MATH 262A: DISCRETE GEOMETRY NOTES

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Contents

1. Sums vs Product

Definition 1.1. The *crossing number* of a graph G, denoted $\mathrm{cr}(G)$, is the minimum number of crossing pair of edges over all possisble drawings of G in the plane.

Lemma 1.2 (Crossing Lemma)**.** *Let $G = (V, E)$ be a graph. If $|E| \geqslant 4|V|$, then*

$$\mathrm{cr}(G) \geqslant \frac{|E|^3}{64|V|^2}.$$

Theorem 1.3. *Let A be a set of n distinct real numbers. Then* $\max\{|A + A|, |A \cdot A|\} = \Omega(n^{5/4})$.

Proof. Denote $A + A = \{s_1, s_2, \ldots, s_x\}$ and $A \cdot A = \{p_1, p_2, \ldots, p_y\}$. Let L be the set of lines $v = a_i(u - a_j)$ for $a_i, a_j \in A$. Construct the graph $G = (V, E)$ with $V = (A + A) \times (A \cdot A)$ and $\{(s_i, p_i), (s_j, p_j)\} \in E$ if and only if there exists a line $l \in L$ such that (s_i, p_i) and (s_j, p_j) are consecutive points on l. Notice that each line passes through at least $n - 1$ points in V, so $|E| \geqslant (n - 1)|L| = \Omega(n^3)$. If $|E| < 4|V|$, then

$$4|A + A| \cdot |A \cdot A| = 4|V| > |E| = \Omega(n^3).$$

But then either $|A + A| = \Omega(n^{3/2})$ or $|A \cdot A| = \Omega(n^{3/2})$. Thus we may assume $|E| \geqslant 4|V|$. By the crossing lemma,

$$\frac{|E|^3}{64|V|^2} \leqslant \mathrm{cr}(G) \leqslant |L|^2 \leqslant n^4.$$

Rearranged, we have

$$|V|^2 \geqslant \frac{|E|^3}{64n^4} = \Omega(n^5).$$

The result now follows. $\square$

2. Crossing Lemma

In this section we prove Lemma 1.2 mentioned in the previous section.

Lemma 2.1. *Let* $G = (V, E)$ *be a graph. Then* $\mathrm{cr}(G) \geqslant |E| - 3|V|$.

Proof. Suppose not. We may assume $|E| \geqslant 3|V|$, otherwise we are done. Remove edges from each crossing until we have a planar graph. Since $\mathrm{cr}(G) < |E| - 3|V|$, we removed less than $|E| - 3|V|$ edges. But then the planar graph has more than $|E| - (|E| - 3|V|) = 3|V|$ edges, contradicting Euler's theorem. □

Proof of Lemma 1.2. For any graph H, define $X_H = \mathrm{cr}(H) - |E(H)| + 3|V(H)|$. By the crossing lemma we know $X_H \geqslant 0$. Consider the drawing of G in $\mathbb{R}^2$ with $\mathrm{cr}(G)$ crossings. Let $S \subseteq V$ be a set vertices where each vertex is chosen independently with probability $p \in [0, 1]$. Let $G' = G[S]$ be the induced subgraph on S. Then

$$\mathbb{E}[X_{G'}] = \mathbb{E}[\mathrm{cr}(G')] - \mathbb{E}[|E(G')|] + 3\mathbb{E}[|V(G')|] = \mathbb{E}[\mathrm{cr}(G')] - p^2|E| + 3p|V| \geqslant 0.$$

Let $C_{G'}$ be the number of crossings in the drawing of G' inherited from G. Obviously, $\mathbb{E}[\mathrm{cr}(G')] \leqslant \mathbb{E}[C_{G'}]$. Since each crossing pair has a probability of p^4 of being in G', we have $\mathbb{E}[C_{G'}] = p^4 \,\mathrm{cr}(G)$, and thus

$$p^4 \,\mathrm{cr}(G) \geqslant \mathbb{E}[\mathrm{cr}(G')] \geqslant p^2|E| - 3p|V|.$$

By setting $p = 4|V|/|E|$, we have

$$\mathrm{cr}(G) \geqslant \frac{|E|}{p^2} - \frac{3|V|}{p^3} \geqslant \frac{|E|^3}{64|V|^2}.$$

□

3. Szemerédi-Trotter Theorem

Definition 3.1. Let P be a set of n points and L be a set of m lines in the plane. We call a pair (p, l) *incidence* if $p \in P$, $l \in L$, and $p \in l$. Define $I(P, L)$ as the number of incidences between P and L, and define $I(m, n)$ as the maximum number of incidences between any m lines and n points.

Definition 3.2. Let P be a set of n points. A line is *generated by* P if it contains at least 2 points from P.

Definition 3.3. For $k \geqslant 2$ and a set of points P, a line l is k-rich if it contains at least k points from P.

Theorem 3.4 (Szemerédi-Trotter Theorem)**.** *For all $m, n \geqslant 1$, we have $I(m, n) = O(m^{2/3}n^{2/3} + m + n)$.*

Proof. We will adopt the same strategy as the proof of Theorem 1.3, which constructs a graph and double counts the number of crossings in it.

Let P be the set of n points in $\mathbb{R}^2$ and L be the set of m lines in $\mathbb{R}^2$. Define graph $G = (V, E)$ where $V = P$ and E is the set of consecutive pairs of vertices along some line in L. We may assume each line in L contains at least one point from P. For $l \in L$, let $|l|$ denote the number of points in P which lies in l. Observe that

$$|E| = \sum_{l \in L} |l| - 1 = |I(P, L)| - m.$$

Hence, it suffices to show that $|E| = O(m^{2/3}n^{2/3} + n)$. We may assume $|E| \geqslant 4|V|$, otherwise we are done. Note that the construction of G gives a natural drawing with points P and lines P in the plane, so we may define C as the number of crossings in this drawing. By the crossing lemma, we have

$$\frac{|E|^3}{64n^2} \leqslant \mathrm{cr}(G) \leqslant C \leqslant \binom{m}{2} = O(m^2).$$

It now follows that

$$|E| = O(n^{2/3}m^{2/3}).$$

This completes the proof. □

Corollary 3.5. *Let P be a set of n points. Then P generates $O(\frac{n^2}{k^3} + \frac{n}{k})$ k-rich lines.*

Proof. Let L_k be the set of k-rich lines generated by P. By the Szemerédi-Trotter theorem,

$$k|L_k| \leqslant I(P, L_k) = c(|L_k|^{2/3}n^{2/3} + |L_k| + n),$$

for some constant c. We may assume $k \geqslant 4c$, otherwise we are done as $|L_k| = O(n^2)$. If $n + |L_k| \geqslant |L_k|^{2/3}n^{2/3}$. Then

$$k|L_k| \leqslant 2c(|L_k| + n) = 2cm + 2c|L_k|.$$

Rearranged,

$$|L_k| \leqslant \frac{2cm}{k - 2c} \leqslant O(m/k).$$

Now suppose $n + |L_k| < |L_k|^{2/3}n^{2/3}$. Then

$$k|L_k| \leqslant 2c|L_k|^{2/3}n^{2/3},$$

and so

$$|L_k| = O(n^2/k^3).$$

□

4. The Cutting Lemma

Lemma 4.1 (Cutting Lemma)**.** *Let L be a set of m lines in $\mathbb{R}^2$ and let $r \in (1, m)$. Then the plane can be subdivied into $t = O(r^2)$ generalized triangles (intersections of three half planes) $\Delta_1, \Delta_2, \ldots, \Delta_t$ such that the interior of each Δ_i is intersected by at most m/r lines of L.*

Lemma 4.2. *Let L be a set of m lines in $\mathbb{R}^2$ and let $r \in (1, m)$. Then the plane can be subdivied into $t = O(r^2 \log^2 n)$ generalized triangles $\Delta_1, \Delta_2, \ldots, \Delta_t$ such that the interior of each Δ_i is intersected by at most m/r lines of L.*

Proof. Put $s = 6r \ln m$. Select a random set of lines $S \subset L$ by making s independent random draws with replacement. Consider the line arrangement of S. Partition any cell that is not a generalized triangle further by adding diagonals that connect vertices. To this end, $\mathbb{R}^2$ is partitioned into t generalized triangles. Consider a box B that contains all bounded triangles Δ_i. Since each line crosses through B two times and each two consecutive lines around B determine an unbounded triangle, the number of unbounded triangles is at most $2s$. Now consider the bounded triangles. View each intersecting point of two lines in S as a vertex of a graph, and each bounded triangle as a face. Let V denote the set of vertices and F the set of faces. We know that $|V| \leqslant \binom{s}{2} = O(s^2)$. By Euler's formula, we have

$$3|F| \leqslant \sum_{f \in F} \deg f = 2|E| = 2(|V| + |F| - 2),$$

and thus

$$|F| \leqslant 2|V| - 4 = O(s^2).$$

Hence, we have $t = O(s^2)$.

We call a (generalized) triangle *horny* if its interior intersects at least m/r lines of L. For any horny triangle T, the probability that no line in S intersects the interior of T is at most $(1 - 1/r)^s$. Using the inequality $1 - x \leqslant e^{-x}$, we have $(1 - 1/r)^s \leqslant e^{-6 \ln m} = m^{-6}$.

Now call a triangle *interesting* if it can appear in a triangulation for some sample $S \subset L$. Notice that each vertex of an interesting triangle is an intersecting point of two lines in the arrangement of L, and thus there are at most $\binom{m}{2}^3 < m^6$ such triangles.

But then the expected number of horny Δ_i's is less than $m^{-6} \cdot m^6 = 1$. It now follows that there exists a set of $S \subseteq L$ such that each Δ_i is intersected by at most m/r lines. $\square$

5. An Aliter for the Szemerédi-Trotter Theorem

Theorem 5.1 (Kővári-Sós-Turán Theorem)**.** *For $s, t \geqslant 2$, let G be an $m \times n$ bipartite graph that does not contain a complete bipartite graph $K_{s,t}$ where the s vertices are from the part of size m. Then,*

$$|E(G)| = O(nm^{1-1/t} + m) \quad \text{and} \quad |E(G)| = O(mn^{1-1/s} + n).$$

Proof. Let M, N be the two parts of the bipartite graph G, with $|M| = m$ and $|N| = n$. Notice that no set of s vertices in M has more than $t-1$ common neighbors in N, so

$$\sum_{v \in M} \binom{d(v)}{t} \leqslant \binom{n}{t}(s-1) \leqslant \frac{sn^t}{t!}.$$

By Jensen's inequality, we have

$$\sum_{v \in M} \binom{d(v)}{t} \geqslant m \binom{\frac{1}{m}\sum_{v \in M} d(v)}{t} \geqslant \frac{m(|E(G)|/m - t)^t}{t!}.$$

The result now follows from the two inequalities. ☐

Corollary 5.2. $|I(m,n)| \leqslant O(n\sqrt{m} + m)$ *and* $|I(m,n)| \leqslant O(m\sqrt{n} + n)$.

Proof. Let P be the set of n points and L be the set of m lines in $\mathbb{R}^2$. Let $G = (P, L)$ be the bipartite graph with parts P and L and (p, l) is an edge if and only if $p \in l$. Since no two points lie on the same line, G is $K_{2,2}$-free. The resulting bounds now follows from the Kővári-Sós-Turán theorem. ☐

We give an alternative proof of a case of the Szemerédi-Trotter theorem with n points and n lines, using the Cutting lemma and the Kővári-Sós-Turán theorem.

Aliter for Theorem 3.4. Let P be the set of n points and L be the set of n lines in $\mathbb{R}^2$. We need to show that there are at most $O(n^{4/3})$ incidences between P and L. We apply the cutting lemma with $r = n^{1/3}$, which divides the plane into $t = O(n^{2/3})$ generalized triangles $\Delta_1, \Delta_2, \ldots, \Delta_t$.

Let V be the points that lie on the vertex of some Δ_i. Since $|V| \leqslant 3t = O(n^{2/3})$, Corollary 5.2 gives us $|I(V, L)| = O(n^{2/3}\sqrt{n} + n^{2/3}) = O(n^{4/3})$.

Let $|L'|$ be the set of lines that borders some triangle Δ_i. Then $|L'| \leqslant 3t = O(n^{2/3})$, and Corollary 5.2 again gives us $|I(P_0, L')| = O(n^{2/3}\sqrt{n} + n^{2/3}) = O(n^{4/3})$.

It remains to count the incidences that occur at the interior of some triangle. Let P_i be the set of points in P that lies in the interior of Δ_i. Let L_i be the set of lines intersecting the interior of Δ_i. By the cutting lemma, $|L_i| \leqslant n/r = O(n^{2/3})$. Hence,

$$\sum_{i=1}^{t} I(P_i, L_i) \leqslant \sum_{i=1}^{t} I(P_i, n^{2/3}) = \sum_{i=1}^{t} O(|P_i|n^{1/3} + n^{2/3}) = O(n^{4/3}).$$

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