

Double Turán Problem

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Turán Problem

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Turán Number

$$\text{ex}(n, F) := \max\{e(G) : |V(G)| = n \text{ and } F \not\subseteq G\}$$

Turán Problem

Turán's Theorem [Tur41]

For $r + 1 \geq 3$,

$$\text{ex}(n, K_{r+1}) = e(T_r(n)),$$

with equality for graph G only if $G = T_r(n)$.

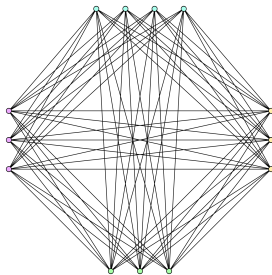
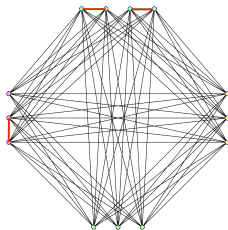


Figure: $T_4(13)$

Erdős-Stone Theorem

Erdős-Stone Theorem [ES46], Simonovits' Theorem [ES66]

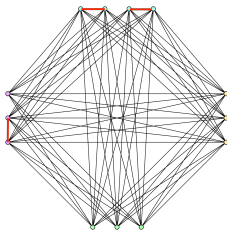
Let F be any graph of chromatic number $r + 1 \geq 2$. Then $\text{ex}(n, F) = e(T_r(n)) + o(n^2)$ as $n \rightarrow \infty$.



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Supersaturation Theorem [ES83]

For all $\epsilon > 0$, there exists $\delta > 0$ such that if G is a n -vertex graph with $\text{ex}(n, F) + \epsilon n^2$ edges, then G contains $\delta n^{v(F)}$ copies of F .

Double Turán Problem: Definition

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Is this tight?

Double Turán Problem: Main Results

Theorem A

Let $n \geq 1$ and F be a non-bipartite graph. Then as $m \rightarrow \infty$

$$\phi(m, n, F) = (1 + o(1))m \cdot \text{ex}(n, F).$$

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But

$$\lim_{n \rightarrow \infty} \frac{\phi(4, n, K_3)}{\binom{n}{2} + 3\lfloor n/4 \rfloor} > 1$$

Double Turán Problem: Main Results

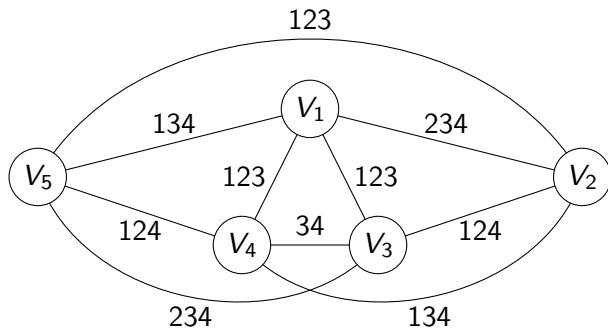


Figure: Construction for four graphs not containing a double- K_3

Double Turán Problem: Main Results

Theorem B (Wilson)

Let F be a graph. If there exists an extremal F -free n -vertex graph with maximum degree at most \sqrt{n}/m^2 , then

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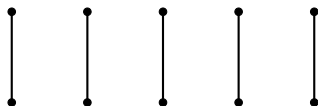
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e.g. The extremal graph for P_2 is a matching, which has maximum degree 1 and $\text{ex}(n, P_2) = \lfloor n/2 \rfloor$.



Double Turán Problem: Motivation

Double Turán problems are closely related to Turán problems for 3-uniform hypergraphs H through **link graphs**.

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Link Graph

For $i \in V(H)$, define graph H_i with

$$V(H_i) = V(H) \setminus \{i\} \quad \text{and} \quad E(H_i) = \{\{j, k\} : \{i, j, k\} \in E(H)\}.$$

Double Turán Problem: Motivation

Example: Octahedron-free 3-uniform hypergraph H

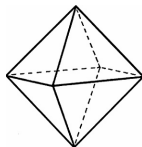


Figure: Octahedron $K_{2,2,2}^{(3)}$

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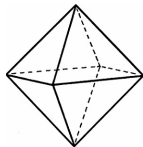


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$$\text{ex}(n, K_{2,2,2}^{(3)}) = \Theta(n^{11/4})$$

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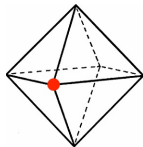


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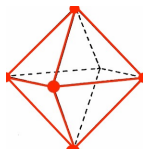


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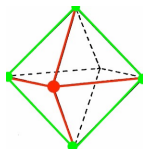


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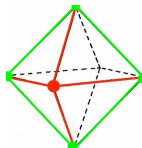


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H is octahedron-free $\implies H_1, H_2, \dots, H_n$ are double $K_{2,2}$ -free.

$$\text{ex}(n, K_{2,2,2}^{(3)}) \leq \phi(n, n, K_{2,2})$$

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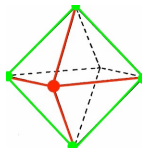


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Let graphs G_1, G_2, \dots, G_m be induced and double F -free.

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Induced Double Turán Problem: Main Results

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For $m \geq 3$ and non-bipartite F , if n is large enough, then

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For $F = K_r$, this theorem applies for all n .

Induced Double Turán Problem: Motivation

Generalized Turán problem

What is the maximum number $\text{ex}(n, F, K_3)$ of triangles in an n -vertex F -free graph G ?

Studied by Alon and Shikhelman [AS16] and Kostochka, Mubayi and Verstraete [KMV15; MM23; MV16].

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Ex. Octahedron-free graph G .

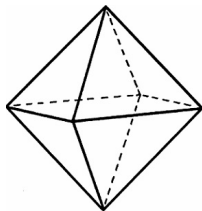


Figure: Octahedron

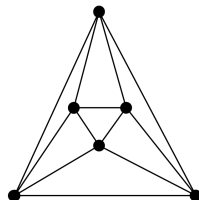


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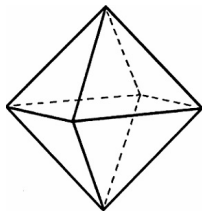


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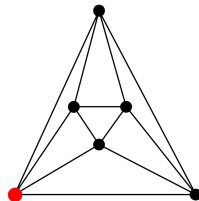


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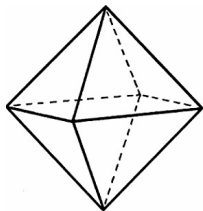


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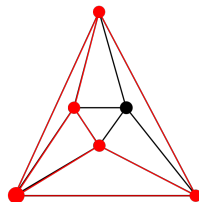


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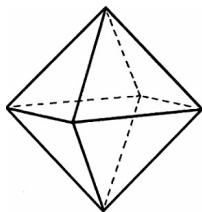


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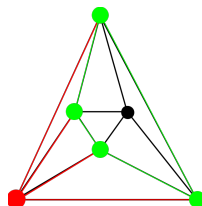


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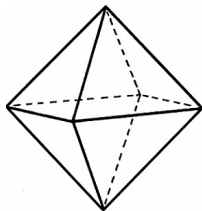


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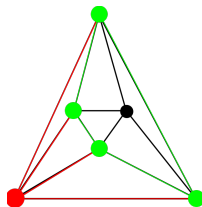


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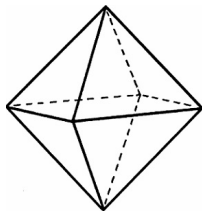


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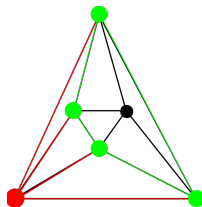


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Conjecture implies

$$\text{ex}(n, K_{2,2,2}, K_3) = \Theta(\phi^*(n, n, K_{2,2})) = \Theta(n^{5/2}).$$

Theorem D

Let P be a path with two edges. Then $\phi(n, n, P) = \Omega(n^{5/2})$, whereas $\phi^*(n, n, P) = o(n^{5/2})$, as $n \rightarrow \infty$. In particular,

$$\lim_{n \rightarrow \infty} \frac{\phi^*(n, n, P)}{\phi(n, n, P)} = 0.$$

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This shows that $\phi(m, n, F)$ and $\phi^*(m, n, F)$ can be very different problems.

Proof of Theorem B: Upper Bound

Theorem B (Wilson)

Let F be a graph. If there exists an extremal F -free n -vertex graph with maximum degree at most \sqrt{n}/m^2 , then

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$$\implies \sum_{i=1}^m e(G_i) = \sum_{S \subseteq [m]} |S| |E_S| \leq \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq 2} (|S| - 1) |E_S|.$$

Proof of Theorem B: Upper Bound

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S| - 1) |E_S| = \sum_{\substack{S \subseteq [m] \\ |S| = 2}} \sum_{T \supseteq S} \frac{(|T| - 1) |E_T|}{\binom{|T|}{2}} \leq \sum_{\substack{S \subseteq [m] \\ |S| = 2}} \sum_{T \supseteq S} |E_T|,$$

as each $T \in [m]$ with $|T| \geq 2$ is counted $\binom{|T|}{2}$ times.

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Observation: If $|S| \geq 2$, the edge set $\bigcup_{T \supseteq S} E_T$ is F -free.

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Proof of Theorem B: Upper Bound

Observation: If $|S| \geq 2$, the edge set $\bigcup_{T \supseteq S} E_T$ is F -free.

$$\left| \bigcup_{T \supseteq S} E_T \right| = \sum_{T \supseteq S} |E_T| \leq \text{ex}(n, F)$$

$$\Rightarrow \sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S| - 1) |E_S| \leq \sum_{\substack{S \subseteq [m], \\ |S|=2}} \sum_{T \supseteq S} |E_T| \leq \binom{m}{2} \text{ex}(n, F)$$

This proves the upper bound.

Proof of Theorem B: Lower Bound

We need to show

$$\phi(m, n, F) \geq \binom{n}{2} + \text{ex}(n, F) \binom{m}{2}.$$

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\implies We can pack these $\binom{m}{2}$ H_{ij} 's into $[n]$ s.t. $E(H_{i_1j_1}) \cap E(H_{i_2j_2}) = \emptyset$

Proof of Theorem B: Lower Bound

For $i \in [m]$, put

$$G_i = H_{1i} \cup H_{2i} \cup \cdots \cup H_{im}$$

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Add edges not in any of H_{ij} to G_1 .

$$\sum_{i=1}^m e(G_i) = 2 \underbrace{\sum e(H_{ij})}_{\binom{m}{2} \text{ex}(n, F)} + \underbrace{\left| E(K_n) \setminus \bigcup E(H_{ij}) \right|}_{\binom{n}{2} - \binom{m}{2} \text{ex}(n, F)} = \binom{n}{2} + \binom{m}{2} \text{ex}(n, F)$$

Proof of Theorem C: Complete Graph Case

Theorem C: Complete Graph Case

Let $m, n, r \geq 3$. Then

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced K_r -free graphs G_1, G_2, \dots, G_m only if $G_1 = G_2 = \dots = G_m = T_{r-1}(n)$.

Proof of Theorem C: Complete Graph Case

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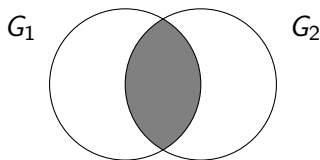
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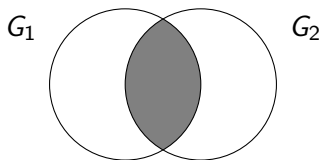
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IDEA: Reduce to $m = 2$.

Proof of Theorem C: Complete Graph Case

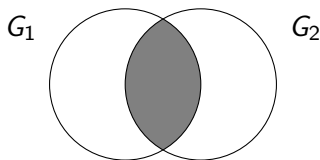


Proof of Theorem C: Complete Graph Case



$G_1 \cap G_2$ (gray area) is double K_r -free.

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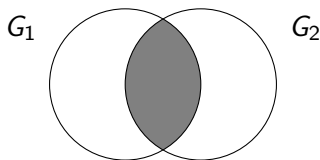


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If G_1, G_2 intersects in t vertices, then

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2T_{r-1}(t)$$

Proof of Theorem C: Complete Graph Case



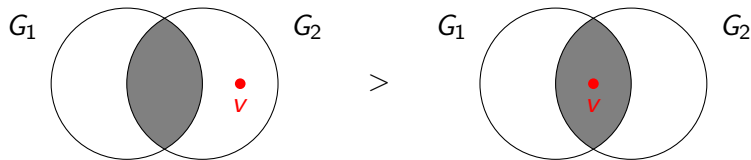
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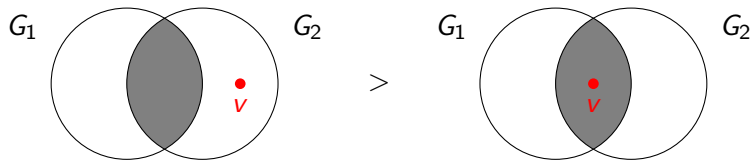
$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2T_{r-1}(t)$$

\implies we are done if the unique maximum occurs at $t = n$.

Proof of Theorem C: Complete Graph Case



Proof of Theorem C: Complete Graph Case

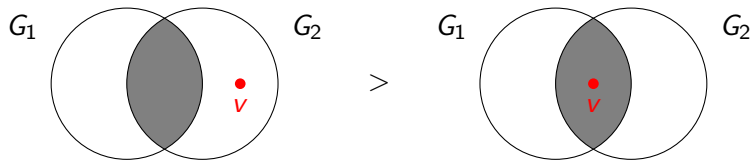


Put $f(t, r) := \binom{n-t}{2} + (n-t)t + 2T_{r-1}(t)$

For example, if $r = 4$,

$$f(t+1, 4) - f(t, 4) = -t + 1 + 2 \underbrace{[T_3(t+1) - T_3(t)]}_{t - \lceil \frac{t}{2} \rceil} = t + 1 - 2 \left\lceil \frac{t}{2} \right\rceil > 0$$

Proof of Theorem C: Complete Graph Case



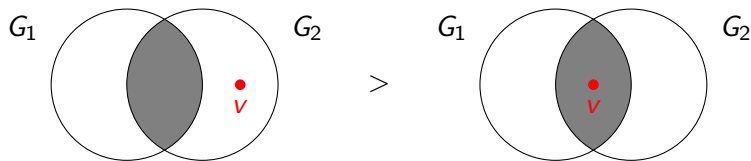
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\implies The larger the intersection, the larger the sum of edges.

\implies The best we can do is $G_1 = G_2 = T_3(n)$.

Concluding Remarks: Induced Double Turán Problem

Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \text{ex}(n, F) + n^2).$$

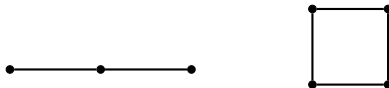
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This conjecture is broadly open for bipartite graphs.



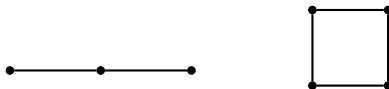
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This conjecture is broadly open for bipartite graphs.



Solving the conjecture would give a solution to $\text{ex}(n, K_{2,2}, K_3)$.

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$$\lim_{n \rightarrow \infty} \frac{\phi(4, n, K_3)}{m \binom{n}{2}} = ??$$

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