MATH 190A: Homework #1

Due on Jan 15, 2025 at 12:00pm

Professor McKernan

Section A02 8:00AM - 8:50AM
Section Leader: Zhiyuan Jiang

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

Find all topologies on the set

$$X = \{a, b, c\}.$$

Proof.

1. $\{\emptyset, X\}$.
2. $\{\emptyset, X, \{a\}\}$.
3. $\{\emptyset, X, \{b\}\}$.
4. $\{\emptyset, X, \{c\}\}$.
5. $\{\emptyset, X, \{a, b\}\}$.
6. $\{\emptyset, X, \{a, c\}\}$.
7. $\{\emptyset, X, \{b, c\}\}$.
8. $\{\emptyset, X, \{b, c\}, \{a\}\}$.
9. $\{\emptyset, X, \{a, c\}, \{b\}\}$.
10. $\{\emptyset, X, \{a, b\}, \{c\}\}$.
11. $\{\emptyset, X, \{a, b\}, \{a\}\}$.
12. $\{\emptyset, X, \{a, b\}, \{b\}\}$.
13. $\{\emptyset, X, \{a, b\}, \{a\}, \{b\}\}$.
14. $\{\emptyset, X, \{a, c\}, \{a\}\}$.
15. $\{\emptyset, X, \{a, c\}, \{c\}\}$.
16. $\{\emptyset, X, \{a, c\}, \{a\}, \{c\}\}$.
17. $\{\emptyset, X, \{b, c\}, \{b\}\}$.
18. $\{\emptyset, X, \{b, c\}, \{c\}\}$.
19. $\{\emptyset, X, \{b, c\}, \{b\}, \{c\}\}$.
20. $\{\emptyset, X, \{a, b\}, \{a, c\}, \{a\}\}$.
21. $\{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$.
22. $\{\emptyset, X, \{a, c\}, \{b, c\}, \{c\}\}$.
23. $\{\emptyset, X, \{a, b\}, \{b, c\}, \{a\}, \{b\}\}$.
24. $\{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}, \{c\}\}$.
25. $\{\emptyset, X, \{a, c\}, \{b, c\}, \{a\}, \{c\}\}$.
26. $\{\emptyset, X, \{a, c\}, \{b, c\}, \{b\}, \{c\}\}$.
27. $\{\emptyset, X, \{a, b\}, \{a, c\}, \{a\}, \{b\}\}$.
28. $\{\emptyset, X, \{a, b\}, \{a, c\}, \{a\}, \{c\}\}$.
29. $\{\emptyset, X, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}\}$.

$\square$

Problem 2

Find a topology $\mathcal{T}$ on the set

$$X = \{a, b, c, d\}$$

such that a subset is open if and only if it is closed and yet $(X, \mathcal{T})$ is neither the trivial nor the discrete topology.

Proof. Consider $\mathcal{T} = \{\emptyset, \{a, b\}, \{c, d\}, X\}$. Note that for $S \subseteq X$, $S \in \mathcal{T}$ if and only if $\mathcal{T} \backslash S \in \mathcal{T}$. Hence, a subset of $\mathcal{T}$ is open if and only if it is closed. $\square$

Problem 3

Let X be a set and define the function

$$d : X \times X \to \mathbb{R}$$

by the rule

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

(i) Show that (X, d) is a metric space.

Proof. Let $x, y, z \in X$. Obviously, $d(x, x) = 0$, $d(x, y) > 0$ if $x \neq y$, and $d(x, y) = d(y, x)$. If $x = z$, then $d(x, z) \leq d(x, y) + d(y, z)$ is satisfied. If $x \neq z$, then either $x \neq y$ or $y \neq z$, so $d(x, z) \leq d(x, y) + d(y, z)$ is satisfied. Hence, (X, d) is a metric space. $\square$

(ii) What is the associated topology?

Proof. The associated topology is the set of all possible unions of open balls in X. Let $S \subseteq X$. Since each singleton set is an open ball in X and $S = \bigcup_{x \in S} \{x\}$, S is an open set in the associated topology. Therefore, the associated topology is the discrete topology. $\square$

Problem 4

True or false? If true, then give a proof and if false, then give a counterexample.

(i) Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same set X. Then we can always compare $\mathcal{T}_1$ and $\mathcal{T}_2$ (that is, either $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ or $\mathcal{T}_1$ is finer than $\mathcal{T}_2$).

Proof. False. Consider the set $X = \{a, b, c\}$ and topologies $T_1 = \{\emptyset, X, \{a\}\}$ and $T_2 = \{\emptyset, X, \{b\}\}$. Since $\{a\} \notin T_2$ and $\{b\} \notin T_1$, we cannot comapre T_1 and T_2. □

(ii) The topology $(X, \mathcal{T})$ associated to a metric space (X, d) is never the trivial topology.

Proof. False. Consider $X = \{a\}$. Then the topology associated to the metric space (X, d) is the trivial topology. □

(iii) If (X, d_1) and (X, d_2) are two metric spaces that give the same topology on X, then $d_1 = d_2$.

Proof. False. Consider $X = \{a, b\}$, $d_1(a, b) = 1$ and $d_2(a, b) = 2$. Then the topologies of both metric spaces are $\{\emptyset, X, \{a\}, \{b\}\}$. □

(iv) The set $[a, b) \subset \mathbb{R}$ is open, in the Euclidean topology, where $a < b$.

Proof. False, as for all $\epsilon > 0$, $(a - \epsilon, a + \epsilon) \notin [a, b)$. □

(v) The set $(a, \infty) \subset \mathbb{R}$ is open, in the Euclidean topology.

Proof. True, as for all $x \in (a, \infty)$, there exists $\epsilon = x - a > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq (a, \infty)$. □

(vi) Let X be any set. Then the set of all infinite subsets of X, union the empty set, is a topology on X.

Proof. False. Consider $X = \mathbb{Z}$. $\{0\} = \mathbb{Z}_{\leq 0} \cap \mathbb{Z}_{\geq 0}$ is not in the topology. □

(vii) The Euclidean topology on $\mathbb{R}$ and the topology on $\mathbb{R}$ given by the metric $d(x, y) = |x - y|$ are the same.

Proof. True. Since the Euclidean topology is the set of all unions of open intervals and the metric topology is the set of all unions of open balls, the two topologies are the same. □

Problem 5

Let $X = \mathbb{R}$ and let

$$\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R} \cup \{-\infty, \infty\}\}.$$

(i) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$.

Proof. First note that $(\infty, \infty) = \emptyset \in \mathcal{T}$, $(-\infty, \infty) = X \in \mathcal{T}$. Let $S = \{(a_i, \infty)\}_{i \in I}$ be a collection of open sets in $\mathcal{T}$. Let $a = \inf_{i \in I} a_i$. Then $\bigcup_{U \in S} U = \bigcup_{i \in I} (a_i, \infty) = (a, \infty) \in \mathcal{T}$. Now suppose $S = \{(a_i, \infty)\}_{i=1}^n$. Then $\bigcap_{i=1}^n (a_i, \infty) = (\max_{1 \le i \le n} a_i, \infty) \in \mathcal{T}$. Hence, $\mathcal{T}$ is a topology on $\mathbb{R}$. $\square$

(ii) Is this topology coarser or finer than the Euclidean topology?

Proof. Since for all $(a, \infty) \in \mathcal{T}$, (a, ∞) is open in the Euclidean topology but $(-1, 1) \notin \mathcal{T}$, $\mathcal{T}$ is coarser than the Euclidean topology. $\square$