

SC9 Probability on Graphs and Lattices: Sheet #2

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Problem 1

Prove that $p_c(1) = 1$.

Proof. Let $p \in [0, 1]$. If $0 \leftrightarrow \infty$, then for any $n \in \mathbb{Z}_{\geq 0}$ we have $0 \leftrightarrow n$ or $0 \leftrightarrow -n$. Notice

$$\mathbb{P}_p(0 \leftrightarrow n) = \mathbb{P}_p(0 \leftrightarrow -n) = p^n.$$

Thus,

$$\theta(p) = \lim_{n \rightarrow \infty} p^n = \begin{cases} 0 & \text{if } p < 1 \\ 1 & \text{if } p = 1 \end{cases}.$$

It now follows that $p_c(1) = \sup\{p \in [0, 1] : \theta(p) = 0\} = 1$. □

Problem 2

An event $A \subset \Omega_G$ is *decreasing* if whenever $\omega \leq \omega'$ and $\omega' \in A$, we also have $\omega \in A$. A function $f : \Omega_G \rightarrow \mathbb{R}$ is *decreasing* if $f(\omega) \geq f(\omega')$ whenever $\omega \leq \omega'$.

- (a) Fix a vertex $v \in V(G)$. Are the following events and variables/functions increasing, decreasing, or neither?

(i) $|C(v)|$

Proof. Increasing, as including more edges does not decrease the size of the largest cluster. \square

(ii) $\{\omega \text{ has no cycles}\}$

Proof. Decreasing, as removing edges would not create a cycle. \square

(iii) $\{v \text{ lies in the largest cluster or } |C(v)| = \infty\}$

Proof. Neither, as adding or removing edges might cause another cluster that does not contain v to become the largest. \square

(iv) $\{\omega \text{ forms a spanning tree of } G\}$

Proof. Neither. Adding edges might create a cycle, and removing edges might disconnect the component. \square

- (b) Prove that the Harris inequality holds for bounded decreasing functions $f, g : \Omega_G \rightarrow \mathbb{R}$, i.e.

$$\mathbb{E}_p[fg] \geq \mathbb{E}_p[f]\mathbb{E}_p[g].$$

Proof. Let $f' = -f$ and $g' = -g$. Then f' and g' are increasing, so by the Harris inequality for increasing functions,

$$\mathbb{E}_p[fg] = \mathbb{E}_p[f'g'] \geq \mathbb{E}_p[f']\mathbb{E}_p[g'] = \mathbb{E}_p[f]\mathbb{E}_p[g].$$

\square

- (c) Suppose f is a bounded increasing function, and A a decreasing event, with $\mathbb{P}_p(A) > 0$. Show that

$$\mathbb{E}_p[f|A] \leq \mathbb{E}_p[f].$$

Proof. Let $f' = -f$ and $g = \mathbb{1}_A$. Then f' and g are decreasing. By (b) we have

$$\mathbb{E}_p[f'g] \geq \mathbb{E}_p[f']\mathbb{P}_p(A).$$

Rearranging and replacing f' with $-f$, we have

$$\mathbb{E}_p[f|A] = \frac{\mathbb{E}_p[fg]}{\mathbb{P}_p(A)} \leq \mathbb{E}_p[f].$$

\square

Problem 3

Given $k \geq 2$ increasing events $A_1, \dots, A_k \subset \Omega_G$, show that

$$\max_{1 \leq i \leq k} \mathbb{P}_p(A_i) \geq 1 - (1 - \mathbb{P}_p(A_1 \cup \dots \cup A_k))^{1/k}.$$

Proof. By induction and the Harris inequality for decreasing events, we have

$$1 - \mathbb{P}_p(A_1 \cup \dots \cup A_k) = \mathbb{P}_p(A_1^C \cap \dots \cap A_k^C) \geq \mathbb{P}_p(A_1^C) \cdots \mathbb{P}_p(A_k^C) = \prod_{i=1}^k (1 - \mathbb{P}_p(A_i)) \geq \left(1 - \max_{1 \leq i \leq k} \mathbb{P}_p(A_i)\right)^k.$$

Rearranging now yields the result. \square

Problem 4

Let \mathbb{T}_∞ be the infinite rooted binary tree, with root ρ , which is exhausted by the sequence (\mathbb{T}_n) , where \mathbb{T}_n is the depth- n binary tree, consisting of all vertices in \mathbb{T}_∞ at graph distance at most n from ρ . In lectures, we showed that the FUSF on \mathbb{T}_∞ is \mathbb{T}_∞ itself.

- (a) Suppose that $(X_k)_{k \geq 0}$ is a SRW on \mathbb{T}_∞ . Let $D_k = d(\rho, X_k)$, where d denotes the graph distance. Describe the process $(D_k)_{k \geq 0}$, and find

$$\theta := \mathbb{P}(D_k > 1 \text{ for all } k \geq 1 \mid D_0 = 1).$$

What is $\mathbb{P}(D_k > m \text{ for all } k \geq 1 \mid D_0 = m)$ for $m > 1$?

Proof. Consider \mathbb{T}_∞ to be a directed graph, with edges pointing from parent to child. Let u be a neighbor of v . Notice that if u is a parent of v , then $d(\rho, u) - d(\rho, v) = -1$, and if u is a child of v , then $d(\rho, u) - d(\rho, v) = 1$. Since each non-root vertex has 1 parent and 2 children, if $X_{k-1} \neq \rho$

$$\mathbb{P}(D_k = D_{k-1} + 1) = \frac{2}{3}, \quad \mathbb{P}(D_k = D_{k-1} - 1) = \frac{1}{3},$$

where as $\mathbb{P}(D_k = D_{k-1} + 1) = 1$ if $D_{k-1} = 0$. For $i \in \mathbb{Z}_{\geq 0}$, define

$$\theta_i := \mathbb{P}(D_k > 1 \text{ for all } k \geq 1 \mid D_0 = i),$$

and note that $\theta_0 = 0$, $\lim_{i \rightarrow \infty} \theta_i = 1$, and

$$\theta_i = \frac{2}{3}\theta_{i+1} + \frac{1}{3}\theta_{i-1} \Rightarrow \theta_{i+1} - \theta_i = \frac{1}{2}(\theta_i - \theta_{i-1}) = \cdots = \frac{1}{2^i}(\theta_1 - \theta_0) = \frac{1}{2^i}\theta.$$

But then

$$\theta_i = \theta_i - \theta_{i-1} + \theta_{i-1} - \theta_{i-2} + \theta_{i-2} - \cdots - \theta_1 + \theta_1 = \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{i-1}}\right)\theta = \left(2 - \frac{1}{2^{i-1}}\right)\theta.$$

Thus,

$$\lim_{i \rightarrow \infty} \theta_i = 2\theta = 1 \Rightarrow \theta = \frac{1}{2}.$$

For $m > 1$, we have $\mathbb{P}(D_k > m \text{ for all } k \geq 1 \mid D_0 = m) = \mathbb{P}(D_k > 1 \text{ for all } k \geq 1 \mid D_0 = 1) = \frac{1}{2}$. \square

- (b) Now write \mathbb{T}_n^W for the wired version of \mathbb{T}_n , with wiring vertex w_n . Let one of the two neighbours of ρ be α . Prove that

$$\liminf_{n \rightarrow \infty} \mathbb{P}(\text{SRW on } \mathbb{T}_n^W \text{ started from } \rho \text{ hits } w_n \text{ before } \alpha) > 0. \quad (1)$$

Proof. Note that w_n are connected to all the leaves of \mathbb{T}_n^W . Let β be the other child of ρ . For $u, v \in V(\mathbb{T}_n^W)$, let $\tau_u(v)$ denote the hitting time of v by the SRW on \mathbb{T}_n^W started from u . Then we have

$$\mathbb{P}(\tau_\rho(w_n) < \tau_\rho(\alpha)) = \frac{1}{2} \cdot \mathbb{P}(\tau_\beta(w_n) < \tau_\beta(\alpha)).$$

Notice that if the SRW started from β , then to hit α it must go through ρ . Let A denote the event where SRW hits started from β hits w_n before α . Let R be the event where the SRW started from β hits ρ before w_n . Then

$$\begin{aligned} \mathbb{P}(\tau_\beta(w_n) < \tau_\beta(\alpha)) &= \mathbb{P}(\tau_\beta(w_n) < \tau_\beta(\alpha) \mid \tau_\beta(w_n) > \tau_\beta(\rho)) \cdot \mathbb{P}(\tau_\beta(w_n) > \tau_\beta(\rho)) \\ &\quad + \mathbb{P}(\tau_\beta(w_n) < \tau_\beta(\alpha) \mid \tau_\beta(w_n) < \tau_\beta(\rho)) \cdot \mathbb{P}(\tau_\beta(w_n) < \tau_\beta(\rho)). \end{aligned}$$

Note that

$$\mathbb{P}(\tau_\beta(w_n) < \tau_\beta(\alpha) | \tau_\beta(w_n) < \tau_\beta(\rho)) = 1 \quad \text{and} \quad \mathbb{P}(\tau_\beta(w_n) < \tau_\beta(\alpha) | \tau_\beta(w_n) < \tau_\beta(\rho)) = \mathbb{P}(\tau_\rho(w_n) < \tau_\rho(\alpha)).$$

Denote $\mathbb{P}(D_k > 1 \text{ for all } n \geq k \geq 1 \mid D_0 = i) = \theta^{(n)}$ and we have

$$\mathbb{P}(\tau_\beta(w_n) < \tau_\beta(\rho)) = \theta^{(n)},$$

and so $\mathbb{P}(\tau_\beta(w_n) > \tau_\beta(\rho)) = 1 - \mathbb{P}(\tau_\beta(w_n) < \tau_\beta(\rho)) = 1 - \theta^{(n)}$. It now follows that

$$\mathbb{P}(\tau_\beta(w_n) < \tau_\beta(\alpha)) = 1 - \theta^{(n)} + \mathbb{P}(\tau_\rho(w_n) < \tau_\rho(\alpha)) \cdot \theta^{(n)}$$

and so

$$\mathbb{P}(\tau_\rho(w_n) < \tau_\rho(\alpha)) = \frac{1 - \theta^{(n)}}{2} + \frac{1}{2} \cdot \mathbb{P}(\tau_\rho(w_n) < \tau_\rho(\alpha)) \cdot \theta^{(n)}.$$

By (a) we have $\lim_{n \rightarrow \infty} \theta^{(n)} = \frac{1}{2}$, and so

$$\lim_{n \rightarrow \infty} \mathbb{P}(\tau_\rho(w_n) < \tau_\rho(\alpha)) = \frac{1}{4} + \frac{1}{4} \cdot \lim_{n \rightarrow \infty} \mathbb{P}(\tau_\rho(w_n) < \tau_\rho(\alpha)) \Rightarrow \lim_{n \rightarrow \infty} \mathbb{P}(\tau_\rho(w_n) < \tau_\rho(\alpha)) = \frac{1}{3} > 0.$$

This completes the proof. \square

- (c) By considering also a version of (1) for a SRW started from α , and using Wilson's algorithm on \mathbb{T}_n^W with initial vertex w_n (or otherwise), prove that the edge $\{\rho, \alpha\}$ is not μ^W -almost surely present, and thus $\text{FUSF} \neq \text{WUSF}$ for \mathbb{T}_∞ .

Proof. By (a) we have

$$\liminf_{n \rightarrow \infty} \mathbb{P}(\text{SRW on } \mathbb{T}_n^W \text{ started from } \alpha \text{ hits } w_n \text{ before } \rho) = \theta = \frac{1}{2}.$$

Now consider running Wilson's algorithm on \mathbb{T}_n^W with an enumeration of the vertices w_n, ρ, α, \dots . Then the probability that the edge $\{\rho, \alpha\}$ is not present is

$$\mathbb{P}(\text{SRW on } \mathbb{T}_n^W \text{ started from } \rho \text{ hits } w_n \text{ before } \alpha) \cdot \mathbb{P}(\text{SRW on } \mathbb{T}_n^W \text{ started from } \alpha \text{ hits } w_n \text{ before } \rho).$$

By the previous results, taking the limit as $n \rightarrow \infty$ yields

$$\lim_{n \rightarrow \infty} \mathbb{P}(\{\rho, \alpha\} \notin E(\mathbb{T}_n^W)) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} > 0.$$

The result now follows. \square

Problem 5

Let σ_n be the number of self-avoiding walks started from the origin in \mathbb{Z}^d with length n .

- (a) Prove that $\sigma_n \geq d^n$.

Proof. Consider any simple walks that only move in the positive direction. Since there are no cycles in such walks, they are self-avoiding walks. There are d^n such walks, and so $\sigma_n \geq d^n$. \square

- (b) Explain why $\sigma_{m+n} \leq \sigma_m \sigma_n$.

Proof. Consider the set of all concatenations of two self-avoiding walks of length m and n . Then the set of self-avoiding walks of length $m+n$ is a subset of this set, and so $\sigma_{m+n} \leq \sigma_m \sigma_n$. \square

- (c) (Fekete's lemma) Let (x_n) be a real sequence satisfying the subadditive property:

$$x_{n+m} \leq x_n + x_m \quad \text{for all } n, m \geq 1.$$

Prove that $\lim_{n \rightarrow \infty} \frac{x_n}{n} \in [-\infty, \infty)$ exists.

Proof. Let $A = \inf_n \frac{x_n}{n}$. By definition, $\liminf_{n \rightarrow \infty} \frac{x_n}{n} \geq A$, so it suffices to show $\limsup_{n \rightarrow \infty} \frac{x_n}{n} \leq A$. Fix $\epsilon > 0$. Let (x_{n_k}) be a convergent subsequence of (x_n) . Pick m such that $x_m/m < A + \epsilon$. Then for $n_k \geq m$, we have $n_k = qm + r$ where $q, r \in \mathbb{N}$ and $0 \leq r < m$. By the subadditive property,

$$\frac{x_{n_k}}{n_k} \leq \frac{qx_m + x_r}{n_k} < \frac{qm(A + \epsilon) + x_r}{n_k} \leq A + \epsilon + \frac{x_r}{n_k}.$$

But then

$$\lim_{k \rightarrow \infty} \frac{x_{n_k}}{n_k} \leq A + \epsilon.$$

Since ϵ is arbitrary, $\limsup_{n \rightarrow \infty} \frac{x_n}{n} \leq A$. This completes the proof. \square

- (d) Hence, or otherwise, prove that there exists $\kappa \in [d, 2d - 1]$ such that for all $\epsilon > 0$,

$$(\kappa - \epsilon)^n \leq \sigma_n \leq (\kappa + \epsilon)^n, \quad \text{for large enough } n.$$

(κ is known as the *connectivity constant* of \mathbb{Z}^d .)

Proof. Consider the sequence (x_n) where $x_n = \frac{\log \sigma_n}{n}$. By (b) we have

$$\frac{\log \sigma_{m+n}}{m+n} \leq \frac{\log \sigma_m \sigma_n}{m+n} = \frac{\log \sigma_m}{m+n} + \frac{\log \sigma_n}{m+n} \leq \frac{\log \sigma_m}{m} + \frac{\log \sigma_n}{n},$$

so (x_n) is subadditive. By (c) we have that $L = \lim_{n \rightarrow \infty} \frac{x_n}{n} \in [-\infty, \infty)$ exists. Thus,

$$\lim_{n \rightarrow \infty} \sigma_n^{\frac{1}{n}} = e^L.$$

Put $\kappa = e^L$. Since $d^n \leq \sigma_n \leq (2d - 1)^n$, we have $\kappa \in [d, 2d - 1]$. \square

Problem 6

Consider the one-dimensional slab $S_n := \mathbb{Z} \times \{1, 2, \dots, n\}$ for any $n \in \mathbb{N}$, considered as a subgraph of the lattice \mathbb{Z}^2 . Show that the critical probability p_c for percolation on S_n is 1.

Hint: you may wish to argue similarly to Question 1.

Proof. Let $p \in [0, 1]$. Let $A_k = \{(k, x) \in S_n : x \in [n]\}$. Let $\{0 \leftrightarrow A_k\} = \bigcup_{v \in A_k} \{0 \leftrightarrow v\}$ and note that $\mathbb{P}(0 \leftrightarrow A_k) = \mathbb{P}(0 \leftrightarrow A_{-k})$. If $0 \leftrightarrow \infty$, then for any $k \in \mathbb{Z}_{\geq 0}$ we have $0 \leftrightarrow A_k$ or $0 \leftrightarrow A_{-k}$. Notice that $0 \leftrightarrow A_k$ implies that for $1 \leq k \leq n$, there is an edge between A_k and A_{k-1} . Put $q = 1 - (1 - p)^n$. We then have,

$$\mathbb{P}(0 \leftrightarrow A_k) \leq q^k.$$

Then

$$\lim_{k \rightarrow \infty} \mathbb{P}(0 \leftrightarrow A_k) \leq \lim_{k \rightarrow \infty} q^k = \begin{cases} 0 & \text{if } p < 1 \\ 1 & \text{if } p = 1 \end{cases}.$$

□

Problem 7

Suppose that \mathbb{T}_d is an infinite rooted tree in which each vertex has d children (so that all vertices except the root ρ have degree $d+1$). Suppose each edge of the tree is open independently with probability p . As usual, let

$$p_c := \sup\{p : \mathbb{P}_p(\rho \leftrightarrow \infty) = 0\}.$$

- (a) Show that $p_c = 1/d$, and that $\mathbb{P}_{p_c}(\rho \leftrightarrow \infty) = 0$.

Hint: recall basic results about branching processes.

Proof. Note that we may view the component of ρ as a result of the branching process with offspring distribution $X \sim \text{Binomial}(d, p)$, where the level of the tree represents the generation of the process. Then the event $\{\rho \leftrightarrow \infty\}$ is the event that the branching process does not extinct. The expected number of offspring for each layer is $\mu = dp$. By the results about branching processes, the branching process will extinct if and only if $\mu \leq 1$. That is,

$$p_c = \sup\{p : \mu \leq 1\} = \frac{1}{d},$$

and that $\mathbb{P}_{p_c}(\rho \leftrightarrow \infty) = 0$. □

- (b) Prove that for $p \leq p_c$,

$$\mathbb{P}_p(\exists v \in V(\mathbb{T}_d) : v \leftrightarrow \infty) = 0.$$

Note that, unlike the case \mathbb{Z}^d treated in the lemma in lectures, \mathbb{T}_d is not vertex-transitive.

Proof. Suppose not. There exists $v \in V(\mathbb{T}_d)$ such that $\mathbb{P}_p(v \leftrightarrow \infty) > 0$. But then

$$\mathbb{P}_p(\rho \leftrightarrow \infty) \geq \mathbb{P}_p(\rho \leftrightarrow v) \cdot \mathbb{P}_p(v \leftrightarrow \infty) > 0,$$

contradiction. □

- (c) Prove that for $p \in (p_c, 1)$,

$$\mathbb{P}_p(\exists v \in V(\mathbb{T}_d) : v \leftrightarrow \infty) = 1,$$

directly, without appeal to Kolmogorov's 0–1 law.

Proof. Let V_n be the set of vertices on layer n of the tree, and note that $|V_n| = d^n$. Let $\{V_n \leftrightarrow \infty\} = \bigcup_{v \in V_n} \{v \leftrightarrow \infty\}$. Since for any $v, v' \in V_n$, the events $\{v \leftrightarrow \infty\}$ and $\{v' \leftrightarrow \infty\}$ are independent and have the same probability,

$$\mathbb{P}(V_n \leftrightarrow \infty) = 1 - \prod_{v \in V_n} (1 - \mathbb{P}_p(v \leftrightarrow \infty)) = 1 - (1 - \mathbb{P}_p(v \leftrightarrow \infty))^{d^n}.$$

Since $p > p_c$, we must have $\mathbb{P}_p(v \leftrightarrow \infty) > 0$ for some $v \in V_n$. But then for any $\epsilon > 0$, there exists large enough n such that $\mathbb{P}_p(\exists v \in V(\mathbb{T}_d) : v \leftrightarrow \infty) \geq \mathbb{P}(V_n \leftrightarrow \infty) > 1 - \epsilon$. This completes the proof. □

- (d) How many infinite clusters are there for $p \in (p_c, 1)$?

Proof. For $v \in V(\mathbb{T}_d)$, let $\mathbb{T}_d^{(v)}$ be the subtree of \mathbb{T}_d rooted at v . By (c), there exists some vertex v_0 such that $v_0 \leftrightarrow \infty$ almost surely. Consider $\mathbb{T}_d^{(v_0)}$. Since $p < 1$, there exists some vertex $v'_0 \in V(\mathbb{T}_d^{(v_0)})$ such that $v_0 \not\leftrightarrow v'_0$ almost surely. By (c), there exists some vertex $v_1 \in V(\mathbb{T}_d^{(v'_0)})$ such that $v_1 \leftrightarrow \infty$ almost surely, and the infinite cluster containing v_1 is disjoint from the infinite cluster containing v_0 . Recursively repeating this process shows that there are infinitely many infinite clusters. □

Problem 8

For any $d \geq 2$, the graph $\mathbb{Z}^d \oplus \mathbb{Z}^d$ is constructed by taking two disjoint copies of \mathbb{Z}^d , and for each labelled vertex $v \in \mathbb{Z}^d$, adding an edge between the two vertices with this label.

For $\mathbb{Z}^2 \oplus \mathbb{Z}^2$, prove that

$$1 - \frac{1}{\sqrt{2}} \leq p_c \leq \frac{1}{2}.$$

Hint: you may find it helpful to observe that $p = 1 - \frac{1}{\sqrt{2}}$ satisfies $2p - p^2 = \frac{1}{2}$.

Proof. By Kesten's theorem, it is obvious that $p_c \leq \frac{1}{2}$, otherwise there exists an infinite cluster in \mathbb{Z}^2 which is also an infinite cluster in $\mathbb{Z}^2 \oplus \mathbb{Z}^2$.

Now suppose $p < 1 - 1/\sqrt{2}$. Consider the dual graph L' of \mathbb{Z}^2 , and define an edge $e \in E(L')$ as open if and only if the corresponding primal edges in at least one of the copies of \mathbb{Z}^2 are open. Then $e \in E(L')$ is open with probability $1 - (1 - p)^2 = 2p - p^2$. Suppose $2p - p^2 < 1/2$. By Kesten's theorem there does not exist an infinite cluster in L' . But then there are no infinite clusters in either copy of \mathbb{Z}^2 . It now follows that $p_c \geq 1 - 1/\sqrt{2}$. \square

Problem 9

Let $\Lambda(n) = [-n, n]^2 \cap \mathbb{Z}^2$. Using the BK inequality, show that, for \mathbb{Z}^2 ,

$$\mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda(n)) \geq \frac{1}{2\sqrt{n}}. \quad (2)$$

Hint: consider horizontal crossings of the box $[-n, n] \times [-n, n-1]$ and think about where such a crossing meets the y -axis in order to split it in two. Then relate the probabilities of each of those smaller crossings to $\mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda(n))$.

Proof. Consider the rectangle $J_n := ([-n, n] \cap \mathbb{Z}) \times ([-n, n-1] \times \mathbb{Z})$. Let L_n, R_n be the left and right boundaries of J_n . Let J'_n be the dual graph of J_n , and denote T_n, B_n the top and bottom boundaries of J'_n . Define $H_n := \{L_n \leftrightarrow R_n\}$ and $V_n := \{T_n \leftrightarrow B_n\}$. Also, let $Y = \{0\} \times ([-n, n-1] \cap \mathbb{Z})$.

Note that if $L_n \leftrightarrow R_n$, then J' must not contain a vertical crossing. On the other hand, if $L_n \not\leftrightarrow R_n$, then there must be a vertical crossing in J' . Thus $\mathbb{P}(H_n) + \mathbb{P}(V_n) = 1$. But then J and J' are isomorphic, and so by symmetry $\mathbb{P}_{1/2}(H_n) = \mathbb{P}_{1/2}(V_n) = \frac{1}{2}$.

Now note that by BK inequality,

$$\mathbb{P}_{1/2}(H_n) = \mathbb{P}_{1/2} \left(\bigcup_{y \in Y} \{L_n \leftrightarrow y\} \circ \{y \leftrightarrow R_n\} \right) \leq \sum_{y \in Y} \mathbb{P}_{1/2}(L_n \leftrightarrow y) \cdot \mathbb{P}_{1/2}(y \leftrightarrow R_n) \leq 2n \cdot \mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda(n))^2.$$

The result now follows from rearranging the above inequality. \square