MATH 140A: Homework #2

Due on Jan 26, 2023 at 23:59pm

Professor Seward

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Problem 1

Prove that no order can be defined in the complex field that turns it into an ordered field.

Proof. Consider i^2. Since i^2 is a square of nonzero element, $-1 = i^2 > 0$, contradiction. Hence, no order can be defined in the complex field. $\square$

Problem 2

Suppose $z = a + bi, w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of complex numbers into an ordered set. Does this ordered set have the least-upper-bound property?

Proof. If $a > c$, then $z > w$. Suppose that $a = c$. If $b > d$, then $z > w$. If $b = d$, then $z = w$. Thus, the order follows the law of trichotomy.

We now show that the order is transitive. Let $x = g + hi$. Suppose that $z > w$ and $w > x$. Since $z > w$, either $a > c$, or $a = c$ and $b > d$. Similarly, since $w > x$, either $c > g$, or $c = g$ and $d > h$. We may assume that $a = c = g$, otherwise $a > g$ and we are done. Then, $b > d > h$, so $z > x$, so the order is indeed transitive.

Note that this ordered set has the least-upper-bound property. Let $B \subset \mathbb{C}$ be non-empty. Since $\mathbb{R}$ has the least-upper-bound property, we know there exists $\alpha = \sup\{k \in \mathbb{R} \mid k + mi \in B\}$ and $\beta = \sup\{m \in \mathbb{R} \mid \alpha + mi \in B\}$. We show that $\alpha + \beta i = \sup B$. Let $a + bi \in \mathbb{C}$. We know $\alpha \geq a$. We may assume that $\alpha = a$. Then, since $\beta \geq b$, we know $\alpha + \beta i \geq a + bi$, so $\alpha + \beta i$ is the upper bound of B. Let $w = c + di$, such that $w < \alpha + \beta i$. If $c < \alpha$, then we may find $p \in \{k \in \mathbb{R} \mid k + mi \in B\}$ such that $p > c$, and thus ther exists $p + qi \in B$ such that $p + qi > w$. If $c = \alpha$ and $d < \beta$, then we may find $t \in \{m \in \mathbb{R} \mid \alpha + mi \in B\}$ such that $t > d$, and thus there exists $s + ti \in B$, such that $s + ti > y$. Therefore, $\alpha + \beta i = \sup B$, so the ordered set does have the least-upper-bound property. $\square$

Problem 3

Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \quad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Proof. If $v \geq 0$, then

$$\begin{aligned}
z^2 &= (a + bi)(a + bi) \\
&= a^2 - b^2 + 2abi \\
&= \frac{|w| + u}{2} - \frac{|w| - u}{2} + 2\left(\frac{|w| + u}{2}\right)^{1/2}\left(\frac{|w| - u}{2}\right)^{1/2} i \\
&= u + 2\left(\frac{|w|^2 - u^2}{4}\right)^{1/2} i \\
&= u + i|v| = u + iv = w.
\end{aligned}$$

If $v \leq 0$, then

$$\begin{aligned}
(\bar{z})^2 &= (a - bi)(a - bi) \\
&= a^2 - b^2 - 2abi \\
&= \frac{|w| + u}{2} - \frac{|w| - u}{2} - 2\left(\frac{|w| + u}{2}\right)^{1/2}\left(\frac{|w| - u}{2}\right)^{1/2} i \\
&= u - 2\left(\frac{|w|^2 - u^2}{4}\right)^{1/2} i \\
&= u - i|v| = u + iv = w.
\end{aligned}$$

Suppose that $w = 0$. Then, $a, b = 0$, so $z = \bar{z} = 0$, which means that $w = 0$ only has one complex root. However, when $w \neq 0$, w has z and $\bar{z}$ as its complex roots. Therefore, every nonzero complex number has two complex roots. $\square$

Problem 4

If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

Proof. On the LHS

$$\begin{aligned}(|x| - |y|)^2 &= |x|^2 + |y|^2 - 2|x||y| \\ &= |x|^2 + |y|^2 - 2|x||\bar{y}| \\ &= |x|^2 + |y|^2 - 2|x\bar{y}|.\end{aligned}$$

On the RHS,

$$\begin{aligned}|x - y|^2 &= (x - y)(\bar{x} - \bar{y}) \\ &= x\bar{x} + y\bar{y} - y\bar{x} - x\bar{y} \\ &= |x|^2 + |y|^2 - (y\bar{x} + x\bar{y}) \\ &= |x|^2 + |y|^2 - (x\bar{y} + \overline{x\bar{y}}) \\ &= |x|^2 + |y|^2 - 2\text{Re } x\bar{y}.\end{aligned}$$

Since $|x\bar{y}| \geq \text{Re } x\bar{y}$, we have $||x| - |y||^2 \leq |x - y|^2$. Since $||x| - |y||, |x - y| \geq 0$, the results follows. $\square$

Problem 5

If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1+z|^2 + |1-z|^2.$$

Proof.

$$\begin{aligned}
|1+z|^2 + |1-z|^2 &= (1+z)(1+\bar{z}) + (1-z)(1-\bar{z}) \\
&= z\bar{z} + 1 + z + \bar{z} + z\bar{z} + 1 - z - \bar{z} \\
&= 2|z|^2 + 2 \\
&= 4.
\end{aligned}$$

□

Problem 6

Under what conditions does equality hold in the Schwarz inequality?

Proof. From the proof of Theorem 1.35, the equality holds when $C^2 = AB$, which is equivalent to $|Ba_j - Cb_j| = 0$, for all j. Hence, the equality holds when $a_j \sum_i^n |b_i|^2 = b_j \sum_i^n a_i \bar{b_i}$, for all j. $\square$

Problem 7

Prove that
$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$$
if $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Proof.

$$\begin{aligned}
|x+y|^2 + |x-y|^2 &= x \cdot x + 2x \cdot y + y \cdot y + x \cdot x - 2x \cdot y + y \cdot y \\
&= 2(x \cdot x + y \cdot y) \\
&= 2|x|^2 + 2|y|^2.
\end{aligned}$$

This implies that in a parallelogram, the square sum of the length of the diagonals equals the the square sum of the length each side. □