MATH 262A: DISCRETE GEOMETRY NOTES

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1. Sums vs Product

Definition 1.1. The *crossing number* of a graph G, denoted $\mathrm{cr}(G)$, is the minimum number of crossing pair of edges over all possisble drawings of G in the plane.

Lemma 1.2 (Crossing Lemma)**.** *Let $G = (V, E)$ be a graph. If $|E| \geqslant 4|V|$, then*

$$\mathrm{cr}(G) \geqslant \frac{|E|^3}{64|V|^2}.$$

Theorem 1.3. *Let A be a set of n distinct real numbers. Then* $\max\{|A + A|, |A \cdot A|\} = \Omega(n^{5/4})$.

Proof. Denote $A + A = \{s_1, s_2, \ldots, s_x\}$ and $A \cdot A = \{p_1, p_2, \ldots, p_y\}$. Let L be the set of lines $v = a_i(u - a_j)$ for $a_i, a_j \in A$. Construct the graph $G = (V, E)$ with $V = (A + A) \times (A \cdot A)$ and $\{(s_i, p_i), (s_j, p_j)\} \in E$ if and only if there exists a line $l \in L$ such that (s_i, p_i) and (s_j, p_j) are consecutive points on l. Notice that each line passes through at least $n - 1$ points in V, so $|E| \geqslant (n - 1)|L| = \Omega(n^3)$. If $|E| < 4|V|$, then

$$4|A + A| \cdot |A \cdot A| = 4|V| > |E| = \Omega(n^3).$$

But then either $|A + A| = \Omega(n^{3/2})$ or $|A \cdot A| = \Omega(n^{3/2})$. Thus we may assume $|E| \geqslant 4|V|$. By the crossing lemma,

$$\frac{|E|^3}{64|V|^2} \leqslant \mathrm{cr}(G) \leqslant |L|^2 \leqslant n^4.$$

Rearranged, we have

$$|V|^2 \geqslant \frac{|E|^3}{64n^4} = \Omega(n^5).$$

The result now follows. □

2. Crossing Lemma

In this section we prove the Crossing lemma mentioned in the previous section.

Lemma 2.1. *Let $G = (V, E)$ be a graph. Then* $\operatorname{cr}(G) \geqslant |E| - 3|V|$.

Proof. Suppose not. We may assume $|E| \geqslant 3|V|$, otherwise we are done. Remove edges from each crossing until we have a planar graph. Since $\operatorname{cr}(G) < |E| - 3|V|$, we removed less than $|E| - 3|V|$ edges. But then the planar graph has more than $|E| - (|E| - 3|V|) = 3|V|$ edges, contradicting Euler's theorem. □

Lemma 2.2 (Crossing Lemma)**.** *Let $G = (V, E)$ be a graph. If $|E| \geqslant 4|V|$, then*

$$\operatorname{cr}(G) \geqslant \frac{|E|^3}{64|V|^2}.$$

Proof. For any graph H, define $X_H = \operatorname{cr}(H) - |E(H)| + 3|V(H)|$. By the crossing lemma we know $X_H \geqslant 0$. Consider the drawing of G in $\mathbb{R}^2$ with $\operatorname{cr}(G)$ crossings. Let $S \subseteq V$ be a set vertices where each vertex is chosen independently with probability $p \in [0, 1]$. Let $G' = G[S]$ be the induced subgraph on S. Then

$$\mathbb{E}[X_{G'}] = \mathbb{E}[\operatorname{cr}(G')] - \mathbb{E}[|E(G')|] + 3\mathbb{E}[|V(G')|] = \mathbb{E}[\operatorname{cr}(G')] - p^2|E| + 3p|V| \geqslant 0.$$

Let $C_{G'}$ be the number of crossings in the drawing of G' inherited from G. Obviously, $\mathbb{E}[\operatorname{cr}(G')] \leqslant \mathbb{E}[C_{G'}]$. Since each crossing pair has a probability of p^4 of being in G', we have $\mathbb{E}[C_{G'}] = p^4 \operatorname{cr}(G)$, and thus

$$p^4 \operatorname{cr}(G) \geqslant \mathbb{E}[\operatorname{cr}(G')] \geqslant p^2|E| - 3p|V|.$$

By setting $p = 4|V|/|E|$, we have

$$\operatorname{cr}(G) \geqslant \frac{|E|}{p^2} - \frac{3|V|}{p^3} \geqslant \frac{|E|^3}{64|V|^2}.$$

□

3. Szemerédi-Trotter Theorem

Definition 3.1. Let P be a set of n points and L be a set of m lines in the plane. We call a pair (p, l) *incidence* if $p \in P$, $l \in L$, and $p \in l$. Define $I(P, L)$ as the number of incidences between P and L, and define $I(m, n)$ as the maximum number of incidences between any m lines and n points.

Definition 3.2. Let P be a set of n points. A line is *generated by P* if it contains at least 2 points from P.

Definition 3.3. For $k \geqslant 2$ and a set of points P, a line l is k-rich if it contains at least k points from P.

Theorem 3.4 (Szemerédi-Trotter Theorem)**.** *For all $m, n \geqslant 1$, we have $I(m, n) = O(m^{2/3}n^{2/3} + m + n)$.*

Proof. We will adopt the same strategy as the proof of Theorem 1.3, which constructs a graph and double counts the number of crossings in it.

Let P be the set of n points in $\mathbb{R}^2$ and L be the set of m lines in $\mathbb{R}^2$. Define graph $G = (V, E)$ where $V = P$ and E is the set of consecutive pairs of vertices along some line in L. We may assume each line in L contains at least one point from P. For $l \in L$, let $|l|$ denote the number of points in P which lies in l. Observe that

$$|E| = \sum_{l \in L} |l| - 1 = |I(P, L)| - m.$$

Hence, it suffices to show that $|E| = O(m^{2/3}n^{2/3} + n)$. We may assume $|E| \geqslant 4|V|$, otherwise we are done. Note that the construction of G gives a natural drawing with points P and lines P in the plane, so we may define C as the number of crossings in this drawing. By the crossing lemma, we have

$$\frac{|E|^3}{64n^2} \leqslant \operatorname{cr}(G) \leqslant C \leqslant \binom{m}{2} = O(m^2).$$

It now follows that

$$|E| = O(n^{2/3}m^{2/3}).$$

This completes the proof. □

Corollary 3.5. *Let P be a set of n points. Then P generates $O(n^2/k^3 + n/k)$ k-rich lines.*

Proof. Let L_k be the set of k-rich lines generated by P. By the Szemerédi-Trotter theorem,

$$k|L_k| \leqslant I(P, L_k) = c(|L_k|^{2/3}n^{2/3} + |L_k| + n),$$

for some constant c. We may assume $k \geqslant 4c$, otherwise we are done as $|L_k| = O(n^2)$. If $n + |L_k| \geqslant |L_k|^{2/3} n^{2/3}$. Then

$$k|L_k| \leqslant 2c(|L_k| + n) = 2cm + 2c|L_k|.$$

Rearranged,

$$|L_k| \leqslant \frac{2cm}{k - 2c} \leqslant O(m/k).$$

Now suppose $n + |L_k| < |L_k|^{2/3} n^{2/3}$. Then

$$k|L_k| \leqslant 2c|L_k|^{2/3} n^{2/3},$$

and so

$$|L_k| = O(n^2/k^3).$$

□

4. The Cutting Lemma

Lemma 4.1 (Cutting Lemma)**.** *Let L be a set of m lines in $\mathbb{R}^2$ and let $r \in (1, m)$. Then the plane can be subdivied into $t = O(r^2)$ generalized triangles (intersections of three half planes) $\Delta_1, \Delta_2, \ldots, \Delta_t$ such that the interior of each Δ_i is intersected by at most m/r lines of L.*

Lemma 4.2. *Let L be a set of m lines in $\mathbb{R}^2$ and let $r \in (1, m)$. Then the plane can be subdivied into $t = O(r^2 \log^2 n)$ generalized triangles $\Delta_1, \Delta_2, \ldots, \Delta_t$ such that the interior of each Δ_i is intersected by at most m/r lines of L.*

Proof. Put $s = 6r \ln m$. Select a random set of lines $S \subset L$ by making s independent random draws with replacement. Consider the line arrangement of S. Partition any cell that is not a generalized triangle further by adding diagonals that connect vertices. To this end, $\mathbb{R}^2$ is partitioned into t generalized triangles. Consider a box B that contains all bounded triangles Δ_i. Since each line crosses through B two times and each two consecutive lines around B determine an unbounded triangle, the number of unbounded triangles is at most $2s$. Now consider the bounded triangles. View each intersecting point of two lines in S as a vertex of a graph, and each bounded triangle as a face. Let V denote the set of vertices and F the set of faces. We know that $|V| \leqslant \binom{s}{2} = O(s^2)$. By Euler's formula, we have

$$3|F| \leqslant \sum_{f \in F} \deg f = 2|E| = 2(|V| + |F| - 2),$$

and thus

$$|F| \leqslant 2|V| - 4 = O(s^2).$$

Hence, we have $t = O(s^2)$.

We call a (generalized) triangle *horny* if its interior intersects at least m/r lines of L. For any horny triangle T, the probability that no line in S intersects the interior of T is at most $(1 - 1/r)^s$. Using the inequality $1 - x \leqslant e^{-x}$, we have $(1 - 1/r)^s \leqslant e^{-6 \ln m} = m^{-6}$.

Now call a triangle *interesting* if it can appear in a triangulation for some sample $S \subset L$. Notice that each vertex of an interesting triangle is an intersecting point of two lines in the arrangement of L, and thus there are at most $\binom{m}{2}^3 < m^6$ such triangles.

But then the expected number of horny Δ_i's is less than $m^{-6} \cdot m^6 = 1$. It now follows that there exists a set of $S \subseteq L$ such that each Δ_i is intersected by at most m/r lines. $\square$

5. An Aliter for the Szemerédi-Trotter Theorem

Theorem 5.1 (Kővári-Sós-Turán Theorem)**.** *For $s, t \geqslant 2$, let G be an $m \times n$ bipartite graph that does not contain a complete bipartite graph $K_{s,t}$ where the s vertices are from the part of size m. Then,*

$$|E(G)| = O(nm^{1-1/t} + m) \quad \text{and} \quad |E(G)| = O(mn^{1-1/s} + n).$$

Proof. Let M, N be the two parts of the bipartite graph G, with $|M| = m$ and $|N| = n$. Notice that no set of s vertices in M has more than $t-1$ common neighbors in N, so

$$\sum_{v \in M} \binom{d(v)}{t} \leqslant \binom{n}{t}(s-1) \leqslant \frac{sn^t}{t!}.$$

By Jensen's inequality, we have

$$\sum_{v \in M} \binom{d(v)}{t} \geqslant m \binom{\frac{1}{m}\sum_{v \in M} d(v)}{t} \geqslant \frac{m(2|E(G)|/m - t)^t}{t!}.$$

The result now follows from the two inequalities. □

Corollary 5.2. *$|I(m,n)| \leqslant O(n\sqrt{m} + m)$ and $|I(m,n)| \leqslant O(m\sqrt{n} + n)$.*

Proof. Let P be the set of n points and L be the set of m lines in $\mathbb{R}^2$. Let $G = (P, L)$ be the bipartite graph with parts P and L and (p, l) is an edge if and only if $p \in l$. Since no two points lie on the same line, G is $K_{2,2}$-free. The resulting bounds now follows from the Kővári-Sós-Turán theorem. □

We give an alternative proof of a case of the Szemerédi-Trotter theorem with n points and n lines, using the Cutting lemma and the Kővári-Sós-Turán theorem.

Aliter for Theorem 3.4. Let P be the set of n points and L be the set of n lines in $\mathbb{R}^2$. We need to show that there are at most $O(n^{4/3})$ incidences between P and L. We apply the cutting lemma with $r = n^{1/3}$, which divides the plane into $t = O(n^{2/3})$ generalized triangles $\Delta_1, \Delta_2, \ldots, \Delta_t$.

Let V be the points that lie on the vertex of some Δ_i. Since $|V| \leqslant 3t = O(n^{2/3})$, Corollary 5.2 gives us $|I(V, L)| = O(n^{2/3}\sqrt{n} + n^{2/3}) = O(n^{4/3})$.

Let $|L'|$ be the set of lines that borders some triangle Δ_i. Then $|L'| \leqslant 3t = O(n^{2/3})$, and Corollary 5.2 again gives us $|I(P_0, L')| = O(n^{2/3}\sqrt{n} + n^{2/3}) = O(n^{4/3})$.

It remains to count the incidences that occur at the interior of some triangle. Let P_i be the set of points in P that lies in the interior of Δ_i. Let L_i be the set of lines intersecting the interior of Δ_i. By the cutting lemma, $|L_i| \leqslant n/r = O(n^{2/3})$. Hence,

$$\sum_{i=1}^{t} I(P_i, L_i) \leqslant \sum_{i=1}^{t} I(P_i, n^{2/3}) = \sum_{i=1}^{t} O(|P_i| n^{1/3} + n^{2/3}) = O(n^{4/3}).$$

$\square$

6. Beck's Theorem

Theorem 6.1 (Beck's Theorem). *Given a set of n points P, there exists $\epsilon \in (0,1)$ such that either P contains ϵn points on a line or P generates at least ϵn^2 distinct lines.*

Proof. We may assume n is large, otherwise we the problem is trivial. Let P be a set of n points in $\mathbb{R}^2$. For $b > a \geqslant 2$, let $L_{[a,b]}$ be the set of lines generated by P with least a but less than b points on it. By Corollary 3.5, $L_{[a,b]} = O(n^2/a^3)$. We first make the following two observations:

For $k \leqslant \sqrt{n}$,

$$\#\{\{p_1,p_2\} : p_1, p_2 \in l,\ l \in L_{[k,\sqrt{n}]}\} \leqslant \sum_{i=0}^{\log_2 \frac{\sqrt{n}}{k}} |L_{[2^i k, 2^{i+1}k]}| \binom{2^{i+1}k}{2} = \sum_{i=0}^{\log_2 \frac{\sqrt{n}}{k}} O(n^2/2^i k) = O(n^2/k).$$

Hence, for $k < \sqrt{n}$, there are $O(n^2/k)$ pair of points in P that lies on a line with at least k but at most $\sqrt{n}$ points.

For $K > \sqrt{n}$,

$$\#\{\{p_1,p_2\} : p_1, p_2 \in l,\ l \in L_{[\sqrt{n},K]}\} \leqslant \sum_{i=0}^{\log_2 \frac{K}{\sqrt{n}}} |L_{[2^i\sqrt{n}, 2^{i+1}\sqrt{n}]}| \binom{2^{i+1}\sqrt{n}}{2} = \sum_{i=0}^{\log_2 \frac{K}{\sqrt{n}}} O(2^i n^{3/2}) = O(Kn).$$

Hence, there are $O(Kn)$ pairs of points from P that lies on a line with at least $\sqrt{n}$ but at most K points.

We now prove the theorem. Let $\epsilon \in (0,1)$ and set $\epsilon' = 4\sqrt{\epsilon}$. Assume that no $\epsilon' n$ points in P are colinear. Let $K = \epsilon' n$ and note that $K > \sqrt{n}$. Then the number of pairs of points in P that lies on a line with at least $\sqrt{n}$ but at most K points is $O(Kn) \leqslant c\epsilon' n^2 \leqslant n^2/10$, for some constant c and suffciently small ϵ. Now let $k = 1/\epsilon'$ and note that $k \leqslant \sqrt{n}$. Then the number of pairs of points in P that lies on a line with at least k but at most $\sqrt{n}$ points is $O(n^2/k) \leqslant c'\epsilon' n^2 \leqslant n^2/10$, for some constant c' and ϵ suffciently small. But then the number of pairs of points in P that lies in a k-rich line is at most $n^2/10 + n^2/10 = n^2/5$. Thus there are at least $\binom{n}{2} - n^2/5 \geqslant n^2/4$ pairs in P that lies on a line with at most k points, and so there are at least $\frac{n^2/4}{\binom{k}{2}} \geqslant \epsilon m^2$ distinct lines generated by P. ∎

7. Simplicial Partition

Theorem 7.1 (Simplicial Partition). *Let P be n points in $\mathbb{R}^2$. There exists partition $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_{2r}$ and generalized triangles $\Delta_1, \Delta_2, \ldots, \Delta_{2r}$, with $P_i \subset \Delta_i$, $|P_i| = n/2r$ for $i < 2r$ and $|P_{2r}| \leqslant n/2r$, such that for any line l generated by P, l will cross the interior of $O(\sqrt{r})$ number of Δ_i's.*

Proof. Pick $r > (\log n)^2$. Let L be the set of lines generated by P. Let $\Delta'_1 \cup \Delta'_2 \cup \cdots \cup \Delta'_r$ be the generalized trianges yielded by the cutting lemma on L with parameter $t = r$. By the pigeonhole principle, there exists Δ_i that contains $\geqslant n/r$ points from P. Let P_1 be some $n/2r$ points selected from Δ_i excluding the corners, and let $\Delta_1 = \Delta'_i$. Set $P' = P \backslash P_1$. For each line that crosses the interior of Δ_1, we double it by creating a copy of the line close to it, and let L' be all the lines after this process. Note that by the cutting lemma, the number of lines that cross the interior of Δ_1 is $c|L|/\sqrt{r}$ for some $c > 0$, and so

$$|L'| \leqslant |L| + \frac{c|L|}{\sqrt{r}} = \left(1 + \frac{c}{\sqrt{r}}\right)|L|.$$

Now apply the cutting lemma again to L' with parameter $t = r(1 - 1/2r)$, and we get a generalized triangle Δ''_i wtih $\geqslant |P'|/t = \frac{|P'|}{r(1-1/2r)} = n/r$ points from P' that lies in Δ''_i. Set P_2 be some $n/2r$ points of P' in Δ''_i excluding the corners, and let $\Delta_2 = \Delta''_i$. Set $P'' = P' \backslash P_2$ and note taht $|P''| = (1 - 1/r)n$. For any line that crosses the interior of Δ_2, we double again it, and let L'' be all the lines after this process. By the same argument,

$$|L''| \leqslant |L'| + \frac{c|L'|}{\sqrt{r(1-1/2r)}} = \left(1 + \frac{c}{\sqrt{r(1-1/2r)}}\right)|L'| \leqslant \left(1 + \frac{c}{\sqrt{r}}\right)\left(1 + \frac{c}{\sqrt{r(1-1/2r)}}\right)|L|.$$

Repeat the above process, and after k iterations we get point sets $P_1, P_2, \ldots, P_k$ and generalized triangles $\Delta_1, \Delta_2, \ldots, \Delta_k$. Set $P^{(k)} = P \backslash (P_1 \cup P_2 \cup \cdots \cup P_k)$. Again, let $L^{(k)}$ be the set of lines after doubling the lines that cross the interior of some $\Delta_i^{(k)}$'s. Then

$$|P^{(k)}| = |P| - \frac{kn}{2r} = \left(1 - \frac{k}{2r}\right)n.$$

$$|L^{(k)}| \leqslant \left(1 + \frac{c}{\sqrt{r}}\right)\left(1 + \frac{c}{\sqrt{r-1/2}}\right) \cdots \left(1 + \frac{c}{\sqrt{r-(k-1)/2}}\right)|L| \leqslant |L| \exp\left(c \sum_{j=0}^{2r-1} \frac{1}{\sqrt{r-j/2}}\right).$$

Iterate this process until there are $< n/2r$ points left, and let P_{2r} be the remaining points and Δ_{2r} be some generalized triangle that contains P_{2r}.

It remains to show that any line $l \in L$ crosses the interior of $O(\sqrt{r})$ Δ_i's. Let x be the number of Δ_i's that some line l crosses. Notice that by the end of the process above,

$$2^x \leqslant \#\text{copies of } l \leqslant |L^{(2r)}| \leqslant |L| \exp\left(c \sum_{j=0}^{2r-1} \frac{1}{\sqrt{r-j/2}}\right) \leqslant n^2 e^{O(\sqrt{r})} = 2^{O(\sqrt{r})}.$$

This proves the theorem. □

8. Triangle Removal Lemma

Definition 8.1. The *density* of edges between two vertex sets A and B is
$$d(A,B) \coloneqq \frac{|E(A,B)|}{|A||B|}.$$

Definition 8.2. Let $\epsilon \in (0,1)$. The pair of vertex sets (A,B) is *ϵ-regular* if for all $A' \leqslant A$ and $B' \leqslant B$ such that $|A'| \geqslant \epsilon|A|$ and $|B'| \geqslant \epsilon|B|$, we have
$$|d(A',B') - d(A,B)| \leqslant \epsilon.$$

Definition 8.3. Given a graph $G=(V,E)$, a partition $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ is a *ϵ-regular* if
$$\sum_{(i,j)\in[k]^2,\,(V_i,V_j)\text{ not }\epsilon\text{-regular}} |V_i||V_j| \leqslant \epsilon|V|^2.$$

Note that we are only interested in dense graphs. This is because if $|E(A,B)| = o(|A||B|)$, the density of 0 and so the pair (A,B) is trivially ϵ-regular.

Theorem 8.4 (Szemerédi's Regularity Lemma)**.** *For all $\epsilon > 0$, there exists $k = k(\epsilon)$ such that for any graph $G=(V,E)$, there exists an ϵ-regular partition $V = V_1 \sqcup \cdots \sqcup V_k$.*

Lemma 8.5 (Counting Lemma)**.** *Let $G=(V,E)$ be a graph, such that V is partitioned into $X \sqcup Y \sqcup Z$ where each pair of them are ϵ-regular, and $d(X,Y)=\alpha, d(X,Z)=\beta, d(Y,Z)=\gamma$, with $\alpha,\beta,\gamma > 2\epsilon$. Then*
$$\#\{K_3 \subseteq G\} \geqslant (1-2\epsilon)(\alpha-\epsilon)(\beta-\epsilon)(\gamma-\epsilon)|X||Y||Z|.$$

Proof. For $x \in X$, denote $d_Y(x) = d(x)\cap Y$ and $d_Z(x) = d(x)\cap Z$. We claim that $d_Y(x) < (\alpha-\epsilon)|Y|$ for at most $\epsilon|X|$ vertices in X. Suppose otherwise. Let $X' \subseteq X$ be the set of vertices with $d_Y(x) < (\alpha-\epsilon)|Y|$. Since (X,Y) is ϵ-regular, $|d(X',Y)-d(X,Y)| \leqslant \epsilon$, and so
$$\alpha - \epsilon < d(X',Y) = \frac{|E(X',Y)|}{|X'||Y|} \leqslant \frac{(\alpha-\epsilon)|X'||Y|}{|X'||Y|} = \alpha - \epsilon.$$
This contradiction proves the claim. By the same argument, we also know that $d_Z(x) < (\gamma-\epsilon)|Y|$ for at most $\epsilon|X|$ vertices in X.

Let $x \in X$ with $d_Y(x) \geqslant (\alpha-\epsilon)|Y|$ and $d_Z(x) \geqslant (\gamma-\epsilon)|Z|$. Let $|Y'| = N(x)\cap Y$ and $|Z'| = N(x)\cap Z$. Then
$$\#\{K_3 \subseteq G, x \in K_3\} = |E(Y',Z')|.$$
Since $|d(Y',Z') - d(Y,Z)| < \epsilon$, we have
$$\beta - \epsilon < d(Y',Z') = \frac{|E(Y',Z')|}{|Y'||Z'|}.$$

Rearranging gives us

$$\#\{K_3 \subseteq G, x \in K_3\} =\geqslant (\beta - \epsilon)|Y'||Z'| \geqslant (\beta - \epsilon)(\alpha - \epsilon)(\gamma - \epsilon)|Y||Z|.$$

Since there are at least $(1 - 2\epsilon)$ such x's in X,

$$\#\{K_3 \subseteq G\} \geqslant (1 - 2\epsilon)(\alpha - \epsilon)(\beta - \epsilon)(\gamma - \epsilon)|X||Y||Z|.$$

□

Theorem 8.6 (Triangle Removal Lemma)**.** *For $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that every graph $G = (V, E)$ with $< \delta n^3$ triangles can be made triangle-free by removing $< \epsilon n^2$ edges.*

Proof. We prove by contrapositive. Suppose G has ϵn^2 edge disjoint triangles. Apply Szemerédi's regularity lemma to G with parameter $\epsilon/4$ to get a partition $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$. For (V_i, V_j), we delete all edges between V_i, V_j if one of the following holds:

(i) V_i, V_j are not $\epsilon/4$-regular. This deletes $< (\epsilon/4)n^2$ edges.
(ii) $d(V_i, V_j) < \epsilon/2$. This deletes $\sum_{(V_i,V_i)} d(V_i, V_j)|V_i||V_i| < (\epsilon/2)n^2$ edges.
(iii) $|V_i|$ or $|V_j|$ is less than $(\epsilon/4k)n$. This deletes $< (\epsilon/4)n^2$ edges.

In total, we delete $< \epsilon n^2$ edges. But then there remains at least 1 triangle in G. Let X, Y, Z be the three parts that contain the vertices of the triangle. By the counting lemma,

$$\#\{K_3 \subseteq G\} \geqslant (1 - \epsilon/2)(\epsilon/2 - \epsilon/4)^3(\epsilon/4k)^3 n^3.$$

The result now follows from setting $\delta = (1 - \epsilon/2)(\epsilon/2 - \epsilon/4)^3(\epsilon/4k)^3$. □

9. Roth's Theorem

Theorem 9.1 (Roth's Theorem). *For all $\epsilon \in (0,1)$, there exists n_0 such that for all $n > n_0(\epsilon)$, any subset of $[n]$ of size $\geqslant \epsilon n$ contains a 3-term arithmetic progression.*

Proof. Let $A \subseteq [n]$ be a set of size $\geqslant \epsilon n$. Consider the grid

$$\mathscr{G} = \{(a,0) : a \in [2n]\} \cup ([2n] \times [2n]) \backslash ([n] \times [2n]).$$

and set lines $l_a : y = x - a$ for $a \in A$. Let $P = \bigcup_{a \in A} l_a \cap \mathscr{G}$. Note that each line l_a intersects n points in $\mathscr{G}$, and so $|P| = |A|n \geqslant \epsilon n^2$. Let $L = L_1 \sqcup L_2 \sqcup L_3$, where L_1 is the set of n vertical lines that cover $\mathscr{G}$, L_2 is the set of $2n$ horizontal lines that cover $\mathscr{G}$, and L_3 is the set of n lines of slope -1 that cover $\mathscr{G}$. Define G as the graph with vertex set L and edges between two lines if they intersect at a point in P. Note that a triangle in G is formed for any three lines that intersect at a point in P, so there are ϵn^2 edge disjoint triangles. By the triangle removal lemma, there are at least δn^3 triangles in G for some $\delta > 0$. But then the only other way to form a triangle in G is for each two of the three lines to intersect at a point in P, and there are $\delta n^3 - \epsilon n^2 > 1$ of them for large enough n. Let $x, y, z \in P$ be the three points that form such triangle, where y is the intersection of the horizontal and vertical sides of the triangle. Let l_a, l_b, l_c be the three lines that pass through x, y, z respectively. Then the distance between l_a and l_b is the same as the distance between l_a and l_c, and so a, b, c form a 3-term arithmetic progression. $\square$

10. Solymosi's Theorem

Theorem 10.1. *Let P be a set of n points and L be a set of n lines in $\mathbb{R}^2$, and let r be a parameter. If the arrangement of P and L does not contain a triangle, then $|I(P,L)| = O(n^{4/3}/\log^* n) = o(n^{4/3})$, where $\log^*$ is the iterated logarithm.*

11. Hyperplane Arrangement

Definition 11.1. A *set system* is a tuple $(V, \mathcal{F})$, where V is a set and $\mathcal{F}$ is a collection of subsets of V.

Definition 11.2. A *hyperplane* in $\mathbb{R}^d$ is a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$.

Definition 11.3. A set H of hyperplanes in $\mathbb{R}^d$ is in *general position* if the intersection of any k members is $(d-k)$-dimensional, for all $k \in \{2, \dots, d\}$.

Theorem 11.4. *The number of cells in an arrangement of n hyperplanes in general position in $\mathbb{R}^d$ is*

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}.$$

Proof. We proceed by induction on n and d. There are $2 = \binom{1}{0} + \binom{1}{1}$ cells when $n = 1$ and $d > 0$, and there are $n + 1 = \binom{n}{0} + \binom{n}{1}$ cells when $d = 1$, so the base case is done. Suppose $d \geqslant 2$. Write $H = \{h_1, \dots, h_n\}$. By induction, the number of cells in the arrangement of $h_1, \dots, h_{n-1}$ is

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n-1}{d}.$$

Given the arrangement of $h_1, \dots, h_{n-1}$, the number of cells that h_n adds to this arrangement is the number of cells in the arrangement of $h_1, \dots, h_{n-1}$ on h_n, which is

$$\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{d-1},$$

by induction. Hence, by Pascal's identity, the total number of cells in the arrangement of $h_1, \dots, h_n$ is

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}.$$

∎

12. VC-Dimension

Definition 12.1. Given set system $(V, \mathcal{F})$, we say $X \subseteq V$ is *shattered* by $\mathcal{F}$ if for each subset $Y \subseteq X$ there exists $F \in \mathcal{F}$ such that $F \cap X = Y$.

Definition 12.2. The *VC-dimension* of $\mathcal{F}$ is the size of the largest subset of V that is shattered by $\mathcal{F}$.

Theorem 12.3 (Sauer-Shelah)**.** *Let $(V, \mathcal{F})$ be a set system with $|V| = n$. If $\mathcal{F}$ has VC-dimension d, then*

$$|\mathcal{F}| \leqslant \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{d}.$$

Proof. We proceed by induction on n and d. If $n = 0$ and $d \geqslant 1$, then we trivially have $|\mathcal{F}| \leqslant 1 = \binom{0}{0}$. Suppose $d = 0$ and $n \geqslant 0$. This implies no nonempty subset of V is shattered by $\mathcal{F}$. If $A, B \in \mathcal{F}$ are distinct and $|A| \geqslant |B|$, then there exists vertex $x \in A \backslash B$. But then $\{x\}$ is shattered by $\mathcal{F}$, and this contradiction shows $|\mathcal{F}| \leqslant 1 = \binom{n}{0}$. Thus the base case is done.

Now suppose $n \geqslant 1$ and $d \geqslant 1$. Fix a vertex $x \in V$ and define the set system $(V\backslash\{x\}, \mathcal{F}_1)$, where $\mathcal{F}_1 = \{A\backslash\{x\} : A \in F\}$. Note that $\mathcal{F}_1$ has VC-dimension $\leqslant d$. By induction,

$$|\mathcal{F}_1| \leqslant \binom{n-1}{0} + \binom{n-1}{1} + \cdots \binom{n-1}{d}.$$

Consider another set system $(V\backslash\{x\}, \mathcal{F}_2)$, where $\mathcal{F}_2 = \{A \in \mathcal{F} : x \notin A, \{x\} \cup A \in \mathcal{F}\}$. We show that $\mathcal{F}_2$ has VC-dimension $\leqslant d-1$. Suppose not. There exists $X \subseteq V$ of size d that is shattered by $\mathcal{F}_2$. That is, there are 2^d subsets of V, say $A_1, \ldots, A_{2^d}$, that shatter X. By definition of $\mathcal{F}_2$, $A_1 \cup \{x\}, \ldots, A_{2^d} \cup \{x\} \in \mathcal{F}$. But then $A_1, \ldots, A_{2^d}$ along with $A_1 \cup \{x\}, \ldots, A_{2^d} \cup \{x\}$ shatter $\{x\} \cup X$, contradiction. Hence, $\mathcal{F}_2$ has VC-dimension $\leqslant d-1$, and by induction,

$$|\mathcal{F}_2| \leqslant \binom{n-1}{0} + \binom{n-1}{1} + \cdots \binom{n-1}{d-1}.$$

It now follows from Pascal's identity that

$$\mathcal{F} \leqslant \mathcal{F}_1 + \mathcal{F}_2 \leqslant \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{d}.$$

□

Definition 12.4. Let $(V, \mathcal{F})$ be a set system and let $S \subseteq V$. Denote $\mathcal{F}|_S = \{A \cap S : A \in \mathcal{F}\}$. The *primal shatter function* is defined as

$$\Pi_{\mathcal{F}}(m) = \max_{|S|=m} |\mathcal{F}|_S|.$$

Definition 12.5. Given a set system $(V, \mathcal{F})$, the *dual set system* of $(V, \mathcal{F})$ is the set system $(V^*, \mathcal{F}^*)$, where $V^* = \mathcal{F}$ and $\mathcal{F}^* = \{A_x \subseteq \mathcal{F} : x \in V, \forall A \in A_x, x \in A\}$

Theorem 12.6. *Let $(V, \mathcal{F})$ be a set system with VC-dimension d. Then $\mathcal{F}^*$ has VC-dimension $< 2^{d+1}$.*

Proof. Suppose not. Then there exists $X^* \subseteq V^*$ of size 2^{d+1} that is shattered by $\mathcal{F}^*$. Hence, there exists $X \subseteq V$ of size $2^{2^{d+1}}$ such that $A_x : x \in X$ shatters X^*. Consider the 0-1 matrix M of size $2^{2^{d+1}} \times 2^{d+1}$, whose rows are indexed by the elements of X and columns are indexed by the elements of X^*, and $M_{v,A} = 1$ if and only if $v \in A$. Since X shatters X^*, each row of M is a unique binary vector of size 2^{d+1}. Let M' denote the $(d+1) \times 2^{d+1}$ matrix whose columns are the binary expansions of the numbers $0, \ldots, 2^{d+1} - 1$ in order. Since the rows of M contain all possible binary vectors of size 2^{d+1}, M' is a submatrix of M. It now follows that the $d+1$ vertices corrsponding to the columns of M' embedded into M are shattered by X^*, and so $\mathcal{F}$ has VC-dimension $d+1$, contradiction. $\square$

13. Packing and Transversal Number

Definition 13.1. Given a set system $(V, \mathcal{F})$, a subset $X \subseteq V$ is called a *transversal* (or *hitting set*) of $\mathcal{F}$ if $X \cap A \neq \emptyset$ for all $A \in \mathcal{F}$. The *transversal number* of $\mathcal{F}$, denoted $\tau(\mathcal{F})$, is the size of the smallest transversal of $\mathcal{F}$.

Definition 13.2. Given a set system $(V, \mathcal{F})$, the *packing number* of $\mathcal{F}$, denoted $\nu(\mathcal{F})$, is the size of the largest subfamily of pairwise disjoint sets in $\mathcal{F}$.

Definition 13.3. Let $(V, \mathcal{F})$ be a set system and let $\epsilon \in [0, 1]$. A set $X \subseteq V$ is called an *ϵ-net* for $(V, \mathcal{F})$ if $X \cap A \neq \emptyset$ for all $A \in \mathcal{F}$ of size $\geqslant \epsilon|V|$.

Lemma 13.4. *Let $(V, \mathcal{F})$ be a set system with n vertices and VC-dimension d, and let $\epsilon > 0$. If each member of $\mathcal{F}$ has size $\geqslant \epsilon n$, then*

$$\tau(\mathcal{F}) \leqslant 4\left(\frac{d}{\epsilon}\right)\log n.$$

Proof. Let $x = (4d/\epsilon)\log n$, and let X be x vertices independently and randomly drawn with replacement from V. Let E denote the event that X is not a transversal of $\mathcal{F}$. Given $A \in \mathcal{F}$, the probability that X does not intersect A is

$$\mathbb{P}(X \cap A = \emptyset) \leqslant (1-\epsilon)^x \leqslant e^{-\epsilon x} = n^{-4d}.$$

By the union bound and the Sauer-Shelah theorem, we now have

$$\mathbb{P}(E) = \mathbb{P}(\exists A \in \mathcal{F},\, X \cap A = \emptyset) \leqslant |\mathcal{F}|\mathbb{P}(X \cap A = \emptyset) \leqslant n^d \cdot n^{-4d} = n^{-3d} < 1.$$

But then there exists $X \subseteq V$ of size x such that X is a transversal of $\mathcal{F}$. Hence, $\tau(\mathcal{F}) \leqslant x = (4d/\epsilon)\log n$. □

Lemma 13.5. *Let $X = X_1 + X_2 + \cdots + X_n$, where X_i are independent random variables with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1-p$. Then $\mathbb{P}(X \geqslant np/2) \geqslant 1/2$, provided that $np \geqslant 8$.*

Proof. Since $\mathbb{E}[X] = np$ and $\mathrm{Var}[X] = np(1-p)$, by the Chebyshev inequality,

$$\mathbb{P}(X < np/2) \leqslant \mathbb{P}(|X - \mathbb{E}[X]| \geqslant np/2) \leqslant \frac{4}{np} \leqslant \frac{1}{2}.$$

□

14. Epsilon-net Theorem

Theorem 14.1 (Epsilon-net Theorem)**.** *Let $(V, \mathcal{F})$ be a set system with n vertices and VC-dimension d. Then $(V, \mathcal{F})$ has an ϵ-net of size $O((d/\epsilon)\log(1/\epsilon))$.*

Proof. We may assume that $A \geqslant \epsilon n$ for all $A \in \mathcal{F}$. Let C be a large enough constant. We need to show $\tau(\mathcal{F}) \leqslant C(d/\epsilon)\log(1/\epsilon)$. Let $s = C(d/\epsilon)\log(1/\epsilon)$. Let N, M each be some s vertices independently and randomly drawn with replacement from V. Let E_0 denote the event that N is not a transversal of $\mathcal{F}$, and let E_1 denote the event that there exists $A \in \mathcal{F}$ such that $N \cap A = \emptyset$ and $|M \cap A| \geqslant \epsilon s/2$. Clearly, $\mathbb{P}(E_1) \leqslant \mathbb{P}(E_0)$, and we will show that $\mathbb{P}(E_1) \leqslant 2\mathbb{P}(E_0)$. In particular, we show that for any N, $\mathbb{P}(E_1|N) \geqslant \mathbb{P}(E_0|N)/2$. If N is a transversal, then $\mathbb{P}(E_1|N) = \mathbb{P}(E_0|N) = 0$. If N is not a transversal, then there exists $A \in \mathcal{F}$ such that $N \cap A = \emptyset$. By Lemma 13.5,

$$\mathbb{P}(E_1|N) = \mathbb{P}(|M \cap A| \geqslant \epsilon s/2|N) > \frac{1}{2} = \frac{\mathbb{P}(E_0|N)}{2}.$$

We now show that $\mathbb{P}(E_0) < 1$ by showing that $\mathbb{P}(E_1) \leqslant 1/2$. Let $Z = \{Z_1, \ldots, Z_{2s}\}$ be $2s$ vertices independently and randomly drawn with replacement from V. Now let N be a random set of s vertices drawn from Z, and let $M = Z \backslash N$. We show that for any Z, we have $\mathbb{P}(E_1|Z) < 1/2$. By definition,

$$\mathbb{P}(E_1|Z) = \mathbb{P}(\exists A \in \mathcal{F} : N \cap A = \emptyset \text{ and } |M \cap A| \geqslant \epsilon s/2|Z).$$

Fix $A \in \mathcal{F}$. If $|A \cap Z| < \epsilon s/2$, then clearly

$$\mathbb{P}(N \cap A = \emptyset \text{ and } |M \cap A| \geqslant \epsilon s/2|Z) = 0.$$

On the other hand, if $|A \cap Z| = k \geqslant \epsilon s/2$, then

$$\mathbb{P}(N \cap A = \emptyset,\ |M \cap A| \geqslant \epsilon s/2|Z) \leqslant \mathbb{P}(N \cap A = \emptyset|Z) \leqslant \frac{\binom{2s-k}{s}}{\binom{2s}{s}} \leqslant \left(1 - \frac{k}{2s}\right)^s \leqslant e^{-k/2} = \epsilon^{Cd/4}.$$

By the Sauer-Shelah theorem, $|\mathcal{F}|_Z| \leqslant (2s)^d$, and so the union bound now yields

$$\mathbb{P}(E_1|Z) \leqslant |\mathcal{F}|_Z|\mathbb{P}(N \cap A = \emptyset|Z) \leqslant (2s)^d \cdot \epsilon^{Cd/4} < 1/2,$$

for large enough C. The completes the proof. □

15. Haussler's Packing Lemma

Definition 15.1. Let $(V, \mathcal{F})$ be a set system and $\delta > 0$. We call $\mathcal{F}$ δ-separated if $|A\Delta B| \geqslant \delta$ for all distinct $A, B \in \mathcal{F}$.

Lemma 15.2. *Let $(V, \mathcal{F})$ be a set system with VC-dimension d, and let $\mathcal{F}' = \{A\Delta B \mid A, B \in \mathcal{F}\}$. Then $\mathcal{F}'$ has VC-dimension D, where D is a constant that only depends on d.*

Proof. Since $\mathcal{F}'$ is the set of symmetric differences of $\mathcal{F}$, for any m members in $\mathcal{F}'$, the number of cells they can create in a Venn-diagram is at most the number of cells some $2m$ members of $\mathcal{F}$ can create. This implies $\Pi^*_{\mathcal{F}'}(m) \leqslant \Pi^*_{\mathcal{F}}(2m)$. But then $\mathcal{F}^*$ has VC-dimension $\leqslant 2^{d+1}$, so $\Pi^*_{\mathcal{F}}(2m) = O((2m)^{2^{d+1}})$ by the Sauer-Shelah theorem. Hence, $\Pi^*_{\mathcal{F}'}(m) = O(m^c)$ for some c that depends on d, and so $\Pi_{\mathcal{F}'}(m) = O(m^{2^{c+1}})$ by Theorem 12.6. Put $D = 2^{c+1}$ and note that D only depends on d. It remains to show that $|F'|$ has VC-dimension at most D. Suppose not. Then there exists $X \subseteq V$ of size $D+1$ that is shattered by $\mathcal{F}'$, which requires 2^{D+1} members of $\mathcal{F}'$. But then $\Pi_{\mathcal{F}'}(D+1) \leqslant \binom{D+1}{0} + \binom{D+1}{1} + \cdots \binom{D+1}{D} < 2^{D+1}$, contradiction. □

Theorem 15.3 (Haussler)**.** *Let $(V, \mathcal{F})$ be a δ-separated set system with VC-dimension d. Then*

$$|\mathcal{F}| = O\left(\left(\frac{n}{\delta}\right)^d\right) \ll O(n^d).$$

Proof. We prove a slightly weaker result which shows $|\mathcal{F}| \leqslant c_d(n\log(n/\delta)/\delta)^d$, for some $c_d > 0$. Let $\mathcal{F}' = \{A\Delta B \mid A, B \in \mathcal{F}, A \neq B\}$ and note that any set in $\mathcal{F}'$ has size $\geqslant \delta$. Let $\epsilon = \delta/n$. By Lemma 15.2, $\mathcal{F}'$ has VC-dimension D, where D only depends on d. By the Epsilon-net theorem, there exists $N \subseteq V$ of size $O((D/\epsilon)\log(1/\epsilon))$, such that $N \cap A \neq \emptyset$ for all $A \in \mathcal{F}$ of size $|A| \geqslant \epsilon n = \delta$. Thus N is a transversal of $\mathcal{F}'$. But then $A \cap N \neq B \cap N$ for all distinct $A, B \in \mathcal{F}$, as $N \cap (A\Delta B) \neq \emptyset$. This implies $|\mathcal{F}| = |\mathcal{F}|_N|$. It now follows from Sauer-Shelah theorem that

$$|\mathcal{F}| = |\mathcal{F}|_N| \leqslant c_d|N|^d = c'_d[(1/\epsilon)\log(1/\epsilon)]^d = c'_d[(n/\delta)\log(n/\delta)]^d,$$

for some c_d, c'_d that only depends on d. □