Math 158 HW2

Question 2.5.2. A tournament is an orientation of a complete graph. Prove that every tournament contains a directed path containing all of its vertices.

Proof. Let T_n be a n-vertex tournament. We will prove by induction on n to show that T_n is traceable for all n. T_1 is obviously traceable as it only contains one vertex. T_2 is traceable, as it contains only one directed edge that connects all the vertices in the graph. Suppose that a T_k contains a directed hamiltonian uv-path P, for some $k \geq 2$. We denote the vertex after x in P as x^+, for some $x \in V(P)$. By adding a vertex w and k directed edges to T_k, we get a T_{k+1}. If $e = (w, u)$ or $(v, w) \in E(T_{k+1})$, we can connect e with P to obtain a hamiltonian path in T_{k+1}. If $(w, u), (v, w) \notin E(T_{k+1})$, we know $(u, w), (w, v) \in E(T_{k+1})$ because $N_{T_{k+1}}(w) = V(P)$, which ensures $d^+_{T_{k+1}}(w), d^-_{T_{k+1}}(w) \geq 1$. Hence, there exist $x \in V(P)$ such that $(x, w), (w, x^+) \in E(T_{k+1})$. We can then add w and $(x, w), (w, x^+)$ to $P - (w, w^+)$ to get a directed hamiltonian path in T_{k+1}. Thus, if T_k is traceable, then T_{k+1} is also traceable. Therefore, all tournaments are traceable. $\square$

Question 2.5.9. The closure of an n-vertex graph G, denoted $C(G)$, consists in adding edges between any two non-adjacent vertices u and v such that $d_G(u) + d_G(v) \geq n$. Prove that a graph G is hamiltonian if and only if $C(G)$ is hamiltonian.

Proof. If G is hamiltonian, G contains a hamiltonian cycle $H \subseteq G$. Since $C(G)$ contains G and $V(C(G)) = V(G)$, we have $H \subseteq G \subseteq C(G)$, and thus $C(G)$ is hamiltonian.
Suppose that $C(G)$ has a hamiltonian cycle F. If F does not contain any edges that are not in G, then G is hamiltonian. Otherwise, there exists $\{u, v\} \in E(F)$ such that $\{u, v\} \notin E(G)$, which implies $d_G(u) + d_G(v) \geq n$. Let $P = F - \{u, v\}$ be a hamiltonian uv-path of $C(G)$, say $v_1 v_2 \dots v_n$, and $N(v)^+ = \{v_{i+1} : v_i \in N_G(v)\}$. We then have $N(v)^+ \cup N(u) \subseteq V(P) \backslash \{u\}$, which shows that $|N(v)^+ \cup N(u)| \leq n - 1$. Since $|N(v)^+| + |N(u)| = d_G(u) + d_G(v) \geq n$, we have

$$
\begin{aligned}
|N(v)^+ \cap N(u)| &= |N(v)^+| + |N(u)| - |N(v)^+ \cup N(u)| && (1) \\
&\geq n - (n-1) = 1. && (2)
\end{aligned}
$$

Hence, $N(v)^+ \cap N(u) \neq \emptyset$. Let $v_k \in N(v)^+ \cap N(u)$, we can then get a new hamiltonian cycle $P - \{v_k, v_{k+1}\} + \{u, v_{k+1}\} + \{v_k, v\}$. This shows that all $e \in E(F) \backslash E(G)$ can be removed from F to obtain a hamiltonian cycle that only consists of edges in G, which shows that G is hamiltonian. Therefore, $C(G)$ is hamiltonian if and only if G is hamiltonian. $\square$

Question 2.5.11. Let G be a hamiltonian bipartite graph of a minimum degree of at least three. Prove that G contains at least two hamiltonian cycles.

Proof. Let C be a hamiltonian cycle in bipartite graph $G(A, B)$, and let $u, v, w \in A$ such that $N_C(u) = \{v, w\}$. Consider the hamiltonian uv-path $P = C - \{u, v\}$. Since G is bipartite and $v \in B$, we know $N(v)^+ \in B$, and thus the vertices of P obtained by all possible rotations are all in B. Let $G'(A', B') \subseteq G$ such that $C \subseteq G'$ and $d_{G'}(b) = 3$ for all $b \in B'$. We know G' exists because $\delta(G) \geq 3$. Let H be a graph whose vertices are hamiltonian paths of G' starting with the edge $\{u, w\}$, where two hamiltonian paths in G' form an edge of H if they are obtained from one another by rotation. If $Q \in H$ is a hamiltonian path that ends at a vertex x, then Q has $3 - 1 = 2$ possible rotations in G' unless $\{u, x\} \in E(G')$, in which case would have $3 - 2 = 1$ rotations instead. In the latter case, Q together with $\{u, w\}$ would form a hamiltonian cycle in G'. By the Handshake Theorem, since the number of vertices with odd degrees is even, there is an even number of paths Q in G' which ends at a neighbor of u. Therefore, G' has an even amount of hamiltonian cycles. Since G' already contains a hamiltonian cycle C, it must contain some other hamiltonian cycle C'. Since $C, C' \subseteq G' \subseteq G$, G has at least two hamiltonian cycles. $\square$

Question 3.8.2. A tiling of an $m \times n$ chess board is a set of dominoes that cover all the squares on the chess board exactly once (each domino covers two adjacent squares).

(a) For which $m \geq 1$ and $n \geq 1$ does an $m \times n$ chess board having a tiling?

Solution. Since the number of squares on the chess board must be even to have a perfect matching, m or n is even. Assume, without loss of generality, that m is even. We will prove by induction on m. If $m = 2$, then we can match each square in one column to one in the adjacent column that is adjacent to it, and this is a perfect matching M_2. Suppose that there is a perfect matching M_k for each $k \times n$ chessboard, where $2 \leq k \leq m$ and is even. We can then split a $(m+2) \times n$ chessboard into a $2 \times n$ and $m \times n$ board. We can then find a perfect matching $M_2 \cup M_m$. This also shows true for even n. Therefore, for all $(m, n) \in \{(a, b) \in \mathbb{N}^2 : ab \text{ is even}\}$, an $m \times n$ chessboard has tiling. □

(b) If we remove two squares from an $m \times n$ chessboard, when do the remaining squares have a tiling?

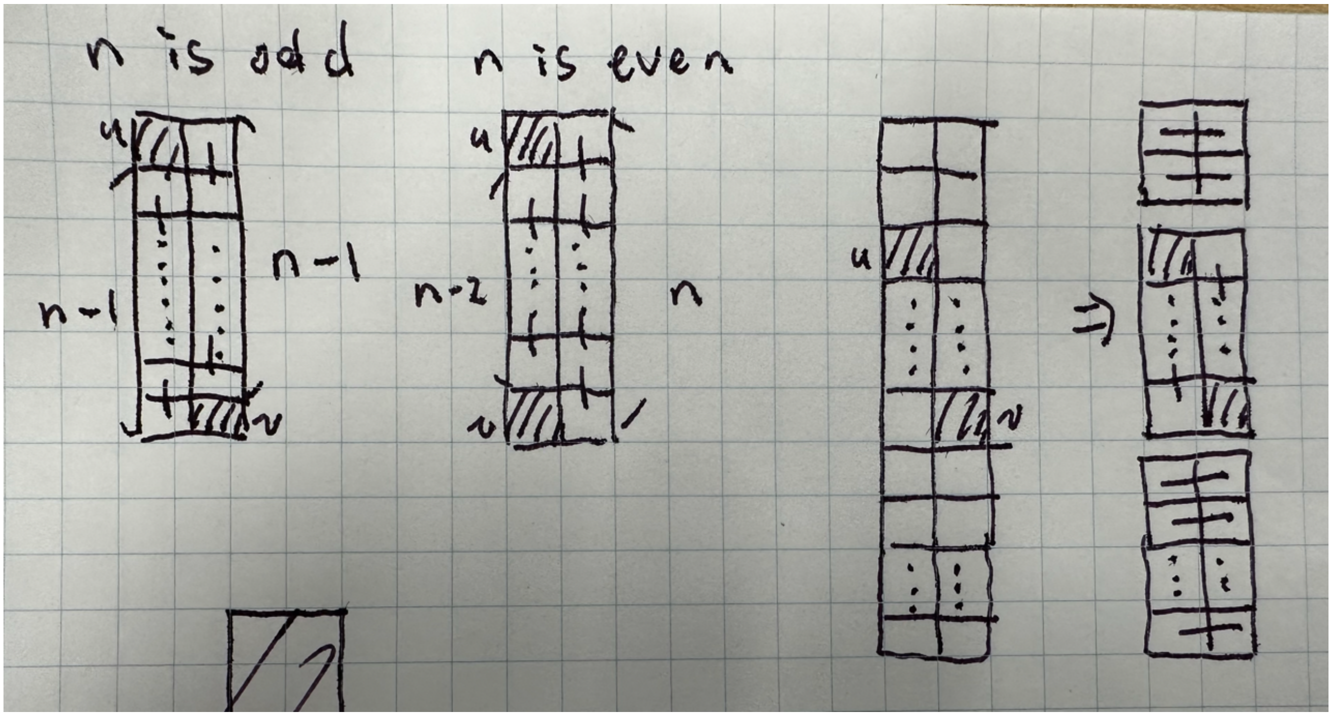
Solution. There must be an even amount of squares to have tiling, so if a $m \times n$ chessboard has tiling after two squares removed, then the $m \times n$ chessboard also has an even amount of squares. Thus, m or n needs to be even.

Assume, without loss of generality, that m is even. Let G be a grid graph whose vertex set contains all squares on a $m \times n$ chessboard, and each pair of vertices forms an edge if they are adjacent to each other on the board. Since m is even, G has a perfect matching. Let v_{xy} correspond to the square in the xth row and yth column, for some $1 \leq x \leq n$, $1 \leq y \leq m$. Let $\{c_1, c_2\}$ be a set of two colors. We color v_{xy} with c_1 if $x + y$ is even and c_2 if $x + y$ is odd. This shows that every square can be colored with no same-colored squares being adjacent. Since each domino covers a c_1 square and a c_2 square, each color must have the same number of squares to have tiling. Therefore, if we remove two squares with the same color, then G does not have a tiling.

Suppose that we remove two squares u, v with different colors. If $n = 1$, then $G - \{u\} - \{v\}$ has a tiling if there are only even components.

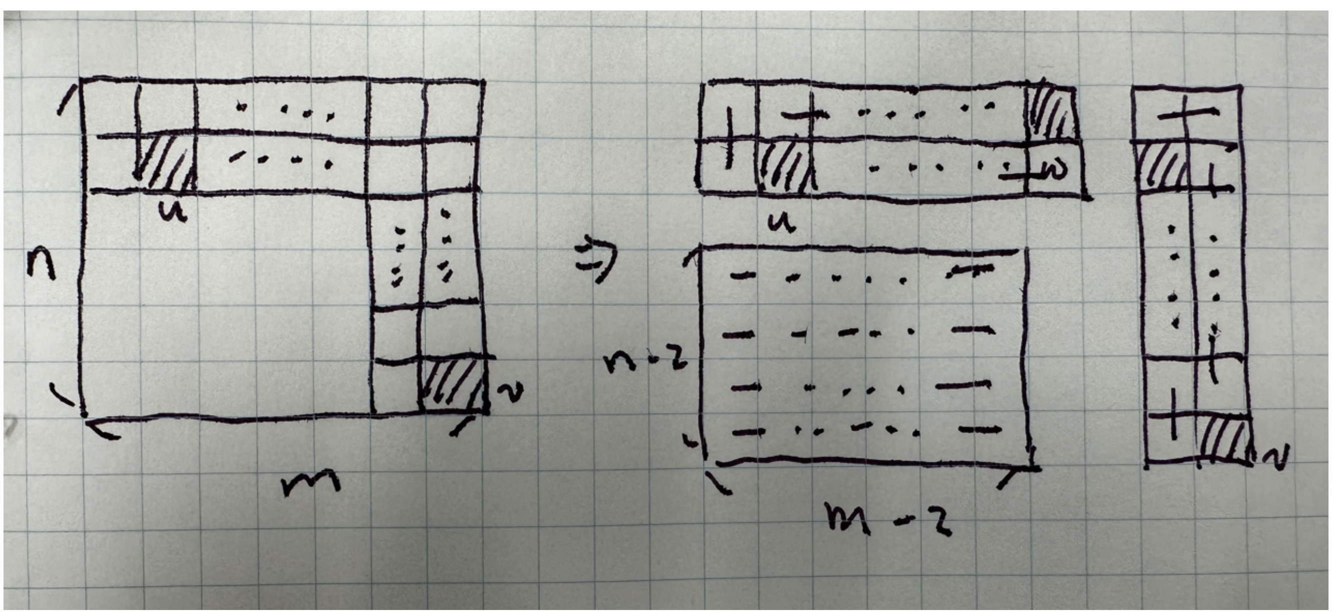
Claim 1. *If $m = 2$, then $G - \{u\} - \{v\}$ has a tiling.*

Suppose that u, v are each in the first and last rows and n is some natural number. Consider the case where n is odd. Since u, v have different colors, u, v are in different columns. This means that the two columns in $G - \{u\} - \{v\}$ both have $n - 1$ number of squares, which is even. Consider the case where n is even. Since u, v has different colors, u, v are in the same column. This means that the two columns in $G - \{u\} - \{v\}$ have n and $n - 2$ numbers of squares respectively, which are also even. Thus, in both cases, we can tile along the columns and cover all squares, which is a tiling. Suppose that u, v are in the ith and jth rows, for some $1 < i \leq j < n$. We can then remove the first $i - 1$ rows and the last $n - j$ rows, as there are two columns so we can find a tiling of them by putting a domino in each row. What is left is a $2 \times (j - i + 1)$ board with two corners on both sides removed, which we just proved to have tiling in the first case. Therefore, $G - \{u\} - \{v\}$ has a tiling if $m = 2$.

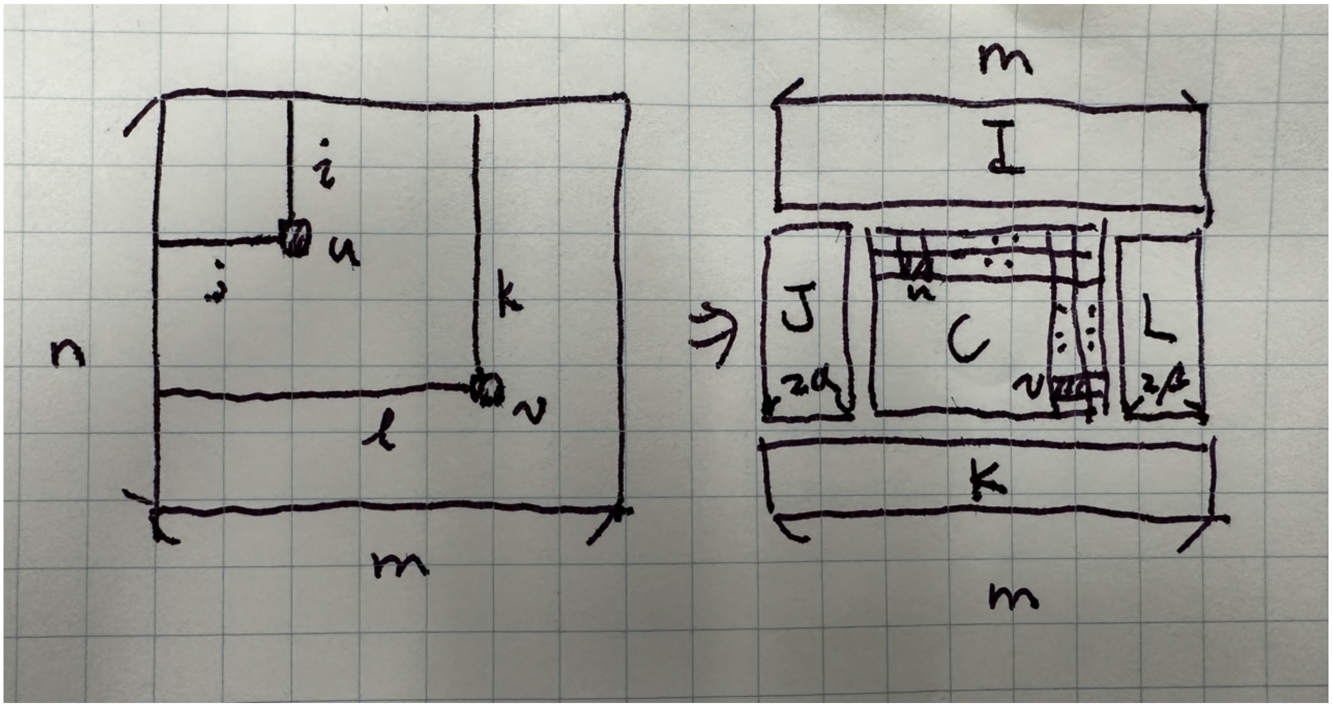


Claim 2. *Suppose that $n \geq 2$. If u, v are each in a 2×2 corner on the opposite side, then $G - \{u\} - \{v\}$ has a perfect matching.*

Assume, without loss of generality, that u is in the top left 2×2 corner and v is in the bottom right 2×2 corner. Since the $(m-2) \times (n-2)$ squares on the bottom left have a perfect matching M_1, as it has an even side, we can first take it out. What is left are the first two rows and last two columns of G, so we can split it into two parts, a $(m-2) \times 2$ board A that contains u and a $(2 \times n)$ board B that contains v. Let $w \in V(B) \cap N_G(V(A))$ such that w shares the same color with u. We remove w from B to A, and we then have two parts, $A + \{w\}$ and $B - \{w\}$. We can view $A + \{w\}$ as a $2 \times (m-1)$ board missing two different color squares u and a square next to w. By Claim 1, since both $A + \{w\}$ and $B - \{w\}$ have exactly two columns and are missing two different-colored squares, $A + \{w\}$ and $B - \{w\}$ each has a perfect matching M_2 and M_3 respectively. Therefore, $G - \{u\} - \{v\}$ has a perfect matching $M_1 \cup M_2 \cup M_3$.



Finally, we will show that $G - \{u\} - \{v\}$ has a perfect matching for all $m, n \geq 2$, m is even. Suppose that u is in the ith row jth column and v is in the kth row lth column of G. Assume, without loss of generality, that $k \geq i$ and $l \geq j$. We can first take out a $m \times (i-1)$ board I that contains the first $(i-1)$ rows of G and a $m \times (n-k)$ board K that contains the last $(n-k)$ rows of G. Since they both have an even side m, they have a perfect matching M_i and M_k respectively. What's left is a $m \times (k-i+1)$ board G'. We can then take out the left-most 2α and right-most 2β columns of G', where α is the greatest integer such that $2\alpha < j$ and β is the greatest integer such that $m-2\beta > l$ and obtain a $2\alpha \times (k-i+1)$ board J and a $2\beta \times (k-i+1)$ board L. Since J and L each have an even side 2α and 2β, they have perfect matching M_J and M_L respectively. What is left is a $(m-2(\alpha+\beta)) \times (k-i+1)$ board C with u, v in opposite side 2×2 corners. By Claim 2, C contains a perfect matching M_C. We now found a perfect matching $M_I \cup M_J \cup M_K \cup M_L \cup M_C$ of $G - \{u\} - \{v\}$. Therefore, if we remove two squares from a $m \times n$ chessboard, it has tiling if and only if mn is even and the two removed squares have different colors and all boards have an even number of squares.



$\square$

Question 3.8.8.

(a) Let G be an n by n bipartite graph of minimum degree more than $n/2$. Prove that G has a perfect matching.

Solution. Suppose that there is a non-hamiltonian n by n bipartite graph of minimum degree at least $n/2$. Amongst all such graphs, let $H(A, B)$ be one with parts A and B and a maximum number of edges. If we add an edge $e = \{v_1, v_{2n}\}$ between non-adjacent vertices in H, we would have a graph with a hamiltonian cycle C, and so $C - e$ is a hamiltonian path in H, say $v_1 v_2 \dots v_{2n}$. Assume, without loss of generality, that $v_1 \in A$ and $v_{2n} \in B$. Let $N(v_1)^+ = \{v_{i+1} : v_i \in N(v_1)\}$. Since $N(v_1)^+ \cup N(v_{2n}) \subseteq A$, we have $|N(v_1)^+ \cup N(v_{2n})| \leq n$. Since $\delta(H) > n/2$, we have $|N(v_1)^+| + |N(v_{2n})| \geq n + 1$. Thus, we have

$$\begin{aligned} |N(v_1)^+ \cap N(v_{2n})| &= |N(v_1)^+| + |N(v_{2n})| - |N(v_1)^+ \cup N(v_{2n})| && (3) \\ &\geq n + 1 - n = 1. && (4) \end{aligned}$$

This shows that $N(v_1)^+ \cap N(v_{2n}) \neq \emptyset$, which proves that H contains a hamiltonian cycle, a contradiction. Therefore, there exists a hamiltonian path P in G such, say $u_1 u_2 \dots u_{2n}$. Let $f = \{(u_i, u_{i+1}) : i \text{ is even}\}$. We can then find a perfect matching $M = P - f$ of G. Hence, G has a perfect matching. □

(b) Let G be a $2n$-vertex graph of minimum degree at least n. Prove that G has a perfect matching.

Solution. If $n = 1$, G itself is a perfect matching for G. Suppose that $n \geq 2$. By Dirac's Theorem, since $\delta(G) \geq |V(G)|/2$, G contains a hamiltonian path P, say $v_1 v_2 \dots v_{2n}$. Let $e = \{(v_i, v_{i+1}) \in E(P) : i \text{ is even}\}$, then $M = P - e$ is a perfect matching in G. Therefore, G has a perfect matching. □