# Double Turán Problem

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# Overview

# Turán Problem

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## Turán Number

$$\mathrm{ex}(n, F) := \max\{e(G) : |V(G)| = n \text{ and } F \not\subseteq G\}$$

# Turán Problem

## Turán's Theorem [Tur41]

For $r+1 \geq 3$,

$$\mathrm{ex}(n, K_{r+1}) = e(T_r(n)),$$

with equality for graph $G$ only if $G = T_r(n)$.

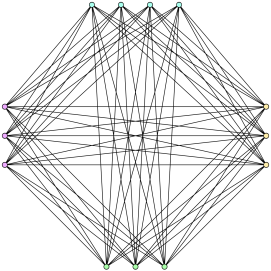
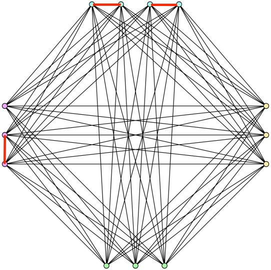


Figure: $T_4(13)$

# Erdős-Stone Theorem

**Erdős-Stone Theorem [ES46], Simonovits' Theorem [ES66]**

Let $F$ be any graph of chromatic number $r + 1 \geq 3$. Then $\mathrm{ex}(n, F) = (1 + o(1))T_r(n)$ as $n \to \infty$.



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Is this tight?

# Double Turán Problem: Main Result

**Theorem A**

Let $n \geq 1$ and $F$ be a non-bipartite graph. Then as $m \to \infty$

$$\phi(m, n, F) = (1 + o(1))m \cdot \mathrm{ex}(n, F).$$

# Double Turán Problem: Main Results

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But

$$\lim_{n\to\infty} \frac{\phi(4, n, K_3)}{\binom{n}{2} + 3\lfloor n/4 \rfloor} > 1$$

# Main Results

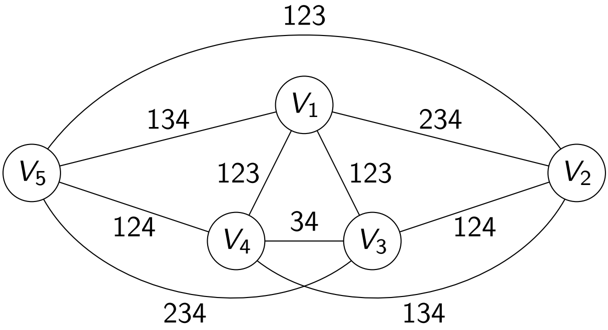


Figure: Construction for four graphs not containing a double-$K_3$

# Double Turán Problem: Main Result

**Theorem B**

Let $F$ be a graph. If there exists an extremal $F$-free $n$-vertex graph with maximum degree at most $\sqrt{n}/m^2$, then

$$\phi(m, n, F) = \binom{n}{2} + \binom{m}{2}\mathrm{ex}(n, F).$$

# Double Turán Problem: Motivation

Double Turán problems are closely related to Turán problems for 3-uniform hypergraphs $H$ through link graphs.

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## Link Graph

For $i \in V(H)$, define graph $H_i$ with

$$V(H_i) = V(H) \backslash \{i\} \quad \text{and} \quad E(H_i) = \{\{j,k\} : \{i,j,k\} \in E(H)\}.$$

# Double Turán Problem: Motivation

Example: Octahedron-free 3-uniform hypergraph $H$

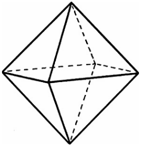

Figure: Octahedron $K^{(3)}_{2,2,2}$

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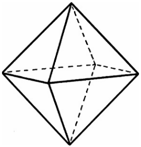

Figure: Octahedron $K_{2,2,2}^{(3)}$

Erdős conjectured that $\mathrm{ex}(n, K_{2,2,2}^{(3)}) = \Theta(n^{11/4})$ [Erd64].

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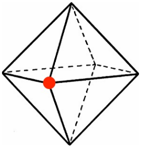

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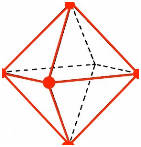

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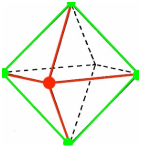

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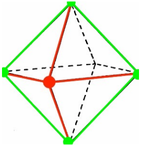

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$H$ is octahedron-free $\implies$ $H_1, H_2, \ldots, H_n$ are double $K_{2,2}$-free.

$$\boxed{\mathrm{ex}(n, K^{(3)}_{2,2,2}) \leq \phi(n, n, K_{2,2})}$$

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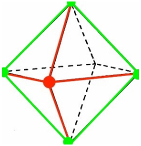

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Let graphs $G_1, G_2, \ldots, G_m$ be induced and double $F$-free.

**Induced Double Turán Problem**

What is the value of $\phi^*(m, n, F) = \max \sum_{i=1}^{m} e(G_i)$?

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## Conjecture

Let $F$ be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

# Main Results

**Theorem C**

For $m \geq 3$ and non-bipartite $F$, if $n$ is large enough, then

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with equality only for identical extremal $n$-vertex $F$-free graphs.

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$\implies$ Conjecture is true for non-bipartite $F$.

**Theorem D**

For $m, n, r \geq 3$,

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced $K_r$-free graphs $G_1, G_2, \ldots, G_m$ only if $G_1 = G_2 = \cdots = G_m = T_{r-1}(n)$.

# Induced Double Turán Problem: Motivation

## Generalized Turán problem

What is the maximum number $\mathrm{ex}(n, F, K_3)$ of triangles in an $n$-vertex $F$-free graph $G$?

Studied by Alon and Shikhelman [AS16] and Kostochka, Mubayi and Verstraete [KMV15; MM23; MV16].

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# Induced Double Turán Problem: Motivation

Ex. Octahedron-free graph $G$.

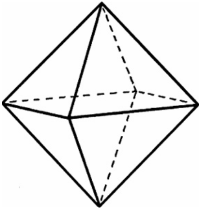

Figure: Octahedron

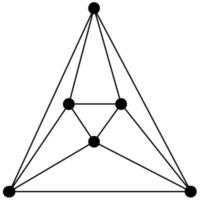

Figure: Octahedron Graph $K_{2,2,2}$

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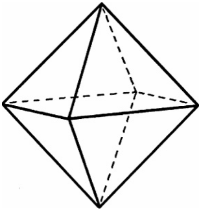

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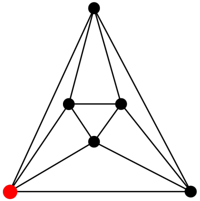

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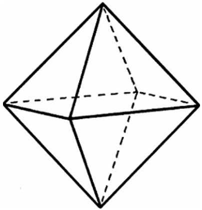

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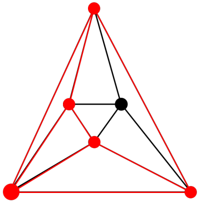

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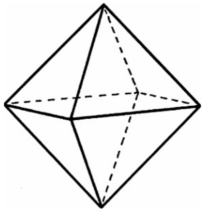

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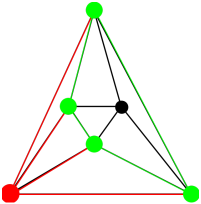

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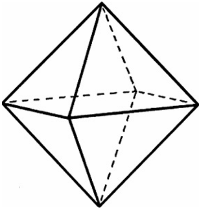

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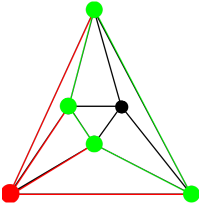

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$G$ is $K_{2,2,2}$-free $\implies$ $G_1, G_2, \ldots, G_n$ are induced and double $K_{2,2}$-free.

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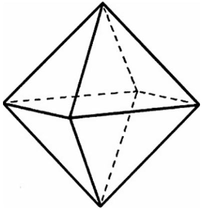

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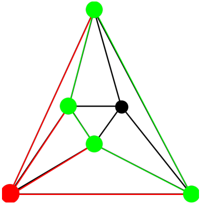

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# Induced Double Turán Problem: Motivation

## Conjecture

Let $F$ be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

Conjecture implies

$$\mathrm{ex}(n, K_{2,2,2}, K_3) = \Theta(\phi^*(n, n, K_{2,2})) = \Theta(n^{5/2}).$$

**Theorem E**

Let $P$ be a path with two edges. Then $\phi(n, n, P) = \Omega(n^{5/2})$, whereas $\phi^*(n, n, P) = o(n^{5/2})$, as $n \to \infty$. In particular,

$$\lim_{n\to\infty} \frac{\phi^*(n, n, P)}{\phi(n, n, P)} = 0.$$

# Main Results

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$$\lim_{n\to\infty} \frac{\phi^*(n, n, P)}{\phi(n, n, P)} = 0.$$

This shows that $\phi(m, n, F)$ and $\phi^*(m, n, F)$ can be very different problems.

# Proof of Theorem B: Upper Bound

## Theorem B

Let $F$ be a graph. If there exists an extremal $F$-free $n$-vertex graph with maximum degree at most $\sqrt{n}/m^2$, then

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Let $E_S$ be the set of edges in exactly $\{G_i\}_{i \in S}$.

$$\implies \sum_{i=1}^{m} e(G_i) = \sum_{S \subseteq [m]} |S||E_S| \leq \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq 2} (|S| - 1)|E_S|.$$

# Proof of Theorem B: Upper Bound

$$\sum_{\substack{S\subseteq[m] \\ |S|\geq 2}} (|S|-1)|E_S| = \sum_{\substack{S\subseteq[m], \\ |S|=2}} \sum_{T\supseteq S} \frac{(|T|-1)|E_T|}{\binom{|T|}{2}} \leq \sum_{\substack{S\subseteq[m], \\ |S|=2}} \sum_{T\supseteq S} |E_T|,$$

as each $T \in [m]$ with $|T| \geq 2$ is counted $\binom{|T|}{2}$ times.

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$$\implies \sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S| - 1)|E_S| \leq \sum_{\substack{S \subseteq [m], \\ |S| = 2}} \sum_{T \supseteq S} |E_T| \leq \binom{m}{2} \mathrm{ex}(n, F)$$

This proves the upper bound.

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$$\phi(m, n, F) \geq \binom{n}{2} + \mathrm{ex}(n, F)\binom{m}{2}.$$

For $1 \leq i < j \leq m$, let $H_{ij}$ be an extremal $F$-free graph on $[n]$ with maximum degree $\Delta(H_{ij}) \leq \sqrt{n}/m^2$

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$\implies$ We can pack these $\binom{m}{2}$ $H_{ij}$'s into $[n]$ s.t. $E(H_{i_1 j_1}) \cap E(H_{i_2 j_2}) = \emptyset$

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Add edges not in any of $H_{ij}$ to $G_1$.

$$\sum_{i=1}^{m} e(G_i) = 2\sum e(H_{ij}) + \left| E(K_n) \setminus \bigsqcup E(H_{ij}) \right| = \binom{n}{2} + \binom{m}{2} \mathrm{ex}(n, F)$$

# Proof of Theorem D

**Theorem D**

Let $m, n, r \geq 3$. Then

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced $K_r$-free graphs $G_1, G_2, \ldots, G_m$ only if $G_1 = G_2 = \cdots = G_m = T_{r-1}(n)$.

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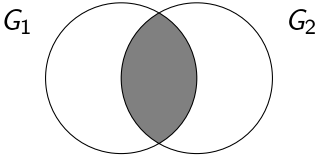
IDEA: Reduce to $m = 2$.

# Proof of Theorem D



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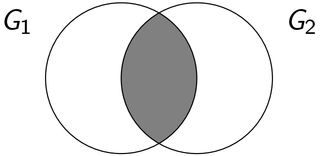


$G_1 \cap G_2$ (gray area) is double $K_r$-free.

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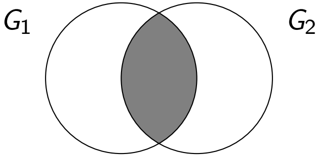


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If $G_1, G_2$ intersects in $t$ vertices, then

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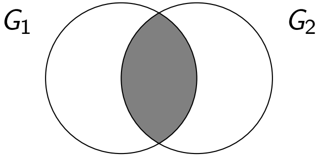


$G_1 \cap G_2$ (gray area) is double $K_r$-free.

If $G_1, G_2$ intersects in $t$ vertices, then

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2T_{r-1}(t)$$

$\implies$ we are done if the unique maximum occurs at $t = n$.

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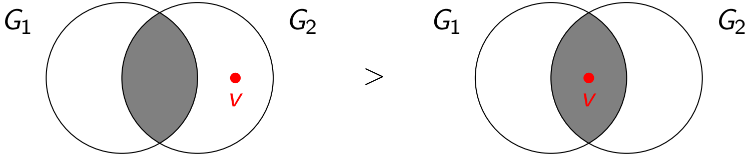

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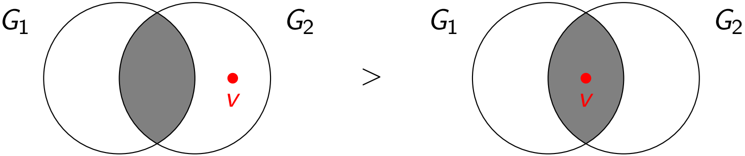


Put $f(t,r) := \binom{n-t}{2} + (n-t)t + 2T_{r-1}(t)$

For example, if $r = 4$,

$$f(t+1,4) - f(t,4) = -t + 1 + 2[\underbrace{T_3(t+1) - T_3(t)}_{t - \lceil \frac{t}{2} \rceil}] = t + 1 - 2\left\lceil \frac{t}{2} \right\rceil > 0$$

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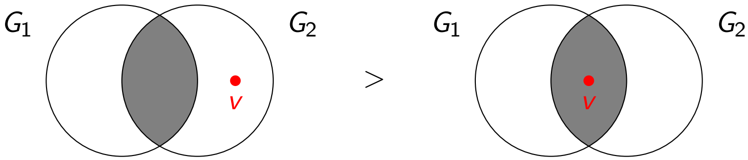


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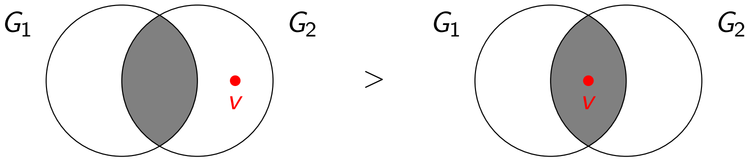


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$\implies$ The larger the intersection, the larger the sum of edges.

$\implies$ The best we can do is $G_1 = G_2 = T_3(n)$.

# Concluding Remarks

- The conjecture we stated is broadly open for bipartite graphs: we cannot even determine the case for $P_2$.

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  $\phi^*(n, n, K_{2,2,2})$ is equivalent (up to constant factors) to the notoriously difficult $\mathrm{ex}(n, K_{2,2,2}, K_3)$ problem. If the conjecture is true, then $\mathrm{ex}(n, K_{2,2,2}, K_3) = \Theta(n^{5/2})$.

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  We only know $\phi(3, n, K_3) = \binom{n}{2} + \lfloor n/2 \rfloor$. The rest is open.

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