MATH 100A: Homework #8

Due on November 30, 2023 at 12:00pm

Professor McKernan

Section A02 5:00PM - 5:50PM
Section Leader: Castellano

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

Find the parity of each of permutation.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 5 & 1 & 3 & 7 & 8 & 9 & 6 \end{pmatrix}$.

Proof. Since

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 5 & 1 & 3 & 7 & 8 & 9 & 6 \end{pmatrix} &= (1,2,4)(3,5)(6,7,8,9) \\ &= (1,2)(1,4)(3,5)(6,7)(6,8)(6,9), \end{aligned}$$

the parity is even. ☐

(b) $(1,2,3,4,5,6)(7,8,9)$.

Proof. Since $(1,2,3,4,5,6)(7,8,9) = (1,2)(1,3)(1,4)(1,5)(1,6)(7,8)(7,9)$, the parity is odd. ☐

(c) $(1,2,3,4,5,6)(1,2,3,4,5,7)$.

Proof. Since

$$(1,2,3,4,5,6)(1,2,3,4,5,7) = (1,2)(1,3)(1,4)(1,5)(1,6)(1,2)(1,3)(1,4)(1,5)(1,7),$$

the parity is even. ☐

(d) $(1,2)(1,2,3)(4,5)(5,6,8)(1,7,9)$.

Proof. Since

$$(1,2)(1,2,3)(4,5)(5,6,8)(1,7,9) = (2,3)(4,5)(5,6)(5,8)(1,7)(1,9),$$

the parity is even. ☐

Problem 2

If σ is a k-cycle, show that σ is an odd permutation if k is even, and is an even permutation if k is odd.

Proof. Since every k-cycle is a product of $k-1$ transposes, the above statement holds. $\square$

Problem 3

Prove that σ and $\tau^{-1}\sigma\tau$, for any σ, $\tau \in S_n$, are of the same parity.

Proof. Since $\tau^{-1}\sigma\tau$ is the conjugate of σ, they are of the same cycle type, and thus they are of the same parity. $\square$

Problem 4

Suppose that you are told that the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & & & 7 & 8 & 9 & 6 \end{pmatrix},$$

in S_9, where the images of 5 and 4 have been lost, is an even permutation. What must the images of 5 and 4 be?

Proof. Notice that the permutation contains $(1, 3, 2)$ and $(6, 7, 8, 9)$, and all the numbers that are not classified to a cycle are 4 and 5. Since the permutation is even, 4 and 5 must form a transposition, otherwise the permutation can be decomposed into $2 + 3 = 5$ transpositions, which forces it to be odd. $\square$

Problem 5

If $n \geq 3$, show that every element in A_n is a product of 3-cycles.

Proof. Note that every element $\sigma \in A_n$ can be decomposed into even number of transpositions. Suppose that σ is the identity. Since the identity permutation can be represented as a $\prod_{1 \leq x < y < z \leq n} (x, y, z)^3$, we may assume σ is not the identity. Since $(a, b)(c, d) = (a, b)(a, c)(c, a)(c, d) = (a, b, c)(c, a, d)$ for any pair of distinct transpositions $(a, b)(c, d)$, we can pair up consecutive transpositions in σ and convert each of them into a product of 3-cycles, which makes σ also a product of 3-cycles. $\square$

Problem 6

Show that every element in A_n is a product of n-cycles.

Proof. Let $\sigma \in A_n$. Since the identity is simply the n-th power of any n-cycle, we may assume that σ is not the identity. Let $(a_1, a_2)(b_1, b_2)$ be a pair of consecutive transpositions in σ. Note that (a_1, a_2) and (b_1, b_2) are distinct, otherwise they may cancell each other. Thus, we may assume $a_1 \neq b_1$. Let τ be a $(n-2)$-cycle $\underbrace{(a_1, \ldots, b_1)}_{n-2 \text{ elements}}$ that only excludes a_2 and b_2. Then,

$$\begin{aligned}
\sigma &= (a_1, a_2)(b_1, b_2) \\
&= (a_1, a_2)\tau\tau^{-1}(b_1, b_2) \\
&= (a_1, a_2)(a_1, \ldots, b_1)(b_1, \ldots, a_1)(b_1, b_2) \\
&= \underbrace{(a_1, a_2, \ldots, b_1)}_{\text{only excludes } b_2}\underbrace{(b_1, \ldots, a_1, b_2)}_{\text{only excludes } a_2} \\
&= (a_2, \ldots, b_1, a_1)(b_2, b_1, \ldots, a_1) \\
&= (a_2, \ldots, b_1, a_1)(a_2, b_2)(b_2, a_2)(b_2, b_1, \ldots, a_1) \\
&= \underbrace{(a_2, \ldots, b_1, a_1, b_2)}_{n\text{-cycle}}\underbrace{(b_2, a_2, b_1, \ldots, a_1)}_{n\text{-cycle}}.
\end{aligned}$$

Since σ is even, we can pair up consecutive transpositions in σ and convert each of them into a product of n-cycles, which makes σ also a product of n-cycles. $\square$

Problem 7

Find a normal subgroup in A_4 of order 4.

Proof. A_4 only contains even permutations, namely the identity, 3-cycles, and the product of 2 disjoint transpositions. 3-cycles cannot be in a subgroup of order 4, so the subgroup can only contain the identity and the product of disjoint transpositions. There are $\frac{1}{2}\binom{4}{2} = 3$ cycles in A_4, so the group we are looking for can only be $S = \{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. Since

$$\begin{aligned}
(1,2)(3,4)(1,3)(2,4) &= (1,4)(2,3) \\
(1,2)(3,4)(1,4)(2,3) &= (1,3)(2,4) \\
(1,3)(2,4)(1,4)(2,3) &= (1,2)(3,4),
\end{aligned}$$

S is a subset of a finite group and is closed under multiplication, so S is a subgroup. Since S contains the identity and all products of disjoint transpositions, S is a union of conjugacy class, which makes S a normal subgroup in A_4. □