MATH 220B: Homework #5

Due on Mar 14, 2025 at 23:59pm

Professor Xiao

Ray Tsai

A16848188

Problem 1

Let f and g be analytic functions on a region G and show that there are analytic functions f_1, g_1, and h on G such that

$$f(z) = h(z)f_1(z) \quad \text{and} \quad g(z) = h(z)g_1(z)$$

for all z in G; and f_1 and g_1 have no common zeros.

Proof. Let Z_f, Z_g be the sets of zeros of f, g respectively counted with multiplicity. Theorem 5.15 yields a analytic function $h(z)$ on G such that h admits zeros on $Z_f \cap Z_g$. Let f_1, g_1 such that

$$f(z) = h(z)f_1(z) \quad \text{and} \quad g(z) = h(z)g_1(z).$$

We know h, f_1 and g_1 are analytic on G. Also, since $h(z)$ contains all the common zeros of $f(z)$ and $g(z)$, f_1 and g_1 have no common zeros. $\square$

Problem 2

(a) Let $0 < |a| < 1$ and $|z| \le r < 1$; show that

$$\left|\frac{a + |a|z}{(1 - \bar{a}z)a}\right| \le \frac{1+r}{1-r}.$$

Proof.

$$\left|\frac{a + |a|z}{(1 - \bar{a}z)a}\right| = \left|\frac{1 + \frac{|a|}{a}z}{1 - \bar{a}z}\right|.$$

By the triangle inequality,

$$\left|1 + \frac{|a|}{a}z\right| \le 1 + |z| \le 1 + r,$$

and

$$|1 - \bar{a}z| \ge 1 - |\bar{a}||z| \ge 1 - |a|r \ge 1 - r.$$

The result now follows. □

(b) Let $\{a_n\}$ be a sequence of complex numbers with $0 < |a_n| < 1$ and

$$\sum(1 - |a_n|) < \infty.$$

Show that the infinite product

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \bar{a_n}z}\right)$$

converges in $H(B(0;1))$ and that $|B(z)| \le 1$. What are the zeros of B? ($B(z)$ is called a *Blaschke Product*.)

Proof. Let K be a compact set. K is contained in $\overline{B}_r(0)$ for some $r < 1$. Let $B_n(z) = \frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \bar{a_n}z}\right)$. By (a),

$$|B_n(z) - 1| = (1 - |a_n|)\left|\frac{a_n + |a_n|z}{(1 - \overline{a_n}z)a_n}\right| \le \frac{1+r}{1-r}(1 - |a_n|)$$

for $z \in K$, and thus $\sum|B_n(z) - 1| \le \frac{1+r}{1-r}\sum(1 - |a_n|) < \infty$. But then $B(z) = \prod B_n(z)$ converges uniformly and absolutely on K. Also note that $B_n(z)$ is an automorphism on the unit disk with a pole at $\frac{1}{\bar{a_n}} \notin \overline{B}_1(0)$ and a zero at a_n. Hence, $B(z) \le \prod 1 = 1$ and $B(z)$ has zeros at a_n. □

(c) Find a sequence $\{a_n\}$ in $B(0;1)$ such that

$$\sum(1 - |a_n|) < \infty$$

and every number $e^{i\theta}$ is a limit point of $\{a_n\}$.

Proof. Consider $a_n = e^{i\pi n/\sqrt{2}}(1 - 2^{-n})$. Then

$$\sum(1 - |a_n|) \le \sum|e^{i\pi n/\sqrt{2}} - a_n| \le \sum 2^{-n} < \infty.$$

Since $\sqrt{2}$ is irrational, the set $\{e^{i\pi n/\sqrt{2}} \mid n \in \mathbb{N}\}$ is dense in the unit circle. That is, for each $e^{i\theta}$, there exists a sequence $\{n_k\}$ such that $e^{i\pi n_k/\sqrt{2}} \to e^{i\theta}$. Hence, $a_{n_k} = e^{i\pi n_k/\sqrt{2}}(1 - 2^{-n_k}) \to e^{i\theta}$. □

Problem 3

Let

$$f = \frac{1}{(z-1)(z-5)}.$$

(a) Prove that there is a sequence of rational functions $R_n(z)$ whose poles can only occur at 2 and 6 such that

$$\lim_{n\to\infty} \sup_{3\leq|z|\leq 4} |f(z) - R_n(z)| = 0. \tag{1}$$

Proof. Pick $\epsilon > 0$. Let $K = \overline{\text{ann}(0;3,4)}$, and let $E = \{2, 6, \infty\}$. Since K is compact and E contains a pole from each component of $\mathbb{C}_\infty \backslash K$, Runge's theorem yields a rational function $R_n(z)$ whose poles can only occur in E and

$$|f(z) - R_n(z)| < \epsilon,$$

for $z \in K$. The result now follows. □

(b) Does there exist a sequence of rational functions $R_n(z)$ whose poles can only occur at 6 such that (1) holds? Justify your answer.

Proof. No. Suppose for the sake of contradiction that there exists such a sequence $\{R_n\}$. Since R_n is analytic on $B_2(0)$, $\int_{|z|=2} R_n(z)\, dz = 0$, and so $\int_{|z|=2} R_n(z)\, dz \to 0$. But then $\int_{|z|=2} f(z)\, dz = -\frac{\pi i}{2}$, contradiction. □

Problem 4

Let

$$G = \{z \in \mathbb{C} : |z| < 1 \text{ and } |z - \frac{1}{3}| > \frac{2}{3}\};$$

and let K be the closure of G:

$$K = \{z \in \mathbb{C} : |z| \leq 1 \text{ and } |z - \frac{1}{3}| \geq \frac{2}{3}\}.$$

Let $A(K)$ be the space of continuous functions on K that are analytic on G equipped with the uniform norm on K. For the purposes of this problem, a Laurent polynomial is a function of the form

$$\sum_{n=-N}^{N} a_n z^n.$$

Determine whether the following are true or false. Justify your answer.

(a) The set of polynomials is dense in $H(G)$.

Proof. True. Since $\mathbb{C}_\infty \backslash G$ is connected, for $f \in H(G)$ there exists a sequence of polynomials $\{p_n\}$ on G such that $p_n \to f$ uniformly, by Corollary 1.15. That is, the set of polynomials is dense in $H(G)$. □

(b) The set of polynomials is dense in $A(K)$.

Proof. False. Consider $f(z) = 1/z$ on K. We know $\int_{|z|=1} f = 2\pi i$. But then $\int_{|z|=1} p_n = 0$ for any polynomial p_n, and there does not exist $\{p_n\}$ that converges to f uniformly on K. Hence, the set of polynomials is not dense in $A(K)$. □

(c) If f is analytic on a neighborhood of K, then f can be uniformly approximated on K by Laurent polynomials.

Proof. True. Let $E = \{0, \infty\}$. Since E meets each component of $\mathbb{C}_\infty \backslash K$, Runge's Theorem furnishes a sequences of rational function $\{R_n\}$ which only have poles in E that converges uniformly to f on K. But then R_n are Laurent polynomials. □