Math 109 HW 7

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1.

Proposition 1. *$P(n)$ is true for all $n \geq 1$.*

Proof. We will show by strong induction on n that for all $n \geq 1$, we have $P(n)$ is true.

Base Case: If $n = 1$, then we have $P(1)$, which is true.

Induction Step: Let $k \geq 1$, $k \in \mathbb{Z}$. Suppose that for all $m \in \mathbb{Z}$, $1 \leq m \leq k$, we have $P(m)$ is true. We will show that $P(k+1)$ is also true.

We know that if $P(k)$ is true, then $P(k+1) istrue$. By the induction hypothesis, we know that $P(k)$ is true, and thus $P(k+1)$ is true. Hence, if $P(1), P(2), \ldots, P(k)$ are all true, then $P(k+1)$ is true.

Therefore, for all $n \geq 1$, we have $P(n)$ is true. □

2.

Proposition 2. *$Q(n)$ is true for all $n \geq 1$.*

Proof. Let $Q(n)$ be the statement "$P(1), P(2), \ldots, P(n)$ are all true." We will show induction on n that for all $n \geq 1$, we have $Q(n)$ is true, which shows that $P(n)$ is true.

Suppose that $Q(1)$ is true and for all $k \geq 1$, if $Q(k)$ is true, then $Q(k+1)$ is also true.

Base Case: If $n = 1$, we have $Q(1)$ is true.

Induction Step: Assume that $Q(k)$ is true for some $k \in \mathbb{N}$. By the Induction Hypothesis, $Q(k+1)$ is true, as $Q(k)$ is true. Since $Q(k+1)$ is true, $P(1), P(2), \ldots, P(n+1)$ are all true, which shows that $P(n+1)$ is true. Thus, if $Q(k)$ is true for some k, $P(n+1)$ is true.

Therefore, for all $n \geq 1$, $P(n)$ is true. □

3.

Proposition 3. $gcd(115, 29) = 1 = -1 \cdot 115 + 4 \cdot 29$.

Proof. Let $gcd(115, 29) = 29m + 115n$, for some $m, n \in \mathbb{Z}$.

$$115 = 29 \cdot 3 + 28 \tag{1}$$
$$29 = 28 \cdot 1 + 1 \tag{2}$$
$$28 = 1 \cdot 28 + 0 \tag{3}$$
$$\tag{4}$$

Therefore, $gcd(115, 29) = 1$.

We then have

$$1 = 29 - 1 \cdot 28 \tag{5}$$
$$= 29 - 1 \cdot (115 - 3 \cdot 29) \tag{6}$$
$$= 29 - 115 + 3 \cdot 29 \tag{7}$$
$$= -1 \cdot 115 + 4 \cdot 29 \tag{8}$$

Therefore, $m = 4, n = -1$. □

4.

Proposition 4. *$gcd(1001, 182) = 91 = -5 \cdot 182 + 1 \cdot 1001$.*

Proof. Let $gcd(1001, 182) = 182m + 1001n$, for some $m, n \in \mathbb{Z}$.

$$1001 = 182 \cdot 5 + 91 \tag{9}$$
$$182 = 91 \cdot 2 + 0 \tag{10}$$

Therefore, $gcd(1001, 182) = 91$.

We then have

$$91 = 1001 - 182 \cdot 5 \tag{11}$$
$$= -5 \cdot 182 + 1 \cdot 1001 \tag{12}$$

Therefore, $m = -5, n = 1$. □

5. (a)

Proposition 5. *$\sim$ is reflexive and symmetric but not transitive.*

Proof. Define $R \subseteq \mathbb{R}^2$ by

$$(x, y) \in R \text{ if } |x - y| \leq 2. \tag{13}$$

Reflexive: Let $x \in \mathbb{R}$. We have $|x - x| = 0 \leq 2$. Thus, for all $x \in \mathbb{R}$, $(x, x) \in R$, and so $\sim$ is reflexive.
Symmetric: Let $(x, y) \in R$. We have $|y - x| = |x - y| \leq 2$. Thus, for all $(x, y) \in R$, $(y, x) \in R$, and so $\sim$ is symmetric.
Transitive: Consider the case $(4, 2), (2, 0) \in R$. $(4, 0) \notin R$ because $|4 - 0| = 4 > 2$, and so $\sim$ is not transitive. □

(b)

Proposition 6. $\sim$ *is reflexive and transitive but not symmetric.*

Proof. Define $R \subseteq \mathbb{Z}^2$ by

$$(m, n) \in R \text{ if } m|n. \tag{14}$$

Reflexive: Let $m \in \mathbb{Z}$. We have $m|m$. Thus, for all $m \in \mathbb{R}$, $(m, m) \in R$, and so $\sim$ is reflexive.

Symmetric: Consider the case $(1, 3) \in R$. $(3, 1) \notin R$ because $3 \nmid 1$, and so $\sim$ is not symmetric.

Transitive: Let $(m, n), (n, k) \in R$. We have $mq = n$ and $np = k$, $q, p \in \mathbb{Z}$. We then have $k = (mq)p = (qp)m$. Since qp is an integer, we have $m|k$. Thus, for all $(m, n), (n, k) \in R$, $(m, k) \in R$, and so $\sim$ is transitive. $\square$

(c)

Proposition 7. $\sim$ *is reflexive, symmetric and transitive.*

Proof. Define $R \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ by

$$((x_1, y_1), (x_2, y_2)) \in R \text{ if } x_1 + 2y_1 = x_2 + 2y_2. \tag{15}$$

Reflexive: Let $(x_1, y_1) \in \mathbb{R}^2$. We have $x_1 + 2y_1 = x_1 + 2y_1$. Thus, for all $(x_1, y_1) \in \mathbb{R}^2$, $((x_1, y_1), (x_1, y_1)) \in R$, and so $\sim$ is reflexive.

Symmetric: Let $((x_1, y_1), (x_2, y_2)) \in R$. We have $x_2 + 2y_2 = x_1 + 2y_1$, . Thus, for all $((x_1, y_1), (x_2, y_2)) \in R$, $((x_2, y_2), (x_1, y_1)) \in R$, and so $\sim$ is symmetric.

Transitive: Let $((x_1, y_1), (x_2, y_2)), ((x_2, y_2), (x_3, y_3)) \in R$. We have $x_1 + 2y_1 = x_2 + 2y_2 = x_3 + 2y_3$.
Thus, for all $((x_1, y_1), (x_2, y_2)), ((x_2, y_2), (x_3, y_3)) \in R$, $((x_1, y_1), (x_3, y_3)) \in R$, and so $\sim$ is transitive. $\square$

(d)

Proposition 8. $\sim$ *symmetric and transitive but not reflexive.*

Proof. Define $R \subseteq \mathbb{R}^2$ by

$$(x, y) \in R \text{ if } \frac{x}{y} = 1. \tag{16}$$

Reflexive: Consider $0 \in \mathbb{R}$. We have $\frac{0}{0}$, which is undefined, and so $\sim$ is not reflexive.

Symmetric: Let $(x, y) \in R$. Since $\frac{x}{y} = 1$, we know that $x = y$. We then have $\frac{y}{x} = 1$. Thus, for all $(x, y) \in R$, $(y, x) \in R$, and so $\sim$ is symmetric.

Transitive: Let $(x, y), (y, z) \in R$. Since $\frac{x}{y} = \frac{y}{z} = 1$, we know that $x = y$ and $y = z$, so $x = z$. We then have $\frac{x}{z} = 1$. Thus, for all $(x, y), (y, z) \in R$, $(x, z) \in R$, and so $\sim$ is transitive. $\square$

6. (a) $x \sim y$ if $|x - y| = 2$, $x, y \in \mathbb{R}$.
   (b) $x \sim y$ if $x|y$, $x, y \in \mathbb{Z}$.
   (c) $x \sim y$ if $\frac{x}{y}$, $x, y \in \mathbb{R}$.

7. The proof is assuming that for all $a \in S$, there exists $b \in S$ such that $a \sim b$. However, $a \in S$ does not imply there exists $b \in S$ such that $a \sim b$ because there could still be elements in S that does not have a relation with any other elements.