MATH 140B: Homework #4

Due on May 3, 2024 at 23:59pm

Professor Seward

Ray Tsai

A16848188

Problem 1

Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercises 6.7 and 6.8. (State the appropriate hypotheses, formulate a theorem, and prove it.) For instance, show that

$$\int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx.$$

Show that one of these integrals converges absolutely, but that the other does not.

Theorem Let F, G be differentiable functions on $[a, \infty)$, where $F' = f \in \mathscr{R}$ and $G' = g \in \mathscr{R}$. Suppose both $\lim_{x\to\infty} F(x)G(x)$ and $\int_a^\infty f(x)G(x)\,dx$ exist. Then

$$\int_a^\infty F(x)g(x)\,dx = \lim_{x\to\infty} F(x)G(x) - F(a)G(a) - \int_a^\infty f(x)G(x)\,dx.$$

Proof. Put $H(x) = F(x)G(x)$. By Theorem 6.13, we know $H' \in \mathscr{R}$. For finite $b > a$, applying Theorem 6.21 to H and its derivative yields

$$H(b) - H(a) = \int_a^b F(x)g(x) + f(x)G(x)\,dx,$$

that is,

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

But then by assumption, $\lim_{x\to\infty} F(x)G(x)$ and $\int_a^\infty f(x)G(x)\,dx$ exist, and thus $\int_a^\infty F(x)g(x)\,dx$ also converges. □

Put $F(x) = \frac{1}{1+x}$ and $G(x) = \sin x$. We know $f(x) = -\frac{1}{(1+x)^2} \in \mathscr{R}$, $g(x) = \cos x$. Note that

$$\lim_{x\to\infty} |F(x)G(x)| = \lim_{x\to\infty} \left|\frac{\sin x}{1+x}\right| \leq \lim_{x\to\infty} \left|\frac{1}{1+x}\right| = 0 = F(0)G(0).$$

By exercise 6.8, we know that $\int_0^\infty \left|\frac{\sin x}{(1+x)^2}\right| dx$ converges as $\sum_{n=0}^\infty \frac{|\sin x|}{(1+n)^2}$ converges by comparison test with $\sum_{n=0}^\infty \frac{1}{(1+n)^2}$. Hence, again by exercise 6.8, $\int_0^\infty \frac{\sin x}{(1+x)^2}\,dx$ also converges, as $\sum_{n=0}^\infty \frac{\sin x}{(1+n)^2}$ converges absolutely. Since the hypothesis holds, we may apply our theorem stated above and get

$$\int_0^\infty \frac{\cos x}{1+x}\,dx = \lim_{x\to\infty} \frac{\sin x}{1+x} - \frac{\sin 0}{1} + \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx.$$

To see that $\int_0^\infty \frac{\cos x}{1+x}\,dx$ does not converge absolutely, we again apply exercise 6.8. Since

$$\sum_{n\geq 0} \left|\frac{\cos x}{1+x}\right| \geq \sum_{n\geq 0} \frac{1}{1+x}$$

diverges, $\int_0^\infty \left|\frac{\cos x}{1+x}\right| dx$ also diverges.

Problem 2

Let α be a fixed increasing function on $[a, b]$. For $u \in \mathscr{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2 \right)^{1/2}.$$

Suppose $f, g, h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof.

$$\begin{aligned}
\|f - h\|_2 &= \|f - g + g - h\|_2 \\
&= \left(\int_a^b |f - g + g - h|^2 \right)^{1/2} \\
&= \left(\int_a^b |f - g|^2 + 2 \int_a^b |(f - g)(g - h)| + \int_a^b |g - h|^2 \right)^{1/2} \\
&\leq \left(\int_a^b |f - g|^2 + 2 \int_a^b |f - g| \int_a^b |g - h| + \int_a^b |g - h|^2 \right)^{1/2} \\
&= \left(\int_a^b |f - g|^2 \right)^{1/2} + \left(\int_a^b |g - h|^2 \right)^{1/2} \\
&= \|f - g\|_2 + \|g - h\|_2.
\end{aligned}$$

$\square$

Problem 3

With the notations of Exercise 6.11, suppose $f \in \mathscr{R}(\alpha)$ and $\epsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \epsilon$.

Proof. Pick $\epsilon > 0$. Since $f \in \mathscr{R}(\alpha)$, there exists a partition $P = \{x_0, \ldots, x_n\}$ on $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon^2/2M.$$

Suppose $|f| < M$. Define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i).$$

if $x_{i-1} \leq t \leq x_i$. Note that g is defined to be linear on every interval $[x_i, x_{i+1}]$, and g remains continuous between neighboring intervals. Hence, g is continuous on $[a, b]$. In addition, on every interval $[x_1, x_{i+1}]$, since $g(t)$ is between $f(x_1)$ and $f(x_{i+1})$, we have $m_i \leq g(t) \leq M_i$ for all $t \in [x_1, x_{i+1}]$. But then

$$\begin{aligned}
\|f - g\|_2^2 &= \int_a^b |f - g|^2 \\
&\leq U(P, |f - g|^2, \alpha) \\
&= \sum_{i=1}^{n} \sup_{x \in [x_i, x_{i+1}]} (f(x) - g(x))^2 \Delta\alpha_i \\
&\leq \sum_{i=1}^{n} (M_i - m_i)^2 \Delta\alpha_i \\
&\leq 2M[U(P, f, \alpha) - L(P, f, \alpha)] < \epsilon^2,
\end{aligned}$$

and thus $\|f - g\|_2 < \epsilon$. $\square$

Problem 4

Define

$$f(x) = \int_x^{x+1} \sin(t^2)\, dt.$$

(a) Prove that $|f(x)| < \frac{1}{x}$ if $x > 0$.

Proof. By Theorem 6.17 and 6.19, we may substitute t^2 by u and get

$$f(x) = \int_{x^2}^{(x+1)^2} \sin(u)\, du^{1/2} = \int_{x^2}^{(x+1)^2} \frac{\sin(u)}{2u^{1/2}}\, du.$$

Put $F(x) = \frac{1}{2u^{1/2}}$ and $G(x) = -\cos(x)$. Applying Theorem 6.22 then yields

$$f(x) = \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \frac{1}{2}\int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}}\, du.$$

But then notice

$$\left|\int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}}\, du\right| \leq \int_{x^2}^{(x+1)^2} \left|\frac{\cos u}{2u^{3/2}}\right| du < \int_{x^2}^{(x+1)^2} \frac{1}{2u^{3/2}} = \frac{1}{x} - \frac{1}{x+1}.$$

Hence,

$$\begin{aligned} |f(x)| &\leq \left|\frac{\cos(x^2)}{2x}\right| + \left|\frac{\cos[(x+1)^2]}{2(x+1)}\right| + \left|\frac{1}{2}\int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}}\, du\right| \\ &< \frac{1}{2x} + \frac{1}{2(x+1)} + \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+1}\right) = \frac{1}{x}. \end{aligned}$$

□

(b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where $|r(x)| < \frac{c}{x}$ and c is a constant.

Proof. By (a),

$$\begin{aligned} 2xf(x) &= \left(\frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \frac{1}{2}\int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}}\, du\right) \\ &= \cos(x^2) - \frac{x}{(x+1)} \cdot \cos[(x+1)^2] - x\int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}}\, du \\ &< \cos(x^2) - \cos[(x+1)^2] + \frac{\cos[(x+1)^2]}{x+1} + \frac{1}{x+1}, \end{aligned}$$

and thus

$$|r(x)| < \left|\frac{\cos[(x+1)^2] + 1}{x+1}\right| < \frac{2}{x}.$$

□

Problem 4 continued on next page. . .

(c) Does $\int_0^\infty \sin(t^2)\,dt$ converge?

Proof.

$$\begin{aligned}
\int_0^\infty \sin(t^2)\,dt &= \sum_{x=0}^\infty f(x) \\
&= f(0) + \sum_{x=1}^\infty \frac{\cos(x^2)}{2x} - \sum_{x=1}^\infty \frac{\cos[(x+1)^2]}{2x} + \sum_{x=1}^\infty \frac{r(x)}{2x} \\
&= f(0) + \sum_{x=1}^\infty \frac{\cos(x^2)}{2x} - \sum_{x=1}^\infty \frac{x+1}{x} \cdot \frac{\cos[(x+1)^2]}{2(x+1)} + \sum_{x=1}^\infty \frac{r(x)}{2x} \\
&= f(0) + \sum_{x=1}^\infty \frac{\cos(x^2)}{2x} - \sum_{x=2}^\infty \frac{x}{x-1} \cdot \frac{\cos(x^2)}{2x} + \sum_{x=1}^\infty \frac{r(x)}{2x} \\
&= f(0) + \frac{\cos 1}{2} + \sum_{x=2}^\infty \frac{\cos(x^2)}{2x(1-x)} + \sum_{x=1}^\infty \frac{r(x)}{2x}.
\end{aligned}$$

But then

$$\sum_{x=1}^\infty \frac{|r(x)|}{2x} < \sum_{x=1}^\infty \frac{1}{x^2},$$

$$\begin{aligned}
\sum_{x=2}^\infty \left|\frac{\cos(x^2)}{2x(1-x)}\right| &< \sum_{x=2}^\infty \left|\frac{1}{2x(1-x)}\right| \\
&= \frac{1}{2}\sum_{x=2}^\infty \frac{1}{x(x-1)} \\
&< \frac{1}{2}\sum_{x=1}^\infty \frac{1}{x^2},
\end{aligned}$$

and thus both series converge by comparison test. Since all terms of $f(0) + \frac{\cos 1}{2} + \sum_{x=2}^\infty \frac{\cos(x^2)}{2x(1-x)} + \sum_{x=1}^\infty \frac{r(x)}{2x}$ converge, $\int_0^\infty \sin(t^2)\,dt$ converges. $\square$

Problem 5

Suppose f is a real, continuously differentiable function on $[a,b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)\,dx = 1.$$

Prove that

$$\int_a^b xf(x)f'(x)\,dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2\,dx \cdot \int_a^b x^2 f^2(x)\,dx \geq \frac{1}{4}.$$

Proof. By Theorem 6.22,

$$\begin{aligned}\int_a^b xf(x)f'(x)\,dx &= bf(b) - af(a) - \int_a^b f(x)(f(x) + xf'(x)) \\ &= bf(b) - af(a) - \int_a^b f^2(x)\,dx - \int_a^b xf(x)f'(x)\,dx.\end{aligned}$$

But then

$$2\int_a^b xf(x)f'(x)\,dx = bf(b) - af(a) - 1 = -1,$$

and the result follows.

It now follows from Hölder's inequality that

$$\int_a^b [f'(x)]^2\,dx \cdot \int_a^b x^2 f^2(x)\,dx \geq \left(\int_a^b xf(x)f'(x)\,dx\right)^2 = \frac{1}{4}.$$

□

Problem 6

Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that

$$\int_a^b \alpha(x)g(x)\,dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,d\alpha.$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition on $[a, b]$. For each segment (x_{i-1}, x_i), the mean value theorem furnishes some $t \in (x_{i-1}, x_i)$ such that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Since α increases monotonically,

$$\begin{aligned}
\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i &= \sum_{i=1}^n \alpha(x_i)(G(x_i) - G(x_{i-1})) \\
&= G(b)\alpha(b) - G(a)\alpha(a) + \sum_{i=1}^n \alpha(x_{i-1})G(x_{i-1}) - \sum_{i=1}^n \alpha(x_i)G(x_{i-1}) \\
&= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i,
\end{aligned}$$

for any partition P. Pick $\epsilon > 0$. There exists P_1, P_2 such that

$$U(P_1, \alpha(x_i)g(t_i)) - L(P_1, \alpha(x_i)g(t_i)) < \epsilon/2,$$

$$U(P_2, G, \alpha) - L(P_2, G, \alpha) < \epsilon/2.$$

Let $P = P_1 \cup P_2$. By Theorem 6.7,

$$\begin{aligned}
&\left| G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,d\alpha - \int_a^b \alpha(x)g(x)\,dx \right| \\
&\leq \left| \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i - \int_a^b G\,d\alpha \right| + \left| \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i - \int_a^b \alpha(x)g(x)\,dx \right| < \epsilon.
\end{aligned}$$

$\square$

Problem 7

Let γ_1 be a curve in $\mathbb{R}^k$ defined on $[a, b]$, let ϕ be a continuous 1-1 mapping of $[c, d]$ into $[a, b]$ such that $\phi(c) = a$; and define $\gamma_2(s) = \gamma_1(\phi(s))$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and only if the same is true of γ_1. Prove that γ_2 and γ_1 have the same length.

Proof. Since ϕ is a continuous 1-1 mapping on a compact space, its inverse mapping ψ from $[a, b]$ into $[c, d]$ is also a continuous mapping, by Theorem 4.17. But then $\gamma_2(s) = \gamma_1(\phi(s))$ and $\gamma_2(\psi(t)) = \gamma_1(t)$, so γ_1 is 1-1 if and only if γ_2 is. Additionally, since ϕ is a continuous bijection with $\phi(a) = c$, we know $\phi(d) = b$, and thus $\gamma_2(c) = \gamma_1(\phi(c)) = \gamma_1(\phi(d)) = \gamma_2(d)$ if and only if $\gamma_1(a) = \gamma_1(b)$. Given a partition $P = \{x_0, \ldots, x_n\}$ on $[c, d]$, γ_1 yields a partition $P' = \{y_0, \ldots, y_n\}$ on $[a, b]$, with $[x_i, x_{i+1}]$ corresponding to $[y_i, y_{i+1}]$ for all i. But then,

$$\Lambda(P, \gamma_1) = \sum_{i=1}^{n} |\gamma_1(x_i) - \gamma_1(x_{i-1})| = \sum_{i=1}^{n} |\gamma_2(\psi(x_i)) - \gamma_2(\psi(x_{i-1}))| = \sum_{i=1}^{n} |\gamma_2(y_i) - \gamma_2(y_{i-1})| = \Lambda(P', \gamma_2).$$

Therefore, γ_1 is a rectifiable curve if and only if γ_2 is, and

$$\Lambda(\gamma_1) = \sup \Lambda(P, \gamma_1) = \sup \Lambda(P', \gamma_2) = \Lambda(\gamma_2),$$

and the result now follows. $\square$