

# MATH 220B: Homework #2

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*Professor Xiao*

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**Problem 1**

Suppose  $f$  is analytic on  $\overline{B}(0; 1)$  and satisfies  $|f(z)| < 1$  for  $|z| = 1$ . Find the number of solutions (counting multiplicities) of the equation  $f(z) = z^n$  where  $n$  is an integer larger than or equal to 1.

*Proof.* Let  $g(z) = z^n$ ,  $h(z) = f(z) - g(z)$ . Since

$$|h(z) + g(z)| = |f(z)| < 1 = |g(z)|$$

for  $|z| = 1$ , by Rouché's theorem,  $h(z)$  has the same number of zeros as  $g(z)$  in  $B(0; 1)$ , that is,  $n$  zeros.  $\square$

## Problem 2

Prove the following Minimum Principle. If  $f$  is a non-constant analytic function on a bounded open set  $G$  and is continuous on  $\overline{G}$ , then either  $f$  has a zero in  $G$  or  $|f|$  assumes its minimum value on  $\partial G$ . (See Exercise IV. 3.6.)

*Proof.* If there exists  $a \in G$  such that  $|f(a)| \leq |f(z)|$  for all  $z \in G$ , then  $f(a) = 0$  by Exercise IV.3.6. Otherwise,  $|f|$  assumes its minimum value on  $\partial G$  as it is continuous on  $\overline{G}$ .  $\square$

### Problem 3

Let  $G$  be a bounded region and suppose  $f$  is continuous on  $\overline{G}$  and analytic on  $G$ . Show that if there is a constant  $c \geq 0$  such that  $|f(z)| = c$  for all  $z$  on the boundary of  $G$  then either  $f$  is a constant function or  $f$  has a zero in  $G$ .

*Proof.* Suppose  $f$  is not constant. By the Maximum Modulus Principle,  $|f(z)| \leq c$  for all  $z \in G$  otherwise  $|f|$  would assume its maximum value in  $G$ . But then by the Minimum Principle we just proved,  $f$  has a zero in  $G$ .  $\square$

## Problem 4

- (a) Let  $f$  be entire and non-constant. For any positive real number  $c$  show that the closure of  $\{z : |f(z)| < c\}$  is the set  $\{z : |f(z)| \leq c\}$ .

*Proof.* Since  $f$  is continuous, it suffices to show that any  $z$  with  $|f(z)| = c$  are in the closure of  $\{z : |f(z)| < c\}$ . Suppose there exists  $z_0$  such that  $|f(z_0)| = c$  and  $z_0$  is not in the closure of  $\{z : |f(z)| < c\}$ . Then there exists  $r > 0$  such that  $B_r(z_0) \cap \{z : |f(z)| < c\} = \emptyset$ . That is,  $|f(z)| \geq c$  for all  $z \in B_r(z_0)$ . But then  $f(B_r(z_0))$  is open by the Open Mapping Theorem, so  $f(B_r(z_0))$  contains an open neighborhood  $U$  of  $f(z_0)$ . This implies  $|f(z_0)| < c$  for some  $z \in B_r(z_0)$ , contradiction.  $\square$

- (b) Let  $p$  be a polynomial and show that each component of  $\{z : |p(z)| < c\}$  contains a zero of  $p$ .

*Proof.* We may assume  $p$  is not constant. Note that each component of  $\{z : |p(z)| < c\}$  is bounded, otherwise  $p$  is constant by the Liouville's Theorem. By (a), the closure of  $\{z : |p(z)| < c\}$  is  $\{z : |p(z)| \leq c\}$ . Since each component  $G$  of  $\{z : |p(z)| \leq c\}$  is bounded and  $|p(z)| = c$  for all  $z \in \partial G$ ,  $p$  has a zero in  $G$  by the previous problem.  $\square$

## Problem 5

Suppose that both  $f$  and  $g$  are analytic on  $\overline{B}(0; R)$  with  $|f(z)| = |g(z)|$  for  $|z| = R$ . Show that if neither  $f$  nor  $g$  vanishes in  $B(0; R)$  then there is a constant  $\lambda$ ,  $|\lambda| = 1$ , such that  $f = \lambda g$ .

*Proof.* We first show that the multiplicities of the zeros of  $f$  and  $g$  on the boundary are the same. Suppose  $f$  has a zero of order  $n$  at  $z_0$  and  $g$  has a zero of order  $m$  at  $z_0$  with  $|z_0| = R$  and  $n \geq m$ . Then  $f(z) = (z - z_0)^n F(z)$  and  $g(z) = (z - z_0)^m G(z)$  for some analytic functions  $F$  and  $G$  with  $F(z_0), G(z_0) \neq 0$ . Since  $|f(z)| = |g(z)|$  for  $|z| = R$ , we have  $|z - z_0|^{n-m} = \left| \frac{G(z)}{F(z)} \right|$ . But then  $n = m$  and  $F(z_0) = G(z_0)$ , otherwise  $G(z_0) = 0$ . Hence, we may define  $h(z) = \frac{g(z)}{f(z)}$  on  $\overline{B}_R(0)$ . Since  $|h(z)| = 1$  for  $|z| = R$  and  $h$  has not zeros in  $B_R(0)$ ,  $|h(z)| = 1$  for all  $z \in \overline{B}_R(0)$  by Exercise VI.1.2. The result now follows.  $\square$

## Problem 6

Let  $f$  be analytic in the disk  $B(0; R)$  and for  $0 \leq r < R$  define  $A(r) = \max\{\operatorname{Re} f(z) : |z| = r\}$ . Show that unless  $f$  is a constant,  $A(r)$  is a strictly increasing function of  $r$ .

*Proof.* Assume that  $f$  is not a constant. Let  $0 \leq r_1 < r_2 < R$ . Consider  $g(z) = e^{f(z)}$  over  $\overline{B}_{r_2}(0)$ . Note that  $|g(z)| = e^{\operatorname{Re} f(z)}$  attains the maximum at the same point as  $\operatorname{Re} f(z)$ . Suppose  $A(r_1) \geq A(r_2)$ . Then  $|g(z)|$  attains a maximum in  $B_{r_2}(0)$ , which makes  $g(z)$  constant by the Maximum Modulus Principle, contradiction.  $\square$

## Problem 7

Does there exist an analytic function  $f : D \rightarrow D$  with  $f(\frac{1}{2}) = \frac{3}{4}$  and  $f'(\frac{1}{2}) = \frac{2}{3}$ ?

*Proof.* By the Schwarz-Pick Lemma,

$$|f'(\frac{1}{2})| \leq \frac{1 - |f(\frac{1}{2})|^2}{1 - |\frac{1}{2}|^2} = \frac{1 - \frac{9}{16}}{1 - \frac{1}{4}} = \frac{7}{12} < \frac{2}{3},$$

and thus such analytic function does not exist. □



## Problem 8

Suppose  $f : D \rightarrow \mathbb{C}$  satisfies  $\operatorname{Re} f(z) \geq 0$  for all  $z$  in  $D$  and suppose that  $f$  is analytic.

- (a) Show that  $\operatorname{Re} f(z) > 0$  for all  $z$  in  $D$ .

*Proof.* Suppose  $z \in D$  such that  $\operatorname{Re} f(z) = 0$ . By the Open Mapping Theorem,  $f(D)$  is open, so there exists  $r > 0$  such that  $B_r(f(z)) \subset f(D)$ . But then  $B_r(f(z))$  contains points with negative real part, contradiction.  $\square$

- (b) By using an appropriate Möbius transformation, apply Schwarz's Lemma to prove that if  $f(0) = 1$  then

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}$$

for  $|z| < 1$ . What can be said if  $f(0) \neq 1$ ?

*Proof.* Let  $\phi(z) = \frac{z-1}{z+1}$  and consider  $g(z) = \phi \circ f(z)$ . Note that  $g(0) = \phi(f(0)) = \phi(1) = 0$ . Since  $\phi$  maps  $\{z : \operatorname{Re}(z) > 0\}$  to  $D$ ,  $g$  maps  $D$  to  $D$ . By Schwarz's Lemma,  $|g(z)| \leq |z|$  for  $z \in D$ . That is,

$$|z| \geq \frac{|f(z) - 1|}{|f(z) + 1|} \geq \frac{|f(z)| - 1}{|f(z)| + 1}.$$

The result now follows from rearranging the inequality. If  $f(0) = \alpha$  for some  $\alpha \neq 1$ , apply the transformation  $\phi(z) = \frac{z-\alpha}{z+\alpha}$  instead.  $\square$

- (c) Show that  $f$  also satisfies

$$|f(z)| \geq \frac{1 - |z|}{1 + |z|}.$$

*Proof.* Note that  $\operatorname{Re} \frac{1}{f(z)} > 0$  for all  $z \in D$ . Hence, consider  $h(z) = \phi \circ (1/f)(z)$ .  $h(0) = 0$  and  $h$  maps  $D$  to  $D$ . By Schwarz's Lemma,  $|h(z)| \leq |z|$  for  $z \in D$ . That is,

$$|z| \geq \frac{|1/f(z) - 1|}{|1/f(z) + 1|} \geq \frac{1 - |f(z)|}{1 + |f(z)|}.$$

The result now follows.  $\square$

## Problem 9

Suppose  $f$  is analytic in some region containing  $\overline{B}(0; 1)$  and  $|f(z)| = 1$  where  $|z| = 1$ . Find a formula for  $f$ . (Hint: First consider the case where  $f$  has no zeros in  $\overline{B}(0; 1)$ .)

*Proof.* Suppose  $f$  has no zeros in  $\overline{B}_1(0)$ . Then by the exercise in the start of this assignment,  $f = c$  with  $|c| = 1$ . Suppose  $f(z_0) = 0$  for some  $z_0 \in \overline{B}_1(0)$ . □

## Problem 10

Is there an analytic function  $f$  on  $B(0; 1)$  such that  $|f(z)| < 1$  for  $|z| < 1$ ,  $f(0) = \frac{1}{2}$ , and  $f'(0) = \frac{3}{4}$ . If so, find such an  $f$ . Is it unique?

*Proof.* Let  $\phi_{\frac{1}{2}}(z) = \frac{z-1/2}{1-z/2}$  be defined as in the textbook, and let  $g = \phi_{\frac{1}{2}} \circ f$ . Since  $g$  maps  $B(0; 1)$  to  $B(0; 1)$ ,  $g(0) = 0$ , and  $|g'(0)| = |\phi'_{\frac{1}{2}} \circ f(0) \cdot f'(0)| = 1$ , by Schwarz's Lemma,  $g(z) = cz$  for some  $|c| = 1$ . Hence,  $f(z) = \frac{\frac{1}{2} + cz}{1 + \frac{c}{2}z}$ . □