Ray Tsai
A16848188

Question 1. The sandwich shop offers 8 different sandwiches. Jamey likes them all equally. He picks one randomly each day for lunch. During a given week of 5 days, let X be the number of times he chooses salami, Y the number of times he chooses falafel, and Z the number of times he chooses veggie. Find the joint probability mass function of (X, Y, Z). Do you recognize some of these distributions?

Solution. For non-negative integers x, y, z such that $x+y+z \leq 5$, the joint probability mass function

$$\begin{aligned}
p_{X,Y,Z}(x, y, z) &= \mathbb{P}(X = x, Y = y, Z = z) \\
&= \binom{5}{x} 8^{-x} \binom{5-x}{y} 8^{-y} \binom{5-x-y}{z} 8^{-z} \left(\frac{5}{8}\right)^{5-x-y-z} \\
&= \frac{\binom{5}{x}\binom{5-x}{y}\binom{5-x-y}{z} 5^{5-x-y-z}}{8^5} \\
&= \frac{5!}{x!y!z!(5-x-y-z)!} \cdot \frac{5^{5-x-y-z}}{8^5}.
\end{aligned}$$

We note that $(X, Y, Z) \sim \text{Multi}(5, 8, \frac{1}{8}, \ldots, \frac{1}{8})$. $\square$

Question 2. Suppose X, Y have joint density function given by $f(x, y) = c(xy + y^2)$ for $0 \le x \le 1$ and $0 \le y \le 1$, and $f(x, y) = 0$ otherwise.

(a) Find c so that f is a joint distribution function.

Solution.

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= c\int_0^1\int_0^1 (xy+y^2)dxdy \\ &= c\int_0^1 \frac{1}{2}y + y^2 dy \\ &= \frac{7c}{12} = 1.\end{aligned}$$

Thus, $c = \frac{12}{7}$. □

(b) Find the marginal densities of X and Y.

Solution.

$$\begin{aligned}f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \frac{12}{7}\int_0^1 (xy+y^2)dy \\ &= \frac{12}{7}\left(\frac{1}{2}x + \frac{1}{3}\right) \\ &= \frac{6}{7}x + \frac{4}{7}.\end{aligned}$$

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \frac{12}{7}\int_0^1 (xy+y^2)dx \\ &= \frac{12}{7}\left(\frac{1}{2}y + y^2\right) \\ &= \frac{6}{7}y + \frac{12}{7}y^2.\end{aligned}$$

□

(c) Compute $\mathbb{P}(X < Y)$.

Solution.

$$
\begin{aligned}
\mathbb{P}(X < Y) &= \frac{12}{7} \int_0^1 \int_0^y (xy + y^2) dx dy \\
&= \frac{12}{7} \int_0^1 \frac{3}{2} y^3 dy \\
&= \frac{9}{14}.
\end{aligned}
$$

□

(d) Compute $\mathbb{E}[XY^2]$.

Solution.

$$
\begin{aligned}
\mathbb{E}[XY^2] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy^2 f(x, y) dx dy \\
&= \frac{12}{7} \int_0^1 \int_0^1 xy^2 (xy + y^2) dx dy \\
&= \frac{12}{7} \int_0^1 \frac{1}{3} y^3 + \frac{1}{2} y^4 dy \\
&= \frac{12}{7} \left(\frac{1}{12} + \frac{1}{10} \right) \\
&= \frac{11}{35}.
\end{aligned}
$$

□

Question 3. Suppose X, Y have joint density function given by $f(x,y) = e^{-x(1+y)}$ for $x > 0$ and $y > 0$, and $f(x,y) = 0$ otherwise.

(a) Find the marginal densities of X and Y.

Solution. For $x > 0$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_0^{\infty} e^{-x(1+y)}dy \\ &= \left[-\frac{1}{x}e^{-x(1+y)}\right]_{y=0}^{\infty} \\ &= \frac{1}{xe^x}. \end{aligned}$$

Otherwise, $f_X(x) = 0$.

For $y > 0$,

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \int_0^{\infty} e^{-x(1+y)}dx \\ &= \left[-\frac{1}{1+y}e^{-x(1+y)}\right]_{x=0}^{\infty} \\ &= \frac{1}{1+y}. \end{aligned}$$

Otherwise, $f_Y(y) = 0$. □

(b) Are X and Y independent?

Solution. Since $f_X(x)f_Y(y) = \frac{1}{x(1+y)e^x} \neq f(x,y)$, X and Y are not independent. □

(c) Compute $\mathbb{E}[XY]$.

Solution.

$$\begin{aligned} \mathbb{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy \\ &= \int_0^{\infty}\int_0^{\infty} xye^{-x(1+y)}dxdy \\ &= \int_0^{\infty} -\left[\frac{y\left(yx+x+1\right)\mathrm{e}^{-(y+1)x}}{\left(y+1\right)^2}\right]_0^{\infty} dy \\ &= \int_0^{\infty} \frac{y}{\left(y+1\right)^2}dy \\ &= \left[\ln\left(|y+1|\right) + \frac{1}{y+1}\right]_0^{\infty} \to \infty. \end{aligned}$$

Thus, $\mathbb{E}[XY]$ is divergent. □

(d) Compute $\mathbb{E}[\frac{X}{1+Y}]$.

Solution.

$$\begin{aligned}
\mathbb{E}\left[\frac{X}{1+Y}\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x}{1+y}f(x,y)dxdy \\
&= \int_{0}^{\infty}\int_{0}^{\infty}\frac{x}{1+y}e^{-x(1+y)}dxdy \\
&= \int_{0}^{\infty}\left[-\frac{(yx+x+1)\,\mathrm{e}^{-(y+1)x}}{(y+1)^3}\right]_0^{\infty}dy \\
&= \int_{0}^{\infty}\frac{1}{(y+1)^3}dy \\
&= \left[-\frac{1}{2\,(y+1)^2}\right]_0^{\infty} = \frac{1}{2}.
\end{aligned}$$

$\square$

Question 4. Suppose that X_1 and X_2 are independent random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$ and $\mathbb{P}(X_2 = 1) = 1 - \mathbb{P}(X_2 = -1) = p$ for some $0 < p < 1$. Let $Y = X_1X_2$. Show that X_2 and Y are independent.

Proof.

$$\begin{aligned} p_{X_2}(x) &= \mathbb{P}(X_2 = x) && (1) \\ &= \begin{cases} p & x = 1 \\ 1 - p & x = -1 \end{cases}. && (2) \end{aligned}$$

$$\begin{aligned} p_Y(y) &= \mathbb{P}(Y = y) && (3) \\ &= \begin{cases} \frac{1}{2}p + \frac{1}{2}(1-p) & y = 1 \\ \frac{1}{2}(1-p) + \frac{1}{2}p & y = -1 \end{cases} && (4) \\ &= \frac{1}{2}. && (5) \end{aligned}$$

$$\begin{aligned} p_{X_2,Y}(x,y) &= \mathbb{P}(X_2 = x, Y = y) && (6) \\ &= \begin{cases} \frac{1}{2}p & x = 1, y = 1 \\ \frac{1}{2}p & x = 1, y = -1 \\ \frac{1}{2}(1-p) & x = -1, y = 1 \\ \frac{1}{2}(1-p) & x = -1, y = -1 \end{cases} && (7) \\ &= \begin{cases} p_{X_2}(1)p_Y(1) & x = 1, y = 1 \\ p_{X_2}(1)p_Y(-1) & x = 1, y = -1 \\ p_{X_2}(-1)p_Y(1) & x = -1, y = 1 \\ p_{X_2}(-1)p_Y(-1) & x = -1, y = -1 \end{cases} && (8) \\ &= p_{X_2}(x)p_Y(y). && (9) \end{aligned}$$

Thus, X_2, Y are independent. $\square$

Question 5. Let $X_1, \dots, X_n$ be independent exponential random variables with parameter λ_i for X_i. Let Y be the minimum of these random variables, that is, $Y = \min(X_1, \dots, X_n)$. Show that $Y \sim Exp(\lambda_1 + \cdots + \lambda_n)$.

Proof. For $y > 0$,

$$\begin{aligned}
\mathbb{P}(Y \geq y) &= \mathbb{P}(X_1, \dots, X_n \geq y) \\
&= \mathbb{P}(X_1 \geq y)\mathbb{P}(X_2 \geq y) \dots \mathbb{P}(X_n \geq y) \\
&= e^{-\lambda_1} e^{-\lambda_2} \dots e^{-\lambda_n} \\
&= e^{-(\lambda_1 + \cdots + \lambda_n)}.
\end{aligned}$$

Thus, the cumulative distribution function of Y is $\mathbb{P}(Y \leq y) = 1 - e^{-(\lambda_1 + \cdots + \lambda_n)}$, which is the same as that of a exponential random variable with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$. Therefore, $Y \sim Exp(\lambda_1 + \cdots + \lambda_n)$. $\square$

Question 6. Let X be a Poisson random variable with parameter $\lambda = 2$, and let Y be a geometric random variable with parameter $p = \frac{2}{3}$. Suppose that X and Y are independent, and let $Z = X + Y$. Find $\mathbb{P}(Z = 3)$.

Solution. Since X, Y are independent,

$$\begin{aligned}\mathbb{P}(Z = 3) &= \mathbb{P}(X = 1, Y = 2) + \mathbb{P}(X = 2, Y = 1) \\ &= 2e^{-2} \cdot \frac{2}{9} + 2e^{-2} \cdot \frac{2}{3} \\ &= \frac{16}{9} e^{-2}.\end{aligned}$$

$\square$

Question 7. Suppose that X and Y are independent exponential random variables with parameters $\lambda \neq \mu$. Find the density function of $X + Y$.

Solution. Let $f_{X+Y}(x)$ be the density function of $X+Y$. Since X, Y are independent, for $z \in [0, \infty)$,

$$\begin{aligned}
f_{X+Y}(z) &= f_X * f_Y(z) \\
&= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\
&= \int_0^z \lambda e^{-\lambda x} \mu e^{-\mu(z-x)} dx \\
&= \lambda \mu e^{-\mu z} \int_0^z e^{(\mu-\lambda)x} dx \\
&= \lambda \mu e^{-\mu z} \left[\frac{1}{\mu - \lambda} e^{(\mu-\lambda)x} \right]_0^z \\
&= \frac{\lambda \mu (e^{-\lambda z} - e^{-\mu z})}{\mu - \lambda}.
\end{aligned}$$

□

Question 8. Let $X_1, \dots, X_n$ be i.i.d. random variables (independent and identical distributed) with $X_i \sim Unif[0,1]$ for each i. Let $Tn = \frac{X1+\cdots+Xn}{n}$. Compute the moment generating function of T_n.

Solution. Since $X_i \sim Unif[0,1]$, $f_{X_i}(x) = 1$. Since $X_1, \dots, X_n$ are independent,

$$
\begin{aligned}
M(t) &= \mathbb{E}[e^{tT_n}] \\
&= \mathbb{E}\left[\prod_{k=1}^{n} e^{\frac{tX_k}{n}}\right] \\
&= \int_{\mathbb{R}^n} \left(\prod_{k=1}^{n} e^{\frac{tx_k}{n}}\right) f_{T_n}(x_1, x_2, \dots, x_n)\, dx_1 dx_2 \dots dx_n \\
&= \int_{\mathbb{R}^n} \left(\prod_{k=1}^{n} e^{\frac{tx_k}{n}} f_{X_k}(x_k)\right)\, dx_1 dx_2 \dots dx_n \\
&= \prod_{k=1}^{n} \int_0^1 e^{\frac{tx_k}{n}} f_{X_k}(x_k) dx_k \\
&= \prod_{k=1}^{n} \int_0^1 e^{\frac{tx_k}{n}}\, dx_k \\
&= \prod_{k=1}^{n} \left(\frac{n}{t} e^{\frac{t}{n}} - \frac{n}{t}\right) \\
&= \left(\frac{n}{t} e^{\frac{t}{n}} - \frac{n}{t}\right)^n
\end{aligned}
$$

□