Superimposed Extremal Graphs

Ray Tsai

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1 Introduction

In this note we talk about *superimposed graphs*. Given graph G with n vertices, let $G_1, \ldots, G_m$ be subgraphs of G. Let F be a graph with at least one edge. Our goal is to determine the maximum sum of the number of edges in each G_i, i.e. $\sum_{i=1}^m e(G_i)$, with the constraint of $E(G_i) \cap E(G_j)$ not including F for all distinct i, j.

2 Objectives

- Examine the case where $G_1, \ldots, G_m$ are induced
  - The case $F = K_3$.
  - Generalize to any F.
- Examine the non-induced case
  - If $F = K_3$, what happens when $m = 3$?

3 Induced Case

In this section, we assume that $G_1, \ldots, G_m$ are induced subgraphs of G.

3.1 Triangle-Free Case

Theorem 3.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j. For $m \geq 2$,*

$$\sum_{i=1}^m e(G_i) \leq m \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = \cdots = G_m = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

We claim that it suffices to show for the case $m = 2$. Suppose the theorem holds for $m = 2$. Put $G_{m+1} = G_1$ and we have

$$\sum_{i=1}^m e(G_i) = \frac{1}{2} \sum_{i=1}^m (e(G_i) + e(G_{i+1})) \leq \frac{1}{2} \sum_{i=1}^m 2 \left\lfloor \frac{n^2}{4} \right\rfloor = m \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality only if $G_i = G_{i+1} = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ for all i. That is, $G_1 = G_2 = \cdots = G_m = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

Proof for $m = 2$. Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both G_1 and G_2. Let $A = V(G_1)\backslash C$, and let $B = V(G_2)\backslash C$. For simplicity, put $a = |A|$, $b = |B|$, and $c = |C|$. We may assume that $a + b + c = n$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to a, b, c. Obviously, $e(G_1[A]) \leq \binom{a}{2}$ and $e(G_2[B]) \leq \binom{b}{2}$. There are at most ac edges in G_1 between A and C, and at most bc edges in G_2 between B and C. Now consider the edges in C. Since G_1, G_2 are induced graphs, we have $\{u, v\} \in E(G_1)$ if and only if $\{u, v\} \in E(G_2)$, for $u, v \in C$. This implies the subgraph of G_1 induced by C is identical to the subgraph of G_2 induced by C. In other words, $E(G_1[C]) = E(G_2[C]) = E(G_i) \cap E(G_j)$, which is triangle-free. By Mantel's Theorem, $e(G_1[C]) \leq \left\lfloor \frac{c^2}{4} \right\rfloor$, with equality if and only if $G_1[C] = K_{\lceil \frac{c}{2} \rceil, \lfloor \frac{c}{2} \rfloor}$. Hence,

$$e(G_1) + e(G_2) \leq \binom{a}{2} + \binom{b}{2} + ac + bc + 2\left\lfloor \frac{c^2}{4} \right\rfloor. \tag{1}$$

Define real differentiable function $f(x, y, z) = \binom{x}{2} + \binom{y}{2} + xz + yz + \frac{z^2}{2}$, with real variables $x, y, z \geq 0$. Given the constraints $x + y + z = n$, we apply the Lagrange multiplier and get

$$x + z - \frac{1}{2} = y + z - \frac{1}{2} = x + y + z = n.$$

Solving it yields $x = y = -\frac{1}{2}$ and $z = n + 1$, which is out of the boundary. Hence, there is no local maximum for $x, y, x > 0$. By compararing the boundary conditions, we conclude that f attains global maximum at $(0, 0, n)$. It now follows that

$$e(G_1) + e(G_2) \leq 2\left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. □

3.2 Generalization to any F