

Double Turán Problem

April 11, 2025

Contents

1	Introduction	2
1.1	Definitions and Notation	2
1.2	Problem Statement	2
2	Induced Version	4
2.1	Complete Graph F	5
2.2	Non-bipartite F	6
3	General Version	8
3.1	Complete F	9
3.2	Bipartite F	12

1 Introduction

Let F be a graph with at least one edge. A graph G is F -free if G does not contain F as a subgraph. The fundamental question in extremal graph theory is to determine the maximum number of edges in an n -vertex F -free graph. These maxima, denoted $\text{ex}(n, F)$, are referred to as the *extremal numbers* or *Turán numbers* for F .

In this thesis, we investigate a closely related problem which we refer to as the *double Turán problem*. Let G_1, G_2, \dots, G_m be graphs on the same vertex set of size n . We are interested in determining the maximum sum of edges over m graphs G_1, G_2, \dots, G_m whose pairwise intersection is F -free. We denote this quantity as $\text{ex}_2(m, n, F)$ and refer to an F in the intersection of two of the graphs G_i as a *double F* .

1.1 Definitions and Notation

Denote the set of first n positive integers as $[n] = \{1, 2, \dots, n\}$. Given a set X , we denote 2^X as the power set of X .

Let $G = (V, E)$ be a graph. Let $V(G) = V$ denote the vertex set and $E(G) = E$ denote the edge set of G . We note by $v(G) = |V|$ the number of vertices and $e(G) = |E|$ the number of edges in G . For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v .

Given graphs G_1, \dots, G_m on some vertex set V , we denote G_{i_1, \dots, i_k} as graph on V with edge set $E(G_{i_1, \dots, i_k}) = \bigcap_{\alpha=1}^k E(G_{i_\alpha})$. Given two graphs G_1, G_2 , we denote $G_1 \cup G_2$ as the graph on $V(G_1) \cup V(G_2)$ with edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Let s

In this thesis, we reserve n to denote the number of vertices in a graph. We call a n -vertex complete graph K_n , and a complete bipartite graph $K_{a,b}$, where a, b are the size of its parts. We denote P_n as a path with n edges, and C_n as a cycle with n edges. Given graph G, H , define $G + H$ as the graph fully connecting G, H , i.e. $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$.

Given graphs G and F , we say that G is F -free if G does not contain F as a subgraph. We denote $\text{ex}(n, F)$ to be the maximum possible number of edges an F -free graph on n vertices, and we call a F -free graph achieving this maximum an extremal graph for F . Given graphs G_1, \dots, G_m on the same set of vertices and F , we say that G_1, \dots, G_m are pairwise F -free if $E(G_i) \cap E(G_j)$ does not contain F for $i \neq j$. Let v be a vertex from G_1, G_2, \dots, G_m . Unless otherwise specified, we denote $d(v)$ as the sum of degree of v over all G_i .

1.2 Problem Statement

Let $\text{ex}_2(m, n, F)$ be the maximum possible number of edges that m pairwise F -free graphs on n vertices can have. Our goal is to determine $\text{ex}_2(m, n, F)$ for different forbidden graphs F . A trivial construction with $G_1 =$

K_n and G_2, \dots, G_m to be extremal graphs for F yields the lower bound $\text{ex}_2(m, n, F) \geq \binom{n}{2} + (m-1)\text{ex}(n, F)$. In this work, we use this bound as a benchmark to either show its tightness or to improve it.

Additionally, we are also interested in a more restrictive version where G_1, \dots, G_m are induced subgraphs of $G_1 \cup \dots \cup G_m$. Let $\text{ex}_2^*(m, n, F)$ as the maximum possible number of edges that m pairwise F -free graphs on n vertices can have, with the constraint that each graph is an induced subgraph of their union. A trivial construction with $G_1 = \dots = G_m$ to be extremal graphs for F yields the lower bound $\text{ex}_2^*(m, n, F) \geq m \cdot \text{ex}(n, F)$. This is the benchmark we use to determine $\text{ex}_2^*(m, n, F)$. Similar to the non-induced case, we will use this bound as a benchmark and base our work on it.

2 Induced Version

In this chapter, we investigate the case where G_1, \dots, G_m are induced subgraphs of $G_1 \cup \dots \cup G_m$ and are pairwise F -free, for some specified F . Unless otherwise specified, when we say G_1, \dots, G_m are induced subgraph, we mean that they are induced subgraphs of $G_1 \cup \dots \cup G_m$.

The following lemma shows that the problem can be reduced to only two graphs.

Lemma 2.1. *Let $n, m, k \geq 2$ with $m \geq k$, F be some graph. Then*

$$\text{ex}_2^*(m, n, F) \leq \frac{m}{k} \cdot \text{ex}_2^*(k, n, F).$$

Moreover, let G_1, \dots, G_m be induced doubly F -free graphs on $[n]$ and suppose that $\sum_{i=1}^k e(G_i) = \text{ex}_2^(k, n, F)$ only if $G_1 = \dots = G_k$. Then $\sum_{i=1}^m e(G_i) = \text{ex}_2^*(m, n, F)$ only if $G_1 = \dots = G_m$.*

Proof. Let G_1, \dots, G_m be induced subgraphs of $G_1 \cup \dots \cup G_m$ with $E(G_i) \cap E(G_j)$ not containing F for $i \neq j$. Put $G_{i+m} = G_i$ for all $i \in [m]$. Then

$$\sum_{i=1}^m e(G_i) = \frac{1}{k} \sum_{i=1}^m [e(G_i) + \dots + e(G_{i+k-1})] \leq \frac{1}{k} \sum_{i=1}^m \text{ex}_2^*(k, n, F) = \frac{m}{k} \cdot \text{ex}_2^*(k, n, F),$$

which establishes the upper bound.

Suppose $\sum_{i=1}^k e(G_i) = \text{ex}_2^*(k, n, F)$. By assumption $G_1 = \dots = G_k$, so $e(G_i) = \text{ex}_2^*(k, n, F)/k$ for $1 \leq i \leq k$. Hence, the construction $G_1 = \dots = G_m$ meets the upperbound. On the other hand, if $G_1 \neq G_2$ then $\sum_{i=1}^k e(G_i) < \text{ex}_2^*(k, n, F)$. Since $\sum_{i=1}^k e(G_{i+j}) \leq \text{ex}_2^*(k, n, F)$ for all $j \geq 1$, we have $\sum_{i=1}^m e(G_i) < \frac{m}{k} \cdot \text{ex}_2^*(k, n, F)$. Thus the extremal condition is met only when $G_1 = \dots = G_m$. \square

Lemma 2.1 allows us to reduce the problem to the case for two subgraphs G_1, G_2 . Let $S = V(G_1) \cap V(G_2)$, $t = |T|$, $s = |V(G_1) \setminus T|$, and $n - t - s = |V(G_2) \setminus T|$. Note that $t, s \in \mathbb{Z}_{\geq 0}$. Since G_1, G_2 are induced subgraphs of $G_1 \cup G_2$, we have $G_1[T] = G_2[T] = G_1 \cap G_2$. But then $G_1 \cap G_2$ is F -free, so $e(G_1[T]) = e(G_2[T]) \leq \text{ex}(t, F)$. Fixing s and t , the optimal construction to maximize the number of edges over G_1, G_2 is thus putting $G_1 \cap G_2$ as an extremal graph for F on c vertices and connect all edges that are not induced in A . This yields the inequality

$$e(G_1) + e(G_2) \leq \binom{s}{2} + \binom{n-s-t}{2} + (n-t)t + 2\text{ex}(t, F).$$

But then notice $\binom{n-t}{2} > \binom{s}{2} + \binom{n-t-s}{2}$ for $0 < s < n-t$. This implies our construction is optimized when $s = 0$ or $s = n-t$, that is, to let G_2 contain G_1 or the other way around. Hence, we may assume $s = 0$ and define the following construction function:

Definition 2.2. For $n \geq t \geq 1$ and F some graph, define

$$\mathcal{C}(n, t, F) := \binom{n-t}{2} + (n-t)t + 2\text{ex}(t, F).$$

The construction described by $\mathcal{C}(n, t, F)$ are graphs G_1, G_2 on $[n]$, such that G_2 is the t -vertex extremal graph for F and $G_1 = G_2 + K_{n-t}$.

The above discussion also yields the following lemma:

Lemma 2.3. Let F be some graph. For $n \geq 1$,

$$\text{ex}_2^*(2, n, F) = \max_{0 \leq t \leq n} \mathcal{C}(n, t, F).$$

Moreover, the equality holds for n -vertex graphs G_1, G_2 only if G_1, G_2 are the construction described by $\mathcal{C}(n, t_{\max}, F)$, where $t_{\max} = \arg \max_{0 \leq t \leq n} \mathcal{C}(n, t, F)$.

The problem is now reduced to maximizing \mathcal{C} over t . In particular, $\mathcal{C}(n, n, F)$ gives our benchmark construction of $G_1 = G_2$ being the extremal graphs for F on n vertices.

2.1 Complete Graph F

Theorem 2.4. For $n, m, r \geq 3$,

$$\text{ex}_2^*(m, n, K_r) = m \cdot \text{ex}(n, K_r),$$

with equality for n -vertex graphs G_1, G_2, \dots, G_m if and only if $G_1 = \dots = G_m = T_{r-1}(n)$.

Proof. By Lemma 2.1, it suffices to show the case for $m = 3$. Let G_1, G_2, G_3 be induced doubly K_r -free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \text{ex}_2^*(3, n, K_r)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\text{ex}_2^*(3, n, K_r) \geq 3\text{ex}(n, K_r)$. Consequently, we must have $e(G_1) + e(G_2) \geq 2\text{ex}(n, K_r)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\text{ex}(n, K_r)$, it suffices to show that $G_1 = G_2 = T_{r-1}(n)$. In particular, we will use Lemma 2.3 to show that G_1, G_2 is an extremal configuration without containing a double K_r .

Let $t = |V(G_1 \cap G_2)|$. By Turán's Theorem,

$$\text{ex}(t, K_r) - \text{ex}(t-1, K_r) = e(T_{r-1}(t)) - e(T_{r-1}(t-1)) = t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, K_r) - \mathcal{C}(n, t-1, K_r) = -t + 1 + 2[\text{ex}(t, K_r) - \text{ex}(t-1, K_r)] = t + 1 - 2 \left\lceil \frac{t}{r-1} \right\rceil. \quad (2.1)$$

For $r \geq 4$, $\mathcal{C}(n, t, K_r)$ is strictly increasing on t , so by Lemma 2.3,

$$\text{ex}_2^*(2, n, K_r) = \mathcal{C}(n, n, K_r) = 2\text{ex}(n, K_r) = e(G_1) + e(G_2)$$

and $G_1 = G_2 = T_{r-1}(n)$, as desired.

Now suppose $r = 3$. Equation (2.1) shows that $\mathcal{C}(n, t, K_r)$ is non-decreasing on t and $\mathcal{C}(n, t, K_r) > \mathcal{C}(n, t, K_r)$ for even t . By Lemma 2.3, we now have

$$\text{ex}_2^*(2, n, K_r) = \max[\mathcal{C}(n, n, K_r), \mathcal{C}(n, n-1, K_r)] = 2\text{ex}(n, K_r) = e(G_1) + e(G_2),$$

and either $G_1 = G_2 = T_{r-1}(n)$, or $G_2 = T_{r-1}(n-1)$ and $G_1 = G_2 + K_1$. If the latter case is true, then $e(G_3) \geq \text{ex}(n, F) > e(G_2)$, and this contradiction completes the proof. \square

2.2 Non-bipartite F

For non-bipartite F , it is hard to determine the extremal graphs for F in general, but their structures becomes more apparent when n is large. More specifically, the Erdős-Stone Theorem tells us that for large n , the extremal graph for F mimics the structure of the Turán graph. With this idea in mind, the following theorem is a generalization of Theorem 2.4 for large n .

Theorem 2.5. *Let $m, r \geq 3$, and F be a r -colorable graph. Then for large enough n ,*

$$\text{ex}_2^*(m, n, F) = m \cdot \text{ex}(n, F),$$

with equality for n -vertex graphs G_1, G_2, \dots, G_m if and only if $G_1 = \dots = G_m$ are extremal F -free graphs.

Proof. By Lemma 2.1, it suffices to show the case for $m = 3$. Let G_1, G_2, G_3 be induced doubly F -free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \text{ex}_2^*(3, n, F)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\text{ex}_2^*(3, n, F) \geq 3\text{ex}(n, F)$. Consequently, we must have $e(G_1) + e(G_2) \geq 2\text{ex}(n, F)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\text{ex}(n, F)$, it suffices to show that $G_1 = G_2$ are n -vertex F -free extremal graphs. In particular, we will use Lemma 2.3 to show that G_1, G_2 is an extremal configuration without containing a double F .

Let $t = |V(G_1 \cap G_2)|$. If $t < \sqrt{n}$, then

$$2\text{ex}(n, F) \geq 2e(T_{r-1}(n)) \geq 2 \left\lfloor \frac{n^2}{4} \right\rfloor \geq \binom{n}{2} + \binom{\sqrt{n}}{2} > \mathcal{C}(n, t, F).$$

Thus $t \geq \sqrt{n}$. But then for large enough t , any extremal t -vertex F -free graph contains a spanning complete $(r-1)$ -partite subgraph $T_{r-1}(t)$, so we may add $\text{ex}(t-1, F) - e(T_{r-1}(t-1))$ edges to $T_{r-1}(t)$ and still avoid F as a subgraph. Hence for large enough t , we have $\text{ex}(t, F) \geq \text{ex}(t-1, F) - e(T_{r-1}(t-1)) + e(T_{r-1}(t))$, and so

$$\text{ex}(t, F) - \text{ex}(t-1, F) \geq e(T_{r-1}(t)) - e(T_{r-1}(t-1)) \geq t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, F) - \mathcal{C}(n, t-1, F) = -t + 1 + 2[\text{ex}(t, F) - \text{ex}(t-1, F)] \geq t + 1 - 2 \left\lceil \frac{t}{r-1} \right\rceil. \quad (2.2)$$

For $r \geq 4$, $\mathcal{C}(n, t, F)$ is strictly increasing on t , so by Lemma 2.3,

$$\text{ex}_2^*(2, n, F) = \mathcal{C}(n, n, F) = 2\text{ex}(n, F) = e(G_1) + e(G_2),$$

and $G_1 = G_2$ are n -vertex F -free extremal graphs, as desired.

Now suppose $r = 3$. Equation (2.2) shows that $\mathcal{C}(n, t, F)$ is strictly increasing for even t and $\mathcal{C}(n, t, F) \geq \mathcal{C}(n, t-1, F)$ for odd t . By Lemma 2.3, we now have

$$\text{ex}_2^*(2, n, F) = \max[\mathcal{C}(n, n, F), \mathcal{C}(n, n-1, F)] = 2\text{ex}(n, F) = e(G_1) + e(G_2),$$

and either $G_1 = G_2$ are n -vertex extremal F -free graphs, or G_2 is an $(n-1)$ -vertex extremal F -free graph and $G_1 = G_2 + K_1$. If the latter case is true, then $e(G_3) \geq \text{ex}(n, F) > e(G_2)$, and this contradiction completes

the proof. □

For small n , we may not be able to achieve the same result. Consider the case when F is the bowtie graph, i.e. the 5-vertex graph with two triangles sharing a vertex. For $n \leq 4$, the n -vertex extremal graph for F is the complete graph K_n . For $n \geq 5$, the n -vertex extremal graph for F is then $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ plus an edge, and so $\text{ex}(n, F) = \lfloor \frac{n^2}{4} \rfloor + 1$. But then in this case when $n = 5$,

$$\mathcal{C}(5, 4, F) = 2e(K_4) + 4 = 16 > \mathcal{C}(5, 5, F) = 2 \left(\left\lfloor \frac{5^2}{4} \right\rfloor + 1 \right) = 14.$$

This yields an instance where the construction $G_1 = K_{v(F)-1}$ and $G_2 = K_n$ beats our benchmark construction. Thus the following lemma gives a lower bound for n to avoid losing to this construction.

Lemma 2.6. *Let $n, r, k \geq 3$, and F be r -colorable with $|V(F)| = k$. If $n > k^2 - 3k + 2$ and $r - 1$ divides n , then*

$$\mathcal{C}(n, n, F) > \mathcal{C}(n, k - 1, F).$$

Proof. We need to show that

$$2\text{ex}(n, F) - \binom{n}{2} > \binom{k-1}{2}.$$

Since $\text{ex}(n, F) \geq e(T_{r-1}(n)) = \left(1 - \frac{1}{r-1}\right) \frac{n^2}{2} \geq \frac{n^2}{4}$,

$$2\text{ex}(n, F) - \binom{n}{2} \geq \frac{n^2}{2} - \binom{n}{2} = \frac{n}{2} > \frac{k^2 - 3k + 2}{2} = \binom{k-1}{2}.$$

□

3 General Version

TODO: add introduction.

Theorem 3.1. *For all n and graph F ,*

$$\text{ex}_2(m, n, F) = m(1 + o(1))\text{ex}(n, F)$$

as $m \rightarrow \infty$.

Proof. Let $r = v(F)$. Pick $\epsilon > 0$. Reorder G_1, \dots, G_m so that $G_1, \dots, G_{m'}$ are all the G_i 's containing at least $(1 + \epsilon)\text{ex}(n, F)$ edges. A theorem of Simonovits states that G contains at least δn^r copies of F for some $\delta = \delta(\epsilon)$. Since there can be at most $\binom{n}{r}$ copies of F across all G_i 's,

$$m' \delta n^r \leq \binom{n}{r} \leq n^r \implies m' \leq \frac{1}{\delta}.$$

It now follows that

$$\begin{aligned} \sum_{i=1}^m e(G_i) &= \sum_{i=1}^{m'} e(G_i) + \sum_{i=m'+1}^m e(G_i) \\ &\leq \frac{1}{\delta} \binom{n}{2} + \left(m - \frac{1}{\delta}\right) (1 + \epsilon) \text{ex}(n, F) \\ &= m \left[1 + \epsilon + \frac{1}{m\delta} \left(\frac{\binom{n}{2}}{\text{ex}(n, F)} - (1 + \epsilon) \right) \right] \text{ex}(n, F). \end{aligned}$$

Since ϵ is arbitrary, the result follows. \square

Theorem 3.2. *For large enough n , suppose that G_1, \dots, G_m are graphs on common vertex set $[n]$ with no copy of F contained in any k of the G_i 's. If there exists extremal F -free subgraph H on n vertices such that $\binom{m}{k} \Delta(H) = o(n^{1/2})$, then*

$$\text{ex}_2(m, n, F) = (k - 1) \binom{n}{2} + \text{ex}(n, F) \binom{m}{k}.$$

Proof. For $S \subseteq [m]$, let E_S denote the set of edges that are contained in exactly $\{G_i\}_{i \in S}$. Then

$$\sum_{i=1}^m e(G_i) = \sum_{S \subseteq [m]} |S| |E_S| \leq (k - 1) \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq k} (|S| - k + 1) |E_S|.$$

Let $A_S = \bigcup_{T \supseteq S} E_T$, i.e. the set of edges that are contained in all G_i with $i \in S$. When $|S| \geq k$, the edge

set A_S is F -free and thus

$$\sum_{T \supseteq S} |E_T| \leq \text{ex}(n, F).$$

Hence,

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq k}} (|S| - k + 1) |E_S| = \sum_{\substack{S \subseteq [m], \\ |S|=k}} \sum_{T \subseteq S} \frac{(|T| - k + 1) |E_T|}{\binom{|T|}{k}} \leq \sum_{\substack{S \subseteq [m], \\ |S|=k}} \sum_{T \subseteq S} |E_T| \leq \binom{m}{k} \text{ex}(n, F),$$

as each $T \in [m]$ with $|T| \geq k$ is counted $\binom{|T|}{k}$ times in total and $|T| - k + 1 \leq \binom{|T|}{k}$. This proves the upper bound.

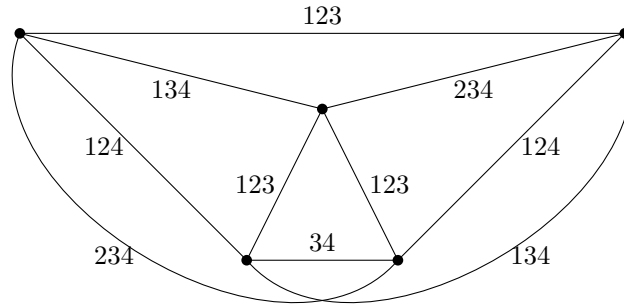
Now we show the bound is tight. In particular, we need to show there exists a construction such that the graph with edge set E_S is an extremal F -free graph, for all $S \subseteq [m]$ of size k . Let $M = \binom{m}{k}$ and H_1, \dots, H_M be copies of an extremal F -free graph on n vertices with $\Delta(H_i) = o(n^{1/2})$ for all i . It suffices to show that we can embed each H_i onto $[n]$ such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each H_i and iteratively decrease the number of intersecting edges. Define a (u, v, i) -swap by swapping the embedding of vertex u and v of H_i , i.e. replacing each edge $\{u, w\} \in E(H_i)$ with the edge $\{v, w\}$ and each edge $\{v, w\} \in E(H_i)$ with the edge $\{u, w\}$. This preserves the type of isomorphism of H_i . Given a vertex v , let $N(v) = N_{H_1}(v) \cup \dots \cup N_{H_M}(v)$. Suppose there exists an intersecting edge $\{u, w\} \in E(H_i) \cap E(H_j)$. Since $|N(u)| \leq M \cdot \Delta(H_i) = o(n^{1/2})$, $|N(u) \cup N(N(u))| = o(n)$ so there exists a vertex $v \notin N(u) \cup N(N(u))$. Since $N(u) \cap N(v) = \emptyset$, performing a (u, v, i) -swap reduces the number of intersecting edges. The result now follows from iterating this process. \square

3.1 Complete F

It turns out that $\text{ex}_2(m, n, K_r)$ can be determined exactly, but the statement of the theorem requires some definitions. Let $k \geq 2$ and sets $S_{ij} \subseteq [m]$, for $1 \leq i < j \leq k$. We call the following type of construction a k -blowup:

Define G_1, G_2, \dots, G_m by partitioning $[n]$ into k sets V_1, V_2, \dots, V_k and letting $\{u, v\} \in E(G_h)$ whenever $u \in V_i, v \in V_j, h \in S_{ij}$. Additionally, for each $i \in [k]$ and $\{u, v\} \subseteq V_i$, we place $\{u, v\}$ in exactly one of G_1 .

Given a k -blowup, we may define m graphs H_1, H_2, \dots, H_m on $[k]$ with edge set $E(H_h) := \{\{i, j\} : h \in S_{ij}\}$ for $h \in [m]$. We call H_1, H_2, \dots, H_m the *pattern* of the k -blowup, and we say that a k -blowup is doubly F -free if its pattern is doubly F -free.



Example of a 5-blowup. A label $i \in \{1, 2, 3, 4\}$ at an edge indicates that the edge is in G_i .

For $M \geq 3$, let $R_M(F)$ denote the M -color Ramsey number for F . That is, the minimum number N such that there exists a monochromatic F in any M -coloring of the edges of K_N .

Let $m \geq 3$ and $M = \binom{m}{2}$. Define $B(m, n, F)$ as the maximum of $\sum_{i=1}^m e(G_i)$ such that G_1, G_2, \dots, G_m form a doubly F -free k -blowup, for some $k \leq R_M(F)$.

We are now ready to state the theorem.

Theorem 3.3. *For $n, m \geq 1$ and $r \geq 2$,*

$$\text{ex}_2(m, n, K_r) = B(m, n, K_r).$$

Proof of Theorem 3.3. Notice that we trivially have $B(m, n, K_r) \leq \text{ex}_2(m, n, K_r)$, so it suffices to show the reverse inequality. That is, we need to show that there exist $k \leq R_M(K_r)$ and S_{ij} for $1 \leq i < j \leq k$ such that the k -blowup construction on meets the desired bound.

Let G_1, G_2, \dots, G_m be graphs on $[n]$ with no double K_r and $\sum_{i=1}^m e(G_i) = \text{ex}_2(m, n, K_r)$. Observe that any pair $\{i, j\} \subseteq [m]$ must be in some G_i , otherwise we may add it to G_1 without creating a double K_r .

We call vertices v, v' *clones* if for all $u \in [n] \setminus \{v, v'\}$ and $i \in [m]$, the edge $\{u, v\} \in E(G_i)$ if and only if $\{u, v'\} \in E(G_i)$. Furthermore, we call $\{v, v'\}$ a *light edge* if $\{v, v'\}$ is in exactly one graph G_i .

We now apply Algorithm 1 to G_1, G_2, \dots, G_m .

Algorithm 1 symmetrization algorithm

```

while  $\exists$  a light edge whose endpoints are not clones do
  among all vertices incident to such an edge, select a vertex  $v$  with maximum degree
   $B_v \leftarrow$  collection of vertices sending a light edge to  $v$  that are not clones of  $v$ 
  while  $B_v \neq \emptyset$  do
    pick  $u \in B_v$ 
     $j \leftarrow$  colour of the light edge from  $u$  to  $v$ 
    for  $1 \leq i \leq m$  do
      if  $i \neq j$  then;
         $N_{G_i}(u) \leftarrow N_{G_i}(v)$ 
      else if  $i = j$  then
         $N_{G_i}(u) \leftarrow (N_{G_i}(v) \setminus \{u\}) \cup \{v\}$ 
      end if
    end for
  end while
end while

```

Claim 3.3.1. *Algorithm 1 terminates.*

Proof. Notice that at the end of the ‘while $B_v \neq \emptyset$ ’ loop, every vertex sending a light edge to v is a clone of v . This implies v along with the set L_v of vertices receiving light edges from v induce a clique of size at least two in some G_i , and an empty graph in every other graph G_j with $j \neq i$. Moreover, any vertex $w \notin L_v$ sends edges to either all or none of the vertices in L_v , and if w is incident to L_v , then w sends edges to L_v in at least two graphs. It now follows that no light edge incident with a vertex in L_v will be picked again in an iteration of the outermost while loop. Thus the algorithm can run through at most $n/2$ such iterations, and so it terminates. \square

Claim 3.3.2. G'_1, G'_2, \dots, G'_m do not contain a double K_r and $\sum_{i=1}^m e(G'_i) = \text{ex}_2(m, n, K_r)$.

Proof. Note that we replace u by a clone of v in the for loop of Algorithm 1. Since $\{u, v\}$ remains to be an light edge in this step, u and v cannot both belong to a double K_r in the modified graphs. Furthermore, any double K_r containing u after the for loop arises from a double K_r containing v prior to the for loop. But then G_1, G_2, \dots, G_m contained no double K_r to begin with, so G'_1, G'_2, \dots, G'_m do not contain a double K_r .

We now show that the algorithm does not reduce the number of edges. By our choice of v , we know $d(v) \geq d(u)$ for all $u \in B_v$ prior to the for loop. Hence, replacing u with a clone of v does not decrease the number of edge over a complete iteration of the inner while loop. Therefore, $\sum_{i=1}^m e(G'_i) = \text{ex}_2(m, n, K_r)$. \square

Hence, the algorithm outputs graphs G'_1, G'_2, \dots, G'_m with $\text{ex}_2(m, n, K_r)$ edges and the additional property that light edges come in ‘clone cliques.’ We may thus partition the vertex set $[n]$ into k disjoint sets V_1, V_2, \dots, V_k , such that each V_i induces a clique of light edges from the same graph. Moreover, for distinct $i, j \in [k]$, define S_{ij} to be the set of all edges between V_i and V_j , and note that any edge in S_{ij} appear in at least two modified graphs. The sets S_{ij} now yield a k -blowup. Notice that if the pattern of the k -blowup contains a double K_r , then the original graphs G_1, G_2, \dots, G_m must have contained a double K_r as well, contradiction. Thus the k -blowup is doubly K_r -free.

It remains to show that $k < R_M(K_r)$. For each edge $\{i, j\} \subseteq [k]$ in the pattern of the k -blowup, we assign an arbitrary distinct pair $\{a, b\} \subseteq L_{ij} \subseteq [m]$ to $\{i, j\}$. If $k \geq R_M(K_r)$, then there exists K_r in the pattern of the k -blowup colored by some distinct pair $\{a, b\} \subseteq [m]$. But then this implies the pattern of the k -blowup contains a double K_r , contradiction. This completes the proof. \square

When $m = 2$,

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_{1,2}) \leq \binom{n}{2} + \text{ex}(n, K_3)$$

which meets the benchmark bound. Surprisingly, our desired bound is also met when $m = 3$:

Theorem 3.4. *For all n ,*

$$\text{ex}_2(3, n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Proof. Define $H_k \subseteq G$ be the graph with edges contained in at least k number of G_i ’s and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \leq \frac{n^2}{2}$. Notice H_2 must not contain any triangles with two edges in H_3 , so

$$e(H_2) + e(H_3) \leq \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}|.$$

Let H'_3 be the graph with the same vertex set as H_3 and edge set $\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}$. It suffices to show that $\frac{n}{2} \geq e(H_3) - e(H'_3)$.

Let $d_1 \geq d_2 \geq \dots \geq d_n$ and $f_1 \geq f_2 \geq \dots \geq f_n$ each be the degree sequence of H_3 and H'_3 , respectively. We show that $f_i \geq d_i - 1$ for all i . Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i . Let $S_i = |N_{H_3}(v_1) \cup \dots \cup N_{H_3}(v_i)|$. Since

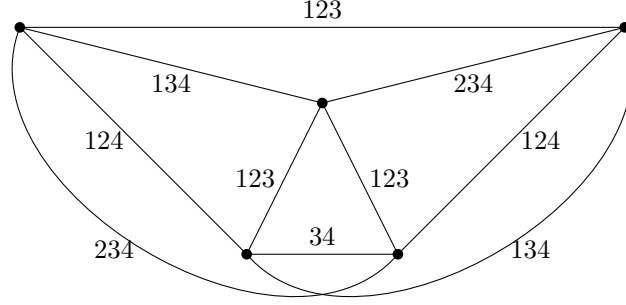
$$\sum_{u \in S_i} d_{H_3}(u) \geq d_1 + \dots + d_i,$$

we have that $|S_i| \geq i$. But then $S_i \setminus \{u_1, \dots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H'_3}(u) \geq d_i - 1$. Hence, $f_i \geq d_i - 1$ for all i , which yields

$$e(H'_3) = \frac{1}{2} \sum_{i=1}^n f_i \geq \frac{1}{2} \sum_{i=1}^n (d_i - 1) = e(H_3) - \frac{n}{2}.$$

□

However, the bound in Proposition 3.1 is not tight for $m \geq 4$, as shown in the following graph:



The number on each edge denotes the set of G_i 's that contain the edge.

The above graph contains 29 edges, which exceeds the bound $\binom{5}{2} + 3\lfloor \frac{5^2}{4} \rfloor = 28$ by 1. By blowing up the above graph, we can construct a graph with $n \in 10\mathbb{Z}$ vertices that contains

$$5 \binom{n/5}{2} + 29 \cdot \frac{(n/5)^2}{4}$$

edges, which exceeds the bound $\binom{n}{2} + 3\lfloor \frac{n^2}{4} \rfloor$ by $n^2/100$.

3.2 Bipartite F

In this section, we discuss the case where F is bipartite. In particular, we focus on the cases where $F \subseteq K_{2,2}$ is P_2 , a path of length 2, or M_2 , a matching with two edges.

Theorem 3.5.

$$\text{ex}_2(m, n, P_2) \leq \left(\frac{1}{2} + o(1) \right) \min\{n^2 \sqrt{m}, mn^{3/2}\},$$

as $n \rightarrow \infty$ or $m \rightarrow \infty$. Moreover,

$$\text{ex}_2(m, n, P_2) = \left(\frac{1}{2} + o(1) \right) mn^{3/2},$$

for $\sqrt{n} \leq m \leq n$.

Proof. Let G_1, \dots, G_m be graphs on $[n]$ not containing a P_2 . We first show the claimed upperbound and then show the tightness of the bound when $\sqrt{n} \leq m \leq n$.

Claim 3.5.1. $\text{ex}_2(m, n, P_2) \leq mn \cdot \frac{1 + \sqrt{4n^2/m + 1}}{4}$.

Proof. Since there are no double P_2 ,

$$\sum_{i=1}^m \#\{P_2 \subseteq G_i\} \leq \#\{P_2 \subseteq G\}.$$

For all G_i , each vertex v in G_i along with two of its neighbors form one unique P_2 , so

$$\#\{P_2 \subseteq G_i\} = \sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2}.$$

By Jensen's inequality,

$$\sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2} \geq n \binom{d_{G_i}(v)/n}{2} = n \binom{2e(G_i)/n}{2} \geq \frac{2(e(G_i))^2}{n} - e(G_i).$$

On the other hand, since each three vertices in G can form at most three P_2 's,

$$\#\{P_2 \subseteq G\} \leq 3 \binom{n}{3} \leq \frac{n^3}{2}.$$

Combining the above inequalities yields and using Jensen's inequality once more yields

$$\frac{2m}{n} \left(\frac{1}{m} \sum_{i=1}^m e(G_i) \right)^2 - \sum_{i=1}^m e(G_i) \stackrel{\text{Jensen's}}{\leq} \sum_{i=1}^m \frac{2(e(G_i))^2}{n} - e(G_i) \leq \frac{n^3}{2}.$$

Solving the quadratic equation gives

$$\sum_{i=1}^m e(G_i) \leq mn \cdot \frac{1 + \sqrt{4n^2/m + 1}}{4}.$$

□

Claim 3.5.2. $\text{ex}_2(m, n, P_2) \leq \frac{1}{2}(mn^{3/2} + n^2)$.

Proof. For each vertex $u \in [n]$, define H_u as the $m \times n$ bipartite graph with edge set $E(H_u) := \{\{v, i\} : \{u, v\} \in E(G_i)\}$. If H_u contains a quadrilateral $\{v, i\}, \{v, j\}, \{w, i\}, \{w, j\}$, then $\{u, v\}, \{u, w\}$ form a double P_2 in $G_i \cap G_j$, contradiction. Thus we conclude that H_u is quadrilateral-free, and therefore $e(H_u) \leq m\sqrt{n} + n$, by the Kővari-Sós-Turán theorem. It now follows that

$$\sum_{i=1}^m e(G_i) = \frac{1}{2} \sum_{u \in V(G)} e(H_u) \leq \frac{1}{2}(mn^{3/2} + n^2).$$

□

The above two claims yield the desired upper bound. We now show the lower bound.

Claim 3.5.3. $\text{ex}_2(m, n, P_2) \geq (1/2 + o(1))mn^{3/2}$, for $\sqrt{n} \leq m \leq n$.

Proof. Suppose G_1, G_2, \dots, G_n are graphs on $[n]$ containing no double P_2 and $\sum_{i=1}^n e(G_i) \geq (1/2 + o(1))n^{5/2}$, with $e(G_1) \geq e(G_2) \geq \dots \geq e(G_n)$. Then G_1, G_2, \dots, G_m are graphs with no double P_2 and $\sum_{i=1}^m e(G_i) \geq (1/2 + o(1))mn^{3/2}$. Hence, it suffices to prove the case for $m = n$.

Consider a finite projective plane with n points and n lines, with prime q chosen so that $n = (1 + o(1))(q^2 + q + 1)$ as $q \rightarrow \infty$. Let $S_1, \dots, S_n \subseteq [n]$ be the n lines of the projective plane. Note that each line S_i contains $q + 1$ points, and the intersection of any two distinct lines S_i, S_j contains $|S_i \cap S_j| = 1$ point.

Define G_1, \dots, G_n to be graphs on $[n]$, each with edge set

$$E(G_i) := \{\{j, k\} \subseteq [n] : j \neq k, j + k \in S_i \pmod n\}.$$

Note that the intersection of distinct G_i, G_j is P_2 free: since $|S_i \cap S_j| = 1$, if $\{a, b\}, \{a, c\} \in E(G_i) \cap E(G_j)$, then $a + b = a + c$ so $b = c$.

We now count the number of edges in G_1, \dots, G_n . Since $|S_i| = q + 1$, for each point $j \in [n]$, there are $q + 1$ choices for $k \in [n]$ such that $j + k \in S_i$. But then we have to avoid counting the same edge twice and loops, so the number of edges in G_i is

$$e(G_i) = \frac{n(q + 1) - \#\text{loops counted for } G_i}{2}.$$

If $j \in [n]$ is even, then $k = j/2$ is the unique number in $[n]$ such that $k + k = j \pmod n$. If $j \in [n]$ is odd, then $k = (n + j)/2$ is the unique number in $[n]$ such that $k + k = j \pmod n$, as n is even. Hence, for each $j \in S_i$, there exists a unique $k \in [n]$ such that $k + k = j \pmod n$, and thus

$$\#\text{loops counted for } G_i = |S_i| = q + 1.$$

Since $q + 1 = (1 + o(1))n^{1/2}$, the number of edges in G_1, \dots, G_n is

$$\sum_{i=1}^n e(G_i) = n \cdot \frac{n(q + 1) - (q + 1)}{2} = \left(\frac{1}{2} + o(1)\right) n^{5/2},$$

as $n \rightarrow \infty$. □

Claim 3.5.4. $\text{ex}_2(m, n, P_2) \geq (1/2 + o(1))\sqrt{mn}n^2$, for $n < m \leq n^2$.

Proof. □

□

Theorem 3.6. For all n, m ,

$$\text{ex}_2(m, n, M_2) \leq n^{5/2}.$$

Proof. Notice that $\#\{M_2 \subseteq G\} = \binom{e(G)}{2}$. On the other hand, each four vertices in G can form at most three M_2 's, so $\#\{M_2 \subseteq G\} \leq 3\binom{n}{4} \leq \frac{n^4}{8}$. By the same argument as in Theorem 3.4, we have

$$\sum_{i=1}^n \binom{e(G_i)}{2} = \sum_{i=1}^n \#\{M_2 \subseteq G_i\} \leq \#\{M_2 \subseteq G\} \leq \frac{n^4}{8}.$$

By Jensen's inequality,

$$\sum_{i=1}^n \binom{e(G_i)}{2} \geq n \binom{\sum_{i=1}^n e(G_i)/n}{2} = \frac{1}{2n} \left[\left(\sum_{i=1}^n e(G_i) \right)^2 - n \sum_{i=1}^n e(G_i)^2 \right].$$

Combining the above inequalities yields

$$\left(\sum_{i=1}^n e(G_i) \right)^2 - n \sum_{i=1}^n e(G_i)^2 \leq \frac{n^5}{4},$$

and solving the quadratic inequality gives

$$\sum_{i=1}^n e(G_i) \leq n^{5/2}.$$

□

We may obtain an exact result if we forbid both P_2 and M_2 at the same time:

Theorem 3.7. *For all n, m ,*

$$\text{ex}_2(m, n, \{P_2, M_2\}) = n^2 - n.$$

Proof. Denote the set of G_i 's as $\{G_i\} = \{G_1, \dots, G_n\}$, and the set of distinct pairs of G_i 's as $\{G_i\}^2 = \{\{G_j, G_k\} : j \neq k\}$. Consider the bipartite graph H with vertex set $V(H) = \{G_i\} \sqcup E(K_n)$ and edge set $E(H) = \{\{G_j, e\} \in \{G_i\} \times E(K_n) : e \in G_j\}$. Define $\phi : \{G_i\}^2 \rightarrow 2^{E(K_n)}$ by sending each $\{G_j, G_k\}$ to their common edge set $E(G_j) \cap E(G_k)$. Notice that each distinct G_j, G_k have at most one edge in common, so $|\phi(G_j, G_k)| \leq 1$. On the other hand, each edge $e \in E(G)$ can be obtained via ϕ by $\binom{d_H(e)}{2}$ possible distinct pairs (G_j, G_k) , and thus $|\phi^{-1}(e)| = \binom{d_H(e)}{2}$. But then

$$\binom{n}{2} \geq \sum_{(G_j, G_k) \in \{G_i\}^2} |\phi(G_j, G_k)| = \sum_{e \in E(K_n)} |\phi^{-1}(e)| = \sum_{e \in E(K_n)} \binom{d_H(e)}{2}.$$

By Jensen's inequality,

$$\sum_{e \in E(K_n)} \binom{d_H(e)}{2} \geq \binom{n}{2} \left(\frac{\sum_{e \in E(K_n)} d_H(e)}{2} / \binom{n}{2} \right) = \binom{n}{2} \left(\frac{\sum_{i=1}^n e(G_i)}{2} / \binom{n}{2} \right).$$

Combining the above inequalities yields

$$2 \binom{n}{2}^2 \geq \left(\sum_{i=1}^n e(G_i) \right)^2 - \binom{n}{2} \sum_{i=1}^n e(G_i),$$

and the result now follows from solving the quadratic inequality.

To see that this bound is tight, consider the construction such that for each distinct $i, j \in [n]$, $E(G_i) \cap E(G_j)$ contains exactly one unique edge $e \in K_n$. The number of edges in this construction is $2 \binom{n}{2} = n^2 - n$. □