CSE 105: Homework #2

Due on Feb 1, 2023 at 23:59pm

Professor Minnes

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Problem 1

Integers can be represented using base b expansions, for a convenient choice of base b: for b an integer greater than 1 and n a positive integer, the **base b expansion of n** is defined to be

$$(a_{k-1} \cdots a_1 a_0)_b$$

where k is a positive integer, $a_0, a_1, \ldots, a_{k-1}$ are nonnegative integers less than b, $a_{k-1} \neq 0$, and

$$n = \sum_{i=0}^{k-1} a_i b^i$$

Notice: *The base b expansion of a positive integer n is a string over the alphabet $\{x \in \mathbb{Z} \mid 0 \leq x < b\}$ whose leftmost character is nonzero.*

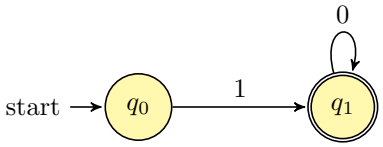
An important property of base b expansions of integers is that, for each integer b greater than 1, each positive integer $n = (a_{k-1} \cdots a_1 a_0)_b$, and each nonnegative integer a less than b,

$$bn + a = (a_{k-1} \cdots a_1 a_0 a)_b$$

In other words, shifting the base b expansion to the left results in multiplying the integer value by the base. In this question we'll explore building deterministic finite automata that recognize languages that correspond to useful sets of integers.

(a) Design a DFA that recognizes the set of binary (base 2) expansions of positive integers that are powers of 2. A complete solution will include the state diagram of your DFA and a brief justification of your construction by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

Solution. Consider the following state diagram of a DFA:



The base 2 representations of all powers of two only have the first digit as 1 and 0 for the rest, 10^* in regular expression, and this the exact language that the above DFA accepts. □

(b) Consider arbitrary positive integer m. Design a DFA that recognizes the set of binary (base 2) expansions of positive integers that are multiples of m. A complete solution will include the formal definition of your DFA (paramterized by m) and a brief justification of your construction by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

Solution. Consider a DFA $(\mathbb{Z}_m, \Sigma = \{0, 1\}, \delta, [0], \{[0]\})$, where the state $[i]$ represents the i-th equivalence class in $\mathbb{Z}_m$ and $\delta : \mathbb{Z}_m \times \Sigma \to \mathbb{Z}_m$ is the transition funtion that maps $([i], \sigma)$ to $[2i + \sigma]$. The states of the DFA records the current remainder of the binary string read. Since the current remainder needs to be multiplied by 2 every times we read a new digit, δ updates the current remainder and add the value of the new digit accordingly. Thus, a string is accepted if and only if the remainder is 0 after the string is fully processed. □

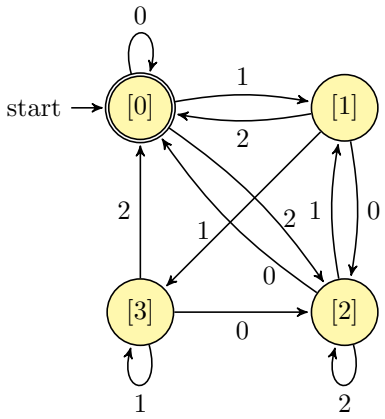
Problem 1 continued on next page...

(c) Choose a positive integer m_0 between 4 and 8 (inclusive) and draw the state diagram of a DFA recognizing the language over $\{0, 1, 2\}$

$$\{w \in \{0,1,2\}^* \mid w \text{ is a base 3 expansion of a positive integer that is a multiple of } m_0\}$$

A complete solution will include the state diagram of your DFA and a brief justification of your construction by explaining the role each state plays in the machine, as well as a brief justification about how the strings accepted and rejected by the machine connect to the specified language.

Solution. Let $m_0 = 4$ and consider the following state diagram of a DFA:



Each state represents an equivalence class in $\mathbb{Z}_4$, and they all together play the role of recording the current remainder of the string, upon dividing by 4. Note that upon reading a new digit σ, state $[i]$ would be mapped to state $[2i + \sigma]$, and a string would get accepted if and only if it ends at state $[0]$ after being copletely processed. In other words, a string would get accepted if and only if it ends it is a multiple of 4. □

Problem 2

For any language $L \subseteq \Sigma^*$, recall that we define its *complement* as

$$\overline{L} := \Sigma^* - L = \{w \in \Sigma^* \mid w \notin L\}$$

That is, the complement of L contains all and only those strings which are not in L. Our notation for regular expressions does not include the complement symbol. However, it turns out that the complement of a language described by a regular expression is guaranteed to also be describable by a (different) regular expression. For example, over the alphabet $\Sigma = \{0, 1\}$, the complement of the language described by the regular expression Σ^*0 is described by the regular expression $\varepsilon \cup \Sigma^*1$ because any string that does not end in 0 must either be the empty string or end in 1.

For each of the regular expressions R over the alphabet $\Sigma = \{a, b\}$ below, write the regular expression for $\overline{L(R)}$. Your regular expressions may use the symbols $\varnothing$, ε, a, b, and the following operations to combine them: union, concatenation, and Kleene star.

Briefly justify why your solution for each part works by giving plain English descriptions of the language described by the regular expression and of its complement and connecting them to the regular expression via relevant definitions. An English description that is more detailed than simply negating the description in the original language will likely be helpful in the justification.

Alternatively, you can justify your solution by first designing a DFA that recognizes $L(R)$, using the construction from class and the book to modify this DFA to get a new DFA that recognizes $\overline{L(R)}$, and then applying the constructions from class and the book to convert this new DFA to a regular expression.

For each part of the question, clearly state which approach you're taking and include enough intermediate steps to illustrate your work.

(a) a^*b^*

Solution. Consider $(a \cup b)^*ba(a \cup b)^*$. Since $L(a^*b^*)$ is the set of strings that has no b preceding a, $\overline{L(a^*b^*)}$ is the set of strings that has at least a b preceding an a, which much contain at least a substring ba. Thus, $(a \cup b)^*ba(a \cup b)^*$ describes the set. □

(b) $(a \cup b)ab^*$

Proof. Consider a disturbingly straight forward example $\epsilon \cup (a \cup b) \cup (a \cup b)b(a \cup b)^* \cup (a \cup b)(a \cup b)^+a(a \cup b)^*$. Since the negation of $(a \cup b)ab^*$ is a string that has length less than 2 or doesn't have a as its second character or contains at least an a after the second character, we take the union of all conditions and obtain the resulting regular expression. □

Problem 3

In this question, you'll practice working with formal general constructions for DFAs and translating between state diagrams and formal definitions. Consider the following general construction: Let $N_1 = (Q, \Sigma, \delta_1, q_1, F_1)$ be a NFA and assume that $q_0 \notin Q$. Define the new NFA $N_2 = (Q \cup \{q_0\}, \Sigma, \delta_2, q_0, \{q_1\})$ where

$$\delta_2 : (Q \cup \{q_0\}) \times \Sigma_\varepsilon \to (Q \cup \{q_0\})$$

is defined by

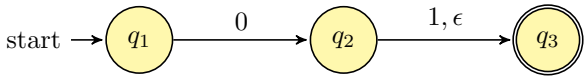
$$\delta_2(q, a) = \begin{cases} \{q' \in Q \mid q \in \delta_1(q', a)\} & \text{if } q \in Q,\ a \in \Sigma_\varepsilon \\ F_1 & \text{if } q = q_0,\ a = \varepsilon \\ \emptyset & \text{if } q = q_0,\ a \in \Sigma \end{cases}$$

(a) Illustrate this construction by defining a specific example NFA N and applying the construction above to create a new NFA. Your example NFA should

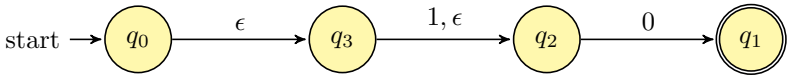
- Have exactly three states (all reachable from the start state),
- Have at least one spontaneous move (arrow labelled ε),
- Accept at least one string and reject at least one string, and
- Not have any states labelled q_0.

Apply the construction above to create the new NFA. A complete submission will include the state diagram of your example NFA N and the state diagram of the NFA resulting from this construction.

Solution. Consider the following state diagram of NFA N:



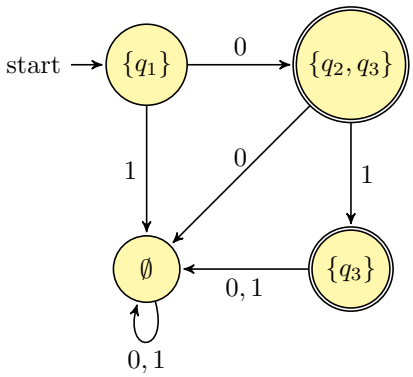
Note that N accepts 0 and rejects 1. The state diagram of the NFA resulting from this construction upon N is:



□

(b) Use Theorem 1.39 on page 55 of the book (see also page 7 in Week 3 notes) to construct a DFA equivalent to your example NFA N from part (a). A complete submission will include the state diagram of your example NFA N and the state diagram of the DFA resulting from this construction, with the correct state labels for this DFA. You may prune the DFA so that only the "macro-states" reachable from the start state are included.

Proof. The DFA equivalent of N from part (a) is



□

(c) Explain the relationship between N_1 and N_2 in the general construction. Give an example string that is accepted by your example NFA N and is rejected by the NFA that results from applying the general construction that illustrates this relationship, or explain why there is no such example string.

Proof. N_2 accepts the reverse of the strings accepted by N_1. Formally, $L(N_2) = \{w \in \Sigma^* \mid w^{\mathcal{R}} \in L(N_1)\}$. For example, 01 is accepted by N but not by the NFA that results from applying the general construction. □

Problem 4

(a) Give an example of a language over the alphabet $\{0, 1\}$ that has cardinality 3 and for which 5 is a pumping length and 4 is not a pumping length. A complete solution will give a clear and precise description of the language, a justification for why 5 is a pumping length, and a justification for why 4 is not a pumping length. Is this language regular?

Solution. Consider the set $L = \{0, 00, 0000\}$. Since L doesn't contain any string of length at least 5, 5 is a valid pumping length. However, 4 is not a valid pumping length, as 0000 cannot be pumped. Since L is of a finite cardinality, there exists a DFA that accepts L, and thus L is regular. □

(b) Consider the following attempted "proof" that the set

$$X = \{uw \mid u \text{ and } w \text{ are strings over } \{0, 1\} \text{ and have the same length}\}$$

is nonregular.

> "Proof" that X_1 is not regular using the Pumping Lemma: Let p be an arbitrary positive integer. We will show that p is not a pumping length for X_1.
>
> Choose s to be the string 1^p0^p, which is in X_1 because we can choose $u = 1^p$ and $w = 0^p$ which each have length p. Since s is in X_1 and has length greater than or equal to p, if p were to be a pumping length for X_1, s ought to be pump'able. That is, there should be a way of dividing s into parts x, y, z where $s = xyz$, $|y| > 0$, $|xy| \leq p$, and for each $i \geq 0$, $xy^iz \in X_1$. Suppose x, y, z are such that $s = xyz$, $|y| > 0$ and $|xy| \leq p$. Since the first p letters of s are all 1 and $|xy| \leq p$, we know that x and y are made up of all 1s. If we let $i = 2$, we get a string xy^iz that is not in X_1 because repeating y twice adds 1s to u but not to w, and strings in X_1 are required to have u and w be the same length. Thus, s is not pumpable (even though it should have been if p were to be a pumping length) and so p is not a pumping length for X_1. Since p was arbitrary, we have demonstrated that X_1 has no pumping length. By the Pumping Lemma, this implies that X_1 is nonregular.

Find the (first and/or most significant) logical error in the "proof" and describe why it's wrong. Then, either prove that the set is actually regular (by finding a regular expression that describes it or a DFA/NFA that recognizes it, and justifying why) **or** fix the proof so that it is logically sound.

Solution. The statement "if we let $i = 2$, we get a string xy^iz that is not in X_1 because repeating y twice adds 1s to u but not to w, and strings in X_1 are required to have u and w be the same length," is incorrect, as u and w are not fixed. Since the modified string remains to have even length, we may pick u to be the first half of the new string and w to be the second half. Since $u, w \in \{0, 1\}^*$ and of the same length, the new string is in X.

To show that X is regular, consider the regular expression $((0 \cup 1)^2)^*$. Since this regular expression describes all strings s of even length, we may choose u to be the first half of the string and w to be the second half. Since u, w are both strings over $\{0, 1\}$ and are of the same length, $s \in X$. Conversely, suppose that s not encompassed by the regular expression. Then, s is of odd length, so there does not exists u, w that are of the same length, and thus $s \notin X$. □

Problem 4 continued on next page...

(c) In class and in the reading so far, we've seen the following examples of nonregular sets:

$$\{0^n1^n \mid n \geq 0\} \qquad \{0^n1^m \mid 0 \leq m \leq n\} \qquad \{0^n1^m0^n \mid n, m \geq 0\}$$
$$\{0^n1^n \mid n \geq 2\} \qquad \{0^i1^{2i} \mid 0 \leq i\} \qquad \{w \in \{0,1\}^* \mid w = w^R\}$$
$$\{0^n1^m \mid 0 \leq n \leq m\} \qquad \{0^i1^{i+1} \mid 0 \leq i\} \qquad \{ww^R \mid w \in \{0,1\}^*\}$$

Modify one of these sets in some way and use the Pumping Lemma to prove that the resulting set is still nonregular.

Proof. Consider $L = \{1^n0^n \mid n \geq 0\}$. Suppose for the sake of contradiction that L is regular. Let p be the pumping length given by the pumping lemma. Consider $s = 1^p0^p$. Since $|s| \geq p$, s is pumpable. By the pumping lemma, we may split s into three parts xyz, where $|xy| \leq p$. Since the first p characters are all 1's, y must consist of all 1's. However, if $i = 2$, we add positive numbers of 1 into the string but no 0, so the numbers of 1 and 0 in the string are not equal, contradiction. Therefore, L is non-regular. □

Problem 5

In Week 2's review quiz, we saw the definition that a set X is said to be **closed under an operation** if, for any elements in X, applying to them gives an element in X. For example, the set of integers is closed under multiplication because if we take any two integers, their product is also an integer.

Prove or disprove each closure claim statement below about the class of regular languages and the class of nonregular languages. Your arguments may refer to theorems proved in the textbook and class, and if they do, should include specific page numbers and references (i.e. write out the claim that was proved in the book and/or class).

Recall the definitions we have:

For languages L_1, L_2 over the alphabet $\Sigma_1 = \{0, 1\}$, we have the associated sets of strings

$$SUBSTRING(L_1) = \{w \in \Sigma_1^* \mid \text{there exist } a, b \in \Sigma_1^* \text{ such that } awb \in L_1\}$$

and

$$L_1 \circ L_2 = \{w \in \Sigma_1^* \mid w = uv \text{ for some strings } u \in L_1 \text{ and } v \in L_2\}$$

1. The set of regular languages over $\{0, 1\}$ is closed under set-wise concatenation.

   Proof. Theorem 1.47 on page 60 of the textbook claims that "the class of regular languages is close under the concatenation operation," and thus the result follows. □

2. The set of nonregular languages over $\{0, 1\}$ is closed under set-wise concatenation.

   Proof. This is false. Consider a nonregular language L and its complement L^c. Since $L \cup L^c = \{0, 1\}^*$ but $\{0, 1\}^*$ is regular, nonregular languages are not closed under set-wise concatenation. □

3. The set of regular languages over $\{0, 1\}$ is closed under the $SUBSTRING$ operation.

   Proof. Let $N = (Q, \Sigma_1, \delta, q_1, F)$ recognize L.

   Construct $N' = (Q', \Sigma_1, \delta', q_0, F')$ to recognize $SUBSTRING(L)$.

   i. $Q' = Q \cup \{q_0\}$

   ii. $F' = \{q \in Q \mid q \in \delta(s), \forall s \in SUBSTRING(L)\}$

   iii. Define δ' so that for any $q \in Q'$ and $a \in \Sigma_\epsilon$,

   $$\delta' = \begin{cases} \delta(q, a), & q \in Q \\ F', & q = q_0, a = \epsilon \\ \emptyset, & q = q_0, a \neq \epsilon. \end{cases}$$

   □

4. The set of nonregular languages over $\{0, 1\}$ is closed under the $SUBSTRING$ operation.

   Proof. This is false. Consider $L = \{0^n 1^n \mid n \in \mathbb{N}\}$. L is non-regular, shown as example 1.73 on page 80 of hte textbook. Since $SUBSTRING(L)$ is simply the set of strings where none of them have 1 proceding 0, it can be described as 0^*1^* in terms of regular expression, and thus $SUBSTRING(L)$ is regular. □