Math 109 HW 9

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1.

Proposition 1. *Suppose that for all $n \geq 1$, $\bigcap_{i=1}^{n} S_i \neq \emptyset$, $i \in \mathbb{Z}_{\geq 1}$. $\bigcap_{i=1}^{\infty} S_i \neq \emptyset$.*

Proof. We will prove by contradiction. Suppose for the sake of contradiction that $\bigcap_{i=1}^{\infty} S_i = \emptyset$. This means that there exists $a > b \geq 1$ such that $S_a \cap S_b = \emptyset$, which means that $\bigcap_{i=1}^{a} S_i = \emptyset$. However, this contradicts our assumption that for all integers $i, n \geq 1$, $\bigcap_{i=1}^{n} S_i \neq \emptyset$.

Therefore, $\bigcap_{i=1}^{\infty} S_i \neq \emptyset$. □

2.

Proposition 2. *Let $f : A \to B$ be a function. If $g, h : B \to A$ are inverse functions of f, then $g(b) = h(b)$ for all $b \in B$.*

Proof. Let g, h be functions such that $f(g(b)) = b$ and $f(h(b)) = b$ for all $b \in B$ and $g(f(a)) = a$ and $h(f(a)) = a$ for all $a \in A$. Let $x \in B$. We will show that $g(x) = h(x)$. Since $f(h(x)) = x$, we have $g(f(h(x))) = g(x)$. In addition, since $g(f(a)) = a$ for all $a \in A$, we then have $g(f(h(x))) = h(x)$. Thus, $g(x) = g(f(h(x))) = h(x)$.

Therefore, the inverse function of f is unique. □

3.

Proposition 3. *If a function $f : A \to B$ has an inverse, then f is bijective.*

Proof. We will prove by contradiction. Let $g : B \to A$ be a function such that $g(f(x)) = x$ and $f(g(y)) = y$, for all $x \in A$, $y \in B$. Suppose for the sake of contradiction that f is not bijective, namely f is not injective or not surjective.

If f is not injective, then there exists $m, n \in A$ such that $m \neq n$ and $f(m) = f(n)$. We then have $g(f(m)) = g(f(n))$. However, since $g(f(m)) = m$ and $g(f(n)) = n$, we have $m = n$, which contradicts our assumption. Thus, f is injective.

If f is not subjective, then there exists $k \in B$ such that for all $l \in A$, $f(l) \neq k$. We then have $f(g(k)) \neq k$. However, this contradicts our assumption that $f(g(y)) = y$, for all $y \in B$. Therefore, f is surjective.

Combining these two cases, our assumption that f is not bijective is contradicted.

Therefore, if there exists an inverse of f, then f is bijective. □

4.

Proposition 4. *If a function $f : A \to B$ is bijective, then it has an inverse.*

Proof. Let $f : A \to B$ be a bijective function, and $g : B \to A$. Let $x \in A, y \in B$, such that $f(x) = y$. We will show that there exists a function $g : B \to A$ such that $g(f(x)) = x$ and $f(g(y)) = x$.

Since f is surjective, we know that there exists a function g such that for all $y \in B$, there exist $z \in A$ such that $g(y) = z$.

Since f is injective and a well-defined function, we know that $f(m) = f(n)$ if and only if $m = n$, $m, n \in A$. Let $m = g(k)$ and $n = g(l)$, for some $k, l \in B$. This shows that there exists g such that if $g(k) = g(l)$, then $k = l$.

This shows that there exists a well-defined function $g : B \to A$ such that $g(y) = x$. We then have $g(f(x)) = g(y)$ and $f(g(y)) = f(x) = y$.

Therefore, if a function is bijective, then it has an inverse. □

5.

Proposition 5. *f is a well-defined function.*

Proof. We will show that f is a well-defined function.

Existence: Let $x \in S$. We will show that there exist $s \in\in S$ such that $f(x) = s$. Let $s = [x] \in S/\sim$. Since $\sim$ is reflexive, we have $x \sim [x]$. This shows that $f(x) = [x] = s$.

Uniqueness: Let $[b_1] = f(a), [b_2] = f(a)$ for some $a, b_1.b_2 \in S$. We will show that $[b_1] = [b_2]$. Since $[b_1] = f(a), [b_2] = f(a)$, we know that $a \sim b_1$ and $a \sim b_2$. Since $\sim$ is symmetric, we have $b_1 \sim a$. Since $\sim$ is transitive, we then have $b_1 \sim b_2$, which shows that $[b_1] = [b_2]$.

Therefore, f is a well-defined function. □