MATH 264A Lecture Notes

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This note is for the graduate combinatorics course MATH 264A at UC San Diego, taught by Professor Lutz Warnke in 2024 Fall. The proofs below are merely my attempts of recreating the contents in lectures, which might not be accurate representations of what was actually taught.

1 Some Basic Tools

This lecture introduces some simple but powerful tools.

Inductive Approaches

This method is done by changing the size of the problem, e.g. adding an vertex or an edge in a graph.

Example 1.1. *Every n-vertex graph with maximum degree Δ has $\geq \beta^n$ valid vertex colorings with $\leq \lceil \Delta + \beta \rceil$ colors.*

Proof. Color the vertices $v_1, v_2, \ldots, v_n$ sequentially. Since v_i has $\leq \Delta$ neighbors already colored, there are $\geq \lceil \Delta + \beta \rceil - \Delta \geq \beta$ choices to color v_i. Define N_i as # valid colorings of $v_1, \ldots, v_i$. Then, the *Telescoping Product* now yields

$$N_n = \frac{N_n}{N_{n-1}} \cdot \frac{N_{n-2}}{N_{n-1}} \cdots \frac{N_1}{N_0} \cdot N_0 \geq \beta^n,$$

as $N_0 = 1$. □

Despite being an extremely basic technique, induction can prove several advanced theorems if used artfully. The following are some exciting theorems which can be proven by induction:

1. Strengthen Lovász Local Lemma (LLL)
2. Chromatic number of triangle-free graph with max-degree Δ is $\leq (1 + o(1))\frac{\Delta}{\log \Delta}$ as $\Delta \to \infty$.
3. Almost all triangle-free graphs are bipartite.

Double Counting/Switching

Also known as the Pertubation method, e.g. change of location of edges.

Example 1.2. *Find the $\#\Pi \in S_n$ without fix-points, i.e. $\Pi(i) \neq i$ for all i.*

Proof. We prove this by a basic approach which consists of several steps:

Step 1: Define the "Switching Operation." Let $S_{n,k}$ be the set of permutations with k fix-points. Define the switching operation to transform $\pi \in S$ to $\pi' \in S_{n,1}$.

Step 2: Consider the auxiliary bipartite graph. Let $S_{n,0}, S_{n,1}$ be parts of the bipartite graph. Connect $\pi \in S_{n,0}$ with $\pi' \in S_{n,1}$ if π' results from π through the switching operation.

Step 3: Double count the degrees.

$$\sum_{\pi \in S_{n,0}} \deg \pi = \sum_{\pi' \in S_{n,1}} \deg \pi'$$

Step 4: Degree essentially transfers to ratio. Suppose $\deg \pi \approx a$ and $\deg \pi' \approx b$, for all $\pi \in S_{n,0}$ and $\pi' \in S_{n,1}$. Then,

$$\frac{|S_{n,0}|}{|S_{n,1}|} \approx \frac{b}{a}.$$

□

This method can be applied to count d-regular graphs with certain properties, i.e. random model without independence.

Asymptotic Methods

Rather than finding the close form of a discrete function, sometimes it is significantly easier to approximate the function in asymptotic settings.

Bootstrapping

Suppose we have an equation $w(z)e^{w(z)} = z$ and we try to extract $w(z)$. By bootstrapping, $w(z) = \ln z - \ln \ln z + o(1)$.

Integral-Approximation

As the title suggests, this method estimates a summation $\sum_{k \in I} f(k)$ with its integral counterpart $\int_I f(x)\, dx$. For example, the the summation derived from the Fibonacci Tiliing Problem can be estimated by the Laplace-Method, i.e.

$$\sum_{0 \le k \le \frac{n}{2}} \binom{n-k}{k} \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} \qquad n \to \infty.$$

2 Inductive Counting

This lecture introduces two exmaples the on inductive counting approach, which often improves the Lovász Local Lemma. The basic approach of this method is to first generalize the problem then use structural induction on, say, the number of vertices or edges. It often involves extending from smaller cases then counting the number of “bad” extensions based on some observed patterns of them.

Non-repetitive Words

A word w is defined as a sequence of symbols from alphabet $\mathcal{A}$. A word w is non-repetitive if no word appears twice in w consecutively. For example, $\underline{abab}$ and $b\underline{abcabc}$ are repetitive words, while ab and aba are non-repetitive words.

The following Theorem is proven by Thue in 1906:

Theorem 2.1. *Let $\mathcal{A}$ be a 3-symbol alphabet. Then for all $n \geq 1$, there exists a non-repetitive word $a_1 \dots a_n$, with $a_i \in \mathcal{A}$ for all i.*

A conjecture on non-repetitive words is that

Conjecture 2.2. *Let $L_1, \dots, L_n$ be subsets of an alphabet $\mathcal{A}$, with $|L_i| = 3$ for all i. Then for $n \geq 1$, there exists a non-repetitive word $a_1 \dots a_n$, with $a_i \in L_i$ for all i.*

This conjecture remains open till this day (2024), but we can prove a slightly weaker version of it using induction:

Theorem 2.3. *Let $L_1, \dots, L_n$ be subsets of an alphabet $\mathcal{A}$, with $|L_i| = 4$ for all i. Then for $n \geq 1$, there exists are at least 2^n non-repetitive words $a_1 \dots a_n$, with $a_i \in L_i$ for all i.*

Proof. We first generalize the problem. Define N_k as the number of non-repetitive words of $a_1 \dots a_k$, with $a_i \in L_i$ for all $i \in [k]$. It suffices to show $N_k \geq 2^k$. Put $\beta = 2$. We proceed by induction on $k \geq 1$ to show that $\frac{N_k}{N_{k-1}} \geq \beta$.

Base Case: We already know $N_0 = 1$. Since there are $|L_1| = 4$ choices for the first symbol, $\frac{N_1}{N_0} = N_1 = 4 \geq \beta$, so the base case is done.

Inductive Step: Suppose $k \geq 2$. Here we use symbols from L_k to (try to) extend each N_{k-1} non-repetitive words of length $k-1$. Hence we may write

$$\#\text{all extensions} = N_{k-1} \cdot |L_k| = N_k + B,$$

where B is the number of repetitive (bad) extensions . By construction, repetition must happen at the end of the extension, say the last j symbols. But then those j symbols uniquely determines the j symbols preceeding them. Summing over all possible j's, we get $B \leq \sum_{j \in [k/2]} N_{k-j}$. By induction, $N_{k-j} \leq N_{k-1} \left(\frac{1}{2}\right)^{j-1}$, and thus

$$B \leq N_{k-1} \sum_{j \in [k/2]} \left(\frac{1}{\beta}\right)^{j-1} \leq N_{k-1} \cdot \frac{\beta}{\beta - 1}.$$

It immediately follows that when $\beta = 2$

$$N_k = N_{k-1} \cdot |L_k| - B \geq N_{k-1} \cdot |L_i| - \frac{\beta}{\beta - 1} \cdot N_{k-1} \geq \beta N_{k-1},$$

and this completes the induction. $\square$

Lower Bound on Size of Intependent Sets

Theorem 2.4. *Let $V_1, \ldots, V_s \subseteq V$ be disjoint sets of size $|V_i| \geq 4\Delta$. Then there exist $(2\Delta)^s$ intependent sets $I \subseteq V$ which contains one vertex from each V_i.*

Proof. Again, we first generalize the problem for $T \subseteq [s]$. Define N_T as the number of independent sets $I \subseteq V_T = \bigcup_{j\in T} V_j$ that contains one vertex from each V_j. It suffices to show that $N_T \geq (2\Delta)^{|T|}$. Put $\beta = 2\Delta$. We proceed by induction on $|T|$ to show that for all $x \in T$, $\frac{N_T}{N_{T-\{x\}}} \geq \beta$.

Base Case: Trivial, as $\frac{N_1}{N_1} = N_1 = |V_i| \geq 4\Delta \geq \beta$.

Inductive Step: Suppose $|T| \geq 2$. Here we try to use vertex from V_x to extend the $N_{T-\{x\}}$ many valid independent sets for $V_{T-\{x\}}$. Hence we write

$$N_{T-\{x\}} \cdot |V_x| = N_T + B,$$

where B is again the number of invalid (bad) extensions. In particular, we need to count the number of possible ways that our extension contains an edge from V_x to an independent set in $V_{T-\{x\}}$. Hence,

$$B \leq |V_x| \cdot \Delta \cdot \max_{z\in T-\{x,z\}} N_{T-\{x,z\}} \leq \frac{4\Delta^2}{\beta} \cdot N_{T-\{x\}},$$

by induction. It now follows that when $\beta = 2\Delta$,

$$N_T = N_{T-\{x\}} \cdot |V_x| - B \geq 4\Delta\left(1 - \frac{\Delta}{\beta}\right) N_{T-\{x\}} \geq \beta N_{T-\{x\}},$$

and this completes the induction. □