Double Turán Problem

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1 Introduction

This thesis focuses on a variation of the *Turán problem* in extremal combinatorics. The fundamental question in extremal hypergraph theory is determining the maximum number of edges in an n-vertex r-uniform graph that does not contain a prescribed r-uniform graph F as a subgraph. These maxima, denoted $\mathrm{ex}(n, F)$, are referred to as the *extremal numbers* or *Turán numbers* for F. One of the cornerstones of extremal graph theory, concerning the case F is a clique, is Turán's Theorem [**?**]. To state the theorem, we need the *Turán graphs* $T_k(n)$, which denotes a complete multipartite graph with n vertices and k parts, which are of size $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$.

Theorem A (Turán's Theorem)**.** *The maximum number of edges in an n-vertex graph containing no clique of order $k+1$ is $e(T_k(n))$, with equality only for $T_k(n)$.*

Simonovits [**?**] observed via the Erdős-Stone Theorem [**?**] that one may thereby obtain the asymptotic value of $\mathrm{ex}(n, F)$ whenever F is a graph of chromatic number $k + 1 \geq 3$:

Theorem B (Erdős-Stone Theorem, Simonovits' Theorem)**.** *Let F be any graph of chromatic number $k+1 \geq 3$. Then $\mathrm{ex}(n, F) = (1 + o_n(1))T_k(n)$ as $n \to \infty$.*

The case F is bipartite is in general wide open, and the order of magnitude of $\mathrm{ex}(n, K_{4,4})$ or $\mathrm{ex}(n, C_8)$ is not known – see Füredi and Simonovits [**?**] for a history of the bipartite Turán problem. There is also no analog of the above theorems for r-uniform hypergraphs; the asymptotic value of $\mathrm{ex}(n, K_k^r)$ is not known for any $k > r \geq 3$, where K_k^r denotes the complete r-uniform hypergraph on k vertices. The asymptotic value of $\mathrm{ex}(n, K_4^3)$ was conjectured by Turán [**?**] to be $\frac{5}{9}\binom{n}{3}$, and this remains open despite decades of intensive research.

In this thesis, we investigate closely related problems which we refer to as *double Turán problems*. To describe these problems, let $G_1, G_2, \ldots, G_m$ be graphs with the same vertex set $V(G_i) = [n]$ for $i \in [m]$. For a graph F, we say that $G_1, G_2, \ldots, G_m$ is *double F-free* if $E(F) \not\subseteq E(G_i) \cap E(G_j)$ for $1 \leq i < j \leq m$. In other words, F does not appear in the intersection of any two of the graphs G_i. We refer to a copy of F in the intersection of two of the graphs G_i as a *double* F. Let $\phi(m, n, F)$ denote the maximum value of $\sum_{i=1}^m e(G_i)$ such that $G_1, G_2, \ldots, G_m$ does not contain a double F. We say that graphs $G_1, G_2, \ldots, G_m$ are *induced* to mean that every G_i is an induced subgraph of $\bigcup_{i=1}^m G_i$. In other words, if $\{u, v\} \in E(G_i)$ and $u, v \in V(G_j)$, then also $\{u, v\} \in E(G_j)$. Let $\phi^*(m, n, F)$ denote the maximum value of $\sum_{i=1}^m e(G_i)$ such that $G_1, G_2, \ldots, G_m$ does not contain a double F and $G_1, G_2, \ldots, G_m$ are induced. Evidently, $\phi^*(m, n, F) \leq \phi(m, n, F)$, and the study of $\phi^*(m, n, F)$ and $\phi(m, n, F)$ is motivated by certain hypergraph extremal problems.

1.1 Link graphs and hypergraphs

Apart from the intrinsic interest in studying $\phi(m, n, F)$, a motivation is that $\phi(m, n, F)$ is closely connected to pure hypergraph extremal problems via the notion of *link graphs*. Let H be a triple system, in other words, a set of three-element subsets of a finite set $[n]$. These three-element subsets form the edge-set $E(H)$ of H, while $V(H) = V$ is the vertex set of H. For $i \in V(H)$, let H_i denote the *link graph* of i, namely $V(H_i) = V(H)\backslash\{i\}$ and $E(H_v) = \{\{j, k\} : \{i, j, k\} \in E(H)\}$. A useful idea in extremal hypergraph theory is to try to reduce an extremal problem for hypergraphs to extremal problems for the link graphs. For instance, a triple system H does not containing three triples on four vertices if and only if all its link graphs are triangle-free.

In the current context, given an graph F, let F^+ denote the triple system with vertex set $V(F^+) = V(F) \cup \{x, y\}$ and edge set $\{e \cup \{x\}, e \cup \{y\} : e \in E(F)\}$. Then $\phi(n, n, F)$ and $\mathrm{ex}(n, F^+)$ are very closely related: if H is an F^+-free triple system with vertex set $[n]$, then clearly the link graphs $H_1, H_2, \ldots, H_n$ are double F-free. Therefore $\mathrm{ex}(n, F^+) \leq \phi(n, n, F)$. This relates the double Turán problem to hypergraph extremal problems.

Now let G be the graph consisting of all pairs contained in triples in F^+. The *generalized Turán problem* asks for the maximum number $\mathrm{ex}(n, G, K_3)$ of triangles in a graph H with vertex set $[n]$ that does not contain G. This problem was studied by Alon and Shikhelman [**?**] and Kostochka, Mubayi and Verstraete [**? ? ?**]. This problem is related to $\phi^*(n, n, F)$ as follows: define $H_i = \{\{j, k\} : \{i, j\}, \{j, k\}, \{i, k\} \in E(H)\}$. Then $H_1, H_2, \ldots, H_n$ are induced and double F-free, so $\phi^*(n, n, F) \geq \mathrm{ex}(n, G, K_3)$. This relates the induced double Turán problem to extremal problems for triangles in graphs.

1.2 Main results : the induced case

The determination of $\phi^*(m, n, F)$ turns out to be fairly straightforward when F is a non-bipartite graph: the extremal objects are m copies of the same extremal graph for F:

Theorem 1. *For $k \geq 3$, there exists $n_0(k)$ such that if $n \geq n_0(k)$ and F is a graph of chromatic number k, then for all $m \geq 3$,*

$$\phi^*(m, n, F) = m \cdot \mathrm{ex}(n, F),$$

with equality only for identical extremal n-vertex F-free graphs.

In the case $F = K_k$, we shall see the theorem is true for all $n \geq 3$:

In this section, we aim to prove the following theorem:

Theorem 2. *Let $m, n, r \geq 1$. Then $\phi^*(m, n, K_{r+1}) = m \cdot e(T_r(n))$ with equality for induced K_{r+1}-free graphs $G_1, G_2, \ldots, G_m$ only if $G_1 = G_2 = \cdots = G_m = T_r(n)$.*

In the case F is a bipartite graph, even the problem of determining the order of magnitude of $\phi^*(m, n, F)$ appears to be difficult, and we do not know the order of magnitude of $\phi^*(m, n, P)$ when P is a path with two edges. In this thesis, we propose the following very broad conjecture:

Conjecture A. *Let F be any non-empty graph and $m, n \geq 1$. Then*

$$\phi^*(m, n, F) = \Theta(m \cdot \mathrm{ex}(n, F) + n^2).$$

It is clear that a single complete graph K_n does not contain a double F, and neither do identical copies $G_1, G_2, \ldots, G_m$ of an extremal n-vertex F-free graph. In particular,

$$\phi^*(m, n, F) \geq \max\left\{\binom{n}{2}, m \cdot \mathrm{ex}(n, F)\right\}.$$

This conjecture is true when F is non-bipartite, by Theorem 1. If F is bipartite, then upper bounds on $\phi^*(m, n, F)$ are more difficult to come by, especially when m is large. For instance, we know

$$\mathrm{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2})$$

and so Conjecture A implies that an n-vertex graph not containing the octahedron graph has $O(n^{5/2})$ triangles. In fact it is also the case that $\mathrm{ex}(2n, K_{2,2,2}, K_3) \geq \phi^*(n, n, K_{2,2})$, for if we have double $K_{2,2}$-free induced graphs $G_1, G_2, \ldots, G_n$ with vertex set $[n]$, then let H be the graph with $V(H) = [2n]$ consisting of all triangles with vertex set $\{i, j, k\}$ such that $n < k \leq 2n$ and $\{i, j\} \in E(G_k)$. The graph H is $K_{2,2,2}$-free and $|E(H)| = \sum_{i=1}^{n/2} e(G_i)$. Similarly, we have

$$\mathrm{ex}(n, K_{1,2,2}, K_3) \leq \phi^*(n, n, K_{1,2})$$

and so Conjecture A implies that an n-vertex graph not containing the octahedron graph has $O(n^2)$ triangles, which is conjectured by Mubayi and Verstraete [**?**]. The conjecture proposes more generally that if F is a tree, then $\phi^*(n, n, F) = O(n^2)$. In fact, it is possible to prove the following theorem using the *removal lemma* as in [**?**] as well as a construction for $\phi(n, n, P)$ in this work:

Theorem 3. *Let P be a path with two edges. Then $\phi(n, n, P) = \Omega(n^{5/2})$ whereas $\phi^*(n, n, P) =$*

$o(n^{5/2})$ *as* $n \to \infty$. *In particular,*

$$\lim_{n\to\infty} \frac{\phi^*(n,n,P)}{\phi(n,n,P)} = 0.$$

If M is a matching with two edges, and M^+ is the graph obtained from two copies of K_4 sharing one edge by deleting that edge, then $\mathrm{ex}(n, M^+, K_3) \leq \phi^*(n,n,M)$. If F is the triple system consisting of all four triangles in M^+, then Füredi [**?**] showed $\mathrm{ex}(n, M^+) = O(n^2)$, answering a conjecture of Erdős [**?**]. It is possible to adapt Füredi's proof to give $\phi^*(n,n,M) = O(n^2)$, so in this case, $\mathrm{ex}(n, M^+, K_3) = \Theta(\phi^*(n,n,M))$. For improvements of the constant factor, see Mubayi and Verstraete [**?**] and Pikhurko and Verstraete [**?**]. We shall see that if F is bipartite and m is not too large relative to n, then Conjecture A is also true.

1.3 Main results : the non-induced case

Determining $\phi(m,n,F)$ even when F is a complete graph is challenging. The second theorem we give is well-suited to the case of bipartite graphs, and is due to Wilson:

Theorem 4. *Let F be a graph. If there exists an extremal F-free n-vertex graph with maximum degree at most $n^{1/2}/m^2$, then*

$$\phi(m,n,F) = \binom{n}{2} + \binom{m}{2}\mathrm{ex}(n,F).$$

[**JV:** It should be true that if m is a constant, then equality holds for every bipartite graph F when n is large enough.] Since $\binom{n}{2} + m - 1 \leq \phi^*(m,n,F) \leq \phi(m,n,F)$ for any graph F with at least two edges, this theorem shows $\phi^*(m,n,F) = (1+o(1))\binom{n}{2}$ whenever the conditions on m in the theorem are satisfied. In particular, if P is the path with two edges, and $m = o(n^{1/4})$ as $n \to \infty$, then for $n \geq 2$,

$$\binom{n}{2} + m - 1 \leq \phi^*(m,n,F) \leq \phi(m,n,F) = \binom{n}{2} + \binom{m}{2}\left\lfloor \frac{n}{2} \right\rfloor.$$

When F is bipartite, the value of $\phi(m,n,F)$ for larger m appears to be difficult to determine. We investigate the case that F is a path or matching with two edges more closely. For a family $\mathcal{F}$ of graphs, we write $\phi(m,n,\mathcal{F})$ for the maximum number of edges in graphs $G_1, G_2, \ldots, G_m$ which are double F-free for all $F \in \mathcal{F}$.

Theorem 5. *Let P be the path with two edges and let M be the matching with two edges.*

*Then as $m, n \to \infty$, **[JV:** Some of these statements need a dependency between m and n, else they are false. Also did we not discuss lower bound on $\phi(m, n, P)$ when $m \geq n$? Like $n^2\sqrt{m}$ perhaps?]*

1. $\phi(m, n, P) \leq \left(\frac{1}{2} + o(1)\right) \min\{n^2\sqrt{m}, mn^{3/2}\}$.
2. $\phi(m, n, P) = \left(\frac{1}{2} + o(1)\right) mn^{3/2}$ *for* $\sqrt{n} \leq m \leq n$.
3. $\phi(m, n, M) \leq n^{5/2}$.
4. $\phi(m, n, \{P, M\}) = n^2 - n$.

Interestingly, while Conjecture A proposes $\phi^*(m, n, P) = O(n^2 + mn)$ for all $m, n \geq 1$, the above theorem shows $\phi(m, n, P)$ is much larger, of order at least $mn^{3/2}$ when $m \geq \sqrt{n}$.

Our first theorem on $\phi(m, n, F)$ for non-bipartite graphs F uses the notion of *supersaturation* – see Erdős and Simonovits [**?**]. We determine the asymptotic value of $\phi(m, n, F)$ as $m \to \infty$ when F is a non-bipartite graph:

Theorem 6. *Let $n \geq 1$ and let F be a non-bipartite graph. Then as $m \to \infty$,*

$$\phi(m, n, F) = (1 + o(1))m \cdot \mathrm{ex}(n, F).$$

The next result we present concerns non-bipartite graphs. To state the theorem, we require the notion of k-color Ramsey numbers. Define $R_k(r)$ to be the *k-color Ramsey number* for the complete graph K_r: that is, the minimum N such that there exists a monochromatic F in any coloring of $E(K_N)$ with k colors. Suppose we have a coloring $c : E(K_N) \to 2^{[m]}$ for some $N < R_k(r)$ where $k \leq \binom{m}{2}$ and $|c(u, w)| \geq 2$ for all $\{u, w\} \in E(K_N)$. For $i \in [m]$, let $H_i = \{\{u, w\} \in E(K_N) : i \in c(u, w)\}$. Then $H_1, H_2, \ldots, H_m$ are double K_r-free. If we replace the vertices of K_N with disjoint sets $V_w : w \in V(K_N)$ whose sizes add up to n, and then let $G_1 = K_n$ and

$$G_i = \{\{x, y\} : (x, y) \in V_u \times V_w, i \in c(u, w)\}$$

then $G_1, G_2, \ldots, G_m$ is also double K_r-free. We call $G_1, G_2, \ldots, G_m$ an (m, n, k)-blowup, and let $f(m, n, r)$ denote the maximum of $e(H_1) + e(H_2) + \cdots + e(H_m)$ such that $H_1, H_2, \ldots, H_m$ is an (m, n, k)-blowup for some $k \leq \binom{m}{2}$. This turns out to be exactly the construction which determines $\phi(m, n, F)$ when F is a complete graph:

Theorem 7. *Let $r \geq 2$ and $m, n \geq 1$. Then*

$$\phi(m, n, K_r) = f(m, n, r).$$

While computing $f(m,n,r)$ is a finite calculation, the Ramsey number $R_k(r)$ unfortunately appears to be intractable in general; it is known that $R_2(3) = 6$ and $R_3(3) = 17$ and $R_2(4) = 18$, but no further multicolor Ramsey numbers are known [**? ?**]. In the special case $r = m = 3$, the following holds:

Theorem 8. *For all $n \geq 1$,*

$$\phi(3,n,K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor .$$

1.4 Definitions and Notation

Denote the set of first n positive integers as $[n] = \{1, 2, \ldots, n\}$. Given a set X, we denote 2^X as the power set of X.

Let $G = (V, E)$ be a graph. Let $V(G)$ denote the vertex set and $E(G)$ denote the edge set of G. Let $e(G) = |E(G)|$ be the number of edges in G. For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v.

Given graphs $G_1, G_2, \ldots, G_m$ on some vertex set V, we denote $G_{i_1,\ldots,i_k}$ as graph on V with edge set $E(G_{i_1,\ldots,i_k}) = \bigcap_{\alpha=1}^{k} E(G_{i_\alpha})$. Given two graphs G_1, G_2, we denote $G_1 \cup G_2$ as the graph on $V(G_1) \cup V(G_2)$ with edge set $E(G_1 \cap G_2) = E(G_1) \cup E(G_2)$. Let s

In this thesis, we reserve n to denote the number of vertices in a graph. We call a n-vertex complete graph K_n, and a complete bipartite graph $K_{a,b}$, where a, b are the sizes of its parts. We denote P_n as a path with n edges, and C_n as a cycle with n edges. Given graph G, H, define $G + H$ as the graph fully connecting G, H, i.e. $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$.

Given graphs G and F, we say that G is F-free if G does not contain F as a subgraph. We denote $\mathrm{ex}(n, F)$ to be the maximum possible number of edges an F-free graph on n vertices, and we call a F-free graph achieving this maximum an extremal graph for F. Given graphs $G_1, \ldots, G_m$ on the same set of vertices and F, we say that $G_1, \ldots, G_m$ are pairwise F-free if $E(G_i) \cap E(G_j)$ does not contain F for $i \neq j$. Let v be a vertex from $G_1, G_2, \ldots, G_m$. Unless otherwise specified, we denote $d(v)$ as the sum of the degree of v over all G_i.

2 The induced double Turán problem

[**JV:** This section could be written more cleanly.] In this chapter, we investigate the case where $G_1, G_2, \ldots, G_m$ are induced subgraphs of $G_1 \cup \cdots \cup G_m$ and are pairwise F-free, for some specified F. The main theorem we prove is Theorem 1 for general non-bipartite graphs F and in the special case of cliques. Unless otherwise specified, when we say $G_1, \dots, G_m$ are induced subgraphs, we mean that they are induced subgraphs of $G_1 \cup \dots \cup G_m$.

The following lemma shows that the problem can be reduced to only two graphs.

Lemma 9. *Let $n, m, k \geq 2$ with $m \geq k$, F be some graph. Then*

$$\phi^*(m, n, F) = \frac{m}{k} \cdot \phi^*(k, n, F).$$

Moreover, let $G_1, \dots, G_m$ be induced double F-free graphs on $[n]$ and suppose $\sum_{i=1}^k e(G_i) = \phi^*(k, n, F)$ only if $G_1 = \dots = G_k$. Then $\sum_{i=1}^m e(G_i) = \phi^*(m, n, F)$ only if $G_1 = \dots = G_m$.

Proof. Let $G_1, \dots, G_m$ be induced double F-free graphs on $[n]$. Put $G_{i+m} = G_i$ for all $i \in [m]$. Then

$$\sum_{i=1}^m e(G_i) = \frac{1}{k} \sum_{i=1}^m [e(G_i) + \dots + e(G_{i+k-1})] \leq \frac{1}{k} \sum_{i=1}^m \phi^*(k, n, F) = \frac{m}{k} \cdot \phi^*(k, n, F),$$

which establishes the upper bound. The lower bound follows from the construction with $G_1 = \dots = G_m$ to be n-vertex extremal graphs for F.

Now suppose $\sum_{i=1}^m e(G_i) = \frac{m}{k} \cdot \phi^*(k, n, F)$ and $G_1 \neq G_2$. By assumption $\sum_{i=1}^k e(G_i) < \phi^*(k, n, F)$. But then $\sum_{i=1}^k e(G_{i+j}) > \phi^*(k, n, F)$ for some $j \geq 1$, contradiction. $\square$

Lemma 9 allows us to reduce the problem to the case for two subgraphs G_1, G_2.

Definition 10. *For $n \geq t \geq 1$ and F some graph, define*

$$\mathcal{C}(n, t, F) := \binom{n-t}{2} + (n-t)t + 2\mathrm{ex}(t, F).$$

The construction described by $\mathcal{C}(n, t, F)$ are graphs G_1, G_2 on $[n]$, such that G_2 is a t-vertex extremal graph for F and $G_1 = G_2 + K_{n-t}$.

Lemma 11. *Let F be some graph. For $n \geq 1$,*

$$\phi^*(2, n, F) = \max_{0 \leq t \leq n} \mathcal{C}(n, t, F).$$

Moreover, the equality holds for graphs G_1, G_2 on $[n]$ only if G_1, G_2 are the construction described by $\mathcal{C}(n, t_{max}, F)$, where $t_{max} \in [n]$ is a maximizer for $\mathcal{C}(n, t, F)$.

Proof. Let G_1, G_2 be induced double F-free graphs on $[n]$. Put $T = V(G_1) \cap V(G_2)$, $t = |T|$, $s = |V(G_1) \backslash T|$, and $n - t - s = |V(G_2) \backslash T|$. Note that $t, s \in \mathbb{Z}_{\geq 0}$. Since G_1, G_2 are induced subgraphs of $G_1 \cup G_2$, we have $G_1[T] = G_2[T] = G_1 \cap G_2$. But then $G_1 \cap G_2$ is F-free, so $e(G_1[T]) = e(G_2[T]) \leq \mathrm{ex}(t, F)$. Notice there can be at most $t(n-t)$ edges between T and $(V(G_1) \cup V(G_2)) \backslash T$. Since $G[V(G_1) \backslash T] \leq \binom{s}{2}$ and $G[V(G_2) \backslash T] \leq \binom{n-t-s}{2}$,

$$e(G_1) + e(G_2) \leq \binom{s}{2} + \binom{n-s-t}{2} + t(n-t) + 2\mathrm{ex}(t, F).$$

But then $\binom{n-t}{2} > \binom{s}{2} + \binom{n-t-s}{2}$ for $0 < s < n - t$, so

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2\mathrm{ex}(t, F) = \mathcal{C}(n, t, F).$$

This establishes the upper bound. From this we also know that $e(G_1) + e(G_2) = \mathcal{C}(n, t, F)$ only if G_1, G_2 are the construction described by $\mathcal{C}(n, t, F)$. The result now follows. □

The problem is now reduced to maximizing $\mathcal{C}$ over t. In particular, $\mathcal{C}(n, n, F)$ gives our benchmark construction of $G_1 = G_2$ being the extremal graphs for F on n vertices.

2.1 Proof of Theorem 2

By Lemma 9, it suffices to prove the theorem for $m = 3$. Let G_1, G_2, G_3 be induced double K_r-free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, K_r)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\phi^*(3, n, K_r) \geq 3\mathrm{ex}(n, K_r)$. Consequently, we must have $e(G_1) + e(G_2) \geq 2\mathrm{ex}(n, K_r)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\mathrm{ex}(n, K_r)$, it suffices to show that $G_1 = G_2 = T_{r-1}(n)$. In particular, we will use Lemma 11 to show that G_1, G_2 is an extremal configuration without containing a double K_r.

Let $t = |V(G_1 \cap G_2)|$. By Turán's Theorem,

$$\mathrm{ex}(t, K_r) - \mathrm{ex}(t-1, K_r) = e(T_{r-1}(t)) - e(T_{r-1}(t-1)) = t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, K_r) - \mathcal{C}(n, t-1, K_r) = -t + 1 + 2[\mathrm{ex}(t, K_r) - \mathrm{ex}(t-1, K_r)] = t + 1 - 2\left\lceil \frac{t}{r-1} \right\rceil. \quad (1)$$

For $r \geq 4$, $\mathcal{C}(n,t,K_r)$ is strictly increasing on t, so by Lemma 11,

$$\phi^*(2,n,K_r) = \mathcal{C}(n,n,K_r) = 2\mathrm{ex}(n,K_r) = e(G_1) + e(G_2)$$

and $G_1 = G_2 = T_{r-1}(n)$, as desired.

Now suppose $r = 3$. Equation (1) shows that $\mathcal{C}(n,t,K_r)$ is non-decreasing on t and $\mathcal{C}(n,t,K_r) > \mathcal{C}(n,t,K_r)$ for even t. By Lemma 11, we now have

$$\phi^*(2,n,K_r) = \max[\mathcal{C}(n,n,K_r), \mathcal{C}(n,n-1,K_r)] = 2\mathrm{ex}(n,K_r) = e(G_1) + e(G_2),$$

and either $G_1 = G_2 = T_{r-1}(n)$, or $G_2 = T_{r-1}(n-1)$ and $G_1 = G_2 + K_1$. If the latter case is true, then $e(G_3) \geq \mathrm{ex}(n,F) > e(G_2)$, and this contradiction completes the proof. □

2.2 Proof of Theorem 1

If F is a graph of chromatic number $r+1 \geq 3$, then Theorem B shows $\mathrm{ex}(n,F) = (1 + o(1))\mathrm{ex}(n,K_{r+1})$ as $n \to \infty$. In this section, we prove Theorem 1.

Proof. By Lemma 9, it suffices to prove the theorem for $m = 3$. Let G_1, G_2, G_3 be induced double F-free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \phi^*(3,n,F)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\phi^*(3,n,F) \geq 3\mathrm{ex}(n,F)$. Consequently, we must have $e(G_1) + e(G_2) \geq 2\mathrm{ex}(n,F)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\mathrm{ex}(n,F)$, it suffices to show that $G_1 = G_2$ are n-vertex F-free extremal graphs. In particular, we will use Lemma 11 to show that G_1, G_2 is an extremal configuration without containing a double F.

Let $t = |V(G_1 \cap G_2)|$. If $t < \sqrt{n}$, then

$$2\mathrm{ex}(n,F) \geq 2e(T_{r-1}(n)) \geq 2\left\lfloor \frac{n^2}{4} \right\rfloor \geq \binom{n}{2} + \binom{\sqrt{n}}{2} > \mathcal{C}(n,t,F).$$

Thus $t \geq \sqrt{n}$. But then for large enough t, any extremal t-vertex F-free graph contains a spanning complete $(r-1)$-partite subgraph $T_{r-1}(t)$, so we may add $\mathrm{ex}(t-1,F) - e(T_{r-1}(t-1))$ egdes to $T_{r-1}(t)$ and still avoid F as a subgraph. Hence for large enough t, we have $\mathrm{ex}(t,F) \geq \mathrm{ex}(t-1,F) - e(T_{r-1}(t-1)) + e(T_{r-1}(t))$, and so

$$\mathrm{ex}(t,F) - \mathrm{ex}(t-1,F) \geq e(T_{r-1}(t)) - e(T_{r-1}(t-1)) \geq t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n,t,F)-\mathcal{C}(n,t-1,F)=-t+1+2[\mathrm{ex}(t,F)-\mathrm{ex}(t-1,F)]\geq t+1-2\left\lceil\frac{t}{r-1}\right\rceil. \quad (2)$$

For $r\geq 4$, $\mathcal{C}(n,t,F)$ is strictly increasing on t, so by Lemma 11,

$$\phi^*(2,n,F)=\mathcal{C}(n,n,F)=2\mathrm{ex}(n,F)=e(G_1)+e(G_2),$$

and $G_1=G_2$ are n-vertex F-free extremal graphs, as desired.

Now suppose $r=3$. Equation (2) shows that $\mathcal{C}(n,t,F)$ is strictly increasing for even t and $\mathcal{C}(n,t,F)\geq\mathcal{C}(n,t-1,F)$ for odd t. By Lemma 11, we now have

$$\phi^*(2,n,F)=\max[\mathcal{C}(n,n,F),\mathcal{C}(n,n-1,F)]=2\mathrm{ex}(n,F)=e(G_1)+e(G_2),$$

and either $G_1=G_2$ are n-vertex extremal F-free graphs, or G_2 is an $(n-1)$-vertex extremal F-free graph and $G_1=G_2+K_1$. If the latter case is true, then $e(G_3)\geq\mathrm{ex}(n,F)>e(G_2)$, and this contradiction completes the proof. $\square$

For small n, we may not be able to achieve the same result. Consider the case when F is the bowtie graph, i.e. the 5-vertex graph with two triangles sharing a vertex. For $n\leq 4$, the n-vertex extremal graph for F is the complete graph K_n. For $n\geq 5$, the n-vertex extremal graph for F is then $K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$ plus an edge, and so $\mathrm{ex}(n,F)=\left\lfloor\frac{n^2}{4}\right\rfloor+1$. But then in this case when $n=5$,

$$\mathcal{C}(5,4,F)=2e(K_4)+4=16>\mathcal{C}(5,5,F)=2\left(\left\lfloor\frac{5^2}{4}\right\rfloor+1\right)=14.$$

This yields an instance where the construction $G_1=K_{k-1}$ and $G_2=K_n$ beats our benchmark construction for $|V(F)|=k$. Thus the following lemma gives a lower bound for n to avoid losing to this construction.

Lemma 12. *Let $n,k\geq 3$ and $r\geq 2$, and let F have chromatic number $r+1$ and $|V(F)|=k$. If $n\geq 2\binom{k-1}{2}+1$ and r divides n, then*

$$\mathcal{C}(n,n,F)>\mathcal{C}(n,k-1,F).$$

In particular, if $n\geq 2\binom{k-1}{2}+1$, then $\phi^*(m,n,F)=m\cdot\mathrm{ex}(n,F)$ for all $m\geq 3$, with equality only for identical extremal n-vertex F-free graphs.

Proof. We need to show that

$$2\mathrm{ex}(n, F) - \binom{n}{2} > \binom{k-1}{2}.$$

Since $\mathrm{ex}(n, F) \geq \lfloor n^2/4 \rfloor$,

$$2\mathrm{ex}(n, F) - \binom{n}{2} \geq 2\left\lfloor \frac{n^2}{4} \right\rfloor - \binom{n}{2} \geq \frac{n}{2} - \frac{1}{4} > \binom{k-1}{2}.$$

This proves the lemma. $\square$

[**JV:** This lemma seems like an afterthought, and should probably go first. Also, can you say something much better if $r \geq 3$?]

2.3 Proof of Theorem 3

[**JV:** This is to be written]

3 The non-induced double Turán problem

In this section, we prove our main theorems on $\phi(m, n, F)$.

3.1 Proof of Theorem 6

We need the following *saturation theorem*, which may be found in [**?**].

Proposition 13. *Let F be any non-empty graph with k vertices. For all $\epsilon > 0$, there exists $\delta > 0$ such that if G is any n-vertex graph with $\mathrm{ex}(n, F) + \epsilon n^2$ edges, then G contains δn^k copies of F.*

Proof of Theorem 6. Let $k = |V(F)|$ and let $\epsilon > 0$. Let $G_1, G_2, \ldots, G_m$ be double F-free. Reorder $G_1, G_2 \ldots, G_m$ so that $e(G_i) \geq \mathrm{ex}(n, F) + \epsilon n^2$ for $1 \leq i \leq \ell$ and $e(G_i) < \mathrm{ex}(n, F) + \epsilon n^2$ for $\ell < i \leq m$. Then each $G_i : 1 \leq i \leq \ell$ contains at least δn^k copies of F, by Proposition 13. On the other hand, there are at most n^k copies of F such that $F \subseteq G_i$ for some $i \in [m]$.

Therefore $\ell \leq 1/\delta$ and

$$\begin{aligned}\sum_{i=1}^{m} e(G_i) &= \sum_{i=1}^{\ell} e(G_i) + \sum_{i=\ell+1}^{m} e(G_i) \\ &\leq \frac{1}{\delta}\binom{n}{2} + (m-\ell)\mathrm{ex}(n,F) + (m-\ell)\epsilon n^2 \\ &\leq m\mathrm{ex}(n,F) + \epsilon m n^2 + \frac{1}{\delta}\binom{n}{2}.\end{aligned}$$

Since F is not bipartite, $\mathrm{ex}(n,F) = \Theta(n^2)$ and so $\phi(m,n,F) \leq m \cdot \mathrm{ex}(n,F) + (\epsilon + 1/\delta m)mn^2$. Since ϵ was arbitrary and δ is a constant depending only on ϵ, we conclude $\phi(m,n,F) \leq (1+o(1))m \cdot \mathrm{ex}(n,F)$ as $m \to \infty$. □

Let F be a bipartite graph with $k \geq 2$ vertices and $j \geq 1$ edges. A strong version of a conjecture of Simonovits [? ?] would suggest that for all $\epsilon > 0$, there exists $\delta > 0$ such that every n-vertex graph G with at least $p\binom{n}{2}(1+\epsilon)\mathrm{ex}(n,F)$ edges contains at least $\delta p^j n^k$ copies of F. For instance, this is known to be true whenever the asymptotic behavior of $\mathrm{ex}(n,F)$ is known, which includes the case $F = K_{2,t}$. If F is bipartite and $m\mathrm{ex}(n,F)/n^2 \to \infty$ as $m,n \to \infty$, then this conjecture with the same proof as above shows $\phi(m,n,F) = (1+o(1))m \cdot \mathrm{ex}(n,F)$. When F contains a cycle, then there exists $\alpha > 0$ such that $\mathrm{ex}(n,F) \geq n^{1+\alpha}$ for large enough n, and we conclude that if F contains a cycle, and the Simonovits conjecture is true for F, then $\phi(m,n,F) = (1+o(1))m \cdot \mathrm{ex}(n,F)$ for $m \geq n$ and $n \to \infty$. In particular, this shows $\phi(m,n,K_{2,t}) = (1+o(1))m \cdot \mathrm{ex}(n,F)$ for $m \geq n$ as $n \to \infty$.

3.2 Proof of Theorem 4

We first show that for all $m,n \geq 1$ and all graphs F,

$$\phi(m,n,F) \leq \binom{n}{2} + \mathrm{ex}(n,F)\binom{m}{2}.$$

Thereafter, we show that if there is an extremal F-free graph with maximum degree at most $n^{1/2}/m^2$, then the above bound is exactly tight.

Proof of the upper bound. For $S \subseteq [m]$, let E_S denote the set of edges that are contained in exactly $\{G_i\}_{i \in S}$. Then

$$\sum_{i=1}^{m} e(G_i) = \sum_{S \subseteq [m]} |S||E_S| \leq (k-1)\binom{n}{2} + \sum_{S \subseteq [m], |S| \geq k} (|S|-k+1)|E_S|.$$

Let $A_S = \bigcup_{T \supseteq S} E_T$, i.e., the set of edges that are contained in all G_i with $i \in S$. When $|S| \geq k$, the edge set A_S is F-free and thus

$$\sum_{T \supseteq S} |E_T| \leq \mathrm{ex}(n, F).$$

Hence,

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq k}} (|S| - k + 1)|E_S| = \sum_{\substack{S \subseteq [m], \\ |S| = k}} \sum_{T \subseteq S} \frac{(|T| - k + 1)|E_T|}{\binom{|T|}{k}} \leq \sum_{\substack{S \subseteq [m], \\ |S| = k}} \sum_{T \subseteq S} |E_T| \leq \binom{m}{k} \mathrm{ex}(n, F),$$

as each $T \in [m]$ with $|T| \geq k$ is counted $\binom{|T|}{k}$ times in total and $|T| - k + 1 \leq \binom{|T|}{k}$. This proves the upper bound.

Proof of the lower bound. We need to show there exists a construction such that the graph with edge set E_S is an extremal F-free graph, for all $S \subseteq [m]$ of size k. Let $M = \binom{m}{k}$ and $H_1, \ldots, H_M$ be copies of an extremal F-free graph on n vertices such that H_i with maximum degree $\triangle \leq n^{1/2}/m^2$ for all $i \in [m]$. It suffices to show that we can embed each H_i onto $[n]$ such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each H_i and iteratively decrease the number of intersecting edges. Define a (u, v, i)-*swap* by swapping the embedding of vertex u and v of H_i, i.e. replacing each edge $\{u, w\} \in E(H_i)$ with the edge $\{u, w\}$ and each edge $\{v, w\} \in E(H_i)$ with the edge $\{v, w\}$. This preserves the type of isomorphism of H_i. Given a vertex v, let $N(v) = N_{H_1}(v) \cup \cdots \cup N_{H_M}(v)$. Suppose there exists an intersecting edge $\{u, w\} \in E(H_i) \cap E(H_j)$. Since $|N(u)| \leq M \cdot \triangle \leq n^{1/2}/2$, $|N(u) \cup N(N(u))| \leq \triangle + \triangle(\triangle - 1) \leq n/4$ so there exists a vertex $v \notin N(u) \cup N(N(u))$. Since $N(u) \cap N(v) = \emptyset$, performing a (u, v, i)-swap reduces the number of intersecting edges. The result now follows from iterating this process. □

3.3 Proof of Theorem 5

The different parts of Theorem 5 give bounds on $\phi(m, n, P)$, $\phi(m, n, M)$ and $\phi(m, n, \{P, M\})$.

Proof of Theorem 5.1 Let $G_1, \ldots, G_m$ be graphs on $[n]$ not containing a double P. We start with the following claim:

Claim 13.1. $\phi(m, n, P) \leq mn \cdot (1 + \sqrt{4n^2/m + 1})/4.$

Proof. Since there is no double P in $G_1, G_2, \ldots, G_m$,

$$\sum_{i=1}^{m} \#\{P \subseteq G_i\} \leq \#\{P \subseteq G\}.$$

For all G_i, each vertex v in G_i along with two of its neighbors form one unique P, so

$$\#\{P \subseteq G_i\} = \sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2}.$$

By Jensen's inequality,

$$\sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2} \geq n \binom{d_{G_i}(v)/n}{2} = n \binom{2e(G_i)/n}{2} \geq \frac{2(e(G_i))^2}{n} - e(G_i).$$

On the other hand, since each three vertices in G can form at most three P's,

$$\#\{P \subseteq G\} \leq 3\binom{n}{3} \leq \frac{n^3}{2}.$$

Combining the above inequalities yields and using Jensen's inequality once more yields

$$\frac{2m}{n}\left(\frac{1}{m}\sum_{i=1}^{m} e(G_i)\right)^2 - \sum_{i=1}^{m} e(G_i) \overset{\text{Jensen}}{\leq} \sum_{i=1}^{m} \frac{2(e(G_i))^2}{n} - e(G_i) \leq \frac{n^3}{2}.$$

Solving the quadratic equation gives

$$\sum_{i=1}^{m} e(G_i) \leq mn \cdot \frac{1 + \sqrt{4n^2/m + 1}}{4}.$$

This proves the claim. $\square$

Claim 13.2. *$\phi(m, n, P) \leq (mn^{3/2} + n^2)/2$.*

Proof. For each vertex $u \in [n]$, define H_u as the $m \times n$ bipartite graph with edge set $E(H_u) := \{\{v, i\} : \{u, v\} \in E(G_i)\}$. If H_u contains a quadrilateral $\{v, i\}, \{v, j\}, \{w, i\}, \{w, j\}$, then $\{u, v\}, \{u, w\}$ form a double P in $G_i \cap G_j$, contradiction. Thus we conclude that H_u is quadrilateral-free, and therefore $e(H_u) \leq m\sqrt{n} + n$, by the Kővari-Sós-Turán Theorem [**?**]. It now follows that

$$\sum_{i=1}^{m} e(G_i) = \frac{1}{2} \sum_{u \in V(G)} e(H_u) \leq \frac{1}{2}(mn^{3/2} + n^2).$$

This proves the claim, and completes Theorem 5.1. $\square$

Proof of Theorem 5.2. We now show

$$\phi(m, n, P) \geq (1/2 + o(1))mn^{3/2}$$

for $\sqrt{n} \leq m \leq n$. Suppose $G_1, G_2, \ldots, G_n$ are graphs on $[n]$ containing no double P and $\sum_{i=1}^{n} e(G_i) \geq (1/2 + o(1))n^{5/2}$, with $e(G_1) \geq e(G_2) \geq \cdots \geq e(G_n)$. Then $G_1, G_2, \ldots, G_m$ are graphs with no double P and $\sum_{i=1}^{m} e(G_i) \geq (1/2 + o(1))mn^{3/2}$. Hence, it suffices to prove the case for $m = n$.

Consider a finite projective plane with n points and n lines, with prime q chosen so that $n = (1 + o(1))(q^2 + q + 1)$ as $q \to \infty$. Let $S_1, \ldots, S_n \subseteq [n]$ be the n lines of the projective plane. Note that each line S_i contains $q + 1$ points, and the intersection of any two distinct lines S_i, S_j contains $|S_i \cap S_j| = 1$ point.

Define $G_1, \ldots, G_n$ to be graphs on $[n]$, each with edge set

$$E(G_i) \coloneqq \{\{j, k\} \subseteq [n] : j \neq k,\ j + k \in S_i \mod n\}.$$

Note that the intersection of distinct G_i, G_j is P free: since $|S_i \cap S_j| = 1$, if $\{a, b\}, \{a, c\} \in E(G_i) \cap E(G_j)$, then $a + b = a + c$ so $b = c$.

We now count the number of edges in $G_1, \ldots, G_n$. Since $|S_i| = q + 1$, for each point $j \in [n]$, there are $q + 1$ choices for $k \in [n]$ such that $j + k \in S_i$. But then we have to avoid counting the same edge twice and loops, so the number of edges in G_i is

$$e(G_i) = \frac{n(q + 1) - \#\text{loops counted for } G_i}{2}.$$

If $j \in [n]$ is even, then $k = j/2$ is the unique number in $[n]$ such that $k + k = j \mod n$. If $j \in [n]$ is odd, then $k = (n + j)/2$ is the unique number in $[n]$ such that $k + k = j \mod n$, as n is even. Hence, for each $j \in S_i$, there exists a unique $k \in [n]$ such that $k + k = j \mod n$, and thus

$$\#\text{loops counted for } G_i = |S_i| = q + 1.$$

Since $q + 1 = (1 + o(1))n^{1/2}$, the number of edges in $G_1, \ldots, G_n$ is

$$\sum_{i=1}^{n} e(G_i) = n \cdot \frac{n(q + 1) - (q + 1)}{2} = \left(\frac{1}{2} + o(1)\right) n^{5/2},$$

as $n \to \infty$. This proves Theorem 5.2. $\square$

Claim 13.3. *$\phi(m,n,P) \geq (1/2 + o(1))\sqrt{m}n^2$, for $n < m \leq n^2$.*

[**JV:** What is missing here]

Proof of Theorem 5.3. We now show for all $m, n \geq 1$, [**JV:** This is false, maybe $m \leq n$ or something? What happens for general m?]

$$\phi(m,n,M) \leq n^{5/2}.$$

Notice that $\#\{M \subseteq G\} = \binom{e(G_i)}{2}$. [**JV:** That statement is false]. On the other hand, each four vertices in G can form at most three M's, so $\#\{M \subseteq G\} \leq 3\binom{n}{4} \leq n^4/8$. By the same argument as in Theorem 3.4, we have

$$\sum_{i=1}^{n} \binom{e(G_i)}{2} = \sum_{i=1}^{n} \#\{M \subseteq G_i\} \leq \#\{M \subseteq G\} \leq \frac{n^4}{8}.$$

By Jensen's inequality,

$$\sum_{i=1}^{n} \binom{e(G_i)}{2} \geq n \binom{\sum_{i=1}^{n} e(G_i)/n}{2} = \frac{1}{2n} \left[\left(\sum_{i=1}^{n} e(G_i) \right)^2 - n \sum_{i=1}^{n} e(G_i) \right].$$

Combining the above inequalities yields

$$\left(\sum_{i=1}^{n} e(G_i) \right)^2 - n \sum_{i=1}^{n} e(G_i) \leq \frac{n^5}{4},$$

and solving the quadratic inequality gives

$$\sum_{i=1}^{n} e(G_i) \leq n^{5/2}.$$

This proves Theorem 5.3.

Proof of Theorem 5.4. We are to show

$$\phi(m,n,\{P,M\}) = n^2 - n.$$

for all $m, n \geq 1$. [**JV:** This is again false. Is $m = n$?] Let $G_1, G_2, \ldots, G_m$ be graphs on $[n]$ not containing a double P or a double M. Denote the set of G_i's as $\{G_i\} = \{G_1, \ldots, G_n\}$, [**JV:** Again, is $m = n$ or what here.] and the set of distinct pairs of G_i's as $\{G_i\}^2 = \{\{G_j, G_k\} : j \neq k\}$. Consider the bipartite graph H with vertex set $V(H) = \{G_i\} \sqcup E(K_n)$

and edge set $E(H) = \{\{G_j, e\} \in \{G_i\} \times E(K_n) : e \in G_j\}$. Define $\phi : \{G_i\}^2 \to 2^{E(K_n)}$ by sending each $\{G_j, G_k\}$ to their common edge set $E(G_j) \cap E(G_k)$. Notice that each distinct G_j, G_k have at most one edge in common, so $|\phi(G_j, G_k)| \leq 1$. On the other hand, each edge $e \in E(G)$ can be obtained via ϕ by $\binom{d_H(e)}{2}$ possible distinct pairs (G_j, G_k), and thus $|\phi^{-1}(e)| = \binom{d_H(e)}{2}$. But then

$$\binom{n}{2} \geq \sum_{(G_j,G_k)\in\{G_i\}^2} |\phi(G_j, G_k)| = \sum_{e\in E(K_n)} |\phi^{-1}(e)| = \sum_{e\in E(K_n)} \binom{d_H(e)}{2}.$$

By Jensen's inequality,

$$\sum_{e\in E(K_n)} \binom{d_H(e)}{2} \geq \binom{n}{2}\binom{\sum_{e\in E(K_n)} d_H(e)/\binom{n}{2}}{2} = \binom{n}{2}\binom{\sum_{i=1}^{n} e(G_i)/\binom{n}{2}}{2}.$$

Combining the above inequalities yields

$$2\binom{n}{2}^2 \geq \left(\sum_{i=1}^{n} e(G_i)\right)^2 - \binom{n}{2}\sum_{i=1}^{n} e(G_i),$$

and the result now follows from solving the quadratic inequality.

To see that this bound is tight, let $V(K_n) = [n]$ and let $G_i = \{\{i, j\} : j \in [n]\backslash\{i\}$. Then $E(G_i) \cap E(G_j) = \{i, j\}$, so $G_1, G_2, \ldots, G_n$ are double P-free and double M-free. Furthermore, $\sum_{i=1}^{n} e(G_i) = n^2 - n$. □

3.4 Proof of Theorem 7

We now prove Theorem 7. Notice that we trivially have $f(m, n, r) \leq \phi(m, n, K_r)$, so it suffices to show the reverse inequality. That is, we need to show that there exists a blowup construction meeting the desired bound.

Let $G_1, G_2, \ldots, G_m$ be graphs on $[n]$ with no double K_r and $\sum_{i=1}^{m} e(G_i) = \phi(m, n, K_r)$. Observe that any pair $\{i, j\} \subseteq [n]$ must be in some G_i, otherwise, we may add it to G_1 without creating a double K_r.

We call vertices v, v' *clones* if for all $u \in [n]\backslash\{v, v'\}$ and $i \in [m]$, the edge $\{u, v\} \in E(G_i)$ if and only if $\{u, v'\} \in E(G_i)$. Furthermore, we call $\{v, v'\}$ a *light edge* if $\{v, v'\}$ is in exactly one graph G_i.

We now apply Algorithm 1 to $G_1, G_2, \ldots, G_m$.

Algorithm 1 symmetrization algorithm

while $\exists$ a light edge whose endpoints are not clones **do**
  among all vertices incident to such an edge, select a vertex v with maximum degree
  $B_v \leftarrow$ collection of vertices sending a light edge to v that are not clones of v
  while $B_v \neq \emptyset$ **do**
    pick $u \in B_v$
    $j \leftarrow$ colour of the light edge from u to v
    for $1 \leq i \leq m$ **do**
      if $i \neq j$ **then**;
        $N_{G_i}(u) \leftarrow N_{G_i}(v)$
      else if $i = j$ **then**
        $N_{G_i}(u) \leftarrow (N_{G_i}(v) \setminus \{u\}) \cup \{v\}$
      end if
    end for
  end while
end while

Claim 13.4. *Algorithm 1 terminates.*

Proof. Notice that at the end of the 'while $B_v \neq \emptyset$' loop, every vertex sending a light edge to v is a clone of v. This implies v along with the set L_v of vertices receiving light edges from v induce a clique of size at least two in some G_i, and an empty graph in every other graph G_j with $j \neq i$. Moreover, any vertex $w \notin L_v$ sends edges to either all or none of the vertices in L_v, and if w is incident to L_v, then w sends edges to L_v in at least two graphs. It now follows that no light edge incident with a vertex in L_v will be picked again in an iteration of the out most while loop. Thus the algorithm can run through at most $n/2$ such iterations, and so it terminates. □

Claim 13.5. *$G'_1, G'_2, \ldots, G'_m$ do not contain a double K_r and $\sum_{i=1}^{m} e(G'_i) = \phi(m, n, K_r)$.*

Proof. Note that we replace u by a clone of v in the for loop of Algorithm 1. Since $\{u, v\}$ remains a light edge in this step, u and v cannot both belong to a double K_r in the modified graphs. Furthermore, any double K_r containing u after the for loop arises from a double K_r containing v prior to the for loop. But then $G_1, G_2, \ldots, G_m$ contained no double K_r to begin with, so $G'_1, G'_2, \ldots, G'_m$ do not contain a double K_r.

We now show that the algorithm does not reduce the number of edges. By our choice of v, we know $d(v) \geq d(u)$ for all $u \in B_v$ prior to the for loop. Hence, replacing u with a clone of v does not decrease the number of edge over a complete iteration of the inner while loop. Therefore, $\sum_{i=1}^{m} e(G_i') = \phi(m, n, K_r)$. □

Hence, the algorithm outputs graphs $G_1', G_2', \ldots, G_m'$ with $\phi(m, n, K_r)$ edges and the additional property that light edges come in 'clone cliques.' We may thus partition the vertex set $[n]$ into k disjoint sets $V_1, V_2, \ldots, V_k$, such that each V_i induces a clique of light edges from the same graph. Moreover, for distinct $i, j \in [k]$, define S_{ij} to be the set of all edges between V_i and V_j, and note that any edge in S_{ij} appears in at least two modified graphs. The sets S_{ij} now yield a k-blowup. Notice that if the pattern of the k-blowup contains a double K_r, then the original graphs $G_1, G_2, \ldots, G_m$ must have contained a double K_r as well, contradiction. Thus the k-blowup is double K_r-free.

It remains to show that $k < R_M(K_r)$. For each edge $\{i, j\} \subseteq [k]$ in the pattern of the k-blowup, we assign an arbitrary distinct pair $\{a, b\} \subseteq L_{ij} \subseteq [m]$ to $\{i, j\}$. If $k \geq R_M(K_r)$, then there exists K_r in the pattern of the k-blowup colored by some distinct pair $\{a, b\} \subseteq [m]$. But then this implies the pattern of the k-blowup contains a double K_r, contradiction. This completes the proof. □

3.5 Proof of Theorem 8

It is not hard to see that $\phi(2, n, K_3) = \binom{n}{2} + \lfloor n^2/4 \rfloor$: if G_1, G_2 is double triangle-free, then we have

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_1 \cap G_2) \leq \binom{n}{2} + \mathrm{ex}(n, K_3)$$

and so $\phi(2, n, K_3) \leq \binom{n}{2} + \lfloor n^2/4 \rfloor$. Taking $G_1 = K_n$ and $G_2 = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ meets this bounds. The main result of this section is to show for all $n \geq 1$,

$$\phi(3, n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Let G_1, G_2, G_3 be double triangle-free. Define $H_k \subseteq G$ to be the graph with edges contained in at least k of the G_i's and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \leq \frac{n^2}{2}$. Notice H_2 must not contain any triangles with two edges in H_3, so

$$e(H_2) + e(H_3) \leq \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}|.$$

Let H_3' be the graph with the same vertex set as H_3 and edge set $\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}$. It suffices to show that $\frac{n}{2} \geq e(H_3) - e(H_3')$.

Let $d_1 \geq d_2 \geq \cdots \geq d_n$ and $f_1 \geq f_2 \geq \cdots \geq f_n$ each be the degree sequence of H_3 and H_3', respectively. We show that $f_i \geq d_i - 1$ for all i. Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i. Let $S_i = |N_{H_3}(v_1) \cup \cdots \cup N_{H_3}(v_i)|$. Since

$$\sum_{u \in S_i} d_{H_3}(u) \geq d_1 + \cdots + d_i,$$

we have that $|S_i| \geq i$. But then $S_i \backslash \{u_1, \ldots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H_3'}(u) \geq d_i - 1$. Hence, $f_i \geq d_i - 1$ for all i, which yields

$$e(H_3') = \frac{1}{2} \sum_{i=1}^{n} f_i \geq \frac{1}{2} \sum_{i=1}^{n} (d_i - 1) = e(H_3) - \frac{n}{2}.$$

This proves Theorem 8. $\square$