MATH 100B: Homework #8

Due on Mar 7, 2024 at 12:00pm

Professor McKernan

Section A02 6:00PM - 6:50PM
Section Leader: Castellano-Macías

Source Consulted: Textbook, Lecture, Discussion, Office Hour

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Problem 1

Let M, N and P be R-modules and let F be a free R-module of rank n. Show that there are isomorphisms, which are all natural (except for the last):

(a) $M \underset{R}{\otimes} N \simeq N \underset{R}{\otimes} M$.

Proof. Let $v : N \times M \to N \underset{R}{\otimes} M$ be the bilinear map associated with $N \underset{R}{\otimes} M$. Define $f : M \times N \to N \underset{R}{\otimes} M$ that sends (m, n) to $v(n, m)$. By the universal property of tensor product, there is an induced module homomorphism $\phi : M \underset{R}{\otimes} N \to N \underset{R}{\otimes} M$. Similarly, there exsits an induced module homomorphism $\psi : N \underset{R}{\otimes} M \to M \underset{R}{\otimes} N$. By the universal property of tensor product, $\mathrm{Hom}(M \oplus N,\ M \oplus N)$ and $\mathrm{Hom}(N \oplus M,\ N \oplus M)$ only contain the identities. But then $\phi \circ \psi \in \mathrm{Hom}(M \oplus N,\ M \oplus N)$ and $\psi \circ \phi \in \mathrm{Hom}(N \oplus M,\ N \oplus M)$, so ϕ and ψ are inverses. It follows that ϕ is a module isomorphism, so $M \underset{R}{\otimes} N \simeq N \underset{R}{\otimes} M$. ☐

(b) $(M \underset{R}{\otimes} N) \underset{R}{\otimes} P \simeq M \underset{R}{\otimes} (N \underset{R}{\otimes} P)$.

Proof. For $m \in M$, define $\psi_m^B : N \times P \to (M \underset{R}{\otimes} N) \underset{R}{\otimes} P$, which sends (n, p) to $(m \otimes n) \otimes p$. Note that ψ_m^B is obvisouly bilinear and well-defined, and thus the universal property gives us a linear mapping $\psi_m : N \otimes P \to (M \underset{R}{\otimes} N) \otimes P$. We now define $\phi^B : M \times (N \otimes P) \to (M \underset{R}{\otimes} N) \otimes P$, which sends $(m, n \otimes p)$ to $\psi_m(n, p)$. We check that ϕ^B is bilinear. Let $m, m' \in M$, $r \in R$, and $v, v' \in N \underset{R}{\otimes} P$, say $v = \sum a_{ij} n_i \otimes p_j$ and $v' = \sum b_{ij} n_i \otimes p_j$. Since

$$\begin{aligned}
\phi^B(m + m', v) &= \psi_{m+m'}(v) \\
&= \sum a_{ij} \psi_{m+m'}(n_i \otimes p_j) \\
&= \sum a_{ij} ((m + m') \otimes n_i) \otimes p_j \\
&= \sum a_{ij} \psi_m(n_i \otimes p_j) + \sum a_{ij} \psi_{m'}(n_i \otimes p_j) \\
&= \psi_m(v) + \psi_{m'}(v) = \phi^B(m, v) + \phi^B(m', v),
\end{aligned}$$

$$\begin{aligned}
\phi^B(m, v + v') &= \psi_m(v + v') \\
&= \sum (a_{ij} + b_{ij}) \psi_m(n_i \otimes p_j) \\
&= \sum a_{ij} (m \otimes n_i) \otimes p_j + \sum b_{ij} (m \otimes n_i) \otimes p_j \\
&= \sum a_{ij} \psi_m(n_i \otimes p_j) + \sum b_{ij} \psi_m(n_i \otimes p_j) \\
&= \psi_m(v) + \psi_m(v') = \phi^B(m, v) + \phi^B(m, v'),
\end{aligned}$$

$$\begin{aligned}
\phi^B(rm, v) &= \psi_{rm}(v) \\
&= \sum a_{ij} \psi_{rm}(n_i \otimes p_j) \\
&= r \sum a_{ij} \psi_m(n_i \otimes p_j) \\
&= r\psi_m(v) = r\phi^B(m, v),
\end{aligned}$$

$$\phi^B(m, rv) = \psi_m(rv) = r\psi_m(v) = r\phi^B(m, v),$$

ϕ^B is indeed bilinear, so we obtain a linear $\phi : M \underset{R}{\otimes} (N \underset{R}{\otimes} P) \to (M \underset{R}{\otimes} N) \underset{R}{\otimes} P$, by the universal property. We may repeat the above process to obtain an induced linear map $\varphi : (M \underset{R}{\otimes} N) \underset{R}{\otimes} P \to M \underset{R}{\otimes} (N \underset{R}{\otimes} P)$, and thus ϕ and φ are inveres of each other, by the standard uniqueness argument. The result now follows. $\square$

(c) $R \underset{R}{\otimes} M \simeq M$.

Proof. Define mapping $f : R \times M \to M$ that sends (r, m) to rm. Note that f is obviously bilinear. The universal property of tensor product gives us a R-linear mapping $\phi : R \underset{R}{\otimes} M \to M$ which sends $r \otimes m$ to $f(rm)$, that is, rm. Since for all $m \in M$, we have $1 \otimes m \in R \underset{R}{\otimes} M$ that is mapped to m via ϕ, so ϕ is surjective. Suppose $r \otimes m$ is in the kernel of ϕ. Then $\phi(r \otimes m) = rm = 0$, so $r = 0$ or $m = 0$. But then $r \otimes m = 0$ in either case, and thus the kernel of ϕ is trivial. The result now follows from the first isomorphism theorem. $\square$

(d) $M \underset{R}{\otimes} (N \oplus P) \simeq (M \underset{R}{\otimes} N) \oplus (M \underset{R}{\otimes} P)$.

Proof. Define mapping $f : M \times (N \oplus P) \to (M \underset{R}{\otimes} N) \oplus (M \underset{R}{\otimes} P)$, which maps $(m, (n \otimes p))$ to $(m \otimes n, m \otimes p)$. This map is obviously well defined. We show that f is bilinear. Suppose $m, m' \in M$, $n, n' \in N$, $p, p' \in P$, and $r \in R$. We then have

$$\begin{aligned} f(m + m', (n, p)) &= ((m + m') \otimes n, (m + m') \otimes p) \\ &= (m \otimes n, m \otimes p) + (m' \otimes n, m' \otimes p) \\ &= f(m, (n, p)) + f(m', (n, p)), \end{aligned}$$

$$\begin{aligned} f(m, (n, p) + (n', p')) &= (m \otimes (n + n'), \otimes(p + p')) \\ &= (m \otimes n, m \otimes p) + (m \otimes n', m \otimes p') \\ &= f(m, (n, p)) + f(m, (n', p')), \end{aligned}$$

$$\begin{aligned} f(rm, (n, p)) &= (rm \otimes n, rm \otimes p) \\ &= (r(m \otimes n), r(m \otimes p)) \\ &= r(m \otimes n, m \otimes p) = rf(m, (n, p)), \end{aligned}$$

$$\begin{aligned} f(m, r(n, p)) &= (m \otimes rn, m \otimes rp) \\ &= (r(m \otimes n), r(m \otimes p)) \\ &= r(m \otimes n, m \otimes p) = rf(m, (n, p)), \end{aligned}$$

and thus f is bilinear. The universal property of tensor product now gives us an induced R-linear mapping

$$\phi : M \underset{R}{\otimes} (N \oplus P) \to (M \underset{R}{\otimes} N) \oplus (M \underset{R}{\otimes} P),$$

which maps $m \otimes (n, p)$ to $f(m, (n, p))$.

It remains to find the inverse of ϕ. Define $\psi_1^B : M \times N \to M \underset{R}{\otimes} (N \oplus P)$ by sending (m, n) to $m \otimes (n, 0)$, and define $\psi_2^B : M \times P \to M \underset{R}{\otimes} (N \oplus P)$ by sending (m, p) to $m \otimes (0, p)$. Note that both ψ_1^B and ψ_2^B are bilinear, so the universal property gives us linear mappings $\psi_1 : M \underset{R}{\otimes} N \to M \underset{R}{\otimes} (N \oplus P)$ and

$\psi_2 : M \underset{R}{\otimes} P \to M \underset{R}{\otimes} (N \oplus P)$. Now define $\psi : (M \underset{R}{\otimes} N) \oplus (M \underset{R}{\otimes} P) \to M \underset{R}{\otimes} (N \oplus P)$ by sending $((m \otimes n), (m' \otimes p))$ to $\psi_1(m \otimes n) + \psi_2(m' \otimes p)$. Note that ψ is linear, as both ψ_1 and ψ_2 are linear.

We now show that ϕ and ψ are inverses of each other. Suppose $v \in M \underset{R}{\otimes} N$, $w \in M \underset{R}{\otimes} P$, and $x \in M \underset{R}{\otimes} (N \oplus P)$, say $v = \sum a_{ij} m_i \otimes n_j$, $w = \sum b_{ij} m_i \otimes p_j$, and $x = \sum c_{ijk} m_i \otimes (n_j, p_k)$. Since both ϕ and ψ are linear,

$$\begin{aligned}
\phi \circ \psi(v, w) &= \phi(\psi_1(v) + \psi_2(w)) \\
&= \phi(\psi_1(v)) + \phi(\psi_2(w)) \\
&= \sum a_{ij} \phi(\psi_1(m_i \otimes n_j)) + \sum b_{ij} \phi(\psi_1(m_i \otimes p_j)) \\
&= \sum a_{ij} \phi(m_i \otimes (n_j, 0)) + \sum b_{ij} \phi(m_i \otimes (0, p_j)) \\
&= \sum a_{ij} (m_i \otimes n_j, 0) + \sum b_{ij} (0, m_i \otimes p_j) = (v, 0) + (0, w) = (v, w),
\end{aligned}$$

$$\begin{aligned}
\psi \circ \phi(x) &= \psi\left(\sum c_{ijk} \phi(m_i \otimes (n_j, p_k))\right) \\
&= \sum c_{ijk} \psi(m_i \otimes n_j, m_i \otimes p_k) \\
&= \sum c_{ijk} (\psi_1(m_i \otimes n_j) + \psi_2(m_i \otimes p_k)) \\
&= \sum c_{ijk} (m_i \otimes (n_j, 0) + (m_i \otimes (0, p_k))) \\
&= \sum c_{ijk} (m_i \otimes (n_j, p_k)) = x,
\end{aligned}$$

and the result follows. □

(e) $F \underset{R}{\otimes} M \simeq M^n$.

Proof. Note that $F \simeq R^n$. We show that $R^n \underset{R}{\otimes} M \simeq M^n$ by induction on n. The base case follows from (c). Suppose $n > 1$. By (d), we have $R^n \underset{R}{\otimes} M \simeq (R \underset{R}{\otimes} M) \oplus (R^{n-1} \underset{R}{\otimes} M) \simeq M \oplus (R^{n-1} \underset{R}{\otimes} M)$. The result now follows from induction. □

Problem 2

Let m and n be integers. Identify $\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n$.

Proof. Let $d = \gcd(m, n)$. We show that $\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \simeq \mathbb{Z}_d$. Let $a, b \in \mathbb{Z}$. We first note that $0 \otimes a = b \otimes 0 = 0$. In addition, since $(ab)(1 \otimes 1) = a \otimes b$, all elements in $\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n$ are multiples of $1 \otimes 1$, so we have a cyclic group.

Define $f : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_d$ that sends (a, b) to ab. Suppose $(a, b) = (a', b')$. We know $a' = a + km$ and $b' = b + ln$, for some $k, l \in \mathbb{Z}$. But then d divides m, n, so $a' = a$ and $b' = b$, mod d. Hence, $f(a, b) = ab = a'b' = f(a', b')$, so f is well-defined. Since f is obviously bilinear, the universal property of tensor product gives us an induced module homomorphism

$$\phi : \mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \to \mathbb{Z}_d,$$

which sends $1 \otimes 1$ to $f(1, 1) = 1$.

Consider $\psi : \mathbb{Z}_d \to \mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n$, which sends k to $k \otimes 1$. Suppose $k' = k + \alpha d$, for some $\alpha \in \mathbb{Z}$. Then,

$$\psi(k') = (k + \alpha d) \otimes 1 = k(1 \otimes 1) + \alpha(d(1 \otimes 1)).$$

Since $d = \gcd(m, n)$, $d = pm + qn$, for some $p, q \in \mathbb{Z}$. But then

$$d(1 \otimes 1) = (pm + qn)(1 \otimes 1) = p(m \otimes 1) + q(1 \otimes n) = 0,$$

so $\psi(k') = k(1 \otimes 1) = \psi(k)$, and thus ψ is well defined. Note that ψ is obviously linear.

Since $\phi \circ \psi(k) = \phi(k(1 \otimes 1)) = k$ and $\psi \circ \phi(a \otimes b) = \psi(ab) = (ab)(1 \otimes 1) = (a \otimes b)$, ϕ is an module isomorphism, and the result follows. $\square$

Problem 3

Show that if M and N are two finitely generated (respectively Noetherian) R-modules (respectively and R is Noetherian) then so is $M \underset{R}{\otimes} N$.

Proof. By Proposition 11.7., it suffices to show that $M \underset{R}{\otimes} N$ is finitely generated. Suppose $m_1, m_2, \ldots, m_k$ and $n_1, n_2, \ldots, n_l$ are the generators of M and N, respectively. Let $m \otimes n \in M \underset{R}{\otimes} N$. Since $m = \sum_i a_i m_i$ and $n = \sum_j b_j n_j$, for some $a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_l \in R$, we have

$$m \otimes n = \sum_i a_i (m_i \otimes n) = \sum_i \sum_j a_i b_j (m_i \otimes n_j) = \sum_{i,j} c_{ij} (m_i \otimes n_j),$$

where $c_{ij} = a_i b_j$. Hence, $M \underset{R}{\otimes} N$ is generated by $m_i \otimes n_j$, for finitely many i, j, and the result now follows. $\square$