Double Turán Problem

January 19, 2025

Contents

1 Introduction **2**

1.1 Definitions and Notation 2

2 Induced Version **3**

2.1 Complete Graph Case 4

2.2 Non-bipartite F 6

3 General Version **9**

3.1 Triangle F 10

3.2 Bipartite F 12

1 Introduction

Let $\mathrm{ex}_2(n, m, F)$ be the maximum possible sum of the number of edges over m subgraphs $G_1, \dots, G_m$ on the same vertex set $[n]$, with the constraint that $E(G_i) \cap E(G_j)$ does not contain graph F for $i \neq j$. Our goal is to determine $\mathrm{ex}_2(n, m, F)$ for different forbidden graphs F. A trivial construction with $G_1 = K_n$ and $G_2, \dots, G_m$ to be extremal graphs for F yields the lower bound $\binom{n}{2} + (m-1)\mathrm{ex}(n, F)$. In this work, we use this bound as a benchmark to either show the tightness of $\mathrm{ex}_2(n, m, F)$ or to provide a better bound.

Additionally, we are also interested in a more restrictive version where $G_1, \dots, G_m$ are induced subgraphs of $G_1 \cup \dots \cup G_m$. We denote $\mathrm{ex}_2^*(n, m, F)$ as the maximum possible sum of the number of edges over m induced subgraphs $G_1, \dots, G_m$ on the same vertex set $[n]$ such that $E(G_i) \cap E(G_j)$ does not contain graph F for $i \neq j$. The trivial construction by taking $G_1, \dots, G_m$ to be extremal graphs for F yields the lower bound $m \cdot \mathrm{ex}(n, F)$. This is the benchmark we use to determine $\mathrm{ex}_2^*(n, m, F)$.

In this work, we will first discuss the induced case, and then shift our focus to the general case. At the end, we will discuss the case where F is bipartite.

1.1 Definitions and Notation

Let $G = (V, E)$ be a graph. Let $V(G) = V$ denote the vertex set and $E(G) = E$ denote the edge set of G. We note by $v(G) = |V|$ the number of vertices and $e(G) = |E|$ the number of edges in G. For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v.

Given $G_1, \dots, G_m$ subgraphs of G, we denote $G_{i_1,\dots,i_k}$ as the subgraph of G with edge set $E(G_{i_1,\dots,i_k}) = \bigcap_{\alpha=1}^{k} E(G_\alpha)$.

In this thesis, we reserve n to denote the number of vertices in a graph. Given a graph F, we denote $\mathrm{ex}(n, F)$ to be the extremal number for F on a graph with n vertices, i.e. the maximum number of edges in a n-vertex graph that does not contain F as a subgraph.

We call a n-vertex complete graph K_n, and a complete bipartite graph $K_{a,b}$, where a, b are the size of its parts. We denote P_n as a path with n edges, and C_n as a cycle with n edges. Given graph G, H, define $G + H$ as the graph fully connecting G, H, i.e. $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$.

We also denote the set of first n positive integers as $[n] = \{1, 2, \dots, n\}$. Given a set X, we denote 2^X as the power set of X.

2 Induced Version

In this section, we assume that $G_1, \dots, G_m$ are induced subgraphs of $G_1 \cup \cdots \cup G_m$ and $E(G_i) \cap E(G_j)$ does not contain F for $i \neq j$. Here, we say that the extremal condition for m subgraphs is met if $\sum_{i=1}^m e(G_i) = \mathrm{ex}_2^*(n, m, F)$.

TODO: add the condition for all G_i's to be extremal graphs for F for all m, and generalize to hypergraph.

The following lemma shows that the problem can be reduceed to only two graphs.

Lemma 2.1. *Let $n, m, k \in \mathbb{N}$ such that $2 \leq k \leq m$, and let F be a graph. Then*

$$\mathrm{ex}_2^*(n, m, F) \leq \frac{m}{k} \cdot \mathrm{ex}_2^*(n, k, F).$$

Moreover, if the extremal condition for k induced subgraphs is met only when $G_1 = \cdots = G_k$, then the above equality holds and the extremal condition for m induced subgraphs is met only when $G_1 = \cdots = G_m$.

Not putting equality because I'm unsure if a construction for k subgraphs can always generalize to m subgraphs. For example, if $F = K_3$ and n is odd, the $G_1 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and $G_2 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$ construction cannot be generalized to $m = n+1$ subgraphs.

Proof. Let $G_1, \dots, G_m$ be induced subgraphs of $G_1 \cup \cdots \cup G_m$ with $E(G_i) \cap E(G_j)$ not containing F for $i \neq j$. Put $G_{i+m} = G_i$ for all $i \in [m]$. Then

$$\sum_{i=1}^m e(G_i) = \frac{1}{k} \sum_{i=1}^m [e(G_i) + \cdots + e(G_{i+k-1})] \leq \frac{1}{k} \sum_{i=1}^m \mathrm{ex}_2^*(n, k, F) = \frac{m}{k} \cdot \mathrm{ex}_2^*(n, k, F),$$

which establishes the upper bound.

Suppose $\sum_{i=1}^k e(G_i) = \mathrm{ex}_2^*(n, k, F)$. By assumption $G_1 = \cdots = G_k$, so $e(G_i) = \mathrm{ex}_2^*(n, k, F)/k$ for $1 \leq i \leq k$. Hence, the construction $G_1 = \cdots = G_m$ meets the upperbound. On the other hand, if $G_1 \neq G_2$ then $\sum_{i=1}^k e(G_i) < \mathrm{ex}_2^*(n, k, F)$. Since $\sum_{i=1}^k e(G_{i+j}) \leq \mathrm{ex}_2^*(n, k, F)$ for all $j \geq 1$, we have $\sum_{i=1}^m e(G_i) < \frac{m}{k} \cdot \mathrm{ex}_2^*(n, k, F)$. Thus the extremal condition is met only when $G_1 = \cdots = G_m$. □

Lemma 2.2. *Let $n \geq 1$, $m \geq 2$ and F be some graph. If $\mathrm{ex}(c, F) - \mathrm{ex}(c-1, F) > \frac{c-1}{2}$ for all $1 \leq c \leq n$, then*

$$\mathrm{ex}_2^*(n, m, F) = m \cdot \mathrm{ex}(n, F)$$

and $G_1 = \cdots = G_m$ are extremal graphs for F when the extremal condition is met.

Not sure if this statement can be strengthened into if and only if. We would need to show there's no F such that $\mathrm{ex}(c, F) - \mathrm{ex}(c-1, F) \leq \frac{c-1}{2}$ for some c but f has a unique maximum at $c = n$. In other words, there can not exist F such that its extremal number increases slowly at some small c but eventually catches up.

Proof. By Lemma 2.1, it suffices to show the case for two subgraphs. Let G_1, G_2 be induced subgraphs of $G_1 \cup G_2$ with $E(G_1) \cap E(G_2)$ not containing F. Moreover, let $C = V(G_1) \cap V(G_2)$ and put $a = |V(G_1) \backslash C|, b = |V(G_2) \backslash C|$, and $c = |C|$. Since G_1, G_2 are induced subgraphs, $G_1[C] = G_2[C] = G_{1,2}$. But then $G_{1,2}$ is F-free, so $e(G_1[C]) = e(G_2[C]) \leq \mathrm{ex}(c, F)$. This yields the inequality

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \mathrm{ex}(c, F)\right].$$

Let $f(a, b, c)$ denote the expression on the right-hand-side of the above inequality.

Claim 1. *f has a unique maximum at $c = n$.*

Proof. Suppose $b < n$. Since $\mathrm{ex}(c, F) - \mathrm{ex}(c-1, F) > \frac{c-1}{2}$,

$$\begin{aligned} f(a, b, c) - f(a, b+1, c-1) &= \binom{a+c}{2} - \binom{a+c-1}{2} \\ &\quad - 2\left[\binom{c}{2} - \binom{c-1}{2} - \mathrm{ex}(c, F) + \mathrm{ex}(c-1, F)\right] \\ &= a - c + 1 + 2[\mathrm{ex}(c+1, F) - \mathrm{ex}(c, F)] > a \geq 0. \end{aligned}$$

Thus, f is strictly increasing with respect to c. By symmetry, f has a unique maximum at $c = n$. □

By the claim,

$$e(G_1) + e(G_2) \leq 2\binom{n}{2} - 2\left[\binom{n}{2} - \mathrm{ex}(n, F)\right] = 2 \cdot \mathrm{ex}(n, F),$$

and and $G_1 = G_2 = G_{1,2}$ are extremal graphs for F on n vertices. □

2.1 Complete Graph Case

Lemma 2.3. *For $n \geq 1$ and $r \geq 2$,*

$$\mathrm{ex}(n, K_{r+1}) - \mathrm{ex}(n-1, K_{r+1}) \geq \frac{n-1}{2},$$

with equality if and only if n is odd and $r = 2$.

Proof. By Turán's Theorem,

$$\mathrm{ex}(n, K_{r+1}) - \mathrm{ex}(n-1, K_{r+1}) = \delta(T_r(n)) = n - \left\lceil \frac{n}{r} \right\rceil \geq n - \left\lceil \frac{n}{2} \right\rceil.$$

The result now follows. □

The following theorem for complete graphs with more than 3 vertices now follows directly from Lemma 2.2 and Lemma 2.3:

Theorem 2.4. *For $n \geq 1$, $m \geq 2$, and $r \geq 3$,*

$$\mathrm{ex}_2^*(n, m, K_{r+1}) = m \cdot e(T_r(n)),$$

and $G_1 = \cdots = G_m = T_r(n)$ when the extremal condition is met.

Surprisingly, the triangle case is more complicated than the case for larger complete graphs. As shown in Lemma 2.3, the condition given by Lemma 2.2 is not satisfied for all n in the triangle case, and there are indeed constructions of induced subgraphs G_1, G_2 that meet the extremal condition but are neither equal nor both complete bipartite graphs. For odd n, consider $G_1 = K_{\frac{n-1}{2},\frac{n-1}{2}}$ and $G_2 = K_{\frac{n-1}{2},\frac{n-1}{2}} + K_1$. The number of edges over G_1, G_2 is $\frac{(n-1)^2}{2} + n - 1 = \frac{n^2-1}{2} = 2\left\lfloor \frac{n^2}{4} \right\rfloor$, which meets the benchmark construction of two complete bipartite graphs. Hence, for the triangle case we have to make some comprimises.

Theorem 2.5. *For $n \geq 1$ and $m \geq 2$,*

$$\mathrm{ex}_2^*(n, m, K_3) = m \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Moreover, if n is even or m is odd, then the extremal condition is met only when $G_1 = G_2$ are complete balanced bipartite graphs on n vertices. Otherwise, the extremal condition is met only when either $G_1 = G_2 = K_{\frac{n+1}{2},\frac{n-1}{2}}$ or $G_1 = K_{\frac{n-1}{2},\frac{n-1}{2}}$ and $G_2 = K_{\frac{n-1}{2},\frac{n-1}{2}} + K_1$.

Proof. By Lemma 2.2 and Lemma 2.3, we are done if n is even, so we may assume that n is odd. We first show the following claim.

Claim 2. $\mathrm{ex}_2^*(n, 2, K_3) = 2\left\lfloor \frac{n^2}{4} \right\rfloor$*, and the extremal condition is met only when either $G_1 = G_2 = K_{\frac{n+1}{2},\frac{n-1}{2}}$ or $G_1 = K_{\frac{n-1}{2},\frac{n-1}{2}}$ and $G_2 = K_{\frac{n-1}{2},\frac{n-1}{2}} + K_1$.*

Proof. Let a, b, c, and $f(a, b, c)$ be defined as in the proof of Lemma 2.2. Then

$$\begin{aligned} f(a,b,c) - f(a,b+2,c-2) &= \binom{a+c}{2} - \binom{a+c-2}{2} \\ &\quad - 2\left[\binom{c}{2} - \binom{c-2}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor + \left\lfloor \frac{(c-2)^2}{4} \right\rfloor\right] \\ &= 2(a+c) + 1 - 2[2c + 1 - (c+1)] \\ &= 2a + 1 > 0. \end{aligned}$$

Hence, f attains its maximum of $2\left\lfloor \frac{n^2}{4} \right\rfloor$ when $c = n$ or $n-1$, that is, $a + b \leq 1$. Thus $\mathrm{ex}_2^*(n, 2, K_3) \leq 2\left\lfloor \frac{n^2}{4} \right\rfloor$. Assume that $a = 0$. When $c = n$, $G_1 = G_2 = G_{1,2}$ are extremal graphs for K_3, which is the complete balanced bipartite graph on n vertices. When $c = n-1$, $G_1 = G_{1,2} = K_{\frac{n-1}{2},\frac{n-1}{2}}$ and G_2 must be a copy of G_1 with all vertices adjacent to the only remaining vertex, i.e. $G_2 = G_1 + K_1$, to meet the extremal number. $\square$

By Lemma 2.1 and the above claim, $\mathrm{ex}_2^*(n,m,K_3) = m\left\lfloor \frac{n^2}{4} \right\rfloor$. It remains to show that for odd m, $G_1 = \cdots = G_m = K_{\frac{n+1}{2},\frac{n-1}{2}}$ when the extremal condition is met. Suppose not. Then the claim guarantees $G_i = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1$ for some i. Put $G_{m+i} = i$. By applying the claim repeatedly,

$$\begin{aligned} G_i &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1 \\ G_{i+1} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} \\ G_{i+2} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor} + K_1 \\ &\vdots \\ G_{i+m} &= K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}, \end{aligned}$$

as m is odd. But then $G_{i+m} = G_i = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$, and this contradiction completes the proof. □

2.2 Non-bipartite F

Theorem 2.6. *Suppose F is $(r+1)$-colorable, with $r \geq 2$. For large enough n,*

$$\mathrm{ex}_2^*(n,n,F) = n \cdot \mathrm{ex}(n,F).$$

In particular, the extremal number is reached only if $G_1 = G_2 = \cdots = G_n$ are n-vertex extremal graphs for F.

By the same argument as in Theorem 2.1, it suffices to prove the following lemma:

Lemma 2.7. *Let F be $(r+1)$-colorable, with $r \geq 2$. Suppose $E(G_1) \cap E(G_2)$ does not include F. For large enough n,*

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n,F),$$

with equality if and only if $G_1 = G_2$ are n-vertex extremal graphs for F, unless n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2 = G_1 + K_1$.

Proof. Let $C = V(G_1) \cap V(G_2)$, $A = V(G_1) \backslash C$, and $B = V(G_2) \backslash C$. Put $a = |A|$, $b = |B|$, $c = |C|$. Since G_1, G_2 are induced graphs, $E(G_1[C]) = E(G_2[C]) = E(G[C]) = E(G_i) \cap E(G_j)$, which is F-free. Hence,

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \mathrm{ex}(c,F)\right]. \tag{2.1}$$

Define $f(a,b,c)$ as the function on the right-hand-side of (2.2). We show that $f(a,b,c)$ attains its maximum at $a = b = 0$ and $c = n$.

Claim 3. *If $c \leq \frac{n}{2}$, then $f(a,b,c) < 2 \cdot \mathrm{ex}(n,F)$.*

Proof. Write $c = kn$ for some $k \in [0, 1/2]$. Since

$$f(a,b,kn) \le 2\binom{(1-k)n/2}{2} - 2\left[\binom{kn}{2} - \mathrm{ex}(kn,F)\right],$$

it suffices to show that

$$\mathrm{ex}(n,F) - \mathrm{ex}(c,F) > \binom{(1-k)n/2}{2} - \binom{kn}{2}$$

for all $k \in [0,1/2]$. By the Erdős-Stone theorem, $\mathrm{ex}(n,F) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} + o(n^2)$ and so the left-hand-side is at least

$$\mathrm{ex}(n,F) - \mathrm{ex}(c,F) \ge \mathrm{ex}(n,F) - \mathrm{ex}(n/2,F) \ge \left(1 - \frac{1}{r}\right)\left(\frac{n^2}{2} - \frac{n^2}{8}\right) - o(n^2) \ge \frac{3n^2}{16} - o(n^2).$$

On the right-hand-side,

$$\binom{(1-k)n/2}{2} - \binom{kn}{2} = (1 - 2k - 3k^2)\frac{n^2}{8} + o(n^2) \le \frac{n^2}{8} + o(n^2).$$

Combining the above inequalities now yields the claim, as n is large. $\square$

Thus we may assume that $c > \frac{n}{2}$. A theorem of Simonovits states that for large enough n, $\mathrm{ex}(n,F) = \mathrm{ex}(n,K_{r+1}) + \mathrm{ex}(n,\tilde{F})$, where $\tilde{F}$ is the family of residue subgraphs of F after F is embedded into $T_r(n)$. This implies

$$\mathrm{ex}(n+1,F) - \mathrm{ex}(n,F) \ge \mathrm{ex}(n,K_{r+1}) - \mathrm{ex}(n+1,K_{r+1}),$$

and so

$$f(a,b-2,c+2) - f(a,b,c) \ge 2a - 2c - 1 + 2[\mathrm{ex}(c+2,K_{r+1}) - \mathrm{ex}(c,K_{r+1})].$$

Since $\mathrm{ex}(c+1,K_{r+1}) - \mathrm{ex}(c,K_{r+1}) \ge c - \left\lfloor \frac{c}{r} \right\rfloor \ge c - \left\lfloor \frac{c}{2} \right\rfloor$, we have

$$\mathrm{ex}(c+2,K_{r+1}) - \mathrm{ex}(c,K_{r+1}) \ge c + 1 - \left\lfloor \frac{c+1}{2} \right\rfloor + c - \left\lfloor \frac{c}{2} \right\rfloor = c + 1,$$

and thus

$$f(a,b-2,c+2) - f(a,b,c) \ge 2a + 1 > 0.$$

By symmetry, we also have $f(a-2,b,c+2) > f(a,b,c)$. Thus, f attains its maximum when c is $n-1$ or n. Equation (2.2) now yields,

$$e(G_1) + e(G_2) \le \max\left[2 \cdot \mathrm{ex}(n,F), 2 \cdot \mathrm{ex}(n-1,F) + n - 1\right].$$

Assume that $a = 0$. Since

$$\begin{aligned} 2 \cdot \mathrm{ex}(n,F) - [2 \cdot \mathrm{ex}(n-1,F) + n - 1]& \\ &\ge 2[\mathrm{ex}(n,K_{r+1}) - \mathrm{ex}(n-1,K_{r+1})] - n + 1 \qquad (2.2)\\ &= 2\left(n - \left\lceil \frac{n}{r} \right\rceil\right) - n + 1 \\ &\ge n + 1 - 2\left\lceil \frac{n}{2} \right\rceil \ge 0, \end{aligned}$$

we have

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n, F).$$

If $c = n$, the equality holds only if $G_1 = G_2$ are n-vertex extramal graphs for F. Suppose $c = n - 1$ and the equality holds. Observe that equation (2.3) is equal to zero only when $r = 2$ and n is odd. Thus if $c = n - 1$, then the equality could only be achieved when $r = 2$, n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2 = G_1 + K_1$. $\square$

3 General Version

TODO: add introduction.

Theorem 3.1. *For all n and graph F,*

$$\mathrm{ex}_2(n,m,F) = m(1+o(1))\mathrm{ex}(n,F)$$

as $m \to \infty$.

Proof. Let $r = v(F)$. Pick $\epsilon > 0$. Reorder $G_1, \ldots, G_m$ so that $G_1, \ldots, G_{m'}$ are all the G_i's containing at least $(1+\epsilon)\mathrm{ex}(n,F)$ edges. A theorem of Simonovits states that G contains at least δn^r copies of F for some $\delta = \delta(\epsilon)$. Since there can be at most $\binom{n}{r}$ copies of F across all G_i's,

$$m'\delta n^r \leq \binom{n}{r} \leq n^r \implies m' \leq \frac{1}{\delta}.$$

It now follows that

$$\begin{aligned}\sum_{i=1}^{m} e(G_i) &= \sum_{i=1}^{m'} e(G_i) + \sum_{i=m'+1}^{m} e(G_i) \\ &\leq \frac{1}{\delta}\binom{n}{2} + \left(m - \frac{1}{\delta}\right)(1+\epsilon)\mathrm{ex}(n,F) \\ &= m\left[1+\epsilon+\frac{1}{m\delta}\left(\frac{\binom{n}{2}}{\mathrm{ex}(n,F)} - (1+\epsilon)\right)\right]\mathrm{ex}(n,F).\end{aligned}$$

Since ϵ is arbitrary, the result follows. $\square$

Theorem 3.2. *For large enough n, suppose that $G_1, \ldots, G_m$ are graphs on common vertex set $[n]$ with no copy of F contained in any k of the G_i's. If there exists extremal F-free subgraph H on n vertices such that $\binom{m}{k}\Delta(H) = o(n^{1/2})$, then*

$$\mathrm{ex}_2(n,m,F) = (k-1)\binom{n}{2} + \mathrm{ex}(n,F)\binom{m}{k}.$$

Proof. For $S \subseteq [m]$, let E_S denote the set of edges that are contained in exactly $\{G_i\}_{i\in S}$. Then

$$\sum_{i=1}^{m} e(G_i) = \sum_{S\subseteq[m]} |S||E_S| \leq (k-1)\binom{n}{2} + \sum_{S\subseteq[m],|S|\geq k} (|S|-k+1)|E_S|.$$

Let $A_S = \bigcup_{T \supseteq S} E_T$, i.e. the set of edges that are contained in all G_i with $i \in S$. When $|S| \geq k$, the edge set A_S is F-free and thus

$$\sum_{T \supseteq S} |E_T| \leq \mathrm{ex}(n, F).$$

Hence,

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq k}} (|S| - k + 1)|E_S| = \sum_{\substack{S \subseteq [m], \\ |S| = k}} \sum_{T \subseteq S} \frac{(|T| - k + 1)|E_T|}{\binom{|T|}{k}} \leq \sum_{\substack{S \subseteq [m], \\ |S| = k}} \sum_{T \subseteq S} |E_T| \leq \binom{m}{k} \mathrm{ex}(n, F),$$

as each $T \in [m]$ with $|T| \geq k$ is counted $\binom{|T|}{k}$ times in total and $|T| - k + 1 \leq \binom{|T|}{k}$. This proves the upper bound.

Now we show the bound is tight. In particular, we need to show there exists a construction such that the graph with edge set E_S is an extremal F-free graph, for all $S \subseteq [m]$ of size k. Let $M = \binom{m}{k}$ and $H_1, \ldots, H_M$ be copies of an extremal F-free graph on n vertices with $\Delta(H_i) = o(n^{1/2})$ for all i. It suffices to show that we can embed each H_i onto $[n]$ such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each H_i and iteratively decrease the number of intersecting edges. Define a (u, v, i)-swap by swapping the embedding of vertex u and v of H_i, i.e. replacing each edge $\{u, w\} \in E(H_i)$ with the edge $\{u, w\}$ and each edge $\{v, w\} \in E(H_i)$ with the edge $\{v, w\}$. This perserves the type of isomorphism of H_i. Given a vertex v, let $N(v) = N_{H_1}(v) \cup \cdots \cup N_{H_M}(v)$. Suppose there exists an intersecting edge $\{u, w\} \in E(H_i) \cap E(H_j)$. Since $|N(u)| \leq M \cdot \Delta(H_i) = o(n^{1/2})$, $|N(u) \cup N(N(u))| = o(n)$ so there exists a vertex $v \notin N(u) \cup N(N(u))$. Since $N(u) \cap N(v) = \emptyset$, performing a (u, v, i)-swap reduces the number of intersecting edges. The result now follows from iterating this process. $\square$

3.1 Triangle F

Consider F to be a triangle. Simply counting the number of triangles in each G_i shows the following:

Theorem 3.3. *For all n, m and $\epsilon > 0$,*

$$\mathrm{ex}_2(n, m, K_3) < \left(m \cdot \frac{1 + \epsilon}{4} + \frac{1}{2\epsilon} - \frac{1}{2} \right) n^2.$$

Claim 4. *There are less than $\frac{2}{\epsilon}$ number of G_i's with at least $(1 + \epsilon)\frac{n^2}{4}$ edges.*

Proof. Suppose $e(G_i) \geq (1 + \epsilon)\frac{n^2}{4}$ for $1 \leq i \leq k$. Let $K_3(G)$ denote the number of triangles in graph G. By the Moon-Moser inequality,

$$K_3(G_i) \geq \epsilon(1 + \epsilon)\frac{n^3}{12}.$$

Since there are no overlapping traingles from different G_i's,

$$\binom{n}{3} \geq \sum_{i=1}^{k} K_3(G_i) \geq \frac{\epsilon(1 + \epsilon)}{12} kn^3.$$

Rearranging yields $k < \frac{2}{\epsilon}$. $\square$

By the claim,

$$\sum_{i=1}^{m} e(G_i) < \frac{2}{\epsilon}\binom{n}{2} + \left(m - \frac{2}{\epsilon}\right)(1+\epsilon)\frac{n^2}{4} \leq m(1+\epsilon)\frac{n^2}{4} + (1-\epsilon)\frac{n^2}{2\epsilon},$$

which proves Theorem 3.3.

When $m = 2$,

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_{1,2}) \leq \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor,$$

which meets the benchmark bound and so $\mathrm{ex}_2(n, 2, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor$.

This result is also true for $m = 3$:

Proposition 3.4. *For all n,*

$$\mathrm{ex}_2(n, 3, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Proof. Define $H_k \subseteq G$ be the graph with edges contained in at least k number of G_i's and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \leq \frac{n^2}{2}$. Notice H_2 must not contain any triangles with two edges in H_3, so

$$e(H_2) + e(H_3) \leq \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}|.$$

Let H_3' be the graph with the same vertex set as H_3 and edge set $\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}$. It suffices to show that $\frac{n}{2} \geq e(H_3) - e(H_3')$.

Let $d_1 \geq d_2 \geq \cdots \geq d_n$ and $f_1 \geq f_2 \geq \cdots \geq f_n$ each be the degree sequence of H_3 and H_3', respectively. We show that $f_i \geq d_i - 1$ for all i. Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i. Let $S_i = |N_{H_3}(v_1) \cup \cdots \cup N_{H_3}(v_i)|$. Since

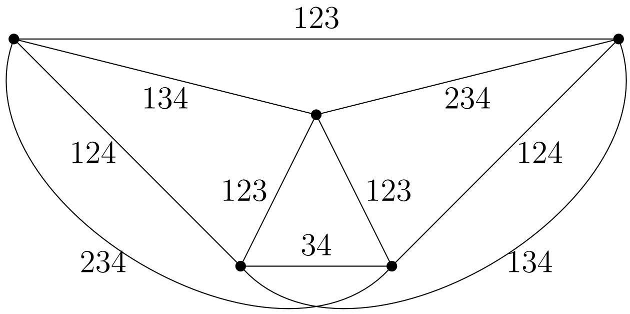
$$\sum_{u \in S_i} d_{H_3}(u) \geq d_1 + \cdots + d_i,$$

we have that $|S_i| \geq i$. But then $S_i \backslash \{u_1, \ldots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H_3'}(u) \geq d_i - 1$. Hence, $f_i \geq d_i - 1$ for all i, which yields

$$e(H_3') = \frac{1}{2}\sum_{i=1}^{n} f_i \geq \frac{1}{2}\sum_{i=1}^{n}(d_i - 1) = e(H_3) - \frac{n}{2}.$$

□

However, the bound in Proposition 3.1 is not tight for $m \geq 4$, as shown in the following graph:



The number on each edge denotes the set of G_i's that contain the edge.

The above graph contains 29 edges, which exceeds the bound $\binom{5}{2} + 3\lfloor \frac{5^2}{4} \rfloor = 28$ by 1. By blowing up the above graph, we can construct a graph with $n \in 10\mathbb{Z}$ vertices that contains

$$5\binom{n/5}{2} + 29 \cdot \frac{(n/5)^2}{4}$$

edges, which exceeds the bound $\binom{n}{2} + 3\lfloor \frac{n^2}{4} \rfloor$ by $n^2/100$.

3.2 Bipartite F

In this section, we discuss the case where F is bipartite. In particular, we focus on the cases where $F \subseteq K_{2,2}$ is P_2, a path of length 2, or M_2, a matching with two edges.

Theorem 3.5.

$$\mathrm{ex}_2(n, m, P_2) \le \left(\frac{1}{2} + o(1)\right) n^2\sqrt{m}$$

as $n \to \infty$ or $m \to \infty$.

Proof. Since there are no overlapping P_2's in different G_i's,

$$\sum_{i=1}^{m} \#\{P_2 \subseteq G_i\} \le \#\{P_2 \subseteq G\}$$

For each G_i, each vertex v in G_i and two of its neighbors form one unique P_2, so

$$\#\{P_2 \subseteq G_i\} = \sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2}.$$

And by Jensen's inequality,

$$\sum_{v \in V(G_i)} \binom{d_{G_i}(v)}{2} \ge n\binom{d_{G_i}(v)/n}{2} = n\binom{2e(G_i)/n}{2} \ge \frac{2(e(G_i))^2}{n} - e(G_i).$$

On the other hand, since each three vertices in G can form at most three P_2's,

$$\#\{P_2 \subseteq G\} \le 3\binom{n}{3} \le \frac{n^3}{2}.$$

Combining the above inequalities yields

$$\frac{2m}{n}\left(\frac{1}{m}\sum_{i=1}^{m} e(G_i)\right)^2 - \sum_{i=1}^{m} e(G_i) \overset{Jensen's}{\leq} \sum_{i=1}^{m} \frac{2(e(G_i))^2}{n} - e(G_i) \leq \frac{n^3}{2},$$

and solving the quadratic equation gives

$$\sum_{i=1}^{m} e(G_i) \leq mn \cdot \frac{1+\sqrt{4n^2/m+1}}{4} = \left(\frac{1}{2}+o(1)\right) n^2\sqrt{m},$$

as $n \to \infty$ or $m \to \infty$. □

When $m = n$, the following projective plane construction shows the above bound is tight asymptotically:

Theorem 3.6.

$$\mathrm{ex}_2(n, n, P_2) = \left(\frac{1}{2}+o(1)\right) n^{5/2},$$

as $n \to \infty$.

Proof. It suffices to show the tightness of the bound in Theorem 3.5. Consider a finite projective plane of order q. The projective plane has $n = q^2 + q + 1$ points and n lines. Let $S_1, \ldots, S_n \subseteq [n]$ be the n lines of the projective plane. Note that each line S_i contains $q + 1$ points, and the intersection of any two distinct lines S_i, S_j contains $|S_i \cap S_j| = 1$ point. Define $G_1, \ldots, G_n$ to be graphs on $[n]$, each with edge set $E(G_i) = \{\{j, k\} \subseteq [n] : j \neq k,\ j + k \in S_i \mod n\}$. Note that the intersection of distinct G_i, G_j is P_2 free: since $|S_i \cap S_j| = 1$, if $\{a, b\}, \{a, c\} \in E(G_i) \cap E(G_j)$, then $a+b = a+c$ so $b = c$. We now count the number of edges in $G_1, \ldots, G_n$. Since $|S_i| = q + 1$, for each point $j \in [n]$, there are $q + 1$ choices for $k \in [n]$ such that $j + k \in S_i$. But then we have to avoid counting the same edge twice and loops, so the number of edges in G_i is

$$e(G_i) = \frac{n(q+1) - \#\text{loops counted for } G_i}{2}.$$

If $j \in [n]$ is even, then $k = j/2$ is the unique number in $[n]$ such that $k + k = j \mod n$. If $j \in [n]$ is odd, then $k = (n + j)/2$ is the unique number in $[n]$ such that $k + k = j \mod n$, as n is even. Hence, for each $j \in S_i$, there exists a unique $k \in [n]$ such that $k + k = j \mod n$, and thus

$$\#\text{loops counted for } G_i = |S_i| = q + 1.$$

Since $q + 1 = (1 + o(1))n^{1/2}$, the number of edges in $G_1, \ldots, G_n$ is

$$\sum_{i=1}^{n} e(G_i) = n \cdot \frac{n(q+1) - (q+1)}{2} = \left(\frac{1}{2}+o(1)\right) n^{5/2},$$

as $n \to \infty$. □

Theorem 3.7. *For all* n, m,

$$\mathrm{ex}_2(n, m, M_2) \leq n^{5/2}.$$

Proof. Notice that $\#\{M_2 \subseteq G\} = \binom{e(G_i)}{2}$. On the other hand, each four vertices in G can form at most three M_2's, so $\#\{M_2 \subseteq G\} \le 3\binom{n}{4} \le \frac{n^4}{8}$. By the same argument as in Theorem 3.4, we have

$$\sum_{i=1}^{n} \binom{e(G_i)}{2} = \sum_{i=1}^{n} \#\{M_2 \subseteq G_i\} \le \#\{M_2 \subseteq G\} \le \frac{n^4}{8}.$$

By Jensen's inequality,

$$\sum_{i=1}^{n} \binom{e(G_i)}{2} \ge n\binom{\sum_{i=1}^{n} e(G_i)/n}{2} = \frac{1}{2n}\left[\left(\sum_{i=1}^{n} e(G_i)\right)^2 - n\sum_{i=1}^{n} e(G_i)\right].$$

Combining the above inequalities yields

$$\left(\sum_{i=1}^{n} e(G_i)\right)^2 - n\sum_{i=1}^{n} e(G_i) \le \frac{n^5}{4},$$

and solving the quadratic inequality gives

$$\sum_{i=1}^{n} e(G_i) \le n^{5/2}.$$

□

We may obtain an exact result if we forbid both P_2 and M_2 at the same time:

Theorem 3.8. *For all n, m,*

$$\mathrm{ex}_2(n, m, \{P_2, M_2\}) = n^2 - n.$$

Proof. Denote the set of G_i's as $\{G_i\} = \{G_1, \ldots, G_n\}$, and the set of distinct pairs of G_i's as $\{G_i\}^2 = \{\{G_j, G_k\} : j \ne k\}$. Consider the bipartite graph H with vertex set $V(H) = \{G_i\} \sqcup E(K_n)$ and edge set $E(H) = \{\{G_j, e\} \in \{G_i\} \times E(K_n) : e \in G_j\}$. Define $\phi : \{G_i\}^2 \to 2^{E(K_n)}$ by sending each $\{G_j, G_k\}$ to their common edge set $E(G_j) \cap E(G_k)$. Notice that each distinct G_j, G_k have at most one edge in common, so $|\phi(G_j, G_k)| \le 1$. On the other hand, each edge $e \in E(G)$ can be obtained via ϕ by $\binom{d_H(e)}{2}$ possible distinct pairs (G_j, G_k), and thus $|\phi^{-1}(e)| = \binom{d_H(e)}{2}$. But then

$$\binom{n}{2} \ge \sum_{(G_j, G_k) \in \{G_i\}^2} |\phi(G_j, G_k)| = \sum_{e \in E(K_n)} |\phi^{-1}(e)| = \sum_{e \in E(K_n)} \binom{d_H(e)}{2}.$$

By Jensen's inequality,

$$\sum_{e \in E(K_n)} \binom{d_H(e)}{2} \ge \binom{n}{2}\binom{\sum_{e \in E(K_n)} d_H(e)/\binom{n}{2}}{2} = \binom{n}{2}\binom{\sum_{i=1}^{n} e(G_i)/\binom{n}{2}}{2}.$$

Combining the above inequalities yields

$$2\binom{n}{2}^2 \ge \left(\sum_{i=1}^{n} e(G_i)\right)^2 - \binom{n}{2}\sum_{i=1}^{n} e(G_i),$$

and the result now follows from solving the quadratic inequality.

To see that this bound is tight, consider the construction such that for each distinct $i, j \in [n]$, $E(G_i) \cap E(G_j)$ contains exactly one unique edge $e \in K_n$. The number of edges in this construction is $2\binom{n}{2} = n^2 - n$. $\square$