Double Turán Problem

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Overview

Introduction

What is the Turán problem?

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Question

Given a graph F, how many edges can an n-vertex graph have while containing no copy of F as a subgraph?

Introduction

What is the hypergraph Turán problem?

Question

Given an r-uniform graph F, how many edges can an n-vertex r-uniform hypergraph have while containing no copy of F as a subgraph?

Introduction

Let F be a graph. We call the following quantity the *Turán number* or *extremal number* of F:

Definition

$$\mathrm{ex}(n, F) := \max\{e(G) : |V(G)| = n \text{ and } F \not\subseteq G\}$$

Introduction

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The *Turán graph* $T_r(n)$ is the complete r-partite graph on n vertices with parts of size $\lfloor n/r \rfloor$ and $\lceil n/r \rceil$.

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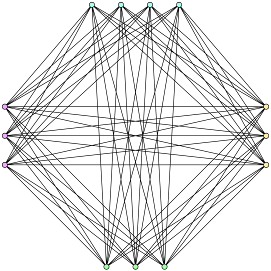

Figure: $T_4(13)$

Introduction

Turán's theorem

The maximum number of edges in an n-vertex graph containing no clique of order $r + 1$ is $e(T_r(n))$, with equality only for $T_r(n)$.

Erdős-Stone Theorem, Simonovits' Theorem

Let F be any graph of chromatic number $r+1 \geq 3$. Then $\mathrm{ex}(n, F) = (1+o(1))T_r(n)$ as $n \to \infty$.

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Double Turán Problem

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Let $G_1, G_2, \ldots, G_m$ be graphs on $[n]$ with $E(F) \not\subseteq E(G_i) \cap E(G_j)$ for distinct $i, j \in [m]$.

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Let $G_1, G_2, \ldots, G_m$ be graphs on $[n]$ with $E(F) \not\subseteq E(G_i) \cap E(G_j)$ for distinct $i, j \in [m]$. (*Double F-free*)

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Question

What is the value of $\phi(m, n, F) = \max \sum_{i=1}^{m} G_i$?

Double Turán Problem

Double Turán problems are closely related to Turán problems for 3-uniform hypergraphs H through *link graphs*.

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Definition

For $i \in V(H)$, define graph H_i with

$$V(H_i) = V(H)\backslash\{i\} \quad \text{and} \quad E(H_i) = \{\{j,k\} : \{i,j,k\} \in E(H)\}.$$

Example: Octahedron-free 3-uniform hypergraph H

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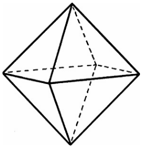

Figure: Octahedron O

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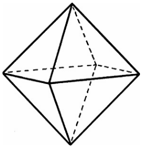

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H is octahedron-free $\implies$ $H_1, H_2, \ldots, H_n$ are double C_4-free.

Double Turán Problem

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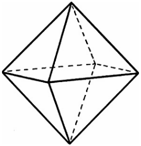

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H is octahedron-free $\implies$ $H_1, H_2, \ldots, H_n$ are double C_4-free.

$$\boxed{\mathrm{ex}(n, O) \leq \phi(n, n, C_4)}$$

Induced Double Turán Problem

What is the *induced* double Turán problem?

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Definition

We call graphs $G_1, G_2, \ldots, G_m$ *induced* if each G_i is an induced subgraph of $\bigcup_{i=1}^{m} G_i$.

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Definition

We call graphs $G_1, G_2, \ldots, G_m$ *induced* if each G_i is an induced subgraph of $\bigcup_{i=1}^m G_i$.

Let graphs $G_1, G_2, \ldots, G_m$ be induced and double F-free.

Question

What is the value of $\phi^*(m, n, F) = \max \sum_{i=1}^m e(G_i)$?

Induced Double Turán Problem

Generalizaed Turán problem

What is the maximum number $\mathrm{ex}(n, F, K_3)$ of triangles in a graph H on $[n]$ with no copy of F as a subgraph?

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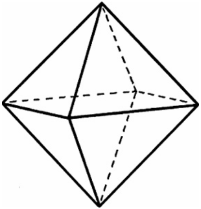
For $i \in V(G)$, define G_i with

$$V(G_i) = V(G) \quad \text{and} \quad E(G_i) = \{\{j,k\} : \{i,j\}, \{j,k\}, \{i,k\} \in E(G)\}$$

Induced Double Turán Problem

Ex. Octahedron-free graph G.



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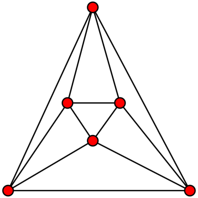

Figure: Octahedron Graph $K_{2,2,2}$

Induced Double Turán Problem

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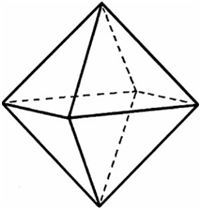

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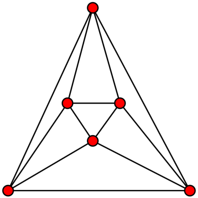

Figure: Octahedron Graph $K_{2,2,2}$

$$G \text{ is } K_{2,2,2}\text{-free} \implies G_1, G_2, \ldots, G_n \text{ are double } K_{2,2}\text{-free.}$$

Induced Double Turán Problem

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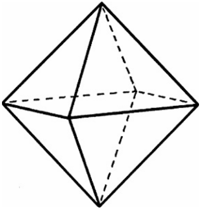

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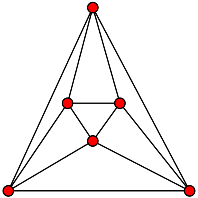

Figure: Octahedron Graph $K_{2,2,2}$

G is $K_{2,2,2}$-free $\implies$ $G_1, G_2, \ldots, G_n$ are double $K_{2,2}$-free.

$$\boxed{\mathrm{ex}(n, K_{2,2,2}, K_3) \leq \phi(n, n, K_{2,2})}$$

Main Results

Theorem A

For $m \geq 3$ and non-bipartite F, if n is large enough, then

$$\phi^*(m, n, F) = m \cdot \mathrm{ex}(n, F),$$

with equality only for identical extremal n-vertex F-free graphs.

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Theorem B

For $m, n, r \geq 3$,

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced K_r-free graphs $G_1, G_2, \ldots, G_m$ only if $G_1 = G_2 = \cdots = G_m = T_{r-1}(n)$.

Main Results

We need the idea of (m, n, k)-*blowup* to state the result for $\phi(m, n, K_r)$.

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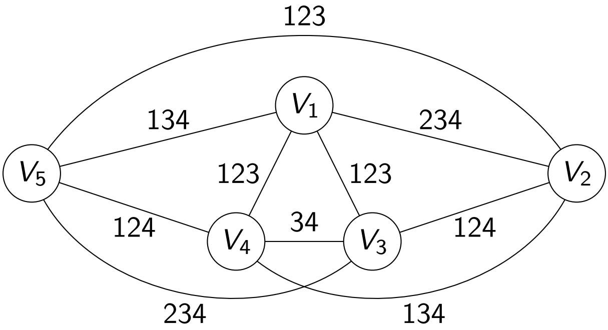

Figure: Example of an $(4, n, 5)$-blowup not containing a double K_3.

Main Results

Let $f(m, n, r)$ denote the maximum possible sum of edges in an double K_r-free (m, n, k)-blowup with $k \leq \binom{m}{2}$.

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Theorem C

For $n \geq 1$,

1. $$\phi(3,n,K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

2. if $r \geq 2$ and $m \geq 1$,

$$\phi(m,n,K_r) = f(m,n,r).$$

In particular,

$$\lim_{n\to\infty} \frac{f(4,n,3)}{\binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor} > 1$$

Main Results

Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

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$$\phi^*(m, n, F) \geq \max\left\{ \binom{n}{2}, m \cdot \mathrm{ex}(n, F) \right\}.$$

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By Theorem 1, the conjecture is true when F is non-bipartite.

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Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

Since

$$\mathrm{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2})$$

the conjecture implies

$$\mathrm{ex}(n, K_{2,2,2}, K_3) \leq O(n^2)$$

Main Results

Theorem D

Let F be a graph. If there exists an extremal F-free n-vertex graph with maximum degree at most $\sqrt{n}/m^2$, then

$$\phi(m, n, F) = \binom{n}{2} + \binom{m}{2}\mathrm{ex}(n, F).$$

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$$\phi(m, n, F) = \binom{n}{2} + \binom{m}{2}\mathrm{ex}(n, F).$$

If P is a path of length 2 and $m = o(n^{1/4})$,

$$\binom{n}{2} + m - 1 \leq \phi^*(m, n, P) \leq \phi(m, n, P) = \binom{n}{2} + \binom{m}{2}\left\lfloor \frac{n}{2} \right\rfloor.$$

Main Results

Theorem E

Let P be a path with two edges. Then $\phi(n, n, P) = \Omega(n^{5/2})$, whereas $\phi^*(n, n, P) = o(n^{5/2})$, as $n \to \infty$. In particular,

$$\lim_{n\to\infty} \frac{\phi^*(n, n, P)}{\phi(n, n, P)} = 0.$$

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$$\lim_{n\to\infty} \frac{\phi^*(n, n, P)}{\phi(n, n, P)} = 0.$$

This shows that $\phi(n, n, P)$ and $\phi^*(n, n, P)$ are very different problems.

Proof of Theorem B

Theorem B

Let $m, n, r \geq 3$. Then

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced K_r-free graphs $G_1, G_2, \ldots, G_m$ only if $G_1 = G_2 = \cdots = G_m = T_{r-1}(n)$.

Proof Roadmap

- Step 1: Reduce to the case of smaller m
- Step 2: Further reduce to an optimization problem
- Step 3: Solve the optimization problem

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Step 1: Reduce to the case of smaller m

Put $G_{i+m} = G_i$ for all $i \in [m]$.

$$\sum_{i=1}^{m} e(G_i) = \frac{1}{k} \sum_{i=1}^{m} [e(G_i) + \cdots + e(G_{i+k-1})]$$

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$$\sum_{i=1}^{m} e(G_i) = \frac{1}{k} \sum_{i=1}^{m} [e(G_i) + \cdots + e(G_{i+k-1})]$$

Since $e(G_i) + \cdots + e(G_{i+k-1}) \leq \phi^*(k, n, F)$,

$$\sum_{i=1}^{m} e(G_i) \leq \frac{m}{k} \cdot \phi^*(k, n, F).$$

Step 2: Further reduce to an optimization problem

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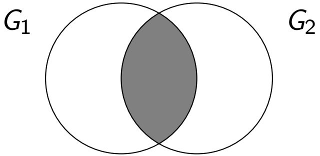

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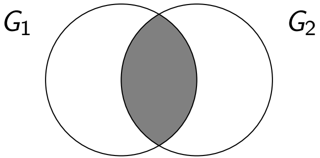

Observation

If G_1, G_2 intersects in t vertices, Then

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2\text{ex}(t, F)$$

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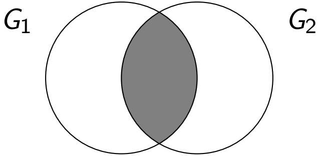

Observation

If G_1, G_2 intersects in t vertices, Then

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2\mathrm{ex}(t, F)$$

$\Longrightarrow$ we are done if the unique maximum is at $t = n$.

Step 3: Solve the optimization problem

Proof Roadmap

- Step 1: Reduce to the case of smaller m
- Step 2: Further reduce to an optimization problem
- **Step 3:** Solve the optimization problem

Extra Ingredients to Prove Theorem A

Theorem A

For $m \geq 3$ and non-bipartite F, if n is large enough, then

$$\phi^*(m, n, F) = m \cdot \mathrm{ex}(n, F),$$

with equality only for identical extremal n-vertex F-free graphs.

- First show that $t = |V(G_1 \cap G_2)| \geq \sqrt{n}$.
- For large enough t, any extremal t-vertex F-free graph contains a spanning $T_{r-1}(t)$.

Proof of Theorem C: Upper Bound

We need to show

$$\phi(m, n, F) \leq \binom{n}{2} + \mathrm{ex}(n, F)\binom{m}{2}.$$

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Let E_S be the set of edges in exactly $\{G_i\}_{i \in S}$.

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$$\phi(m, n, F) \leq \binom{n}{2} + \mathrm{ex}(n, F)\binom{m}{2}.$$

Let E_S be the set of edges in exactly $\{G_i\}_{i \in S}$.

$$\implies \sum_{i=1}^{m} e(G_i) = \sum_{S \subseteq [m]} |S||E_S| \leq \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq 2} (|S| - 1)|E_S|.$$

Proof of Theorem C: Upper Bound

$$\sum_{\substack{S\subseteq[m]\\|S|\geq 2}}(|S|-1)|E_S| = \sum_{\substack{S\subseteq[m],\\|S|=2}}\sum_{T\supseteq S}\frac{(|T|-1)|E_T|}{\binom{|T|}{2}} \leq \sum_{\substack{S\subseteq[m],\\|S|=2}}\sum_{T\supseteq S}|E_T|,$$

as each $T \in [m]$ with $|T| \geq 2$ is counted $\binom{|T|}{2}$ times.

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Observation: If $|S| \geq 2$, the edge set $\bigcup_{T \supseteq S} E_T$ is F-free.

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$$\left| \bigcup_{T \supseteq S} E_T \right| = \sum_{T \supseteq S} |E_T| \leq \mathrm{ex}(n, F)$$

$$\implies \sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S| - 1)|E_S| \leq \sum_{\substack{S \subseteq [m], \\ |S| = 2}} \sum_{T \supseteq S} |E_T| \leq \binom{m}{2} \mathrm{ex}(n, F)$$

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Let $H_1, \ldots, H_{\binom{m}{2}}$ be extremal F-free graphs on $[n]$ with $\Delta(H_i) \leq \sqrt{n}/m^2$

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n.t.s. we can embed each H_i onto $[n]$ s.t. edges are pairwise disjoint.

IDEA: start with any embedding and iteratively decrease overlapping edges

Proof of Theorem C: Lower Bound

Define a (u, v, i)-*swap* by swapping the embedding of vertex u and v of H_i

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Resulting graph remains isomorphic