# MATH 140B: Homework #3

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*Professor Seward*

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# Problem 1

Suppose $f$ is a bounded real function on $[a, b]$ and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

*Proof.* $f$ is not necessarily integrable. Consider

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}.$$

$f^2(x) = 1$ is obviously continuous, as it is constant. Since both rationals and irrationals are dense in $\mathbb{R}$,

$$U(P, f) = \sum_{i=0}^{n} \Delta x_i = b - a, \quad L(P, f) = \sum_{i=0}^{n} -\Delta x_i = a - b,$$

for any partition $P$. But then $U(P, f) - L(P, f) = 2(b - a)$, and thus $f \notin \mathscr{R}$.

Suppose $f^3 \in \mathscr{R}$. Since $f$ is bounded, we may assume $|f| < M$. Define $\phi(x) = \sqrt[3]{x}$. Note that $\phi(f^3(x)) = f(x)$. Since $x^3$ is a continuous 1-1 mapping on $[-M^{1/3}, M^{1/3}]$, its inverse $\phi = \sqrt[3]{x}$ is continuous on $[-M, M]$, by Theorem 4.17. But then by Theorem 6.11, $f(x) = \phi(f^3(x)) \in \mathscr{R}$ on $[a, b]$. $\square$

# Problem 2

Let $P$ be the Cantor set constructed in Theorem 2.44. Let $f$ be a bounded real function on $[0,1]$ which is continuous at every point outside $P$. Prove that $f \in \mathscr{R}$ on $[0,1]$.

*Proof.* We first show that $P$ can be covered by finitely many segments whose total length can be made as small as desired. Pick $\epsilon > 0$. Note that $P = \bigcap_{n=1}^{\infty} E_n$, where $E_n$ is the union of $2^n$ intervals, each of length $3^{-n}$. Pick $n$ large enough such that $\frac{2^n}{3^n} < \epsilon$. We know $P$ can be covered by $E_n$. Put $\nu \in (0, \epsilon - \frac{2^n}{3^n})$. Let $C$ be the union of segments, where each segment corresponds to an interval in $E_n$ with both endpoints extended by $\frac{\nu}{2^{n+1}}$. Then, $C$ is a open cover of $P$ and the total length of all segments in $C$ is $\frac{2^n}{3^n} + \nu < \epsilon$.

We may assume that $C = \bigcup_{i=1}^{2^n}(u_i, v_i)$ and the intervals $[u_i, v_i]$ are pairwise disjoint. Let $M = \sup|f(x)|$. Put $K = [0,1] \backslash C$. Since $K$ is compact, $f$ is uniformly continuous on $K$. Hence, there exists $\delta > 0$ such that $|f(s) - f(t)| < \epsilon$ whenever $s, t \in K$ and $|s - t| < \delta$.

Now consider a partition $\rho = \{x_0, x_1, \ldots, x_k\}$ of $[0,1]$ such that each $u_i, v_i$ occurs in $\rho$ and no point of any segment $(u_i, v_i)$ occurs in $\rho$. Additionally, if $x_{i-1}$ is not one of the $u_j$, then $\Delta x_i < \delta$.

Note that $M_i - m_i \leq 2M$ for every $i$, and that $M_i - m_i \leq \epsilon$ unless $x_{i-1}$ is one of the $u_j$. Hence,

$$\begin{aligned} U(\rho, f) - L(\rho, f) &= \sum_{i=1}^{k}(M_i - m_i)\Delta x_i \\ &= \sum_{x_{i-1}=u_j}^{k}(M_i - m_i)\Delta x_i + \sum_{x_{i-1}\neq u_j}^{k}(M_i - m_i)\Delta x_i \\ &< 2M\epsilon + \epsilon = (2M+1)\epsilon, \end{aligned}$$

and the result follows from Theorem 6.6. □

# Problem 3

Suppose $f$ is a real function on $(0,1]$ and $f \in \mathscr{R}$ on $[c,1]$ for every $c > 0$. Define

$$\int_0^1 f(x)\,dx = \lim_{c\to 0}\int_c^1 f(x)\,dx$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0,1]$, show that this definition of the integral agrees with the old one.

*Proof.*

$$\lim_{c\to 0}\int_c^1 f(x)\,dx = \int_0^1 f(x)\,dx - \lim_{c\to 0}\int_0^c f(x)\,dx,$$

so it remains to show that $\lim_{c\to 0}\int_0^c f(x)\,dx = 0$. Since $f \in \mathscr{R}$, we may assume $|f(x)| \leq M$ for $x \in [0,1]$. Pick $\epsilon > 0$. Then, given any partition $P = \{x_0, \ldots, x_n\}$, we have $\delta = \frac{\epsilon}{nM}$ such that,

$$U(P,f) = \sum_{i=1}^n M_i \Delta x_i \leq nMc < \epsilon, \quad L(P,f) = \sum_{i=1}^n m_i \Delta x_i > -nMc > -\epsilon,$$

for all $c \in (0,\delta)$. But then

$$\left|\int_0^c f(x)\right| < \epsilon,$$

and the result follows. □

(b) Construct a function $f$ such that the above limit exists, although it fails to exist with $|f|$ in place of $f$.

*Proof.* Define $f(x) = \frac{(-1)^n}{n}$ if $x \in (\frac{1}{n+1}, \frac{1}{n}]$, for $n \in \mathbb{N}$. Suppose $c = \frac{1}{n+1}$. Then,

$$\int_c^1 f(x)\,dx = \sum_{k=1}^n \frac{(-1)^k}{k}, \quad \int_c^1 |f(x)|\,dx = \sum_{k=1}^n \frac{1}{k}.$$

As $c \to 0$, $n \to \infty$, and thus $\int_c^1 f(x)\,dx$ converges but not $\int_c^1 |f(x)|\,dx$. □

# Problem 4

Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where $a$ is fixed. Define

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after $f$ has been replaced by $|f|$, it is said to converge *absolutely*.

Assume that $f(x) \geq 0$ and that $f$ decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)\,dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series.)

*Proof.* Consider the partition $P = \{1, \ldots, n\}$. Note that as $f \geq 0$, both the $\int_1^n f(x)\,dx$ and $\sum_{k=1}^n f(k)$ are monotonically increasing with respect to $n$. Hence, it suffices to show that the integral and summation are bounded together. Since $f$ is monotonically decreasing,

$$U(P, f) = \sum_{k=1}^{n-1} f(k), \quad L(P, f) = \sum_{k=2}^{n} f(k).$$

Note that since $f$ is at least 0 and monotonically decreasing, $\lim_{n \to \infty} f(n) \in \mathbb{R}$. We then get

$$f(n) + \sum_{k=2}^n f(k) \leq f(n) + \int_1^n f(x)\,dx \leq \sum_{k=1}^n f(k) \leq f(1) + \int_2^n f(x)\,dx.$$

But then $\int_1^\infty f(x)\,dx$ and $\sum_{n=1}^\infty f(n)$ are bounded together, and the result follows. $\square$

# Problem 5

Let $p$ and $q$ be positive real numbers such that

$$\frac{1}{p}+\frac{1}{q}=1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p}+\frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

*Proof.* Fix $u$. Define $f(v) = \frac{u^p}{p}+\frac{v^q}{q} - uv$. Then, $f'(v) = v^{q-1} - u, f''(v) = (q-1)v^{q-2} \geq 0$, for $v \geq 0$. Hence, $f(v)$ reaches minimum at $v = u^{\frac{1}{q-1}}$. Note that $p = \frac{q}{q-1}$. But then

$$\begin{aligned} f(v) &= \frac{u^p}{p}+\frac{v^q}{q} - uv \\ &\geq \frac{u^p}{p}+\frac{u^{\frac{q}{q-1}}}{q} - u^{\frac{q}{q-1}} \\ &= \left(\frac{1}{p}+\frac{1}{q}-1\right)u^p = 0, \end{aligned}$$

and the result follows. □

(b) If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fgd\alpha \leq 1.$$

*Proof.* By (a),

$$\int_a^b \frac{f^p}{p}d\alpha + \int_a^b \frac{g^q}{q}d\alpha = \int_a^b \frac{f^p}{p}+\frac{g^q}{q}d\alpha \geq \int_a^b fgd\alpha.$$

But then

$$1 = \frac{1}{p}+\frac{1}{q} = \int_a^b fgd\alpha.$$

□

(c) If $f$ and $g$ are complex functions in $\mathscr{R}(\alpha)$, then

$$\left|\int_a^b fgd\alpha\right| \leq \left(\int_a^b |f|^p d\alpha\right)^{1/p}\left(\int_a^b |g|^q d\alpha\right)^{1/q}.$$

This is Hölder's inequality. When $p = q = 2$, it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

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*Proof.* Put $F = \int_a^b |f|^p d\alpha$, $G = \int_a^b |g|^p d\alpha$. Since $f, g \in \mathscr{R}(\alpha)$, $|f|, |g| < M \in \mathbb{R}$. Note that $F = 0$ implies $\int_a^b |f| d\alpha = 0$. Thus

$$0 = M \int_a^b |f| \, d\alpha \geq \int_a^b |f||g| \, d\alpha \geq \left| \int_a^b fg d\alpha \right|,$$

and the inequality holds.

Hence, we may assume $F, G > 0$. Substituting $f$ as $\frac{|f|}{F^{1/p}}$ and $g$ as $\frac{|g|}{G^{1/q}}$, we get

$$\int_a^b \frac{|f||g|}{F^{1/p} G^{1/q}} \, d\alpha \leq= 1.$$

But then

$$\left| \int_a^b |fg| \, d\alpha \right| \leq \int_a^b |f||g| \, d\alpha \leq F^{1/p} G^{1/q}.$$

□

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 6.7 and 6.8.

*Proof.* Since the equality holds for any finite interval, and thus the inequality also holds if the improper integrals converge.

Suppose the improper integral of $f$ or $g$ diverge, the right-hand side of the inequality tends to infinity, and thus the inequality still holds. □