Double Turán Problem

Ray Tsai

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Overview

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Turán Number

$$\mathrm{ex}(n, F) := \max\{e(G) : |V(G)| = n \text{ and } F \not\subseteq G\}$$

Turán's Theorem [Tur41]

For $r+1 \geq 3$,

$$\mathrm{ex}(n, K_{r+1}) = e(T_r(n)),$$

with equality for graph G only if $G = T_r(n)$.

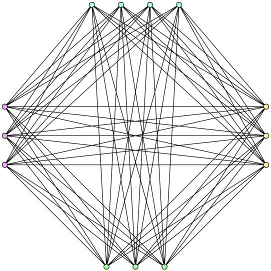
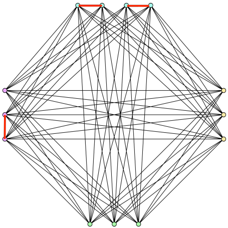


Figure: $T_4(13)$

Erdős-Stone Theorem

Erdős-Stone Theorem [ES46], Simonovits' Theorem [ES66]

Let F be any graph of chromatic number $r+1 \geq 2$. Then $\mathrm{ex}(n, F) = e(T_r(n)) + o(n^2)$ as $n \to \infty$.



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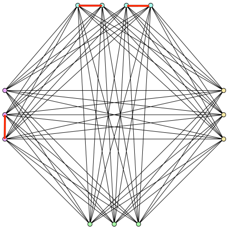

Supersaturation Theorem [ES83]

For all $\epsilon > 0$, there exists $\delta > 0$ such that if G is a n-vertex graph with $\mathrm{ex}(n, F) + \epsilon n^2$ edges, then G contains $\delta n^{v(F)}$ copies of F.

Double Turán Problem: Definition

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Is this tight?

Double Turán Problem: Main Result

Theorem A

Let $n \geq 1$ and F be a non-bipartite graph. Then as $m \to \infty$

$$\phi(m, n, F) = (1 + o(1))m \cdot \text{ex}(n, F).$$

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$$\phi(3, n, K_3) = \binom{n}{2} + \lfloor n/2 \rfloor$$

But

$$\lim_{n \to \infty} \frac{\phi(4, n, K_3)}{\binom{n}{2} + 3\lfloor n/4 \rfloor} > 1$$

Main Results

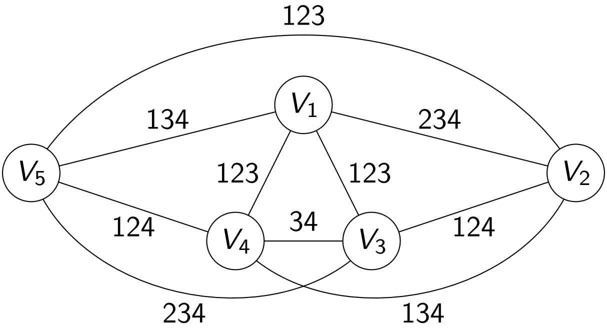

Figure: Construction for four graphs not containing a double-K_3

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Theorem B (Wilson)

Let F be a graph. If there exists an extremal F-free n-vertex graph with maximum degree at most $\sqrt{n}/m^2$, then

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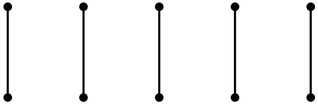
e.g. The extremal graph for P_2 is a matching, which has maximum degree 1 and $\mathrm{ex}(n, P_2) = \lfloor n/2 \rfloor$.



Double Turán Problem: Motivation

Double Turán problems are closely related to Turán problems for 3-uniform hypergraphs H through link graphs.

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Link Graph

For $i \in V(H)$, define graph H_i with

$$V(H_i) = V(H)\backslash\{i\} \quad \text{and} \quad E(H_i) = \{\{j,k\} : \{i,j,k\} \in E(H)\}.$$

Double Turán Problem: Motivation

Example: Octahedron-free 3-uniform hypergraph H

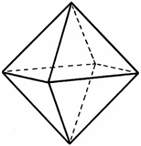

Figure: Octahedron $K_{2,2,2}^{(3)}$

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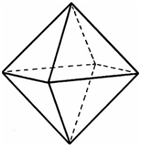

Figure: Octahedron $K^{(3)}_{2,2,2}$

Conjecture [Erd64]

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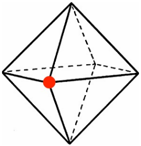

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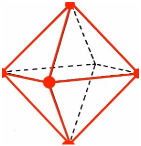

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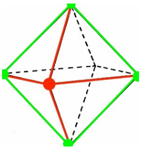

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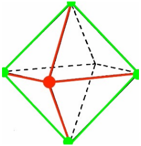

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H is octahedron-free $\implies$ $H_1, H_2, \ldots, H_n$ are double $K_{2,2}$-free.

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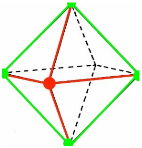

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We call graphs $G_1, G_2, \ldots, G_m$ induced if each G_i is an induced subgraph of $\bigcup_{i=1}^{m} G_i$.

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Let graphs $G_1, G_2, \ldots, G_m$ be induced and double F-free.

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$$\phi(m, n, F) \geq \phi^*(m, n, F)$$

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Main Results

Theorem C

For $m \geq 3$ and non-bipartite F, if n is large enough, then

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$\implies$ Conjecture is true for non-bipartite F.

For $F = K_r$, this theorem applies for all n.

Induced Double Turán Problem: Motivation

Generalized Turán problem

What is the maximum number $\mathrm{ex}(n, F, K_3)$ of triangles in an n-vertex F-free graph G?

Studied by Alon and Shikhelman [AS16] and Kostochka, Mubayi and Verstraete [KMV15; MM23; MV16].

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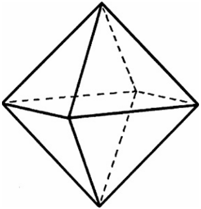
Ex. Octahedron-free graph G.



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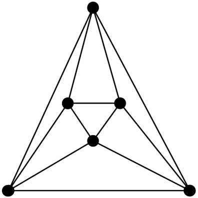

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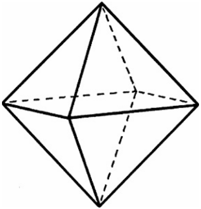

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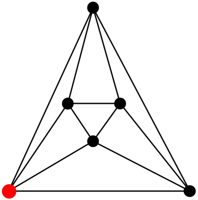

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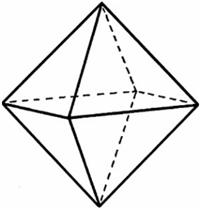

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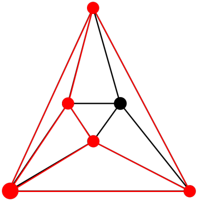

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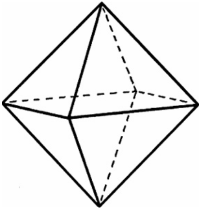

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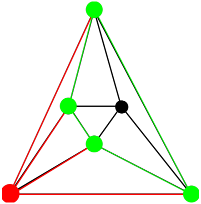

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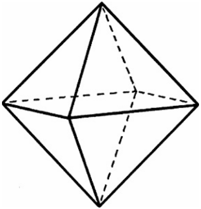

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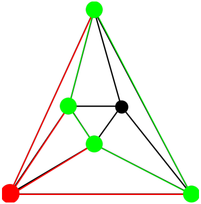

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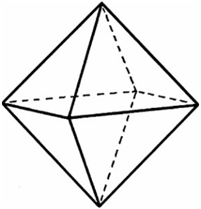

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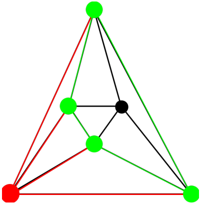

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Induced Double Turán Problem: Motivation

Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

Conjecture implies

$$\mathrm{ex}(n, K_{2,2,2}, K_3) = \Theta(\phi^*(n, n, K_{2,2})) = \Theta(n^{5/2}).$$

Main Results

Theorem D

Let P be a path with two edges. Then $\phi(n,n,P) = \Omega(n^{5/2})$, whereas $\phi^*(n,n,P) = o(n^{5/2})$, as $n \to \infty$. In particular,

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This shows that $\phi(m, n, F)$ and $\phi^*(m, n, F)$ can be very different problems.

Proof of Theorem B: Upper Bound

Theorem B (Wilson)

Let F be a graph. If there exists an extremal F-free n-vertex graph with maximum degree at most $\sqrt{n}/m^2$, then

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Let E_S be the set of edges in exactly $\{G_i\}_{i \in S}$.

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$$\phi(m, n, F) \leq \binom{n}{2} + \mathrm{ex}(n, F)\binom{m}{2}.$$

Let E_S be the set of edges in exactly $\{G_i\}_{i \in S}$.

$$\implies \sum_{i=1}^{m} e(G_i) = \sum_{S \subseteq [m]} |S||E_S| \leq \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq 2} (|S| - 1)|E_S|.$$

Proof of Theorem B: Upper Bound

$$\sum_{\substack{S\subseteq[m]\\|S|\geq 2}}(|S|-1)|E_S| = \sum_{\substack{S\subseteq[m],\\|S|=2}}\sum_{T\supseteq S}\frac{(|T|-1)|E_T|}{\binom{|T|}{2}} \leq \sum_{\substack{S\subseteq[m],\\|S|=2}}\sum_{T\supseteq S}|E_T|,$$

as each $T\in[m]$ with $|T|\geq 2$ is counted $\binom{|T|}{2}$ times.

Proof of Theorem B: Upper Bound

Observation: If $|S| \geq 2$, the edge set $\bigcup_{T \supseteq S} E_T$ is F-free.

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$$\implies \sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S| - 1)|E_S| \leq \sum_{\substack{S \subseteq [m], \\ |S| = 2}} \sum_{T \supseteq S} |E_T| \leq \binom{m}{2} \mathrm{ex}(n, F)$$

This proves the upper bound.

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$$\phi(m, n, F) \geq \binom{n}{2} + \mathrm{ex}(n, F)\binom{m}{2}.$$

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$\implies$ We can pack these $\binom{m}{2}$ H_{ij}'s into $[n]$ s.t. $E(H_{i_1 j_1}) \cap E(H_{i_2 j_2}) = \emptyset$

Proof of Theorem B: Lower Bound

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Add edges not in any of H_{ij} to G_1.

$$\sum_{i=1}^{m} e(G_i) = 2\underbrace{\sum e(H_{ij})}_{\binom{m}{2}\mathrm{ex}(n,F)} + \underbrace{\left|E(K_n) \setminus \bigsqcup E(H_{ij})\right|}_{\binom{n}{2}-\binom{m}{2}\mathrm{ex}(n,F)} = \binom{n}{2} + \binom{m}{2}\mathrm{ex}(n,F)$$

Proof of Theorem D

Theorem D

Let $m, n, r \geq 3$. Then

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced K_r-free graphs $G_1, G_2, \ldots, G_m$ only if $G_1 = G_2 = \cdots = G_m = T_{r-1}(n)$.

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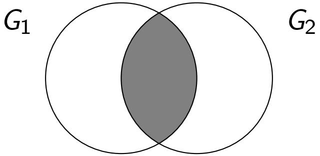
IDEA: Reduce to $m = 2$.

Proof of Theorem D



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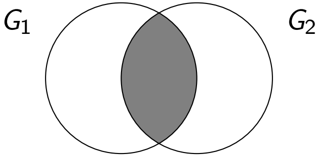

$G_1 \cap G_2$ (gray area) is double K_r-free.

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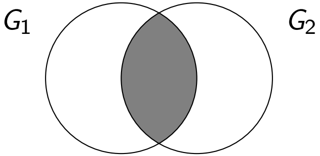

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If G_1, G_2 intersects in t vertices, then

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2T_{r-1}(t)$$

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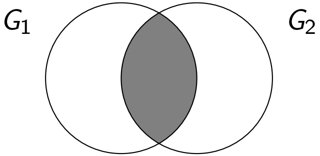

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If G_1, G_2 intersects in t vertices, then

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$\implies$ we are done if the unique maximum occurs at $t = n$.

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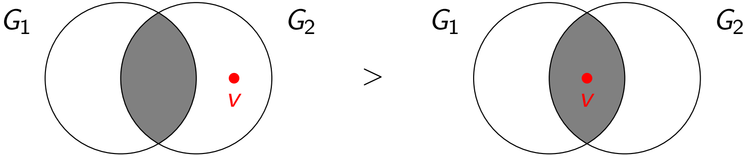

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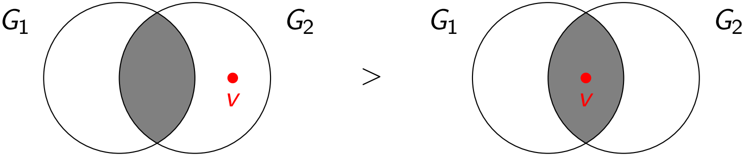

Put $f(t,r) := \binom{n-t}{2} + (n-t)t + 2T_{r-1}(t)$

For example, if $r = 4$,

$$f(t+1,4) - f(t,4) = -t + 1 + 2[\underbrace{T_3(t+1) - T_3(t)}_{t - \lceil \frac{t}{2} \rceil}] = t + 1 - 2\left\lceil \frac{t}{2} \right\rceil > 0$$

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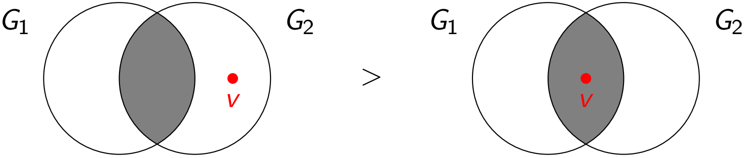

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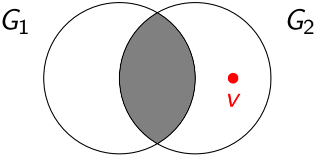

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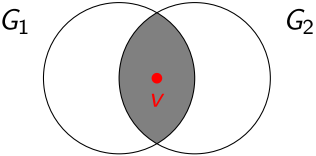

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$\implies$ The larger the intersection, the larger the sum of edges.

$\implies$ The best we can do is $G_1 = G_2 = T_3(n)$.

Concluding Remarks: Induced Double Turán Problem

Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

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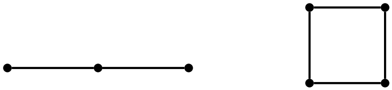
This conjecture is broadly open for bipartite graphs.



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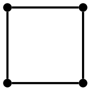

Solving the conjecture would give a solution to $\mathrm{ex}(n, K_{2,2,2}, K_3)$.

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$$\lim_{n\to\infty} \frac{\phi(4, n, K_3)}{m\binom{n}{2}} = ??$$

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