

# Double Turán Problem

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## Question

Given a graph  $F$ , how many edges can an  $n$ -vertex graph have while containing no copy of  $F$  as a subgraph?

What is the hypergraph Turán problem?

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Given an  $r$ -uniform graph  $F$ , how many edges can an  $n$ -vertex  $r$ -uniform hypergraph have while containing no copy of  $F$  as a subgraph?

Let  $F$  be a graph. We call the following quantity the *Turán number* or *extremal number* of  $F$ :

## Definition

$$\text{ex}(n, F) := \max\{e(G) : |V(G)| = n \text{ and } F \not\subseteq G\}$$

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examples of turan graphs

## Turán's theorem

The maximum number of edges in an  $n$ -vertex graph containing no clique of order  $r + 1$  is  $e(T_r(n))$ , with equality only for  $T_r(n)$ .



## Erdős-Stone Theorem, Simonovits' Theorem

Let  $F$  be any graph of chromatic number  $r + 1 \geq 3$ . Then  
 $\text{ex}(n, F) = (1 + o(1)) T_r(n)$  as  $n \rightarrow \infty$ .

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What is the value of  $\phi(m, n, F) = \max \sum_{i=1}^m |G_i|$ ?

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We call graphs  $G_1, G_2, \dots, G_m$  *induced* if each  $G_i$  is an induced subgraph of  $\bigcup_{j=1}^m G_j$ .

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Let graphs  $G_1, G_2, \dots, G_m$  be induced and double  $F$ -free.

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What is the value of  $\phi^*(m, n, F) = \max \sum_{i=1}^m e(G_i)$ ?



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# Motivation

Double Turán problems are closely related to hypergraph Turán problems through *link graphs*.

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## Definition

For  $i \in V(H)$ , the *link graph* of  $i$ , denoted  $H_i$ , is the graph with  $V(H_i) = V(H) \setminus \{i\}$  and  $E(H_i) = \{\{j, k\} : \{i, j, k\} \in E(H)\}$ .

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examples of link graphs

# Motivation

Let  $F^+$  denote the triple system on  $[n]$  with vertex set  $V(F) \cup \{x, y\}$  and edge set  $\{e \cup \{x\}, e \cup \{y\} : e \in E(F)\}$ .

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If  $H$  is an  $F^+$ -free triple system on  $[n]$ , then the link graphs  $H_1, \dots, H_n$  are double  $F$ -free.



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$$\text{ex}(n, F^+) \leq \phi(n, n, F)$$

# Motivation

Let  $\bar{F}$  be the graph consisting of all pairs contained in triples in  $F^+$ .

## Generalized Turán problem

What is the maximum number  $\text{ex}(n, \bar{F}, K_3)$  of triangles in a graph  $H$  on  $[n]$  with no copy of  $\bar{F}$  as a subgraph?

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## Observation

$H_1, H_2, \dots, H_n$  are induced and double  $F$ -free.

$$\text{ex}(n, \bar{F}, K_3) \leq \phi^*(n, n, F)$$

# The Induced Case

## Conjecture

Let  $F$  be any non-empty graph and  $m, n \geq 1$ . Then

$$\phi^*(m, n, F) = \Theta(m \cdot \text{ex}(n, F) + n^2).$$

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$$\phi^*(m, n, F) \geq \max\left\{\binom{n}{2}, m \cdot \text{ex}(n, F)\right\}.$$

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Let  $F$  be any non-empty graph and  $m, n \geq 1$ . Then

$$\phi^*(m, n, F) = \Theta(m \cdot \text{ex}(n, F) + n^2).$$

Since

$$\text{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2})$$

the conjecture implies

$$\text{ex}(n, K_{2,2,2}, K_3) \leq O(n^{5/2})$$



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Let  $F$  be any non-empty graph and  $m, n \geq 1$ . Then

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Conjecture holds for non-bipartite  $F$ :

## Theorem A

For all  $m \geq 3$  and non-bipartite  $F$ , if  $n$  is large enough, then

$$\phi^*(m, n, F) = m \cdot \text{ex}(n, F),$$

with equality only for identical extremal  $n$ -vertex  $F$ -free graphs.

# The Induced Case

## Theorem B

Let  $m, n, r \geq 3$ . Then

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced  $K_r$ -free graphs  $G_1, G_2, \dots, G_m$  only if  $G_1 = G_2 = \dots = G_m = T_{r-1}(n)$ .

# Proof of Theorem B

## Proof Roadmap

- Step 1: Reduce to the case of smaller  $m$
- Step 2: Further reduce to an optimization problem
- Step 3: Solve the optimization problem

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## Step 1: Reduce to the case of smaller $m$

Let  $G_1, \dots, G_m$  be induced double  $F$ -free graphs on  $[n]$ .

### Lemma 1

For  $2 \leq k \leq m$ ,

$$\phi^*(m, n, F) \leq \frac{m}{k} \cdot \phi^*(k, n, F).$$

Moreover, suppose  $\sum_{i=1}^k e(G_i) = \phi^*(k, n, F)$  only if  $G_1 = \dots = G_k$ . Then  $\sum_{i=1}^m e(G_i) = \phi^*(m, n, F)$  only if  $G_1 = \dots = G_m$ .

## Step 1: Reduce to the case of smaller $m$

Put  $G_{i+m} = G_i$  for all  $i \in [m]$ .

$$\sum_{i=1}^m e(G_i) = \frac{1}{k} \sum_{i=1}^m [e(G_i) + \cdots + e(G_{i+k-1})]$$

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Since  $e(G_i) + \cdots + e(G_{i+k-1}) \leq \phi^*(k, n, F)$ ,

$$\sum_{i=1}^m e(G_i) \leq \frac{m}{k} \cdot \phi^*(k, n, F).$$

## Step 1: Reduce to the case of smaller $m$

Suppose  $\sum_{i=1}^m e(G_i) = \frac{m}{k} \cdot \phi^*(k, n, F)$  and  $G_1 \neq G_2$ .



## Step 1: Reduce to the case of smaller $m$

Suppose  $\sum_{i=1}^m e(G_i) = \frac{m}{k} \cdot \phi^*(k, n, F)$  and  $G_1 \neq G_2$ .

By assumption  $\sum_{i=1}^k e(G_i) < \phi^*(k, n, F)$ .

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By assumption  $\sum_{i=1}^k e(G_i) < \phi^*(k, n, F)$ .

$\implies e(G_i) + \cdots + e(G_{i+k-1}) > \phi^*(k, n, F)$  for some  $i > 1$ , contradiction.

## Step 2: Further reduce to an optimization problem

### Proof Roadmap

- Step 1: Reduce to the case of smaller  $m$
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[Show the picture for  $G_1, G_2$ .]

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### Observation

If  $G_1, G_2$  intersects in  $t$  vertices, Then

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2\text{ex}(t, F)$$

## Step 2: Further reduce to an optimization problem

Define  $f(n, t, F) := \binom{n-t}{2} + (n-t)t + 2\text{ex}(t, F)$ .

### Lemma 2

Let  $F$  be some graph. For  $n \geq 1$ ,

$$\phi^*(2, n, F) = \max_{0 \leq t \leq n} f(n, t, F).$$

# Step 3: Solve the optimization problem

## Proof Roadmap

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By Lemma 1, it suffices prove the case  $m = 3$ .



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Let  $G_1, G_2, G_3$  be induced double  $K_r$ -free graphs, such that  $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, K_r)$  and  $e(G_1) \geq e(G_2) \geq e(G_3)$ .

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Since  $\phi^*(3, n, K_r) \geq 3\text{ex}(n, K_r)$ , we must have  $e(G_1) + e(G_2) \geq 2\text{ex}(n, K_r)$ .

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Let  $G_1, G_2, G_3$  be induced double  $K_r$ -free graphs, such that  $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, K_r)$  and  $e(G_1) \geq e(G_2) \geq e(G_3)$ .

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Since  $G_1, G_2, G_3$  are induced, we only need to show  $G_1 = G_2 = T_{r-1}(n)$ .

## Step 3: Solve the optimization problem

Let  $t = |V(G_1 \cap G_2)|$ . By Turán's Theorem,

$$\text{ex}(t, K_r) - \text{ex}(t-1, K_r) = t - \left\lceil \frac{t}{r-1} \right\rceil.$$

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It follows that

$$\begin{aligned} f(n, t, K_r) - f(n, t-1, K_r) &= -t + 1 + 2[\text{ex}(t, K_r) - \text{ex}(t-1, K_r)] \\ &= t + 1 - 2 \left\lceil \frac{t}{r-1} \right\rceil. \end{aligned} \tag{1}$$

## Step 3: Solve the optimization problem

For  $r \geq 4$ ,

$$f(n, t, K_r) - f(n, t - 1, K_r) = t + 1 - 2 \left\lceil \frac{t}{r - 1} \right\rceil > 0$$

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$\implies t = n$  so that  $e(G_1) + e(G_2) \geq 2\text{ex}(n, F)$ .



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$\implies t = n$  so that  $e(G_1) + e(G_2) \geq 2\text{ex}(n, F)$ .

$\implies \phi^*(2, n, K_r) = 2\text{ex}(n, F)$  and  $G_1 = G_2 = T_{r-1}(n)$ .

This solves the case  $r \geq 4$ .

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For  $r = 3$ ,

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$\implies f(n, t, K_3)$  is non-decreasing on  $t$  and  $f(n, t, K_3) > f(n, t, K_3)$  for even  $t$ .

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$\implies t = n$  or  $t = n - 1$  so that  $e(G_1) + e(G_2) \geq 2\text{ex}(n, F)$ .

$\implies \phi^*(2, n, K_3) = 2\text{ex}(n, K_3)$ , and either  $G_1 = G_2 = T_2(n)$  or  $G_2 = T_2(n - 1)$  and  $G_1 = G_2 + K_1$ .

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Suppose  $G_2 = T_2(n-1)$  and  $G_1 = G_2 + K_1$ .

Since  $e(G_1) + e(G_2) + e(G_3) \geq 3\text{ex}(n, K_3)$ ,

$$e(G_3) = \text{ex}(n, K_3) > T_2(n-1) = e(G_2),$$

contradiction.



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contradiction.

This solves Theorem B.

# Extra Ingredients to Prove Theorem A

- First show that  $t = |V(G_1 \cap G_2)| \geq \sqrt{n}$ .
- For large enough  $t$ , any extremal  $t$ -vertex  $F$ -free graph contains a spanning  $T_{r-1}(t)$ .