MATH 220B: Homework #1

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Professor Xiao

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Problem 1

Each of the following functions f has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity define $f(0)$ so that f is analytic at $z = 0$; if it is a pole find the singular part; if it is an essential singularity determine $f(\{z : 0 < |z| < \delta\})$ for arbitrarily small values of δ.

(a) $f(z) = \frac{\sin z}{z}$

Proof. f has a removable singularity at $z = 0$. Since the power series expansion of $\sin z$ is $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$,

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

and so defining $f(0) = \lim_{z\to 0} f(z) = 1$ makes f analytic. □

(b) $f(z) = \frac{\cos z}{z}$

Proof. Since the power series expansion of $\cos z$ is $1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$,

$$f(z) = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \cdots,$$

and so the singluar part of f is $\frac{1}{z}$. □

(j) $f(z) = z^n \sin\left(\frac{1}{z}\right)$

Proof. Note that the Laurent expansion

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{n-k} = \sum_{k=-\infty}^{n} \frac{(-1)^{n-k}}{[2(n-k)+1]!} z^k$$

has infinitely many terms with negative powers of z and so f has an essential singularity at $z = 0$.

□

Problem 2

Let $f(z) = \frac{1}{z(z-1)(z-2)}$; give the Laurent Expansion of $f(z)$ in each of the following annuli:

(b) $\text{ann}(0; 1, 2)$

Proof. By partial fractions decomposition,

$$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}.$$

Since $|z| > 1$,

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

Since $0 < |z| < 2$,

$$\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.$$

Hence, the Laurent expansion of f in $\text{ann}(0; 1, 2)$ is

$$f(z) = \frac{1/2}{z} - \sum_{n=1}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{1}{2^{n+2}} z^n.$$

□

(c) $\text{ann}(0; 2, \infty)$

Proof. By partial fractions decomposition,

$$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}.$$

Since $|z| > 2$,

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

and

$$\frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\frac{2}{z}} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{z^n}.$$

Hence, the Laurent expansion of f in $\text{ann}(0; 1, 2)$ is

$$f(z) = \frac{1/2}{z} - \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{2^{n-2}}{z^n} = \frac{1/2}{z} + \sum_{n=1}^{\infty} \frac{2^{n-2}-1}{z^n}.$$

□

Problem 3

If $f : G \to \mathbb{C}$ is analytic except for poles, show that the poles of f cannot have a limit point in G.

Proof. Suppose $a \in G$ is a limit point of poles of f. Then a cannot be a pole, as there does not exist $R > 0$ such that $B_R(a) \backslash \{a\}$ is analytic. By the Open Mapping Theorem, there exists $r > 0$ such that $B_r(f(a)) \subseteq f(G \backslash \{\text{poles}\})$. Since f is continuous, $f^{-1}(B_r(f(a)))$ is open in $G \backslash \{\text{poles}\}$ and contains a. But then a is a limit point of poles, so there does not exist an open neighborhood of a that is contained in $G \backslash \{\text{poles}\}$, contradiction. $\square$

Problem 4

Suppose that f has an essential singularity at $z = a$. Prove the following strengthened version of the Casorati-Weierstrass Theorem. If $c \in \mathbb{C}$ and $\epsilon > 0$ are given, then for each $\delta > 0$ there is a number α, $|c - \alpha| < \epsilon$, such that $f(z) = \alpha$ has infinitely many solutions in $B(a;\delta)$.

Proof. We may assume that $a = 0$, otherwise we work with $f(z - a)$ instead. For $n \in \mathbb{N}$, let $S_n = f[\text{ann}(0, 0, \delta/(n+1))]$, the image of $f(z)$ for all $0 < |z| < \delta/(n+1)$. Let $\alpha_1 = c$ and $r_1 = \epsilon$. For $n \in \mathbb{N}$, since S_n is dense by the Casorati-Weierstrass Theorem, there exists $\alpha_{n+1} \in S_n \cap B_\epsilon(c)$. By the Open Mapping Theorem, S_n is open, so there exists $r_{n+1} \in (0, \delta/(n+1))$ such that $\overline{B}_{r_{n+1}}(\alpha_{n+1}) \subseteq S_n \cap B_{r_n}(\alpha_n)$. By iterating this process, we obtain sequences $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$. Since $r_n \to 0$, α_n is Cauchy and thus converges to some $\alpha \in B_\epsilon(c) \cap \bigcap_{n=1}^{\infty} S_n$, and the result follows. □

Problem 5

Let $R > 0$ and $G = \{z : |z| > R\}$; a function $f : G \to \mathbb{C}$ has a *removable singularity*, a *pole*, or an *essential singularity at infinity* if $f(z^{-1})$ has, respectively, a removable singularity, a pole, or an essential singularity at $z = 0$. If f has a pole at ∞, then the order of the pole is the order of the pole of $f(z^{-1})$ at $z = 0$.

(a) Prove that an entire function has a removable singularity at infinity iff it is a constant.

Proof. If an entire function f has a removable singularity at ∞, then $f(z^{-1})$ has a removable singularity at $z = 0$. But then $f(z^{-1})$ is bounded around 0, which implies f is bounded in any neighborhood of ∞. By Liouville's Theorem, f is constant.

If $f(z) = c$, then $f(z^{-1}) = c$ and so f has a removable singularity at ∞. □

(b) Prove that an entire function has a pole at infinity of order m iff it is a polynomial of degree m.

Proof. If an entire function f has a pole at ∞ of order m, then $f(z^{-1})z^m$ has a removable singularity at $z = 0$. But then $f(z^{-1})z^m$ is bounded in any neighborhood of 0, so $f(z^{-1})z^m$, which implies $f(z)z^{-m}$ is bounded in any neighborhood of ∞. By Liouville's Theorem, $f(z)z^{-m}$ is constant, and thus $f(z)$ is a polynomial of degree m.

If $f(z) = a_m z^m + \cdots + a_0$ with $a_m \neq 0$, then $z^m f(z^{-1}) = \lim_{z \to 0} z^m f(z^{-1}) = a_0 z^m + \cdots + a_m$, which has a removable singularity at $z = 0$. Thus, f has a pole at ∞ of order m. □

Problem 6

Calculate the following integrals:

(a) $\int_0^\infty \frac{x^2\,dx}{x^4+x^2+1}$

Proof. Put $f(z) = \frac{z^2}{z^4+z^2+1}$. Since f is even, $\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^\infty f(x)\,dx$. Note that f has poles at $a_1 = e^{i\pi/3}$ and $a_2 = e^{2i\pi/3}$. For $R > 1$, let $\gamma_R = Re^{it}$, $0 \le t \le \pi$. Since a_1, a_2 are enclosed by $\gamma_R \cup [-R, R]$, by the Residue Theorem,

$$\int_{-R}^R f(x)\,dx = 2\pi i[\operatorname{Res}(f,a_1) + \operatorname{Res}(f,a_2)] - \int_{\gamma_R} f(z)\,dz.$$

Calculating the residues, we have

$$\begin{aligned}\operatorname{Res}(f,a_1) &= \frac{e^{2i\pi/3}}{4e^{i\pi}+2e^{i\pi/3}} = \frac{-\frac{1}{2}+\frac{i\sqrt{3}}{2}}{-4+1+i\sqrt{3}} = \frac{1}{4} - \frac{\sqrt{3}}{12}i,\\ \operatorname{Res}(f,a_2) &= \frac{e^{4i\pi/3}}{4e^{2i\pi}+2e^{2i\pi/3}} = \frac{-\frac{1}{2}-\frac{i\sqrt{3}}{2}}{4-1+i\sqrt{3}} = -\frac{1}{4} - \frac{\sqrt{3}}{12}i.\end{aligned}$$

As $R \to \infty$,

$$\left|\int_{\gamma_R} f(z)\,dz\right| \le \int_{\gamma_R} \frac{|z|^2|}{|z^4+z^2+1|}\,|dz| \le \int_{\gamma_R} \frac{|z|^2}{|z|^4-|z|^2-1}\,|dz| = \frac{\pi R^3}{R^4-R^2-1} \to 0.$$

Hence, combining the above results,

$$\int_0^\infty \frac{x^2\,dx}{x^4+x^2+1} = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R f(x)\,dx = \pi i\left(\frac{1}{4} - \frac{\sqrt{3}}{12}i - \frac{1}{4} - \frac{\sqrt{3}}{12}i\right) = \frac{\pi}{2\sqrt{3}}.$$

□

(b) $\int_0^\infty \frac{\cos x - 1}{x^2}\,dx$

Proof. Put $f(z) = \frac{e^{iz}-1}{z^2}$. Let $R > r > 0$ and define $\gamma_r(t) = re^{-it}$, $0 \le t \le \pi$, and $\gamma_R(t) = Re^{it}$, $0 \le t \le \pi$. Let $\gamma = [-R,-r] \cup \gamma_r \cup [r,R] \cup \gamma_R$. Since f is analytic on $\mathbb{C}\backslash\{0\}$,

$$\int_\gamma f(z)\,dz = \int_{-R}^{-r} f(z)\,dz + \int_{\gamma_r} f(z)\,dz + \int_r^R f(z)\,dz + \int_{\gamma_R} f(z)\,dz = 0$$

Note that

$$\int_{-R}^{-r} \frac{e^{iz}-1}{z^2}\,dz = \int_r^R \frac{e^{-iz}-1}{z^2}\,dz,$$

and so

$$\int_{-R}^{-r} f(z)\,dz + \int_r^R f(z)\,dz = \int_r^R \frac{e^{iz}+e^{-iz}-2}{z^2}\,dz = 2\int_r^R \frac{\cos(z)-1}{z^2}\,dz.$$

Hence,

$$\int_r^R \frac{\cos(z)-1}{z^2}\,dz = -\frac{1}{2}\int_{\gamma_r} f(z)\,dz - \frac{1}{2}\int_{\gamma_R} f(z)\,dz.$$

As $R \to \infty$,

$$\left|\int_{\gamma_R} f(z)\,dz\right| \le \int_{\gamma_R} \frac{|e^{iz}-1|}{|z|^2}\,|dz| \le \frac{1}{R^2}\left(\int_{\gamma_R} |e^{iz}|\,|dz| + \int_{\gamma_R} |dz|\right) = \frac{2}{R^2}\int_{\gamma_R} |dz| = \frac{2\pi}{R} \to 0.$$

Problem 6 continued on next page...

On the other hand, since $f(z) = \frac{i}{z} - \frac{1}{2} - \frac{iz}{6} + \cdots$,

$$\begin{aligned}
\lim_{r \to 0} \int_{\gamma_r} f(z)\, dz &= \lim_{r \to 0} i \int_{\gamma_r} \frac{1}{z}\, dz + \left(- \lim_{r \to 0} \int_{\gamma_r} \frac{1}{2}\, dz + \lim_{r \to 0} \int_{\gamma_r} \frac{iz}{6}\, dz - \cdots \right) \\
&= \lim_{r \to 0} i \int_{\gamma_r} \frac{1}{z}\, dz \\
&= \lim_{r \to 0} i \int_0^{\pi} \frac{-ire^{-it}}{re^{-it}}\, dt = \pi.
\end{aligned}$$

Thus,

$$\int_0^{\infty} \frac{\cos(z) - 1}{z^2}\, dz = -\frac{1}{2} \lim_{r \to 0} \int_{\gamma_r} f(z)\, dz = -\frac{\pi}{2}.$$

□

Problem 7

Verify the following equation:

$$\int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta} = \frac{\pi}{2\sqrt{a(a+1)}}, \quad \text{if } a > 0;$$

Proof. Note that

$$\int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta} = \int_0^{\pi} \frac{1}{2a+1-\cos\theta}\, d\theta,$$

and since $\cos\theta = \cos -\theta$,

$$\int_0^{\pi} \frac{1}{2a+1-\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{2a+1-\cos\theta}\, d\theta.$$

Put $z = e^{i\theta}$ and we have

$$\int_0^{2\pi} \frac{1}{2a+1-\cos\theta}\, d\theta = \int_{|z|=1} \frac{1}{2a+1-\frac{z+z^{-1}}{2}} \frac{dz}{iz} = 2i\int_{|z|=1} \frac{1}{z^2-(4a+2)z+1}\, dz.$$

Let $f(z) = \frac{1}{z^2-(4a+2)z+1}$. $f(z)$ have simple poles at $z = 2a+1\pm 2\sqrt{a(a+1)}$. Since $|2a+1+2\sqrt{a(a+1)}| > 1$, by the Residue Theorem,

$$\int_{|z|=1} \frac{1}{z^2-(4a+2)z+1}\, dz = 2\pi i \operatorname{Res}(f, 2a+1-2\sqrt{a(a+1)}).$$

Since

$$\operatorname{Res}(f, 2a+1-2\sqrt{a(a+1)}) = \left.\frac{1}{2z-(4a+2)}\right|_{z=2a+1-2\sqrt{a(a+1)}} = -\frac{1}{4\sqrt{a(a+1)}},$$

we have

$$\int_0^{\pi} \frac{1}{2a+1-\cos\theta}\, d\theta = \frac{1}{2}\cdot 2i\cdot 2\pi i\cdot -\frac{1}{4\sqrt{a(a+1)}} = \frac{\pi}{2\sqrt{a(a+1)}},$$

for $a > 0$. □

Problem 8

Suppose that f has a simple pole at $z = a$ and let g be analytic in an open set containing a. Show that

$$\operatorname{Res}(fg; a) = g(a) \operatorname{Res}(f; a).$$

Proof.

$$\operatorname{Res}(fg; a) = \lim_{z \to a} (z-a) f(z) g(z) = g(a) \lim_{z \to a} (z-a) f(z) = g(a) \operatorname{Res}(f; a).$$

□

Problem 9

Use the previous result to show that if G is a region and f is analytic in G except for simple poles at $a_1, \ldots, a_n$; and if g is analytic in G, then

$$\frac{1}{2\pi i}\int_\gamma fg = \sum_{k=1}^{n} n(\gamma; a_k)\, g(a_k)\, \mathrm{Res}(f; a_k)$$

for any closed rectifiable curve γ not passing through $a_1, \ldots, a_n$ such that $\gamma \approx 0$ in G.

Proof. By the Residue Theorem and the last problem,

$$\frac{1}{2\pi i}\int_\gamma fg = \sum_{k=1}^{n} n(\gamma; a_k)\, \mathrm{Res}(fg; a_k) = \sum_{k=1}^{n} n(\gamma; a_k)\, g(a_k)\, \mathrm{Res}(f; a_k).$$

□