

Double Turán Problem

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- 3 Induced Double Turán Problem
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What is the Turán problem?

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Question

Given a graph F , how many edges can an n -vertex graph have while containing no copy of F as a subgraph?

Let F be a graph. We call the following quantity the *Turán number* or *extremal number* of F :

Definition

$$\text{ex}(n, F) := \max\{e(G) : |V(G)| = n \text{ and } F \not\subseteq G\}$$

Turán's theorem

The maximum number of edges in an n -vertex graph containing no clique of order $r + 1$ is $e(T_r(n))$, with equality only for $T_r(n)$.

Erdős-Stone Theorem, Simonovits' Theorem

Let F be any graph of chromatic number $r + 1 \geq 3$. Then
 $\text{ex}(n, F) = (1 + o(1)) T_r(n)$ as $n \rightarrow \infty$.

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Question

What is the value of $\phi(m, n, F) = \max \sum_{i=1}^m |G_i|$?

Double Turán Problem

Double Turán problems are closely related to Turán problems for 3-uniform hypergraphs H through *link graphs*.

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Definition

For $i \in V(H)$, define graph H_i with

$$V(H_i) = V(H) \setminus \{i\} \quad \text{and} \quad E(H_i) = \{\{j, k\} : \{i, j, k\} \in E(H)\}.$$

Double Turán Problem

Example: Octahedron-free 3-uniform hypergraph H

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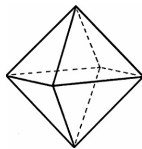


Figure: Octahedron O

Double Turán Problem

Example: Octahedron-free 3-uniform hypergraph H

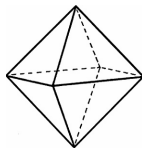


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H is octahedron-free $\implies H_1, H_2, \dots, H_n$ are double C_4 -free.

Double Turán Problem

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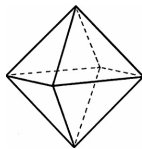


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H is octahedron-free $\implies H_1, H_2, \dots, H_n$ are double C_4 -free.

$$\boxed{\text{ex}(n, O) \leq \phi(n, n, C_4)}$$

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Let graphs G_1, G_2, \dots, G_m be induced and double F -free.

Question

What is the value of $\phi^*(m, n, F) = \max \sum_{i=1}^m e(G_i)$?

Induced Double Turán Problem

Generalized Turán problem

What is the maximum number $ex(n, F, K_3)$ of triangles in a graph H on $[n]$ with no copy of F as a subgraph?

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$$V(G_i) = V(G) \quad \text{and} \quad E(G_i) = \{\{j, k\} : \{i, j\}, \{j, k\}, \{i, k\} \in E(G)\}$$

Induced Double Turán Problem

Ex. Octahedron-free graph G .

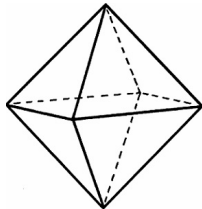


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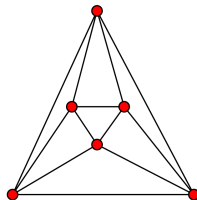


Figure: Octahedron Graph $K_{2,2,2}$

Induced Double Turán Problem

Ex. Octahedron-free graph G .

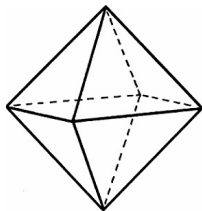


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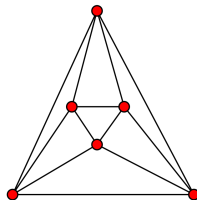


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G is $K_{2,2,2}$ -free $\implies G_1, G_2, \dots, G_n$ are double $K_{2,2}$ -free.

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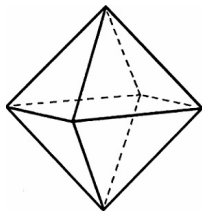


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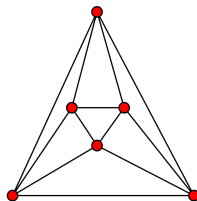


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G is $K_{2,2,2}$ -free $\implies G_1, G_2, \dots, G_n$ are double $K_{2,2}$ -free.

$$\text{ex}(n, K_{2,2,2}, K_3) \leq \phi(n, n, K_{2,2})$$

Theorem A

For $m \geq 3$ and non-bipartite F , if n is large enough, then

$$\phi^*(m, n, F) = m \cdot \text{ex}(n, F),$$

with equality only for identical extremal n -vertex F -free graphs.

Main Results

Theorem A

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$$\phi^*(m, n, F) = m \cdot \text{ex}(n, F),$$

with equality only for identical extremal n -vertex F -free graphs.

Theorem B

For $m, n, r \geq 3$,

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced K_r -free graphs G_1, G_2, \dots, G_m only if $G_1 = G_2 = \dots = G_m = T_{r-1}(n)$.

Main Results

We need the idea of (m, n, k) -blowup to state the result for $\phi(m, n, K_r)$.

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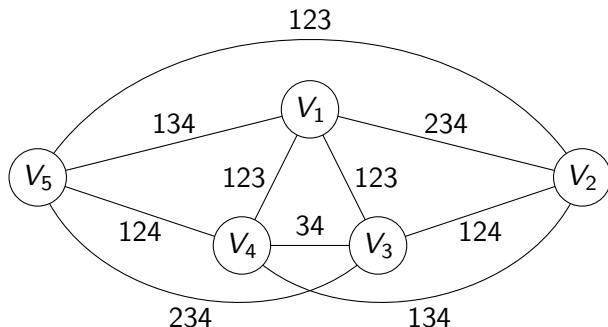


Figure: Example of an $(4, n, 5)$ -blowup not containing a double K_3 .

Main Results

Let $f(m, n, r)$ denote the maximum possible sum of edges in an double K_r -free (m, n, k) -blowup with $k \leq \binom{m}{2}$.

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Theorem C

For $n \geq 1$,

①

$$\phi(3, n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

② if $r \geq 2$ and $m \geq 1$,

$$\phi(m, n, K_r) = f(m, n, r).$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{f(4, n, 3)}{\binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor} > 1$$

Conjecture

Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \text{ex}(n, F) + n^2).$$

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$$\phi^*(m, n, F) \geq \max\left\{\binom{n}{2}, m \cdot \text{ex}(n, F)\right\}.$$

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By Theorem 1, the conjecture is true when F is non-bipartite.

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Let F be any non-empty graph and $m, n \geq 1$. Then

$$\phi^*(m, n, F) = \Theta(m \cdot \text{ex}(n, F) + n^2).$$

Since

$$\text{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2})$$

the conjecture implies

$$\text{ex}(n, K_{2,2,2}, K_3) \leq O(n^2)$$

Theorem D

Let F be a graph. If there exists an extremal F -free n -vertex graph with maximum degree at most \sqrt{n}/m^2 , then

$$\phi(m, n, F) = \binom{n}{2} + \binom{m}{2} \text{ex}(n, F).$$

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$$\phi(m, n, F) = \binom{n}{2} + \binom{m}{2} \text{ex}(n, F).$$

If P is a path of length 2 and $m = o(n^{1/4})$,

$$\binom{n}{2} + m - 1 \leq \phi^*(m, n, P) \leq \phi(m, n, P) = \binom{n}{2} + \binom{m}{2} \left\lfloor \frac{n}{2} \right\rfloor.$$

Theorem E

Let P be a path with two edges. Then $\phi(n, n, P) = \Omega(n^{5/2})$, whereas $\phi^*(n, n, P) = o(n^{5/2})$, as $n \rightarrow \infty$. In particular,

$$\lim_{n \rightarrow \infty} \frac{\phi^*(n, n, P)}{\phi(n, n, P)} = 0.$$

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This shows that $\phi(n, n, P)$ and $\phi^*(n, n, P)$ are very different problems.

Theorem B

Let $m, n, r \geq 3$. Then

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)),$$

with equality for induced K_r -free graphs G_1, G_2, \dots, G_m only if $G_1 = G_2 = \dots = G_m = T_{r-1}(n)$.

Proof Roadmap

- Step 1: Reduce to the case of smaller m
- Step 2: Further reduce to an optimization problem
- Step 3: Solve the optimization problem

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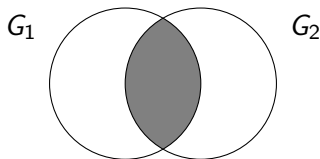
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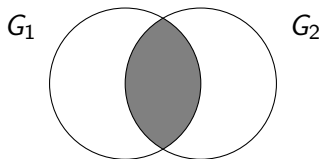
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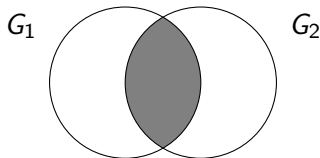


Observation

If G_1, G_2 intersects in t vertices, Then

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2\text{ex}(t, F)$$

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\implies we are done if the unique maximum is at $t = n$.

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We need to show

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$$\implies \sum_{i=1}^m e(G_i) = \sum_{S \subseteq [m]} |S| |E_S| \leq \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq 2} (|S| - 1) |E_S|.$$

Proof of Theorem C: Upper Bound

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S| - 1) |E_S| = \sum_{\substack{S \subseteq [m], \\ |S| = 2}} \sum_{T \supseteq S} \frac{(|T| - 1) |E_T|}{\binom{|T|}{2}} \leq \sum_{\substack{S \subseteq [m], \\ |S| = 2}} \sum_{T \supseteq S} |E_T|,$$

as each $T \in [m]$ with $|T| \geq 2$ is counted $\binom{|T|}{2}$ times.

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$$\Rightarrow \sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S| - 1) |E_S| \leq \sum_{\substack{S \subseteq [m] \\ |S|=2}} \sum_{T \supseteq S} |E_T| \leq \binom{m}{2} \text{ex}(n, F)$$

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IDEA: start with any embedding and iteratively decrease overlapping edges

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$\implies (u, v, i)$ -swap preserves graph isomorphism.

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Keep swapping until no overlapping edges left