

Double Turán Problem

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1 Introduction

1.1 Definitions and Notation

Let $G = (V, E)$ be a graph. Let $V(G) = V$ denote the vertex set and $E(G) = E$ denote the edge set of G . We note by $v(G) = |V|$ the number of vertices and $e(G) = |E|$ the number of edges in G . For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v .

Given G_1, \dots, G_m subgraphs of G , we denote G_{i_1, \dots, i_k} as the subgraph of G with edge set $E(G_{i_1, \dots, i_k}) = \bigcap_{\alpha=1}^k E(G_{i_\alpha})$.

In this thesis, we reserve n to denote the number of vertices in a graph. Given a graph F , we denote $\text{ex}(n, F)$ to be the extremal number for F on a graph with n vertices, i.e. the maximum number of edges in a n -vertex graph that does not contain F as a subgraph.

We call a n -vertex complete graph K_n , and a complete bipartite graph $K_{a,b}$, where a, b are the size of its parts. We denote P_n as a path with n edges, and C_n as a cycle with n edges. Given graph G, H , define $G + H$ as the graph fully connecting G, H , i.e. $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$.

We also denote the set of first n positive integers as $[n] = \{1, 2, \dots, n\}$. Given a set X , we denote 2^X as the power set of X .

1.2 Problem Statement

Given graph G with n vertices, let G_1, \dots, G_m be subgraphs of G . Let F be a graph with at least one edge. Our goal is to determine the maximum sum of the number of edges over all G_i 's, i.e. $\sum_{i=1}^m e(G_i)$, with the constraint of $E(G_i) \cap E(G_j)$ not including some graph F for all distinct i, j .

In this thesis, we first consider the case where G_1, \dots, G_m are induced subgraphs, and then shift our focus to the general case. At the end, we will discuss the case where F is bipartite.

2 Induced Case

In this section, we assume that G_1, \dots, G_m are induced subgraphs of G . We first show a simpler case where F is a triangle.

2.1 Triangle F

Theorem 2.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j . Then*

$$\sum_{i=1}^n e(G_i) \leq n \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = \dots = G_n = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

First we prove the case for two graphs:

Lemma 2.1.1. *Suppose $E(G_1) \cap E(G_2)$ does not include K_3 . Then*

$$e(G_1) + e(G_2) \leq 2 \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$, unless n is odd and $G_1 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and $G_2 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$.

Proof. Let $C = V(G_1) \cap V(G_2)$, $A = V(G_1) \setminus C$, and $B = V(G_2) \setminus C$. Put $a = |A|$, $b = |B|$, and $c = |C|$. We may assume that $a + b + c = n$.

We find an upper bound for $e(G_1) + e(G_2)$ with respect to a, b, c . Since G_1, G_2 are induced graphs, $\{u, v\} \in E(G_1)$ if and only if $\{u, v\} \in E(G_2)$, for $u, v \in C$, and thus $E(G_1[C]) = E(G_2[C]) = E(G_i) \cap E(G_j)$. But then $E(G_i) \cap E(G_j)$ is triangle-free, so $e(G_1[C]) \leq \left\lfloor \frac{c^2}{4} \right\rfloor$, with equality if and only if $G_1[C] = K_{\lceil \frac{c}{2} \rceil, \lfloor \frac{c}{2} \rfloor}$. Hence,

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2 \left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor \right]. \quad (2.1)$$

Define $f(a, b, c)$ as the function on the right-hand-side of (2.1). We show that $f(a, b, c)$

attains its maximum $2 \left\lfloor \frac{n^2}{4} \right\rfloor$ at $a = b = 0$ and $c = n$. For $b \geq 2$,

$$\begin{aligned} f(a, b-2, c+2) - f(a, b, c) &= \binom{a+c+2}{2} - \binom{a+c}{2} \\ &\quad - 2 \left[\binom{c+2}{2} - \binom{c}{2} - \left\lfloor \frac{(c+2)^2}{4} \right\rfloor + \left\lfloor \frac{c^2}{4} \right\rfloor \right] \\ &= 2(a+c) + 1 - 2[2c+1 - (c+1)] \\ &= 2a + 1 > 0. \end{aligned}$$

By symmetry, $f(a-2, b, c+2) > f(a, b, c)$, and thus f attains its maximum when c is $n-1$ or n , that is, $a+b \leq 1$. Equation (2.1) now yields,

$$e(G_1) + e(G_2) \leq 2 \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Assume that $a = 0$. When $c = n$, the equality holds only if $G_1 = G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. If $c = n-1$, then the equality holds only if n is odd, $G_1 = G[C] = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and G_2 is a copy of G_1 with all vertices adjacent to the only remaining vertex, i.e. $G_2 = G_1 + K_1$. \square

We now prove Theorem 2.1.

Proof of Theorem 2.1. Assume that $n > 1$. Put $G_{n+i} = G_i$. By Lemma 3.2,

$$\sum_{i=1}^n e(G_i) = \frac{1}{2} \sum_{i=1}^n (e(G_i) + e(G_{i+1})) \leq \frac{1}{2} \sum_{i=1}^n 2 \left\lfloor \frac{n^2}{4} \right\rfloor = n \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Suppose the equality holds. By Lemma 3.2, we are done if n is even. Suppose n is odd and $G_i = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$ for some i . By applying Lemma 3.2 repeatedly,

$$\begin{aligned} G_i &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1 \\ G_{i+1} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} \\ G_{i+2} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1 \\ &\vdots \end{aligned}$$

and the alternation proceeds. But then $G_{n+i} = G_i = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ as n is odd, and this contradiction completes the proof. \square

2.2 Non-bipartite F

Theorem 2.2. *Let F be $(r+1)$ -colorable, with $r \geq 2$. Suppose that $E(G_i) \cap E(G_j)$ is F -free for distinct i, j . For large enough n ,*

$$\sum_{i=1}^n e(G_i) \leq n \cdot \text{ex}(n, F),$$

with equality if and only if $G_1 = G_2 = \dots = G_n$ are n -vertex extremal graphs for F .

By the same argument as in Theorem 2.1, it suffices to prove the following lemma:

Lemma 2.2.1. *Let F be $(r+1)$ -colorable, with $r \geq 2$. Suppose $E(G_1) \cap E(G_2)$ does not include F . For large enough n ,*

$$e(G_1) + e(G_2) \leq 2 \cdot \text{ex}(n, F),$$

with equality if and only if $G_1 = G_2$ are n -vertex extremal graphs for F , unless n is odd, G_1 is an $(n-1)$ -vertex extremal graph for F , and $G_2 = G_1 + K_1$.

Proof. Let $C = V(G_1) \cap V(G_2)$, $A = V(G_1) \setminus C$, and $B = V(G_2) \setminus C$. Put $a = |A|$, $b = |B|$, $c = |C|$. Since G_1, G_2 are induced graphs, $E(G_1[C]) = E(G_2[C]) = E(G[C]) = E(G_i) \cap E(G_j)$, which is F -free. Hence,

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2 \left[\binom{c}{2} - \text{ex}(c, F) \right]. \quad (2.2)$$

Define $f(a, b, c)$ as the function on the right-hand-side of (2.2). We show that $f(a, b, c)$ attains its maximum at $a = b = 0$ and $c = n$.

Claim 1. *If $c \leq \frac{n}{2}$, then $f(a, b, c) < 2 \cdot \text{ex}(n, F)$.*

Proof. Write $c = kn$ for some $k \in [0, 1/2]$. Since

$$f(a, b, kn) \leq 2 \binom{(1-k)n/2}{2} - 2 \left[\binom{kn}{2} - \text{ex}(kn, F) \right],$$

it suffices to show that

$$\text{ex}(n, F) - \text{ex}(c, F) > \binom{(1-k)n/2}{2} - \binom{kn}{2}$$

for all $k \in [0, 1/2]$. By the Erdős-Stone theorem, $\text{ex}(n, F) = \left(1 - \frac{1}{r}\right) \frac{n^2}{2} + o(n^2)$ and so the left-hand-side is at least

$$\text{ex}(n, F) - \text{ex}(c, F) \geq \text{ex}(n, F) - \text{ex}(n/2, F) \geq \left(1 - \frac{1}{r}\right) \left(\frac{n^2}{2} - \frac{n^2}{8}\right) - o(n^2) \geq \frac{3n^2}{16} - o(n^2)$$

On the right-hand-side,

$$\binom{(1-k)n/2}{2} - \binom{kn}{2} = (1 - 2k - 3k^2) \frac{n^2}{8} + o(n^2) \leq \frac{n^2}{8} + o(n^2).$$

Combining the above inequalities now yields the claim, as n is large. \square

Thus we may assume that $c > \frac{n}{2}$. A theorem of Simonovits states that for large enough n , $\text{ex}(n, F) = \text{ex}(n, K_{r+1}) + \text{ex}(n, \tilde{F})$, where \tilde{F} is the family of residue subgraphs of F after F is embedded into $T_r(n)$. This implies

$$\text{ex}(n+1, F) - \text{ex}(n, F) \geq \text{ex}(n, K_{r+1}) - \text{ex}(n+1, K_{r+1}),$$

and so

$$f(a, b-2, c+2) - f(a, b, c) \geq 2a - 2c - 1 + 2[\text{ex}(c+2, K_{r+1}) - \text{ex}(c, K_{r+1})].$$

Since $\text{ex}(c+1, K_{r+1}) - \text{ex}(c, K_{r+1}) \geq c - \lfloor \frac{c}{r} \rfloor \geq c - \lfloor \frac{c}{2} \rfloor$, we have

$$\text{ex}(c+2, K_{r+1}) - \text{ex}(c, K_{r+1}) \geq c+1 - \left\lfloor \frac{c+1}{2} \right\rfloor + c - \left\lfloor \frac{c}{2} \right\rfloor = c+1,$$

and thus

$$f(a, b-2, c+2) - f(a, b, c) \geq 2a+1 > 0.$$

By symmetry, we also have $f(a-2, b, c+2) > f(a, b, c)$. Thus, f attains its maximum when c is $n-1$ or n . Equation (2.2) now yields,

$$e(G_1) + e(G_2) \leq \max[2 \cdot \text{ex}(n, F), 2 \cdot \text{ex}(n-1, F) + n-1].$$

Assume that $a = 0$. Since

$$\begin{aligned} 2 \cdot \text{ex}(n, F) - [2 \cdot \text{ex}(n-1, F) + n-1] & \\ & \geq 2[\text{ex}(n, K_{r+1}) - \text{ex}(n-1, K_{r+1})] - n+1 \quad (2.3) \\ & = 2\left(n - \left\lceil \frac{n}{r} \right\rceil\right) - n+1 \\ & \geq n+1 - 2\left\lceil \frac{n}{2} \right\rceil \geq 0, \end{aligned}$$

we have

$$e(G_1) + e(G_2) \leq 2 \cdot \text{ex}(n, F).$$

If $c = n$, the equality holds only if $G_1 = G_2$ are n -vertex extremal graphs for F . Suppose $c = n-1$ and the equality holds. Observe that equation (2.3) equals to zero only when $r = 2$ and n is odd. Hence, if $c = n-1$, equality could only be achieved when $r = 2$, n is odd, G_1 is an $(n-1)$ -vertex extremal graph for F , and $G_2 = G_1 + K_1$. \square

3 General Case

We now relax the assumption that G_1, \dots, G_m are induced subgraphs. The trivial construction of putting $G_1 = K_n$ and G_2, \dots, G_m to be extremal graphs for F yields the lower bound

$$\sum_{i=1}^m e(G_i) = \binom{n}{2} + (m-1)\text{ex}(n, F). \quad (3.1)$$

In this section we examine whether this bound is tight. The following is an asymptotic result on the number of G_i 's:

Theorem 3.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include r -vertex graph F for distinct i, j . Then for large enough n ,*

$$\sum_{i=1}^m e(G_i) \leq m(1 + o(1))\text{ex}(n, F),$$

as $m \rightarrow \infty$.

Proof. Pick $\epsilon > 0$. Reorder G_1, \dots, G_m so that $G_1, \dots, G_{m'}$ are all the G_i 's containing at least $(1 + \epsilon)\text{ex}(n, F)$ edges. A theorem of Simonovits states that G contains at least δn^r copies of F for some $\delta = \delta(\epsilon)$. Since there can be at most $\binom{n}{r}$ copies of F across all G_i 's,

$$m'\delta n^r \leq \binom{n}{r} \leq n^r \implies m' \leq \frac{1}{\delta}.$$

It now follows that

$$\begin{aligned} \sum_{i=1}^m e(G_i) &= \sum_{i=1}^{m'} e(G_i) + \sum_{i=m'+1}^m e(G_i) \\ &\leq \frac{1}{\delta} \binom{n}{2} + \left(m - \frac{1}{\delta}\right) (1 + \epsilon)\text{ex}(n, F) \\ &= m \left[1 + \epsilon + \frac{1}{m\delta} \left(\frac{\binom{n}{2}}{\text{ex}(n, F)} - (1 + \epsilon) \right) \right] \text{ex}(n, F). \end{aligned}$$

Since ϵ is arbitrary, the result follows. □

3.1 Triangle F

Consider F to be a triangle. Simply counting the number of triangles in each G_i shows the following:

Theorem 3.2. *For any $\epsilon > 0$, if $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j , then*

$$\sum_{i=1}^m e(G_i) < m(1 + \epsilon) \frac{n^2}{4} + (1 - \epsilon) \frac{n^2}{2\epsilon}.$$

Claim 2. *There are less than $\frac{2}{\epsilon}$ number of G_i 's with $e(G_i) \geq (1 + \epsilon) \frac{n^2}{4}$.*

Proof. Suppose $e(G_i) \geq (1 + \epsilon) \frac{n^2}{4}$ for $1 \leq i \leq k$. Let $K_3(G)$ denote the number of triangles in graph G . By the Moon-Moser inequality,

$$K_3(G_i) \geq \epsilon(1 + \epsilon) \frac{n^3}{12}.$$

Since there are no overlapping triangles from different G_i 's,

$$\binom{n}{3} \geq \sum_{i=1}^k K_3(G_i) \geq \frac{\epsilon(1 + \epsilon)}{12} k n^3.$$

Rearranging yields $k < \frac{2}{\epsilon}$. □

By the claim,

$$\sum_{i=1}^m e(G_i) < \frac{2}{\epsilon} \binom{n}{2} + \left(m - \frac{2}{\epsilon}\right) (1 + \epsilon) \frac{n^2}{4} \leq m(1 + \epsilon) \frac{n^2}{4} + (1 - \epsilon) \frac{n^2}{2\epsilon},$$

which proves Theorem 3.2.

It can be easily shown that the bound in Theorem 3.2 is tight when $m = 2$, as

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_{1,2}) \leq \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor.$$

This result is also true for $m = 3$:

Proposition 3.3. *Let G_1, G_2, G_3 be subgraphs of some G such that no triangle is contained in any two graphs, then*

$$e(G_1) + e(G_2) + e(G_3) \leq \binom{n}{2} + \frac{n^2}{2}.$$

Proof. Define $H_k \subseteq G$ be the graph with edges contained in at least k number of G_i 's and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \leq \frac{n^2}{2}$. Notice H_2 must not contain any triangles with two edges in H_3 , so

$$e(H_2) + e(H_3) \leq \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}|.$$

Let H'_3 be the graph with the same vertex set as H_3 and edge set $\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}$. It suffices to show that $\frac{n}{2} \geq e(H_3) - e(H'_3)$.

Let $d_1 \geq d_2 \geq \dots \geq d_n$ and $f_1 \geq f_2 \geq \dots \geq f_n$ each be the degree sequence of H_3 and H'_3 , respectively. We show that $f_i \geq d_i - 1$ for all i . Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i . Let $S_i = |N_{H_3}(v_1) \cup \dots \cup N_{H_3}(v_i)|$. Since

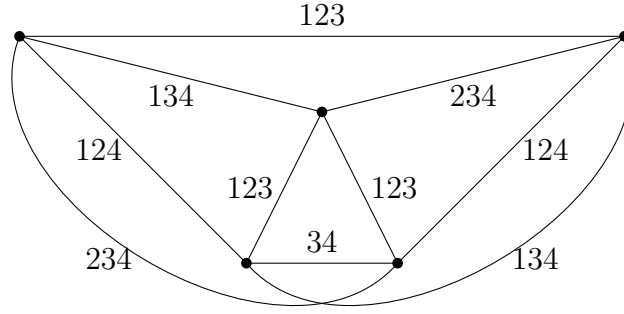
$$\sum_{u \in S_i} d_{H_3}(u) \geq d_1 + \dots + d_i,$$

we have that $|S_i| \geq i$. But then $S_i \setminus \{u_1, \dots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H'_3}(u) \geq d_i - 1$. Hence, $f_i \geq d_i - 1$ for all i , which yields

$$e(H'_3) = \frac{1}{2} \sum_{i=1}^n f_i \geq \frac{1}{2} \sum_{i=1}^n (d_i - 1) = e(H_3) - \frac{n}{2}.$$

□

However, the bound in Proposition 3.1 is not tight for $m \geq 4$, as shown in the following graph:



The number on each edge denotes the set of G_i 's that contain the edge.

The above graph contains 29 edges, which exceeds the bound $\binom{5}{2} + 3\lfloor \frac{5^2}{4} \rfloor = 28$ by 1. By blowing up the above graph, we can construct a graph with $n \in 10\mathbb{Z}$ vertices that contains

$$5 \binom{n/5}{2} + 29 \cdot \frac{(n/5)^2}{4}$$

edges, which exceeds the bound $\binom{n}{2} + 3\lfloor \frac{n^2}{4} \rfloor$ by $n^2/100$.

3.2 Bipartite F

In this section, we discuss the case where F is bipartite. In particular, we focus on the cases where $F \subseteq K_{2,2}$ is P_2 , a path of length 2, or M_2 , a matching with two edges.

Theorem 3.4. *Given graph G and G_1, \dots, G_n subgraphs of G with $E(G_i) \cap E(G_j)$ not containing P_2 for distinct i, j . Then*

$$\sum_{i=1}^m e(G_i) \leq \frac{n^{5/2}}{2}.$$

Proof. Since there are no overlapping P_2 's in different G_i 's,

$$\sum_{i=1}^n \#\{P_2 \subseteq G_i\} \leq \#\{P_2 \subseteq G\}$$

For each G_i , each vertex v in G_i and two of its neighbors form one unique P_2 , so

$$\#\{P_2 \subseteq G_i\} = \sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2}.$$

And by Jensen's inequality,

$$\sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2} \geq n \binom{d_{G_i}(v)/n}{2} = n \binom{2e(G_i)/n}{2} \geq \frac{2(e(G_i))^2}{n}.$$

On the other hand, since each three vertices in G can form at most three P_2 's,

$$\#\{P_2 \subseteq G\} \leq 3 \binom{n}{3} = \frac{n^3}{2}.$$

Combining the above inequalities yields

$$\sum_{i=1}^n \frac{2(e(G_i))^2}{n} \leq \frac{n^3}{2} \xrightarrow{\text{Jensen's}} \sum_{i=1}^n e(G_i) \leq \frac{n^{5/2}}{2}.$$

□

Theorem 3.5. *Given graph G and G_1, \dots, G_n subgraphs of G with $E(G_i) \cap E(G_j)$ not containing M_2 for distinct i, j . Then*

$$\sum_{i=1}^n e(G_i) \leq n^{5/2}.$$

Proof. Notice that $\#\{M_2 \subseteq G\} = \binom{e(G)}{2}$. On the other hand, each four vertices in G can form at most three M_2 's, so $\#\{M_2 \subseteq G\} \leq 3 \binom{n}{4} \leq \frac{n^4}{8}$. By the same argument as in Theorem 3.4, we have

$$\sum_{i=1}^n \binom{e(G_i)}{2} = \sum_{i=1}^n \#\{M_2 \subseteq G_i\} \leq \#\{M_2 \subseteq G\} \leq \frac{n^4}{8}.$$

By Jensen's inequality,

$$\sum_{i=1}^n \binom{e(G_i)}{2} \geq n \binom{\sum_{i=1}^n e(G_i)/n}{2} = \frac{1}{2n} \left[\left(\sum_{i=1}^n e(G_i) \right)^2 - n \sum_{i=1}^n e(G_i) \right].$$

Combining the above inequalities yields

$$\left(\sum_{i=1}^n e(G_i) \right)^2 - n \sum_{i=1}^n e(G_i) \leq \frac{n^5}{4},$$

and solving the quadratic inequality gives

$$\sum_{i=1}^n e(G_i) \leq n^{5/2}.$$

□

We may obtain a tight bound if we forbid both P_2 and M_2 at the same time:

Theorem 3.6. *Given graphs $G_1, \dots, G_n \subseteq K_n$ with $E(G_i) \cap E(G_j)$ not containing P_2 or M_2 for distinct i, j . Then*

$$\sum_{i=1}^n e(G_i) \leq n^2 - n,$$

and this bound is tight.

Proof. Denote the set of G_i 's as $\{G_i\} = \{G_1, \dots, G_n\}$, and the set of distinct pairs of G_i 's as $\{G_i\}^2 = \{\{G_j, G_k\} : j \neq k\}$. Consider the bipartite graph H with vertex set $V(H) = \{G_i\} \sqcup E(K_n)$ and edge set $E(H) = \{\{G_j, e\} \in \{G_i\} \times E(K_n) : e \in G_j\}$. Define $\phi : \{G_i\}^2 \rightarrow 2^{E(K_n)}$ by sending each $\{G_j, G_k\}$ to their common edge set $E(G_j) \cap E(G_k)$. Notice that each distinct G_j, G_k have at most one edge in common, so $|\phi(G_j, G_k)| \leq 1$. On the other hand, each edge $e \in E(K_n)$ can be obtained via ϕ by $\binom{d_H(e)}{2}$ possible distinct pairs (G_j, G_k) , and thus $|\phi^{-1}(e)| = \binom{d_H(e)}{2}$. But then

$$\binom{n}{2} \geq \sum_{(G_j, G_k) \in \{G_i\}^2} |\phi(G_j, G_k)| = \sum_{e \in E(K_n)} |\phi^{-1}(e)| = \sum_{e \in E(K_n)} \binom{d_H(e)}{2}.$$

By Jensen's inequality,

$$\sum_{e \in E(K_n)} \binom{d_H(e)}{2} \geq \binom{n}{2} \left(\frac{\sum_{e \in E(K_n)} d_H(e)}{2} / \binom{n}{2} \right) = \binom{n}{2} \left(\frac{\sum_{i=1}^n e(G_i)}{2} / \binom{n}{2} \right).$$

Combining the above inequalities yields

$$2 \binom{n}{2}^2 \geq \left(\sum_{i=1}^n e(G_i) \right)^2 - \binom{n}{2} \sum_{i=1}^n e(G_i),$$

and the result now follows from solving the quadratic inequality.

To see that this bound is tight, consider the construction such that for each distinct $i, j \in [n]$, $E(G_i) \cap E(G_j)$ contains exactly one unique edge $e \in K_n$. The number of edges in this construction is $2 \binom{n}{2} = n^2 - n$. \square

Theorem 3.7. *Let F be a graph such that there exists extremal F -free subgraph H on n vertices that have maximum degree $\Delta(H) = o(n^{1/2})$. For large enough n , if G_1, \dots, G_m are graphs on common vertex set $[n]$ such that no copy of F is contained in any k of the G_i 's, then*

$$\sum_{i=1}^m e(G_i) = (k-1) \binom{n}{2} + \text{ex}(n, F) \binom{m}{k},$$

and the bound is tight.

Proof. For $S \subseteq [m]$, let E_S denote the set of edges that are contained in exactly $\{G_i\}_{i \in S}$. Then

$$\sum_{i=1}^m e(G_i) = \sum_{S \subseteq [m]} |S| |E_S| \leq (k-1) \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq k} (|S| - k + 1) |E_S|.$$

Let $A_S = \bigcup_{T \supseteq S} E_T$, i.e. the set of edges that are contained in all G_i with $i \in S$. When $|S| \geq k$, the edge set A_S is F -free and thus

$$\sum_{T \supseteq S} |E_T| \leq \text{ex}(n, F).$$

Hence,

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq k}} (|S| - k + 1) |E_S| = \sum_{\substack{S \subseteq [m], T \subseteq S \\ |S|=k}} \sum_{T \supseteq S} \frac{(|T| - k + 1) |E_T|}{\binom{|T|}{k}} \leq \sum_{\substack{S \subseteq [m], T \subseteq S \\ |S|=k}} \sum_{T \supseteq S} |E_T| \leq \binom{m}{k} \text{ex}(n, F),$$

as each $T \in [m]$ with $|T| \geq k$ is counted $\binom{|T|}{k}$ times in total and $|T| - k + 1 \leq \binom{|T|}{k}$. This proves the upper bound.

Now we show the bound is tight. In particular, we need to show there exists a construction such that the graph with edge set E_S is an extremal F -free graph, for all $S \subseteq [m]$ of size k . Let $M = \binom{m}{k}$ and H_1, \dots, H_M be copies of an extremal F -free graph on n vertices. It suffices to show that we can embed each H_i onto $[n]$ such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each H_i and iteratively decrease the number of intersecting edges. Define a (u, v, i) -swap by swapping the embedding of vertex u and v of H_i , i.e. replacing each edge $\{u, w\} \in E(H_i)$ with the edge $\{v, w\}$ and each edge $\{v, w\} \in E(H_i)$ with the edge $\{u, w\}$. This preserves the type of isomorphism of H_i . Given a vertex v , let $N(v) = N_{H_1}(v) \cup \dots \cup N_{H_M}(v)$. Suppose there exists an intersecting edge $\{u, w\} \in E(H_i) \cap E(H_j)$. Since the maximum degree of each H_i is $o(n^{1/2})$, $|N(u) \cup N(N(u))| = o(n)$ so there exists a vertex $v \notin N(u) \cup N(N(u))$. Since $N(u) \cap N(v) = \emptyset$, performing a (u, v, i) -swap reduces the number of intersecting edges. The result now follows by iterating this process. \square