Double Turán Problem

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April 21, 2025

Introduction

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Question

Given a graph F, how many edges can an n-vertex graph have while containing no copy of F as a subgraph?

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What is the hypergraph Turán problem?

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Given an r-uniform graph F, how many edges can an n-vertex r-uniform hypergraph have while containing no copy of F as a subgraph?

Introduction

Let F be a graph. We call the following quantity the *Turán number* or *extremal number* of F:

Definition

$$\mathrm{ex}(n, F) := \max\{e(G) : |V(G)| = n \text{ and } F \not\subseteq G\}$$

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The *Turán graph* $T_r(n)$ is the complete r-partite graph on n vertices with parts of size $\lfloor n/r \rfloor$ and $\lceil n/r \rceil$.

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examples of turan graphs

Introduction

Turán's theorem

The maximum number of edges in an n-vertex graph containing no clique of order $r+1$ is $e(T_r(n))$, with equality only for $T_r(n)$.

Erdős-Stone Theorem, Simonovits' Theorem

Let F be any graph of chromatic number $r+1 \geq 3$. Then $\mathrm{ex}(n,F) = (1+o(1))T_r(n)$ as $n \to \infty$.

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Let $G_1, G_2, \ldots, G_m$ be graphs on $[n]$ with $E(F) \not\subseteq E(G_i) \cap E(G_j)$ for distinct $i, j \in [m]$. (*Double F-free*)

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What is the value of $\phi(m, n, F) = \max \sum_{i=1}^{m} G_i$?

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Let graphs $G_1, G_2, \ldots, G_m$ be induced and double F-free.

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Let graphs $G_1, G_2, \ldots, G_m$ on $[n]$ be induced and double F-free.

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Motivation

Double Turán problems are closely related to hypergraph Turán problems through *link graphs*.

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Let H be a triple system, i.e. a 3-uniform hypergraph.

Definition

For $i \in V(H)$, the *link graph* of i, denoted H_i, is the graph with $V(H_i) = V(H) \backslash \{i\}$ and $E(H_i) = \{\{j,k\} : \{i,j,k\} \in E(H)\}$.

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examples of link graphs

Motivation

Let F^+ denote the triple system on $[n]$ with vertex set $V(F) \cup \{x, y\}$ and edge set $\{e \cup \{x\}, e \cup \{y\} : e \in E(F)\}$.

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Observation

If H is an F^+-free triple system on $[n]$, then the link graphs $H_1, \ldots, H_n$ are double F-free.

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If H is an F^+-free triple system on $[n]$, then the link graphs $H_1, \ldots, H_n$ are double F-free.

$$\mathrm{ex}(n, F^+) \leq \phi(n, n, F)$$

Motivation

Let F' be the graph consisting of all pairs contained in triples in F^+.

Generalizaed Turán problem

What is the maximum number $\text{ex}(n, F', K_3)$ of triangles in a graph H on $[n]$ with no copy of F' as a subgraph?

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Observation

$H_1, H_2, \ldots, H_n$ are induced and double F-free.

$$\text{ex}(n, F', K_3) \leq \phi^*(n, n, F)$$

The Induced Case

Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

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$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

$$\phi^*(m, n, F) \geq \max\left\{ \binom{n}{2}, m \cdot \mathrm{ex}(n, F) \right\}.$$

Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

Since

$$\mathrm{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2})$$

the conjecture implies

$$\mathrm{ex}(n, K_{2,2,2}, K_3) \leq O(n^{5/2})$$

The Induced Case

Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

Conjecture holds for non-bipartite F:

Theorem A

For all $m \geq 3$ and non-bipartite F, if n is large enough, then

$$\phi^*(m, n, F) = m \cdot \mathrm{ex}(n, F),$$

with equality only for identical extremal n-vertex F-free graphs.

The Induced Case

Theorem B

Let $m, n, r \geq 3$. Then

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced K_r-free graphs $G_1, G_2, \ldots, G_m$ only if $G_1 = G_2 = \cdots = G_m = T_{r-1}(n)$.

Proof of Theorem B

Proof Roadmap

- Step 1: Reduce to the case of smaller m
- Step 2: Further reduce to an optimization problem
- Step 3: Solve the optimization problem

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Step 1: Reduce to the case of smaller m

Let $G_1, \dots, G_m$ be induced double F-free graphs on $[n]$.

Lemma 1

For $2 \le k \le m$,

$$\phi^*(m, n, F) \le \frac{m}{k} \cdot \phi^*(k, n, F).$$

Moreover, suppose $\sum_{i=1}^{k} e(G_i) = \phi^*(k, n, F)$ only if $G_1 = \cdots = G_k$. Then $\sum_{i=1}^{m} e(G_i) = \phi^*(m, n, F)$ only if $G_1 = \cdots = G_m$.

Step 1: Reduce to the case of smaller m

Put $G_{i+m} = G_i$ for all $i \in [m]$.

$$\sum_{i=1}^{m} e(G_i) = \frac{1}{k} \sum_{i=1}^{m} [e(G_i) + \cdots + e(G_{i+k-1})]$$

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$$\sum_{i=1}^{m} e(G_i) = \frac{1}{k}\sum_{i=1}^{m}[e(G_i) + \cdots + e(G_{i+k-1})]$$

Since $e(G_i) + \cdots + e(G_{i+k-1}) \leq \phi^*(k, n, F)$,

$$\sum_{i=1}^{m} e(G_i) \leq \frac{m}{k} \cdot \phi^*(k, n, F).$$

Step 1: Reduce to the case of smaller m

Suppose $\sum_{i=1}^{m} e(G_i) = \frac{m}{k} \cdot \phi^*(k, n, F)$ and $G_1 \neq G_2$.

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Suppose $\sum_{i=1}^{m} e(G_i) = \frac{m}{k} \cdot \phi^*(k, n, F)$ and $G_1 \neq G_2$.

By assumption $\sum_{i=1}^{k} e(G_i) < \phi^*(k, n, F)$.

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Suppose $\sum_{i=1}^{m} e(G_i) = \frac{m}{k} \cdot \phi^*(k, n, F)$ and $G_1 \neq G_2$.

By assumption $\sum_{i=1}^{k} e(G_i) < \phi^*(k, n, F)$.

But then $e(G_i) + \cdots + e(G_{i+k-1}) > \phi^*(k, n, F)$ for some $j \geq 1$, contradiction.

Step 2: Further reduce to an optimization problem

Proof Roadmap

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- **Step 2** Further reduce to an optimization problem
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Step 2: Further reduce to an optimization problem

[Show the picture for G_1, G_2.]

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Observation

If G_1, G_2 intersects in t vertices, Then

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2\mathrm{ex}(t, F)$$

Step 2: Further reduce to an optimization problem

Define $f(n,t,F) := \binom{n-t}{2} + (n-t)t + 2\mathrm{ex}(t,F)$.

Lemma 2

Let F be some graph. For $n \geq 1$,

$$\phi^*(2,n,F) = \max_{0 \leq t \leq n} f(n,t,F).$$

Step 3: Solve the optimization problem

Proof Roadmap

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By Lemma 1, it suffices prove the case $m = 3$.

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Let G_1, G_2, G_3 be induced double K_r-free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, K_r)$ and $e(G_1) \geq e(G_2) \geq e(G_3)$.

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By Lemma 1, it suffices prove the case $m = 3$.

Let G_1, G_2, G_3 be induced double K_r-free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, K_r)$ and $e(G_1) \geq e(G_2) \geq e(G_3)$.

Since $\phi^*(3, n, K_r) \geq 3\mathrm{ex}(n, K_r)$, we must have $e(G_1) + e(G_2) \geq 2\mathrm{ex}(n, K_r)$.

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Since $\phi^*(3, n, K_r) \geq 3\text{ex}(n, K_r)$, we must have $e(G_1) + e(G_2) \geq 2\text{ex}(n, K_r)$.

Since G_1, G_2, G_3 are induce and $e(G_1) + e(G_2) + e(G_3) \geq 3\text{ex}(n, K_r)$, we only need to show $G_1 = G_2 = T_{r-1}(n)$.

Step 3: Solve the optimization problem

Let $t = |V(G_1 \cap G_2)|$. By Turán's Theorem,

$$\mathrm{ex}(t, K_r) - \mathrm{ex}(t-1, K_r) = t - \left\lceil \frac{t}{r-1} \right\rceil .$$

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$$\mathrm{ex}(t, K_r) - \mathrm{ex}(t-1, K_r) = t - \left\lceil \frac{t}{r-1} \right\rceil .$$

It follows that

$$\begin{aligned} f(n, t, K_r) - f(n, t-1, K_r) &= -t + 1 + 2[\mathrm{ex}(t, K_r) - \mathrm{ex}(t-1, K_r)] \\ &= t + 1 - 2\left\lceil \frac{t}{r-1} \right\rceil . \end{aligned} \tag{1}$$

Step 3: Solve the optimization problem

For $r \geq 4$,

$$f(n, t, K_r) - f(n, t-1, K_r) = t + 1 - 2\left\lceil \frac{t}{r-1} \right\rceil > 0$$

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$\implies f(n, t, K_r)$ is strictly increasing on t.

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$$f(n, t, K_r) - f(n, t-1, K_r) = t + 1 - 2\left\lceil \frac{t}{r-1} \right\rceil > 0$$

$\implies$ $f(n, t, K_r)$ is strictly increasing on t.

$\implies$ $t = n$ so that $e(G_1) + e(G_2) \geq 2\mathrm{ex}(n, F)$.

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For $r \geq 4$,

$$f(n, t, K_r) - f(n, t-1, K_r) = t + 1 - 2\left\lceil \frac{t}{r-1} \right\rceil > 0$$

$\implies f(n, t, K_r)$ is strictly increasing on t.

$\implies t = n$ so that $e(G_1) + e(G_2) \geq 2\text{ex}(n, F)$.

$\implies \phi^*(2, n, K_r) = 2\text{ex}(n, F)$ and $G_1 = G_2 = T_{r-1}(n)$.

This solves the case $r \geq 4$.

Step 3: Solve the optimization problem

For $r = 3$,

$$f(n, t, K_r) - f(n, t-1, K_r) = t + 1 - 2\left\lceil \frac{t}{2} \right\rceil \geq 0$$

Step 3: Solve the optimization problem

For $r = 3$,

$$f(n, t, K_r) - f(n, t-1, K_r) = t + 1 - 2\left\lceil \frac{t}{2} \right\rceil \geq 0$$

$\implies f(n, t, K_3)$ is non-decreasing on t and $f(n, t, K_3) > f(n, t, K_3)$ for even t.

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$\implies t = n$ or $t = n - 1$ so that $e(G_1) + e(G_2) \geq 2\text{ex}(n, F)$.

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$\implies f(n, t, K_3)$ is non-decreasing on t and $f(n, t, K_3) > f(n, t, K_3)$ for even t.

$\implies t = n$ or $t = n - 1$ so that $e(G_1) + e(G_2) \geq 2\text{ex}(n, F)$.

$\implies \phi^*(2, n, K_3) = 2\text{ex}(n, K_3)$, and either $G_1 = G_2 = T_2(n)$ or $G_2 = T_2(n-1)$ and $G_1 = G_2 + K_1$.

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Suppose $G_2 = T_2(n-1)$ and $G_1 = G_2 + K_1$.

Since $e(G_1) + e(G_2) + e(G_3) \geq 3\mathrm{ex}(n, K_3)$,

$$e(G_3) = \mathrm{ex}(n, K_3) > T_2(n-1) = e(G_2),$$

contradiction.

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Suppose $G_2 = T_2(n-1)$ and $G_1 = G_2 + K_1$.

Since $e(G_1) + e(G_2) + e(G_3) \geq 3\mathrm{ex}(n, K_3)$,

$$e(G_3) = \mathrm{ex}(n, K_3) > T_2(n-1) = e(G_2),$$

contradiction.

This solves Theorem B.

Extra Ingredients to Prove Theorem A

- First show that $t = |V(G_1 \cap G_2)| \geq \sqrt{n}$.
- For large enough t, any extremal t-vertex F-free graph contains a spanning $T_{r-1}(t)$.