Double Turán Problem

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Contents

1 Introduction

Let $\mathrm{ex}_2(n, m, F)$ be the maximum possible sum of the number of edges over m subgraphs $G_1, \dots, G_m$ on the same vertex set $[n]$, with the constraint that $E(G_i) \cap E(G_j)$ does not contain graph F for $i \neq j$. Our goal is to determine $\mathrm{ex}_2(n, m, F)$ for different forbidden graphs F. A trivial construction with $G_1 = K_n$ and $G_2, \dots, G_m$ to be extremal graphs for F yields the lower bound $\binom{n}{2} + (m-1)\mathrm{ex}(n, F)$. In this work, we use this bound as a benchmark to either show the tightness of $\mathrm{ex}_2(n, m, F)$ or to provide a better bound.

Additionally, we are also interested in a more restrictive version where $G_1, \dots, G_m$ are induced subgraphs of $G_1 \cup \dots \cup G_m$. We denote $\mathrm{ex}_2^*(n, m, F)$ as the maximum possible sum of the number of edges over m induced subgraphs $G_1, \dots, G_m$ on the same vertex set $[n]$ such that $E(G_i) \cap E(G_j)$ does not contain graph F for $i \neq j$. The trivial construction by taking $G_1, \dots, G_m$ to be extremal graphs for F yields the lower bound $m \cdot \mathrm{ex}(n, F)$. This is the benchmark we use to determine $\mathrm{ex}_2^*(n, m, F)$.

In this work, we will first discuss the induced case, and then shift our focus to the general case. At the end, we will discuss the case where F is bipartite.

1.1 Definitions and Notation

Let $G = (V, E)$ be a graph. Let $V(G) = V$ denote the vertex set and $E(G) = E$ denote the edge set of G. We note by $v(G) = |V|$ the number of vertices and $e(G) = |E|$ the number of edges in G. For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v.

Given $G_1, \dots, G_m$ subgraphs of G, we denote $G_{i_1,\dots,i_k}$ as the subgraph of G with edge set $E(G_{i_1,\dots,i_k}) = \bigcap_{\alpha=1}^{k} E(G_\alpha)$.

In this thesis, we reserve n to denote the number of vertices in a graph. Given a graph F, we denote $\mathrm{ex}(n, F)$ to be the extremal number for F on a graph with n vertices, i.e. the maximum number of edges in a n-vertex graph that does not contain F as a subgraph.

We call a n-vertex complete graph K_n, and a complete bipartite graph $K_{a,b}$, where a, b are the size of its parts. We denote P_n as a path with n edges, and C_n as a cycle with n edges. Given graph G, H, define $G + H$ as the graph fully connecting G, H, i.e. $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$.

We also denote the set of first n positive integers as $[n] = \{1, 2, \dots, n\}$. Given a set X, we denote 2^X as the power set of X.

2 Induced Version

In this section, we assume that $G_1, \ldots, G_m$ are induced subgraphs of $G_1 \cup \cdots \cup G_m$ and $E(G_i) \cap E(G_j)$ does not contain F for $i \neq j$. Here, we say that the extremal condition for m subgraphs is met if $\sum_{i=1}^{m} e(G_i) = \mathrm{ex}_2^*(n, m, F)$.

TODO: add the condition for all G_i's to be extremal graphs for F for all m, and generalize to hypergraph.

The following lemma shows that the problem can be reduceed to only two graphs.

Lemma 2.1. *Let $n, m, k \geq 1$ such that $2 \leq k \leq m$, and let F be some graph. Then*

$$\mathrm{ex}_2^*(n, m, F) \leq \frac{m}{k} \cdot \mathrm{ex}_2^*(n, k, F).$$

Moreover, if the extremal condition for k induced subgraphs is met only when $G_1 = \cdots = G_k$, then the above equality holds and the extremal condition for m induced subgraphs is met if and only if $G_1 = \cdots = G_m$.

Not putting equality because I'm unsure if a construction for k subgraphs can always generalize to m subgraphs. For example, if $F = K_3$ and n is odd, the $G_1 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and $G_2 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$ construction cannot be generalized to $m = n+1$ subgraphs.

Proof. Let $G_1, \ldots, G_m$ be induced subgraphs of $G_1 \cup \cdots \cup G_m$ with $E(G_i) \cap E(G_j)$ not containing F for $i \neq j$. Put $G_{i+m} = G_i$ for all $i \in [m]$. Then

$$\sum_{i=1}^{m} e(G_i) = \frac{1}{k} \sum_{i=1}^{m} [e(G_i) + \cdots + e(G_{i+k-1})] \leq \frac{1}{k} \sum_{i=1}^{m} \mathrm{ex}_2^*(n, k, F) = \frac{m}{k} \cdot \mathrm{ex}_2^*(n, k, F),$$

which establishes the upper bound.

Suppose $\sum_{i=1}^{k} e(G_i) = \mathrm{ex}_2^*(n, k, F)$. By assumption $G_1 = \cdots = G_k$, so $e(G_i) = \mathrm{ex}_2^*(n, k, F)/k$ for $1 \leq i \leq k$. Hence, the construction $G_1 = \cdots = G_m$ meets the upperbound. On the other hand, if $G_1 \neq G_2$ then $\sum_{i=1}^{k} e(G_i) < \mathrm{ex}_2^*(n, k, F)$. Since $\sum_{i=1}^{k} e(G_{i+j}) \leq \mathrm{ex}_2^*(n, k, F)$ for all $j \geq 1$, we have $\sum_{i=1}^{m} e(G_i) < \frac{m}{k} \cdot \mathrm{ex}_2^*(n, k, F)$. Thus the extremal condition is met only when $G_1 = \cdots = G_m$. □

Lemma 2.2. *Let $n \geq 1$, $m \geq 2$ and F be some graph. If $\mathrm{ex}(c, F) - \mathrm{ex}(c-1, F) > \frac{c-1}{2}$ for all $1 \leq c \leq n$, then*

$$\mathrm{ex}_2^*(n, m, F) = m \cdot \mathrm{ex}(n, F)$$

and the extremal condition is met if and only if $G_1 = \cdots = G_m$ are extremal graphs for F.

Not sure if this statement can be strengthened into if and only if. We would need to show there's no F such that $\mathrm{ex}(c,F)-\mathrm{ex}(c-1,F)\le\frac{c-1}{2}$ for some c but f has a unique maximum at $c=n$. In other words, there can not exist F such that its extremal number increases slowly at some small c but eventually catches up.

Proof. By Lemma 2.1, it suffices to show the case for two subgraphs. Let G_1, G_2 be induced subgraphs of $G_1\cup G_2$ with $E(G_1)\cap E(G_2)$ not containing F. Moreover, let $C=V(G_1)\cap V(G_2)$ and put $a=|V(G_1)\backslash C|, b=|V(G_2)\backslash C|$, and $c=|C|$. Since G_1, G_2 are induced subgraphs, $G_1[C]=G_2[C]=G_{1,2}$. But then $G_{1,2}$ is F-free, so $e(G_1[C])=e(G_2[C])\le \mathrm{ex}(c,F)$. This yields the inequality

$$e(G_1)+e(G_2)\le\binom{a+c}{2}+\binom{b+c}{2}-2\left[\binom{c}{2}-\mathrm{ex}(c,F)\right].$$

Let $f(a,b,c)$ denote the expression on the right-hand-side of the above inequality.

Claim 1. *f has a unique maximum at $c=n$.*

Proof. Suppose $b<n$. Since $\mathrm{ex}(c,F)-\mathrm{ex}(c-1,F)>\frac{c-1}{2}$,

$$\begin{aligned}f(a,b,c)-f(a,b+1,c-1)&=\binom{a+c}{2}-\binom{a+c-1}{2}\\&\quad-2\left[\binom{c}{2}-\binom{c-1}{2}-\mathrm{ex}(c,F)+\mathrm{ex}(c-1,F)\right]\\&=a-c+1+2[\mathrm{ex}(c,F)-\mathrm{ex}(c-1,F)]>a\ge 0.\end{aligned}$$

Thus, f is strictly increasing with respect to c. By symmetry, f has a unique maximum at $c=n$. □

By the claim,

$$e(G_1)+e(G_2)\le 2\binom{n}{2}-2\left[\binom{n}{2}-\mathrm{ex}(n,F)\right]=2\cdot\mathrm{ex}(n,F),$$

and and $G_1=G_2=G_{1,2}$ are extremal graphs for F on n vertices. □

2.1 Complete Graph Case

Lemma 2.3. *For $n\ge 1$ and $r\ge 2$,*

$$\mathrm{ex}(n,K_{r+1})-\mathrm{ex}(n-1,K_{r+1})\ge\frac{n-1}{2},$$

with equality if and only if n is odd and $r=2$.

Proof. By Turán's Theorem,

$$\mathrm{ex}(n,K_{r+1})-\mathrm{ex}(n-1,K_{r+1})=\delta(T_r(n))=n-\left\lceil\frac{n}{r}\right\rceil\ge n-\left\lceil\frac{n}{2}\right\rceil.$$

The result now follows. □

The following theorem for complete graphs with more than 3 vertices now follows directly from Lemma 2.2 and Lemma 2.3:

Theorem 2.4. *For $n \geq 1$, $m \geq 2$, and $r \geq 3$,*

$$\mathrm{ex}_2^*(n, m, K_{r+1}) = m \cdot e(T_r(n)),$$

and the extremal condition is met if and only if $G_1 = \cdots = G_m = T_r(n)$.

Surprisingly, the triangle case is more complicated than the case for larger complete graphs. As shown in Lemma 2.3, the condition given by Lemma 2.2 is not satisfied for all n in the triangle case, and there are indeed constructions of induced subgraphs G_1, G_2 that meet the extremal condition but are neither equal nor both complete bipartite graphs. For odd n, consider $G_1 = K_{\frac{n-1}{2},\frac{n-1}{2}}$ and $G_2 = K_{\frac{n-1}{2},\frac{n-1}{2}} + K_1$. The number of edges over G_1, G_2 is $\frac{(n-1)^2}{2} + n - 1 = \frac{n^2-1}{2} = 2\left\lfloor \frac{n^2}{4} \right\rfloor$, which meets the benchmark construction of two complete bipartite graphs. Hence, for the triangle case we have to make some comprimises.

Theorem 2.5. *For $n \geq 1$ and $m \geq 2$,*

$$\mathrm{ex}_2^*(n, m, K_3) = m \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Moreover, if n is even or m is odd, then the extremal condition is met if and only if $G_1 = \cdots = G_m$ are complete balanced bipartite graphs on n vertices. Otherwise, the extremal condition is met if and only if when either $G_1 = \cdots = G_m = K_{\frac{n+1}{2},\frac{n-1}{2}}$ or $G_{2k+1} = K_{\frac{n-1}{2},\frac{n-1}{2}}$ and $G_{2k} = K_{\frac{n-1}{2},\frac{n-1}{2}} + K_1$ for all $k \in [m/2]$.

Proof. We first show the following claim.

Claim 2. $\mathrm{ex}_2^*(n, 2, K_3) = 2\left\lfloor \frac{n^2}{4} \right\rfloor$, *and the extremal condition is met only when $G_1 = G_2 = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, unless n is odd, $G_1 = K_{\frac{n-1}{2},\frac{n-1}{2}}$, and $G_2 = K_{\frac{n-1}{2},\frac{n-1}{2}} + K_1$.*

Proof. Let a, b, c, and $f(a, b, c)$ be defined as in the proof of Lemma 2.2. Then

$$\begin{aligned} f(a, b, c) - f(a, b + 2, c - 2) &= \binom{a+c}{2} - \binom{a+c-2}{2} \\ &\quad - 2\left[\binom{c}{2} - \binom{c-2}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor + \left\lfloor \frac{(c-2)^2}{4} \right\rfloor\right] \\ &= 2(a + c) + 1 - 2[2c + 1 - (c + 1)] \\ &= 2a + 1 > 0. \end{aligned}$$

This shows that f has a maximum of $2\left\lfloor \frac{n^2}{4} \right\rfloor$ at $c = n - 1$ and $c = n$, which implies $\mathrm{ex}_2^*(n, 2, K_3) = 2\left\lfloor \frac{n^2}{4} \right\rfloor$. We are done if $c = n$, so assume that $c = n - 1$ and $a = 0$. Then $G_1 = G_{1,2} = K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ and

$$e(G_1) + e(G_2) = 2\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + \deg(v),$$

where v is the only vertex not in C. But then to meet the extremal condition,

$$\deg(v) = 2\left\lfloor \frac{n^2}{4} \right\rfloor - 2\left\lfloor \frac{(n-1)^2}{4} \right\rfloor = \begin{cases} n & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Hence, n must be odd and G_2 must be a copy of G_1 with all vertices adjacent to the only remaining vertex, i.e. $G_2 = G_1 + K_1$. $\square$

By Lemma 2.1 and the above claim, it remains to show that for odd m, $G_1 = \cdots = G_m = K_{\frac{n+1}{2},\frac{n-1}{2}}$ when the extremal condition is met. Suppose not. the above claim then guarantees $G_i = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$ for some i. Put $G_{j+m} = G_j$ for all j. By applying the claim repeatedly,

$$\begin{aligned} G_i &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1 \\ G_{i+1} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} \\ G_{i+2} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1 \\ &\vdots \\ G_{i+m} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}, \end{aligned}$$

as m is odd. But then $G_{i+m} = G_i = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$, and this contradiction completes the proof. $\square$

Since n cannot be both even and odd, we have the following corollary:

Corollary 2.6. *For $n \geq 1$,*

$$\mathrm{ex}_2^*(n, n, K_3) = n \left\lfloor \frac{n^2}{4} \right\rfloor,$$

and the extremal condition is met if and only if $G_1 = \cdots = G_n$ are complete balanced bipartite graphs on n vertices.

Combining Theorem 2.4 and Theorem 2.5, we have the following theorem for complete graphs:

Theorem 2.7. *For $n \geq 1$ and $m, r \geq 2$,*

$$\mathrm{ex}_2^*(n, m, K_{r+1}) = m \cdot \mathrm{ex}(n, K_{r+1}).$$

Moreover, the extremal condition is met if and only if $G_1 = \cdots = G_m = T_r(n)$, unless $r = 2$, n is odd, and m is even, in which case the extremal condition is met if and only if when either $G_1 = \cdots = G_m = K_{\frac{n+1}{2},\frac{n-1}{2}}$ or $G_{2k-1} = K_{\frac{n-1}{2},\frac{n-1}{2}}$ and $G_{2k} = K_{\frac{n-1}{2},\frac{n-1}{2}} + K_1$ for all $k \in [m/2]$.

2.2 Non-bipartite F

For non-bipartite F, it is hard to determine the extremal graphs for F in general, but their structures becomes more apparent when n is large. More specifically, the Erdős-Stone Theorem tells us that for large n, the extremal graph for F mimics the structure of the Turán graph. With this idea in mind, the following theorem is a generalization of Theorem 2.7 for large n.

Theorem 2.8. *For $n \geq 1$, and $m, r \geq 2$, if F is $(r+1)$-colorable, then large enough n*

$$\mathrm{ex}_2^*(n, m, F) = m \cdot \mathrm{ex}(n, F),$$

and the extremal condition is met if and only if $G_1 = G_2 = \cdots = G_n$ are extremal graphs for F, unless $r = 2$, n is odd, and m is even, in which case the extremal condition is met if and only if when either $G_1 = \cdots = G_m = K_{\frac{n+1}{2}, \frac{n-1}{2}}$ or $G_{2k-1} = K_{\frac{n-1}{2}, \frac{n-1}{2}}$ and $G_{2k} = K_{\frac{n-1}{2}, \frac{n-1}{2}} + K_1$ for all $k \in [m/2]$.

Proof. It suffices to show for $m = 2$ by Lemma 2.1. Let a, b, c and $f(a, b, c)$ be defined as in the proof of Lemma 2.2. We first show that f fails to meet the desired bound for small c.

Claim 3. *If $c \leq \frac{n}{2}$, then $f(a, b, c) < 2 \cdot \mathrm{ex}(n, F)$.*

Proof. Write $c = kn$ for some $k \in [0, 1/2]$. Since

$$f(a, b, kn) \leq 2\binom{(1-k)n/2}{2} - 2\left[\binom{kn}{2} - \mathrm{ex}(kn, F)\right],$$

it suffices to show that

$$\mathrm{ex}(n, F) - \mathrm{ex}(kn, F) > \binom{(1-k)n/2}{2} - \binom{kn}{2}$$

for all $k \in [0, 1/2]$. By the Erdős-Stone theorem, $\mathrm{ex}(n, F) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} + o(n^2)$ and so the left-hand-side is at least

$$\mathrm{ex}(n, F) - \mathrm{ex}(kn, F) \geq \mathrm{ex}(n, F) - \mathrm{ex}(n/2, F) \geq \left(1 - \frac{1}{r}\right)\left(\frac{n^2}{2} - \frac{n^2}{8}\right) - o(n^2) \geq \frac{3n^2}{16} - o(n^2).$$

On the right-hand-side,

$$\binom{(1-k)n/2}{2} - \binom{kn}{2} = (1 - 2k - 3k^2)\frac{n^2}{8} + o(n^2) \leq \frac{n^2}{8} + o(n^2).$$

Combining the above inequalities now yields the claim, as n is large. □

Thus we may assume that $c > \frac{n}{2}$. A theorem of Simonovits states that for large enough n, $\mathrm{ex}(n, F) = \mathrm{ex}(n, K_{r+1}) + \mathrm{ex}(n, \tilde{F})$, where $\tilde{F}$ is the family of residue subgraphs of F after F is embedded into $T_r(n)$. This implies

$$\mathrm{ex}(n, F) - \mathrm{ex}(n-1, F) \geq \mathrm{ex}(n, K_{r+1}) - \mathrm{ex}(n-1, K_{r+1}).$$

Thus by Lemma 2.2 and Lemma 2.3, we are done for $r > 3$ or n is even. For $r = 2$ and n odd, the proof is follows the argument for the triangle case. TODO: maybe write this out. □

Lemma 2.9. *Let F be $(r+1)$-colorable, with $r \geq 2$. Suppose $E(G_1) \cap E(G_2)$ does not include F. For large enough n,*

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n, F),$$

with equality if and only if $G_1 = G_2$ are n-vertex extremal graphs for F, unless n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2 = G_1 + K_1$.

Proof. Let $C = V(G_1) \cap V(G_2)$, $A = V(G_1) \backslash C$, and $B = V(G_2) \backslash C$. Put $a = |A|$, $b = |B|$, $c = |C|$. Since G_1, G_2 are induced graphs, $E(G_1[C]) = E(G_2[C]) = E(G[C]) = E(G_i) \cap E(G_j)$, which is F-free. Hence,

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \operatorname{ex}(c, F)\right]. \tag{2.1}$$

Define $f(a,b,c)$ as the function on the right-hand-side of (2.2). We show that $f(a,b,c)$ attains its maximum at $a = b = 0$ and $c = n$.

Claim 4. *If $c \leq \frac{n}{2}$, then $f(a,b,c) < 2 \cdot \operatorname{ex}(n, F)$.*

Proof. Write $c = kn$ for some $k \in [0, 1/2]$. Since

$$f(a,b,kn) \leq 2\binom{(1-k)n/2}{2} - 2\left[\binom{kn}{2} - \operatorname{ex}(kn, F)\right],$$

it suffices to show that

$$\operatorname{ex}(n,F) - \operatorname{ex}(c,F) > \binom{(1-k)n/2}{2} - \binom{kn}{2}$$

for all $k \in [0, 1/2]$. By the Erdős-Stone theorem, $\operatorname{ex}(n,F) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} + o(n^2)$ and so the left-hand-side is at least

$$\operatorname{ex}(n,F) - \operatorname{ex}(c,F) \geq \operatorname{ex}(n,F) - \operatorname{ex}(n/2,F) \geq \left(1 - \frac{1}{r}\right)\left(\frac{n^2}{2} - \frac{n^2}{8}\right) - o(n^2) \geq \frac{3n^2}{16} - o(n^2).$$

On the right-hand-side,

$$\binom{(1-k)n/2}{2} - \binom{kn}{2} = (1 - 2k - 3k^2)\frac{n^2}{8} + o(n^2) \leq \frac{n^2}{8} + o(n^2).$$

Combining the above inequalities now yields the claim, as n is large. □

Thus we may assume that $c > \frac{n}{2}$. A theorem of Simonovits states that for large enough n, $\operatorname{ex}(n,F) = \operatorname{ex}(n, K_{r+1}) + \operatorname{ex}(n, \tilde{F})$, where $\tilde{F}$ is the family of residue subgraphs of F after F is embedded into $T_r(n)$. This implies

$$\operatorname{ex}(n+1, F) - \operatorname{ex}(n, F) \geq \operatorname{ex}(n, K_{r+1}) - \operatorname{ex}(n+1, K_{r+1}),$$

and so

$$f(a, b-2, c+2) - f(a,b,c) \geq 2a - 2c - 1 + 2[\operatorname{ex}(c+2, K_{r+1}) - \operatorname{ex}(c, K_{r+1})].$$

Since $\operatorname{ex}(c+1, K_{r+1}) - \operatorname{ex}(c, K_{r+1}) \geq c - \left\lfloor \frac{c}{r} \right\rfloor \geq c - \left\lfloor \frac{c}{2} \right\rfloor$, we have

$$\operatorname{ex}(c+2, K_{r+1}) - \operatorname{ex}(c, K_{r+1}) \geq c + 1 - \left\lfloor \frac{c+1}{2} \right\rfloor + c - \left\lfloor \frac{c}{2} \right\rfloor = c + 1,$$

and thus

$$f(a, b-2, c+2) - f(a,b,c) \geq 2a + 1 > 0.$$

By symmetry, we also have $f(a-2,b,c+2) > f(a,b,c)$. Thus, f attains its maximum when c is $n-1$ or n. Equation (2.2) now yields,

$$e(G_1)+e(G_2) \leq \max\left[2\cdot \mathrm{ex}(n,F), 2\cdot \mathrm{ex}(n-1,F)+n-1\right].$$

Assume that $a=0$. Since

$$\begin{aligned} 2\cdot \mathrm{ex}(n,F) &- [2\cdot \mathrm{ex}(n-1,F)+n-1] \\ &\geq 2[\mathrm{ex}(n,K_{r+1})-\mathrm{ex}(n-1,K_{r+1})]-n+1 \qquad (2.2)\\ &= 2\left(n-\left\lceil \frac{n}{r}\right\rceil\right)-n+1 \\ &\geq n+1-2\left\lceil\frac{n}{2}\right\rceil \geq 0, \end{aligned}$$

we have

$$e(G_1)+e(G_2) \leq 2\cdot \mathrm{ex}(n,F).$$

If $c=n$, the equality holds only if $G_1=G_2$ are n-vertex extramal graphs for F. Suppose $c=n-1$ and the equality holds. Observe that equation (2.3) is equal to zero only when $r=2$ and n is odd. Thus if $c=n-1$, then the equality could only be achieved when $r=2$, n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2=G_1+K_1$. $\square$

3 General Version

TODO: add introduction.

Theorem 3.1. *For all n and graph F,*

$$\mathrm{ex}_2(n, m, F) = m(1 + o(1))\mathrm{ex}(n, F)$$

as $m \to \infty$.

Proof. Let $r = v(F)$. Pick $\epsilon > 0$. Reorder $G_1, \dots, G_m$ so that $G_1, \dots, G_{m'}$ are all the G_i's containing at least $(1 + \epsilon)\mathrm{ex}(n, F)$ edges. A theorem of Simonovits states that G contains at least δn^r copies of F for some $\delta = \delta(\epsilon)$. Since there can be at most $\binom{n}{r}$ copies of F across all G_i's,

$$m'\delta n^r \leq \binom{n}{r} \leq n^r \implies m' \leq \frac{1}{\delta}.$$

It now follows that

$$\begin{aligned}
\sum_{i=1}^{m} e(G_i) &= \sum_{i=1}^{m'} e(G_i) + \sum_{i=m'+1}^{m} e(G_i) \\
&\leq \frac{1}{\delta}\binom{n}{2} + \left(m - \frac{1}{\delta}\right)(1 + \epsilon)\mathrm{ex}(n, F) \\
&= m\left[1 + \epsilon + \frac{1}{m\delta}\left(\frac{\binom{n}{2}}{\mathrm{ex}(n, F)} - (1 + \epsilon)\right)\right]\mathrm{ex}(n, F).
\end{aligned}$$

Since ϵ is arbitrary, the result follows. □

Theorem 3.2. *For large enough n, suppose that $G_1, \dots, G_m$ are graphs on common vertex set $[n]$ with no copy of F contained in any k of the G_i's. If there exists extremal F-free subgraph H on n vertices such that $\binom{m}{k}\Delta(H) = o(n^{1/2})$, then*

$$\mathrm{ex}_2(n, m, F) = (k - 1)\binom{n}{2} + \mathrm{ex}(n, F)\binom{m}{k}.$$

Proof. For $S \subseteq [m]$, let E_S denote the set of edges that are contained in exactly $\{G_i\}_{i \in S}$. Then

$$\sum_{i=1}^{m} e(G_i) = \sum_{S \subseteq [m]} |S||E_S| \leq (k - 1)\binom{n}{2} + \sum_{S \subseteq [m], |S| \geq k} (|S| - k + 1)|E_S|.$$

Let $A_S = \bigcup_{T \supseteq S} E_T$, i.e. the set of edges that are contained in all G_i with $i \in S$. When $|S| \geq k$, the edge set A_S is F-free and thus

$$\sum_{T \supseteq S} |E_T| \leq \mathrm{ex}(n, F).$$

Hence,

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq k}} (|S| - k + 1)|E_S| = \sum_{\substack{S \subseteq [m], \\ |S|=k}} \sum_{T \subseteq S} \frac{(|T| - k + 1)|E_T|}{\binom{|T|}{k}} \leq \sum_{\substack{S \subseteq [m], \\ |S|=k}} \sum_{T \subseteq S} |E_T| \leq \binom{m}{k} \mathrm{ex}(n, F),$$

as each $T \in [m]$ with $|T| \geq k$ is counted $\binom{|T|}{k}$ times in total and $|T| - k + 1 \leq \binom{|T|}{k}$. This proves the upper bound.

Now we show the bound is tight. In particular, we need to show there exists a construction such that the graph with edge set E_S is an extremal F-free graph, for all $S \subseteq [m]$ of size k. Let $M = \binom{m}{k}$ and $H_1, \ldots, H_M$ be copies of an extremal F-free graph on n vertices with $\Delta(H_i) = o(n^{1/2})$ for all i. It suffices to show that we can embed each H_i onto $[n]$ such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each H_i and iteratively decrease the number of intersecting edges. Define a (u, v, i)-swap by swapping the embedding of vertex u and v of H_i, i.e. replacing each edge $\{u, w\} \in E(H_i)$ with the edge $\{u, w\}$ and each edge $\{v, w\} \in E(H_i)$ with the edge $\{v, w\}$. This perserves the type of isomorphism of H_i. Given a vertex v, let $N(v) = N_{H_1}(v) \cup \cdots \cup N_{H_M}(v)$. Suppose there exists an intersecting edge $\{u, w\} \in E(H_i) \cap E(H_j)$. Since $|N(u)| \leq M \cdot \Delta(H_i) = o(n^{1/2})$, $|N(u) \cup N(N(u))| = o(n)$ so there exists a vertex $v \notin N(u) \cup N(N(u))$. Since $N(u) \cap N(v) = \emptyset$, performing a (u, v, i)-swap reduces the number of intersecting edges. The result now follows from iterating this process. $\square$

3.1 Triangle F

Consider F to be a triangle. Simply counting the number of triangles in each G_i shows the following:

Theorem 3.3. *For all n, m and $\epsilon > 0$,*

$$\mathrm{ex}_2(n, m, K_3) < \left(m \cdot \frac{1+\epsilon}{4} + \frac{1}{2\epsilon} - \frac{1}{2} \right) n^2.$$

Claim 5. *There are less than $\frac{2}{\epsilon}$ number of G_i's with at least $(1+\epsilon)\frac{n^2}{4}$ edges.*

Proof. Suppose $e(G_i) \geq (1+\epsilon)\frac{n^2}{4}$ for $1 \leq i \leq k$. Let $K_3(G)$ denote the number of triangles in graph G. By the Moon-Moser inequality,

$$K_3(G_i) \geq \epsilon(1+\epsilon)\frac{n^3}{12}.$$

Since there are no overlapping traingles from different G_i's,

$$\binom{n}{3} \geq \sum_{i=1}^{k} K_3(G_i) \geq \frac{\epsilon(1+\epsilon)}{12} kn^3.$$

Rearranging yields $k < \frac{2}{\epsilon}$. $\square$

By the claim,

$$\sum_{i=1}^{m} e(G_i) < \frac{2}{\epsilon}\binom{n}{2} + \left(m - \frac{2}{\epsilon}\right)(1+\epsilon)\frac{n^2}{4} \leq m(1+\epsilon)\frac{n^2}{4} + (1-\epsilon)\frac{n^2}{2\epsilon},$$

which proves Theorem 3.3.

When $m = 2$,

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_{1,2}) \leq \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor,$$

which meets the benchmark bound and so $\mathrm{ex}_2(n, 2, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor$.

This result is also true for $m = 3$:

Proposition 3.4. *For all n,*

$$\mathrm{ex}_2(n, 3, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Proof. Define $H_k \subseteq G$ be the graph with edges contained in at least k number of G_i's and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \leq \frac{n^2}{2}$. Notice H_2 must not contain any triangles with two edges in H_3, so

$$e(H_2) + e(H_3) \leq \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}|.$$

Let H_3' be the graph with the same vertex set as H_3 and edge set $\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}$. It suffices to show that $\frac{n}{2} \geq e(H_3) - e(H_3')$.

Let $d_1 \geq d_2 \geq \cdots \geq d_n$ and $f_1 \geq f_2 \geq \cdots \geq f_n$ each be the degree sequence of H_3 and H_3', respectively. We show that $f_i \geq d_i - 1$ for all i. Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i. Let $S_i = |N_{H_3}(v_1) \cup \cdots \cup N_{H_3}(v_i)|$. Since

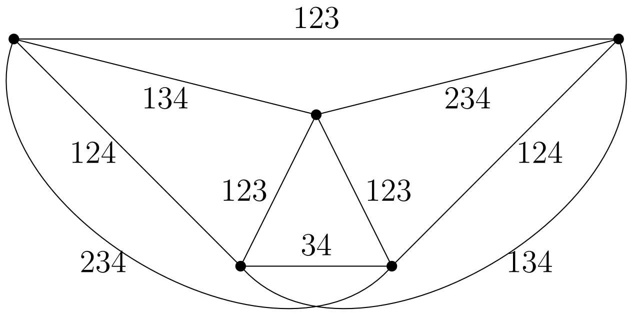
$$\sum_{u \in S_i} d_{H_3}(u) \geq d_1 + \cdots + d_i,$$

we have that $|S_i| \geq i$. But then $S_i \backslash \{u_1, \ldots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H_3'}(u) \geq d_i - 1$. Hence, $f_i \geq d_i - 1$ for all i, which yields

$$e(H_3') = \frac{1}{2}\sum_{i=1}^{n} f_i \geq \frac{1}{2}\sum_{i=1}^{n}(d_i - 1) = e(H_3) - \frac{n}{2}.$$

$\square$

However, the bound in Proposition 3.1 is not tight for $m \geq 4$, as shown in the following graph:



The number on each edge denotes the set of G_i's that contain the edge.

The above graph contains 29 edges, which exceeds the bound $\binom{5}{2} + 3\lfloor \frac{5^2}{4} \rfloor = 28$ by 1. By blowing up the above graph, we can construct a graph with $n \in 10\mathbb{Z}$ vertices that contains

$$5\binom{n/5}{2} + 29 \cdot \frac{(n/5)^2}{4}$$

edges, which exceeds the bound $\binom{n}{2} + 3\lfloor \frac{n^2}{4} \rfloor$ by $n^2/100$.

3.2 Bipartite F

In this section, we discuss the case where F is bipartite. In particular, we focus on the cases where $F \subseteq K_{2,2}$ is P_2, a path of length 2, or M_2, a matching with two edges.

Theorem 3.5.

$$\mathrm{ex}_2(n, m, P_2) \leq \left(\frac{1}{2} + o(1)\right) n^2 \sqrt{m}$$

as $n \to \infty$ or $m \to \infty$.

Proof. Since there are no overlapping P_2's in different G_i's,

$$\sum_{i=1}^{m} \#\{P_2 \subseteq G_i\} \leq \#\{P_2 \subseteq G\}$$

For each G_i, each vertex v in G_i and two of its neighbors form one unique P_2, so

$$\#\{P_2 \subseteq G_i\} = \sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2}.$$

And by Jensen's inequality,

$$\sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2} \geq n\binom{d_{G_i}(v)/n}{2} = n\binom{2e(G_i)/n}{2} \geq \frac{2(e(G_i))^2}{n} - e(G_i).$$

On the other hand, since each three vertices in G can form at most three P_2's,

$$\#\{P_2 \subseteq G\} \leq 3\binom{n}{3} \leq \frac{n^3}{2}.$$

Combining the above inequalities yields

$$\frac{2m}{n}\left(\frac{1}{m}\sum_{i=1}^{m} e(G_i)\right)^2 - \sum_{i=1}^{m} e(G_i) \overset{Jensen's}{\leq} \sum_{i=1}^{m} \frac{2(e(G_i))^2}{n} - e(G_i) \leq \frac{n^3}{2},$$

and solving the quadratic equation gives

$$\sum_{i=1}^{m} e(G_i) \leq mn \cdot \frac{1+\sqrt{4n^2/m+1}}{4} = \left(\frac{1}{2}+o(1)\right) n^2\sqrt{m},$$

as $n \to \infty$ or $m \to \infty$. □

When $m = n$, the following projective plane construction shows the above bound is tight asymptotically:

Theorem 3.6.

$$\mathrm{ex}_2(n, n, P_2) = \left(\frac{1}{2}+o(1)\right) n^{5/2},$$

as $n \to \infty$.

Proof. It suffices to show the tightness of the bound in Theorem 3.5. Consider a finite projective plane of order q. The projective plane has $n = q^2 + q + 1$ points and n lines. Let $S_1, \ldots, S_n \subseteq [n]$ be the n lines of the projective plane. Note that each line S_i contains $q + 1$ points, and the intersection of any two distinct lines S_i, S_j contains $|S_i \cap S_j| = 1$ point. Define $G_1, \ldots, G_n$ to be graphs on $[n]$, each with edge set $E(G_i) = \{\{j, k\} \subseteq [n] : j \neq k,\ j + k \in S_i \mod n\}$. Note that the intersection of distinct G_i, G_j is P_2 free: since $|S_i \cap S_j| = 1$, if $\{a, b\}, \{a, c\} \in E(G_i) \cap E(G_j)$, then $a + b = a + c$ so $b = c$. We now count the number of edges in $G_1, \ldots, G_n$. Since $|S_i| = q + 1$, for each point $j \in [n]$, there are $q + 1$ choices for $k \in [n]$ such that $j + k \in S_i$. But then we have to avoid counting the same edge twice and loops, so the number of edges in G_i is

$$e(G_i) = \frac{n(q+1) - \#\text{loops counted for } G_i}{2}.$$

If $j \in [n]$ is even, then $k = j/2$ is the unique number in $[n]$ such that $k + k = j \mod n$. If $j \in [n]$ is odd, then $k = (n + j)/2$ is the unique number in $[n]$ such that $k + k = j \mod n$, as n is even. Hence, for each $j \in S_i$, there exists a unique $k \in [n]$ such that $k + k = j \mod n$, and thus

$$\#\text{loops counted for } G_i = |S_i| = q + 1.$$

Since $q + 1 = (1 + o(1))n^{1/2}$, the number of edges in $G_1, \ldots, G_n$ is

$$\sum_{i=1}^{n} e(G_i) = n \cdot \frac{n(q+1) - (q+1)}{2} = \left(\frac{1}{2}+o(1)\right) n^{5/2},$$

as $n \to \infty$. □

Theorem 3.7. *For all* n, m,

$$\mathrm{ex}_2(n, m, M_2) \leq n^{5/2}.$$

Proof. Notice that $\#\{M_2 \subseteq G\} = \binom{e(G_i)}{2}$. On the other hand, each four vertices in G can form at most three M_2's, so $\#\{M_2 \subseteq G\} \leq 3\binom{n}{4} \leq \frac{n^4}{8}$. By the same argument as in Theorem 3.4, we have

$$\sum_{i=1}^{n} \binom{e(G_i)}{2} = \sum_{i=1}^{n} \#\{M_2 \subseteq G_i\} \leq \#\{M_2 \subseteq G\} \leq \frac{n^4}{8}.$$

By Jensen's inequality,

$$\sum_{i=1}^{n} \binom{e(G_i)}{2} \geq n\binom{\sum_{i=1}^{n} e(G_i)/n}{2} = \frac{1}{2n}\left[\left(\sum_{i=1}^{n} e(G_i)\right)^2 - n\sum_{i=1}^{n} e(G_i)\right].$$

Combining the above inequalities yields

$$\left(\sum_{i=1}^{n} e(G_i)\right)^2 - n\sum_{i=1}^{n} e(G_i) \leq \frac{n^5}{4},$$

and solving the quadratic inequality gives

$$\sum_{i=1}^{n} e(G_i) \leq n^{5/2}.$$

□

We may obtain an exact result if we forbid both P_2 and M_2 at the same time:

Theorem 3.8. *For all n, m,*

$$\mathrm{ex}_2(n, m, \{P_2, M_2\}) = n^2 - n.$$

Proof. Denote the set of G_i's as $\{G_i\} = \{G_1, \ldots, G_n\}$, and the set of distinct pairs of G_i's as $\{G_i\}^2 = \{\{G_j, G_k\} : j \neq k\}$. Consider the bipartite graph H with vertex set $V(H) = \{G_i\} \sqcup E(K_n)$ and edge set $E(H) = \{\{G_j, e\} \in \{G_i\} \times E(K_n) : e \in G_j\}$. Define $\phi : \{G_i\}^2 \to 2^{E(K_n)}$ by sending each $\{G_j, G_k\}$ to their common edge set $E(G_j) \cap E(G_k)$. Notice that each distinct G_j, G_k have at most one edge in common, so $|\phi(G_j, G_k)| \leq 1$. On the other hand, each edge $e \in E(G)$ can be obtained via ϕ by $\binom{d_H(e)}{2}$ possible distinct pairs (G_j, G_k), and thus $|\phi^{-1}(e)| = \binom{d_H(e)}{2}$. But then

$$\binom{n}{2} \geq \sum_{(G_j,G_k)\in\{G_i\}^2} |\phi(G_j, G_k)| = \sum_{e\in E(K_n)} |\phi^{-1}(e)| = \sum_{e\in E(K_n)} \binom{d_H(e)}{2}.$$

By Jensen's inequality,

$$\sum_{e\in E(K_n)} \binom{d_H(e)}{2} \geq \binom{n}{2}\binom{\sum_{e\in E(K_n)} d_H(e)/\binom{n}{2}}{2} = \binom{n}{2}\binom{\sum_{i=1}^{n} e(G_i)/\binom{n}{2}}{2}.$$

Combining the above inequalities yields

$$2\binom{n}{2}^2 \geq \left(\sum_{i=1}^{n} e(G_i)\right)^2 - \binom{n}{2}\sum_{i=1}^{n} e(G_i),$$

and the result now follows from solving the quadratic inequality.

To see that this bound is tight, consider the construction such that for each distinct $i, j \in [n]$, $E(G_i) \cap E(G_j)$ contains exactly one unique edge $e \in K_n$. The number of edges in this construction is $2\binom{n}{2} = n^2 - n$. $\square$