

MATH 100A: Homework #2

Due on October 12, 2023 at 12:00pm

Professor McKernan

Section A02 5:00PM - 5:50PM

Section Leader: Castellano

Source Consulted: Textbook, Lecture, Discussion, Office Hour, ChatGPT

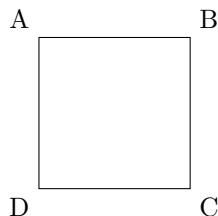
Ray Tsai

A16848188

Problem 1

Give a description of D_4 , the group of symmetries of the square, similar to the one given in class, and find all of its subgroups.

Proof. Let there be a square S with vertices A, B, C, D labeled clockwise, as shown below.



D_4 contains all the symmetries of S , including rotations, flips, and the identity I . There are three types of rotations, R, R^2, R^3 , which rotates S counter-clockwise by $90^\circ, 180^\circ, 270^\circ$, respectively. Note that the 360° rotation $R^4 = I$. Additionally, there are four types of flips, F_1, F_2, F_3, F_4 . F_1 flips S vertically through its center. F_2 flips S horizontally through its center. F_3 flips S diagonally through vertices A, C . F_4 flips S diagonally through vertices B, D . Note that each flip is its own inverse.

We now show that there can be at most 8 configurations of the positions of vertices. Since A, C must be at the opposite position, A 's position determines C 's. We pick A 's position first, then pick B and D 's, so there are at most $\binom{4}{1}\binom{2}{1} = 8$ symmetries. Since the identity, rotations, and flips we mentioned are all distinct, they account for all 8 symmetries. Thus, we conclude that $D_4 = \{I, R, R^2, R^3, F_1, F_2, F_3, F_4\}$.

We now find all the subgroups of D_4 . We first note $\{I\}$ and D_4 itself are subgroups of D_4 . There are also the cyclic subgroups $\langle R \rangle = \langle R^3 \rangle, \langle R^2 \rangle, \langle F_1 \rangle, \langle F_2 \rangle, \langle F_3 \rangle, \langle F_4 \rangle$. Note that $\langle R \rangle = \langle R^3 \rangle = \{I, R, R^2, R^3\}$ is the subgroup that contains all rotations. Suppose that we include some flip, say F_1 to $\langle R \rangle$. Then, we would also have $F_2 = R^2 F_1, F_3 = R F_1, F_4 = R^3 F_1$ in the group, which becomes D_4 .

We also observe that $F_1 F_3 = F_2 F_4 = R$, and $F_2 F_3 = F_1 F_4 = R^3$. This implies that if we include any of those pairs of flips in the same group, the group ultimately becomes D_4 , by the result we obtained above.

Note that $F_1 F_2 = F_2 F_1 = F_3 F_4 = F_4 F_3 = R^2$, so we attempt to construct subgroups with $\langle R^2 \rangle$. Suppose we include F_1 to $\langle R^2 \rangle$, then we get $F_2 = F_1 R^2$. Since each of $\{I, R^2, F_1, F_2\}$ is its own inverse and any two elements' product is still in the group, it is a subgroup. Suppose we include F_3 to $\langle R^2 \rangle$, then we get $F_4 = F_3 R^2$. Since each of $\{I, R^2, F_3, F_4\}$ is its own inverse and any two elements' product is still in the group, it is a subgroup.

Since no more combination of elements in D_4 can be used to generate a new group, we have exhausted all subgroups of D_4 , namely

$$\langle I \rangle, \langle R \rangle, \langle R^2 \rangle, \langle F_1 \rangle, \langle F_2 \rangle, \langle F_3 \rangle, \langle F_4 \rangle, \{I, R^2, F_1, F_2\}, \{I, R^2, F_3, F_4\}, D_4.$$

□

Problem 2

Suppose that G is a set closed under an associative operation such that

1. given $a, y \in G$, there is an $x \in G$ such that $ax = y$, and
2. given $a, w \in G$, there is a $u \in G$ such that $ua = w$.

Show that G is a group.

Proof. Let $b, c \in G$. We know that there exists $a, d \in G$ such that $ab = b$ and $bd = c$. Then, we get $abd = bd = ac = c$, and so a is a left identity element. Similarly, we can also find a right identity element f using the above approach. This follows that since $af = a = f$, the left and right inverse are the same element, and so G contains an identity element $e = a = f$.

Let $\alpha \in G$. We know that there exists $\beta, \gamma \in G$ such that $\alpha\beta = e$ and $\beta\gamma = e$. This follows that since $\alpha\beta\gamma = \alpha = \gamma$, $\alpha\beta = e$ and $\beta\alpha = e$, and thus all elements in G has an inverse. Therefore, G is a group. \square

Problem 3

If G is a finite set closed under an associative operation such that $ax = ay$ forces $x = y$ and $ua = wa$ forces $u = w$, for every $a, x, y, u, w \in G$, prove that G is a group.

Proof. Let $a \in G$. Define $f : G \rightarrow G$ to be $f(g) = ag$. Since $ax = ay$ implies $x = y$, we know f is injective. This follows that f is also surjective since G is a finite set, and so for each $c \in G$, there exists $b \in G$ such that $ab = c$. Similarly, we can define $h : G \rightarrow G$ to be $h(g) = ga$ and show that there exists $x \in G$ such that $xa = c$, and thus the rest of the proof follows the previous problem. \square

Problem 4

Verify that $Z(G)$, the center of G , is a subgroup of G .

Proof. We first verify that $Z(G)$ is closed under the operation of G . Let $a, b \in Z(G)$, and let $x \in G$. Since $abg = agb = gab$, $ab \in Z(G)$, and thus $Z(G)$ fulfills the closed property.

We now check the inverse property. Let $e \in G$ be the identity element, and let $c \in Z(G)$. Then, for all $x \in G$,

$$\begin{aligned} cx &= xc \\ x &= c^{-1}xc \\ xc^{-1} &= c^{-1}x, \end{aligned}$$

and thus $c^{-1} \in Z(G)$. Therefore, $Z(G)$ is a subgroup of G . □

Problem 5

If $C(a)$ is the centralizer of a in G , prove that $Z(G) = \bigcap_{a \in G} C(a)$.

Proof. Let $z \in Z(G)$, and let $a \in G$. Since $za = az$, $z \in C(a)$, and so $z \in \bigcap_{a \in G} C(a)$. Therefore, $Z(G) \subseteq \bigcap_{a \in G} C(a)$.

Let $c \in \bigcap_{a \in G} C(a)$. Since for all $a \in G$, $c \in C(a)$, and so $ca = ac$. Therefore, $c \in Z(G)$, which means that $\bigcap_{a \in G} C(a) \subseteq Z(G)$. Combining two results, we conclude that $Z(G) = \bigcap_{a \in G} C(a)$. \square

Problem 6

If G is an abelian group and if $H = \{a \in G \mid a^2 = e\}$, show that H is a subgroup of G .

Proof. Let $a, b \in H$. Since $(ab)^2 = abab = a^2b^2 = e$, we get $ab \in H$, and thus H is closed under the operation of G . Since $a^2a^{-2} = e = (a^{-1})^2$, $a^{-1} \in H$ for all $a \in H$. Therefore, H is a subgroup of G . \square

Problem 7

Prove that a cyclic group is abelian.

Proof. Let $G = \langle a \rangle$ be a cyclic group. Let $b = a^k, c = a^j \in G$. Since $bc = a^{k+j} = cb$, G is abelian. \square

Problem 8

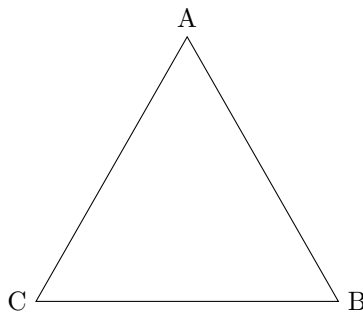
If A, B are subgroups of an abelian group G , let $AB = \{ab \mid a \in A, b \in B\}$. Prove that AB is a subgroup of G .

Proof. Let $c = a_1b_1, d = a_2b_2 \in AB$, for $a_1, a_2, b_1, b_2 \in G$. Since $a_1a_2 \in A$ and $b_1b_2 \in B$, we get $cd = a_1b_1a_2b_2 = (a_1a_2)(b_1b_2) \in AB$, and so AB is closed under the operation of G . Note that since A, B are subgroups of G , we know $a^{-1} \in A$ and $b^{-1} \in B$. Since $c^{-1} = (a_1b_1)^{-1} = b_1^{-1}a_1^{-1} = a^{-1}b^{-1}$, we get $c^{-1} \in AB$. Therefore, AB is a subgroup of G . \square

Problem 9

Give an example of a group G and two subgroups A, B of G such that AB is not a subgroup of G .

Proof. Consider D_3 , the group of symmetries of a triangle. Let F_1, F_2, F_3 be the flips through vertex A, B, C respectively.



We take subgroups $A = \langle F_1 \rangle$ and $B = \langle R \rangle$. Consider F_3 and R . Since $F_3 = F_1 R$ and $R = I R$, we know $F_3, R \in AB$. However, $F_2 = F_3 R \neq ab$ for all $a \in A, b \in B$. This implies that AB does not have the closed property, and thus it is not a subgroup of G . \square