MATH 220A: Homework #5

Due on Nov 1, 2024 at 23:59pm

Professor Ebenfelt

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Problem 1

Suppose $f : G \to \mathbb{C}$ is analytic and that G is connected. Show that if $f(z)$ is real for all z in G then f is constant.

Proof. Put $f(x + iy) = u(x, y) + iv(x, y)$, where u, v are real-valued functions. Since f is real-valued, $v(x, y) = 0$ for all $x, y \in G$. By the Cauchy-Riemann equations,

$$u_x = v_y = 0 \qquad u_y = -v_y = 0.$$

But then u is constant, and thus f is constant. $\square$

Problem 2

Find an open connected set $G \subset \mathbb{C}$ and two continuous functions f and g defined on G such that $f(z)^2 = g(z)^2 = 1 - z^2$ for all z in G. Can you make G maximal? Are f and g analytic?

Proof. Let $G = (\mathbb{C} \backslash \mathbb{R}) \cup [-1, 1]$. Consider $f(z) = \exp(\frac{1}{2} Log(1 - z^2))$ and $g(z) = \exp(\frac{1}{2} Log(1 - z^2))$. Then $f(z)^2 = g(z)^2 = 1 - z^2$ for all $z \in G$. Notice that G is maximal in $\mathbb{C}$, as any larger set would make $1 - z^2 \in \mathbb{R}_{\leq 0}$, which makes $Log(1 - z^2)$ undefined. Since f, g are compositions of analytic functions, they are analytic. $\square$

Problem 3

Let G be a region and define $G^* = \{z : \overline{z} \in G\}$. If $f : G \to \mathbb{C}$ is analytic, prove that $f^* : G^* \to \mathbb{C}$, defined by $f^*(z) = \overline{f(\overline{z})}$, is also analytic.

Proof. Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. Then $f^*(z) = u(x, -y) - iv(x, -y)$. By the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$, and so

$$\partial_x u(x, -y) = -\partial_y v(x, -y) = \partial_y[-v(x, -y)], \quad \partial_y u(x, -y) = \partial_x v(x, -y) = -\partial_x[-v(x, -y)].$$

Thus, f^* is analytic. $\square$

Problem 4

Prove that there is no branch of the logarithm defined on $G = \mathbb{C} \setminus \{0\}$. (Hint: Suppose such a branch exists and compare this with the principal branch.)

Proof. Denote Log as the principal branch of the logarithm and let H be its domain. Suppose there exists a branch of the logarithm f defined on G. There exists $k \in \mathbb{Z}$ such that $f(z) = Log(z) + i2\pi k$, for all $z \in H$. Consider the limit of Log at $z = -1$. Approaching from above and below the real axis, we get

$$\lim_{\theta \to \pi} \log|z| + i\theta = i\pi \neq -i\pi = \lim_{\theta \to -\pi} \log|z| + i\theta,$$

so $\lim_{z \to -1} Log(z)$ does not exist. But then $\lim_{z \to -1} f(z)$ does not exist, contradicting the continuity of f. $\square$