

# MATH 140B: Homework #2

Due on Apr 19, 2024 at 23:59pm

*Professor Seward*

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## Problem 1

Suppose  $f$  is defined in a neighborhood of  $x$ , and suppose  $f''(x)$  exists. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by example that the limit may exist even if  $f''(x)$  does not.

*Proof.* Put  $g(h) = f(x+h) + f(x-h) - 2f(x)$ . Since  $g$  is differentiable in a neighborhood of  $x$  and  $g(h) \rightarrow 0$  as  $h \rightarrow 0$ , we may apply the L'Hospital's Rule and get

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{g(h)}{h^2} &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} \\ &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{2h} - \lim_{h \rightarrow 0} \frac{f'(x-h) - f'(x)}{2h} \\ &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{2h} - \lim_{k \rightarrow 0} \frac{f'(x+k) - f'(x)}{-2k} \\ &= \frac{f''(x)}{2} + \frac{f''(x)}{2} = f''(x). \end{aligned}$$

Consider  $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$ .  $f$  is not continuous at 0, so  $f''(0)$  does not exist. But then  $f(h) + f(-h) - 2f(0) = 0$  for all  $h > 0$ , so  $\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$  exists.  $\square$

## Problem 2

Suppose  $a \in \mathbb{R}^1$ ,  $f$  is a twice-differentiable real function on  $(a, \infty)$ , and  $M_0, M_1, M_2$  are the least upper bounds of  $|f(x)|, |f'(x)|, |f''(x)|$ , respectively, on  $(a, \infty)$ . Prove that

$$M_1^2 \leq 4M_0M_2. \quad (1)$$

Does  $M_1^2 \leq 4M_0M_2$  hold for vector-valued functions too?

*Proof.* Let  $x \in (a, \infty)$ . Put  $h > 0$ . By Taylor's Theorem, there exists  $t \in (x, x+2h)$  such that

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2f''(t),$$

that is,

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] + hf''(t).$$

But then

$$-\frac{M_0}{h} - hM_2 \leq f'(x) \leq \frac{M_0}{h} + hM_2.$$

It follows that

$$M_1^2 \leq \left(\frac{M_0}{h} + hM_2\right)^2 = \left(\frac{M_0^2}{h^2} + h^2M_2^2\right) + 2M_0M_2 \leq 4M_0M_2,$$

as  $\frac{M_0^2}{h^2} + h^2M_2^2 \geq 2M_0M_2$  by AM-GM.

To show that  $M_1^2 = 4M_0M_2$  can actually happen, take  $a = -1$ , define

$$f(x) = \begin{cases} 2x^2 - 1 & x \in (-1, 0) \\ \frac{x^2-1}{x^2+1} & x \in [0, \infty) \end{cases}.$$

we know

$$f'(x) = \begin{cases} 4x & x \in (-1, 0) \\ \frac{4x}{(x^2+1)^2} & x \in [0, \infty) \end{cases}, \quad f''(x) = \begin{cases} 4 & x \in (-1, 0) \\ \frac{4(-3x^2+1)}{(x^2+1)^3} & x \in [0, \infty) \end{cases}, \quad f'''(x) = \begin{cases} 0 & x \in (-1, 0) \\ \frac{48x(x^2-1)}{(x^2+1)^4} & x \in [0, \infty) \end{cases}$$

Since  $f' < 0$  when  $x < 0$  but  $f' > 0$  when  $x > 0$ ,  $f(x)$  monotonically decreases from 1 to  $-1$  then monotonically approaches 1, and thus  $M_0 = 1$ .

When  $x < 0$ , since  $f'' > 0$ ,  $f'$  monotonically increases from  $-4$  to 0. Notice that  $\frac{4(-3x^2+1)}{(x^2+1)^3} = 0$  has a single positive root at  $x = \frac{1}{\sqrt{3}}$ . Since  $f'(0) = 0$ ,  $f'(1/\sqrt{3}) = \frac{3\sqrt{3}}{4}$ , and  $\lim_{x \rightarrow \infty} f'(x) = 0$ ,  $|f'(x)| \leq \frac{3\sqrt{3}}{4} < 4$  for nonnegative  $x$ . Hence,  $M_1 = 4$ .

Notice that  $f'''(x) = 0$  has a single positive root at  $x = 1$ . But then  $f''(0) = 4$ ,  $f''(1) = -1$ ,  $\lim_{x \rightarrow \infty} f''(x) = 0$ , so  $M_2 = 4$ .

Therefore, the equality of (1) holds for this example.  $\square$

### Problem 3

Suppose  $f$  is a real function on  $(-\infty, \infty)$ . Call  $x$  a fixed point of  $f$  if  $f(x) = x$ .

- (a) If  $f$  is differentiable and  $f'(t) \neq 1$  for every real  $t$ , prove that  $f$  has at most one fixed point.

*Proof.* Suppose for contradiction that  $f$  has multiple fixed points, say  $x, y$ ,  $x < y$ . By MVT, there exists  $t \in (x, y)$  such that

$$f(y) - f(x) = x - y = (x - y)f'(t).$$

But then  $f'(t) = 1$ , contradiction.  $\square$

- (b) Show that the function  $f$  defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although  $0 < f'(t) < 1$  for all real  $t$ .

*Proof.* We can easily see that

$$f'(t) = 1 + \frac{-e^t}{(1 + e^t)^2}.$$

Since  $e^t, (1 + e^t)^2 > 0$  and  $e^t < (1 + e^t)^2$ , we have  $0 < \frac{e^t}{(1 + e^t)^2} < 1$ , and so  $0 < f'(t) < 1$ .

Suppose  $t$  is a fixed point of  $f$ , which implies  $t + (1 + e^t)^{-1} = t$ . But then  $(1 + e^t)^{-1} = 0$ , contradiction.  $\square$

- (c) However, if there is a constant  $A < 1$  such that  $|f'(t)| \leq A$  for all real  $t$ , prove that a fixed point of  $f$  exists, and that  $x = \lim_{n \rightarrow \infty} x_n$ , where  $x_1$  is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for  $n = 1, 2, 3, \dots$

*Proof.* Since  $x_{n+1} = f(x_n)$  and  $x_n = f(x_{n-1})$ , by MVT,

$$|f(x_n) - f(x_{n-1})| = |x_{n+1} - x_n| = |f'(t)(x_n - x_{n-1})| \leq |f'(t)| |x_n - x_{n-1}| \leq A |x_n - x_{n-1}|,$$

for some  $t$ , and thus  $|x_{n+1} - x_n| \leq A^{n-1} |x_2 - x_1|$ . But then for  $m, n \geq N$ ,

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \dots + |x_{n+1} - x_n| \\ &= (x_2 - x_1) \sum_{k=n-1}^{m-2} A_k \\ &\leq (x_2 - x_1) \sum_{k=N}^{\infty} A_k \leq \frac{|x_2 - x_1| A^N}{1 - A}. \end{aligned}$$

As  $A < 1$ ,  $|x_m - x_n| \rightarrow 0$  as  $N \rightarrow \infty$ . Therefore,  $(x_n)$  is a Cauchy sequence in the reals, which converges to some  $x$ . But then  $f(x) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x$ , so  $x$  is a fixed point.  $\square$

- (d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \rightarrow (x_2, x_2) \rightarrow (x_2, x_3) \rightarrow (x_3, x_3) \rightarrow (x_3, x_4) \rightarrow \dots$$

*Proof.*  $\square$

## Problem 4

Suppose  $\alpha$  increases on  $[a, b]$ ,  $a \leq x_0 \leq b$ ,  $\alpha$  is continuous at  $x_0$ ,  $f(x_0) = 1$ , and  $f(x) = 0$  if  $x \neq x_0$ . Prove that  $f \in \mathcal{R}(\alpha)$  and that  $\int f d\alpha = 0$ .

*Proof.* Pick arbitrary  $\epsilon > 0$ . We first note that the infimum of  $f(x)$  over any interval in  $[a, b]$  is 0, so  $L(P, f, \alpha) = 0$ . Since  $\alpha$  is continuous at  $x_0$ , there exists  $\delta > 0$  such that  $|\alpha(x) - \alpha(x_0)| < \epsilon/2$  whenever  $|x - x_0| < \delta$ . Consider the partition  $P = \{a, x_0 - \delta', x_0 + \delta', b\}$ , where  $0 < \delta' < \min\{\delta, x_0 - a, b - x_0\}$ . We then have

$$\begin{aligned} U(P, f, \alpha) &= \alpha(x_0 + \delta') - \alpha(x_0 - \delta') \\ &= (\alpha(x_0 + \delta') - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta')) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence,  $U(P, f, \alpha) - L(P, f, \alpha) = \epsilon$ , and so  $f \in \mathcal{R}(\alpha)$  by Theorem 6.6. Since  $L(P, f, \alpha) \leq \int f d\alpha \leq U(P, f, \alpha)$ , we have  $\int f d\alpha = 0$ .  $\square$

## Problem 5

Suppose  $f \geq 0$ ,  $f$  is continuous on  $[a, b]$ , and  $\int_a^b f(x) dx = 0$ . Prove that  $f(x) = 0$  for all  $x \in [a, b]$ .

*Proof.* Suppose for the sake of contradiction that there exists some  $x_0 \in [a, b]$  with  $f(x_0) = \epsilon$ , for some  $\epsilon > 0$ . Since  $f$  is continuous, there exists  $\delta > 0$  such that  $|f(x) - f(x_0)| < \epsilon$  for all  $|x - x_0| < \delta$ . Consider the partition  $P = \{a, x_0 - \delta', x_0 + \delta', b\}$ , where  $0 < \delta' < \min\{\delta, x_0 - a, b - x_0\}$ . We know  $m = \inf f(x) > 0$ , for  $x \in (x_0 - \delta', x_0 + \delta')$ . But then  $L(P, f) \geq 2\delta'm > 0$ , which forces  $\int_a^b f(x) dx > 0$ , contradiction.  $\square$

## Problem 6

Define three functions  $\beta_1, \beta_2, \beta_3$  as follows:  $\beta_j(x) = 0$  if  $x < 0$ ,  $\beta_j(x) = 1$  if  $x > 0$  for  $j = 1, 2, 3$ ; and  $\beta_1(0) = 0$ ,  $\beta_2(0) = 1$ ,  $\beta_3(0) = 1/2$ . Let  $f$  be a bounded function on  $[-1, 1]$ .

- (a) Prove that  $f \in \mathcal{R}(\beta_1)$  if and only if  $f(0+)$  equals  $f(0)$  and that then

$$\int f d\beta_1 = f(0).$$

*Proof.* Suppose  $f \in \mathcal{R}(\beta_1)$ . Pick  $\epsilon > 0$ . There exists partition  $P$  such that

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon.$$

Let  $P^*$  be a refinement which contains 0. Let  $\delta \in P^*$  such that  $[0, \delta]$  is an interval given by the partition  $P$ . Then,  $U(P^*, f, \beta_1) - L(P^*, f, \beta_1) = \sup f(x) - \inf f(x) < \epsilon$ ,  $x \in [0, \delta]$ . But then,  $|f(t) - f(0)| < \epsilon$  whenever  $t \in (0, \delta)$ . Hence,  $f(0+) = f(0)$ .

We now suppose  $f(0+) = f(0)$ . Pick  $\epsilon > 0$ . There exists  $\delta > 0$  such that  $|f(t) - f(0)| < \epsilon/2$  whenever  $t \in (0, \delta)$ . Let  $\delta' < \min(1, \delta)$  be positive. Consider the partition  $P = \{-1, 0, \delta', 1\}$ . Then,

$$U(P, f, \beta_1) - L(P, f, \beta_1) = f(s) - f(t) \leq |f(s) - f(0)| + |f(t) - f(0)| < \epsilon,$$

for some  $s, t \in [0, \delta']$ . Hence, by Theorem 6.6,  $f \in \mathcal{R}(\beta_1)$ . Note that for any  $P$  which contains 0, we have  $U(P, f, \beta_1) = M$  and  $L(P, f, \beta_1) = m$ , where  $M = \sup_{x \in (0, \delta')} f(x)$  and  $m = \inf_{x \in (0, \delta')} f(x)$ . Since  $f(0+) = f(0)$ ,  $f(x) \rightarrow f(0)$  as  $\delta \rightarrow 0$ , and so  $\int f d\beta_1 = f(0)$ .  $\square$

- (b) State and prove a similar result for  $\beta_2$ .

*Proof.* We show that  $f \in \mathcal{R}(\beta_2)$  if and only if  $f(0-)$  equals  $f(0)$  and that then  $\int f d\beta_2 = f(0)$ .

Suppose  $f \in \mathcal{R}(\beta_2)$ . Pick  $\epsilon > 0$ . There exists partition  $P$  such that

$$U(P, f, \beta_2) - L(P, f, \beta_2) < \epsilon.$$

Let  $P^*$  be a refinement which contains 0. Let  $-\delta \in P^*$  such that  $[-\delta, 0]$  is an interval given by the partition  $P$ . Then,  $U(P^*, f, \beta_2) - L(P^*, f, \beta_2) = \sup f(x) - \inf f(x) < \epsilon$ ,  $x \in [-\delta, 0]$ . But then,  $|f(t) - f(0)| < \epsilon$  whenever  $t \in (-\delta, 0)$ . Hence,  $f(0-) = f(0)$ .

We now suppose  $f(0-) = f(0)$ . Pick  $\epsilon > 0$ . There exists  $\delta > 0$  such that  $|f(t) - f(0)| < \epsilon/2$  whenever  $t \in (-\delta, 0)$ . Let  $\delta' < \min(1, \delta)$  be positive. Consider the partition  $P = \{-1, -\delta', 0, 1\}$ . Then,

$$U(P, f, \beta_2) - L(P, f, \beta_2) = f(s) - f(t) \leq |f(s) - f(0)| + |f(t) - f(0)| < \epsilon,$$

for some  $s, t \in [-\delta', 0]$ . Hence, by Theorem 6.6,  $f \in \mathcal{R}(\beta_2)$ . Note that for any  $P$  which contains 0, we have  $U(P, f, \beta_2) = M$  and  $L(P, f, \beta_2) = m$ , where  $M = \sup_{x \in (-\delta', 0)} f(x)$  and  $m = \inf_{x \in (-\delta', 0)} f(x)$ . Since  $f(0-) = f(0)$ ,  $f(x) \rightarrow f(0)$  as  $\delta' \rightarrow 0$ , and so  $\int f d\beta_2 = f(0)$ .  $\square$

- (c) Prove that  $f \in \mathcal{R}(\beta_3)$  if and only if  $f$  is continuous at 0.

*Proof.* Suppose  $f \in \mathcal{R}(\beta_3)$ . Pick  $\epsilon > 0$ . There exists partition  $P$  such that

$$U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon.$$

Let  $P^*$  be a refinement which contains 0. Let  $[x_i, 0]$ ,  $[0, x_{i+1}]$  be the intervals given by  $P^*$  which contains 0. Then,

$$U(P^*, f, \beta_3) - L(P^*, f, \beta_3) = \frac{1}{2} \left( \sup_{x \in [x_i, 0]} f(x) - \inf_{x \in [x_i, 0]} f(x) + \sup_{x \in [0, x_{i+1}]} f(x) - \inf_{x \in [0, x_{i+1}]} f(x) \right) < \epsilon/2$$

But then,  $|f(t) - f(0)| < \epsilon$  whenever  $t \in (-\delta, \delta)$ , where  $\delta = \min(|x_i|, |x_{i+1}|)$ . Hence,  $f$  is continuous at 0.

We now suppose  $f$  is continuous at 0. Pick  $\epsilon > 0$ . There exists  $\delta > 0$  such that  $|f(t) - f(0)| < \epsilon/2$  whenever  $t \in (-\delta, \delta)$ . Let  $\delta' < \min(1, \delta)$  be positive. Consider the partition  $P = \{-1, -\delta', \delta', 1\}$ . Then,

$$U(P, f, \beta_3) - L(P, f, \beta_3) = f(s) - f(t) \leq |f(s) - f(0)| + |f(t) - f(0)| < \epsilon,$$

for some  $s, t \in [-\delta', \delta']$ . Hence, by Theorem 6.6,  $f \in \mathcal{R}(\beta_2)$ . □

(d) If  $f$  is continuous at 0 prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

*Proof.* We have already shown  $\int f d\beta_1 = \int f d\beta_2 = f(0)$ , from (a), (b). It remains show  $\int f d\beta_3 = f(0)$ .

Pick  $\epsilon > 0$ . There exists  $\delta > 0$  such that  $|f(t) - f(0)| < \epsilon/2$  whenever  $|t| < \delta$ . But then for any  $P$  which contains  $-\delta, 0, \delta$ , we have  $U(P, f, \beta_3) < f(0) + \epsilon$  and  $L(P, f, \beta_3) > f(0) - \epsilon$ . Hence,

$$f(0) - \epsilon < L(P, f, \beta_3) \leq \int f d\beta_3 \leq U(P, f, \beta_3) < f(0) + \epsilon,$$

for arbitrary  $\epsilon$ , and the result follows. □



## Problem 7

If  $f(x) = 0$  for all irrational  $x$ ,  $f(x) = 1$  for all rational  $x$ , prove that  $f \notin \mathcal{R}$  on  $[a, b]$  for any  $a < b$ .

*Proof.* Take any partition  $P = \{x_0 = a, \dots, x_n = b\}$ . Notice that there exists an irrational in any interval, so  $L(P, f) = 0$ . But then  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , so for any distinct  $x_i, x_{i+1}$ , there exists  $q \in \mathbb{Q}$  such that  $x_i < q < x_{i+1}$ . Hence,  $U(P, f) = \sum_{i=1}^n (x_i - x_{i-1}) = b - a > 0$ . Hence,  $\inf U(P, f) = b - a \neq 0 = \sup L(P, f)$ , and the result now follows.  $\square$