Ray Tsai
A16848188

Question 1. Let X be an exponential random variable with parameter $\lambda = 1/2$.

(a) Use Markov's inequality to find an upper bound for $\mathbb{P}(X > 6)$.

Solution. By Markov's inequality,

$$\mathbb{P}(X > 6) \leq \frac{1}{3}.$$

□

(b) Use Chebyshev's inequality to find an upper bound for $\mathbb{P}(X > 6)$.

Solution. We first note that $\mathbb{E}[X] = 2$, $\mathrm{Var}(X) = 4$, and $X > 0$. By Chebyshev's inequality,

$$\mathbb{P}(X > 6) \leq \mathbb{P}(|X - 2| > 4) \leq \frac{4}{16} = \frac{1}{4}.$$

□

(c) Explicitly compute $\mathbb{P}(X > 6)$ and compare it with the upper bounds you derived.

Solution.

$$\begin{aligned} \mathbb{P}(X > 6) &= 1 - \mathbb{P}(X \leq 6) \\ &= 1 - (1 - e^{-\frac{6}{2}}) \\ &= e^{-3} \approx \frac{1}{20}. \end{aligned}$$

Comparing with the upper bounds, $\frac{\frac{1}{20}}{\frac{1}{3}} = 15\%$, $\frac{\frac{1}{20}}{\frac{1}{4}} = 20\%$, both of which are not tight. □

Question 2. Suppose we roll a die 3600 times. Let X_i be the number showing on the ith roll. Let $S_n = X_1 + \cdots + X_n$. By the law of large numbers, we know that S_n/n will be close to 3.5. Approximate the probability that S_n/n differs from 3.5 by more than 0.05. Write a numerical answer or leave it in terms of Φ if you use a normal approximation.

Solution. Let $n = 3600$. We know that $\mathbb{E}[S_n/n] = 3.5$.

$$\mathrm{Var}(S_n/n) = \frac{1}{n^2}\mathrm{Var}(S_n) = \frac{1}{n}\mathrm{Var}(X_1) = \frac{35}{43200}.$$

Let $Z = \frac{S_n/n - 3.5}{\sqrt{\frac{35}{43200}}}$. By normal approximation,

$$\begin{aligned}\mathbb{P}(|S_n/n - 3.5| > 0.05) &= 2\mathbb{P}\left(Z < \frac{-0.05}{\sqrt{\frac{35}{43200}}}\right) \\ &\approx 2\Phi(-1.757).\end{aligned}$$

$\square$

Question 3. Let $X_1, \dots, X_{100}$ be i.i.d. exponential random variable with parameter $\lambda = 1$. Approximate

$$\mathbb{P}\left(\sum_{i=1}^{100} X_i > 90\right).$$

Solution. Let $Y = \sum_{i=1}^{100} X_i$. Then $\mathbb{E}[Y] = 100\mathbb{E}[X_1] = 100$, $\mathrm{Var}(Y) = 100\mathrm{Var}(X_1) = 100$. Let $Z = \frac{Y-100}{10}$.

$$\begin{aligned}\mathbb{P}(Y > 90) &= \mathbb{P}\left(Z > -1\right)\\ &= 1 - \Phi(-1).\end{aligned}$$

$\square$

Question 4. Suppose that the checkout time at the Art of Espresso has a mean of 5 minutes and a standard deviation of 2 minutes. Estimate the probability to serve at least 36 customers during a 3-hour and a-half shift.

Solution. Let T be the time spent on serving 36 customers. Then $\mathbb{E}[T] = 180$ and $\mathrm{Var}(T) = 144$. Let $Z = \frac{T-180}{\sqrt{144}}$ Thus,

$$\begin{aligned}
\mathbb{P}(T \leq 210) &= \mathbb{P}\left(Z \leq \frac{30}{\sqrt{144}}\right) \\
&= \Phi\left(\frac{5}{2}\right).
\end{aligned}$$

□

Question 5. Suppose the random variable X is positive and has moment generating function

$$M_X(t) = (1-2t)^{-3/2}.$$

(a) Use Markov's inequality to to bound $\mathbb{P}(X > 8)$.

Solution.

$$\begin{aligned}\mathbb{P}(X > 8) &\le \frac{\mathbb{E}[X]}{8} \\ &= \frac{M'_X(0)}{8} \\ &= \frac{3}{8}.\end{aligned}$$

□

(b) Use Chebyshev's inequality to to bound $\mathbb{P}(X > 8)$.

Solution. We note that $\mathbb{E}[X] = M'_X(0) = 3$.

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= M''_X(0) - M'_X(0)^2 \\ &= 15(1-0)^{-7/2} - 9(1-0)^{-5} \\ &= 6.\end{aligned}$$

$$\begin{aligned}\mathbb{P}(X > 8) &\le \mathbb{P}(|X - \mathbb{E}[X]| > 8 - \mathbb{E}[X]) \\ &\le \frac{\mathrm{Var}(X)}{(8-\mathbb{E}[X])^2} \\ &= \frac{6}{25}.\end{aligned}$$

□

Question 6. Every morning I take either bus number 5 or bus number 8 to work. Every morning the waiting time for the number 5 is exponential with mean 10 minutes, while the waiting time for the number 8 is exponential with a mean of 20 minutes. Assume all waiting times are independent of each other. Let S_n be the total amount of bus waiting (in minutes) that I have done during n mornings. Compute

$$\lim_{x\to\infty} \mathbb{P}(S_n \leq 7n).$$

Solution. Let A be the waiting time for bus number 5, B be the waiting time for bus number 8, and T be the bus waiting time in one morning. We note that $T = \min(A, B)$. The cumulative distribution function

$$\begin{aligned}
F_T(x) &= 1 - \mathbb{P}(T > x) \\
&= 1 - \mathbb{P}(A > x, B > x) \\
&= 1 - e^{-\frac{x}{10}} e^{-\frac{x}{20}} \\
&= 1 - e^{-\frac{3x}{20}}.
\end{aligned}$$

Therefore, $T \sim Exp(\frac{3}{20})$, and thus $\mathbb{E}[T] = \frac{20}{3}$, $\mathrm{Var}(T) = \frac{400}{9}$. Let $T_1, \ldots, T_n$ be random variables such that each of them is i.i.d with T. Since $S_n = \sum_{i=1}^n T_i$, $\mathbb{E}[S_n] = \frac{20}{3}n$ and $\mathrm{Var}(T) = \frac{400}{9}n$. Let $Z_n = \frac{S_n - \frac{20}{3}n}{\sqrt{\frac{400}{9}n}}$. By normal approximation,

$$\begin{aligned}
\mathbb{P}(S_n \leq 7n) &= \mathbb{P}\left(Z_n \leq \frac{7n - \frac{20}{3}n}{\sqrt{\frac{400}{9}n}}\right) \\
&= \mathbb{P}\left(Z_n \leq \frac{\sqrt{n}}{20}\right).
\end{aligned}$$

Therefore,

$$\lim_{x\to\infty} \mathbb{P}(S_n \leq 7n) = \lim_{x\to\infty} \mathbb{P}\left(Z_n \leq \frac{\sqrt{n}}{20}\right) \to 1.$$

□

Question 7. Let X be a continuous random variable with pdf $f(x) = \frac{5}{x^6}$ for $x \geq 1$ and 0 otherwise.

(a) Use Chebyshev's inequality to bound $\mathbb{P}(X \geq 2.5)$.

Solution. We first note that

$$\begin{aligned}
\mathbb{E}[X] &= \int_1^\infty \frac{5}{x^5} dx = \frac{5}{4}, \\
\mathbb{E}[X^2] &= \int_1^\infty \frac{5}{x^4} dx = \frac{5}{3}, \\
\mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \frac{5}{3} - \frac{25}{16} = \frac{5}{48}.
\end{aligned}$$

By Chebyshev's inequality

$$\begin{aligned}
\mathbb{P}(X \geq 2.5) &= \mathbb{P}\left(X - \frac{5}{4} \geq 2.5 - \frac{5}{4}\right) && (1)\\
&\leq \mathbb{P}\left(|X - \frac{5}{4}| \geq \frac{5}{4}\right) && (2)\\
&\leq \frac{\mathrm{Var}(X)}{\left(\frac{5}{4}\right)^2} && (3)\\
&= \frac{1}{15}. && (4)
\end{aligned}$$

$\square$

(b) For what value of a can we say that $\mathbb{P}(X \geq a) \leq 15\%$.

Solution. Suppose that $\mathbb{P}(X \geq a) \leq 15\%$. The cumulative distribution function of X is $F(x) = \int_{-\infty}^{x} \frac{5}{s^6} ds = -x^{-5} + 1$ for $x \geq 1$ and 0 otherwise. We note that $a \geq 1$. Thus, for $x \geq 1$,

$$\begin{aligned}
F(x) &= -a^{-5} + 1 \\
&= \mathbb{P}(X < a) > 85\%.
\end{aligned}$$

Thus, for $a \geq \left(\frac{20}{3}\right)^{\frac{1}{5}}$, $\mathbb{P}(X \geq a) \leq 15\%$. $\square$