MATH 140B: Homework #6

Due on May 17, 2024 at 23:59pm

Professor Seward

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Problem 1

Suppose g and f_n ($n = 1, 2, 3, \ldots$) are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^\infty g(x)\,dx < \infty.$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx.$$

Proof. Since $|f_n| \leq g$ and $\int_t^\infty g(x)\,dx < \infty$, we know $\int_t^\infty f_n(x)\,dx$ exists. Define $h_k = \int_{1/k}^\infty f(x)\,dx$. Pick $\epsilon > 0$. Since $\int_0^\infty g(x)\,dx$ exists, there exists N such that $\int_0^{1/n} g(x)\,dx < \epsilon/2$ for all $n \geq N$. Let $\beta > \alpha \geq N$. There exists large enough n such that $|f(x) - f_n(x)| < \epsilon(\beta - \alpha)/2\alpha\beta$ for all $x \in [1/\beta, 1/\alpha]$. But then,

$$\begin{aligned}
|h_\beta - h_\alpha| &= \left|\int_{1/\beta}^{1/\alpha} f(x)\,dx\right| \\
&\leq \int_{1/\beta}^{1/\alpha} |f(x) - f_n(x)| + |f_n(x)|\,dx \\
&\leq \int_{1/\beta}^{1/\alpha} |f(x) - f_n(x)|\,dx + \int_{1/\beta}^{1/\alpha} g(x)\,dx \\
&< \epsilon/2 + \epsilon/2 = \epsilon,
\end{aligned}$$

and thus h_k converges uniformly by Cauchy criterion. Let $I_n = \int_0^\infty f_n(x)\,dx$ and let $I = \int_0^\infty f_n(x)\,dx$. It remains to show that $\int_0^\infty f(x)\,dx$ converges to $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx$.

$$\left|\int_0^\infty f_n(x) - f(x)\,dx\right| < \epsilon,$$

for large enough n. $\square$