

# Double Turán Problem

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# 1 Introduction

TODO: Add introduction and motivation.

## 1.1 Definitions and Notation

Denote the set of first  $n$  positive integers as  $[n] = \{1, 2, \dots, n\}$ . Given a set  $X$ , we denote  $2^X$  as the power set of  $X$ .

Let  $G = (V, E)$  be a graph. Let  $V(G) = V$  denote the vertex set and  $E(G) = E$  denote the edge set of  $G$ . We note by  $v(G) = |V|$  the number of vertices and  $e(G) = |E|$  the number of edges in  $G$ . For vertex  $v \in V(G)$ , we denote by  $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$  the neighborhood of  $v$ .

Given graphs  $G_1, \dots, G_m$  on some vertex set  $V$ , we denote  $G_{i_1, \dots, i_k}$  as graph on  $V$  with edge set  $E(G_{i_1, \dots, i_k}) = \bigcap_{\alpha=1}^k E(G_{i_\alpha})$ . Given two graphs  $G_1, G_2$ , we denote  $G_1 \cup G_2$  as the graph on  $V(G_1) \cup V(G_2)$  with edge set  $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ . Let  $s$

In this thesis, we reserve  $n$  to denote the number of vertices in a graph. We call a  $n$ -vertex complete graph  $K_n$ , and a complete bipartite graph  $K_{a,b}$ , where  $a, b$  are the size of its parts. We denote  $P_n$  as a path with  $n$  edges, and  $C_n$  as a cycle with  $n$  edges. Given graph  $G, H$ , define  $G + H$  as the graph fully connecting  $G, H$ , i.e.  $V(G + H) = V(G) \cup V(H)$  and  $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$ .

Given graphs  $G$  and  $F$ , we say that  $G$  is  $F$ -free if  $G$  does not contain  $F$  as a subgraph. We denote  $\text{ex}(n, F)$  to be the maximum possible number of edges an  $F$ -free graph on  $n$  vertices, and we call a  $F$ -free graph achieving this maximum an extremal graph for  $F$ . Given graphs  $G_1, \dots, G_m$  on the same set of vertices and  $F$ , we say that  $G_1, \dots, G_m$  are pairwise  $F$ -free if  $E(G_i) \cap E(G_j)$  does not contain  $F$  for  $i \neq j$ .

## 1.2 Problem Statement

Let  $\text{ex}_2(n, m, F)$  be the maximum possible number of edges that  $m$  pairwise  $F$ -free graphs on  $n$  vertices can have. Our goal is to determine  $\text{ex}_2(n, m, F)$  for different forbidden graphs  $F$ . A trivial construction with  $G_1 = K_n$  and  $G_2, \dots, G_m$  to be extremal graphs for  $F$  yields the lower bound  $\text{ex}_2(n, m, F) \geq \binom{n}{2} + (m-1)\text{ex}(n, F)$ . In this work, we use this bound as a benchmark to either show its tightness or to improve it.

Additionally, we are also interested in a more restrictive version where  $G_1, \dots, G_m$  are induced subgraphs of  $G_1 \cup \dots \cup G_m$ . Let  $\text{ex}_2^*(n, m, F)$  as the maximum possible number of

edges that  $m$  pairwise  $F$ -free graphs on  $n$  vertices can have, with the constraint that each graph is an induced subgraph of their union. A trivial construction with  $G_1 = \dots = G_m$  to be extremal graphs for  $F$  yields the lower bound  $\text{ex}_2^*(n, m, F) \geq m \cdot \text{ex}(n, F)$ . This is the benchmark we use to determine  $\text{ex}_2^*(n, m, F)$ . Similar to the non-induced case, we will use this bound as a benchmark and base our work on it.

## 2 Induced Version

In this section, we investigate the case where  $G_1, \dots, G_m$  are induced subgraphs of  $G_1 \cup \dots \cup G_m$  and are pairwise  $F$ -free, for some specified  $F$ . Unless otherwise specified, when we say  $G_1, \dots, G_m$  are induced subgraph, we mean that they are induced subgraphs of  $G_1 \cup \dots \cup G_m$ .

The following lemma shows that the problem can be reduced to only two graphs.

**Lemma 2.1.** *Let  $n, m, k \geq 1$  such that  $2 \leq k \leq m$ ,  $F$  be some graph, and  $G_1, \dots, G_m$  be pairwise  $F$ -free induced subgraphs on  $n$  vertices. Then*

$$\text{ex}_2^*(n, m, F) \leq \frac{m}{k} \cdot \text{ex}_2^*(n, k, F).$$

Moreover, if  $\sum_{i=1}^k e(G_i) = \text{ex}_2^*(n, k, F)$  only if  $G_1 = \dots = G_k$ , then  $\sum_{i=1}^m e(G_i) = \text{ex}_2^*(n, m, F)$  only if  $G_1 = \dots = G_m$  and  $\text{ex}_2^*(n, m, F) = \frac{m}{k} \cdot \text{ex}_2^*(n, k, F)$ .

Not putting equality because I'm unsure if a construction for  $k$  subgraphs can always generalize to  $m$  subgraphs. For example, if  $F = K_3$  and  $n$  is odd, the  $G_1 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$  and  $G_2 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$  construction cannot be generalized to  $m = n+1$  subgraphs.

*Proof.* Let  $G_1, \dots, G_m$  be induced subgraphs of  $G_1 \cup \dots \cup G_m$  with  $E(G_i) \cap E(G_j)$  not containing  $F$  for  $i \neq j$ . Put  $G_{i+m} = G_i$  for all  $i \in [m]$ . Then

$$\sum_{i=1}^m e(G_i) = \frac{1}{k} \sum_{i=1}^m [e(G_i) + \dots + e(G_{i+k-1})] \leq \frac{1}{k} \sum_{i=1}^m \text{ex}_2^*(n, k, F) = \frac{m}{k} \cdot \text{ex}_2^*(n, k, F),$$

which establishes the upper bound.

Suppose  $\sum_{i=1}^k e(G_i) = \text{ex}_2^*(n, k, F)$ . By assumption  $G_1 = \dots = G_k$ , so  $e(G_i) = \text{ex}_2^*(n, k, F)/k$  for  $1 \leq i \leq k$ . Hence, the construction  $G_1 = \dots = G_m$  meets the upperbound. On the other hand, if  $G_1 \neq G_2$  then  $\sum_{i=1}^k e(G_i) < \text{ex}_2^*(n, k, F)$ . Since  $\sum_{i=1}^k e(G_{i+j}) \leq \text{ex}_2^*(n, k, F)$  for all  $j \geq 1$ , we have  $\sum_{i=1}^m e(G_i) < \frac{m}{k} \cdot \text{ex}_2^*(n, k, F)$ . Thus the extremal condition is met only when  $G_1 = \dots = G_m$ .  $\square$

Lemma 2.1 allows us to reduce the problem to the case for two subgraphs  $G_1, G_2$ . Let  $C = V(G_1) \cap V(G_2)$ ,  $c = |C|$ ,  $d = |V(G_1) \setminus C|$ , and  $n - c - d = |V(G_2) \setminus C|$ . Note that  $c, d \in \mathbb{Z}_{\geq 0}$ . Since  $G_1, G_2$  are induced subgraphs of  $G_1 \cup G_2$ ,  $G_1[C] = G_2[C] = G_{1,2}$ . But then  $G_{1,2}$  is  $F$ -free, so  $e(G_1[C]) = e(G_2[C]) \leq \text{ex}(c, F)$ . Thus, given  $c, d$ , the optimal construction to maximize the number of edges over  $G_1, G_2$  is to put  $G_{1,2}$  as an extremal

graph for  $F$  on  $c$  vertices and connect all edges that are not induced in  $A$ . This yields the inequality

$$e(G_1) + e(G_2) \leq \binom{d}{2} + \binom{n-c-d}{2} + (n-c)c + 2\text{ex}(c, F).$$

But then notice that  $\binom{n-c}{2} > \binom{d}{2} + \binom{n-c-d}{2}$  for  $0 < d < n-c$ . This implies our construction is optimized when  $d = 0$  or  $d = n-c$ , that is, to let  $G_2$  contain  $G_1$  or the other way around. Hence, we may assume  $d = 0$  and define the construction function as

$$\mathcal{C}(n, c, F) := \binom{n-c}{2} + (n-c)c + 2\text{ex}(c, F),$$

i.e. the number of edges over two induced graphs in the above construction. Since  $e(G_1) + e(G_2) \leq \mathcal{C}(n, c, F)$ ,

$$\text{ex}_2^*(n, 2, F) = \max_{0 \leq c \leq n} \mathcal{C}(n, c, F),$$

and thus the problem is reduced to maximizing  $\mathcal{C}$  over  $c$ . In particular,  $\mathcal{C}(n, n, F)$  gives our benchmark construction of  $G_1 = G_2$  being the extremal graphs for  $F$  on  $n$  vertices. For  $0 \leq k \leq c \leq n$ , define

$$\Delta_k \mathcal{C}(n, c, F) := \mathcal{C}(n, c, F) - \mathcal{C}(n, c-k, F) = \frac{1}{2}k(k-2c+1) + 2[\text{ex}(c, F) - \text{ex}(c-k, F)]$$

and denote  $\Delta \mathcal{C} = \Delta_1 \mathcal{C}$ . Most of the work in this section will show that the maximum of  $\mathcal{C}$  happens when  $c \geq n-k$  by proving that  $\Delta_k \mathcal{C}(n, c, F) > 0$  for all  $c \leq n-k$ .

**Lemma 2.2.** *Let  $n, c_0 \geq 1$ ,  $m \geq 2$ , and  $F$  be some graph. If  $\mathcal{C}(n, c, F) < 2 \cdot \text{ex}(n, F)$  for  $0 \leq c < c_0$  and  $\text{ex}(c, F) - \text{ex}(c-1, F) > \frac{c-1}{2}$  for  $c_0 \leq c \leq n$ , then*

$$\text{ex}_2^*(n, m, F) = m \cdot \text{ex}(n, F)$$

*and the extremal condition is met if and only if all  $m$  induced pairwise  $F$ -free subgraphs are equal and extremal graphs for  $F$ .*

**This should be if and only if and I will strengthen it shortly.**

*Proof.* It suffices to show  $\mathcal{C}(n, c, F)$  has a unique maximum of  $2\text{ex}(n, F)$  at  $c = n$ . We may assume  $c \geq c_0$  by assumption. Suppose  $c < n$ . Since  $\text{ex}(c, F) - \text{ex}(c-1, F) > \frac{c-1}{2}$ ,

$$\Delta \mathcal{C}(n, c, F) = -c + 1 + 2[\text{ex}(c, F) - \text{ex}(c-1, F)] > 0.$$

Thus,  $\mathcal{C}$  is strictly increasing with respect to  $c$  for  $c \geq c_0$ , so  $\mathcal{C}$  has a unique maximum of  $2 \cdot \text{ex}(n, F)$  at  $c = n$ , which yields the unique extremal construction of  $G_1 = G_2$  being extremal graphs for  $F$  on  $n$  vertices.  $\square$

## 2.1 Complete Graph $F$

**Lemma 2.3.** *For  $n \geq 1$  and  $r \geq 2$ ,*

$$\text{ex}(n, K_{r+1}) - \text{ex}(n-1, K_{r+1}) \geq \frac{n-1}{2},$$

*with equality if and only if  $n$  is odd and  $r = 2$ .*

*Proof.* By Turán's Theorem,

$$\text{ex}(n, K_{r+1}) - \text{ex}(n-1, K_{r+1}) = \delta(T_r(n)) = n - \left\lceil \frac{n}{r} \right\rceil \geq n - \left\lceil \frac{n}{2} \right\rceil.$$

The result now follows.  $\square$

The following theorem for complete graphs with more than 3 vertices now follows directly from Lemma 2.2 and Lemma 2.3:

**Theorem 2.4.** *Let  $n \geq 1$ ,  $m \geq 2$ , and  $r \geq 3$ , and let  $G_1, \dots, G_m$  be pairwise  $K_{r+1}$ -free induced subgraphs on  $n$  vertices. Then*

$$\text{ex}_2^*(n, m, K_{r+1}) = m \cdot e(T_r(n)),$$

*and  $\sum_i e(G_i) = \text{ex}_2^*(n, m, K_{r+1})$  if and only if  $G_1 = \dots = G_m = T_r(n)$ .*

Surprisingly, the triangle case is more complicated than the case for larger complete graphs. As shown in Lemma 2.3, the condition given by Lemma 2.2 is not satisfied for all  $n$  in the triangle case, and there are indeed constructions of induced subgraphs  $G_1, G_2$  that meet the extremal condition but are neither equal nor both complete bipartite graphs. For odd  $n$ , consider  $G_1 = K_{\frac{n-1}{2}, \frac{n-1}{2}}$  and  $G_2 = K_{\frac{n-1}{2}, \frac{n-1}{2}} + K_1$ . The number of edges over  $G_1, G_2$  is  $\frac{(n-1)^2}{2} + n - 1 = \frac{n^2-1}{2} = 2 \left\lfloor \frac{n^2}{4} \right\rfloor$ , which meets the benchmark construction of two complete bipartite graphs. We will show that this is the only deviant construction for the triangle case.

**Theorem 2.5.** *Let  $n \geq 1$ ,  $m \geq 2$ , and let  $G_1, \dots, G_m$  be pairwise  $K_3$ -free induced subgraphs on  $n$  vertices. Then*

$$\text{ex}_2^*(n, m, K_3) = m \left\lfloor \frac{n^2}{4} \right\rfloor.$$

*Moreover, if  $n$  is even or  $m$  is odd, then  $\sum_i e(G_i)$  if and only if  $G_1 = \dots = G_m = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ . Otherwise, the extremal condition is met if and only if either  $G_1 = \dots = G_m = K_{\frac{n+1}{2}, \frac{n-1}{2}}$  or  $G_{2k+1} = K_{\frac{n-1}{2}, \frac{n-1}{2}}$  and  $G_{2k} = K_{\frac{n-1}{2}, \frac{n-1}{2}} + K_1$  for all  $k \in [m/2]$ .*

*Proof.* We first show the following claim.

**Claim 2.5.1.**  $\text{ex}_2^*(n, 2, K_3) = 2 \left\lfloor \frac{n^2}{4} \right\rfloor$ , and the extremal condition is met only when  $G_1 = G_2 = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ , unless  $n$  is odd,  $G_1 = K_{\frac{n-1}{2}, \frac{n-1}{2}}$ , and  $G_2 = K_{\frac{n-1}{2}, \frac{n-1}{2}} + K_1$ .

*Proof.* Consider  $\Delta_2 \mathcal{C}(n, c, K_3)$ . Since

$$\Delta_2 \mathcal{C}(n, c, K_3) = -2c + 3 + 2 \left[ \left\lfloor \frac{c^2}{4} \right\rfloor - \left\lfloor \frac{(c-2)^2}{4} \right\rfloor \right] = -2c + 3 + 2(c-1) = 1 > 0,$$

$\mathcal{C}(n, c, K_3)$  has a maximum of  $2 \left\lfloor \frac{n^2}{4} \right\rfloor$  when  $c \geq n-1$ , which implies  $\text{ex}_2^*(n, 2, K_3) = 2 \left\lfloor \frac{n^2}{4} \right\rfloor$ . We are done if  $c = n$ , so assume that  $c = n-1$ . Then in the extremal condition,  $G_1 = G_{1,2} = K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$  and

$$e(G_1) + e(G_2) = 2 \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \deg(v),$$

where  $v$  is the only vertex not in  $G_{1,2}$ . But then to meet the extremal condition,

$$\deg(v) = 2 \left\lfloor \frac{n^2}{4} \right\rfloor - 2 \left\lfloor \frac{(n-1)^2}{4} \right\rfloor = \begin{cases} n & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Hence,  $n$  must be odd and  $G_2$  must be a copy of  $G_1$  with all vertices adjacent to the only remaining vertex, i.e.  $G_2 = G_1 + K_1$ .  $\square$

By Lemma 2.1 and the above claim, it remains to show that for odd  $m$ ,  $G_1 = \dots = G_m = K_{\frac{n+1}{2}, \frac{n-1}{2}}$  when the extremal condition is met. Suppose not. the above claim then guarantees  $G_i = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$  for some  $i$ . Put  $G_{j+m} = G_j$  for all  $j$ . By applying the claim repeatedly,

$$\begin{aligned} G_i &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1 \\ G_{i+1} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} \\ G_{i+2} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1 \\ &\vdots \\ G_{i+m} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}, \end{aligned}$$

as  $m$  is odd. But then  $G_{i+m} = G_i = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$ , and this contradiction completes the proof.  $\square$

Combining Theorem 2.4 and Theorem 2.5, we have the following theorem for complete graphs:

**Theorem 2.6.** *For  $n \geq 1$  and  $m, r \geq 2$ , let  $G_1, \dots, G_m$  be pairwise  $K_{r+1}$ -free induced subgraphs on  $n$  vertices. Then*

$$\text{ex}_2^*(n, m, K_{r+1}) = m \cdot e(T_r(n)).$$

Moreover,  $\sum_i e(G_i) = \text{ex}_2^*(n, m, K_{r+1})$  if and only if  $G_1 = \dots = G_m = T_r(n)$ , unless  $r = 2$ ,  $n$  is odd, and  $m$  is even, in which case the  $\sum_i e(G_i) = \text{ex}_2^*(n, m, K_{r+1})$  if and only if when either  $G_1 = \dots = G_m = K_{\frac{n+1}{2}, \frac{n-1}{2}}$  or  $G_{2k-1} = K_{\frac{n-1}{2}, \frac{n-1}{2}}$  and  $G_{2k} = K_{\frac{n-1}{2}, \frac{n-1}{2}} + K_1$  for all  $k \in [m/2]$ .

Since  $n$  cannot be both even and odd, we also have the following corollary:

**Corollary 2.7.** *For  $n \geq 2$ , let  $G_1, \dots, G_n$  be pairwise  $K_{r+1}$ -free induced subgraphs on  $n$  vertices. Then*

$$\text{ex}_2^*(n, n, K_{r+1}) = n \cdot e(T_r(n)).$$

and  $\sum_i e(G_i) = \text{ex}_2^*(n, n, K_3)$  if and only if  $G_1 = \dots = G_n = T_r(n)$ .

## 2.2 Non-bipartite $F$

For non-bipartite  $F$ , it is hard to determine the extremal graphs for  $F$  in general, but their structures becomes more apparent when  $n$  is large.



More specifically, the Erdős-Stone Theorem tells us that for large  $n$ , the extremal graph for  $F$  mimics the structure of the Turán graph. With this idea in mind, the following theorem is a generalization of Theorem 2.6 for large  $n$ .

**Theorem 2.8.** *Let  $n \geq 1$ ,  $m, r \geq 2$ ,  $F$  be a  $(r+1)$ -colorable graph, and  $G_1, \dots, G_m$  be pairwise  $F$ -free induced subgraphs on  $n$  vertices. Then for large enough  $n$ ,*

$$\text{ex}_2^*(n, m, F) = m \cdot \text{ex}(n, F),$$

*and  $\sum_i e(G_i) = \text{ex}_2^*(n, m, F)$  if and only if  $G_1 = G_2 = \dots = G_m$  are  $n$ -vertex extremal graphs for  $F$ , unless  $r = 2$ ,  $n$  is odd, and  $m$  is even, in which case  $\sum_i e(G_i) = \text{ex}_2^*(n, m, F)$  if and only if when either  $G_1 = \dots = G_m$  are  $n$ -vertex extremal graph for  $F$ , or  $G_{2k-1}$  is the  $(n-1)$ -vertex extremal graph for  $F$  and  $G_{2k} = G_1 + K_1$  for all  $k \in [m/2]$ .*

**This proof only works for  $r \geq 3$ . Ignore the case  $r = 2$  for now.**

*Proof.* It suffices to show for  $m = 2$  by Lemma 2.1. We first show that  $\mathcal{C}(n, c, F)$  fails to meet the desired bound for small  $c$ .

**Claim 2.8.1.** *If  $c \leq \frac{n}{2}$ , then  $\mathcal{C}(n, c, F) < 2\text{ex}(n, F)$ .*

*Proof.* Write  $c = kn$  for some  $k \in [0, 1/2]$ . Since

$$\mathcal{C}(n, kn, F) = \binom{(1-k)n}{2} + k(1-k)n^2 + 2\text{ex}(kn, F),$$

it suffices to show that

$$\text{ex}(n, F) - \text{ex}(kn, F) > \frac{1}{2} \binom{(1-k)n}{2} + \frac{k(1-k)}{2} n^2$$

for all  $k \in [0, 1/2]$ . By the Erdős-Stone theorem,  $\text{ex}(n, F) = (1 - \frac{1}{r}) \frac{n^2}{2} + o(n^2)$  and so the left-hand-side is at least

$$\text{ex}(n, F) - \text{ex}(kn, F) \geq \text{ex}(n, F) - \text{ex}(n/2, F) \geq \left(1 - \frac{1}{r}\right) \left(\frac{n^2}{2} - \frac{n^2}{8}\right) - o(n^2) \geq \frac{3n^2}{16} - o(n^2).$$

On the right-hand-side,

$$\frac{1}{2} \binom{(1-k)n}{2} + \frac{k(1-k)}{2} n^2 = \frac{1-k^2}{4} n^2 + o(n^2) \leq \frac{n^2}{4} + o(n^2)$$

**The problem is here.** If  $r = 2$ , there does not exist  $\alpha \in (0, 1]$  such that for  $c \leq \alpha n$  the claim works: Erdős-Stone gives us  $\text{ex}(n, F) - \text{ex}(kn, F) \geq \frac{1}{4} (1 - \alpha^2) n^2 + o(n^2)$ , which exceeds the bound  $\frac{n^2}{4} + o(n^2)$  for the right-hand-side for any  $\alpha > 0$ .  $\square$

Thus we may assume that  $c > \frac{n}{2}$ . A theorem of Simonovits states that for large enough  $c$ ,  $\text{ex}(c, F) = \text{ex}(c, K_{r+1}) + \text{ex}(c, \tilde{F})$ , where  $\tilde{F}$  is the family of residue subgraphs of  $F$  after  $F$  is embedded into  $T_r(c)$ . This implies

$$\text{ex}(c, F) - \text{ex}(c-1, F) \geq \text{ex}(c, K_{r+1}) - \text{ex}(c-1, K_{r+1}),$$

as we assume  $n$  is sufficiently large. Thus by Lemma 2.2 and Lemma 2.3, we are done if  $r > 3$ .

The remaining proof is for  $r = 3$ .

The above inequality also implies that for  $r = 2$ ,

$$\Delta_2 \mathcal{C}(n, c, F) \geq \Delta_2 \mathcal{C}(n, c, K_3),$$

which is positive by the proof of Claim 2.5.1. Thus when  $c$  is  $n - 1$  or  $n$ ,  $\mathcal{C}(n, c, F)$  attains its maximum, and plugging in  $c = n$  and  $c = n - 1$  yields

$$\mathcal{C}(n, c, F) \leq \max [2 \cdot \text{ex}(n, F), 2 \cdot \text{ex}(n - 1, F) + n - 1].$$

By Lemma 2.3,

$$2 \cdot \text{ex}(n, F) - [2 \cdot \text{ex}(n - 1, F) + n - 1] \geq 2[\text{ex}(n, K_3) - \text{ex}(n - 1, K_3)] - n + 1 \geq 0,$$

with equality only if  $n$  is odd. Hence,  $\mathcal{C}(n, c, F) \leq 2 \cdot \text{ex}(n, F)$ . We may assume that  $n$  is odd and  $c = n - 1$ , otherwise we are done by Lemma 2.1. Then in the extremal condition,  $G_1 = G_{1,2}$  is the extremal graph for  $F$  on  $n - 1$  vertices, and  $G_2$  must be  $G_1 + K_1$ . It remains to show that for odd  $m$ ,  $G_1 = \dots = G_m$  are extremal graphs for  $F$  when the extremal condition is met, and this follows from the parity argument in the proof of Theorem 2.5.  $\square$

Similar to Corollary 2.7, we have the following corollary:

This also only works for  $r \geq 3$  because of the theorem above.

**Corollary 2.9.** *Let  $n, r \geq 2$ ,  $F$  be a  $(r + 1)$ -colorable graph, and  $G_1, \dots, G_n$  be pairwise  $F$ -free induced subgraphs on  $n$  vertices. Then for large enough  $n$ ,*

$$\text{ex}_2^*(n, n, F) = n \cdot \text{ex}(n, F),$$

and  $\sum_i e(G_i) = \text{ex}_2^*(n, n, F)$  if and only if  $G_1 = G_2 = \dots = G_n$  are extremal graphs for  $F$ .

For small  $n$ , we may not be able to achieve the same result. Consider the case when  $F$  is the bowtie graph, i.e. the 5-vertex graph with two triangles sharing a vertex. For  $n \leq 4$ , the  $n$ -vertex extremal graph for  $F$  is the complete graph  $K_n$ . For  $n \geq 5$ , the  $n$ -vertex extremal graph for  $F$  is then  $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$  plus an edge, and so  $\text{ex}(n, F) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$ . But then in this case when  $n = 5$ ,

$$\mathcal{C}(5, 4, F) = 2e(K_4) + 4 = 16 > \mathcal{C}(5, 5, F) = 2 \left( \left\lfloor \frac{5^2}{4} \right\rfloor + 1 \right) = 14.$$

This yields an instance where the construction  $G_1 = K_{v(F)-1}$  and  $G_2 = K_n$  beats our benchmark construction. Thus the following lemma gives a lower bound for  $n$  to avoid losing to this construction.

**Lemma 2.10.** *Let  $n \geq 1$ ,  $r \geq 2$ , and  $F$  be  $(r + 1)$ -colorable with  $|V(F)| = t \geq 3$ . If  $n > t^2 - 3t + 2$  and  $r$  divides  $n$ , then*

$$\mathcal{C}(n, n, F) > \mathcal{C}(n, t - 1, F).$$

*Proof.* We need to show that

$$2\text{ex}(n, F) - \binom{n}{2} > \binom{t-1}{2}.$$

Since  $\text{ex}(n, F) \geq e(T_r(n)) = (1 - \frac{1}{r}) \frac{n^2}{2} \geq \frac{n^2}{4}$ ,

$$2\text{ex}(n, F) - \binom{n}{2} \geq \frac{n^2}{2} - \binom{n}{2} = \frac{n}{2} > \frac{t^2 - 3t + 2}{2} = \binom{t-1}{2}.$$

□

## 2.3 Bipartite $F$

## 2.4 Hypergraph $F$

### 3 General Version

TODO: add introduction.

**Theorem 3.1.** *For all  $n$  and graph  $F$ ,*

$$\text{ex}_2(n, m, F) = m(1 + o(1))\text{ex}(n, F)$$

as  $m \rightarrow \infty$ .

*Proof.* Let  $r = v(F)$ . Pick  $\epsilon > 0$ . Reorder  $G_1, \dots, G_m$  so that  $G_1, \dots, G_{m'}$  are all the  $G_i$ 's containing at least  $(1 + \epsilon)\text{ex}(n, F)$  edges. A theorem of Simonovits states that  $G$  contains at least  $\delta n^r$  copies of  $F$  for some  $\delta = \delta(\epsilon)$ . Since there can be at most  $\binom{n}{r}$  copies of  $F$  across all  $G_i$ 's,

$$m' \delta n^r \leq \binom{n}{r} \leq n^r \implies m' \leq \frac{1}{\delta}.$$

It now follows that

$$\begin{aligned} \sum_{i=1}^m e(G_i) &= \sum_{i=1}^{m'} e(G_i) + \sum_{i=m'+1}^m e(G_i) \\ &\leq \frac{1}{\delta} \binom{n}{2} + \left(m - \frac{1}{\delta}\right) (1 + \epsilon) \text{ex}(n, F) \\ &= m \left[ 1 + \epsilon + \frac{1}{m\delta} \left( \frac{\binom{n}{2}}{\text{ex}(n, F)} - (1 + \epsilon) \right) \right] \text{ex}(n, F). \end{aligned}$$

Since  $\epsilon$  is arbitrary, the result follows.  $\square$

**Theorem 3.2.** *For large enough  $n$ , suppose that  $G_1, \dots, G_m$  are graphs on common vertex set  $[n]$  with no copy of  $F$  contained in any  $k$  of the  $G_i$ 's. If there exists extremal  $F$ -free subgraph  $H$  on  $n$  vertices such that  $\binom{m}{k} \Delta(H) = o(n^{1/2})$ , then*

$$\text{ex}_2(n, m, F) = (k - 1) \binom{n}{2} + \text{ex}(n, F) \binom{m}{k}.$$

*Proof.* For  $S \subseteq [m]$ , let  $E_S$  denote the set of edges that are contained in exactly  $\{G_i\}_{i \in S}$ . Then

$$\sum_{i=1}^m e(G_i) = \sum_{S \subseteq [m]} |S| |E_S| \leq (k - 1) \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq k} (|S| - k + 1) |E_S|.$$

Let  $A_S = \bigcup_{T \supseteq S} E_T$ , i.e. the set of edges that are contained in all  $G_i$  with  $i \in S$ . When  $|S| \geq k$ , the edge set  $A_S$  is  $F$ -free and thus

$$\sum_{T \supseteq S} |E_T| \leq \text{ex}(n, F).$$

Hence,

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq k}} (|S| - k + 1) |E_S| = \sum_{\substack{S \subseteq [m], T \subseteq S \\ |S|=k}} \sum_{\substack{|T| \geq k}} \frac{(|T| - k + 1) |E_T|}{\binom{|T|}{k}} \leq \sum_{\substack{S \subseteq [m], T \subseteq S \\ |S|=k}} \sum_{|T| \geq k} |E_T| \leq \binom{m}{k} \text{ex}(n, F),$$

as each  $T \in [m]$  with  $|T| \geq k$  is counted  $\binom{|T|}{k}$  times in total and  $|T| - k + 1 \leq \binom{|T|}{k}$ . This proves the upper bound.

Now we show the bound is tight. In particular, we need to show there exists a construction such that the graph with edge set  $E_S$  is an extremal  $F$ -free graph, for all  $S \subseteq [m]$  of size  $k$ . Let  $M = \binom{m}{k}$  and  $H_1, \dots, H_M$  be copies of an extremal  $F$ -free graph on  $n$  vertices with  $\Delta(H_i) = o(n^{1/2})$  for all  $i$ . It suffices to show that we can embed each  $H_i$  onto  $[n]$  such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each  $H_i$  and iteratively decrease the number of intersecting edges. Define a  $(u, v, i)$ -swap by swapping the embedding of vertex  $u$  and  $v$  of  $H_i$ , i.e. replacing each edge  $\{u, w\} \in E(H_i)$  with the edge  $\{v, w\}$  and each edge  $\{v, w\} \in E(H_i)$  with the edge  $\{u, w\}$ . This preserves the type of isomorphism of  $H_i$ . Given a vertex  $v$ , let  $N(v) = N_{H_1}(v) \cup \dots \cup N_{H_M}(v)$ . Suppose there exists an intersecting edge  $\{u, w\} \in E(H_i) \cap E(H_j)$ . Since  $|N(u)| \leq M \cdot \Delta(H_i) = o(n^{1/2})$ ,  $|N(u) \cup N(N(u))| = o(n)$  so there exists a vertex  $v \notin N(u) \cup N(N(u))$ . Since  $N(u) \cap N(v) = \emptyset$ , performing a  $(u, v, i)$ -swap reduces the number of intersecting edges. The result now follows from iterating this process.  $\square$

### 3.1 Triangle $F$

Consider  $F$  to be a triangle. Simply counting the number of triangles in each  $G_i$  shows the following:

**Theorem 3.3.** *For all  $n, m$  and  $\epsilon > 0$ ,*

$$\text{ex}_2(n, m, K_3) < \left( m \cdot \frac{1 + \epsilon}{4} + \frac{1}{2\epsilon} - \frac{1}{2} \right) n^2.$$

**Claim 3.3.1.** *There are less than  $\frac{2}{\epsilon}$  number of  $G_i$ 's with at least  $(1 + \epsilon)\frac{n^2}{4}$  edges.*

*Proof.* Suppose  $e(G_i) \geq (1 + \epsilon)\frac{n^2}{4}$  for  $1 \leq i \leq k$ . Let  $K_3(G)$  denote the number of triangles in graph  $G$ . By the Moon-Moser inequality,

$$K_3(G_i) \geq \epsilon(1 + \epsilon)\frac{n^3}{12}.$$

Since there are no overlapping triangles from different  $G_i$ 's,

$$\binom{n}{3} \geq \sum_{i=1}^k K_3(G_i) \geq \frac{\epsilon(1 + \epsilon)}{12} kn^3.$$

Rearranging yields  $k < \frac{2}{\epsilon}$ .  $\square$

By the claim,

$$\sum_{i=1}^m e(G_i) < \frac{2}{\epsilon} \binom{n}{2} + \left(m - \frac{2}{\epsilon}\right) (1 + \epsilon) \frac{n^2}{4} \leq m(1 + \epsilon) \frac{n^2}{4} + (1 - \epsilon) \frac{n^2}{2\epsilon},$$

which proves Theorem 3.3.

When  $m = 2$ ,

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_{1,2}) \leq \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor,$$

which meets the benchmark bound and so  $\text{ex}_2(n, 2, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor$ .

This result is also true for  $m = 3$ :

**Proposition 3.4.** *For all  $n$ ,*

$$\text{ex}_2(n, 3, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

*Proof.* Define  $H_k \subseteq G$  be the graph with edges contained in at least  $k$  number of  $G_i$ 's and note that  $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$ . Thus it suffices to show that  $e(H_2) + e(H_3) \leq \frac{n^2}{2}$ . Notice  $H_2$  must not contain any triangles with two edges in  $H_3$ , so

$$e(H_2) + e(H_3) \leq \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}|.$$

Let  $H'_3$  be the graph with the same vertex set as  $H_3$  and edge set  $\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}$ . It suffices to show that  $\frac{n}{2} \geq e(H_3) - e(H'_3)$ .

Let  $d_1 \geq d_2 \geq \dots \geq d_n$  and  $f_1 \geq f_2 \geq \dots \geq f_n$  each be the degree sequence of  $H_3$  and  $H'_3$ , respectively. We show that  $f_i \geq d_i - 1$  for all  $i$ . Let  $v_i$  denote the vertex in  $H$  with degree  $d_i$  and  $u_i$  be the vertex in  $H$  with degree  $f_i$ . Let  $S_i = |N_{H_3}(v_1) \cup \dots \cup N_{H_3}(v_i)|$ . Since

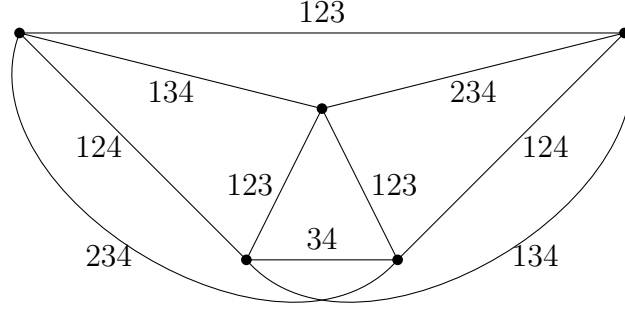
$$\sum_{u \in S_i} d_{H_3}(u) \geq d_1 + \dots + d_i,$$

we have that  $|S_i| \geq i$ . But then  $S_i \setminus \{u_1, \dots, u_{i-1}\}$  is non-empty, and every  $u \in S_i$  has degree  $d_{H'_3}(u) \geq d_i - 1$ . Hence,  $f_i \geq d_i - 1$  for all  $i$ , which yields

$$e(H'_3) = \frac{1}{2} \sum_{i=1}^n f_i \geq \frac{1}{2} \sum_{i=1}^n (d_i - 1) = e(H_3) - \frac{n}{2}.$$

□

However, the bound in Proposition 3.1 is not tight for  $m \geq 4$ , as shown in the following graph:



The number on each edge denotes the set of  $G_i$ 's that contain the edge.

The above graph contains 29 edges, which exceeds the bound  $\binom{5}{2} + 3\lfloor \frac{5^2}{4} \rfloor = 28$  by 1. By blowing up the above graph, we can construct a graph with  $n \in 10\mathbb{Z}$  vertices that contains

$$5 \binom{n/5}{2} + 29 \cdot \frac{(n/5)^2}{4}$$

edges, which exceeds the bound  $\binom{n}{2} + 3\lfloor \frac{n^2}{4} \rfloor$  by  $n^2/100$ .

## 3.2 Bipartite $F$

In this section, we discuss the case where  $F$  is bipartite. In particular, we focus on the cases where  $F \subseteq K_{2,2}$  is  $P_2$ , a path of length 2, or  $M_2$ , a matching with two edges.

**Theorem 3.5.**

$$\text{ex}_2(n, m, P_2) \leq \left( \frac{1}{2} + o(1) \right) n^2 \sqrt{m}$$

as  $n \rightarrow \infty$  or  $m \rightarrow \infty$ .

*Proof.* Since there are no overlapping  $P_2$ 's in different  $G_i$ 's,

$$\sum_{i=1}^m \# \{P_2 \subseteq G_i\} \leq \# \{P_2 \subseteq G\}$$

For each  $G_i$ , each vertex  $v$  in  $G_i$  and two of its neighbors form one unique  $P_2$ , so

$$\# \{P_2 \subseteq G_i\} = \sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2}.$$

And by Jensen's inequality,

$$\sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2} \geq n \binom{d_{G_i}(v)/n}{2} = n \binom{2e(G_i)/n}{2} \geq \frac{2(e(G_i))^2}{n} - e(G_i).$$

On the other hand, since each three vertices in  $G$  can form at most three  $P_2$ 's,

$$\# \{P_2 \subseteq G\} \leq 3 \binom{n}{3} \leq \frac{n^3}{2}.$$

Combining the above inequalities yields

$$\frac{2m}{n} \left( \frac{1}{m} \sum_{i=1}^m e(G_i) \right)^2 - \sum_{i=1}^m e(G_i) \stackrel{\text{Jensen's}}{\leq} \sum_{i=1}^m \frac{2(e(G_i))^2}{n} - e(G_i) \leq \frac{n^3}{2},$$

and solving the quadratic equation gives

$$\sum_{i=1}^m e(G_i) \leq mn \cdot \frac{1 + \sqrt{4n^2/m + 1}}{4} = \left( \frac{1}{2} + o(1) \right) n^2 \sqrt{m},$$

as  $n \rightarrow \infty$  or  $m \rightarrow \infty$ . □

When  $m = n$ , the following projective plane construction shows the above bound is tight asymptotically:

**Theorem 3.6.**

$$\text{ex}_2(n, n, P_2) = \left( \frac{1}{2} + o(1) \right) n^{5/2},$$

as  $n \rightarrow \infty$ .

*Proof.* It suffices to show the tightness of the bound in Theorem 3.5. Consider a finite projective plane of order  $q$ . The projective plane has  $n = q^2 + q + 1$  points and  $n$  lines. Let  $S_1, \dots, S_n \subseteq [n]$  be the  $n$  lines of the projective plane. Note that each line  $S_i$  contains  $q + 1$  points, and the intersection of any two distinct lines  $S_i, S_j$  contains  $|S_i \cap S_j| = 1$  point. Define  $G_1, \dots, G_n$  to be graphs on  $[n]$ , each with edge set  $E(G_i) = \{\{j, k\} \subseteq [n] : j \neq k, j + k \in S_i \pmod n\}$ . Note that the intersection of distinct  $G_i, G_j$  is  $P_2$  free: since  $|S_i \cap S_j| = 1$ , if  $\{a, b\}, \{a, c\} \in E(G_i) \cap E(G_j)$ , then  $a + b = a + c$  so  $b = c$ . We now count the number of edges in  $G_1, \dots, G_n$ . Since  $|S_i| = q + 1$ , for each point  $j \in [n]$ , there are  $q + 1$  choices for  $k \in [n]$  such that  $j + k \in S_i$ . But then we have to avoid counting the same edge twice and loops, so the number of edges in  $G_i$  is

$$e(G_i) = \frac{n(q + 1) - \#\text{loops counted for } G_i}{2}.$$

If  $j \in [n]$  is even, then  $k = j/2$  is the unique number in  $[n]$  such that  $k + k = j \pmod n$ . If  $j \in [n]$  is odd, then  $k = (n + j)/2$  is the unique number in  $[n]$  such that  $k + k = j \pmod n$ , as  $n$  is even. Hence, for each  $j \in S_i$ , there exists a unique  $k \in [n]$  such that  $k + k = j \pmod n$ , and thus

$$\#\text{loops counted for } G_i = |S_i| = q + 1.$$

Since  $q + 1 = (1 + o(1))n^{1/2}$ , the number of edges in  $G_1, \dots, G_n$  is

$$\sum_{i=1}^n e(G_i) = n \cdot \frac{n(q + 1) - (q + 1)}{2} = \left( \frac{1}{2} + o(1) \right) n^{5/2},$$

as  $n \rightarrow \infty$ . □

**Theorem 3.7.** For all  $n, m$ ,

$$\text{ex}_2(n, m, M_2) \leq n^{5/2}.$$



*Proof.* Notice that  $\#\{M_2 \subseteq G\} = \binom{e(G)}{2}$ . On the other hand, each four vertices in  $G$  can form at most three  $M_2$ 's, so  $\#\{M_2 \subseteq G\} \leq 3\binom{n}{4} \leq \frac{n^4}{8}$ . By the same argument as in Theorem 3.4, we have

$$\sum_{i=1}^n \binom{e(G_i)}{2} = \sum_{i=1}^n \#\{M_2 \subseteq G_i\} \leq \#\{M_2 \subseteq G\} \leq \frac{n^4}{8}.$$

By Jensen's inequality,

$$\sum_{i=1}^n \binom{e(G_i)}{2} \geq n \binom{\sum_{i=1}^n e(G_i)/n}{2} = \frac{1}{2n} \left[ \left( \sum_{i=1}^n e(G_i) \right)^2 - n \sum_{i=1}^n e(G_i) \right].$$

Combining the above inequalities yields

$$\left( \sum_{i=1}^n e(G_i) \right)^2 - n \sum_{i=1}^n e(G_i) \leq \frac{n^5}{4},$$

and solving the quadratic inequality gives

$$\sum_{i=1}^n e(G_i) \leq n^{5/2}.$$

□

We may obtain an exact result if we forbid both  $P_2$  and  $M_2$  at the same time:

**Theorem 3.8.** *For all  $n, m$ ,*

$$\text{ex}_2(n, m, \{P_2, M_2\}) = n^2 - n.$$

*Proof.* Denote the set of  $G_i$ 's as  $\{G_i\} = \{G_1, \dots, G_n\}$ , and the set of distinct pairs of  $G_i$ 's as  $\{G_i\}^2 = \{\{G_j, G_k\} : j \neq k\}$ . Consider the bipartite graph  $H$  with vertex set  $V(H) = \{G_i\} \sqcup E(K_n)$  and edge set  $E(H) = \{\{G_j, e\} \in \{G_i\} \times E(K_n) : e \in G_j\}$ . Define  $\phi : \{G_i\}^2 \rightarrow 2^{E(K_n)}$  by sending each  $\{G_j, G_k\}$  to their common edge set  $E(G_j) \cap E(G_k)$ . Notice that each distinct  $G_j, G_k$  have at most one edge in common, so  $|\phi(G_j, G_k)| \leq 1$ . On the other hand, each edge  $e \in E(G)$  can be obtained via  $\phi$  by  $\binom{d_H(e)}{2}$  possible distinct pairs  $(G_j, G_k)$ , and thus  $|\phi^{-1}(e)| = \binom{d_H(e)}{2}$ . But then

$$\binom{n}{2} \geq \sum_{(G_j, G_k) \in \{G_i\}^2} |\phi(G_j, G_k)| = \sum_{e \in E(K_n)} |\phi^{-1}(e)| = \sum_{e \in E(K_n)} \binom{d_H(e)}{2}.$$

By Jensen's inequality,

$$\sum_{e \in E(K_n)} \binom{d_H(e)}{2} \geq \binom{n}{2} \binom{\sum_{e \in E(K_n)} d_H(e)/n}{2} = \binom{n}{2} \binom{\sum_{i=1}^n e(G_i)/n}{2}.$$

Combining the above inequalities yields

$$2 \binom{n}{2}^2 \geq \left( \sum_{i=1}^n e(G_i) \right)^2 - \binom{n}{2} \sum_{i=1}^n e(G_i),$$

and the result now follows from solving the quadratic inequality.

To see that this bound is tight, consider the construction such that for each distinct  $i, j \in [n]$ ,  $E(G_i) \cap E(G_j)$  contains exactly one unique edge  $e \in K_n$ . The number of edges in this construction is  $2\binom{n}{2} = n^2 - n$ .  $\square$