(G'_i) = \mathrm{ex}_2(m, n, K_r)$. □

Hence, the algorithm outputs graphs $G'_1, G'_2, \ldots, G'_m$ with $\mathrm{ex}_2(m, n, K_r)$ edges and the additional property that light edges come in 'clone cliques.' We may thus partition the vertex set $[n]$ into k disjoint sets $V_1, V_2, \ldots, V_k$, such that each V_i induces a clique of light edges from the same graph. Moreover, for distinct $i, j \in [k]$, define S_{ij} to be the set of all edges between V_i and V_j, and note that any edge in S_{ij} appear in at least two modified graphs. The sets S_{ij} now yield a k-blowup. Notice that if the pattern of the k-blowup contains a double K_r, then the original graphs $G_1, G_2, \ldots, G_m$ must have contained a double K_r as well, contradiction. Thus the k-blowup is doubly K_r-free.

It remains to show that $k < R_M(K_r)$. For each edge $\{i, j\} \subseteq [k]$ in the pattern of the k-blowup, we assign an arbitrary distinct pair $\{a, b\} \subseteq L_{ij} \subseteq [m]$ to $\{i, j\}$. If $k \geq R_M(K_r)$, then there exists K_r in the pattern of the k-blowup colored by some distinct pair $\{a, b\} \subseteq [m]$. But then this implies the pattern of the k-blowup contains a double K_r, contradiction. This completes the proof. □

When $m = 2$,

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_{1,2}) \leq \binom{n}{2} + \mathrm{ex}(n, K_3)$$

which meets the benchmark bound. Surprisingly, our desired bound is also met when $m = 3$:

Theorem 3.4. *For all n,*

$$\mathrm{ex}_2(3, n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Proof. Define $H_k \subseteq G$ be the graph with edges contained in at least k number of G_i's and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \leq \frac{n^2}{2}$. Notice H_2 must not contain any triangles with two edges in H_3, so

$$e(H_2) + e(H_3) \leq \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}|.$$

Let H'_3 be the graph with the same vertex set as H_3 and edge set $\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}$. It suffices to show that $\frac{n}{2} \geq e(H_3) - e(H'_3)$.

Let $d_1 \geq d_2 \geq \cdots \geq d_n$ and $f_1 \geq f_2 \geq \cdots \geq f_n$ each be the degree sequence of H_3 and H'_3, respectively. We show that $f_i \geq d_i - 1$ for all i. Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i. Let $S_i = |N_{H_3}(v_1) \cup \cdots \cup N_{H_3}(v_i)|$. Since

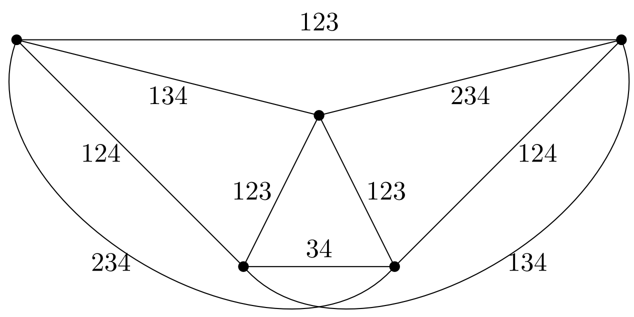
$$\sum_{u \in S_i} d_{H_3}(u) \geq d_1 + \cdots + d_i,$$

we have that $|S_i| \geq i$. But then $S_i \backslash \{u_1, \ldots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H'_3}(u) \geq d_i - 1$. Hence, $f_i \geq d_i - 1$ for all i, which yields

$$e(H'_3) = \frac{1}{2} \sum_{i=1}^{n} f_i \geq \frac{1}{2} \sum_{i=1}^{n} (d_i - 1) = e(H_3) - \frac{n}{2}.$$

□

However, the bound in Proposition 3.1 is not tight for $m \geq 4$, as shown in the following graph:



The number on each edge denotes the set of G_i's that contain the edge.

The above graph contains 29 edges, which exceeds the bound $\binom{5}{2} + 3\lfloor \frac{5^2}{4} \rfloor = 28$ by 1. By blowing up the above graph, we can construct a graph with $n \in 10\mathbb{Z}$ vertices that contains

$$5\binom{n/5}{2} + 29 \cdot \frac{(n/5)^2}{4}$$

edges, which exceeds the bound $\binom{n}{2} + 3\lfloor \frac{n^2}{4} \rfloor$ by $n^2/100$.

3.2 Bipartite F

In this section, we discuss the case where F is bipartite. In particular, we focus on the cases where $F \subseteq K_{2,2}$ is P_2, a path of length 2, or M_2, a matching with two edges.

Theorem 3.5.
$$\mathrm{ex}_2(m, n, P_2) \leq \left(\frac{1}{2} + o(1)\right) \min\{n^2\sqrt{m}, mn^{3/2}\},$$

as $n \to \infty$ or $m \to \infty$. Moreover,

$$\mathrm{ex}_2(m, n, P_2) = \left(\frac{1}{2} + o(1)\right) mn^{3/2},$$

for $\sqrt{n} \leq m \leq n$.

Proof. Let $G_1, \ldots, G_m$ be graphs on $[n]$ not containing a P_2. We first show the claimed upperbound and then show the tightness of the bound when $\sqrt{n} \leq m \leq n$.

Claim 3.5.1. $\mathrm{ex}_2(m, n, P_2) \leq mn \cdot \frac{1+\sqrt{4n^2/m+1}}{4}$.

Proof. Since there are no double P_2,

$$\sum_{i=1}^{m} \#\{P_2 \subseteq G_i\} \leq \#\{P_2 \subseteq G\}.$$

For all G_i, each vertex v in G_i along with two of its neighbors form one unique P_2, so

$$\#\{P_2 \subseteq G_i\} = \sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2}.$$

By Jensen's inequality,

$$\sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2} \geq n \binom{d_{G_i}(v)/n}{2} = n \binom{2e(G_i)/n}{2} \geq \frac{2(e(G_i))^2}{n} - e(G_i).$$

On the other hand, since each three vertices in G can form at most three P_2's,

$$\#\{P_2 \subseteq G\} \leq 3\binom{n}{3} \leq \frac{n^3}{2}.$$

Combining the above inequalities yields and using Jensen's inequality once more yields

$$\frac{2m}{n}\left(\frac{1}{m}\sum_{i=1}^{m} e(G_i)\right)^2 - \sum_{i=1}^{m} e(G_i) \overset{Jensen's}{\leq} \sum_{i=1}^{m} \frac{2(e(G_i))^2}{n} - e(G_i) \leq \frac{n^3}{2}.$$

Solving the quadratic equation gives

$$\sum_{i=1}^{m} e(G_i) \leq mn \cdot \frac{1 + \sqrt{4n^2/m + 1}}{4}.$$

□

Claim 3.5.2. $\mathrm{ex}_2(m, n, P_2) \leq \frac{1}{2}(mn^{3/2} + n^2)$.

Proof. For each vertex $u \in [n]$, define H_u as the $m \times n$ bipartite graph with edge set $E(H_u) \coloneqq \{\{v, i\} : \{u, v\} \in E(G_i)\}$. If H_u contains a quadrilateral $\{v, i\}, \{v, j\}, \{w, i\}, \{w, j\}$, then $\{u, v\}, \{u, w\}$ form a double P_2 in $G_i \cap G_j$, contradiction. Thus we conclude that H_u is quadrilateral-free, and therefore $e(H_u) \leq m\sqrt{n}+n$, by the Kővari-Sós-Turán theorem. It now follows that

$$\sum_{i=1}^{m} e(G_i) = \frac{1}{2} \sum_{u \in V(G)} e(H_u) \leq \frac{1}{2}(mn^{3/2} + n^2).$$

□

The above two claims yield the desired upper bound. We now show the lower bound.

Claim 3.5.3. $\mathrm{ex}_2(m, n, P_2) \geq (1/2 + o(1))mn^{3/2}$*, for* $\sqrt{n} \leq m \leq n$.

Proof. Suppose $G_1, G_2, \ldots, G_n$ are graphs on $[n]$ containing no double P_2 and $\sum_{i=1}^{n} e(G_i) \geq (1/2+o(1))n^{5/2}$, with $e(G_1) \geq e(G_2) \geq \cdots \geq e(G_n)$. Then $G_1, G_2, \ldots, G_m$ are graphs with no double P_2 and $\sum_{i=1}^{m} e(G_i) \geq (1/2 + o(1))mn^{3/2}$. Hence, it suffices to prove the case for $m = n$.

Consider a finite projective plane with n points and n lines, with prime q chosen so that $n = (1 + o(1))(q^2 + q + 1)$ as $q \to \infty$. Let $S_1, \ldots, S_n \subseteq [n]$ be the n lines of the projective plane. Note that each line S_i contains $q + 1$ points, and the intersection of any two distinct lines S_i, S_j contains $|S_i \cap S_j| = 1$ point.

Define $G_1, \dots, G_n$ to be graphs on $[n]$, each with edge set

$$E(G_i) \coloneqq \{\{j,k\} \subseteq [n] : j \neq k,\, j+k \in S_i \mod n\}.$$

Note that the intersection of distinct G_i, G_j is P_2 free: since $|S_i \cap S_j| = 1$, if $\{a,b\},\{a,c\} \in E(G_i) \cap E(G_j)$, then $a+b = a+c$ so $b = c$.

We now count the number of edges in $G_1, \dots, G_n$. Since $|S_i| = q+1$, for each point $j \in [n]$, there are $q+1$ choices for $k \in [n]$ such that $j+k \in S_i$. But then we have to avoid counting the same edge twice and loops, so the number of edges in G_i is

$$e(G_i) = \frac{n(q+1) - \#\text{loops counted for } G_i}{2}.$$

If $j \in [n]$ is even, then $k = j/2$ is the unique number in $[n]$ such that $k + k = j \mod n$. If $j \in [n]$ is odd, then $k = (n+j)/2$ is the unique number in $[n]$ such that $k+k = j \mod n$, as n is even. Hence, for each $j \in S_i$, there exists a unique $k \in [n]$ such that $k + k = j \mod n$, and thus

$$\#\text{loops counted for } G_i = |S_i| = q+1.$$

Since $q + 1 = (1 + o(1))n^{1/2}$, the number of edges in $G_1, \dots, G_n$ is

$$\sum_{i=1}^{n} e(G_i) = n \cdot \frac{n(q+1) - (q+1)}{2} = \left(\frac{1}{2} + o(1)\right) n^{5/2},$$

as $n \to \infty$. □

Claim 3.5.4. $\mathrm{ex}_2(m,n,P_2) \geq (1/2 + o(1))\sqrt{m}n^2$*, for* $n < m \leq n^2$*.*

Proof. □

□

Theorem 3.6. *For all* n, m*,*

$$\mathrm{ex}_2(m,n,M_2) \leq n^{5/2}.$$

Proof. Notice that $\#\{M_2 \subseteq G\} = \binom{e(G_i)}{2}$. On the other hand, each four vertices in G can form at most three M_2's, so $\#\{M_2 \subseteq G\} \leq 3\binom{n}{4} \leq \frac{n^4}{8}$. By the same argument as in Theorem 3.4, we have

$$\sum_{i=1}^{n} \binom{e(G_i)}{2} = \sum_{i=1}^{n} \#\{M_2 \subseteq G_i\} \leq \#\{M_2 \subseteq G\} \leq \frac{n^4}{8}.$$

By Jensen's inequality,

$$\sum_{i=1}^{n} \binom{e(G_i)}{2} \geq n \binom{\sum_{i=1}^{n} e(G_i)/n}{2} = \frac{1}{2n}\left[\left(\sum_{i=1}^{n} e(G_i)\right)^2 - n\sum_{i=1}^{n} e(G_i)\right].$$

Combining the above inequalities yields

$$\left(\sum_{i=1}^{n} e(G_i)\right)^2 - n\sum_{i=1}^{n} e(G_i) \leq \frac{n^5}{4},$$

and solving the quadratic inequality gives

$$\sum_{i=1}^{n} e(G_i) \leq n^{5/2}.$$

$\square$

We may obtain an exact result if we forbid both P_2 and M_2 at the same time:

Theorem 3.7. *For all n, m,*

$$\mathrm{ex}_2(m, n, \{P_2, M_2\}) = n^2 - n.$$

Proof. Denote the set of G_i's as $\{G_i\} = \{G_1, \ldots, G_n\}$, and the set of distinct pairs of G_i's as $\{G_i\}^2 = \{\{G_j, G_k\} : j \neq k\}$. Consider the bipartite graph H with vertex set $V(H) = \{G_i\} \sqcup E(K_n)$ and edge set $E(H) = \{\{G_j, e\} \in \{G_i\} \times E(K_n) : e \in G_j\}$. Define $\phi : \{G_i\}^2 \to 2^{E(K_n)}$ by sending each $\{G_j, G_k\}$ to their common edge set $E(G_j) \cap E(G_k)$. Notice that each distinct G_j, G_k have at most one edge in common, so $|\phi(G_j, G_k)| \leq 1$. On the other hand, each edge $e \in E(G)$ can be obtained via ϕ by $\binom{d_H(e)}{2}$ possible distinct pairs (G_j, G_k), and thus $|\phi^{-1}(e)| = \binom{d_H(e)}{2}$. But then

$$\binom{n}{2} \geq \sum_{(G_j, G_k) \in \{G_i\}^2} |\phi(G_j, G_k)| = \sum_{e \in E(K_n)} |\phi^{-1}(e)| = \sum_{e \in E(K_n)} \binom{d_H(e)}{2}.$$

By Jensen's inequality,

$$\sum_{e \in E(K_n)} \binom{d_H(e)}{2} \geq \binom{n}{2} \binom{\sum_{e \in E(K_n)} d_H(e) / \binom{n}{2}}{2} = \binom{n}{2} \binom{\sum_{i=1}^{n} e(G_i) / \binom{n}{2}}{2}.$$

Combining the above inequalities yields

$$2\binom{n}{2}^2 \geq \left(\sum_{i=1}^{n} e(G_i)\right)^2 - \binom{n}{2} \sum_{i=1}^{n} e(G_i),$$

and the result now follows from solving the quadratic inequality.

To see that this bound is tight, consider the construction such that for each distinct $i, j \in [n]$, $E(G_i) \cap E(G_j)$ contains exactly one unique edge $e \in K_n$. The number of edges in this construction is $2\binom{n}{2} = n^2 - n$. $\square$