MATH 180C: Homework #1

Due on Apr 12, 2024 at 23:59pm

Professor Carfagnini

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Problem 1

A pure birth process starting from $X(0) = 0$ has birth parameters $\lambda_0 = 1$, $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 5$. Let W_3 be the random time that it takes the process to reach state 3.

(a) Write W_3 as a sum of sojourn times and thereby deduce that the mean time is $E[W_3] = \frac{11}{6}$.

Proof. We write $W_k = \sum_{i=0}^{k-1} S_i$, where S_i's are the sojourn times, with $S_i \sim \text{Exp}(\lambda_i)$. Hence,

$$E[W_3] = \sum_{i=0}^{2} E[S_i] = \sum_{i=0}^{2} \frac{1}{\lambda_i} = \frac{11}{6}.$$

□

(b) Determine the mean of $W_1 + W_2 + W_3$.

Proof.

$$E[W_1 + W_2 + W_3] = E[W_1] + E[W_2] + E[W_3] = \frac{1}{\lambda_2} + \frac{2}{\lambda_1} + \frac{3}{\lambda_0} = \frac{25}{6}.$$

□

(c) What is the variance of W_3?

Proof.

$$Var(W_3) = Var(S_0) + Var(S_1) + Var(S_2) = \sum_{i=0}^{2} \frac{1}{\lambda_i^2} = \frac{49}{36}.$$

□

Problem 2

Consider an experiment in which a certain event will occur with probability αh and will not occur with probability $1 - \alpha h$, where α is a fixed positive parameter and h is a small ($h < \frac{1}{\alpha}$) positive variable. Suppose that n independent trials of the experiment are carried out, and the total number of times that the event occurs is noted. Show that

(a) The probability that the event never occurs during the n trials is $1 - n\alpha h + o(h)$;

Proof.

$$\begin{aligned}
P\{\text{never occurs during the } n \text{ trials}\} &= \prod_{i=1}^{n} P\{\text{doesn't occur in the } i\text{th trial}\} \\
&= (1 - \alpha h)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} (-\alpha h)^k \\
&= 1 - n\alpha h + h^2 \sum_{k=2}^{n} \binom{n}{k} (-\alpha)^k h^{k-2} \\
&= 1 - n\alpha h + o(h).
\end{aligned}$$

□

(b) The probability that the event occurs exactly once is $n\alpha h + o(h)$;

Proof.

$$\begin{aligned}
P\{\text{occurs exactly once in } n \text{ trials}\} &= \sum_{i=1}^{n} P\{\text{occurs only in the } i\text{th trial}\} \\
&= \sum_{i=1}^{n} \alpha h (1 - \alpha h)^{n-1} \\
&= n\alpha h \left(1 - h \sum_{k=1}^{n-1} \binom{n}{k} (-\alpha)^k h^{k-1} \right)^{n-1} \\
&= n\alpha \left(h^{\frac{1}{n-1}} - h^{\frac{n}{n-1}} \sum_{k=1}^{n-1} \binom{n}{k} (-\alpha)^k h^{k-1} \right)^{n-1} \\
&= n\alpha (h^{\frac{1}{n-1}} + o(h))^{n-1} \\
&= n\alpha (h + o(h)) \\
&= n\alpha h + o(h).
\end{aligned}$$

□

(c) The probability that the event occurs twice or more is $o(h)$.

Proof.

$$\begin{aligned}
P\{\text{occurs more than once}\} &= 1 - (P\{\text{never occurs in } n \text{ trials}\} + P\{\text{occurs exactly once in } n \text{ trials}\}) \\
&= 1 - (1 - n\alpha h + n\alpha h + o(h)) = o(h).
\end{aligned}$$

□

Problem 3

Let W_k be the time to the kth birth in a pure birth process starting from $X(0) = 0$. Establish the equivalence

$$\Pr\{W_1 > t, W_2 > t + s\} = P_0(t)[P_0(s) + P_1(s)].$$

From this relation together with equation (6.7), determine the joint density for W_1 and W_2, and then the joint density of $S_0 = W_1$ and $S_1 = W_2 - W_1$.

Proof.

$$\begin{aligned}\Pr\{W_1 > t, W_2 > t+s\} &= \Pr\{W_2 > t+s | W_1 > t\} P_0(t) \\ &= (\Pr\{W_2 > t+s | W_1 > t, W_1 < t+s\} \Pr\{S_1 > s | W_1 > t\} + \Pr\{W_1 \geq t+s | W_1 > t\}) P_0(t) \\ &= P_1(s)(1 - P_0(s)) + P_0(s)P_0(t)\end{aligned}$$

□