

MATH 188: Homework #4

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Professor Kunnawalkam Elayavalli

Ray Tsai

A16848188

Problem 1

- (a) Let r be a fixed nonnegative integer. Show that both $S(n+r, n)$ and $c(n+r, n)$ are polynomial functions of n of degree $2r$ for $n \geq 0$.

Proof. We first prove the case for $S(n+r, n)$. Consider the number k of non-singleton blocks in a partition of $[n+r]$ with n blocks. To count the number of partitions with exactly k non-singleton blocks, we first pick the $n-k$ elements from $[n+r]$ that are in singletons, and then we calculate the number $a_{k,r}$ of possible orientations of the remaining $k+r$ elements. Note that $a_{k,r}$ is not dependent on n . Hence, summing over all possible k , we have

$$S(n+r, n) = \sum_{k=1}^r \binom{n+r}{n-k} a_{k,r} = \sum_{k=1}^r \binom{n+r}{r+k} a_{k,r}.$$

But then $\binom{n+r}{r+k} a_{k,r}$ is a polynomial of n of degree $r+k$. Since $r+k$ goes up to $2r$ exactly once, $S(n+r, n)$ is a polynomial of n of degree $2r$.

The similar argument works for $c(n+r, n)$. Consider the number k of non-trivial cycles in a permutation of size $n+r$ with n disjoint cycles. To count the number of permutation with exactly k non-trivial cycles, we first pick the $n-k$ elements from $[n+r]$ such that each of them are cycles on its own, and then we calculate the number $b_{k,r}$ of possible cycle formations of the remaining $k+r$ elements. Note that $b_{k,r}$ is not dependent on n . Hence, summing over all possible k , we have

$$c(n+r, n) = \sum_{k=1}^r \binom{n+r}{n-k} b_{k,r} = \sum_{k=1}^r \binom{n+r}{r+k} b_{k,r}.$$

But then $\binom{n+r}{r+k} b_{k,r}$ is a polynomial of n of degree $r+k$. Since $r+k$ goes up to $2r$ exactly once, $c(n+r, n)$ is a polynomial of n of degree $2r$. \square

- (b) Compute these polynomials for $r = 2, 3$.

Proof. We first compute $S(n+r, n)$ for $r = 2, 3$. When $r = 2$, there are either 1 or 2 non-singleton blocks. If there is only one non-singleton block, then 3 elements are in a block and the remaining elements each form a singleton, which has $\binom{n+2}{3}$ possibilities. If there are 2 non-singleton blocks, then there are 2 blocks of size 2 and $n-2$ singletons, which has $3\binom{n+2}{4}$ possibilities. Hence, $S(n+2, n) = \binom{n+2}{3} + 3\binom{n+2}{4}$. When $r = 3$, the number of non-singleton blocks ranges from 1 to 3. If there is only one non-singleton block, then 4 elements are in a block and the remaining elements each form a singleton, which has $\binom{n+3}{4}$ possibilities. If there are 2 non-singleton blocks, then there is a block of size 2, a block of size 3, and $n-2$ singletons, which has $\binom{5}{2}\binom{n+3}{5} = 10\binom{n+3}{5}$ possibilities. If there are 3 non-singleton blocks, then there are 3 blocks of size 2 and all singletons for the rest, which has $\frac{1}{3!}\binom{6}{2}\binom{4}{2}\binom{n+3}{6} = 15\binom{n+3}{6}$ possibilities. Hence, $S(n+3, n) = \binom{n+3}{4} + 10\binom{n+3}{5} + 15\binom{n+3}{6}$.

We now compute $c(n+r, n)$ for $r = 2, 3$. When $r = 2$, there are either 1 or 2 non-trivial cycles. If there is only one non-trivial cycle, then there is a 3-cycle and $n-1$ singletons, which has $2\binom{n+2}{3}$ possibilities. If there are 2 non-trivial cycles, then there are 2 transpositions and $n-2$ singletons, which has $3\binom{n+2}{4}$ possibilities. Hence, $c(n+2, n) = 2\binom{n+2}{3} + 3\binom{n+2}{4}$. When $r = 3$, the number of non-trivial cycles ranges from 1 to 3. If there is only one non-trivial cycle, then there is a 4-cycle and $n-1$ singletons, which has $3!\binom{n+3}{4}$ possibilities. If there are 2 non-trivial cycles, then there is a transposition, a 3-cycle, and $n-2$ singletons, which has $2\binom{5}{2}\binom{n+3}{5} = 20\binom{n+3}{5}$ possibilities. If there are 3 non-trivial cycles, then there are 3 transpositions and all singletons for the rest, which has $\frac{1}{3!}\binom{6}{2}\binom{4}{2}\binom{n+3}{6} = 15\binom{n+3}{6}$ possibilities. Hence, $c(n+3, n) = 6\binom{n+3}{4} + 20\binom{n+3}{5} + 15\binom{n+3}{6}$. \square

Problem 2

For $n > 0$, let a_n be the number of partitions of n such that every part appears at most twice, and let b_n be the number of partitions of n such that no part is divisible by 3. Set $a_0 = b_0 = 1$. Show that $a_n = b_n$ for all n .

Proof. Let $A(x)$ be the generating function of a_n and $B(x)$ be the generating function of b_n . Since a_n is the number of partitions of n such that every part appears at most twice,

$$A(x) = \sum_{n \geq 0} a_n x^n = \prod_{i \geq 1} (1 + x^i + x^{2i}),$$

as we either choose 1, x^i , or x^{2i} from the i th term, when multiplying out the right side. What we get then is x^N where N is the sum of the i where we chose x^i or x^{2i} . But we get x^N one time for every partition of N into parts which repeat at most once, so the coefficient is a_N .

On the other hand, since b_n is the number of partitions of n such that no part is divisible by 3,

$$B(x) = \sum_{n \geq 0} b_n x^n = \prod_{i \geq 1, 3 \nmid i} \frac{1}{1 - x^i} = \frac{\prod_{i \geq 1} \frac{1}{1 - x^i}}{\prod_{i \geq 1} \frac{1}{1 - x^{3i}}} = \prod_{i \geq 1} \frac{1 - x^{3i}}{1 - x^i}.$$

But then notice that $1 + x^i + x^{2i} = \frac{1 - x^{3i}}{1 - x^i}$ for all i . Hence,

$$A(x) = \prod_{i \geq 1} \frac{1 - x^{3i}}{1 - x^i} = B(x),$$

and the result now follows. □

Problem 3

Let y be a variable. Prove the following generalization of Example 3.27:

$$\prod_{i \geq 0} (1 + x^{2i+1}y) = \sum_{r \geq 0} \frac{x^{r^2} y^r}{(1-x^2)(1-x^4) \cdots (1-x^{2r})}$$

Proof. Notice that $[y^k x^n] \prod_{i \geq 0} (1 + x^{2i+1}y)$ is counting the number of partitions of n with k distinct odd parts, as the exponent of the y term indicates the number of times we picked the $x^{2i+1}y$ term when expanding the multiplication. On the other hand, from Example 3.27 we know

$$[y^k x^n] \sum_{r \geq 0} \frac{x^{r^2} y^r}{(1-x^2)(1-x^4) \cdots (1-x^{2r})} = [y^k x^n] \sum_{r \geq 0} y^r \left(x^{r^2} \sum_{n \geq 0} p_{\leq r}(n) x^{2n} \right)$$

is counting the number of self-conjugate partitions of n with a Durfee square of size k . We now show that there is a bijection between the set of self-conjugate partitions with Durfee square of size r and the set of partition with r distinct odd parts. Given a self-conjugate partition of n which has a Durfee square of size r , we may use the reversible transformation described in Theorem 3.26 to obtain a new partition of n with r distinct odd parts, and thus the bijection. The result now follows from the bijection. \square

Problem 4

- (a) Use the following q -analogue of Pascal's identity (you don't need to prove it)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \quad \text{for } n \geq k > 0$$

to show that if d is a non-negative integer, then

$$\sum_{n \geq 0} \begin{bmatrix} n+d \\ n \end{bmatrix}_q x^n = \prod_{i=0}^d (1 - q^i x)^{-1} = \frac{1}{(1-x)(1-qx) \cdots (1-q^d x)}$$

Proof. We proceed by induction on d . If $d = 0$, then $\sum_{n \geq 0} \begin{bmatrix} n \\ n \end{bmatrix}_q x^n = \sum_{n \geq 0} x^n = \frac{1}{1-x}$, and the base case is done. Suppose $d \geq 1$. Then,

$$\begin{aligned} \sum_{n \geq 0} \begin{bmatrix} n+d \\ n \end{bmatrix}_q x^n &= 1 + \sum_{n \geq 1} q^n \begin{bmatrix} n+(d-1) \\ n \end{bmatrix}_q x^n + \sum_{n \geq 1} \begin{bmatrix} (n-1)+d \\ n-1 \end{bmatrix}_q x^n \\ &= \sum_{n \geq 0} \begin{bmatrix} n+(d-1) \\ n \end{bmatrix}_q (qx)^n + x \sum_{n \geq 0} \begin{bmatrix} n+d \\ n \end{bmatrix}_q x^n \\ &= \frac{1}{1-x} \sum_{n \geq 0} \begin{bmatrix} n+(d-1) \\ n \end{bmatrix}_q (qx)^n \\ &= \frac{1}{(1-x)(1-qx) \cdots (1-q^d x)}, \end{aligned}$$

where the last equality follows from induction. \square

- (b) Give a direct explanation (i.e., independent of the Schubert decomposition explanation from lecture) for why the coefficient of x^n of the right side is the sum $\sum q^{|\lambda|}$ over all integer partitions λ whose Young diagram fits in the $n \times d$ rectangle.

Proof. Note that

$$[x^n] \prod_{i=0}^d (1 - q^i x)^{-1} = \sum_{\substack{(a_0, \dots, a_d) \in \mathbb{Z}^d \\ a_0 + \dots + a_d = n}} q^{a_1 + \dots + da_d}.$$

We now show the bijection between the weak compositions of n with $d+1$ parts and the integer partitions λ whose Young diagram fits in the $n \times d$ rectangle.

Given an integer partitions λ whose Young diagram fits in the $n \times d$ rectangle, let a_i be the number of parts of λ that are equal to $i \geq 1$ and put $a_0 = n - a_1 - \dots - a_d$. Then, (a_0, a_1, \dots, a_d) is a weak composition of n .

On the other hand, given a weak compositions (a_0, \dots, a_d) of n , there is an integer partition λ with a_i number of i 's for all $i \geq 1$. Since each part of λ is at most d and $\ell(\lambda) \leq n$, the Young diagram of λ fits in the $n \times d$ rectangle.

But then $a_1 + \dots + da_d = |\lambda|$, and thus

$$\sum_{\substack{(a_0, \dots, a_d) \in \mathbb{Z}^d \\ a_0 + \dots + a_d = n}} q^{a_1 + \dots + da_d} = \sum q^{|\lambda|}.$$

\square

Problem 5

Let V, W be \mathbf{F}_q -vector spaces with $\dim V = n$ and $\dim W = m$.

- (a) How many linear maps $V \rightarrow W$ are there?

Proof. Consider the number of ways we can map the canonical basis vectors e_1, \dots, e_n of V to some vectors in W . Since there are q^m choices of vectors for each e_i to be sent to, there are q^{mn} choices in total. Hence, there are q^{mn} linear maps $V \rightarrow W$. \square

- (b) Suppose $n \geq m$. How many surjective linear maps $V \rightarrow W$ are there?

Proof. By the universal property of a quotient and the First Isomorphism Theorem, any surjective linear map $\phi : V \rightarrow W$ corresponds to a unique induced isomorphism $u : V/\text{Ker } \phi \rightarrow W$. Note that $\text{Ker } \phi$ is of $(n - m)$ -dimension and the number of isomorphisms $V/\text{Ker } \phi \rightarrow W$ is equal to $|\mathbf{GL}_m(\mathbf{F}_q)|$. Hence, there is a bijection between the set of surjective linear maps $V \rightarrow W$ and $\mathbf{Gr}_{n-m}(\mathbf{F}_q^n) \times \mathbf{GL}_m(\mathbf{F}_q)$. But then by Theorem 3.34 and 3.35,

$$|\mathbf{Gr}_{n-m}(\mathbf{F}_q^n)| = \begin{bmatrix} n \\ m \end{bmatrix}_q, \quad |\mathbf{GL}_m(\mathbf{F}_q)| = \prod_{i=0}^{m-1} (q^m - q^i),$$

and thus there are $\begin{bmatrix} n \\ m \end{bmatrix}_q \prod_{i=0}^{m-1} (q^m - q^i) = \prod_{i=0}^{m-1} (q^n - q^i)$ surjective linear maps $V \rightarrow W$. \square

- (c) Pick $k \leq \min(m, n)$. How many rank k linear maps $V \rightarrow W$ are there?

Proof. By the universal property of a quotient and the First Isomorphism Theorem, any linear map $\phi : V \rightarrow W$ of rank k corresponds to a unique induced isomorphism from V/K to some k -dimensional subspace U of W , where K is the kernel of ϕ . Note that K is of $(n - k)$ -dimension and the number of isomorphisms $V/K \rightarrow W$ is equal to $|\mathbf{GL}_k(\mathbf{F}_q)|$. Since there are $|\mathbf{Gr}_{n-k}(\mathbf{F}_q^n)|$ choices for K , $|\mathbf{Gr}_k(\mathbf{F}_q^m)|$ choices for U , and $|\mathbf{GL}_k(\mathbf{F}_q)|$ choices for isomorphisms $V/K \rightarrow W$, there are

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} (q^k - q^i) = \begin{bmatrix} m \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} (q^n - q^i)$$

rank k linear maps $V \rightarrow W$, by Theorem 3.34 and 3.35. \square

Problem 6

Prove

$$\sum_{n \geq 1} x^{n(n-1)/2} = \prod_{n \geq 1} \frac{1 - x^{2n}}{1 - x^{2n-1}}.$$

Proof. By the Jacobi triple product,

$$\prod_{n \geq 1} (1 - x^{2n})(1 + x^{2n-1}y)(1 + x^{2n-1}y^{-1}) = \sum_{n=-\infty}^{\infty} x^{n^2} y^n.$$

Hence, substituting both x and y as \sqrt{x} , we have

$$\begin{aligned} \prod_{n \geq 1} (1 - x^n)(1 + x^n)(1 + x^{n-1}) &= \sum_{n=-\infty}^{\infty} x^{n(n+1)/2} \\ &= 1 + \sum_{n \geq 1} x^{n(n+1)/2} + x^{n(n-1)/2} \\ &= \sum_{n \geq 0} x^{n(n+1)/2} + \sum_{n \geq 1} x^{n(n-1)/2} \\ &= 2 \sum_{n \geq 0} x^{n(n+1)/2}. \end{aligned}$$

It now follows that

$$\begin{aligned} \sum_{n \geq 0} x^{n(n+1)/2} &= \frac{1}{2} \prod_{n \geq 1} (1 - x^n)(1 + x^n)(1 + x^{n-1}) \\ &= \frac{1}{2} \prod_{n \geq 1} (1 - x^{2n})(1 + x^{n-1}) \\ &= \left(\prod_{n \geq 1} (1 - x^{2n}) \right) \left(\frac{1}{2} \prod_{n \geq 0} (1 + x^n) \right) \\ &= \left(\prod_{n \geq 1} (1 - x^{2n}) \right) \left(\prod_{n \geq 1} (1 + x^n) \right) \\ &= \prod_{n \geq 1} \frac{1 - x^{2n}}{1 - x^{2n-1}}, \end{aligned}$$

where the last step follows from Theorem 3.25. □

Source cited: <https://www.math.uwaterloo.ca/~dmjacks/C0630/JTPID.pdf>

Problem 7

Pick integers satisfying $1 \leq k_1 < k_2 < \cdots < k_r \leq n$. Let X be the set of subspaces W_1, \dots, W_r of F_q^n such that $\dim W_i = k_i$ for all i and $W_i \subset W_{i+1}$ for $i < r$.

- (a) Find a formula for $|X|$ by generalizing Example 3.39, i.e., use a q -analogue of a multinomial coefficient.

Proof. For any $n \geq k_r > \cdots > k_1 \geq 1$, we show that there are $\begin{bmatrix} n \\ k_1, k_2 - k_1, \dots, n - k_r \end{bmatrix}_q$ ways of picking subspaces W_1, \dots, W_r of F_q^n by induction on r . If $r = 1$, it is obvious that there are $\begin{bmatrix} n \\ k_1 \end{bmatrix}_q$ ways of picking W_1 . Suppose $r \geq 2$. There are $\begin{bmatrix} n \\ k_r \end{bmatrix}_q$ ways of picking W_r . But then by induction, there are $\begin{bmatrix} k_r \\ k_1, k_2 - k_1, \dots, k_r - k_{r-1} \end{bmatrix}_q$ ways of picking W_1, \dots, W_{r-1} which are contained in W_r . It now follows that there are

$$\begin{bmatrix} n \\ k_r \end{bmatrix}_q \begin{bmatrix} k_r \\ k_1, k_2 - k_1, \dots, k_r - k_{r-1} \end{bmatrix}_q = \begin{bmatrix} n \\ k_1, k_2 - k_1, \dots, n - k_r \end{bmatrix}_q$$

ways of picking W_1, \dots, W_r of F_q^n . \square

- (b) $|X|$ is also a polynomial in q ; find an explicit description of this polynomial using a generalization of the Schubert decomposition of the Grassmannian.

Proof. By the Schubert decomposition of $\mathbf{Gr}_k(F_q^n)$, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = |\mathbf{Gr}_k(F_q^n)| = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|},$$

where λ is any integer partition whose Young diagram fits into the $k \times (n - k)$ box. It now follows that,

$$\begin{aligned} |X| &= \begin{bmatrix} n \\ k_1, k_2 - k_1, \dots, n - k_r \end{bmatrix}_q \\ &= \begin{bmatrix} n \\ k_r \end{bmatrix}_q \begin{bmatrix} k_r \\ k_{r-1} \end{bmatrix}_q \cdots \begin{bmatrix} k_2 \\ k_1 \end{bmatrix}_q \\ &= \left(\sum_{\lambda \subseteq k_r \times (n - k_r)} q^{|\lambda|} \right) \left(\sum_{\lambda \subseteq k_{r-1} \times (k_r - k_{r-1})} q^{|\lambda|} \right) \cdots \left(\sum_{\lambda \subseteq k_1 \times (k_2 - k_1)} q^{|\lambda|} \right). \end{aligned}$$

\square