MATH 264B: Homework

Professor Rhoades

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Problem 1

Let $c_n = \frac{1}{n+1}\binom{2n}{n}$ be the nth Catalan number. Prove that

$$c_n = \sum_{i=1}^{n} c_{i-1} \cdot c_{n-i}$$

for all $n > 0$.

Proof. Note that c_n is the number of Dyck paths of size n. Observe that a Dyck path of size n consists of a prime Dyck path of size i starting at the origin and a Dyck path of size $n - i$, for some $i \in [n]$. Let p_n denote the number of prime Dyck paths of size n. Then

$$c_n = \sum_{i=1}^{n} p_i \cdot c_{n-i}.$$

Since we may bijectively map prime Dyck paths of size i to Dyck paths of size $i - 1$ by removing the first and last steps, we have $p_i = c_{i-1}$. The result now follows. $\square$

Problem 2

Let $\tau \in \mathfrak{S}_k$ be a permutation (thought of as a 'pattern'). A permutation $w \in \mathfrak{S}_n$ is called τ-*avoiding* if its one-line notation $[w(1), \ldots, w(n)]$ does not contain a subsequence

$$w(i_1), \ldots, w(i_k) \quad \text{with } 1 \leq i_1 < \cdots < i_k \leq n$$

order-isomorphic to τ.

Prove that the number of 312-avoiding permutations $w \in \mathfrak{S}_n$ equals the Catalan number c_n. (Hint: Consider $w^{-1}(1)$.) Explain why c_n also counts τ-avoiding permutations in $\mathfrak{S}_n$ for the patterns $\tau = 132, 213, 231$.

Proof. Let A_n be the set of 312-avoiding permutations in $\mathfrak{S}_n$. For $i \in [n]$, we show a bijection between $\{w \in A_n : w^{-1}(1) = i\}$ and $A_{i-1} \times A_{n-i}$. Let $w \in A_n$ with $w^{-1}(1) = i$. Write $L = [w(1), \ldots, w(i-1)]$ and $R = [w(i+1), \ldots, w(n)]$. Notice that if $w(k) > w(m)$ for some $1 \leq k < i < m \leq n$, then $w(k), w(i), w(m)$ is a 312-pattern, contradiction. Hence, $\max L < \min R$. Now let $w_L = [w(1) - 1, \ldots, w(i-1) - 1] \in A_{i-1}$ and $w_R = [w(i+1) - i, \ldots, w(n) - i] \in A_{n-i}$ and map w to (w_L, w_R), and the above operation is reversible. This gives the desired bijection. Therefore, A_n has the recursion formula

$$|A_n| = \sum_{i=1}^{n} |A_{i-1}| \cdot |A_{n-i}|,$$

which is the same as the recursion for the Catalan numbers.

Since permutations 312 and 231 are isomorphic, we only need to consider 132 and 213. Consider the involutions $\mathrm{inv}(w)$ and $\mathrm{rev}(w)$, which maps w to w^{-1} and the reverse of w, respectively. Since $\mathrm{inv}(312) = 132$ and $\mathrm{rev}(312) = 213$, inv maps 312-avoiding permutations to 132-avoiding permutations, and rev maps 312-avoiding permutations to 213-avoiding permutations. Hence, we may the composition of the above bijection and inv or rev gives the desired bijection. Therefore, c_n also counts 132-avoiding and 213-avoiding permutations. $\square$

Problem 3

Let $S_{n,k}$ be the Stirling number of the second kind counting set partitions of $[n]$ into k blocks. Find an expression for the exponential generating function

$$\sum_{n \geq 0} S_{n,k} \cdot \frac{x^n}{n!}$$

where k is fixed.

Proof. Since $e^x - 1 = \sum_{n \geq 1} x^n/n!$ is the exponential generating function for non-empty sets, $(e^x - 1)^k$ counts the number of ways to partition number of ways to $[n]$ into k ordered blocks. Hence, $(e^x - 1)^k/k!$ counts the number of ways to partition $[n]$ into k unordered blocks, which is just $S_{n,k}$. Therefore,

$$\sum_{n \geq 0} S_{n,k} \cdot \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}.$$

$\square$

Problem 4

Each of n distinguishable telephone poles is painted red, white, blue, or yellow. An odd number are painted blue and an even number are painted yellow. In how many ways can this be done?

Proof. Write $E(x) = \sum_{n \text{ even}} x^n/n!$ and $O(x) = \sum_{n \text{ odd}} x^n/n!$. Then

$$E(x) = \frac{e^x + 1}{2} \quad \text{and} \quad O(x) = \frac{e^x - 1}{2}.$$

Note that $E(x)$ and $O(x)$ are the exponential generating function for even and odd sets, respectively. Then the exponential generating funtion for the ways to paint the poles is

$$e^x \cdot e^x \cdot E(x) \cdot O(x) = e^{2x} \cdot \frac{e^x - 1}{2} \cdot \frac{e^x + 1}{2} = \frac{e^{4x} - e^{2x}}{4}.$$

$\square$

Problem 5

Let G be a finitely generated group and write

$$\mathrm{Hom}(G, \mathfrak{S}_n) := \{\text{all homomorphisms } G \to \mathfrak{S}_n\}.$$

For any $n \geq 0$, the set $\mathrm{Hom}(G, \mathfrak{S}_n)$ is finite (why?). We consider the exponential generating function

$$F(x) := \sum_{n \geq 0} \#\mathrm{Hom}(G, \mathfrak{S}_n) \frac{x^n}{n!}.$$

If g_n denotes the number of *transitive* actions of G on $[n]$, prove that

$$F(x) = \exp\left(\sum_{d \geq 1} g_d \cdot \frac{x^d}{d!}\right).$$

Proof. Since a homomorphism $G \to \mathfrak{S}_n$ is equivalent to a G-action on $[n]$, we have $\#\mathrm{Hom}(G, \mathfrak{S}_n) = \#\{G\text{-actions on } [n]\}$. Given an G-action α on $[n]$, α induces a partition of $[n]$ into orbits $O_1 \sqcup \cdots \sqcup O_k$, and the restriction of α to O_i is a transitive action of G on O_i. In particular, α may be determined by k transitive actions on each orbit. Applying the exponential generating function, we get

$$F(x) = \exp\left(\sum_{d \geq 1} g_d \cdot \frac{x^d}{d!}\right).$$

$\square$

Problem 6

In the context of the previous problem, if $j_d(G)$ is the number of subgroups of G having index d, show that

$$g_d = j_d(G) \cdot (d-1)!.$$

Proof. $\qquad$ □

Problem 7

For each of the following sets of graphs, let $f(n)$ be the number of graphs on the labeled vertex set $[n]$ such that every connected component is isomorphic to some graph in the set. In each case, find the exponential generating function of $f(n)$, where $f(0) := 1$.

(a) Cycles C_i of length $i \geq k$ where $k \geq 3$ is fixed.

Proof. Given n vertices, the number of ways to order them into an undirected cycle is $(n-1)!/2$. Hence, let

$$C(x) = \sum_{n \geq k} \frac{(n-1)!}{2} \cdot \frac{x^n}{n!} = \frac{1}{2} \sum_{n \geq k} \frac{x^n}{n} = -\frac{1}{2} \ln(1-x) - \frac{1}{2} \sum_{i=1}^{k-1} \frac{x^i}{i}$$

be the exponential generating function that counts cycles of length k or more. Since the connected components of a graph on $[n]$ may be viewed as a partition of $[n]$,

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \exp(C(x)) = (1-x)^{-1/2} \exp\left(-\frac{1}{2} \sum_{i=1}^{k-1} \frac{x^i}{i}\right).$$

$\square$

(b) Stars $K_{i,1}$ for some $i \geq 1$. (Here $K_{r,s}$ is the complete bipartite graph.)

Proof. Given n vertices, the number of ways to order them into a $K_{n-1,1}$ is n. Hence, let

$$S(x) = \sum_{n \geq 1} n \cdot \frac{x^n}{n!} = \sum_{n \geq 1} \frac{x^n}{(n-1)!} = x(e^x - 1)$$

be the exponential generating function that counts stars. Since the connected components of a graph on $[n]$ may be viewed as a partition of $[n]$,

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \exp(S(x)) = \exp[x(e^x - 1)].$$

$\square$

(c) Wheels W_i for $i \geq 4$. (Here W_i is obtained from C_{i-1} by adding a new vertex connected to every other vertex.)

Proof. Given n vertices, the number of ways to order them into a W_n is $n \cdot (n-2)!/2$. Hence, let

$$W(x) = \sum_{n \geq 4} \frac{n(n-2)!}{2} \cdot \frac{x^n}{n!} = \sum_{n \geq 4} \frac{x^n}{2(n-1)} = \frac{x}{2} \sum_{n \geq 3} \frac{x^n}{n} = \frac{x}{2}\left(-\ln(1-x) - x - \frac{x^2}{2}\right)$$

be the exponential generating function that counts wheels. Since the connected components of a graph on $[n]$ may be viewed as a partition of $[n]$,

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \exp(W(x)) = \exp\left[\frac{x}{2}\left(-\ln(1-x) - x - \frac{x^2}{2}\right)\right].$$

$\square$

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(d) Paths P_i with $i \geq 1$ vertices. (So that P_1 is a single vertex and P_2 is a single edge.)

Proof. Given $n \geq 2$ vertices, the number of ways to order them into an undirected path is $n!/2$. Hence, let

$$P(x) = x + \sum_{n \geq 2} \frac{n!}{2} \cdot \frac{x^n}{n!} = x + \frac{1}{2} \sum_{n \geq 2} x^n = \frac{x^2 - 2x + 2}{2 - 2x}$$

be the exponential generating function that counts paths. Since the connected components of a graph on $[n]$ may be viewed as a partition of $[n]$,

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \exp(P(x)) = \exp(x^2 - 2x + 2) - \exp(2 - 2x).$$

$\square$