

# Double Turán Problem

Ray Tsai

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# 1 Introduction

This thesis focuses on a variation of the *Turán problem* in extremal combinatorics. The fundamental question in extremal hypergraph theory is determining the maximum number of edges in an  $n$ -vertex  $r$ -uniform graph that does not contain a prescribed  $r$ -uniform graph  $F$  as a subgraph. These maxima, denoted  $\text{ex}(n, F)$ , are referred to as the *extremal numbers* or *Turán numbers* for  $F$ . One of the cornerstones of extremal graph theory, concerning the case  $F$  is a clique, is Turán's Theorem [18]. To state the theorem, we need the *Turán graphs*  $T_k(n)$ , which denotes a complete multipartite graph with  $n$  vertices and  $k$  parts of size  $\lfloor n/k \rfloor$  or  $\lceil n/k \rceil$ .

**Theorem A** (Turán's Theorem). *The maximum number of edges in an  $n$ -vertex graph containing no clique of order  $k + 1$  is  $e(T_k(n))$ , with equality only for  $T_k(n)$ .*

Simonovits [5] observed via the Erdős-Stone Theorem [3] that one may obtain the asymptotic value of  $\text{ex}(n, F)$  whenever  $F$  is a graph of chromatic number  $k + 1 \geq 3$ :

**Theorem B** (Erdős-Stone Theorem, Simonovits' Theorem). *Let  $F$  be any graph of chromatic number  $k + 1 \geq 3$ . Then  $\text{ex}(n, F) = (1 + o_n(1))T_k(n)$  as  $n \rightarrow \infty$ .*

The case where  $F$  is bipartite is in general wide open, and the order of magnitude of  $\text{ex}(n, K_{4,4})$  or  $\text{ex}(n, C_8)$  is not known – see Füredi and Simonovits [8] for a history of the bipartite Turán problem. There is also no analog of the above theorems for  $r$ -uniform hypergraphs; the asymptotic value of  $\text{ex}(n, K_k^r)$  is not known for any  $k > r \geq 3$ , where  $K_k^r$  denotes the complete  $r$ -uniform hypergraph on  $k$  vertices. The asymptotic value of  $\text{ex}(n, K_4^3)$  was conjectured by Turán [18] to be  $\frac{5}{9}\binom{n}{3}$ , and this remains open despite decades of intensive research.

In this thesis, we investigate closely related problems which we refer to as *double Turán problems*. To describe these problems, let  $G_1, G_2, \dots, G_m$  be graphs with the same vertex set  $V(G_i) = [n]$  for  $i \in [m]$ . For a graph  $F$ , we say that  $G_1, G_2, \dots, G_m$  is *double  $F$ -free* if  $E(F) \not\subseteq E(G_i) \cap E(G_j)$  for  $1 \leq i < j \leq m$ . In other words,  $F$  does not appear in the intersection of any two of the graphs  $G_i$ . We refer to a copy of  $F$  in the intersection of two of the graphs  $G_i$  as a *double  $F$* . Let  $\phi(m, n, F)$  denote the maximum value of  $\sum_{i=1}^m e(G_i)$  such that  $G_1, G_2, \dots, G_m$  does not contain a double  $F$ . We say that graphs  $G_1, G_2, \dots, G_m$  are *induced* to mean that every  $G_i$  is an induced subgraph of  $\bigcup_{i=1}^m G_i$ . In other words, if  $\{u, v\} \in E(G_i)$  and  $u, v \in V(G_j)$ , then  $\{u, v\} \in E(G_j)$ . Let  $\phi^*(m, n, F)$  denote the maximum value of  $\sum_{i=1}^m e(G_i)$  such that  $G_1, G_2, \dots, G_m$  does not contain a double  $F$  and  $G_1, G_2, \dots, G_m$  are induced. Evidently,  $\phi^*(m, n, F) \leq \phi(m, n, F)$ , and the study of  $\phi^*(m, n, F)$  and  $\phi(m, n, F)$  is motivated by certain hypergraph extremal problems.

## 1.1 Link graphs and hypergraphs

Apart from the intrinsic interest in studying  $\phi(m, n, F)$ , a motivation is that  $\phi(m, n, F)$  is closely connected to pure hypergraph extremal problems via the notion of *link graphs*. Let  $H$  be a triple system, that is, a set of three-element subsets of a finite set  $[n]$ . These three-element subsets form the edge-set  $E(H)$  of  $H$ , while  $V(H) = V$  is the vertex set of  $H$ . For  $i \in V(H)$ , let  $H_i$  denote the *link graph* of  $i$ , with  $V(H_i) = V(H) \setminus \{i\}$  and  $E(H_i) = \{\{j, k\} : \{i, j, k\} \in E(H)\}$ . A useful idea in extremal hypergraph theory is to try to reduce an extremal problem for hypergraphs to extremal problems for the link graphs. For instance, a triple system  $H$  does not contain three triples on four vertices if and only if all its link graphs are triangle-free.

In the current context, given a graph  $F$ , let  $F^+$  denote the triple system with vertex set  $V(F^+) = V(F) \cup \{x, y\}$  and edge set  $\{e \cup \{x\}, e \cup \{y\} : e \in E(F)\}$ . Then  $\phi(n, n, F)$  and  $\text{ex}(n, F^+)$  are intimately related: if  $H$  is an  $F^+$ -free triple system with vertex set  $[n]$ , then clearly the link graphs  $H_1, H_2, \dots, H_n$  are double  $F$ -free, which implies  $\text{ex}(n, F^+) \leq \phi(n, n, F)$ . This relates the double Turán problem to hypergraph extremal problems.

Now let  $G$  be the graph consisting of all pairs contained in triples in  $F^+$ . The *generalized Turán problem* asks for the maximum number  $\text{ex}(n, G, K_3)$  of triangles in a graph  $H$  with vertex set  $[n]$  that does not contain  $G$ . This problem was studied by Alon and Shikhelman [1] and Kostochka, Mubayi and Verstraete [10, 12, 14]. This problem is related to  $\phi^*(n, n, F)$  as follows: define  $H_i = \{\{j, k\} : \{i, j\}, \{j, k\}, \{i, k\} \in E(H)\}$ . Then  $H_1, H_2, \dots, H_n$  are induced and double  $F$ -free, so  $\phi^*(n, n, F) \geq \text{ex}(n, G, K_3)$ . This relates the induced double Turán problem to extremal problems for triangles in graphs.

## 1.2 Main results : the induced case

The determination of  $\phi^*(m, n, F)$  turns out to be fairly straightforward when  $F$  is a non-bipartite graph: the extremal objects are  $m$  copies of the same extremal graph for  $F$ :

**Theorem 1.** *For  $r \geq 3$ , there exists  $n_0(r)$  such that if  $n \geq n_0(r)$  and  $F$  is a graph of chromatic number  $r$ , then for all  $m \geq 3$ ,*

$$\phi^*(m, n, F) = m \cdot \text{ex}(n, F),$$

*with equality only for identical extremal  $n$ -vertex  $F$ -free graphs.*

In the case  $F = K_r$ , we shall see the theorem is true for all  $n \geq 3$ :

**Theorem 2.** *Let  $m, n, r \geq 3$ . Then  $\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n))$  with equality for induced  $K_r$ -free graphs  $G_1, G_2, \dots, G_m$  only if  $G_1 = G_2 = \dots = G_m = T_{r-1}(n)$ .*

In the case  $F$  is a bipartite graph, even the problem of determining the order of magnitude of  $\phi^*(m, n, F)$  appears to be difficult, and we do not know the order of magnitude of  $\phi^*(m, n, P)$  when  $P$  is a path with two edges. In this thesis, we propose the following very broad conjecture:

**Conjecture A.** *Let  $F$  be any non-empty graph and  $m, n \geq 1$ . Then*

$$\phi^*(m, n, F) = \Theta(m \cdot \text{ex}(n, F) + n^2).$$

It is clear that a single complete graph  $K_n$  does not contain a double  $F$ , and neither do identical copies  $G_1, G_2, \dots, G_m$  of an extremal  $n$ -vertex  $F$ -free graph. In particular,

$$\phi^*(m, n, F) \geq \max\left\{\binom{n}{2}, m \cdot \text{ex}(n, F)\right\}.$$

This conjecture is true when  $F$  is non-bipartite, by Theorem 1. If  $F$  is bipartite, then upper bounds on  $\phi^*(m, n, F)$  are more difficult to come by, especially when  $m$  is large. For instance, we know

$$\text{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2})$$

and so Conjecture A implies that an  $n$ -vertex graph not containing the octahedron graph has  $O(n^{5/2})$  triangles. In fact, it is also the case that  $\text{ex}(2n, K_{2,2,2}, K_3) \geq \phi^*(n, n, K_{2,2})$ : if we have double  $K_{2,2}$ -free induced graphs  $G_1, G_2, \dots, G_n$  with vertex set  $[n]$ , then let  $H$  be the graph with  $V(H) = [2n]$  consisting of all triangles with vertex set  $\{i, j, k\}$  such that  $n < k \leq 2n$  and  $\{i, j\} \in E(G_k)$ . The graph  $H$  is  $K_{2,2,2}$ -free and  $|E(H)| = \sum_{i=1}^{n/2} e(G_i)$ . Similarly, we have

$$\text{ex}(n, K_{1,2,2}, K_3) \leq \phi^*(n, n, K_{1,2})$$

and so Conjecture A implies that an  $n$ -vertex graph not containing the octahedron graph has  $O(n^2)$  triangles, which is conjectured by Mubayi and Verstraete [14]. The conjecture proposes more generally that if  $F$  is a tree, then  $\phi^*(n, n, F) = O(n^2)$ . In fact, it is possible to prove the following theorem using the *removal lemma* as in [12] as well as a construction for  $\phi(n, n, P)$  in this work:

**Theorem 3.** *Let  $P$  be a path with two edges. Then  $\phi(n, n, P) = \Omega(n^{5/2})$ , whereas  $\phi^*(n, n, P) = o(n^{5/2})$ , as  $n \rightarrow \infty$ . In particular,*

$$\lim_{n \rightarrow \infty} \frac{\phi^*(n, n, P)}{\phi(n, n, P)} = 0.$$

If  $M$  is a matching with two edges, and  $M^+$  is the graph obtained from two copies of  $K_4$  sharing one edge by removing that edge, then  $\text{ex}(n, M^+, K_3) \leq \phi^*(n, n, M)$ . If  $F$  is the triple system consisting of all four triangles in  $M^+$ , then Füredi [7] showed  $\text{ex}(n, M^+) = O(n^2)$ , answering a conjecture of Erdős [4]. It is possible to adapt Füredi's proof to give  $\phi^*(n, n, M) = O(n^2)$ , so in this case,  $\text{ex}(n, M^+, K_3) = \Theta(\phi^*(n, n, M))$ . For improvements of the constant factor, see Mubayi and Verstraete [13] and Pikhurko and Verstraete [15]. We shall see that if  $F$  is bipartite and  $m$  is not too large relative to  $n$ , then Conjecture A is also true.

### 1.3 Main results : the non-induced case

Determining  $\phi(m, n, F)$  even when  $F$  is a complete graph is challenging. The second theorem we give is well-suited to the case of bipartite graphs, and is due to Wilson:

**Theorem 4.** *Let  $F$  be a graph. If there exists an extremal  $F$ -free  $n$ -vertex graph with maximum degree at most  $n^{1/2}/m^2$ , then*

$$\phi(m, n, F) = \binom{n}{2} + \binom{m}{2} \text{ex}(n, F).$$

[JV: It should be true that if  $m$  is a constant, then equality holds for every bipartite graph  $F$  when  $n$  is large enough.] Since  $\binom{n}{2} + m - 1 \leq \phi^*(m, n, F) \leq \phi(m, n, F)$  for any graph  $F$  with at least two edges, this theorem shows  $\phi^*(m, n, F) = (1 + o(1))\binom{n}{2}$  whenever the conditions on  $m$  in the theorem are satisfied. In particular, if  $P$  is the path with two edges, and  $m = o(n^{1/4})$  as  $n \rightarrow \infty$ , then for  $n \geq 2$ ,

$$\binom{n}{2} + m - 1 \leq \phi^*(m, n, F) \leq \phi(m, n, F) = \binom{n}{2} + \binom{m}{2} \left\lfloor \frac{n}{2} \right\rfloor.$$

When  $F$  is bipartite, the value of  $\phi(m, n, F)$  for larger  $m$  appears to be difficult to determine. We investigate the case that  $F$  is a path or matching with two edges more closely. For a family  $\mathcal{F}$  of graphs, we write  $\phi(m, n, \mathcal{F})$  for the maximum number of edges in graphs  $G_1, G_2, \dots, G_m$  which are double  $F$ -free for all  $F \in \mathcal{F}$ .

**Theorem 5.** *Let  $P$  be the path with two edges and let  $M$  be the matching with two edges. Then as  $m, n \rightarrow \infty$ , [JV: Some of these statements need a dependency between  $m$  and  $n$ , else they are false. Also did we not discuss lower bound on  $\phi(m, n, P)$  when  $m \geq n$ ? Like  $n^2\sqrt{m}$  perhaps? ]*

[RT: For the lower bound on  $\phi(m, n, P)$  when  $m \geq n$ , we did discuss the construction of

putting a smaller projective plane in each line of the projective plane. I just have problem with the sizes of the projective planes that I could not recreate the proof.]

1.  $\phi(m, n, P) \leq (\frac{1}{2} + o(1)) \min\{n^2\sqrt{m}, mn^{3/2}\}$ , and this bound is tight for  $\sqrt{n} \leq m \leq n^2$ .
3.  $\phi(m, n, M) \leq \frac{m + \sqrt{m^2 + mn^4 + 4m^2n^3}}{2}$ .
4.  $\phi(m, n, \{P, M\}) \leq n^2 + \frac{mn}{\sqrt{2}}$ .

Interestingly, while Conjecture A proposes  $\phi^*(m, n, P) = O(n^2 + mn)$  for all  $m, n \geq 1$ , the above theorem shows  $\phi(m, n, P)$  is much larger, of order at least  $mn^{3/2}$  when  $m \geq \sqrt{n}$ .

Our first theorem on  $\phi(m, n, F)$  for non-bipartite graphs  $F$  uses the notion of *supersaturation* – see Erdős and Simonovits [6]. We determine the asymptotic value of  $\phi(m, n, F)$  as  $m \rightarrow \infty$  when  $F$  is a non-bipartite graph:

**Theorem 6.** *Let  $n \geq 1$  and let  $F$  be a non-bipartite graph. Then as  $m \rightarrow \infty$ ,*

$$\phi(m, n, F) = (1 + o(1))m \cdot \text{ex}(n, F).$$

The next result we present concerns non-bipartite graphs. To state the theorem, we require the notion of  $k$ -color Ramsey numbers. Define  $R_k(r)$  to be the  $k$ -color Ramsey number for the complete graph  $K_r$ : that is, the minimum  $N$  such that there exists a monochromatic  $F$  in any coloring of  $E(K_N)$  with  $k$  colors. Suppose we have a coloring  $c : E(K_N) \rightarrow 2^{[m]}$  for some  $N < R_k(r)$  where  $k \leq \binom{m}{2}$  and  $|c(u, w)| \geq 2$  for all  $\{u, w\} \in E(K_N)$ . For  $i \in [m]$ , let  $H_i = \{\{u, w\} \in E(K_N) : i \in c(u, w)\}$ . Then  $H_1, H_2, \dots, H_m$  are double  $K_r$ -free. If we replace the vertices of  $K_N$  with disjoint sets  $V_w : w \in V(K_N)$  whose sizes add up to  $n$ , and then let  $G_1 = K_n$  and

$$G_i = \{\{x, y\} : (x, y) \in V_u \times V_w, i \in c(u, w)\}$$

then  $G_1, G_2, \dots, G_m$  is also double  $K_r$ -free. We call  $G_1, G_2, \dots, G_m$  an  $(m, n, k)$ -blowup, and let  $f(m, n, r)$  denote the maximum of  $e(H_1) + e(H_2) + \dots + e(H_m)$  such that  $H_1, H_2, \dots, H_m$  is an  $(m, n, k)$ -blowup for some  $k \leq \binom{m}{2}$ . This turns out to be exactly the construction which determines  $\phi(m, n, F)$  when  $F$  is a complete graph:

**Theorem 7.** *Let  $r \geq 2$  and  $m, n \geq 1$ . Then*

$$\phi(m, n, K_r) = f(m, n, r).$$

[RT: We also have this generalization of the blowup theorem proved. I didn't include it because I haven't understood the proof.]

We may generalize this result to any family of graphs up to asymptotic error:

**Theorem 8.** *Let  $m, n \geq 1$ , and let  $\mathcal{F}$  be a family of graphs. Then*

$$\phi(m, n, K_r) = f(m, n, \mathcal{F}).$$

[RT: need to define  $f(m, n, \mathcal{F})$ .]

While computing  $f(m, n, r)$  is a finite calculation, the Ramsey number  $R_k(r)$  unfortunately appears to be intractable in general; it is known that  $R_2(3) = 6$  and  $R_3(3) = 17$  and  $R_2(4) = 18$ , but no further multicolor Ramsey numbers are known [2, 11]. In the special case  $r = m = 3$ , the following holds:

**Theorem 9.** *For all  $n \geq 1$ ,*

$$\phi(3, n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

## 1.4 Definitions and Notation

Denote the set of first  $n$  positive integers as  $[n] = \{1, 2, \dots, n\}$ . Given a set  $X$ , we denote  $2^X$  as the power set of  $X$ .

Let  $G = (V, E)$  be a graph. Let  $V(G)$  denote the vertex set and  $E(G)$  denote the edge set of  $G$ . Let  $e(G) = |E(G)|$  be the number of edges in  $G$ . For vertex  $v \in V(G)$ , we denote by  $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$  the neighborhood of  $v$ .

Given graphs  $G_1, G_2, \dots, G_m$  on some vertex set  $V$ , we denote  $G_{i_1, \dots, i_k}$  as graph on  $V$  with edge set  $E(G_{i_1, \dots, i_k}) = \bigcap_{\alpha=1}^k E(G_{i_\alpha})$ . Given two graphs  $G_1, G_2$ , we denote  $G_1 \cup G_2$  as the graph on  $V(G_1) \cup V(G_2)$  with edge set  $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ . Let  $s$

In this thesis, we reserve  $n$  to denote the number of vertices in a graph. We call a  $n$ -vertex complete graph  $K_n$ , and a complete bipartite graph  $K_{a,b}$ , where  $a, b$  are the sizes of its parts. We denote  $P_n$  as a path with  $n$  edges, and  $C_n$  as a cycle with  $n$  edges. Given graph  $G, H$ , define  $G + H$  as the graph fully connecting  $G, H$ , i.e.  $V(G + H) = V(G) \cup V(H)$  and  $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$ .

Given graphs  $G$  and  $F$ , we say that  $G$  is  $F$ -free if  $G$  does not contain  $F$  as a subgraph. We denote  $\text{ex}(n, F)$  to be the maximum possible number of edges an  $F$ -free graph on  $n$  vertices, and we call a  $F$ -free graph achieving this maximum an extremal graph for  $F$ . Given graphs

$G_1, \dots, G_m$  on the same set of vertices and  $F$ , we say that  $G_1, \dots, G_m$  are pairwise  $F$ -free if  $E(G_i) \cap E(G_j)$  does not contain  $F$  for  $i \neq j$ . Let  $v$  be a vertex from  $G_1, G_2, \dots, G_m$ . Unless otherwise specified, we denote  $d(v)$  as the sum of the degree of  $v$  over all  $G_i$ .

## 2 The induced double Turán problem

[JV: This section could be written more cleanly.]

We prove the theorems for  $\phi^*(m, n, F)$  in this chapter. In particular, the main theorem we prove is Theorem 1 for general non-bipartite graphs  $F$  and in the special case of cliques. We will first introduce two observations that significantly simplify the problem.

The first observation is that the determination of  $\phi^*(m, n, F)$  can be reduced down to the case of two graphs, which is stated in the following lemma:

**Lemma 10.** *Let  $n, m, k \geq 2$  with  $m \geq k$ ,  $F$  be some graph. Then*

$$\phi^*(m, n, F) \leq \frac{m}{k} \cdot \phi^*(k, n, F).$$

Moreover, let  $G_1, \dots, G_m$  be induced double  $F$ -free graphs on  $[n]$  and suppose  $\sum_{i=1}^k e(G_i) = \phi^*(k, n, F)$  only if  $G_1 = \dots = G_k$ . Then  $\sum_{i=1}^m e(G_i) = \phi^*(m, n, F)$  only if  $G_1 = \dots = G_m$ .

*Proof.* Let  $G_1, \dots, G_m$  be induced double  $F$ -free graphs on  $[n]$ . Put  $G_{i+m} = G_i$  for all  $i \in [m]$ . Then

$$\sum_{i=1}^m e(G_i) = \frac{1}{k} \sum_{i=1}^m [e(G_i) + \dots + e(G_{i+k-1})] \leq \frac{1}{k} \sum_{i=1}^m \phi^*(k, n, F) = \frac{m}{k} \cdot \phi^*(k, n, F),$$

which establishes the upper bound. The lower bound follows from the construction with  $G_1 = \dots = G_m$  to be  $n$ -vertex extremal graphs for  $F$ .

Now suppose  $\sum_{i=1}^m e(G_i) = \frac{m}{k} \cdot \phi^*(k, n, F)$  and  $G_1 \neq G_2$ . By assumption  $\sum_{i=1}^k e(G_i) < \phi^*(k, n, F)$ . But then  $\sum_{i=1}^k e(G_{i+j}) > \phi^*(k, n, F)$  for some  $j \geq 1$ , contradiction.  $\square$

Now that we may determine  $\phi^*(m, n, F)$  by examining  $\phi^*(m, n, F)$ , the second observation is that  $\phi^*(m, n, F)$  can be further reduced to a finite optimization problem on a single variable. To state the lemma, we introduce the following construction function:

**Definition 11.** *For  $n \geq t \geq 1$  and  $F$  some graph, define*

$$\mathcal{C}(n, t, F) := \binom{n-t}{2} + (n-t)t + 2\text{ex}(t, F).$$



The construction described by  $\mathcal{C}(n, t, F)$  are graphs  $G_1, G_2$  on  $[n]$ , such that  $G_2$  is a  $t$ -vertex extremal graph for  $F$  and  $G_1 = G_2 + K_{n-t}$ .

**Lemma 12.** *Let  $F$  be some graph. For  $n \geq 1$ ,*

$$\phi^*(2, n, F) = \max_{0 \leq t \leq n} \mathcal{C}(n, t, F).$$

Moreover, the equality holds for graphs  $G_1, G_2$  on  $[n]$  only if  $G_1, G_2$  are the construction described by  $\mathcal{C}(n, t_{\max}, F)$ , where  $t_{\max} \in [n]$  is a maximizer for  $\mathcal{C}(n, t, F)$ .

*Proof.* Let  $G_1, G_2$  be induced double  $F$ -free graphs on  $[n]$ . Put  $T = V(G_1) \cap V(G_2)$ ,  $t = |T|$ ,  $s = |V(G_1) \setminus T|$ , and  $n - t - s = |V(G_2) \setminus T|$ . Note that  $t, s \in \mathbb{Z}_{\geq 0}$ . Since  $G_1, G_2$  are induced subgraphs of  $G_1 \cup G_2$ , we have  $G_1[T] = G_2[T] = G_1 \cap G_2$ . But then  $G_1 \cap G_2$  is  $F$ -free, so  $e(G_1[T]) = e(G_2[T]) \leq \text{ex}(t, F)$ . Notice there can be at most  $t(n - t)$  edges between  $T$  and  $(V(G_1) \cup V(G_2)) \setminus T$ . Since  $G[V(G_1) \setminus T] \leq \binom{s}{2}$  and  $G[V(G_2) \setminus T] \leq \binom{n-t-s}{2}$ ,

$$e(G_1) + e(G_2) \leq \binom{s}{2} + \binom{n-s-t}{2} + t(n-t) + 2\text{ex}(t, F).$$

But then  $\binom{n-t}{2} > \binom{s}{2} + \binom{n-t-s}{2}$  for  $0 < s < n - t$ , so

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2\text{ex}(t, F) = \mathcal{C}(n, t, F).$$

This establishes the upper bound. From this we also know that  $e(G_1) + e(G_2) = \mathcal{C}(n, t, F)$  only if  $G_1, G_2$  are the construction described by  $\mathcal{C}(n, t, F)$ . The result now follows.  $\square$

## 2.1 Proof of Theorem 2

By Lemma 10, it suffices to prove the theorem for  $m = 3$ . Let  $G_1, G_2, G_3$  be induced double  $K_r$ -free graphs, such that  $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, K_r)$ . We may assume  $e(G_1) \geq e(G_2) \geq e(G_3)$ , and we already know  $\phi^*(3, n, K_r) \geq 3\text{ex}(n, K_r)$ . Consequently, we must have  $e(G_1) + e(G_2) \geq 2\text{ex}(n, K_r)$ . Since  $G_1, G_2, G_3$  are induced and  $e(G_1) + e(G_2) + e(G_3) \geq 3\text{ex}(n, K_r)$ , it suffices to show that  $G_1 = G_2 = T_{r-1}(n)$ . In particular, we will use Lemma 12 to show that  $G_1, G_2$  is an extremal configuration without containing a double  $K_r$ .

Let  $t = |V(G_1 \cap G_2)|$ . By Turán's Theorem,

$$\text{ex}(t, K_r) - \text{ex}(t-1, K_r) = e(T_{r-1}(t)) - e(T_{r-1}(t-1)) = t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, K_r) - \mathcal{C}(n, t-1, K_r) = -t + 1 + 2[\text{ex}(t, K_r) - \text{ex}(t-1, K_r)] = t + 1 - 2 \left\lceil \frac{t}{r-1} \right\rceil. \quad (1)$$

For  $r \geq 4$ ,  $\mathcal{C}(n, t, K_r)$  is strictly increasing on  $t$ , so by Lemma 12,

$$\phi^*(2, n, K_r) = \mathcal{C}(n, n, K_r) = 2\text{ex}(n, K_r) = e(G_1) + e(G_2)$$

and  $G_1 = G_2 = T_{r-1}(n)$ , as desired.

Now suppose  $r = 3$ . Equation (1) shows that  $\mathcal{C}(n, t, K_r)$  is non-decreasing on  $t$  and  $\mathcal{C}(n, t, K_r) > \mathcal{C}(n, t, K_r)$  for even  $t$ . By Lemma 12, we now have

$$\phi^*(2, n, K_r) = \max[\mathcal{C}(n, n, K_r), \mathcal{C}(n, n-1, K_r)] = 2\text{ex}(n, K_r) = e(G_1) + e(G_2),$$

and either  $G_1 = G_2 = T_{r-1}(n)$ , or  $G_2 = T_{r-1}(n-1)$  and  $G_1 = G_2 + K_1$ . If the latter case is true, then  $e(G_3) \geq \text{ex}(n, F) > e(G_2)$ , and this contradiction completes the proof.  $\square$

## 2.2 Proof of Theorem 1

If  $F$  is a graph of chromatic number  $r+1 \geq 3$ , then Theorem B shows  $\text{ex}(n, F) = (1 + o(1))\text{ex}(n, K_{r+1})$  as  $n \rightarrow \infty$ . In this section, we prove Theorem 1.

*Proof of Theorem 1.* By Lemma 10, it suffices to prove the theorem for  $m = 3$ . Let  $G_1, G_2, G_3$  be induced double  $F$ -free graphs, such that  $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, F)$ . We may assume  $e(G_1) \geq e(G_2) \geq e(G_3)$ , and we already know  $\phi^*(3, n, F) \geq 3\text{ex}(n, F)$ . Consequently, we must have  $e(G_1) + e(G_2) \geq 2\text{ex}(n, F)$ . Since  $G_1, G_2, G_3$  are induced and  $e(G_1) + e(G_2) + e(G_3) \geq 3\text{ex}(n, F)$ , it suffices to show that  $G_1 = G_2$  are  $n$ -vertex  $F$ -free extremal graphs. In particular, we will use Lemma 12 to show that  $G_1, G_2$  is an extremal configuration without containing a double  $F$ .

Let  $t = |V(G_1 \cap G_2)|$ . If  $t < \sqrt{n}$ , then

$$2\text{ex}(n, F) \geq 2e(T_{r-1}(n)) \geq 2 \left\lfloor \frac{n^2}{4} \right\rfloor \geq \binom{n}{2} + \binom{\sqrt{n}}{2} > \mathcal{C}(n, t, F).$$

Thus  $t \geq \sqrt{n}$ . But then for large enough  $t$ , any extremal  $t$ -vertex  $F$ -free graph contains a spanning complete  $(r-1)$ -partite subgraph  $T_{r-1}(t)$ , so we may add  $\text{ex}(t-1, F) - e(T_{r-1}(t-1))$  edges to  $T_{r-1}(t)$  and still avoid  $F$  as a subgraph. Hence for large enough  $t$ , we have  $\text{ex}(t, F) \geq$

$\text{ex}(t-1, F) - e(T_{r-1}(t-1)) + e(T_{r-1}(t))$ , and so

$$\text{ex}(t, F) - \text{ex}(t-1, F) \geq e(T_{r-1}(t)) - e(T_{r-1}(t-1)) \geq t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, F) - \mathcal{C}(n, t-1, F) = -t + 1 + 2[\text{ex}(t, F) - \text{ex}(t-1, F)] \geq t + 1 - 2 \left\lceil \frac{t}{r-1} \right\rceil. \quad (2)$$

For  $r \geq 4$ ,  $\mathcal{C}(n, t, F)$  is strictly increasing on  $t$ , so by Lemma 12,

$$\phi^*(2, n, F) = \mathcal{C}(n, n, F) = 2\text{ex}(n, F) = e(G_1) + e(G_2),$$

and  $G_1 = G_2$  are  $n$ -vertex  $F$ -free extremal graphs, as desired.

Now suppose  $r = 3$ . Equation (2) shows that  $\mathcal{C}(n, t, F)$  is strictly increasing for even  $t$  and  $\mathcal{C}(n, t, F) \geq \mathcal{C}(n, t-1, F)$  for odd  $t$ . By Lemma 12, we now have

$$\phi^*(2, n, F) = \max[\mathcal{C}(n, n, F), \mathcal{C}(n, n-1, F)] = 2\text{ex}(n, F) = e(G_1) + e(G_2),$$

and either  $G_1 = G_2$  are  $n$ -vertex extremal  $F$ -free graphs, or  $G_2$  is an  $(n-1)$ -vertex extremal  $F$ -free graph and  $G_1 = G_2 + K_1$ . If the latter case is true, then  $e(G_3) \geq \text{ex}(n, F) > e(G_2)$ , and this contradiction completes the proof.  $\square$

For small  $n$ , we may not be able to achieve the same result. Consider the case when  $F$  is the bowtie graph, i.e. the 5-vertex graph with two triangles sharing a vertex. For  $n \leq 4$ , the  $n$ -vertex extremal graph for  $F$  is the complete graph  $K_n$ . For  $n \geq 5$ , the  $n$ -vertex extremal graph for  $F$  is then  $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$  plus an edge, and so  $\text{ex}(n, F) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$ . But then in this case when  $n = 5$ ,

$$\mathcal{C}(5, 4, F) = 2e(K_4) + 4 = 16 > \mathcal{C}(5, 5, F) = 2 \left( \left\lfloor \frac{5^2}{4} \right\rfloor + 1 \right) = 14.$$

This yields an instance where the construction  $G_1 = K_{k-1}$  and  $G_2 = K_n$  beats our benchmark construction for  $|V(F)| = k$ . Thus the following lemma gives a lower bound for  $n$  to avoid losing to this construction.

**Lemma 13.** *Let  $n, k \geq 3$  and  $r \geq 2$ , and let  $F$  have chromatic number  $r+1$  and  $|V(F)| = k$ . If  $n \geq 2 \binom{k-1}{2} + 1$  and  $r$  divides  $n$ , then*

$$\mathcal{C}(n, n, F) > \mathcal{C}(n, k-1, F).$$

In particular, if  $n \geq 2\binom{k-1}{2} + 1$ , then  $\phi^*(m, n, F) = m \cdot \text{ex}(n, F)$  for all  $m \geq 3$ , with equality only for identical extremal  $n$ -vertex  $F$ -free graphs.

*Proof.* We need to show that

$$2\text{ex}(n, F) - \binom{n}{2} > \binom{k-1}{2}.$$

Since  $\text{ex}(n, F) \geq \lfloor n^2/4 \rfloor$ ,

$$2\text{ex}(n, F) - \binom{n}{2} \geq 2 \left\lfloor \frac{n^2}{4} \right\rfloor - \binom{n}{2} \geq \frac{n}{2} - \frac{1}{4} > \binom{k-1}{2}.$$

This proves the lemma. □

[JV: This lemma seems like an afterthought, and should probably go first. Also, can you say something much better if  $r \geq 3$ ?]

## 2.3 Proof of Theorem 3

[JV: This is to be written]

# 3 The non-induced double Turán problem

In this section, we prove our main theorems on  $\phi(m, n, F)$ .

## 3.1 Proof of Theorem 6

We need the following *saturation theorem*, which may be found in [6].

**Proposition 14.** *Let  $F$  be any non-empty graph with  $k$  vertices. For all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $G$  is any  $n$ -vertex graph with  $\text{ex}(n, F) + \epsilon n^2$  edges, then  $G$  contains  $\delta n^k$  copies of  $F$ .*

*Proof of Theorem 6.* Let  $k = |V(F)|$  and let  $\epsilon > 0$ . Let  $G_1, G_2, \dots, G_m$  be double  $F$ -free. Reorder  $G_1, G_2, \dots, G_m$  so that  $e(G_i) \geq \text{ex}(n, F) + \epsilon n^2$  for  $1 \leq i \leq \ell$  and  $e(G_i) < \text{ex}(n, F) + \epsilon n^2$  for  $\ell < i \leq m$ . Then each  $G_i : 1 \leq i \leq \ell$  contains at least  $\delta n^k$  copies of  $F$ , by Proposition

14. On the other hand, there are at most  $n^k$  copies of  $F$  such that  $F \subseteq G_i$  for some  $i \in [m]$ . Therefore  $\ell \leq 1/\delta$  and

$$\begin{aligned} \sum_{i=1}^m e(G_i) &= \sum_{i=1}^{\ell} e(G_i) + \sum_{i=\ell+1}^m e(G_i) \\ &\leq \frac{1}{\delta} \binom{n}{2} + (m - \ell) \text{ex}(n, F) + (m - \ell) \epsilon n^2 \\ &\leq m \cdot \text{ex}(n, F) + \epsilon m n^2 + \frac{1}{\delta} \binom{n}{2}. \end{aligned}$$

Since  $F$  is not bipartite,  $\text{ex}(n, F) = \Theta(n^2)$  and so  $\phi(m, n, F) \leq m \cdot \text{ex}(n, F) + (\epsilon + 1/\delta m) m n^2$ . Since  $\epsilon$  was arbitrary and  $\delta$  is a constant depending only on  $\epsilon$ , we conclude  $\phi(m, n, F) \leq (1 + o(1)) m \cdot \text{ex}(n, F)$  as  $m \rightarrow \infty$ .  $\square$

Let  $F$  be a bipartite graph with  $k \geq 2$  vertices and  $j \geq 1$  edges. A strong version of a conjecture of Simonovits [16, 17] would suggest that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that every  $n$ -vertex graph  $G$  with at least  $p(\frac{n}{2})(1 + \epsilon) \text{ex}(n, F)$  edges contains at least  $\delta p^j n^k$  copies of  $F$ . For instance, this is known to be true whenever the asymptotic behavior of  $\text{ex}(n, F)$  is known, which includes the case  $F = K_{2,t}$ . If  $F$  is bipartite and  $m \cdot \text{ex}(n, F)/n^2 \rightarrow \infty$  as  $m, n \rightarrow \infty$ , then this conjecture with the same proof as above shows  $\phi(m, n, F) = (1 + o(1)) m \cdot \text{ex}(n, F)$ . When  $F$  contains a cycle, then there exists  $\alpha > 0$  such that  $\text{ex}(n, F) \geq n^{1+\alpha}$  for large enough  $n$ . Thus, we conclude that if  $F$  contains a cycle and the Simonovits conjecture is true for  $F$ , then  $\phi(m, n, F) = (1 + o(1)) m \cdot \text{ex}(n, F)$  for  $m \geq n$  and  $n \rightarrow \infty$ . In particular, this shows  $\phi(m, n, K_{2,t}) = (1 + o(1)) m \cdot \text{ex}(n, F)$  for  $m \geq n$  as  $n \rightarrow \infty$ .

### 3.2 Proof of Theorem 4

We first show that for all  $m, n \geq 1$  and graph  $F$ ,

$$\phi(m, n, F) \leq \binom{n}{2} + \text{ex}(n, F) \binom{m}{2}.$$

Thereafter, we show that if there is an extremal  $F$ -free graph with maximum degree at most  $n^{1/2}/m^2$ , then the above bound is exactly tight.

*Proof of the upper bound.* For  $S \subseteq [m]$ , let  $E_S$  denote the set of edges that are contained in exactly  $\{G_i\}_{i \in S}$ . Then

$$\sum_{i=1}^m e(G_i) = \sum_{S \subseteq [m]} |S| |E_S| \leq (k-1) \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq k} (|S| - k + 1) |E_S|.$$

Let  $A_S = \bigcup_{T \supseteq S} E_T$ , i.e., the set of edges that are contained in all  $G_i$  with  $i \in S$ . When  $|S| \geq k$ , the edge set  $A_S$  is  $F$ -free and thus

$$\sum_{T \supseteq S} |E_T| \leq \text{ex}(n, F).$$

Hence,

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq k}} (|S| - k + 1) |E_S| = \sum_{\substack{S \subseteq [m], \\ |S| = k}} \sum_{T \supseteq S} \frac{(|T| - k + 1) |E_T|}{\binom{|T|}{k}} \leq \sum_{\substack{S \subseteq [m], \\ |S| = k}} \sum_{T \supseteq S} |E_T| \leq \binom{m}{k} \text{ex}(n, F),$$

as each  $T \in [m]$  with  $|T| \geq k$  is counted  $\binom{|T|}{k}$  times in total and  $|T| - k + 1 \leq \binom{|T|}{k}$ . This proves the upper bound.

*Proof of the lower bound.* We need to show there exists a construction such that the graph with edge set  $E_S$  is an extremal  $F$ -free graph, for all  $S \subseteq [m]$  of size  $k$ . Let  $M = \binom{m}{k}$  and  $H_1, \dots, H_M$  be copies of an extremal  $F$ -free graph on  $n$  vertices such that  $H_i$  with maximum degree  $\Delta \leq n^{1/2}/m^2$  for all  $i \in [m]$ . It suffices to show that we can embed each  $H_i$  onto  $[n]$  such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each  $H_i$  and iteratively decrease the number of intersecting edges. Define a  $(u, v, i)$ -swap by swapping the embedding of vertex  $u$  and  $v$  of  $H_i$ , i.e. replacing each edge  $\{u, w\} \in E(H_i)$  with the edge  $\{v, w\}$  and each edge  $\{v, w\} \in E(H_i)$  with the edge  $\{u, w\}$ . This preserves the type of isomorphism of  $H_i$ . Given a vertex  $v$ , let  $N(v) = N_{H_1}(v) \cup \dots \cup N_{H_M}(v)$ . Suppose there exists an intersecting edge  $\{u, w\} \in E(H_i) \cap E(H_j)$ . Since  $|N(u)| \leq M \cdot \Delta \leq n^{1/2}/2$ ,  $|N(u) \cup N(N(u))| \leq \Delta + \Delta(\Delta - 1) \leq n/4$  so there exists a vertex  $v \notin N(u) \cup N(N(u))$ . Since  $N(u) \cap N(v) = \emptyset$ , performing a  $(u, v, i)$ -swap reduces the number of intersecting edges. The result now follows from iterating this process.  $\square$

### 3.3 Proof of Theorem 5

The different parts of Theorem 5 give bounds on  $\phi(m, n, P)$ ,  $\phi(m, n, M)$  and  $\phi(m, n, \{P, M\})$ .

*Upperbound for Theorem 5.1* Let  $G_1, \dots, G_m$  be graphs on  $[n]$  not containing a double  $P$ . We start with the following claim:

**Claim 14.1.**  $\phi(m, n, P) \leq mn \cdot (1 + \sqrt{4n^2/m + 1})/4$ .

*Proof.* Since there is no double  $P$  in  $G_1, G_2, \dots, G_m$ ,

$$\sum_{i=1}^m \#\{P \subseteq G_i\} \leq \#\{P \subseteq K_n\}.$$

For all  $G_i$ , each vertex  $v$  in  $G_i$  along with two of its neighbors form one unique  $P$ , so

$$\#\{P \subseteq G_i\} = \sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2}.$$

By Jensen's inequality,

$$\sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2} \geq n \binom{\sum_{v \in V(G_i)} d_{G_i}(v)/n}{2} = n \binom{2e(G_i)/n}{2} \geq \frac{2(e(G_i))^2}{n} - e(G_i).$$

On the other hand, since each three vertices in  $G$  can form at most three  $P$ 's,

$$\#\{P \subseteq K_n\} \leq 3 \binom{n}{3} \leq \frac{n^3}{2}.$$

Combining the above inequalities yields and using Jensen's inequality once more yields

$$\frac{2m}{n} \left( \frac{1}{m} \sum_{i=1}^m e(G_i) \right)^2 - \sum_{i=1}^m e(G_i) \stackrel{\text{Jensen}}{\leq} \sum_{i=1}^m \frac{2(e(G_i))^2}{n} - e(G_i) \leq \frac{n^3}{2}.$$

Solving the quadratic equation gives

$$\sum_{i=1}^m e(G_i) \leq mn \cdot \frac{1 + \sqrt{4n^2/m + 1}}{4}.$$

This proves the claim. □

**Claim 14.2.**  $\phi(m, n, P) \leq (mn^{3/2} + n^2)/2$ .

*Proof.* For each vertex  $u \in [n]$ , define  $H_u$  as the  $m \times n$  bipartite graph with edge set  $E(H_u) := \{\{v, i\} : \{u, v\} \in E(G_i)\}$ . If  $H_u$  contains a quadrilateral  $\{v, i\}, \{v, j\}, \{w, i\}, \{w, j\}$ , then  $\{u, v\}, \{u, w\}$  form a double  $P$  in  $G_i \cap G_j$ , contradiction. Thus we conclude that  $H_u$  is quadrilateral-free, and therefore  $e(H_u) \leq m\sqrt{n} + n$ , by the Kővari-Sós-Turán Theorem [9]. It now follows that

$$\sum_{i=1}^m e(G_i) = \frac{1}{2} \sum_{u \in V(G)} e(H_u) \leq \frac{1}{2} (mn^{3/2} + n^2).$$

This proves the claim. □

*Lowerbound for 5.1.* We now show

$$\phi(m, n, P) \geq \begin{cases} (1/2 + o(1))mn^{3/2}, & \text{if } \sqrt{n} \leq m \leq n, \\ (1/2 + o(1))\sqrt{mn}n^2, & \text{if } n < m \leq n^2. \end{cases}$$

We first prove the case  $\sqrt{n} \leq m \leq n$ . Suppose  $G_1, G_2, \dots, G_n$  are graphs on  $[n]$  containing no double  $P$  and  $\sum_{i=1}^n e(G_i) \geq (1/2 + o(1))n^{5/2}$ , with  $e(G_1) \geq e(G_2) \geq \dots \geq e(G_n)$ . Then  $G_1, G_2, \dots, G_m$  are graphs with no double  $P$  and  $\sum_{i=1}^m e(G_i) \geq (1/2 + o(1))mn^{3/2}$ . Hence, it suffices to prove the case for  $m = n$ .

Consider a finite projective plane with  $n$  points and  $n$  lines, with prime  $q$  chosen so that  $n = (1 + o(1))(q^2 + q + 1)$  as  $q \rightarrow \infty$ . Let  $S_1, \dots, S_n \subseteq [n]$  be the  $n$  lines of the projective plane. Note that each line  $S_i$  contains  $q + 1$  points, and the intersection of any two distinct lines  $S_i, S_j$  contains  $|S_i \cap S_j| = 1$  point.

Define  $G_1, \dots, G_n$  to be graphs on  $[n]$ , each with edge set

$$E(G_i) := \{\{j, k\} \subseteq [n] : j \neq k, j + k \in S_i \pmod n\}.$$

Note that the intersection of distinct  $G_i, G_j$  is  $P$  free: since  $|S_i \cap S_j| = 1$ , if  $\{a, b\}, \{a, c\} \in E(G_i) \cap E(G_j)$ , then  $a + b = a + c$  so  $b = c$ .

We now count the number of edges in  $G_1, \dots, G_n$ . Since  $|S_i| = q + 1$ , for each point  $j \in [n]$ , there are  $q + 1$  choices for  $k \in [n]$  such that  $j + k \in S_i$ . But then we have to avoid counting the same edge twice and loops, so the number of edges in  $G_i$  is

$$e(G_i) = \frac{n(q + 1) - \#\text{loops counted for } G_i}{2}.$$

If  $j \in [n]$  is even, then  $k = j/2$  is the unique number in  $[n]$  such that  $k + k = j \pmod n$ . If  $j \in [n]$  is odd, then  $k = (n + j)/2$  is the unique number in  $[n]$  such that  $k + k = j \pmod n$ , as  $n$  is even. Hence, for each  $j \in S_i$ , there exists a unique  $k \in [n]$  such that  $k + k = j \pmod n$ , and thus

$$\#\text{loops counted for } G_i = |S_i| = q + 1.$$

Since  $q + 1 = (1 + o(1))n^{1/2}$ , the number of edges in  $G_1, \dots, G_n$  is

$$\sum_{i=1}^n e(G_i) = n \cdot \frac{n(q + 1) - (q + 1)}{2} = \left(\frac{1}{2} + o(1)\right) n^{5/2},$$



as  $n \rightarrow \infty$ .

The case for  $n < m \leq n^2$  is similar. Consider the finite projective plane  $P$  with  $n$  points defined above. Since  $|S_i| = q + 1 > \sqrt{n} \geq n^2/m$ , we may further place a smaller projective plane  $P_i$  with  $n^2/m$  points inside each line  $S_i$ . Since each line of  $P_i$  has size roughly  $n/\sqrt{m}$ , each  $S_i$  contains roughly  $m/n$  lines, and thus we now have  $m$  small lines in total. Define  $G'_i$  on each small line the same way we defined  $G_i$  on  $S_i$ . Following the same line of calculations above, the construction of  $G'_1, \dots, G'_m$  now gives  $\sum_{i=1}^m e(G_i) = (1/2 + o(1))\sqrt{mn}^2$ , provided  $m \leq n^2$ .  $\square$

**[JV: What is missing here]**

*Proof of Theorem 5.2.* We now show for all  $m, n \geq 1$ , **[JV: This is false, maybe  $m \leq n$  or something? What happens for general  $m$ ?]****[RT: I have rewrote this proof. The original bound of  $n^{5/2}$  works for  $m \leq n$ .]**

$$\phi(m, n, M) \leq \frac{m + \sqrt{m^2 + mn^4 + 4m^2n^3}}{2}$$

Notice that

$$\#\{M \subseteq G_i\} \geq \binom{e(G_i)}{2} - \#\{P \subseteq K_n\} \geq \binom{e(G_i)}{2} - 3\binom{n}{3}$$

**[JV: That statement is false].** On the other hand, each four vertices in  $G$  can form at most three  $M$ 's, so  $\#\{M \subseteq K_n\} \leq 3\binom{n}{4} \leq n^4/8$ . Using the same argument in Theorem 5.2, we have

$$\sum_{i=1}^m \binom{e(G_i)}{2} - 3m\binom{n}{3} = \sum_{i=1}^m \#\{M \subseteq G_i\} \leq \#\{M \subseteq K_n\} \leq \frac{n^4}{8}.$$

By Jensen's inequality,

$$\sum_{i=1}^m \binom{e(G_i)}{2} \geq m \binom{\sum_{i=1}^m e(G_i)/m}{2} = \frac{1}{2m} \left[ \left( \sum_{i=1}^m e(G_i) \right)^2 - m \sum_{i=1}^m e(G_i)^2 \right].$$

Combining the above inequalities yields

$$\left( \sum_{i=1}^m e(G_i) \right)^2 - m \sum_{i=1}^m e(G_i)^2 \leq mn^4/4 + m^2n^3,$$

and solving the quadratic inequality gives

$$\sum_{i=1}^m e(G_i) \leq \frac{m + \sqrt{m^2 + mn^4 + 4m^2n^3}}{2}$$

This proves Theorem 5.2.

*Proof of Theorem 5.3.* We are to show

$$\phi(m, n, \{P, M\}) \leq n^2 + \frac{mn}{\sqrt{2}}.$$

For all  $m, n \geq 1$ , let  $G_1, \dots, G_m$  be graphs on  $[n]$  that contain neither a double  $P$  nor a double  $M$ . Construct the bipartite graph  $H$ , where one part consists of the set  $[m]$ , the other part consists of all edges of  $K_n$ , and edge set given by  $E(H) = \{\{i, e\} : e \in E(G_i)\}$ . Notice that  $H$  is  $K_{2,2}$ -free: if  $\{i, e\}, \{j, e\}, \{i, f\}, \{j, f\}$  are edges in  $H$ , then  $G_i \cap G_j$  contains a double  $P$  or a double  $M$ , contradiction. It now follows from the Kővari-Sós-Turán Theorem that

$$\sum_{i=1}^m e(G_i) = e(H) \leq 2 \binom{n}{2} + m \binom{n}{2}^{\frac{n}{2}} \leq n^2 + \frac{mn}{\sqrt{2}},$$

as desired. □

### 3.4 Proof of Theorem 7

We now prove Theorem 7. Notice that we trivially have  $f(m, n, r) \leq \phi(m, n, K_r)$ , so it suffices to show the reverse inequality. That is, we need to show that there exists a blowup construction meeting the desired bound.

Let  $G_1, G_2, \dots, G_m$  be graphs on  $[n]$  with no double  $K_r$  and  $\sum_{i=1}^m e(G_i) = \phi(m, n, K_r)$ . Observe that any pair  $\{i, j\} \subseteq [n]$  must be in some  $G_i$ , otherwise, we may add it to  $G_1$  without creating a double  $K_r$ .

We call vertices  $v, v'$  *clones* if for all  $u \in [n] \setminus \{v, v'\}$  and  $i \in [m]$ , the edge  $\{u, v\} \in E(G_i)$  if and only if  $\{u, v'\} \in E(G_i)$ . Furthermore, we call  $\{v, v'\}$  a *light edge* if  $\{v, v'\}$  is in exactly one graph  $G_i$ .

We now apply Algorithm 1 to  $G_1, G_2, \dots, G_m$ .

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**Algorithm 1** symmetrization algorithm

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```
while  $\exists$  a light edge whose endpoints are not clones do
    among all vertices incident to such an edge, select a vertex  $v$  with maximum degree
     $B_v \leftarrow$  collection of vertices sending a light edge to  $v$  that are not clones of  $v$ 
    while  $B_v \neq \emptyset$  do
        pick  $u \in B_v$ 
         $j \leftarrow$  colour of the light edge from  $u$  to  $v$ 
        for  $1 \leq i \leq m$  do
            if  $i \neq j$  then;
                 $N_{G_i}(u) \leftarrow N_{G_i}(v)$ 
            else if  $i = j$  then
                 $N_{G_i}(u) \leftarrow (N_{G_i}(v) \setminus \{u\}) \cup \{v\}$ 
            end if
        end for
    end while
end while
```

---

**Claim 14.3.** *Algorithm 1 terminates.*

*Proof.* Notice that at the end of the ‘while  $B_v \neq \emptyset$ ’ loop, every vertex sending a light edge to  $v$  is a clone of  $v$ . This implies  $v$  along with the set  $L_v$  of vertices receiving light edges from  $v$  induce a clique of size at least two in some  $G_i$ , and an empty graph in every other graph  $G_j$  with  $j \neq i$ . Moreover, any vertex  $w \notin L_v$  sends edges to either all or none of the vertices in  $L_v$ , and if  $w$  is incident to  $L_v$ , then  $w$  sends edges to  $L_v$  in at least two graphs. It now follows that no light edge incident with a vertex in  $L_v$  will be picked again in an iteration of the out most while loop. Thus the algorithm can run through at most  $n/2$  such iterations, and so it terminates.  $\square$

**Claim 14.4.**  $G'_1, G'_2, \dots, G'_m$  do not contain a double  $K_r$  and  $\sum_{i=1}^m e(G'_i) = \phi(m, n, K_r)$ .

*Proof.* Note that we replace  $u$  by a clone of  $v$  in the for loop of Algorithm 1. Since  $\{u, v\}$  remains a light edge in this step,  $u$  and  $v$  cannot both belong to a double  $K_r$  in the modified graphs. Furthermore, any double  $K_r$  containing  $u$  after the for loop arises from a double  $K_r$  containing  $v$  prior to the for loop. But then  $G_1, G_2, \dots, G_m$  contained no double  $K_r$  to begin with, so  $G'_1, G'_2, \dots, G'_m$  do not contain a double  $K_r$ .

We now show that the algorithm does not reduce the number of edges. By our choice of  $v$ , we know  $d(v) \geq d(u)$  for all  $u \in B_v$  prior to the for loop. Hence, replacing  $u$  with a clone

of  $v$  does not decrease the number of edge over a complete iteration of the inner while loop. Therefore,  $\sum_{i=1}^m e(G'_i) = \phi(m, n, K_r)$ .  $\square$

Hence, the algorithm outputs graphs  $G'_1, G'_2, \dots, G'_m$  with  $\phi(m, n, K_r)$  edges and the additional property that light edges come in ‘clone cliques.’ We may thus partition the vertex set  $[n]$  into  $k$  disjoint sets  $V_1, V_2, \dots, V_k$ , such that each  $V_i$  induces a clique of light edges from the same graph. Moreover, for distinct  $i, j \in [k]$ , define  $S_{ij}$  to be the set of all edges between  $V_i$  and  $V_j$ , and note that any edge in  $S_{ij}$  appears in at least two modified graphs. The sets  $S_{ij}$  now yield a  $k$ -blowup. Notice that if the pattern of the  $k$ -blowup contains a double  $K_r$ , then the original graphs  $G_1, G_2, \dots, G_m$  must have contained a double  $K_r$  as well, contradiction. Thus the  $k$ -blowup is double  $K_r$ -free.

It remains to show that  $k < R_M(K_r)$ . For each edge  $\{i, j\} \subseteq [k]$  in the pattern of the  $k$ -blowup, we assign an arbitrary distinct pair  $\{a, b\} \subseteq L_{ij} \subseteq [m]$  to  $\{i, j\}$ . If  $k \geq R_M(K_r)$ , then there exists  $K_r$  in the pattern of the  $k$ -blowup colored by some distinct pair  $\{a, b\} \subseteq [m]$ . But then this implies the pattern of the  $k$ -blowup contains a double  $K_r$ , contradiction. This completes the proof.  $\square$

### 3.5 Proof of Theorem 9

It is not hard to see that  $\phi(2, n, K_3) = \binom{n}{2} + \lfloor n^2/4 \rfloor$ : if  $G_1, G_2$  is double triangle-free, then we have

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_1 \cap G_2) \leq \binom{n}{2} + \text{ex}(n, K_3)$$

and so  $\phi(2, n, K_3) \leq \binom{n}{2} + \lfloor n^2/4 \rfloor$ . Taking  $G_1 = K_n$  and  $G_2 = K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$  meets this bounds. The main result of this section is to show for all  $n \geq 1$ ,

$$\phi(3, n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Let  $G_1, G_2, G_3$  be double triangle-free. Define  $H_k \subseteq G$  to be the graph with edges contained in at least  $k$  of the  $G_i$ ’s and note that  $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$ . Thus it suffices to show that  $e(H_2) + e(H_3) \leq \frac{n^2}{2}$ . Notice  $H_2$  must not contain any triangles with two edges in  $H_3$ , so

$$e(H_2) + e(H_3) \leq \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}|.$$

Let  $H'_3$  be the graph with the same vertex set as  $H_3$  and edge set  $\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}$ . It suffices to show that  $\frac{n}{2} \geq e(H_3) - e(H'_3)$ .

Let  $d_1 \geq d_2 \geq \dots \geq d_n$  and  $f_1 \geq f_2 \geq \dots \geq f_n$  each be the degree sequence of  $H_3$  and  $H'_3$ , respectively. We show that  $f_i \geq d_i - 1$  for all  $i$ . Let  $v_i$  denote the vertex in  $H$  with degree  $d_i$  and  $u_i$  be the vertex in  $H$  with degree  $f_i$ . Let  $S_i = |N_{H_3}(v_1) \cup \dots \cup N_{H_3}(v_i)|$ . Since

$$\sum_{u \in S_i} d_{H_3}(u) \geq d_1 + \dots + d_i,$$

we have that  $|S_i| \geq i$ . But then  $S_i \setminus \{u_1, \dots, u_{i-1}\}$  is non-empty, and every  $u \in S_i$  has degree  $d_{H'_3}(u) \geq d_i - 1$ . Hence,  $f_i \geq d_i - 1$  for all  $i$ , which yields

$$e(H'_3) = \frac{1}{2} \sum_{i=1}^n f_i \geq \frac{1}{2} \sum_{i=1}^n (d_i - 1) = e(H_3) - \frac{n}{2}.$$

This proves Theorem 9. □

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