

# MATH 264A: Homework

Due on Nov 2, 2024 at 23:59pm

*Professor Warnke*

**Ray Tsai**

A16848188

## Problem 1

- (a) Let  $G = (V, E)$  be an  $n$ -vertex graph and suppose that each vertex  $v \in V$  is associated with a list  $S(v)$  of colors of size at least  $4r$ , where  $r$  is a positive integer. Suppose also that for each  $v \in V$  and each  $c \in S(v)$ , there are at most  $r$  neighbors  $u$  of  $v$  such that  $c \in S(u)$ . Using induction, prove that there are at least  $(2r)^n$  proper colorings of  $G$  under which each vertex  $v$  receives a color from its list  $S(v)$ .

*Remark:* a proof using the classical Lovász Local Lemma (LLL) requires  $2er \approx 5.44r$  instead of  $4r$ .

*Proof.* For  $S \subseteq V$ , define  $N_S$  as the number of proper colorings of  $G$  under which  $v$  receives a color from its list  $S(v)$  for all  $v \in S$ . It suffices to show that for all  $T \subseteq V$  and  $x \in T$ ,  $N_T/N_{T \setminus \{x\}} \geq 2r$ . We proceed by induction on  $|T|$ . When  $T = \{x\}$ ,  $N_T = S(x)$  and  $N_{T \setminus \{x\}} = N_\emptyset = 1$ , we have  $N_T/N_{T \setminus \{x\}} \geq 4r \geq 2r$ . Suppose  $|T| > 2$ . Then,

$$|S(x)| \cdot N_{T \setminus \{x\}} = N_T + B,$$

where  $B$  is the number of improper colorings of  $T$  which becomes proper if we ignore the color of  $x$ . Notice that any element counted by  $B$  contains a vertex  $u$  which is a neighbor of  $x$  that shares the same color  $c$  as  $x$  and a proper coloring of  $T \setminus \{u, x\}$ . This yields the upper bound

$$B \leq |S(x)| \cdot (\# \text{choice of vertex } u) \cdot N_{T \setminus \{u, x\}} \leq 4r \cdot r \cdot N_{T \setminus \{u, x\}}.$$

By induction,  $N_{T \setminus \{u, x\}} \leq \frac{1}{2r} \cdot N_{T \setminus \{x\}}$ , and thus

$$N_T = |S(x)| \cdot N_{T \setminus \{x\}} - B \geq 4r N_{T \setminus \{x\}} - 4r \cdot r \cdot \frac{1}{2r} \cdot N_{T \setminus \{x\}} = 2r N_{T \setminus \{x\}},$$

and this completes the induction.  $\square$

- (b) A  $k$ -SAT formula is an expression such as

$$(x_1 \text{ OR } x_4 \text{ OR } \overline{x_6}) \text{ AND } (x_1 \text{ OR } \overline{x_2} \text{ OR } x_5),$$

where the variables  $x_i$  take values true or false,  $\overline{x_i}$  means not  $x_i$ , and  $k$  distinct variables or their negations are OR-ed together in each clause. A formula is called satisfiable if there is an assignment of values to the variables making the expression true. Suppose that in a given  $k$ -SAT formula  $\Phi$  no variable lies in more than  $2^k/(ek)$  clauses. Using induction, prove that  $\Phi$  has at least  $(2 - 2/k)^n$  many satisfying assignments (which in fact remains true if we relax the assumed  $2^k/(ek)$  upper bound to  $2^k/k \cdot (1 - 1/k)^{k-1}$ ).

*Proof.* Let  $\Phi_i$  denote the  $k$ -SAT formula which AND's together all the clauses of  $\Phi$  that involve only the first  $i$  variables. Define  $N_i$  as the number of satisfying assignments of  $\Phi_i$ . We show that  $N_i \geq (2 - 2/k)^i$  by induction on  $i \geq k$ . And then I'm stuck.  $\square$

- (c) Let  $\mathcal{A}$  be an alphabet, and let  $\mathcal{F}$  be a set of forbidden strings. Assume that there exists  $\beta > 0$  such that

$$|\mathcal{A}| - \sum_{f \in \mathcal{F}} \beta^{1-|f|} \geq \beta.$$

Using induction, prove that there exists at least  $\beta^n$  words of length  $n$  over alphabet  $\mathcal{A}$  that avoid all the substrings in  $\mathcal{F}$ .

*Proof.* Define  $N_k$  to be the set of words of length  $n$  over alphabet  $\mathcal{A}$  that avoid all the substrings in  $\mathcal{F}$ . We show that  $|N_k|/|N_{k-1}| \geq \beta$  by induction on  $k \geq 1$ . Since  $|N_0| = 1$ ,

$$\frac{|N_1|}{|N_0|} = |N_1| \geq |\mathcal{A}| - |\{f \in \mathcal{F} \mid |f| = 1\}| \geq \beta,$$

as  $|\{f \in F \mid |f| = 1\}| \leq \sum_{f \in \mathcal{F}} \beta^{1-|f|}$ . Hence, the base case holds. Suppose  $k > 1$ . Let  $B$  denote the set of words over  $\mathcal{A}$  of the form  $a_1 \dots a_k$  that contains some substring in  $F$ , with  $a_1 \dots a_{k-1} \in N_{k-1}$ . Then

$$|\mathcal{A}| \cdot |N_{k-1}| = |N_k| + |B|.$$

By construction, any word in  $B$  consists of a forbidden string  $f$  at the end and some word in  $N_{k-|f|}$  at the beginning. Summing over all  $f$ , we have the bound

$$|B| \leq \sum_{f \in \mathcal{F}} |N_{k-|f|}|.$$

By induction,  $|N_{k-|f|}| \leq \beta^{1-|f|} \cdot |N_{k-1}|$ , and thus

$$|B| \leq |N_{k-1}| \sum_{f \in \mathcal{F}} \beta^{1-|f|}.$$

Therefore,

$$|N_k| = |\mathcal{A}| \cdot |N_{k-1}| - |B| \geq |N_{k-1}| \left( |\mathcal{A}| - \sum_{f \in \mathcal{F}} \beta^{1-|f|} \right) \geq \beta |N_{k-1}|,$$

and this completes the induction. It now follows that

$$|N_n| = \frac{|N_n|}{|N_{n-1}|} \cdot \frac{|N_{n-1}|}{|N_{n-2}|} \dots \frac{|N_1|}{|N_0|} \geq \beta^n.$$

□

## Problem 2

In the inductive proof of the ‘almost all triangle-free graphs are 2-colorable’ result, we defined the following sets (using  $\Gamma(v)$  and  $\Gamma(S)$  to denote the set of neighbors of a vertex  $v$  or set of vertices  $S$ ):

- $\mathcal{Col}_2(n)$  is the set of all 2-colorable graphs on  $n$  vertices.
- $\mathcal{T}(n)$  is the set of all triangle-free graphs on  $n$  vertices.
- $\mathcal{A}(n) \subseteq \mathcal{T}(n)$  is the subset of graphs containing a vertex  $v$  such that  $|\Gamma(v)| \leq \log_2 n$ .
- $\mathcal{B}(n) \subseteq \mathcal{T}(n)$  is the subset of graphs containing a vertex set  $Q$  of size  $|Q| = \log_2 n$ , such that  $|\Gamma(Q)| \leq (1/2 - 1/1000)n$ .
- $\mathcal{D}(n) \subseteq \mathcal{T}(n) \setminus (\mathcal{A}(n) \cup \mathcal{B}(n))$  is the subset of graphs containing an edge  $\{x, y\}$  and vertex sets  $Q_x \subseteq \Gamma(x)$  and  $Q_y \subseteq \Gamma(y)$ , such that  $|Q_x| = \log_2 n$ ,  $|Q_y| = \log_2 n$ , and  $|\Gamma(Q_x) \cap \Gamma(Q_y)| \geq n/100$ .

(a) Prove that  $|\mathcal{D}(n)|/|\mathcal{T}(n-2)| \leq 2^{(1-1/2000)n}$  for all sufficiently large  $n \geq n_0$ .

*Proof.* To generate a graph in  $\mathcal{D}(n)$ , we first pick two vertices  $x, y$  to be adjacent and then place a triangle free graph on the remaining  $n-2$  vertices. Lastly, we pick two subsets from the  $n-2$  vertices to be  $x$  and  $y$ 's neighbors respectively. Since the graph is not in  $\mathcal{A}(n)$ ,  $|\Gamma(x)|, |\Gamma(y)| > \log_2 n$ . But then the graph is also not in  $\mathcal{B}(n)$ , so  $|\Gamma(\Gamma(x))|, |\Gamma(\Gamma(y))|$  each have size greater than  $(1/2 - 1/1000)n$ . Since the graph is triangle-free,  $x$  cannot be adjacent to any vertex in  $\Gamma(\Gamma(x))$  and similarly for  $y$ , and thus  $|\Gamma(x)|, |\Gamma(y)| \leq (1/2 + 1/1000)n$ . This yields the bound

$$|\mathcal{D}(n)| \leq \binom{n}{2} \cdot |\mathcal{T}(n-2)| \cdot 2^{(1/2-1/1000)n} \cdot 2^{(1/2-1/1000)n} \leq 2^{(1-1/2000)n} \cdot |\mathcal{T}(n-2)|,$$

for large enough  $n$ . □

(b) Prove that  $|\mathcal{Col}_2(n)|/|\mathcal{Col}_2(n-1)| \geq 2^{\frac{1}{2}(n-1)}$  for all sufficiently large  $n \geq n_0$ .

*Proof.* Each graph in  $\mathcal{Col}_2(n)$  consists of vertex  $n$ , a graph  $H \in \mathcal{Col}_2(n-1)$ , and edges between  $n$  and  $H$ . Since  $H$  is bipartite,  $H$  contains an independent set  $I_H$  of size at least  $2^{\frac{1}{2}(n-1)}$ . By picking a graph  $H$  from  $\mathcal{Col}_2(n-1)$  and adding some edges between  $n$  and  $I_H$ , we may uniquely generate a graph in  $\mathcal{Col}_2(n)$ . Therefore,

$$|\mathcal{Col}_2(n)| \geq |\mathcal{Col}_2(n-1)| \cdot 2^{\frac{1}{2}(n-1)},$$

and the result now follows. □

### Problem 3

In this problem we discuss in more detail one calculation in the proof of the Bregman's Theorem from class. Given a bipartite graph  $G = (L \cup R, E)$  with  $|L| = |R| = n$ , fix a perfect matching  $M$  of  $G$ . For each vertex  $i \in L$ , there thus is a unique vertex  $X_i$  such that  $\{i, X_i\} \in M$  is a matching edge. Now write  $R_i$  for the set of  $j \in L$  for which  $\{j, X_j\} \in M$  and  $\{i, X_j\} \in E$ , i.e., the set of vertices  $j \in L$  for which there is a matching edge in  $M$  that contains  $j$  and a neighbor of  $i$ . By construction, we have  $|R_i| = \deg_G(i)$ . Using a (permutation) counting argument, show that for each vertex  $i \in L$  and  $1 \leq j \leq \deg_G(i)$  we have

$$\mathbb{P}(\text{vertex } i \text{ appears in } \pi \text{ in the } j\text{th position among the vertices in } R_i) = \frac{1}{\deg_G(i)}.$$

*Proof.*

$$\begin{aligned} & \mathbb{P}(\text{vertex } i \text{ in the } j\text{th position among the vertices in } R_i \text{ in } \pi) \\ &= \frac{\#\{\text{permutations of } R_i \text{ with } i \text{ appearing in the } j\text{th position}\}}{\#\{\text{permutations of } R_i\}} \\ &= \frac{\#\{\text{permutations of } R_i \setminus \{i\}\}}{\#\{\text{permutations of } R_i\}} \\ &= \frac{(|R_i| - 1)!}{|R_i|!} = \frac{1}{\deg_G(i)}. \end{aligned}$$

□

## Problem 4

- (a) An order  $n$  Latin square is an  $n \times n$  matrix in which each row and column is a permutation of  $[n]$ . Using the entropy method, prove that the number of order  $n$  Latin squares is at most

$$L(n) \leq \left( (1 + o(1)) \frac{n}{e^2} \right)^{n^2}$$

as  $n \rightarrow \infty$ .

*Proof.* Choose  $X$  uniformly at random from all order  $n$  Latin squares. Let  $X_i$  denote the  $i$ th row and let  $X_{ij}$  denote the  $i$ th row  $j$ th column of  $X$ . Independently and uniformly take a random  $a_{ij}$  from  $[0, 1]$  for all  $i, j \in [n]$ , and denote  $A = (a_{ij})$ . Let  $R_{ij}(A) = \{X_{il} : a_{il} < a_{ij}\}$  and  $C_{ij}(A) = \{X_{kj} : a_{kj} < a_{ij}\}$ . Note that  $\mathbb{P}(a_{ij} = a_{kl}) = 0$  for all  $(i, j) \neq (k, l)$ . By the Chain rule,

$$\log_2 L(n) = H(X) = \mathbb{E}_A \sum_{i,j \in [n]} H(X_{ij} \mid X_{kl} : a_{kl} < a_{ij}).$$

Let  $N_{ij}(A, X) = |[n] - R_{ij}(A) \cup C_{ij}(A)|$ . Rewriting the entropy in expectation form,

$$H(X_{ij} \mid X_{kl} : a_{kl} < a_{ij}) = \mathbb{E}_X [\log_2 N_{ij}(A, X)],$$

and thus

$$H(X) = \mathbb{E}_A \sum_{i,j \in [n]} \mathbb{E}_X [\log_2 N_{ij}(A, X)] = \mathbb{E}_X \sum_{i,j \in [n]} \mathbb{E}_A [\log_2 N_{ij}(A, X) \mid X].$$

By Jensen's inequality,

$$\mathbb{E}_A [\log_2 N_{ij}(A, X) \mid X] \leq \mathbb{E}_{a_{ij}} [\log_2 \mathbb{E}_{A \setminus \{a_{ij}\}} [N_{ij}(A, X) \mid X, a_{ij}]].$$

Note that each row and column of a given  $X$  is a fixed permutation of  $[n]$ , so each  $m \in [n]$  must be assigned to some square in a given row or column. Let  $m \in [n]$ . Since we are conditioning on  $a_{ij}$ , if  $m = X_{ij}$ , then  $\mathbb{P}[m \notin (R_{ij}(A) \cup C_{ij}(A))] = 1$ . Otherwise, if  $m \neq X_{ij}$ ,

$$\mathbb{P}[m \notin (R_{ij}(A) \cup C_{ij}(A))] = \mathbb{P}(m \notin R_{ij}(A)) \mathbb{P}(m \notin C_{ij}(A)) = (1 - a_{ij})^2,$$

and thus

$$\mathbb{E}_{A \setminus \{a_{ij}\}} [N_{ij}(A, X) \mid X, a_{ij}] = \sum_{m \in [n]} \mathbb{P}[m \notin (R_{ij}(A) \cup C_{ij}(A))] = 1 + (n-1)(1 - a_{ij})^2.$$

This yields

$$\mathbb{E}_{a_{ij}} [\log_2 \mathbb{E}_{A \setminus \{a_{ij}\}} [N_{ij}(A, X) \mid X, a_{ij}]] = \int_0^1 \log_2(1 + (n-1)(1-x)^2) dx \leq \log_2(n) - 2 + o(1).$$

It now follows that

$$H(X) \leq \mathbb{E}_X \sum_{i,j \in [n]} \mathbb{E}_{a_{ij}} [\log_2(n) - 2 + o(1)] \leq n^2(\log_2(n) - 2 + o(1)),$$

and thus the result.  $\square$

- (b) Sudoku puzzles are order 9 Latin squares, divided into 9 smaller  $3 \times 3$  blocks, with the additional constraint that each block must contain all the symbols  $\{1, \dots, 9\}$ . Sudoku squares of order  $N$  can be defined similarly for any square number  $N = n^2$  (to clarify: normal Sudoku puzzles are simply order 9 Sudoku squares). Using the entropy method, prove that the number of order  $N$  Sudoku squares is at most

$$S(n) \leq \left( (1 + o(1)) \frac{N}{e^3} \right)^{N^2}$$

as  $N \rightarrow \infty$ .

*Proof.* Choose  $X$  uniformly at random from all order  $n$  Latin squares. Let  $X_i$  denote the  $i$ th row and let  $X_{ij}$  denote the  $i$ th row  $j$ th column of  $X$ . Independently and uniformly take a random  $a_{ij}$  from  $[0, 1]$  for all  $i, j \in [n]$ , and denote  $A = (a_{ij})$ . Define

- $R_{ij}(A) = \{X_{il} : a_{il} < a_{ij}\}$
- $C_{ij}(A) = \{X_{kj} : a_{kj} < a_{ij}\}$
- $B_{ij}(A) = \{X_{kl} : a_{kl} < a_{ij} \text{ and } X_{kl} \text{ in the same block as } X_{ij}\}.$

Note that  $\mathbb{P}(a_{ij} = a_{kl}) = 0$  for all  $(i, j) \neq (k, l)$ . By the Chain rule,

$$\log_2 L(n) = H(X) = \mathbb{E}_A \sum_{i,j \in [n]} H(X_{ij} \mid X_{kl} : a_{kl} < a_{ij}).$$

Let  $N_{ij}(A, X) = |[N] - R_{ij}(A) \cup C_{ij}(A) \cup B_{ij}(A)|$ . Rewriting the entropy in expectation form,

$$H(X_{ij} \mid X_{kl} : a_{kl} < a_{ij}) = \mathbb{E}_X [\log_2 N_{ij}(A, X)],$$

and thus

$$H(X) = \mathbb{E}_A \sum_{i,j \in [n]} \mathbb{E}_X [\log_2 N_{ij}(A, X)] = \mathbb{E}_X \sum_{i,j \in [n]} \mathbb{E}_A [\log_2 N_{ij}(A, X) \mid X].$$

By Jensen's inequality,

$$\mathbb{E}_A [\log_2 N_{ij}(A, X) \mid X] \leq \mathbb{E}_{a_{ij}} [\log_2 \mathbb{E}_{A \setminus \{a_{ij}\}} [N_{ij}(A, X) \mid X, a_{ij}]].$$

Note that each row, column, and block of a given  $X$  is a fixed permutation of  $[n]$ , so each  $m \in [n]$  must be assigned to some square in a given row, column, or block. Let  $m \in [n]$ . Since we are conditioning on  $a_{ij}$ , if  $m = X_{ij}$ , then  $\mathbb{P}[m \notin (R_{ij}(A) \cup C_{ij}(A) \cup B_{ij}(A))] = 1$ . Otherwise, if  $m \neq X_{ij}$ ,

$$\mathbb{P}[m \notin (R_{ij}(A) \cup C_{ij}(A) \cup B_{ij}(A))] = \mathbb{P}(m \notin R_{ij}(A)) \mathbb{P}(m \notin C_{ij}(A)) \mathbb{P}(m \notin B_{ij}(A)) = (1 - a_{ij})^3,$$

and thus

$$\mathbb{E}_{A \setminus \{a_{ij}\}} [N_{ij}(A, X) \mid X, a_{ij}] = \sum_{m \in [N]} \mathbb{P}[m \notin (R_{ij}(A) \cup C_{ij}(A) \cup B_{ij}(A))] = 1 + (N - 1)(1 - a_{ij})^3.$$

Hence,

$$\mathbb{E}_{a_{ij}} [\log_2 \mathbb{E}_{A \setminus \{a_{ij}\}} [N_{ij}(A, X) \mid X, a_{ij}]] = \int_0^1 \log_2(1 + (N - 1)(1 - x)^3) dx \leq \log_2(N) - 3 + o(1).$$

It now follows that

$$H(X) \leq \mathbb{E}_X \sum_{i,j \in [n]} \mathbb{E}_{a_{ij}} [\log_2(1 + (N - 1)(1 - a_{ij})^3)] \leq N^2(\log_2(N) - 3) + o(1),$$

and thus the result.  $\square$