

Double Turán Problem

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- 4 Proof of Theorem B
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Turán Problem

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Turán Number

$$\text{ex}(n, F) := \max\{e(G) : |V(G)| = n \text{ and } F \not\subseteq G\}$$

Turán Problem

Turán's Theorem [Tur41]

For $r + 1 \geq 3$,

$$\text{ex}(n, K_{r+1}) = e(T_r(n)),$$

with equality for graph G only if $G = T_r(n)$.

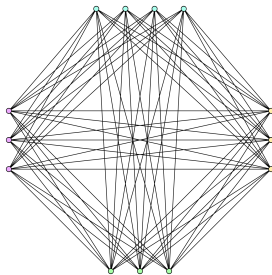
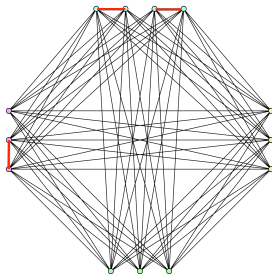


Figure: $T_4(13)$

Erdős-Stone Theorem

Erdős-Stone Theorem [ES46], Simonovits' Theorem [ES66]

Let F be any graph of chromatic number $r + 1 \geq 3$. Then $\text{ex}(n, F) = (1 + o(1))T_r(n)$ as $n \rightarrow \infty$.



Double Turán Problem: Definition

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Is this tight?

Double Turán Problem: Main Result

Theorem A

Let $n \geq 1$ and F be a non-bipartite graph. Then as $m \rightarrow \infty$

$$\phi(m, n, F) = (1 + o(1))m \cdot \text{ex}(n, F).$$

Double Turán Problem: Main Results

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$$\phi(3, n, K_3) = \binom{n}{2} + \lfloor n/2 \rfloor$$

But

$$\lim_{n \rightarrow \infty} \frac{\phi(4, n, K_3)}{\binom{n}{2} + 3\lfloor n/4 \rfloor} > 1$$

Main Results

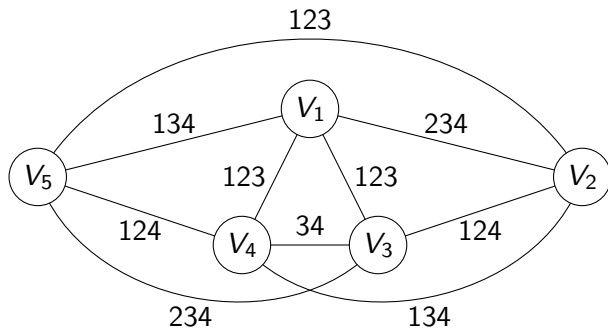


Figure: Construction for four graphs not containing a double- K_3

Double Turán Problem: Main Result

Theorem B

Let F be a graph. If there exists an extremal F -free n -vertex graph with maximum degree at most \sqrt{n}/m^2 , then

$$\phi(m, n, F) = \binom{n}{2} + \binom{m}{2} \text{ex}(n, F).$$

Double Turán Problem: Motivation

Double Turán problems are closely related to Turán problems for 3-uniform hypergraphs H through **link graphs**.

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Link Graph

For $i \in V(H)$, define graph H_i with

$$V(H_i) = V(H) \setminus \{i\} \quad \text{and} \quad E(H_i) = \{\{j, k\} : \{i, j, k\} \in E(H)\}.$$

Double Turán Problem: Motivation

Example: Octahedron-free 3-uniform hypergraph H

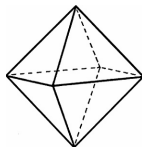


Figure: Octahedron $K_{2,2,2}^{(3)}$

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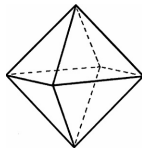


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Erdős conjectured that $\text{ex}(n, K_{2,2,2}^{(3)}) = \Theta(n^{11/4})$ [Erd64].

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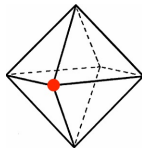


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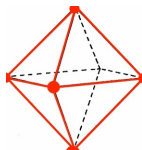


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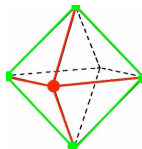


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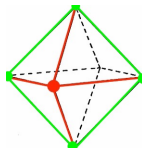


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H is octahedron-free $\implies H_1, H_2, \dots, H_n$ are double $K_{2,2}$ -free.

$$\text{ex}(n, K_{2,2,2}^{(3)}) \leq \phi(n, n, K_{2,2})$$

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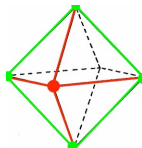


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Let graphs G_1, G_2, \dots, G_m be induced and double F -free.

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$$\phi(m, n, F) \geq \phi^*(m, n, F)$$

Induced Double Turán Problem: Conjecture

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What is the value of $\phi^*(m, n, F) = \max \sum_{i=1}^m e(G_i)$?

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Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \text{ex}(n, F) + n^2).$$

Theorem C

For $m \geq 3$ and non-bipartite F , if n is large enough, then

$$\phi^*(m, n, F) = m \cdot \text{ex}(n, F),$$

with equality only for identical extremal n -vertex F -free graphs.

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\implies Conjecture is true for non-bipartite F .

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Theorem D

For $m, n, r \geq 3$,

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced K_r -free graphs G_1, G_2, \dots, G_m only if $G_1 = G_2 = \dots = G_m = T_{r-1}(n)$.

Induced Double Turán Problem: Motivation

Generalized Turán problem

What is the maximum number $\text{ex}(n, F, K_3)$ of triangles in a graph H on $[n]$ with no copy of F as a subgraph?

Studied by Alon and Shikhelman [AS16] and Kostochka, Mubayi and Verstraete [KMV15; MM23; MV16].

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Induced Double Turán Problem: Motivation

Ex. Octahedron-free graph G .

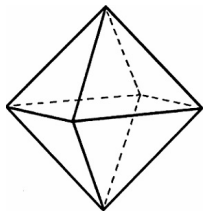


Figure: Octahedron

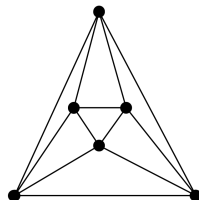


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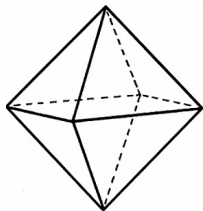


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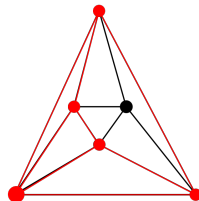


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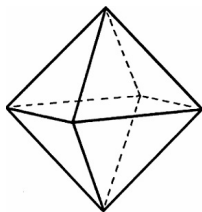


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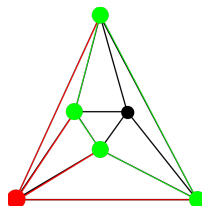


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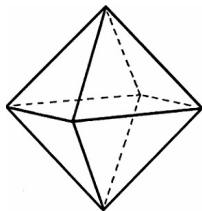


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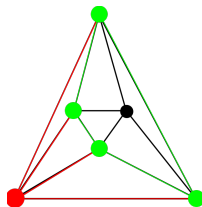


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G is $K_{2,2,2}$ -free $\implies G_1, G_2, \dots, G_n$ are induced and double $K_{2,2}$ -free.

$$\text{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2})$$

Induced Double Turán Problem: Motivation

Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \text{ex}(n, F) + n^2).$$

Conjecture implies

$$\text{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2}) \leq O(n^{5/2}),$$

which solves a conjecture of

Theorem E

Let P be a path with two edges. Then $\phi(n, n, P) = \Omega(n^{5/2})$, whereas $\phi^*(n, n, P) = o(n^{5/2})$, as $n \rightarrow \infty$. In particular,

$$\lim_{n \rightarrow \infty} \frac{\phi^*(n, n, P)}{\phi(n, n, P)} = 0.$$

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This shows that $\phi(m, n, F)$ and $\phi^*(m, n, F)$ can be very different problems.

Proof of Theorem B: Upper Bound

Theorem B

Let F be a graph. If there exists an extremal F -free n -vertex graph with maximum degree at most \sqrt{n}/m^2 , then

$$\phi(m, n, F) = \binom{n}{2} + \binom{m}{2} \text{ex}(n, F).$$

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Let E_S be the set of edges in exactly $\{G_i\}_{i \in S}$.

$$\Rightarrow \sum_{i=1}^m e(G_i) = \sum_{S \subseteq [m]} |S| |E_S| \leq \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq 2} (|S| - 1) |E_S|.$$

Proof of Theorem B: Upper Bound

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S| - 1) |E_S| = \sum_{\substack{S \subseteq [m] \\ |S|=2}} \sum_{T \supseteq S} \frac{(|T| - 1) |E_T|}{\binom{|T|}{2}} \leq \sum_{\substack{S \subseteq [m] \\ |S|=2}} \sum_{T \supseteq S} |E_T|,$$

as each $T \in [m]$ with $|T| \geq 2$ is counted $\binom{|T|}{2}$ times.

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Observation: If $|S| \geq 2$, the edge set $\bigcup_{T \supseteq S} E_T$ is F -free.

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$$\Rightarrow \sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S| - 1) |E_S| \leq \sum_{\substack{S \subseteq [m], \\ |S|=2}} \sum_{T \supseteq S} |E_T| \leq \binom{m}{2} \text{ex}(n, F)$$

This proves the upper bound.

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We need to show

$$\phi(m, n, F) \geq \binom{n}{2} + \text{ex}(n, F) \binom{m}{2}.$$

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For $1 \leq i < j \leq m$, let H_{ij} be an extremal F -free graph on $[n]$ with maximum degree $\Delta(H_{ij}) \leq \sqrt{n}/m^2$

\implies We can pack these $\binom{m}{2}$ H_{ij} 's into $[n]$ s.t. $E(H_{i_1j_1}) \cap E(H_{i_2j_2}) = \emptyset$

Proof of Theorem B: Lower Bound

For $i \in [m]$, put

$$G_i = H_{1i} \cup H_{2i} \cup \cdots \cup H_{im}$$

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Then $G_i \cap G_j = H_{ij} \implies F$ -free

Add edges not in any of H_{ij} to G_1 .

$$\sum_{i=1}^m e(G_i) = 2 \sum e(H_{ij}) + \left| E(K_n) \setminus \bigcup E(H_{ij}) \right| = \binom{n}{2} + \binom{m}{2} \text{ex}(n, F)$$

Theorem D

Let $m, n, r \geq 3$. Then

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced K_r -free graphs G_1, G_2, \dots, G_m only if $G_1 = G_2 = \dots = G_m = T_{r-1}(n)$.

Theorem D

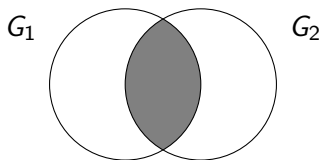
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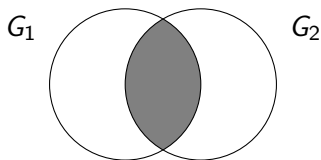
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IDEA: Reduce to $m = 2$.

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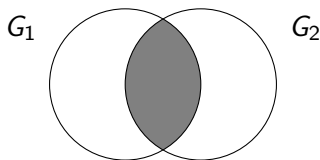


Proof of Theorem D



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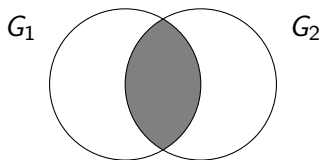


$G_1 \cap G_2$ (gray area) is double K_r -free.

If G_1, G_2 intersects in t vertices, then

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Proof of Theorem D



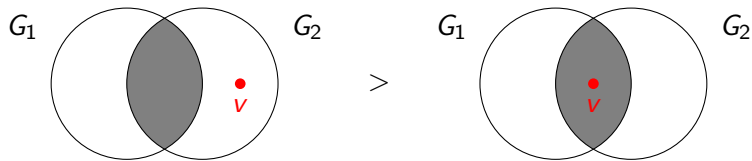
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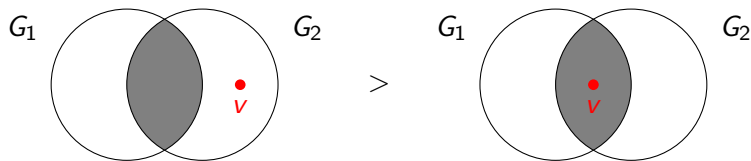
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\implies we are done if the unique maximum occurs at $t = n$.

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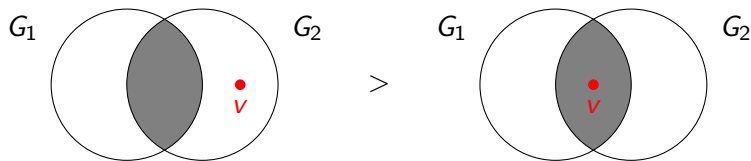


Put $f(t, r) := \binom{n-t}{2} + (n-t)t + 2T_{r-1}(t)$

For example, if $r = 4$,

$$f(t+1, 4) - f(t, 4) = -t + 1 + 2 \underbrace{[T_3(t+1) - T_3(t)]}_{t - \lceil \frac{t}{2} \rceil} = t + 1 - 2 \left\lceil \frac{t}{2} \right\rceil > 0$$

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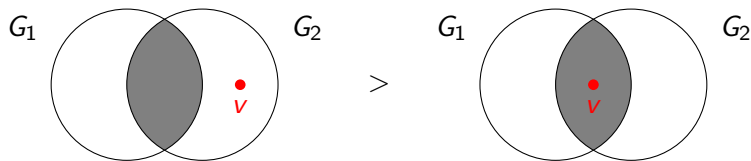
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\implies The best we can do is $G_1 = G_2 = T_3(n)$.

Concluding Remarks

- The conjecture we stated is broadly open for bipartite graphs: we cannot even determine the case for P_2 .

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We only know $\phi(3, n, K_3) = \binom{n}{2} + \lfloor n/2 \rfloor$. The rest is open.

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