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Math 180A **Homework 1** **Professor Carfagnini**

Question 1 We roll a fair die four times

(a) Describe the sample space Ω and the probability measure $\mathbb{P}$ that model this experiment. To describe $\mathbb{P}$, give the value $\mathbb{P}(\Omega)$ for each outcome $\omega \in \Omega$.

Solution.

$$\Omega = \{(x, y, w, z) \,|\, x, y, w, z \in [6]\}.$$

For all $\omega \in \Omega$,

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|} = \frac{1}{6^4} = \frac{1}{1296}.$$

□

(b) Let A be the event that there are at least two fives among the four rolls. Let B be the event that there is at most one five among the four rolls. Find the probabilities $\mathbb{P}(A)$ and $\mathbb{P}(B)$ by finding the ratio of the number of favorable outcomes to the total.

Solution.

$$\begin{aligned}
\mathbb{P}(A) &= P(\text{two fives}) + P(\text{three fives}) + P(\text{four fives}) && (1)\\
&= \frac{5^2\binom{4}{2}}{|\Omega|} + \frac{5\binom{4}{3}}{|\Omega|} + \frac{1}{|\Omega|} && (2)\\
&= \frac{171}{1296}. && (3)\\
& && (4)\\
\mathbb{P}(B) &= P(\text{no five}) + P(\text{one five}) && (5)\\
&= \frac{5^4}{|\Omega|} + \frac{5^3\binom{4}{1}}{|\Omega|} && (6)\\
&= \frac{1125}{1296}. && (7)
\end{aligned}$$

□

(c) What is the set $A \cup B$? Check if $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

Solution. Since $B = A^c$, $A \cup B = \Omega$, and thus $P(A \cup B) = 1$. We also know that $\mathbb{P}(A) + \mathbb{P}(B) = \frac{171}{1296} + \frac{1125}{1296} = 1$. Therefore, the statement holds true. □

Question 2 Every day a kindergarten class chooses randomly one of the 50 state flags to hang on the wall, without regard to previous choices. We are interested in the flags that are chosen on Monday, Tuesday and Wednesday of next week.

(a) Describe the sample space Ω and the probability measure $\mathbb{P}$ that model this experiment.

Solution.

$$\Omega = \{(m,t,w) \mid m,t,w \in [50]\}.$$

For all $\omega \in \Omega$,

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|} = \frac{1}{50^3} = \frac{1}{125000}.$$

□

(b) What is the probability that the class hangs Wisconsin's flag on Monday, Michigan's flag on Tuesday, and California's flag on Wednesday?

Solution. Let ω be the event where the class hangs Winconsin's flag on Monday, Michigan's flag on Tuesday, and California's flag on Wednesday. Since $\omega \in \Omega$, $\mathbb{P}(\omega) = \frac{1}{125000}$. □

(c) What is the probability that Wisconsin's flag will be hung at least two of the three days?

Solution.

$$\mathbb{P}(\text{Wisconsin's flag hung at least two days}) = \frac{49\binom{3}{2}+1}{125000} = \frac{148}{125000}.$$

□

Question 3 10 women, 5 nonbinary folks, and 5 men are meeting in a conference room. Four people are chosen at random from the 20 to form a committee.

(a) What is the probability that the committee consists of 2 women, 1 nonbinary person, and 1 man?

Solution. Let A be the event where the committee consists of 2 women, 1 nonbinary person, and 1 man.

$$\mathbb{P}(A) = \frac{\binom{10}{2}\binom{5}{1}\binom{5}{1}}{\binom{20}{4}} = \frac{75}{323}.$$

□

(b) Among the 20 is a couple of friends, Alex (who identifies as a woman) and Pete (who identifies as nonbinary). What is the probability that Alex and Pete both end up on the committee?

Solution. Let A be the event where Alex and Pete both end up on the committee.

$$\mathbb{P}(A) = \frac{\binom{18}{2}}{\binom{20}{4}} = \frac{3}{95}.$$

□

(c) What is the probability that Alex ends up on the committee but Pete doesn't?

Solution. Let A be the event where Alex ends up on the committee but Pete doesn't.

$$\mathbb{P}(A) = \frac{\binom{18}{3}}{\binom{20}{4}} = \frac{16}{95}.$$

□

Question 4 Pick a uniformly chosen random point inside a unit square (a square of side length 1) and draw a circle of radius $\frac{1}{3}$ around the point. Find the probability that the circle lies entirely inside the square.

Solution. A circle C with radius $\frac{1}{3}$ would lie entirely inside the unit square S if the distance from its center to any side of S is greater or equal to $\frac{1}{3}$, which requires the center of C to be in the smaller square S' with side length $\frac{1}{3}$ centered in S. Therefore,

$$\mathbb{P}(C \text{ lies entirely in } S) = \frac{\text{Area}(S')}{\text{Area}(S)} = \frac{1}{9}.$$

$\square$

Question 5 An urn contains 1 green ball, 1 red ball, 1 yellow ball, and 1 white ball. I draw 4 balls with replacements. What is the probability that there is at least one color that is repeated exactly twice?

Solution. Let $A = \{$ at least one color repeat exactly twice $\}, O = \{$ only one color repeat exactly twice $\}$, $T = \{$ two colors repeat exactly twice $\}$. Since O and T are disjoint to each other,

$$\mathbb{P}(A) = \mathbb{P}(O) + \mathbb{P}(T) = \frac{\binom{4}{2} \cdot \frac{4!}{1!}}{4^4} + \frac{\binom{4}{2} \cdot \frac{4!}{2!2!}}{4^4} = \frac{45}{64}.$$

$\square$

Question 6 Assume that $\mathbb{P}(A) = \frac{2}{5}$ and $\mathbb{P}(B) = \frac{7}{10}$. Making no further assumptions on A and B, show that $\mathbb{P}(AB)$ satisfies $\frac{1}{10} \leq \mathbb{P}(AB) \leq \frac{2}{5}$.

Solution. By inclusion-exclusion,

$$\mathbb{P}(AB) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = \frac{11}{10} - \mathbb{P}(A \cup B).$$

Since $\mathbb{P}(B) = \frac{7}{10} \leq \mathbb{P}(A \cup B) \leq 1$, we get $\frac{1}{10} \leq \mathbb{P}(AB) \leq \frac{2}{5}$. $\square$

Question 7 Show that for any events $A_1, A_2, \ldots, A_m$

$$\mathbb{P}(A_1 \cup \cdots \cup A_m) \leq \sum_{k=1}^{m} \mathbb{P}(A_k).$$

Solution. We will prove by induction on m. When $m = 2$,

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$$

by inclusion-exclusion, and thus $\mathbb{P}(A_1 \cup A_2) \leq \mathbb{P}(A_1) + \mathbb{P}(A_2)$ because $\mathbb{P}(A_1 \cap A_2) \geq 0$. For $m > 2$,

$$\mathbb{P}(A_1 \cup \cdots \cup A_m) \leq \mathbb{P}(A_m) + \mathbb{P}(A_1 \cup \cdots \cup A_{m-1})$$

by inclusion-exclusion. By induction, $\mathbb{P}(A_1 \cup \cdots \cup A_{m-1}) \leq \sum_{k=1}^{m-1} \mathbb{P}(A_k)$, and thus

$$\mathbb{P}(A_1 \cup \cdots \cup A_m) \leq \mathbb{P}(A_m) + \sum_{k=1}^{m-1} \mathbb{P}(A_k) = \sum_{k=1}^{m} \mathbb{P}(A_k).$$

$\square$