

Double Turán Problems



Ray Tsai

Advisor: Professor Jacques Verstraëte

Department of Mathematics

University of California San Diego

In partial fulfillment of the requirements
for the Mathematics Honors Program

June 2025

Acknowledgements

I would like to thank Professor Jacques Verstraëte for introducing me to the world of mathematical research and for his invaluable guidance over these past two years. His patience and unwavering support, especially during moments when I struggled, have been instrumental in my growth.

I also thank my family for their utmost support in all aspects of my academic journey. Their belief in me has allowed me to focus on my studies and pursue my aspirations without distraction. I would not be where I am today without them.

Contents

1	Introduction	1
1.1	Problem Statement	2
1.2	Motivation : Link Graphs and Hypergraphs	2
1.3	Main Results : The Induced Case	3
1.4	Main Results : The Non-induced Case	5
1.5	Definitions and Notations	7
2	The Induced Double Turán Problem	8
2.1	Proof of Theorem 2	9
2.2	Proof of Theorem 1	10
2.3	Proof of Theorem 3	11
3	The Non-induced Double Turán Problem	14
3.1	Proof of Theorem 4	14
3.2	Proof of Theorem 5	15
3.3	Proof of Theorem 6	16
3.4	Proof of Theorem 7	18
4	Concluding Remarks	19

1 Introduction

This thesis focuses on a variation of the *Turán problem* in extremal combinatorics. The fundamental question in extremal hypergraph theory is determining the maximum number of edges in an n -vertex r -uniform graph that does not contain a prescribed r -uniform graph F as a subgraph. These maxima, denoted $\text{ex}(n, F)$, are referred to as the *extremal numbers* or *Turán numbers* for F . One of the cornerstones of extremal graph theory, concerning the case F is a clique, is Turán's Theorem [19]. To state the theorem, we need the *Turán graphs* $T_k(n)$, which denotes a complete multipartite graph with n vertices and k parts of size $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$.

Theorem A (Turán's Theorem). *The maximum number of edges in an n -vertex graph G containing no clique of order $r + 1$ is $e(T_r(n))$, with equality only if $G = T_r(n)$.*

Simonovits [5] observed via the Erdős-Stone Theorem [3] that the asymptotic value of $\text{ex}(n, F)$ may be obtained whenever F is non-bipartite:

Theorem B (Erdős-Stone Theorem, Simonovits' Theorem). *Let F be any graph of chromatic number $r + 1 \geq 3$. Then $\text{ex}(n, F) = (1 + o(1))T_r(n)$ as $n \rightarrow \infty$.*

There are a number of proofs of the Erdős-Stone Theorem. A very general framework involves *Szemerédi's Regularity Lemma*, which may be stated as follows. A pair (U, V) of disjoint sets of vertices in a graph G is called ϵ -regular if for any $X \subseteq U$ and $Y \subseteq V$ of size at least $\epsilon|U|$ and $\epsilon|V|$ respectively,

$$\left| \frac{e(X, Y)}{|X||Y|} - \frac{e(U, V)}{|U||V|} \right| < \epsilon.$$

The following was proved by Szemerédi [18]:

Theorem C (Szemerédi's Regularity Lemma). *For all $\epsilon > 0$, there exist m and M such that for every graph G , there exists a partition (V_1, V_2, \dots, V_k) of $V(G)$ such that $m \leq k \leq M$ and $|V_1| \leq |V_2| \leq \dots \leq |V_k| \leq |V_1| + 1$ and all but at most ϵk^2 pairs (V_i, V_j) are ϵ -regular.*

The value of $\text{ex}(n, F)$ for bipartite F is in general wide open, and the order of magnitude of $\text{ex}(n, K_{4,4})$ or $\text{ex}(n, C_8)$ is not known – see Füredi and Simonovits [8] for a history of the bipartite Turán problem. There is also no analog of the above theorems for r -uniform hypergraphs. The asymptotic value of $\text{ex}(n, K_k^r)$ is not known for any $k > r \geq 3$, where K_k^r denotes the complete r -uniform hypergraph on k vertices. The asymptotic value of $\text{ex}(n, K_4^3)$ was conjectured by Turán [19] to be $\frac{5}{9}\binom{n}{3}$, and this remains open despite decades of intensive research.

1.1 Problem Statement

In this thesis, we investigate closely related problems which we refer to as *double Turán problems*. To describe these problems, let G_1, G_2, \dots, G_m be graphs with the same vertex set $V(G_i) = [n]$ for $i \in [m]$.

Definition 1. For a graph F , we say that G_1, G_2, \dots, G_m is *double F -free* if $E(F) \not\subseteq E(G_i) \cap E(G_j)$ for $1 \leq i < j \leq m$. Moreover, we call a copy of F in $G_i \cap G_j$ a *double F in G_i, G_j* .

The double Turán problem asks for the maximum sum of edges over G_1, G_2, \dots, G_m such that G_1, G_2, \dots, G_m are double F -free.

Definition 2. For a graph F , the *double Turán number* for F , denoted $\phi(m, n, F)$, is the maximum value of $\sum_{i=1}^m e(G_i)$ such that G_1, G_2, \dots, G_m are double F -free.

We would also like to study a special case of the double Turán problem, which we refer to as *induced double Turán problem*.

Definition 3. We call G_1, G_2, \dots, G_m *induced* if for each $i \in [m]$, G_i is an induced subgraph of $\bigcup_{i=1}^m G_i$.

In other words, if $\{u, v\} \in E(G_i)$ and $u, v \in V(G_j)$, then $\{u, v\} \in E(G_j)$.

Definition 4. For a graph F , the *induced double Turán number*, denoted $\phi^*(m, n, F)$, is the maximum value of $\sum_{i=1}^m e(G_i)$ such that G_1, G_2, \dots, G_m are induced and double F -free.

The induced double Turán problem is then to determine $\phi^*(m, n, F)$. Clearly, $\phi(m, n, F) \geq \phi^*(m, n, F)$. Similar to the Turán problem, the induced and non-induced double Turán problems behave differently depending on whether F is bipartite or non-bipartite. Thus we will study these two cases separately in this thesis. We shall see that the study of $\phi(m, n, F)$ and $\phi^*(m, n, F)$ is motivated by certain hypergraph extremal problems.

1.2 Motivation : Link Graphs and Hypergraphs

Apart from the intrinsic interest in investigating $\phi(m, n, F)$, one motivation to study is that $\phi(m, n, F)$ is closely connected to pure hypergraph extremal problems via the notion of *link graphs*. Let H be a triple system, that is, a set of three-element subsets of a finite set $[n]$. We may view H as a 3-uniform hypergraph, where the edges are the three-element subsets of $[n]$.

Definition 5. For $i \in V(H)$, the link graph of i , denoted H_i , is the graph with $V(H_i) = V(H) \setminus \{i\}$ and $E(H_i) = \{\{j, k\} : \{i, j, k\} \in E(H)\}$.

A handy idea in extremal hypergraph theory is to reduce a hypergraph extremal problem to extremal problems for the link graphs. For instance, a triple system H does not contain a tetrahedron, i.e. four triples on four vertices, if and only if all its link graphs are triangle-free.

In the current context, given a graph F , let F^+ denote the triple system with $V(F^+) = V(F) \cup \{x, y\}$ and $E(F^+) = \{e \cup \{x\}, e \cup \{y\} : e \in E(F)\}$. For example, if F is a 4-cycle $K_{2,2}$, then F^+ is the hypergraph for octahedron. We can see that $\phi(n, n, F)$ and $\text{ex}(n, F^+)$ are intimately related: if H is an F^+ -free triple system with vertex set $[n]$, then the link graphs H_1, H_2, \dots, H_n are double F -free, otherwise there exists a double F in some $H_i \cap H_j$ and the vertices i, j along with that copy of F form a F^+ in H , contradiction. Thus, $\text{ex}(n, F^+) \leq \phi(n, n, F)$, and this relates the double Turán problem to hypergraph extremal problems.

On the other hand, the study of the induced double Turán problem is motivated by a special case of the *generalized Turán problem*, which asks for the maximum number $\text{ex}(n, \bar{F}, K_3)$ of triangles in a graph G with vertex set $[n]$ that does not contain some graph \bar{F} . This problem was studied by Alon and Shikhelman [1] and Kostochka, Mubayi and Verstraete [10, 12, 14]. Similar to how link graphs relate to hypergraph extremal problems, the generalized Turán problem is related to $\phi^*(n, n, F)$ as follows: Let \bar{F} be the graph consisting of all pairs contained in triples in F^+ . For example, if $F = K_{2,2}$ then $\bar{F} = K_{2,2,2}$. For $i \in [n]$, define $E(G_i) = \{\{j, k\} : \{i, j\}, \{j, k\}, \{i, k\} \in E(G)\}$. Then G_1, G_2, \dots, G_n are induced and double F -free, so $\phi^*(n, n, F) \geq \text{ex}(n, \bar{F}, K_3)$. This relates the induced double Turán problem to extremal problems for triangles in graphs.

1.3 Main Results : The Induced Case

The determination of $\phi^*(m, n, F)$ turns out to be fairly straightforward when F is a non-bipartite graph: the extremal objects are simply m copies of the same extremal graph for F .

Theorem 1. Let $r, m \geq 3$. There exists $n_0(r)$ such that if $n \geq n_0(r)$ and F is a graph of chromatic number r , then

$$\phi^*(m, n, F) = m \cdot \text{ex}(n, F).$$

Moreover, G_1, \dots, G_m are induced double F -free graphs on $[n]$ that sum up to $m \cdot \text{ex}(n, F)$ edges only if $G_1 = \dots = G_m$ are identical extremal n -vertex F -free graphs.

In the case $F = K_r$, we shall see the theorem is true for all $n \geq 3$:

Theorem 2. *Let $m, n, r \geq 3$. Then*

$$\phi^*(m, n, K_r) = m \cdot e(T_{r-1}(n)).$$

Moreover, G_1, \dots, G_m are induced double K_r -free graphs on $[n]$ that sum up to $m \cdot e(T_{r-1}(n))$ edges only if $G_1 = \dots = G_m = T_{r-1}(n)$.

In the case F is a bipartite graph, even determining the order of magnitude of $\phi^*(m, n, F)$ appears to be difficult. In fact, we do not even know the order of magnitude of $\phi^*(m, n, P)$ when P is a path with two edges. In this thesis, we propose the following very broad conjecture:

Conjecture A. *Let F be any non-empty graph and $m, n \geq 1$. Then*

$$\phi^*(m, n, F) = \Theta(m \cdot \text{ex}(n, F) + n^2).$$

It is clear that a single complete graph K_n does not contain a double F , and neither do identical copies G_1, G_2, \dots, G_m of an extremal n -vertex F -free graph. Thus we have the trivial lower bound

$$\phi^*(m, n, F) \geq \max \left\{ \binom{n}{2}, m \cdot \text{ex}(n, F) \right\}.$$

This conjecture is true when F is non-bipartite, by Theorem 1. If F is bipartite, then the upper bounds on $\phi^*(m, n, F)$ are more difficult to come by, especially when m is large. For instance, from our discussion in the previous section, we know

$$\text{ex}(n, K_{2,2,2}, K_3) \leq \phi^*(n, n, K_{2,2}),$$

and so Conjecture A implies that an n -vertex graph not containing the octahedron graph has $O(n^{5/2})$ triangles. In fact, it is also the case that $\text{ex}(2n, K_{2,2,2}, K_3) \geq \phi^*(n, n, K_{2,2})$: if we have double $K_{2,2}$ -free induced graphs G_1, G_2, \dots, G_n with vertex set $[n]$, then let H be the graph with $V(H) = [2n]$ consisting of all triangles with vertex set $\{i, j, k\}$ such that $n < k \leq 2n$ and $\{i, j\} \in E(G_k)$. The graph H is $K_{2,2,2}$ -free and $|E(H)| = \sum_{i=1}^{n/2} e(G_i)$. This shows that $\text{ex}(n, K_{2,2,2}, K_3)$ and $\phi^*(n, n, K_{2,2})$ are equivalent up to a constant factor. Similarly, we have

$$\text{ex}(n, K_{1,2,2}, K_3) \leq \phi^*(n, n, K_{1,2}),$$

and so Conjecture A implies that an n -vertex graph not containing the wheel graph has $O(n^2)$ triangles, which is conjectured by Mubayi and Verstraete [14]. The conjecture proposes

more generally that if F is a tree, then $\phi^*(n, n, F) = O(n^2)$. In fact, it is possible to prove the following theorem using the *removal lemma* as in [12] as well as a projective plane construction for $\phi(n, n, P)$:

Theorem 3. *Let P be a path with two edges. Then $\phi(n, n, P) = \Omega(n^{5/2})$, whereas $\phi^*(n, n, P) = o(n^{5/2})$, as $n \rightarrow \infty$. In particular,*

$$\lim_{n \rightarrow \infty} \frac{\phi^*(n, n, P)}{\phi(n, n, P)} = 0.$$

Apart from determining the order of magnitude of $\phi^*(m, n, F)$, this above theorem also shows that the order of magnitude of $\phi(n, n, P)$ and $\phi^*(n, n, P)$ can differ significantly. This suggests that the induced and non-induced double Turán problems may be fundamentally different.

A special bipartite case where Conjecture A is true is when F is a matching with two edges. Let M denote a matching with two edges, and let M^+ denote the graph obtained from two copies of K_4 sharing one edge by removing that edge. Then $\text{ex}(n, M^+, K_3) \leq \phi^*(n, n, M)$. If F is the triple system consisting of all four triangles in M^+ , then Füredi [7] showed $\text{ex}(n, M^+) = O(n^2)$, answering a conjecture of Erdős [4]. It is possible to adapt Füredi's proof to give $\phi^*(n, n, M) = O(n^2)$, so in this case Conjecture A is true and $\text{ex}(n, M^+, K_3) = \Theta(\phi^*(n, n, M))$. For improvements of the constant factor, see Mubayi and Verstraëte [13] and Pikhurko and Verstraëte [15]. We shall see that for some bipartite F , if m is not too large relative to n , then Conjecture A is also true.

1.4 Main Results : The Non-induced Case

Determining $\phi(m, n, F)$ even when F is a complete graph is challenging. The forth theorem we give is well-suited to the case of certain bipartite graphs, and is due to Wilson:

Theorem 4. *Let F be a graph. If there exists an extremal F -free n -vertex graph with maximum degree at most $n^{1/2}/m^2$, then*

$$\phi(m, n, F) = \binom{n}{2} + \binom{m}{2} \text{ex}(n, F).$$

By the Erdős-Stone Theorem, the extremal number for any non-bipartite F is $\Theta(n^2)$, which implies that the maximum degree of an extremal F -free graph exceeds $n^{1/2}$. Thus the condition on the maximum degree in the above theorem can only be satisfied for bipartite graphs, for instance, a path of two edges. An example of a bipartite graph that does not apply

to this theorem would be a matching of two edges, whose extremal graph has a maximum degree of $n - 1$.

Since $\binom{n}{2} + m - 1 \leq \phi^*(m, n, F) \leq \phi(m, n, F)$ for any graph F with at least two edges, Theorem 4 shows $\phi^*(m, n, F) = (1 + o(1))\binom{n}{2}$ whenever the conditions on m in the theorem are satisfied, proving Conjecture A to be true in this case.

Our first theorem on $\phi(m, n, F)$ for non-bipartite graphs F uses the notion of *supersaturation* – see Erdős and Simonovits [6]. We determine the asymptotic value of $\phi(m, n, F)$ as $m \rightarrow \infty$ when F is a non-bipartite graph:

Theorem 5. *Let $n \geq 1$ and let F be a non-bipartite graph. Then as $m \rightarrow \infty$,*

$$\phi(m, n, F) = (1 + o(1))m \cdot \text{ex}(n, F).$$

The next result we present concerns non-bipartite graphs. To state the theorem, we require the notion of the M -color Ramsey number.

Definition 6. *For $M \geq 2$, the M -color Ramsey number, denoted $R_M(r)$, is the smallest integer N such that any coloring of the edges of the complete graph K_N with M colors contains a monochromatic complete subgraph on r vertices.*

Suppose we have a monochromatic K_r -free coloring $c : E(K_N) \rightarrow 2^{[m]}$. For $i \in [m]$, let $H_i = \{\{u, w\} \in E(K_N) : i \in c(u, w)\}$. Then H_1, H_2, \dots, H_m are double K_r -free. If we replace the vertices of K_N with disjoint sets V_1, V_2, \dots, V_N whose sizes add up to n , and then let

$$G_i = \{\{x, y\} : (x, y) \in V_u \times V_w, i \in c(u, w), 1 \leq u < w \leq N\}$$

and make each V_i cliques in G_1 , then G_1, G_2, \dots, G_m is also double K_r -free. We call G_1, G_2, \dots, G_m an (m, n, N) -blowup.

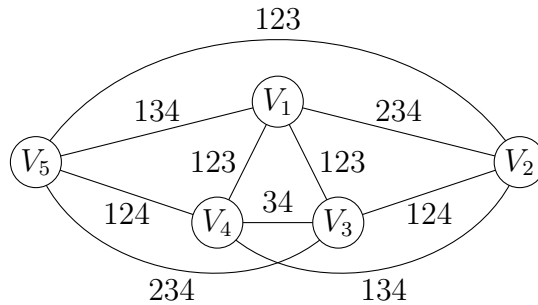


Figure 1: Example of an $(4, n, 5)$ -blowup not containing a double K_3 .

Definition 7. Let $f(m, n, r)$ denote the maximum of $e(G_1) + e(G_2) + \cdots + e(G_m)$ such that G_1, G_2, \dots, G_m is a double K_r -free (m, n, N) -blowup for some $N < R_{\binom{m}{2}}(r)$.

This turns out to be exactly the construction which determines $\phi(m, n, F)$ when F is a complete graph:

Theorem 6. For $m, n, r \geq 3$,

$$\phi(m, n, K_r) = f(m, n, r).$$

While computing $f(m, n, r)$ is a finite calculation, the Ramsey number $R_{\binom{m}{2}}(r)$ unfortunately appears to be intractable in general. It is known that $R_2(3) = 6$ and $R_3(3) = 17$ and $R_2(4) = 18$, but no further multicolor Ramsey numbers are known [2, 11]. In the special case $r = m = 3$, the following holds:

Theorem 7. For $n \geq 1$,

$$\phi(3, n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

The same problem immediately becomes difficult when m is increased by 1. The blowup construction in Figure 1 shows $\phi(4, n, K_3) - [\binom{n}{2} + 3\text{ex}(n, K_3)] \geq n^2/100$ as $n \rightarrow \infty$. This suggests the actual values of $\phi(m, n, K_r)$ can be significantly larger than our trivial lower bound. We leave it as an open problem to determine $\phi(m, n, K_r)$ for $r, m \geq 3$ and $(r, m) \neq (3, 3)$.

1.5 Definitions and Notations

Denote the set of first n positive integers as $[n] = \{1, 2, \dots, n\}$. Given a set X , we denote 2^X as the power set of X . Given graph $G = (V, E)$, let $V(G)$ denote the vertex set and $E(G)$ denote the edge set of G . Let $e(G) = |E(G)|$ be the number of edges in G . For vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ the neighborhood of v . Given two graphs G_1, G_2 , we denote $G_1 \cup G_2$ as the graph on $V(G_1) \cup V(G_2)$ with edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Similarly, we define $G_1 \cap G_2$ as the graph on $V(G_1) \cap V(G_2)$ with edge set $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$. In this thesis, we reserve n to denote the number of vertices in a graph. We call a n -vertex complete graph K_n , and a complete bipartite graph $K_{a,b}$, where a, b are the sizes of its parts. Given graph G, H , define $G + H$ as the graph fully connecting G, H , i.e. $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$. Let v be a vertex from G_1, G_2, \dots, G_m . Unless otherwise specified, we denote $d(v)$ as the sum of the degree of v over all G_i .

2 The Induced Double Turán Problem

We prove the theorems for $\phi^*(m, n, F)$ in this chapter. In particular, the main theorem we prove is Theorem 1 for general non-bipartite graphs F and in the special case of cliques. We will first introduce two observations that simplify the problem.

The first observation is that the determination of $\phi^*(m, n, F)$ can be reduced to smaller values of m :

Lemma 8. *Let $n, m, k \geq 2$ with $m \geq k$, and let F be some graph. Then*

$$\phi^*(m, n, F) \leq \frac{m}{k} \cdot \phi^*(k, n, F).$$

Moreover, let G_1, \dots, G_m be induced double F -free graphs on $[n]$ and suppose $\sum_{i=1}^k e(G_i) = \phi^(k, n, F)$ only if $G_1 = \dots = G_k$. Then $\sum_{i=1}^m e(G_i) = \phi^*(m, n, F)$ only if $G_1 = \dots = G_m$.*

Proof. Let G_1, \dots, G_m be induced double F -free graphs on $[n]$. Put $G_{i+m} = G_i$ for all $i \in [m]$. Then

$$\sum_{i=1}^m e(G_i) = \frac{1}{k} \sum_{i=1}^m [e(G_i) + \dots + e(G_{i+k-1})] \leq \frac{1}{k} \sum_{i=1}^m \phi^*(k, n, F) = \frac{m}{k} \cdot \phi^*(k, n, F),$$

which establishes the upper bound. The lower bound follows from the construction with $G_1 = \dots = G_m$ to be n -vertex extremal graphs for F .

Now suppose $\sum_{i=1}^m e(G_i) = (m/k)\phi^*(k, n, F)$ and $G_1 \neq G_2$. By assumption $\sum_{i=1}^k e(G_i) < \phi^*(k, n, F)$. But then $\sum_{i=1}^k e(G_{i+j}) > \phi^*(k, n, F)$ for some $j \geq 1$, contradiction. \square

The second observation shows that the determination of $\phi^*(2, n, F)$ can be reduced to an optimization problem over the number of vertices in the intersection of the two graphs:

Lemma 9. *Let $n \geq 1$. For graph F , define $\mathcal{C}(n, t, F) := \binom{n-t}{2} + (n-t)t + 2\text{ex}(t, F)$. Let G_1, G_2 be induced double F -free graphs on $[n]$. Then*

$$e(G_1) + e(G_2) \leq \max_{0 \leq t \leq n} \mathcal{C}(n, t, F),$$

with equality only if G_2 is an extremal graph for F with t_{\max} vertices and $G_1 = G_2 + K_{n-t}$, where t_{\max} is a maximizer of $\mathcal{C}(n, t, F)$ over $0 \leq t \leq n$.

Proof. Let G_1, G_2 be induced double F -free graphs on $[n]$. Put $T = V(G_1) \cap V(G_2)$, $t = |T|$, $s = |V(G_1) \setminus T|$, and $n - t - s = |V(G_2) \setminus T|$. Note that $t, s \in \mathbb{Z}_{\geq 0}$. Since G_1, G_2 are induced

subgraphs of $G_1 \cup G_2$, we have $G_1[T] = G_2[T] = G_1 \cap G_2$. But then $G_1 \cap G_2$ is F -free, so $e(G_1[T]) = e(G_2[T]) \leq \text{ex}(t, F)$. Notice there can be at most $t(n-t)$ edges between T and $(V(G_1) \cup V(G_2)) \setminus T$. Since $G[V(G_1) \setminus T] \leq \binom{s}{2}$ and $G[V(G_2) \setminus T] \leq \binom{n-t-s}{2}$,

$$e(G_1) + e(G_2) \leq \binom{s}{2} + \binom{n-s-t}{2} + t(n-t) + 2\text{ex}(t, F).$$

But then $\binom{n-t}{2} > \binom{s}{2} + \binom{n-t-s}{2}$ for $0 < s < n-t$, so

$$e(G_1) + e(G_2) \leq \binom{n-t}{2} + (n-t)t + 2\text{ex}(t, F) = \mathcal{C}(n, t, F).$$

This establishes the upper bound. From this we also know that $e(G_1) + e(G_2) = \mathcal{C}(n, t, F)$ only if G_2 is the t -vertex extremal graph for F and $G_1 = G_2 + K_{n-t}$. The result now follows. \square

2.1 Proof of Theorem 2

By Lemma 8, it suffices to prove the theorem for $m = 3$. Let G_1, G_2, G_3 be induced double K_r -free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, K_r)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\phi^*(3, n, K_r) \geq 3\text{ex}(n, K_r)$. Consequently, we must have $e(G_1) + e(G_2) \geq 2\text{ex}(n, K_r)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\text{ex}(n, K_r)$, it suffices to show that $G_1 = G_2 = T_{r-1}(n)$. In particular, we will use Lemma 9 to show that G_1, G_2 is an extremal configuration without containing a double K_r .

Let $t = |V(G_1 \cap G_2)|$. By Turán's Theorem,

$$\text{ex}(t, K_r) - \text{ex}(t-1, K_r) = e(T_{r-1}(t)) - e(T_{r-1}(t-1)) = t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, K_r) - \mathcal{C}(n, t-1, K_r) = -t + 1 + 2[\text{ex}(t, K_r) - \text{ex}(t-1, K_r)] = t + 1 - 2 \left\lceil \frac{t}{r-1} \right\rceil. \quad (1)$$

For $r \geq 4$, $\mathcal{C}(n, t, K_r)$ is strictly increasing on t , so by Lemma 9,

$$\phi^*(2, n, K_r) = \mathcal{C}(n, n, K_r) = 2\text{ex}(n, K_r) = e(G_1) + e(G_2)$$

and $G_1 = G_2 = T_{r-1}(n)$, as desired.

Now suppose $r = 3$. Equation (1) shows that $\mathcal{C}(n, t, K_r)$ is non-decreasing on t and

$\mathcal{C}(n, t, K_r) > \mathcal{C}(n, t, K_r)$ for even t . By Lemma 9, we now have

$$\phi^*(2, n, K_r) = \max[\mathcal{C}(n, n, K_r), \mathcal{C}(n, n-1, K_r)] = 2\text{ex}(n, K_r) = e(G_1) + e(G_2),$$

and either $G_1 = G_2 = T_{r-1}(n)$, or $G_2 = T_{r-1}(n-1)$ and $G_1 = G_2 + K_1$. If the latter case is true, then $e(G_3) \geq \text{ex}(n, F) > e(G_2)$, and this contradiction completes the proof. \square

2.2 Proof of Theorem 1

If F is a graph of chromatic number $r+1 \geq 3$, then Theorem B shows $\text{ex}(n, F) = (1 + o(1))\text{ex}(n, K_{r+1})$ as $n \rightarrow \infty$. In this section, we prove Theorem 1 following the same line of reasoning as in the proof of Theorem 2.

Proof of Theorem 1. By Lemma 8, it suffices to prove the theorem for $m = 3$. Let G_1, G_2, G_3 be induced double F -free graphs, such that $e(G_1) + e(G_2) + e(G_3) = \phi^*(3, n, F)$. We may assume $e(G_1) \geq e(G_2) \geq e(G_3)$, and we already know $\phi^*(3, n, F) \geq 3\text{ex}(n, F)$. Consequently, we must have $e(G_1) + e(G_2) \geq 2\text{ex}(n, F)$. Since G_1, G_2, G_3 are induced and $e(G_1) + e(G_2) + e(G_3) \geq 3\text{ex}(n, F)$, it suffices to show that $G_1 = G_2$ are n -vertex F -free extremal graphs. In particular, we will use Lemma 9 to show that G_1, G_2 is an extremal configuration without containing a double F .

Let $t = |V(G_1 \cap G_2)|$. If $t < \sqrt{n}$, then

$$2\text{ex}(n, F) \geq 2e(T_{r-1}(n)) \geq 2 \left\lfloor \frac{n^2}{4} \right\rfloor \geq \binom{n}{2} + \binom{\sqrt{n}}{2} > \mathcal{C}(n, t, F).$$

Thus $t \geq \sqrt{n}$. But then for large enough t , any extremal t -vertex F -free graph contains a spanning complete $(r-1)$ -partite subgraph $T_{r-1}(t)$, so we may add $\text{ex}(t-1, F) - e(T_{r-1}(t-1))$ edges to $T_{r-1}(t)$ and still avoid F as a subgraph. Hence for large enough t , we have $\text{ex}(t, F) \geq \text{ex}(t-1, F) - e(T_{r-1}(t-1)) + e(T_{r-1}(t))$, and so

$$\text{ex}(t, F) - \text{ex}(t-1, F) \geq e(T_{r-1}(t)) - e(T_{r-1}(t-1)) \geq t - \left\lceil \frac{t}{r-1} \right\rceil.$$

It immediately follows that

$$\mathcal{C}(n, t, F) - \mathcal{C}(n, t-1, F) = -t + 1 + 2[\text{ex}(t, F) - \text{ex}(t-1, F)] \geq t + 1 - 2 \left\lceil \frac{t}{r-1} \right\rceil. \quad (2)$$

For $r \geq 4$, $\mathcal{C}(n, t, F)$ is strictly increasing on t , so by Lemma 9,

$$\phi^*(2, n, F) = \mathcal{C}(n, n, F) = 2\text{ex}(n, F) = e(G_1) + e(G_2),$$

and $G_1 = G_2$ are n -vertex F -free extremal graphs, as desired.

Now suppose $r = 3$. Equation (2) shows that $\mathcal{C}(n, t, F)$ is strictly increasing for even t and $\mathcal{C}(n, t, F) \geq \mathcal{C}(n, t-1, F)$ for odd t . By Lemma 9, we now have

$$\phi^*(2, n, F) = \max[\mathcal{C}(n, n, F), \mathcal{C}(n, n-1, F)] = 2\text{ex}(n, F) = e(G_1) + e(G_2),$$

and either $G_1 = G_2$ are n -vertex extremal F -free graphs, or G_2 is an $(n-1)$ -vertex extremal F -free graph and $G_1 = G_2 + K_1$. If the latter case is true, then $e(G_3) \geq \text{ex}(n, F) > e(G_2)$, and this contradiction completes the proof. \square

2.3 Proof of Theorem 3

We need to show that $\phi(n, n, P) = \Omega(n^{5/2})$ and $\phi^*(n, n, P) = o(n^{5/2})$, as $n \rightarrow \infty$.

Claim 1. For $\sqrt{n} < m \leq n$,

$$\phi(n, n, P) = (1/2 + o(1))mn^{3/2},$$

as $n \rightarrow \infty$.

We first show that $\phi(m, n, P) \leq (mn^{3/2} + n^2)/2$. For each vertex $u \in [n]$, define H_u as the $m \times n$ bipartite graph with edge set $E(H_u) := \{\{v, i\} : \{u, v\} \in E(G_i)\}$. If H_u contains a quadrilateral $\{v, i\}, \{v, j\}, \{w, i\}, \{w, j\}$, then $\{u, v\}, \{u, w\}$ form a double P in $G_i \cap G_j$, contradiction. Thus we conclude that H_u is quadrilateral-free, and therefore $e(H_u) \leq m\sqrt{n} + n$, by the Kővári-Sós-Turán Theorem [9]. It now follows that

$$\sum_{i=1}^m e(G_i) = \frac{1}{2} \sum_{u \in V(G)} e(H_u) \leq \frac{1}{2}(mn^{3/2} + n^2).$$

We now show the upperbound is tight asymptotically by giving a finite projective plane construction. Suppose G_1, G_2, \dots, G_n are graphs on $[n]$ containing no double P and $\sum_{i=1}^n e(G_i) \geq (1/2 + o(1))n^{5/2}$, with $e(G_1) \geq e(G_2) \geq \dots \geq e(G_n)$. Then G_1, G_2, \dots, G_m are graphs with no double P and $\sum_{i=1}^m e(G_i) \geq (1/2 + o(1))mn^{3/2}$. Hence, it suffices to prove the case for $m = n$.

Consider a finite projective plane with n points and n lines, with prime q chosen so that $n = (1 + o(1))(q^2 + q + 1)$ as $q \rightarrow \infty$. Let $S_1, \dots, S_n \subseteq [n]$ be the n lines of the projective plane. Note that each line S_i contains $q + 1$ points, and the intersection of any two distinct lines S_i, S_j contains $|S_i \cap S_j| = 1$ point.

Define G_1, \dots, G_n to be graphs on $[n]$, each with edge set

$$E(G_i) := \{\{j, k\} \subseteq [n] : j \neq k, j + k \in S_i \pmod n\}.$$

Note that the intersection of distinct G_i, G_j is P free: since $|S_i \cap S_j| = 1$, if $\{a, b\}, \{a, c\} \in E(G_i) \cap E(G_j)$, then $a + b = a + c$ so $b = c$.

We now count the number of edges in G_1, \dots, G_n . Since $|S_i| = q + 1$, for each point $j \in [n]$, there are $q + 1$ choices for $k \in [n]$ such that $j + k \in S_i$. But then we have to avoid counting the same edge twice and loops, so the number of edges in G_i is

$$e(G_i) = \frac{n(q + 1) - \#\text{loops counted for } G_i}{2}.$$

If $j \in [n]$ is even, then $k = j/2$ is the unique number in $[n]$ such that $k + k = j \pmod n$. If $j \in [n]$ is odd, then $k = (n + j)/2$ is the unique number in $[n]$ such that $k + k = j \pmod n$, as n is even. Hence, for each $j \in S_i$, there exists a unique $k \in [n]$ such that $k + k = j \pmod n$, and thus

$$\#\text{loops counted for } G_i = |S_i| = q + 1.$$

Since $q + 1 = (1 + o(1))n^{1/2}$, the number of edges in G_1, \dots, G_n is

$$\sum_{i=1}^n e(G_i) = n \cdot \frac{n(q + 1) - (q + 1)}{2} = \left(\frac{1}{2} + o(1)\right) n^{5/2},$$

as $n \rightarrow \infty$. This proves the first claim.

Claim 2. $\phi^*(n, n, P) = o(n^{5/2})$, as $n \rightarrow \infty$.

Let G_1, G_2, \dots, G_n be induced and double P -free and let $\epsilon > 0$. Let $d_i(v)$ be the degree of vertex v in the graph G_i . Let I be the set of pairs (i, v) such that $d_i(v) \geq \sqrt{n}/\epsilon + 1$. Since G_1, G_2, \dots, G_n do not contain a double P ,

$$\sum_{(i,v) \in I} \binom{d_i(v)}{2} \leq n^3.$$

The maximum possible value of $\sum_{(i,v) \in I} d_i(v)$ subject to this constraint is when $d_i(v) =$

$\sqrt{n}/\epsilon + 1$ for all (i, v) , in which case $|I| \leq 2\epsilon^2 n^2$ and so

$$\sum_{(i,v) \in I} d_i(v) \leq (2\epsilon^2 n^2) \cdot \left(\frac{\sqrt{n}}{\epsilon} + 1 \right) = 3\epsilon n^{5/2}$$

for large enough n . Remove all edges of G_i on vertex v such that $(i, v) \in I$. The total number of edges removed is at most $3\epsilon n^{5/2}$. Let G'_1, G'_2, \dots, G'_n be the remaining subgraphs of G_1, G_2, \dots, G_n . If $e(G'_i) \leq \epsilon n^{3/2}$, then remove all edges of G'_i . The number of edges removed in this process is at most $\epsilon n^{5/2}$. The remaining graphs $G''_1, G''_2, \dots, G''_m$ have each at least $\epsilon n^{3/2}$ edges and maximum degree at most \sqrt{n}/ϵ . In particular, each G''_i contains a matching M_i of size at least $e(G''_i)/2\Delta(G''_i) = \epsilon^2 n/2$. If $m \leq \epsilon n$, then

$$\sum_{i=1}^n e(G_i) \leq 4\epsilon n^{5/2} + \sum_{i=1}^m e(G''_i) \leq 4\epsilon n^{5/2} + \phi(m, n, P) \leq 5\epsilon n^{5/2}$$

by Claim 1. If $m > \epsilon n$, then we apply Szemerédi's Regularity Lemma to find, for some $\delta > 0$ depending only on ϵ , a matching M_1 in G''_1 such that for some pair of set $X, Y \subseteq V(M_1)$ of size at least δn each, there is a set E of at least $\delta^3 n^2$ edges $\{x, y\}$ of $G''_1 \cup G''_2 \cup \dots \cup G''_m$ such that $x \in X$ and $y \in Y$. Since G''_1 is induced, $E \subseteq E(G_1)$. In particular, there are at least $\delta^5 n^3/4$ copies of P in G_1 . We can repeat the argument in the remaining graphs $G''_i : i \in [2, m]$ to get say M_2 in G''_2 as above, which gives $\delta^5 n^3/4$ copies of P in G_2 . If we do this $4\delta^{-5}$ times, then we have found n^3 copies of P in the first $4\delta^{-5}$ graphs, and two of them have the same edge-set. We conclude $\sum_{i=1}^n e(G_i) \leq 5\epsilon n^{5/2}$ if n is large enough. Since ϵ is arbitrary, this completes the proof. \square

3 The Non-induced Double Turán Problem

In this section, we prove our main theorems on $\phi(m, n, F)$.

3.1 Proof of Theorem 4

We first show that for all $m, n \geq 1$ and graph F ,

$$\phi(m, n, F) \leq \binom{n}{2} + \text{ex}(n, F) \binom{m}{2}.$$

Thereafter, we show that if there is an extremal F -free graph with maximum degree at most $n^{1/2}/m^2$, then the above bound is tight.

Proof of the upper bound. For $S \subseteq [m]$, let E_S denote the set of edges that are contained in exactly $\{G_i\}_{i \in S}$, and note that $E_S \cap E_{S'} = \emptyset$ if $S \neq S'$. Then

$$\sum_{i=1}^m e(G_i) = \sum_{S \subseteq [m]} |S| |E_S| \leq \binom{n}{2} + \sum_{S \subseteq [m], |S| \geq 2} (|S| - 1) |E_S|.$$

Note that $\bigsqcup_{T \supseteq S} E_T = \bigcap_{i \in S} E(G_i)$, which is F -free for $|S| \geq 2$ and so

$$\sum_{T \supseteq S} |E_T| \leq \text{ex}(n, F).$$

It now follows that

$$\sum_{\substack{S \subseteq [m] \\ |S| \geq 2}} (|S| - 1) |E_S| = \sum_{\substack{S \subseteq [m], \\ |S| = 2}} \sum_{T \supseteq S} \frac{(|T| - 1) |E_T|}{\binom{|T|}{2}} \leq \sum_{\substack{S \subseteq [m], \\ |S| = 2}} \sum_{T \supseteq S} |E_T| \leq \binom{m}{2} \text{ex}(n, F),$$

as each $T \in [m]$ with $|T| \geq 2$ is counted $\binom{|T|}{2}$ times in total and $|T| - 1 \leq \binom{|T|}{2}$. This proves the upper bound.

Proof of the lower bound. We need to show there exists a construction such that the graph with edge set E_S is an extremal F -free graph, for all $S \subseteq [m]$ of size 2. Let $M = \binom{m}{2}$ and H_1, \dots, H_M be copies of an extremal F -free graph on n vertices such that H_i with maximum degree $\Delta \leq n^{1/2}/m^2$ for all $i \in [m]$. It suffices to show that we can embed each H_i onto $[n]$ such that their edge sets are pairwise disjoint. We begin by an arbitrary embedding of each H_i and iteratively decrease the number of intersecting edges. Define a (u, v, i) -swap by swapping the embedding of vertex u and v of H_i , i.e. replacing each edge $\{u, w\} \in E(H_i)$

with the edge $\{u, w\}$ and each edge $\{v, w\} \in E(H_i)$ with the edge $\{v, w\}$. This preserves the type of isomorphism of H_i . Given a vertex v , let $N(v) = N_{H_1}(v) \cup \dots \cup N_{H_M}(v)$. Suppose there exists an intersecting edge $\{u, w\} \in E(H_i) \cap E(H_j)$. Since $|N(u)| \leq M \cdot \Delta \leq n^{1/2}/2$, $|N(u) \cup N(N(u))| \leq \Delta + \Delta(\Delta - 1) \leq n/4$, so there exists a vertex $v \notin N(u) \cup N(N(u))$. Since $N(u) \cap N(v) = \emptyset$, performing a (u, v, i) -swap reduces the number of intersecting edges. The result now follows from iterating this process. \square

3.2 Proof of Theorem 5

We need the following *saturation theorem*, which may be found in [6].

Proposition 10. *Let F be any non-empty graph with k vertices. For all $\epsilon > 0$, there exists $\delta > 0$ such that if G is any n -vertex graph with $\text{ex}(n, F) + \epsilon n^2$ edges, then G contains δn^k copies of F .*

Proof of Theorem 5. Let $k = |V(F)|$ and let $\epsilon > 0$. Let G_1, G_2, \dots, G_m be double F -free. Reorder G_1, G_2, \dots, G_m so that $e(G_i) \geq \text{ex}(n, F) + \epsilon n^2$ for $1 \leq i \leq \ell$ and $e(G_i) < \text{ex}(n, F) + \epsilon n^2$ for $\ell < i \leq m$. Then each $G_i : 1 \leq i \leq \ell$ contains at least δn^k copies of F , by Proposition 10. On the other hand, there are at most n^k copies of F such that $F \subseteq G_i$ for some $i \in [m]$. Therefore $\ell \leq 1/\delta$ and

$$\begin{aligned} \sum_{i=1}^m e(G_i) &= \sum_{i=1}^{\ell} e(G_i) + \sum_{i=\ell+1}^m e(G_i) \\ &\leq \frac{1}{\delta} \binom{n}{2} + (m - \ell) \text{ex}(n, F) + (m - \ell) \epsilon n^2 \\ &\leq m \cdot \text{ex}(n, F) + \epsilon m n^2 + \frac{1}{\delta} \binom{n}{2}. \end{aligned}$$

Since F is not bipartite, $\text{ex}(n, F) = \Theta(n^2)$ and so $\phi(m, n, F) \leq m \cdot \text{ex}(n, F) + (\epsilon + 1/\delta m) m n^2$. Since ϵ was arbitrary and δ is a constant depending only on ϵ , we conclude $\phi(m, n, F) \leq (1 + o(1)) m \cdot \text{ex}(n, F)$ as $m \rightarrow \infty$. \square

Let F be a bipartite graph with $k \geq 2$ vertices and $j \geq 1$ edges. A strong version of a conjecture of Simonovits [16, 17] would suggest that for all $\epsilon > 0$, there exists $\delta > 0$ such that every n -vertex graph G with at least $p\binom{n}{2}(1 + \epsilon)\text{ex}(n, F)$ edges contains at least $\delta p^j n^k$ copies of F . For instance, this is known to be true whenever the asymptotic behavior of $\text{ex}(n, F)$ is known, which includes the case $F = K_{2,t}$. If F is bipartite and $m \cdot \text{ex}(n, F)/n^2 \rightarrow \infty$ as $m, n \rightarrow \infty$, then this conjecture with the same proof as above shows $\phi(m, n, F) = (1 + o(1)) m \cdot \text{ex}(n, F)$. When F contains a cycle, then there exists $\alpha > 0$ such that $\text{ex}(n, F) \geq n^{1+\alpha}$ for

large enough n . Thus, we conclude that if F contains a cycle and the Simonovits conjecture is true for F , then $\phi(m, n, F) = (1 + o(1))m \cdot \text{ex}(n, F)$ for $m \geq n$ and $n \rightarrow \infty$. In particular, this shows $\phi(m, n, K_{2,t}) = (1 + o(1))m \cdot \text{ex}(n, F)$ for $m \geq n$ as $n \rightarrow \infty$.

3.3 Proof of Theorem 6

We now prove Theorem 6. Notice that we trivially have $f(m, n, r) \leq \phi(m, n, K_r)$, so it suffices to show the reverse inequality. That is, we need to show that there exists a blowup construction meeting the desired bound.

Let G_1, G_2, \dots, G_m be graphs on $[n]$ with no double K_r and $\sum_{i=1}^m e(G_i) = \phi(m, n, K_r)$. Observe that any pair $\{i, j\} \subseteq [n]$ must be in some G_i , otherwise, we may add it to G_1 without creating a double K_r .

We call vertices v, v' *clones* if for all $u \in [n] \setminus \{v, v'\}$ and $i \in [m]$, the edge $\{u, v\} \in E(G_i)$ if and only if $\{u, v'\} \in E(G_i)$. Furthermore, we call $\{v, v'\}$ a *light edge* if $\{v, v'\}$ is in exactly one graph G_i .

We now apply Algorithm 1 to G_1, G_2, \dots, G_m .

Algorithm 1 symmetrization algorithm

```

while  $\exists$  a light edge whose endpoints are not clones do
  among all vertices incident to such an edge, select a vertex  $v$  with maximum degree
   $B_v \leftarrow$  collection of vertices sending a light edge to  $v$  that are not clones of  $v$ 
  while  $B_v \neq \emptyset$  do
    pick  $u \in B_v$ 
     $j \leftarrow$  colour of the light edge from  $u$  to  $v$ 
    for  $1 \leq i \leq m$  do
      if  $i \neq j$  then;
         $N_{G_i}(u) \leftarrow N_{G_i}(v)$ 
      else if  $i = j$  then
         $N_{G_i}(u) \leftarrow (N_{G_i}(v) \setminus \{u\}) \cup \{v\}$ 
      end if
    end for
  end while
end while

```

Claim 3. *Algorithm 1 terminates.*

Notice that at the end of the ‘while $B_v \neq \emptyset$ ’ loop, every vertex sending a light edge to v is a clone of v . This implies v along with the set L_v of vertices receiving light edges from v induce a clique of size at least two in some G_i , and an empty graph in every other graph G_j

with $j \neq i$. Moreover, any vertex $w \notin L_v$ sends edges to either all or none of the vertices in L_v , and if w is incident to L_v , then w sends edges to L_v in at least two graphs. It now follows that no light edge incident with a vertex in L_v will be picked again in an iteration of the out most while loop. Thus the algorithm can run through at most $n/2$ such iterations, and so it terminates.

Claim 4. *The resulting graphs G'_1, G'_2, \dots, G'_m do not contain a double K_r and $\sum_{i=1}^m e(G'_i) = \phi(m, n, K_r)$.*

Note that we replace u by a clone of v in the for loop of Algorithm 1. Since $\{u, v\}$ remains a light edge in this step, u and v cannot both belong to a double K_r in the modified graphs. Furthermore, any double K_r containing u after the for loop arises from a double K_r containing v prior to the for loop. But then G_1, G_2, \dots, G_m contained no double K_r to begin with, so G'_1, G'_2, \dots, G'_m do not contain a double K_r .

We now show that the algorithm does not reduce the number of edges. By our choice of v , we know $d(v) \geq d(u)$ for all $u \in B_v$ prior to the for loop. Hence, replacing u with a clone of v does not decrease the number of edge over a complete iteration of the inner while loop. Therefore, $\sum_{i=1}^m e(G'_i) = \phi(m, n, K_r)$. The proof of the claim is now complete.

Hence, the algorithm results in graphs G'_1, G'_2, \dots, G'_m with $\phi(m, n, K_r)$ edges and the additional property that light edges come in ‘clone cliques.’ We may thus partition the vertex set $[n]$ into k disjoint sets V_1, V_2, \dots, V_k , such that each V_i induces a clique of light edges from the same graph. Moreover, for distinct $i, j \in [k]$, define S_{ij} to be the set of all edges between V_i and V_j , and note that any edge in S_{ij} appears in at least two modified graphs. The sets S_{ij} now yield a k -blowup. Notice that if the pattern of the k -blowup contains a double K_r , then the original graphs G_1, G_2, \dots, G_m must have contained a double K_r as well, contradiction. Thus the k -blowup is double K_r -free.

It remains to show that $k < R_M(K_r)$. For each edge $\{i, j\} \subseteq [k]$ in the pattern of the k -blowup, we assign an arbitrary distinct pair $\{a, b\} \subseteq L_{ij} \subseteq [m]$ to $\{i, j\}$. If $k \geq R_M(K_r)$, then there exists K_r in the pattern of the k -blowup colored by some distinct pair $\{a, b\} \subseteq [m]$. But then this implies the pattern of the k -blowup contains a double K_r , contradiction. This completes the proof. \square

3.4 Proof of Theorem 7

It is not hard to see that $\phi(2, n, K_3) = \binom{n}{2} + \lfloor n^2/4 \rfloor$: if G_1, G_2 is double triangle-free, then we have

$$e(G_1) + e(G_2) \leq \binom{n}{2} + e(G_1 \cap G_2) \leq \binom{n}{2} + \text{ex}(n, K_3)$$

and so $\phi(2, n, K_3) \leq \binom{n}{2} + \lfloor n^2/4 \rfloor$. Taking $G_1 = K_n$ and $G_2 = K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$ meets this bounds. The main result of this section is to show for all $n \geq 1$,

$$\phi(3, n, K_3) = \binom{n}{2} + \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Let G_1, G_2, G_3 be double triangle-free. Define $H_k \subseteq G$ to be the graph with edges contained in at least k of the G_i 's and note that $e(G_1) + e(G_2) + e(G_3) = e(H_1) + e(H_2) + e(H_3)$. Thus it suffices to show that $e(H_2) + e(H_3) \leq n^2/2$. Notice H_2 must not contain any triangles with two edges in H_3 , so

$$e(H_2) + e(H_3) \leq \binom{n}{2} + e(H_3) - |\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}|.$$

Let H'_3 be the graph with the same vertex set as H_3 and edge set $\{\{u, v\} : u \neq v, N_{H_3}(u) \cap N_{H_3}(v) \neq \emptyset\}$. It suffices to show that $n/2 \geq e(H_3) - e(H'_3)$.

Let $d_1 \geq d_2 \geq \dots \geq d_n$ and $f_1 \geq f_2 \geq \dots \geq f_n$ each be the degree sequence of H_3 and H'_3 , respectively. We show that $f_i \geq d_i - 1$ for all i . Let v_i denote the vertex in H with degree d_i and u_i be the vertex in H with degree f_i . Let $S_i = |N_{H_3}(v_1) \cup \dots \cup N_{H_3}(v_i)|$. Since

$$\sum_{u \in S_i} d_{H_3}(u) \geq d_1 + \dots + d_i,$$

we have that $|S_i| \geq i$. But then $S_i \setminus \{u_1, \dots, u_{i-1}\}$ is non-empty, and every $u \in S_i$ has degree $d_{H'_3}(u) \geq d_i - 1$. Hence, $f_i \geq d_i - 1$ for all i , which yields

$$e(H'_3) = \frac{1}{2} \sum_{i=1}^n f_i \geq \frac{1}{2} \sum_{i=1}^n (d_i - 1) = e(H_3) - \frac{n}{2}.$$

This proves Theorem 7. □

4 Concluding Remarks

- For Theorem 1, we may not be able to achieve the same result with smaller n . For example, consider F to be the bowtie graph, i.e. the 5-vertex graph with two triangles sharing a vertex. The n -vertex extremal graph for F is given by $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ plus an edge when $n \geq 5$, otherwise it is the complete graph. For $n = 5$, the construction $G_1 = K_4, G_2 = K_5$ then shows that $\phi^*(2, 5, F) > 2 \cdot \text{ex}(5, F)$. Fortunately, for non-bipartite F with $|V(F)| = k$, it is not hard to show $n \geq k^2$ is sufficient to avoid this issue.
- We note that Theorem 6 may be generalized to any family of non-bipartite graphs up to asymptotic error via Szemerédi's Regularity lemma
- One could ask for the analogous results for hypergraphs. That is, if F is an r -uniform hypergraph, let $\phi(m, n, F)$ be the maximum number of edges over m double F -free r -uniform hypergraphs on $[n]$. Again, we have $\phi(m, n, F) \geq \binom{n}{r} + (m - 1) \cdot \text{ex}(n, F)$. Another direction of generalization is to relax the constraint to no copies of F contained in the intersection of k of the graphs G_1, G_2, \dots, G_m . Many of the theorems and proofs also hold in this case. For example, the proof of Theorem 4 applies for this generalization by merely changing the numbers.

References

- [1] N. Alon and C. Shikhelman. Many T copies in H -free graphs. *J. Combin. Theory Ser. B*, 121:146–172, 2016. ISSN 0095-8956,1096-0902. doi: 10.1016/j.jctb.2016.03.004. URL <https://doi.org/10.1016/j.jctb.2016.03.004>.
- [2] D. Conlon and A. Ferber. Lower bounds for multicolor Ramsey numbers. *Adv. Math.*, 378:Paper No. 107528, 5, 2021. ISSN 0001-8708,1090-2082. doi: 10.1016/j.aim.2020.107528. URL <https://doi.org/10.1016/j.aim.2020.107528>.
- [3] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946. ISSN 0002-9904. doi: 10.1090/S0002-9904-1946-08715-7. URL <https://doi.org/10.1090/S0002-9904-1946-08715-7>.
- [4] P. Erdős. Problems and results in combinatorial analysis. In *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977)*, volume No. XIX of *Congress. Numer.*, pages 3–12. Utilitas Math., Winnipeg, MB, 1977. ISBN 0-919628-19-2.
- [5] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar*, 1:51–57, 1966. ISSN 0081-6906.
- [6] P. Erdős and M. Simonovits. Supersaturated graphs and hypergraphs. *Combinatorica*, 3(2):181–192, 1983. ISSN 0209-9683. doi: 10.1007/BF02579292. URL <https://doi.org/10.1007/BF02579292>.
- [7] Z. Füredi. Hypergraphs in which all disjoint pairs have distinct unions. *Combinatorica*, 4(2-3):161–168, 1984. ISSN 0209-9683. doi: 10.1007/BF02579216. URL <https://doi.org/10.1007/BF02579216>.
- [8] Z. Füredi and M. Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdős centennial*, volume 25 of *Bolyai Soc. Math. Stud.*, pages 169–264. János Bolyai Math. Soc., Budapest, 2013. doi: 10.1007/978-3-642-39286-3_7. URL https://doi.org/10.1007/978-3-642-39286-3_7.
- [9] T. Kővari, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954. doi: 10.4064/cm-3-1-50-57. URL <https://doi.org/10.4064/cm-3-1-50-57>.
- [10] A. Kostochka, D. Mubayi, and J. Verstraëte. Turán problems and shadows III: expansions of graphs. *SIAM J. Discrete Math.*, 29(2):868–876, 2015. ISSN 0895-4801,1095-7146. doi: 10.1137/140977138. URL <https://doi.org/10.1137/140977138>.

- [11] H. Lefmann. A note on Ramsey numbers. *Studia Sci. Math. Hungar.*, 22(1-4):445–446, 1987. ISSN 0081-6906,1588-2896.
- [12] D. Mubayi and S. Mukherjee. Triangles in graphs without bipartite suspensions. *Discrete Math.*, 346(6):Paper No. 113355, 19, 2023. ISSN 0012-365X,1872-681X. doi: 10.1016/j.disc.2023.113355. URL <https://doi.org/10.1016/j.disc.2023.113355>.
- [13] D. Mubayi and J. Verstraëte. A hypergraph extension of the bipartite Turán problem. *J. Combin. Theory Ser. A*, 106(2):237–253, 2004. ISSN 0097-3165,1096-0899. doi: 10.1016/j.jcta.2004.02.002. URL <https://doi.org/10.1016/j.jcta.2004.02.002>.
- [14] D. Mubayi and J. Verstraëte. A survey of Turán problems for expansions. In *Recent trends in combinatorics*, volume 159 of *IMA Vol. Math. Appl.*, pages 117–143. Springer, [Cham], 2016. ISBN 978-3-319-24296-5; 978-3-319-24298-9. doi: 10.1007/978-3-319-24298-9_5. URL https://doi.org/10.1007/978-3-319-24298-9_5.
- [15] O. Pikhurko and J. Verstraëte. The maximum size of hypergraphs without generalized 4-cycles. *J. Combin. Theory Ser. A*, 116(3):637–649, 2009. ISSN 0097-3165,1096-0899. doi: 10.1016/j.jcta.2008.09.002. URL <https://doi.org/10.1016/j.jcta.2008.09.002>.
- [16] A. Sidorenko. A correlation inequality for bipartite graphs. *Graphs Combin.*, 9(2):201–204, 1993. ISSN 0911-0119. doi: 10.1007/BF02988307. URL <https://doi.org/10.1007/BF02988307>.
- [17] M. Simonovits. Extremal graph problems, degenerate extremal problems, and super-saturated graphs. In *Progress in graph theory (Waterloo, Ont., 1982)*, pages 419–437. Academic Press, Toronto, ON, 1984.
- [18] E. Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978.
- [19] P. Turán. Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941.