MATH 220B: Homework #4

Due on Mar 1, 2025 at 23:59pm

Professor Xiao

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Problem 1

(a) Let G be a region, let $a \in G$ and suppose that $f : (G - \{a\}) \to \mathbb{C}$ is an analytic function such that $f(G - \{a\}) = \Omega$ is bounded. Show that f has a removable singularity at $z = a$. If f is one-one, show that $f(a) \in \partial\Omega$.

Proof. Since Ω is bounded, $\lim_{z \to a} |f(z)| < \infty$. Hence, $\lim_{z \to a}(z - a)f(z) = 0$, and so f has a removable singularity at $z = a$.

Suppose f is injective. Let $w = f(a)$. Since f is analytic on G, $\lim_{z \to a} f(z) = f(a) = w$. This shows that $w \in \overline{\Omega}$. Since f is injective, $f(z_n) \neq w$ for all n. Hence, $f(a) \in \partial\Omega$. $\square$

(b) Show that there is no one-one analytic function which maps $G = \{z : 0 < |z| < 1\}$ onto an annulus $\Omega = \{z : r < |z| < R\}$ where $r > 0$.

Proof. Suppose there exists a injective analytic function $f : G \to \Omega$. Since Ω is bounded, f has a removable singularity at $z = 0$ and $|f(0)|$ is either r or R. But then Ω is not simply connected and conformally equivalent to $B_1(0)$, contradiction. $\square$

Problem 2

Find an analytic function f which maps $\{z : |z| < 1, \operatorname{Re} z > 0\}$ onto $B(0;1)$ in a one-one fashion.

Proof. Consider the Mobius transformation $h(z) = \frac{1+z}{1-z}$. h maps $B_1(0)$ to the right half plane bijectively. Additionally, the interval $(-1, 1)$ is mapped to the real axis and $h(i) = i$, so h maps the upper half circle to the first quadrant. Since $z \mapsto z^2$ maps the first quadrant to the upper half plane bijectively and $g(z) = \frac{z-i}{z+i}$ maps the upper half plane to the unit disk bijectively, $f = g \circ h^2$ is the function we desire. $\square$

Problem 3

Let G_1 and G_2 be simply connected regions, neither of which is the whole plane. Let f be a one-one analytic mapping of G_1 onto G_2. Let $a \in G_1$ and put $\alpha = f(a)$. Prove that for any one-one analytic map h of G_1 into G_2 with $h(a) = \alpha$, it follows that $|h'(a)| \leq |f'(a)|$. Suppose h is not assumed to be one-one; what can be said?

Proof. Let $g = f^{-1} \circ h$. Then g is an analytic function from G_1 to G_1 with $g(a) = a$. By the Riemann Mapping Theorem, there exists bijective analytic functions $\phi : G_1 \to D$ and $\phi(a) = 0$. Then $\bar{g} = \phi \circ g \circ \phi^{-1}$ is an analytic function from D to D with $\bar{g}(0) = 0$. By Schwarz's Lemma, $|g'(a)| = |\bar{g}'(0)| \leq 1$. But then $g'(a) = (f^{-1})'(h(a)) \cdot h'(a) = \frac{h'(a)}{f'(a)}$, and so $|h'(a)| \leq |f'(a)|$. If h is not assumed to be one-one, then the resulting $\bar{g}$ would not be a rotation. In this case, $|\bar{g}'(0)| < 1$ and so $|h'(a)| < |f'(a)|$. ☐

Problem 4

Let r_1, r_2, R_1, R_2 be positive numbers such that $R_1/r_1 = R_2/r_2$; show that $\text{ann}(0; r_1, R_1)$ and $\text{ann}(0; r_2, R_2)$ are conformally equivalent. (The converse of this is presented in Exercise X.4.)

Proof. Let $m = R_1/r_1 = R_2/r_2$, and consider $f(z) = mz$. f is aa Mobius transformation, and so f is injective and analytic. We first note that $f(0) = 0$. Suppose $|z| = r$. Then $|f(z)| = |mz| = mr$. Hence, f maps $B_r(0)$ to $B_{mr}(0)$. Put r as r_1 and r_2 and the result follows. □

Problem 5

Show that there is an analytic function f defined on $G = \text{ann}(0; 0, 1)$ such that f' never vanishes and $f(G) = B(0; 1)$.

Proof. $\square$