MATH 140B: Homework #5

Due on May 10, 2024 at 23:59pm

Professor Seward

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Problem 1

If (f_n) and (g_n) converge uniformly on a set E, prove that $(f_n + g_n)$ converges uniformly on E. If, in addition, (f_n) and (g_n) are sequences of bounded functions, prove that $(f_n g_n)$ converges uniformly on E.

Proof. Pick $\epsilon > 0$. Since (f_n) and (g_n) converge uniformly, there exists N, M such that for all $x \in E$, $|f_{n_1}(x) - f_{n_2}(x)| \leq \epsilon/2$ and $|g_{m_1}(x) - g_{m_2}(x)| \leq \epsilon/2$, for all $n_1, n_2 \geq N$ and $m_1, m_2 \geq M$. Put $L = \max(N, M)$. For all $m, n \geq L$,

$$\begin{aligned} |(f_n + g_n)(x) - (f_m + g_m)(x)| &= |(f_n(x) - f_m(x)) + (g_n(x) - g_m(x))| \\ &\leq |(f_n(x) - f_m(x))| + |(g_n(x) - g_m(x))| \leq \epsilon, \end{aligned}$$

for all $x \in E$. Hence, $(f_n + g_n)$ converges uniformly.

Now suppose that there exists $B > 0$ such that $\sup_x |f_n(x)| < B$ and $\sup_x |g_n(x)| < B$ for all n. Since (f_n) and (g_n) converge uniformly, there exists N, M such that for all $x \in E$, $|f_{n_1}(x) - f_{n_2}(x)| \leq \epsilon/2B$ and $|g_{m_1}(x) - g_{m_2}(x)| \leq \epsilon/2B$, for all $n_1, n_2 \geq N$ and $m_1, m_2 \geq M$. Put $L = \max(N, M)$. For all $m, n \geq L$. Then, for all $m, n \geq L$,

$$\begin{aligned} |(f_n g_n)(x) - (f_m g_m)(x)| &= |(f_n g_n)(x) - (f_m g_n)(x) + (f_m g_n)(x) - (f_m g_m)(x)| \\ &\leq |f_n(x) g_n(x) - f_m(x) g_n(x)| + |f_m(x) g_n(x) - f_m(x) g_m(x)| \\ &< B|(f_n(x) - f_m(x))| + B|(g_n(x) - g_m(x))| \leq \epsilon, \end{aligned}$$

for all $x \in E$. Hence, $(f_n g_n)$ converges uniformly. $\square$

Problem 2

Construct sequences (f_n), (g_n) which converge uniformly on some set E, but such that $(f_n g_n)$ does not converge uniformly on E (of course, $(f_n g_n)$ must converge on E).

Proof. Consider $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$ on $\mathbb{R}^+$. Since f_n remains the same for all n, so it converges uniformly to $f(x) = x$. Given any $\epsilon > 0$, $|g_n| < \epsilon$ for $n > \frac{1}{\epsilon}$, and thus g_n converges uniformly to 0. But then there always exists $x > n$ such that $(f_n g_n)(x) > 1$. Hence, $\sup_x |(f_n g_n)(x) - 0| > 1$, $(f_n g_n)$ does not converge uniformly. $\square$

Problem 3

Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Proof. Notice that when $x = 0$, $f(x) = \sum_{n=1}^{\infty} 1$ diverges. Additionally, when $x = -\frac{1}{n^2}$ for some n, the nth term in $f(x)$ is not well-defined, and thus the infinite sum is also not well-defined. For $x \neq 0$ and $x \neq -\frac{1}{n^2}$ for all n,

$$\sum_{n=1}^{\infty} \left| \frac{1}{1+n^2x} \right| \leq \sum_{n=1}^{\infty} \left| \frac{1}{xn^2} \right| = \frac{1}{|x|} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Hence, $f(x)$ converges absolutely if and only of $x \neq 0$ and $x \neq -\frac{1}{n^2}$ for any n.

Let $E = \mathbb{R} \backslash (\{0\} \cup \{-\frac{1}{n^2} \mid n \in \mathbb{N}\})$. We show that $f(x)$ converges uniformly on $E_\delta = E \backslash (-\delta, \delta)$, for any $\delta > 0$. Pick $\epsilon > 0$. Let $M_n = \frac{2}{\delta n^2}$. For $x > 0$,

$$\left| \frac{1}{1+n^2x} \right| \leq \frac{1}{n^2} \cdot \frac{1}{\delta} < M_n.$$

For $x < 0$,

$$\left| \frac{1}{1+n^2x} \right| < \frac{1}{\frac{1}{2}|x|} \cdot \frac{1}{n^2} \leq \frac{1}{\frac{1}{2}\delta} \cdot \frac{1}{n^2} = M_n,$$

when $n \geq N$ for some N. Hence, each term of $f(x)$ is bounded by M_n when $n \geq N$. Define $f'(x) = \sum_{n=N}^{\infty} \frac{1}{1+n^2x}$. Since each term of $f'(x)$ is bounded by M_n and $\sum M_n$ converges, $f'(x)$ converges uniformly on E_δ, and thus $f(x) = \sum_{n=1}^{N-1} \frac{1}{1+n^2x} + f'(x)$ also converges uniformly on E_δ.

$f(x)$ trivially fails to converge uniformly on any interval which contains either 0 or $-\frac{1}{n^2}$ for some n. We now show that $f(x)$ fails to converge uniformly on $(0, \delta]$ for any $\delta > 0$. Let $f_m(x) = \sum_{n=1}^{m} \frac{1}{1+n^2x}$. Suppose for the sake of contradiction that there eixsts M such that for all $m \geq M$,

$$|f(x) - f_m(x)| < \frac{1}{4}.$$

Pick $x_0 \in (0, \delta]$ small enough such that $\lceil \frac{1}{\sqrt{x_0}} \rceil > M$. Let $N = \lceil \frac{1}{\sqrt{x_0}} \rceil$. Then, $N^2 \geq \frac{1}{x} \geq (N-1)^2$. But then

$$\begin{aligned}
|f(x) - f_{N-1}(x)| &= \left| \sum_{n=N}^{\infty} \frac{1}{1+n^2x} \right| \\
&\geq \left| \sum_{n=N}^{\infty} \frac{1}{2n^2x} \right| \\
&= \frac{1}{2x} \sum_{n=N}^{\infty} \frac{1}{n^2} \\
&\geq \frac{(N-1)^2}{2N^2} \geq \frac{1}{4},
\end{aligned}$$

contradiction.

Since $\frac{1}{1+n^2x}$ is continuous on E for all $n \in \mathbb{N}$, the partial sums of $f(x)$ is continuous on E. Given any point $x \in E$, pick $\delta \in (0, |x|)$. By Theorem 7.12, since f uniformly converges on E_δ, f is continuous on E_δ, and thus f is continuous on x. Hence, f is continuous whenever the series converges.

f is not bounded. Given any $M > 0$, pick $x = \frac{1}{4M^2}$. Then,

$$\begin{aligned}
|f(x)| &= \left| \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \right| \\
&= \left| \sum_{n=1}^{2M} \frac{1}{1+n^2x} + \sum_{n=2M+1}^{\infty} \frac{1}{1+n^2x} \right| \\
&> 2M \left(\frac{1}{1+(2M)^2x} \right) = M.
\end{aligned}$$

□

Problem 4

For $n = 1, 2, 3, \dots,$ and x real, put
$$f_n(x) = \frac{x}{1+nx^2}.$$
Show that (f_n) converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

Proof. We show that (f_n) converges to $f(x) = 0$. Pick $\epsilon > 0$. Put $N > \frac{1}{4\epsilon^2}$. Note that

$$\left|\frac{x}{1+nx^2}\right| = \left|\frac{1}{\frac{1}{x}+nx}\right|.$$

By AM-GM, $\frac{1}{x} + nx \geq 2\sqrt{n}$. It follows that for $n \geq N$,

$$\left|\frac{x}{1+nx^2}\right| \leq \frac{1}{2\sqrt{n}} < \epsilon,$$

and thus (f_n) converges to 0 uniformly.

Note that $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$. In particular, $f_n'(0) = 1$. When $x \neq 0$, $\lim_{n\to\infty} f_n'(x) = 0 = f'(x)$. But then if $x = 0$, $\lim_{n\to\infty} f_n'(0) = 1 \neq f'(x)$. $\square$

Problem 5

Let (f_n) be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

Proof. By Theorem 7.12, since f_n is continuous for all n, f is continuous, which implies $\lim_{n\to\infty} f(x_n) = f(x)$. Hence, it suffices to show that

$$\lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} f(x_n).$$

Pick $\epsilon > 0$. Since (f_n) uniformly converges to f, there exists N such that

$$|f_n(x) - f(x)| < \epsilon,$$

for all $n \geq N$. But then

$$|f_n(x_n) - f(x_n)| < \epsilon,$$

for all $n \geq N$, and the result now follows.

However, the converse to this is not true. Consider $f_n(x) = x^n$ on $[0, 1)$ and $f(x) = 0$. Let (x_n) be a sequence in $[0, 1)$ which converges to some $x \in E$. Since $|x_n| < 1$,

$$\lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} x_n^n = 0 = f(x).$$

But then (f_n) does not converge uniformly, as for any $\epsilon \in (0, 1)$, there exists $x > \sqrt[n]{\epsilon}$ in $[0, 1)$ such that $x^n > \epsilon$. $\square$

Problem 6

Letting (x) denote the fractional part of the real number x (see Exercise 4.16 for the definition), consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$$

for x real. Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

Proof. We show that $f(x)$ is discontinuous for all $x \in \mathbb{Q}$, which is obviously a countable dense set. We first note that the partial sums $f(x)$ converges uniformly as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by Theorem 7.10.

Notice that (nx) is discontinuous if and only if $nx \in \mathbb{Z}$ if and only if $x = p/q$, where n is a multiple of q. Hence, for any irrational x, since the partial sums of $f(x)$ is continuous, $f(x)$ is continuous on x, by Theorem 7.12.

Now suppose $x \in \mathbb{Q}$, say $x = p/q$. Define $f_q'(x) = \sum_{k=1}^{\infty} \frac{(kqx)}{[kq]^2}$ and consider $f_q(x) = f(x) - f_q'(x)$. Note that

$$f_q'(x-) = \sum_{k=1}^{\infty} \frac{1}{[kq]^2} \neq 0 = f_q'(x),$$

and thus $f_q'(x)$ is discontinuous on x. Since f_q' contains all terms which are discontinuous on x, all terms of $f_q(x)$ are continuous on x, and thus the partial sum of $f_q(x)$ is continuous on x. Again we know that the partial sums of $f_q(x)$ converge uniformly, by Theorem 7.10. By Theorem 7.12, $f_q(x)$ is continuous on x. But then $f_q'(x) = f_q(x) - f(x)$ is discontinuous on x, so $f(x)$ is discontinuous on x. Hence, $f(x)$ is discontinuous on x if and only if $x \in \mathbb{Q}$.

Since $(nx)/n^2$ is piece-wise continuous, $(nx)/n^2 \in \mathscr{R}$, and thus $\sum_{n=1}^{m}(nx)/n^2 \in \mathscr{R}$. It now follows that the partial sums of $f(x)$ converges uniformly on any given bounded interval, so f is Riemann-integrable on every bounded interval, by Theorem 7.16. $\square$

Problem 7

Let f be a continuous real function on $\mathbb{R}^1$ with the following properties: $0 \leq f(t) \leq 1$, and

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{2}{3}, \\ 1 & \text{for } \frac{2}{3} \leq t \leq 1. \end{cases}$$

$f(t+2) = f(t)$ for every t, and

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \quad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

Prove that Φ is continuous and that Φ maps $I = [0,1]$ onto the unit square $I^2 \subseteq \mathbb{R}^2$. In fact, show that Φ maps the Cantor set onto I^2.

Proof. We first note that both $x(t)$ and $y(t)$ converges uniformly as $\sum_{n=1}^{\infty} 2^{-n} = 1$ converges, by Theorem 7.10. Since f is continuous, the partial sums of both $x(t)$ and $y(t)$ are continuous, and thus $x(t)$ and $y(t)$ are continuous, by Theorem 7.16. It now follows from Theorem 4.10 that Φ is continuous.

We now show that Φ maps $I = [0,1]$ onto I^2. Notice thay each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \quad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n},$$

where each a_i is 0 or 1. Let $t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$. By Exercise 3.19, $t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$ is in the Cantor set. Since

$$3^k t_0 = \sum_{i=1}^{\infty} 3^{-i+k-1}(2a_i) = 2\sum_{i=1}^{k-1} 3^{-i+k-1} a_i + \sum_{i=0}^{\infty} 3^{-i-1}(2a_{i+k}),$$

we know $f(3^k t_0) = f\left(\sum_{i=0}^{\infty} 3^{-i-1}(2a_{i+k})\right)$. But then

$$\sum_{i=0}^{\infty} 3^{-i-1}(2a_{i+k}) = \frac{2}{3}a_k + \frac{2}{3}\sum_{i=1}^{\infty} 3^{-i} a_{i+k},$$

and

$$0 \leq \frac{2}{3}\sum_{i=1}^{\infty} 3^{-i} a_{i+k} \leq \frac{2}{3} \cdot \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{3},$$

so $\sum_{i=0}^{\infty} 3^{-i-1}(2a_{i+k}) \in [0, \frac{2}{3}]$ if $a_k = 0$ and $\sum_{i=0}^{\infty} 3^{-i-1}(2a_{i+k}) \in [\frac{2}{3}, 1]$ otherwise. Hence, $f(3^k t_0) = a_k$. It now follows that $x(t_0) = x_0$ and $y(t_0) = y_0$, and so Φ maps the Cantor set $C \subset I$ onto I^2. $\square$