

MATH 190A: Homework #0

Due on Jan 7, 2025 at 12:00pm

Professor McKernan

Section A02 8:00AM - 8:50AM

Section Leader: Zhiyuan Jiang

Source Consulted: Textbook, Lecture, Discussion

Ray Tsai

A16848188

Problem 1

This problem illustrates how variants of the usual switching arguments can sometimes lead to fairly simple proofs (if one is content with bounds instead of asymptotics). Let \mathcal{N}_i denote the set of permutations of $[n]$ with exactly i fixed points. Picking a random permutation $\pi_n \in S_n$ uniformly at random, below we give a simple and robust proof of the non-trivial bound

$$\mathbb{P}(\pi_n \text{ has no fixed points}) \geq \frac{1}{3} - o(1) \quad \text{as } n \rightarrow \infty$$

- (a) Using a basic switching argument, show that $|\mathcal{N}_1|/|\mathcal{N}_0| \leq 1 + O(1/n)$.

Proof. Define switching operation $\phi : \mathcal{N}_1 \rightarrow 2^{\mathcal{N}_0}$ by sending $\pi \in \mathcal{N}_1$ to the set of all possible permutations π' obtained by switching the fixed point k with any other element in $[n] \setminus \{k\}$.

Forward Switching:

Let $\pi \in \mathcal{N}_1$ with $\pi(k) = k$. Notice if permutation π' is defined by

$$\pi'(i) = \begin{cases} \pi(j) & \text{if } i = k \\ \pi(k) = k & \text{if } i = j \\ \pi(i) & \text{otherwise} \end{cases}$$

for some $j \neq k$, then $\pi' \in \mathcal{N}_0$. Since there are $n - 1$ choices for j , we have $|\phi(\pi)| = n - 1$.

Reverse Switching:

Number of ways $\pi' \in \mathcal{N}_0$ can be obtained by switching some $\pi \in \mathcal{N}_1$ is

$$|\phi^{-1}(\pi')| = (\text{\#choices for fixed point } k)(\text{\#valid points in } \pi' \text{ to swap with } k)$$

There are n choices for k . Notice that we need to swap k with $\pi'(k)$ to possibly obtain a permutation with k as its only fixed point. If $\pi'(k) = j$ and $\pi'(j) = k$, then swapping k with j will yield a permutation with both k and j as fixed points. Hence, the number of valid points to swap with k is ≤ 1 , and thus $|\phi^{-1}(\pi')| \leq n$.

Double Counting:

$$\sum_{\pi \in \mathcal{N}_1} |\phi(\pi)| = \sum_{\pi' \in \mathcal{N}_0} |\phi^{-1}(\pi')| \implies |\mathcal{N}_1|/|\mathcal{N}_0| \leq \frac{n-1}{n} = 1 + O(1/n).$$

□

- (b) Using a simple counting argument, show that in a random permutation, the expected number of fixed points is equal to one.

Proof. Let X be the number of fixed points in a random permutation $\pi_n \in S_n$, and let X_i be the indicator for the event that $\pi_n(i) = i$. Then,

$$\mathbb{E}_{\pi_n}[X] = \sum_{i=1}^n \mathbb{E}_{\pi_n}[X_i] = \sum_{i=1}^n \mathbb{P}(i \text{ is a fixed point}) = \sum_{i=1}^n \frac{\#\pi \in S_n \text{ with } \pi(i) = i}{|S_n|} = \sum_{i=1}^n \frac{1}{n} = 1.$$

□

(c) By combining these estimates, conclude that

$$\mathbb{P}(\pi_n \text{ has no fixed points}) = \frac{|\mathcal{N}_0|}{n!} \geq \frac{1}{3} - O(1/n).$$

Proof. Since $\sum_{i=0}^n \frac{|\mathcal{N}_i|}{n!} = 1$, combining (b) we have

$$2 \left(1 - \frac{|\mathcal{N}_0|}{n!} - \frac{|\mathcal{N}_1|}{n!} \right) = 2 \sum_{i=2}^n \frac{|\mathcal{N}_i|}{n!} \leq \sum_{i=2}^n i \cdot \frac{|\mathcal{N}_i|}{n!} = \mathbb{E}_{\pi_n}[X] - \frac{|\mathcal{N}_1|}{n!} = 1 - \frac{|\mathcal{N}_1|}{n!}.$$

Rearranging yields

$$2 \cdot \frac{|\mathcal{N}_0|}{n!} \geq 1 - \frac{|\mathcal{N}_1|}{n!}$$

It now follows $|\mathcal{N}_1| \leq (1 + O(1/n))|\mathcal{N}_0|$ from (a) that

$$2 \cdot \frac{|\mathcal{N}_0|}{n!} \geq 1 - (1 + O(1/n)) \frac{|\mathcal{N}_0|}{n!} \implies \frac{|\mathcal{N}_0|}{n!} \geq \frac{1}{3} - O(1/n).$$

□