MATH 140B: Homework #9

Due on Jun 7, 2024 at 23:59pm

Professor Seward

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Problem 1

Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x + 2\pi) = f(x)$ for all x.

(a) Compute the Fourier coefficients of f.

Proof. Let c_n denote the nth fourier coefficient of f. We first note that

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{\delta}{\pi}.$$

For $n \neq 0$,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx}\, dx = \frac{1}{2in\pi}(e^{in\delta} - e^{-in\delta}) = \frac{\sin(n\delta)}{n\pi}.$$

□

(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

Proof. Since $f(t) = 1$ for all $t \in (-\delta, \delta)$, it follows from Theorem 8.14 that

$$\sum_{-\infty}^{\infty} c_n = f(0) = 1.$$

Since $\frac{\sin(-n\delta)}{-n\pi} = \frac{\sin(n\delta)}{n\pi}$,

$$\pi = \delta + \sum_{n \neq 0} \frac{\sin(n\delta)}{n} = \delta + 2 \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n},$$

and the result now follows from rearranging the equation. □

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

Proof. Note that $\frac{\sin^2(n\delta)}{(n\pi)^2}$ is an even function with respect to n. By Parseval's theorem

$$\frac{\delta^2}{\pi^2} + 2 \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{(n\pi)^2} = \sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{\delta}{\pi}.$$

The result now follows from rearranging the equation. □

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(d) Let $\delta \to 0$ and prove that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

Proof. We first show that the improper integral exists. Pick $\epsilon > 0$. By L'Hopital's rule,

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x \to 0} \cos x = 1,$$

and thus there exists $\nu > 0$ such that $\left|(\frac{\sin x}{x})^2 - 1\right| < \epsilon$ whenever $|x| < \nu$. Hence,

$$\nu(1-\epsilon) \leq \int_0^\nu \left(\frac{\sin x}{x}\right)^2 dx \leq \nu(1+\epsilon),$$

and so the the improper integral $\int_0^1 \left(\frac{\sin x}{x}\right)^2 dx$ exists. On the other hand,

$$\left|\int_1^n \left(\frac{\sin x}{x}\right)^2 dx\right| \leq \int_1^n \frac{1}{x^2}\, dx = 1 - \frac{1}{n},$$

and thus $\int_1^n \left(\frac{\sin x}{x}\right)^2 dx \to 1$ as $n \to \infty$.

Pick $\epsilon > 0$. Since the improper integral exists, there exists $A > 0, B > \max(A, 3/\epsilon)$, such that

$$\left|\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx - \int_a^b \left(\frac{\sin x}{x}\right)^2 dx\right| < \epsilon/3,$$

for all $a \in (0, A]$ and $b \geq B$.

We now prove 2 lemmas:

Lemma 1 Let $f : [a, b] \to \mathbb{R}$ be continuous. For all $\epsilon > 0$, there exists $\delta_1 > 0$ such that, for any partition $P = \{x_1, \ldots, x_r\}$ on $[a, b]$ with $\max_i \Delta x_i < \delta_1$, we have

(i) $U(P, f) - L(P, f) < \epsilon$

(ii) $\left|\int_a^b f - \sum_{i=1}^n f(s_i)\Delta x_i\right| < \epsilon$, for any $s_i \in [x_{i-1}, x_i]$.

Proof. (a) immediately follows from the proof of Theorem 6.8. (b) is by Theorem 6.7(c). □

Lemma 2 With the setups of Lemma, let $[c, d] \subseteq [a, b]$. For any partition $Q = \{y_1, \ldots, y_m\}$ on $[c, d]$ with $\max_i \Delta y_i < \delta_1$,

(i) $U(Q, f|_{[c,d]}) - L(Q, f|_{[c,d]}) < \epsilon$

(ii) $\left|\int_c^d f - \sum_{i=1}^n f(s_i)\Delta y_i\right| < \epsilon$, for any $s_i \in [y_{i-1}, y_i]$.

Proof. Consider the partition $Q^+ = Q \cup Q'$ on $[a, b]$, where the length of every interval in Q' is also lesser than δ_1. By Lemma 1, $U(Q, f|_{[c,d]}) - L(Q, f|_{[c,d]}) \leq U(Q^+, f) - L(Q^+, f) < \epsilon$. (b) again follows from Theorem 6.7(c). □

Now consider $g : [0, B+1] \to \mathbb{R}$ with $g(x) = \begin{cases} \left(\frac{\sin x}{x}\right)^2 & x > 0 \\ 1 & x = 0 \end{cases}$. Note that g is continuous on $[0, B+1]$.
For $T \in (0, B+1]$, we have

$$\begin{aligned} \left| \int_0^T g(x)\,dx - \int_0^T \left(\frac{\sin x}{x}\right)^2 dx \right| &\le \int_0^T \left| g(x) - \left(\frac{\sin x}{x}\right)^2 \right| dx \\ &= \int_0^\eta \left| g(x) - \left(\frac{\sin x}{x}\right)^2 \right| dx \end{aligned}$$

for arbitrary $\eta > 0$. Since $\left(\frac{\sin x}{x}\right)^2$ is bounded, we have $\int_0^T g(x)\,dx = \int_0^T \left(\frac{\sin x}{x}\right)^2 dx$ for all $T \in (0, B+1]$.
Applying Lemma 1 with $\epsilon/3$ gives a δ_1 for g. Let $\delta_0 = \min(A, \delta_1, 1)$.
Fix arbitrary $\delta \in (0, \delta_0)$. Let $N = \lfloor \frac{B}{\delta} \rfloor$. Note that $\frac{B}{\delta} \le N < \frac{B}{\delta} + 1$, and so $B \le N\delta < B + \delta \le B + 1$.
Since $\delta < \delta_0 \le A$ and $N\delta \ge B$, we have

$$\left| \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx - \int_\delta^{N\delta} \left(\frac{\sin x}{x}\right)^2 dx \right| < \epsilon/3.$$

Since $[\delta, N\delta] \subseteq [0, B+1]$, Lemma 2 tells us

$$\left| \int_\delta^{N\delta} \left(\frac{\sin x}{x}\right)^2 dx - \sum_{n=1}^N \frac{\sin(n\delta)}{(n\delta)^2}\delta \right| < \epsilon/3.$$

Since

$$\begin{aligned} \left| \sum_{n=1}^N \frac{\sin(n\delta)}{(n\delta)^2}\delta - \sum_{n=1}^\infty \frac{\sin(n\delta)}{n^2\delta} \right| &\le \sum_{n=N+1}^\infty \frac{|\sin(n\delta)|}{n^2\delta} \\ &\le \frac{1}{\delta} \sum_{n=N+1}^\infty \frac{1}{n^2} \\ &\le \frac{1}{\delta} \int_N^\infty \frac{1}{x^2}\,dx = \frac{1}{N\delta} \le \frac{1}{B} < \epsilon/3. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left| \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx - \frac{\pi - \delta}{2} \right| \\ &\le \left| \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx - \int_\delta^{N\delta} \left(\frac{\sin x}{x}\right)^2 dx \right| \\ &\quad + \left| \int_\delta^{N\delta} \left(\frac{\sin x}{x}\right)^2 dx - \sum_{n=1}^N \frac{\sin(n\delta)}{(n\delta)^2}\delta \right| + \left| \sum_{n=1}^N \frac{\sin(n\delta)}{(n\delta)^2}\delta - \sum_{n=1}^\infty \frac{\sin(n\delta)}{n^2\delta} \right| < \epsilon. \end{aligned}$$

But then δ is arbitrary, so $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$. □

(e) Put $\delta = \frac{\pi}{2}$ in (c). What do you get?

Proof.

$$\sum_{n=1}^\infty \frac{\sin^2(n\pi/2)}{n^2\pi/2} = \frac{2}{\pi} \sum_{k=1}^\infty \frac{1}{(2k+1)^2} = \frac{\pi}{4},$$

and thus

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

□

Problem 2

Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. For $x \in \mathbb{R}$, define $f(x + 2\pi) = f(x)$.

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, dx = \pi$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} xe^{-inx}\, dx = -\frac{1}{in} e^{-2\pi in} - \frac{1}{2\pi(in)^2}(e^{-in2\pi} - 1) = \frac{i}{n}.$$

By Parseval's theorem,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

On the left-hand-side, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2\, dx = \frac{4\pi^2}{3}.$$

On the right-hand-side, since $|c_n|^2 = \frac{1}{n^2} = |c_{-n}|^2$,

$$\sum_{-\infty}^{\infty} |c_n|^2 = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hence, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\left(\frac{4\pi^2}{3} - \pi^2\right) = \frac{\pi^2}{6}.$$

□

Problem 3

If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Proof. Let $x, t \in [-\pi, \pi]$. By MVT,

$$|f(x+t) - f(x)| = |f'(s)||t| = 2|t||\pi - |s|| \leq 2\pi|t|,$$

for some $s \in (-\pi, \pi)$. Hence, the Fourier series converges to f for all x, by Theorem 8.14. The Fourier coefficients of $f(x)$ for $n = 0$ is

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 \, dx = \frac{1}{\pi} \int_{-\pi}^{0} x^2 \, dx = \frac{\pi^2}{3},$$

and for $n \neq 0$ is

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx} \, dx = \frac{1}{2\pi} \left(\pi^2 \int_{-\pi}^{\pi} e^{-inx} \, dx - 2\pi \int_{-\pi}^{\pi} |x| e^{-inx} \, dx + \int_{-\pi}^{\pi} x^2 e^{-inx} \, dx \right).$$

We calculate each integral separately:

$$\int_{-\pi}^{\pi} e^{-inx} \, dx = \frac{1}{in}(e^{in\pi} - e^{-in\pi}) = \frac{2}{n} \sin(n\pi) = 0.$$

$$\int_{-\pi}^{\pi} |x| e^{-inx} \, dx = \int_{-\pi}^{\pi} |x| \cos(nx) \, dx - i \int_{-\pi}^{\pi} |x| \sin(nx) \, dx = 2 \int_{0}^{\pi} x \cos(nx) \, dx = \frac{2(\cos(nx) - 1)}{n^2}.$$

$$\int_{-\pi}^{\pi} x^2 e^{-inx} \, dx = \int_{-\pi}^{\pi} x^2 \cos(nx) \, dx - i \int_{-\pi}^{\pi} x^2 \sin(nx) \, dx = 2 \int_{0}^{\pi} x^2 \cos(nx) \, dx = \frac{4\pi \cos(nx)}{n^2}.$$

Hence, $c_n = \frac{2 - 2\cos(nx)}{n^2} + \frac{2\cos(nx)}{n^2} = \frac{2}{n^2}$. It now follows that

$$f(x) = \frac{\pi^2}{3} + \sum_{n \neq 0} \frac{2}{n^2} e^{-inx} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2}(e^{-inx} + e^{-inx}) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx.$$

When $x = 0$, we have

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

By Parseval's Theorem,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{-\infty}^{\infty} |c_n|^2,$$

and thus

$$\frac{1}{\pi} \int_{0}^{\pi} (x - \pi)^4 \, dx = \frac{\pi^4}{5} = \frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

$\square$

Problem 4

Let γ be a continuously differentiable closed curve in the complex plane, with parameter interval $[a, b]$, and assume that $\gamma(t) \neq 0$ for every $t \in [a, b]$. Define the *index* of γ to be

$$\operatorname{Ind}(\gamma) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)}\, dt.$$

Prove that $\operatorname{Ind}(\gamma)$ is always an integer. Compute $\operatorname{Ind}(\gamma)$ when $\gamma(t) = e^{int}$, $a = 0$, $b = 2\pi$. Explain why $\operatorname{Ind}(\gamma)$ is often called the winding number of γ around 0.

Proof. Define $\phi(x) = \int_a^x \frac{\gamma'(t)}{\gamma(t)}\, dt$. Since $\phi' = \frac{\gamma'}{\gamma}$ and $\phi(a) = 0$,

$$(\gamma(x)\exp(-\phi(x)))' = \gamma'(x)\exp(-\phi(x)) - \gamma(x)\phi'(x)\exp(-\phi(x)) = 0,$$

and thus $\gamma(x)\exp(-\phi(x)) = \gamma(a)$. Since $\gamma(a) = \gamma(b)$, we have $\exp\phi(b) = \frac{\gamma(b)}{\gamma(a)} = 1$, and so $\phi(b) = 2n\pi i$. But then $\phi(b) = \int_a^b \frac{\gamma'(t)}{\gamma(t)}\, dt = 2\pi i \operatorname{Ind}(\gamma)$, an thus $\operatorname{Ind}(\gamma) = n$ for some integer n.

Now consider $\operatorname{Ind}(\gamma)$ when $\gamma(t) = e^{int}$, $a = 0$, $b = 2\pi$.

$$\operatorname{Ind}(\gamma) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{ine^{int}}{e^{int}}\, dt = \frac{1}{2\pi i} \int_0^{2\pi} in\, dt = n.$$

Since $\operatorname{Ind}(\gamma)$ represents the number of rotations $\gamma(t)$ goes around 0, it makes sense to call $\operatorname{Ind}(\gamma)$ the winding number of γ around 0. $\square$

Problem 5

Let γ be as in Exercise 8.23, and assume in addition that the range of γ does not intersect the negative real axis. Prove that $\text{Ind}(\gamma) = 0$.

Proof. For any $c \geq 0$, define $\gamma_c(t) = \gamma(t) + c$. Consider the function

$$f(c) = \text{Ind}(\gamma_c) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) + c} \, dt$$

on $[0, \infty)$. We show that $f(c)$ is continuous and integer-valued. Since γ' is continuous on $[-\pi, \pi]$, $|\gamma'| < M$ for some M. Since γ does not intersect the negative real axis, $\gamma_c \neq 0$. But then $|\gamma_c|$ is continuous on compact set, so $\min_t |\gamma_c(t)|$ exists for all $c \geq 0$. Consider $\min_{c,t} |\gamma_c(t)| = |\gamma(t) + c|$. If $\text{Re}(\gamma(t)) > 0$, then $|\gamma_c(t)| = |(\text{Re}(\gamma(t)) + c) + i\text{Im}(\gamma(t))| \geq |\gamma(t)| > 0$. If $\text{Re}(\gamma(t)) \leq 0$, then $\text{Im}(\gamma(t)) \neq 0$ by assumption, and thus $|\gamma_c(t)| > 0$. Hence, we know $\min_{c,t} |\gamma_c(t)| \geq m > 0$ for some m. Pick $\epsilon > 0$. Let $\delta = \frac{2\pi\epsilon m^2}{(b-a)M}$. Then,

$$\begin{aligned} |f(x) - f(y)| &\leq \frac{1}{2\pi} \int_a^b \left| \frac{\gamma'(t)}{\gamma(t) + x} - \frac{\gamma'(t)}{\gamma(t) + y} \right| dt \\ &\leq \frac{1}{2\pi} \int_a^b \frac{M|y - x|}{|\gamma(t) + x||\gamma(t) + y|} \, dt \\ &\leq \frac{(b-a)M|y - x|}{2\pi m^2} < \epsilon, \end{aligned}$$

whenever $x, y \geq 0$ and $|x - y| < \delta$, and so f is continuous.

Put $\phi_c(x) = \int_a^x \frac{\gamma'(t)}{\gamma_c(t)} \, dt$. With the exact same argument as in Exercise 8.23, we get that $f(c) = \text{Ind}(\gamma_c)$ is integer valued.

Given sequence of function $\left\{ \frac{\gamma'}{\gamma_n} \right\}_{n \in \mathbb{N}}$, for all $n \geq \frac{M}{\epsilon} - m$, we have

$$\left| \frac{\gamma'(t)}{\gamma_n(t)} \right| = \frac{M}{m + n} < \epsilon,$$

for all $t \in [a, b]$. Hence, $\frac{\gamma'}{\gamma_n} \to 0$ uniformly. By Theorem 7.16,

$$\lim_{c \to \infty} \text{Ind}(\gamma_c) = \frac{1}{2\pi i} \int_a^b \lim_{c \to \infty} \frac{\gamma'(t)}{\gamma(t) + c} \, dt = 0.$$

Since $\text{Ind}(\gamma_c)$ is integer-valued and continuous, it now follows that $\text{Ind}(\gamma_c) = 0$, for all c, which includes $c = 0$. $\square$

Problem 6

Suppose γ_1 and γ_2 are curves as in Exercise 8.23, and $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$ for $a \leq t \leq b$. Prove that $\text{Ind}(\gamma_1) = \text{Ind}(\gamma_2)$.

Proof. Put $\gamma = \frac{\gamma_2}{\gamma_1}$. Note that γ is well-defined, as $\gamma_1 \neq 0$. Then $|1 - \gamma| = \frac{|\gamma_1(t) - \gamma_2(t)|}{|\gamma_1(t)|} < 1$, and so γ does not intersect with the negative real axis. By Exercise 8.24, $\text{Ind}(\gamma) = 0$. Also,

$$\frac{\gamma'}{\gamma} = \frac{\frac{\gamma_2'\gamma_1 - \gamma_1'\gamma_2}{\gamma_1^2}}{\frac{\gamma_2}{\gamma_1}} = \frac{\gamma_2'\gamma_1 - \gamma_1'\gamma_2}{\gamma_1\gamma_2} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1}.$$

Hence,

$$\text{Ind}(\gamma) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)}\,dt = \frac{1}{2\pi i}\int_a^b \frac{\gamma_2'(t)}{\gamma_2(t)}\,dt - \frac{1}{2\pi i}\int_a^b \frac{\gamma_1'(t)}{\gamma_1(t)}\,dt = 0,$$

and thus

$$\text{Ind}(\gamma_1) = \frac{1}{2\pi i}\int_a^b \frac{\gamma_2'(t)}{\gamma_2(t)}\,dt = \frac{1}{2\pi i}\int_a^b \frac{\gamma_1'(t)}{\gamma_1(t)}\,dt = \text{Ind}(\gamma_2).$$

□