MATH 190A: Homework #7

Due on Feb 26, 2025 at 12:00pm

Professor McKernan

Section A02 8:00AM - 8:50AM
Section Leader: Zhiyuan Jiang

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

Let X be the topological space whose closed sets are the finite sets plus the whole of X. Show that X is compact.

Proof. Let $\{U_\alpha\}_\alpha$ be an open cover of X. We may assue that X is infinite, otherwise we are done. Pick U_1 from $\{U_\alpha\}_\alpha$. Since U_1 is open, $X \backslash U_1$ is finite. For $x_i \in X \backslash U_1$, pick U_i from $\{U_\alpha\}_\alpha$ such that $x_i \in U_i$. Then $\{U_1, U_2, \ldots, U_i\}$ is a finite subcover of $\{U_\alpha\}_\alpha$. Therefore, X is compact. $\square$

Problem 2

Let X be a topological space. Show that X is compact if and only if for every collection of closed sets

$$\mathcal{F} = \{F_\alpha \mid \alpha \in \Lambda\}$$

such that every finite subcollection

$$\mathcal{F}_0 = \{F_\beta \mid \beta \in M\}$$

has non-empty intersection, then the intersection of every element of $\mathcal{F}$ is non-empty.

Proof. We first show the forward direction. Suppose for contradiction that $\bigcap_{F \in \mathcal{F}} F = \emptyset$. Let $\mathcal{G}$ be the collection of $X \backslash F$ for all $F \in \mathcal{F}$. Then $\bigcup_{G \in \mathcal{G}} G = X \backslash \bigcap_{F \in \mathcal{F}} F = X$. But then $\mathcal{G}$ is an open cover of X, and so there exists a finite subcover $\mathcal{G}_0 = \{X \backslash F_i \mid 1 \leq i \leq n\}$, for some $F_1, \ldots, F_n \in \mathcal{F}$. That is, $\bigcap_{G \in \mathcal{G}_0} G = X \backslash \bigcap_{i=1}^n F_i$. But then $\bigcap_{i=1}^n F_i \neq \emptyset$, so $\bigcap_{G \in \mathcal{G}_0} G \neq X$, a contradiction.

Now we show the converse. Let $\{U_\alpha\}$ be an open cover of X. Let $\mathcal{F}$ be the collection of $X \backslash U_\alpha$ for all α. Then $\mathcal{F}$ is a collection of closed sets. Since $\bigcap_{F \in \mathcal{F}} F = \bigcap_\alpha (X \backslash U_\alpha) = X \backslash \bigcup_\alpha U_\alpha = \emptyset$, there exists a finite subcollection $\mathcal{F}_0 = \{X \backslash U_{\alpha_1}, \ldots, X \backslash U_{\alpha_n}\}$ such that $\bigcap_{F \in \mathcal{F}_0} F = X \backslash \bigcap_{i=1}^n U_{\alpha_i} = \emptyset$. But then $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of X. $\square$

Problem 3

True or false? If true then give a proof and if false then give a counterexample.

(i) If X is a compact topological space and $f : X \to \mathbb{R}$ is continuous, then f achieves its maximum and minimum value.

Proof. True. Since f is continuous and X is compact, $f(X)$ is compact and so $f(X) = [a, b]$ for some $a < b$. Therefore, f achieves its maximum and minimum value. $\square$

(ii) Let $f : X \to Y$ be continuous and injective. If Y is Hausdorff, then X is Hausdorff.

Proof. True. Let $x_1, x_2 \in X$ and $y_1 = f(x_1)$, $y_2 = f(x_2)$. Since Y is Hausdorff, there exists disjoint open sets U_1 and U_2 such that $y_1 \in U_1$, $y_2 \in U_2$. Since f is continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open in X. Since f is injective, $f(f^{-1}(U_1)) \subseteq U_1$ and $f(f^{-1}(U_2)) \subseteq U_2$. Therefore, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are disjoint open sets in X that contain x_1 and x_2 respectively. $\square$

(iii) If A and B are compact subspaces of a topological space X then $A \cup B$ is compact.

Proof. True. Let $\{U_\alpha\}$ be an open cover of $A \cup B$. Then $\{U_\alpha \cap A\}_\alpha$ and $\{U_\alpha \cap B\}_\alpha$ are open covers of A and B respectively. Since A and B are compact, there exists finite subcovers $\{U_1, U_2, \ldots, U_n\}$ and $\{V_1, V_2, \ldots, V_m\}$ of A and B respectively. Then $\{U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_m\}$ is a finite subcover of $A \cup B$. $\square$

(iv) If A and B are compact subspaces of a topological space X then $A \cap B$ is compact.

Proof. False. Consider $X = \mathbb{R} \cup \{a, b\}$ whose open sets are the canonical open sets of $\mathbb{R}$ and $\mathbb{R} \cup U$, for $U \subseteq \{a, b\}$. Let $A = \mathbb{R} \cup \{a\}$ and $B = \mathbb{R} \cup \{b\}$. Let $\{U_\alpha\}$ be an open cover of X. Then there exists U_1 such that $a \in U_1$, and so $\mathbb{R} \subseteq U_1$. Hence, A is compact. Similarly, B is compact. However, $A \cap B = \mathbb{R}$ is not compact. $\square$

Problem 4

Let X be a compact topological space and let Y be a Hausdorff topological space. If $f : X \to Y$ is continuous and a bijection, then show that f is a homeomorphism.

Proof. It suffices to show that f^{-1} is continuous. Let U be closed in X. Since X is compact, U is compact, and thus $f(U)$ is compact. But then Y is Hausdorff, so $f(U)$ is closed. Therefore, f^{-1} is continuous. $\square$