MATH 100A: Homework #7

Due on November 21, 2023 at 12:00pm

Professor McKernan

Section A02 5:00PM - 5:50PM
Section Leader: Castellano

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

If G_1 and G_2 are groups, prove that $G_1 \times G_2 \simeq G_2 \times G_1$.

Proof. Define $\phi : G_1 \times G_2 \to G_2 \times G_1$ as $\phi(a, b) = (b, a)$. ϕ is obviously a well-defined. Define $\psi : G_2 \times G_1 \to G_1 \times G_2$ as $\psi(b, a) = (a, b)$. Since $\phi(\psi(b, a)) = \phi(a, b) = (b, a)$ and $\psi(\phi(a, b)) = \psi(b, a) = (a, b)$, ψ is an inverse of ϕ, so ϕ is bijective. Since $\phi(a, b)\phi(a', b') = (bb', aa') = \phi(aa', bb')$, ϕ is an isomorphism, and thus $G_1 \times G_2 \simeq G_2 \times G_1$. $\square$

Problem 2

If G_1 and G_2 are cyclic groups of orders m and n, respectively, prove that $G_1 \times G_2$ is cyclic if and only if m and n are relatively prime.

Proof. Suppose that $G_1 \times G_2$ is cyclic. Then $G_1 \times G_2 = \{(a^i, b^i) \mid i \in \mathbb{Z}\}$, for some $a \in G_1$, $b \in G_2$. Since $G_1 \times G_2$ is of order mn, we know m, n is relatively prime, otherwise we can find $k < mn$ such that $(a^k, b^k) = (e_1, e_2)$, which contradicts that $G_1 \times G_2$ is of order mn. Suppose that m, n are relatively prime. Let $c \in G_1, d \in G_2$ each be the generator of their respective group. Let $(x, y) = (c^j, d^l) \in G_1 \times G_2$, and let $d = l - j$. Since m, n are relatively prime, there exists $m\alpha + n\beta = 1$. Multiplying both sides by d, we get $md\alpha + nd\beta = l - j$, and so there exists $x = (d\alpha)m + j = (-d\beta)n + l$. Thus, $(x, y) = (c^j, d^l) = (c^x, d^x)$, and so $G_1 \times G_2$ is cyclic. $\square$

Problem 3

Let G be a group, $A = G \times G$. In A let $T = \{(g, g) \mid g \in G\}$.

(a) Prove that $T \simeq G$.

Proof. Let $\phi : T \to G$ be the natural projection. Then, ϕ is well-defined and surjective. Since $\phi(g, g)\phi(g', g') = gg' = \phi(gg', gg')$, ϕ is a homomorphism. Let $(a, a) \in \text{Ker } \phi$. $\phi(a, a) = a = e$, and so Ker ϕ is trivial. Therefore, ϕ is an isomorphism, and thus $T \simeq G$. $\square$

(b) Prove that $T \lhd A$ if and only if G is abelian.

Proof. Suppose that $T \lhd A$. For $(g, h) \in A$, $(g, h)(g, g)(g^{-1}, h^{-1}) = (g, hgh^{-1}) \in T$. This implies that for all $g, h \in G$, $g = hgh^{-1}$. Rearranged, we get $gh = hg$, which makes G abelian. Suppose that G is abelian. Let $(g, g) \in T$, $(a, b) \in A$. Since $(a, b)(g, g)(a^{-1}, b^{-1}) = (aga^{-1}, bgb^{-1}) = (g, g) \in T$, T is normal in A. $\square$

Problem 4

Let H and K be two normal subgroups of a group G, whose intersection is the trivial subgroup. Prove that every element of H commutes with every element of K.

Proof. Let $h \in H$, $k \in K$. Since H is normal, $h^{-1}k^{-1}hk = h^{-1}h'k^{-1}k = h^{-1}h' \in H$. By symmetry, $h^{-1}k^{-1}hk \in K$, which makes $h^{-1}k^{-1}hk \in H \cap K = \{e\}$. Thus, we know $h^{-1}k^{-1}hk$ must be the identity element, and thus $hk = kh$. $\square$

Problem 5

Prove that a group G is isomorphic to the product of two groups H' and K' if and only if G contains two normal subgroups H and K, such that

1. H is isomorphic to H' and K is isomorphic to K'.
2. $H \cap K = \{e\}$.
3. $G = H \vee K$.

Proof. Suppose that $G \simeq H' \times K'$. Let $\phi : H' \times K' \to G$ be an isomorphism, $G_{H'} = \{(h, e_{k'}) \mid h \in H'\}$, and $G_{K'} = \{(e_{h'}, k) \mid k \in K'\}$, where $e_{h'} \in H', e_{k'} \in K'$ are the identity element of their corresponding groups. Let $H = \phi(G_{H'})$ and $K = \phi(G_{K'})$. From Homework 6 question 2.7.4, we have shown that $H' \simeq G_{H'}$ and $K' \simeq G_{K'}$, and $G_{H'}, G_{K'}$ are normal subgroups of $H' \times K'$. Thus, we know $H \simeq G_{H'} \simeq H'$ and $K \simeq G_{K'} \simeq K'$ are both normal subgroups of G. Let $\psi : G \to H' \times K'$ be the inverse of ϕ. Then, $\psi(H \cap K) = G_{H'} \cap G_{K'} = \{(e_{h'}, e_{k'})\}$, which contains only the identity element of $H' \times K'$. Since ψ is an isomorphism, $H \cap K = \{e\}$. Note that for all $x \in H' \times K'$, $x = ab$, for some $a \in G_{H'}$, $b \in G_{K'}$. Thus, $\phi(x) = \phi(ab) = \phi(a)\phi(b) = hk$, where $h \in H$ and $k \in K$. This implies that $G = HK$, and so $G = H \vee K$, by the Second Isomorphism Theorem.

We now suppose that conditions 1-3 hold. Since H, K are normal, by the Second Isomorphism Theorem, $G = H \vee K = HK$. Let $\alpha : H \to H'$ and $\beta : K \to K'$ be isomorphisms. Define $\varphi : G \to H' \times K'$ as $\varphi(hk) = (\alpha(h), \beta(k))$, for $h \in H$, $k \in K$. Suppose $hk = h_0k_0 \in G$, for $h, h_0 \in H$ and $k, k_0 \in K$. Then, $\varphi(hk) = (\alpha(h), \beta(k)) = (\alpha(h_0), \beta(k_0)) = \varphi(h'k')$, so φ is well-defined. Define $\theta : H' \times K' \to G$ as $\theta(h', k') = \alpha^{-1}(h')\beta^{-1}(k')$, where α^{-1}, β^{-1} are the inverses of α, β, respectively. We then get $\varphi(\theta(h', k')) = \varphi(\alpha^{-1}(h')\beta^{-1}(k')) = (\alpha(\alpha^{-1}(h')), \beta(\beta^{-1}(k'))) = (h', k')$ and $\theta(\varphi(hk)) = \theta(\alpha(h), \beta(k)) = \alpha^{-1}(\alpha(h))\beta^{-1}(\beta(k)) = hk$. Thus, θ is the inverse of φ, so φ is a bijective mapping. Finally, we check that φ is a homomorphism. Let $m = hk, n = h_1k_1 \in G$, where $h, h_1 \in H$ and $k, k_1 \in K$. Note that since H, K are both normal and $H \cap K = \{e\}$, every element of H commutes with every element of K, by result we obtained in the previous problem. Thus,

$$\begin{aligned}
\varphi(mn) &= \varphi(hkh_1k_1) \\
&= \varphi(hh_1kk_1) \\
&= (\alpha(hh_1), \beta(kk_1)) \\
&= (\alpha(h)\alpha(h_1), \beta(k)\beta(k_1)) \\
&= (\alpha(h), \beta(k))(\alpha(h_1), \beta(k_1)) \\
&= \varphi(hk)\varphi(h_1k_1) \\
&= \varphi(m)\varphi(n).
\end{aligned}$$

Therefore, φ is an isomorphism, and so $G \simeq H' \times K'$. $\square$