Math 109 HW 4

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1.

Proposition 1. *1 is not even.*

Proof. We will show that 1 is not even by contradiction. For the sake of contradiction, by HW4 fact 2, assume 1 is an even integer $2k$ for some integer k. By HW4 fact 1, we know that 1 is the smallest positive integer, so $k > 0$ is equivalent to $k \geq 1$ if k is an integer. Thus, we can split the situation into two cases, $k \leq 0$ and $k \geq 1$.

If $k \leq 0$, then $2k \leq 0$. This suggests that there does not exist integer $k \leq 0$ such that $2k = 1$.

If $k \geq 1$, then $2k \geq 2$. This suggests that there does not exist integer $k \geq 1$ such that $2k = 1$.

Hence, it is shown that there does not exist integer k such that $2k = 1$, which contradicts our assumption.

Therefore, 1 is not even. $\square$

2.

Proposition 2. *If n is an odd integer, then n is not even.*

Proof. We will show that if n is an odd integer, then n is not even by contradiction. For the sake of contradiction, by HW4 fact 2 and 3, assume that n is both an odd integer $2k+1$ and even integer $2l$, for some integers k, l.

$$n = 2k + 1 = 2l \tag{1}$$
$$2k + 1 + (-2k) = 2l + (-2k) \tag{2}$$
$$1 = 2(l - k) \tag{3}$$

Let $l - k$ be some integer m.

$$1 = 2(l - k) = 2m \tag{4}$$

By HW4 fact 2, 1 is even. However, it contradicts the fact that 1 is not even, which we proved in HW4 Q1.
Therefore, if n is an odd integer, then n is not even.
In addition, "if n is an even integer, then n is not odd" is also true because it is the contrapositive of the proposition we just proved. □

3.

Proposition 3. *If $5n$ is odd, then n is odd.*

Proof. We will prove by using the contrapositive.

By HW4 fact 2, let n be some even integer $2k$ for some integer k. We will show that if n is not odd, then $5n$ is not odd, which means that if n is even, then $5n$ is even by HW4 Q2.

$$5n = 5(2k) \tag{5}$$
$$= 2(5k) \tag{6}$$

Let $5k$ be some integer l.

$$2(5k) = 2l \tag{7}$$

Hence, if n is even, then $5n$ is even if n is even by HW4 fact 4. Therefore, if 5n is odd, then n is odd. □

4.

Proposition 4. *if a, b are positive real numbers with $a \neq b$, then*

$$\frac{1}{a} + \frac{1}{b} \neq \frac{4}{a+b}. \tag{8}$$

Proof. We will prove by contradiction. Suppose for the sake of contradiction that, for some positive integers a, b, $a \neq b$ and

$$\frac{1}{a} + \frac{1}{b} = \frac{4}{a+b}. \tag{9}$$

We can do some arithmetic operations to the equation.

$$\frac{a+b}{ab} = \frac{4}{a+b} \tag{10}$$
$$(a+b)^2 = 4ab \tag{11}$$
$$a^2 + 2ab + b^2 = 4ab \tag{12}$$
$$a^2 - 2ab + b^2 = 0 \tag{13}$$
$$(a-b)^2 = 0 \tag{14}$$
$$\tag{15}$$

This suggests that $a - b = 0$, which implies that $a = b$. However, this contradicts our assumption $a \neq b$.

Therefore, if a, b are positive real numbers with $a \neq b$, then

$$\frac{1}{a} + \frac{1}{b} \neq \frac{4}{a+b}. \tag{16}$$

□

5.

Proposition 5. $\sqrt[4]{2}$ *is irrational.*

Proof. We will prove by contradiction. Suppose for the sake of contradiction that $\sqrt[4]{2}$ is rational.

By HW4 fact 7, let $\sqrt[4]{2}$ be some rational number $\frac{m}{n}$, such that the greatest common divisor of m, n is 1.

$$\sqrt{2} = (\sqrt[4]{2})^2 \tag{17}$$

$$= (\frac{m}{n})^2 \tag{18}$$

$$= \frac{m^2}{n^2} \tag{19}$$

Let m^2 and n^2 be some integers k, l.

$$\frac{m^2}{n^2} = \frac{k}{l} \tag{20}$$

This shows that if $\sqrt[4]{2}$ is rational, then $\sqrt{2}$ is rational by HW4 fact 7. However, this contradicts the fact that $\sqrt{2}$ is irrational.

Therefore, $\sqrt[4]{2}$ is irrational. □

6.

Proposition 6. *There does not exist the smallest positive real number x such that for all positive real number y, we have $x \leq y$.*

Proof. We will prove by contradiction. Suppose for the sake of contradiction that there exists the smallest positive real number x such that for all positive real number y, we have $x \leq y$.

Let $y = \frac{x}{2}$. $\frac{x}{2}$ is a positive number, since $x > 0$ so $\frac{x}{2} > \frac{1}{2} \cdot 0 = 0$, which is positive by HW4 fact 5. This shows that $x > \frac{x}{2} = y$, which contradicts our assumption that for all positive real number y, we have $x \leq y$.

Therefore, there does not exist the smallest positive real number. □

7. (a)

Proposition 7. *$A \subseteq A \cup B$.*

Proof. Let $x \in A$. We will show that $x \in A \cup B$. $A \cup B$ implies that $(\forall y)[(y \in A \vee y \in B) \to (y \in A \cup B)]$. This shows that $(\forall x \in A)(x \in A \cup B)$. Therefore, $A \subseteq A \cup B$. $\square$

(b)

Proposition 8. *$A \cap B \subseteq A$.*

Proof. Let $x \in A \cap B$. We will show that $x \in A$. $A \cap B$ implies that $(\forall y)[(y \in A \cup B) \to (y \in A \wedge y \in B)]$. This shows that $(\forall x \in A \cap B)(x \in A)$. Therefore, $A \cap B \subseteq A$. $\square$

(c)

Proposition 9. *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. Let $x \in A$. We will show that $x \in C$. $A \subseteq B$ means that $(\forall x \in A)(x \in B)$. $B \subseteq C$ means that $(\forall x \in B)(x \in C)$. This shows that $(\forall x \in A)[(x \in B) \to (x \in C)]$. Since $(\forall x \in A)(x \in B)$, $(\forall x \in A)(x \in C)$. Therefore, $A \subseteq C$. $\square$

8.

Proposition 10. *If $A \cap B^c = \emptyset$, then $A \subseteq B$.*

Proof. Let $x \in A$. We will show that $(\forall x \in A)[(A \cap B^c = \emptyset) \to (x \in B)]$. $(x \in A \cap B^c)$ is equivalent to $(x \in A \wedge x \notin B)$. Hence, $A \cap B^c = \emptyset$ means that $(\nexists x \in A)(x \notin B)$, which is equivalent to $(\forall x \in A)(x \in B)$. Therefore, by definition, $A \subseteq B$. $\square$