CSE 101: Homework #2

Due on Apr 17, 2024 at 23:59pm

Professor Jones

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Problem 1

Run the SCC algorithm on the following directed graph G. When doing DFS on G^R: whenever there is a choice of vertices to explore, always pick the one that is alphabetically first.

$A : D, G$

$B : F, G, L$

$C : B, E$

$D : G, H$

$E : A, J$

$F : B$

$G : H$

$H : B, L$

$I : K$

$J : F, L$

$K : D$

$L : E, H, K$

(a) In what order are the strongly connected components (SCCs) found?

Proof. We first run DFS on G^R and get the `post` numbers:

A	B	C	D	E	F	G	H	I	J	K	L
24	21	4	17	23	10	19	20	15	9	16	22

Then, we run the undirected connected components algorithm on G by descending `post` order:

A	B	C	D	E	F	G	H	I	J	K	L
1	1	3	1	1	1	1	1	2	1	1	1

Hence, we find the following strongly connected components in the following sequence:

$$\{A, B, D, E, F, G, H, J, K, L\}, \{I\}, \{C\}.$$

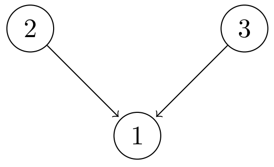
□

(b) Which are source SCCs and which are sink SCCs?

Proof. Since $\{A, B, D, E, F, G, H, J, K, L\}$ does not have outgoing edges, it is a sink. Since SCCs $\{I\}$ and $\{C\}$ don't have incoming edges, they are sources. □

(c) Draw the "metagraph" (each meta-node is an SCC of G)

Proof. The following is the metagraph of G, where node 1 represents $\{A, B, D, E, F, G, H, J, K, L\}$, node 2 represents $\{I\}$, and node 2 represents $\{C\}$.



□

Problem 2

Consider the following problem:

Given a strongly connected simple *directed* graph G, determine the total number of cycles in the graph.

Consider the following algorithm that claims to compute the total number of cycles in the graph.

For each algorithm,

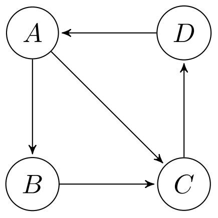
- Provide a runtime analysis (Based on $|V|$ and $|E|$)
- identify if it correctly solves the problem.
- If it is correct, provide a correctness proof. If it is not correct, provide a counterexample.

1. **Algorithm1**(G; a strongly connected simple directed graph G.)

```
1. Run DFS($G$)
2. c = 0
3. for each edge $(u, v)$ then
4.    if $post(v) > post(u)$ then
5.       $c = c + 1$
6. return c
```

Proof. We first give a runtime analysis to this algorithm. We already know DFS takes $O(|V| + |E|)$ time. Following the DFS is a loop which iterates over all edges, which takes an additional $O(|E|)$ time. Hence, the runtime complexity of this algorithm is $O(|V| + 2|E|) = O(|V| + |E|)$.

However, the algorithm is incorrect. Consider the following graph:



Note that the graph is simple and strongly connected, as there is a directed hamiltonian cycle. Performing DFS on this graph starting from A yields the following `post` numbers:

A	B	C	D
8	7	6	5

(D, A) is the only edge that meets the condition at step 4, so the algorithm outputs 1. But then there are two cycles in the graph, namely $A \to C \to D \to A$ and $A \to B \to C \to D \to A$. ☐

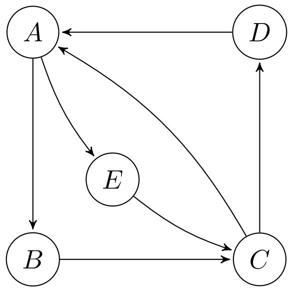
2. **Algorithm2**(G; a strongly connected simple directed graph G.)

   [[run graphsearch and every time you encounter a vertex you have already seen before, increment your counter, c.]]

```
1.  $c = 0$
2.  **for all** $v \in V$:
3.    Status($v$) = **U**
4.  Pick any vertex $s$
5.  Status($s$) = **F**
6.  Initialize a Stack: $F = [s]$
7.  **while** $|F| > 0$
8.    $w = pop(F)$.
9.    For each outgoing neighbor $y$ of $w$ (for each $(w, y) \in E$):
10.     **if** Status($y$) $\neq$ **U**:
11.       $c = c + 1$
12.     **else:**
13.       Status($y$) = **F**
14.       $push(F, y)$
15.   Status($w$) = **X**
16. **return** $c$
```

Proof. Since this algorithm is just graphsearch with a counter, the time complexity of the algorithm is the same as a standard graphsearch, which takes $O(|V| + |E|)$ time.

However, this algorithm is incorrect. Consider the following graph:



The graph is obviously simple and strongly connected. We first note that the graph has 4 cycles, namely $A \to B \to C \to A$, $A \to B \to C \to D \to A$, $A \to E \to C \to A$, $A \to E \to C \to D \to A$.

Suppose that the algorithm starts at vertex A. Here is the order of edges the for loop in the algorithm will check:

$$(A, B) \to (A, E) \to (B, C) \to (C, \mathbf{A}) \to (C, D) \to (D, \mathbf{A}) \to (E, \mathbf{C}).$$

The bolded vertices represents the already visited vertices, which causes the counter to increment. But then the algorithm returns 3, which is not the number of cycles in the graph. □

Problem 3

You are given a simple directed graph G with vertex set V, edge set E and vertex labels $L(v) \in \{0, 1\}$ as well as a starting and ending vertex s, t.

Design a reasonably efficient algorithm that determines if there is a walk from s to t such that the sequence of vertex labels in the walk have exactly one occurrence of two 1's in a row.

Proof. Consider the following algorithm:

> Create a graph G' in the following way:
>
> for each vertex v in G, create two copies v', v'' in G'. For each edge (x, y) in G,
>
> - If $L(x) == 1$ and $L(y) == 1$, create an edge (x', y'') in G'
> - otherwise, create 2 edges $(x', y'), (x'', y'')$ in G'.
>
> Then, run explore in G' from s'. Return TRUE if t'' is visited, and return FALSE otherwise.

We now give a justification of correctness. Suppose that the algorithm returns TRUE. Then, there is a path s' to t'' in G'. By construction, there is no edge between v' and v'' in G', so P' is of the form $P' = (s', v'_1, \ldots, v'_k, u''_1, \ldots, u''_j, t'')$. We now map each vertex in the path back to the corresponding vertex in $V(G)$ (by removing the apostrophes) and obtain a new path $P \subseteq G$ from s to t, with exactly one edge (v_k, u_1) having two vertices labelled 1.

We now prove the converse. Suppose that there is a walk in G from s to t with exactly one occurrence of two 1's in a row. Then, the walk can be condensed to a path P of the form $(s, v_1, \ldots, v_k, u_1, \ldots, u_j, t)$, with (v_k, u_1) being the only edge such that $L(v_k) == L(u_1) == 1$. By construction, each edge (s', v'_1), $\ldots$, (v_{k-1}, v_k), (u''_1, u''_2), $\ldots$, (u''_j, t'') are in G'. But then since $L(v_k) == L(u_1) == 1$, (v'_k, u''_1) is also in G', which makes $P' = (s', v'_1, \ldots, v'_k, u''_1, \ldots, u''_j, t'')$ a path in G'. Hence, the algorithm returns TRUE.

We now give a runtime analysis of the algorithm. It takes $O(|V|)$ time to create copies of vertices and $O(|E|)$ to create the edges in G'. Running explore on G' has runtime $O(|V| + |E|)$. Hence, the total runtime of the algorithm is $O(2|V| + 2|E|) = O(|V| + |E|)$. □

Problem 4

You are given a directed graph.

Design a reasonably efficient algorithm that *determines* if there exists a walk that goes through each vertex at least once.

Proof. Consider the following algorithm:

```
1. hasWalk = 1
2. cc = 0
3. Run **SCC**($G$)
4. **for** $i = 1; i <= cc; i++$ **then**
5.     **for** $j = 1; j <= cc; j++$ **then**
6.         Meta[$i$][$j$] = 0
7. **for** each edge $(u, v)$ **then**
8.     Meta[ccnum[$u$], ccnum[$v$]] = 1
9. **for** $i = 1; i <= cc; i++$ **then**
10.     hasWalk *= Meta[$i + 1$, $i$]
11. **return** hasWalk == 1
```

Given directed graph G, the algorithm constructs the metagraph M of G and check if there exists a path $P \subseteq M$ which chains all vertices of M in decreasing ccnum value.

We now check that if the algorithm correctly determines the existence of the desired walk. Note that a strongly connect component contains such a walk, as there exists a path between any ordered pair of vertices. In particular, for any s, t in a strongly connected component, there exists a path from s to t which passes through all vertices in the component. Hence, there exists such a walk in G if and only if there exists a walk which passes through all vertices in the metagraph M. But then the M has no cycles, so there exists such a walk in M if and only if there is a path P which passes through all vertices in M. It remains to show that P exists if and only if $(i+1, i) \in E(M)$ for all $i \in V(M)$, $i \neq cc$. One direction is obvious, so we only need to show the existence of P implies $(i+1, i) \in E(M)$ for all $i \in V(M)$, $i \neq cc$. Notice that the sink of P must be $u = \min V(M)$, otherwise u has an incoming edge, contradicting the nature of the SCC algorithm. Remove u from P. By the nature of the SCC algorithm, the sink of $P \setminus \{u\}$ is the next smallest element in $V(M)$, namely $u + 1$. Hence, we may recursively remove the sink of P to get the next smallest element in $V(M)$, and thus $(i, i+1) \in E(M)$, for all $1 \leq i < cc$. Therefore, the algorithm returns TRUE if there exists a wwalk that goes through each vertex at least once and returns FALSE otherwise.

We now give a runtime analysis of the algorithm. We already know the SCC algorithm takes $O(|V| + |E|)$ time. The nested loop at step 4-6 iterates through all pairs of components, which is $O(|V|^2)$ at the very worst. The loop at step 7 iterates through all edges, so it takes $O(|E|)$ time. Finally, the last loop simply loops through all components, so it is $O(|V|)$ at the very worst. Hence, the algorithm has a runtime of $O(|V|^2)$. □