Math 109 HW 5

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1.

Proposition 1. *If $A \subseteq B$ then $B^c \subseteq A^c$.*

Proof. Suppose that $A \subseteq B$. We will prove by the contrapositive. If $B^c = \emptyset$, then $B^c \subseteq A^c$, as $\emptyset$ is a subset of any set. Suppose that B is not empty. Let $x \notin B$. We will show that $x \notin A$.

Since $A \subseteq B$, we have $(\forall y \in U)[(y \in A) \to (y \in B)]$. The contrapositive of this statement is that $(\forall y \in U)[(y \notin B) \to (y \notin A)]$. Thus, since $x \notin B$, we have $x \notin A$.

Therefore $B^c \subseteq A^c$ by definition. $\square$

2.

Proposition 2. *If A, B are disjoint and $C \subseteq B$, then A, C are disjoint.*

Proof. Suppose that $(\forall y \in A)(y \notin B)$ and $C \subseteq B$. Let $x \in A$. We will show that $x \notin C$.

Since A, B are disjoint, $x \notin B$ because $x \in A$. Since $C \subseteq B$, we have $(\forall z \in C)(z \in B)$, which is equivalent to $(\forall z \notin B)(z \notin C)$. Thus, since $x \notin B$, we have $x \notin C$.

Therefore, A, C are disjoint by definition. $\square$

3. (a)

Proposition 3. *If $A \subseteq B$ and $A \neq \emptyset$, then A, B are not disjoint.*

Proof. Let $x \in A$. We will show that $x \in B$.

Since $A \subseteq B$, $x \in B$ because $x \in A$. Therefore, A, B are not disjoint by definition. $\square$

(b)

Proposition 4. *If $A \subseteq B$, then A, B can be disjoint.*

Proof. Consider $A = \emptyset$. Since an empty set is a subset of all sets, we have $A \subseteq B$. Since $A = \emptyset$, A, B are disjoint, as they do not share any common elements. $\square$

4.

Proposition 5. *$A, B, C \subseteq U$ and are not empty. If $(A \cap B)^c \subseteq C$, then $A \subseteq B \cup C$.*

Proof. Suppose that $(A \cap B)^c \subseteq C$. Let $x \in A$. We will show that $x \in B \cup C$.

We can separate the situation into two cases, $x \in B$ and $x \notin B$.

If $x \in B$, then $x \in B \cup C$.

If $x \notin B$, then $x \notin (A \cap B)$ beacause $x \notin A \vee x \notin B$, which means that $x \in (A \cap B)^c$. Since $(A \cap B)^c \subseteq C$, we have $x \in C$. Thus, $x \in B \cup C$.

Therefore, $x \in B \cup C$. $\square$

5. (a)

Proposition 6. *f is not injective.*

Proof. Consider $f(1)$ and $f(3)$, $1, 3 \in \mathbb{Z}$. $f(1) = f(3) = 1$, but $1 \neq 3$. Therefore, f is not injective by definition. $\square$

Proposition 7. *f is surjective.*

Proof. Let $b \in \{0, 1\}$. We will prove that there exist $a \in \mathbb{Z}$ such that $f(a) = b$.

We can separate it into 2 cases, $b = 0$ and $b = 1$.
If $b = 0$, There exist $a = 0$ such that $f(a) = f(0) = 0 = b$.
If $b = 1$, There exist $a = 1$ such that $f(a) = f(1) = 1 = b$.

Therefore, we have exhausted all possibilities of b and shown that f is surjective by definition. $\square$

(b)

Proposition 8. *g is injective.*

Proof. We will prove by the contrapositive of the definition of an injective function. Let $a_1, a_2 \in \{0, 1\}$. We will show that if $a_1 \neq a_2$, then $g(a_1) \neq g(a_2)$.

Let $a_1 = 0, a_2 = 1$. $a_1 = 0 \neq 1 = a_2$ and $g(a_1) = 1 \neq -1 = g(a_2)$.

Therefore, since there are no elements other than $0, 1$ in $\{0, 1\}$, we have exhausted all the possibilities and proved that g is injective. $\square$

Proposition 9. *g is surjective.*

Proof. Let $b \in \{1, -1\}$. We will prove that there exist $a \in \{0, 1\}$ such that $g(a) = b$.

We can separate it into 2 cases, $b = 1$ and $b = -1$.
If $b = 1$, There exist $a = 0$ such that $f(a) = f(0) = 1 = b$.
If $b = -1$, There exist $a = 1$ such that $f(a) = f(1) = -1 = b$.

Therefore, g is surjective by definition. $\square$

(c)

Proposition 10. *h is injective.*

Proof. Let $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$, $a_1b_1 = a_2b_2 = 1$. We will show that if $h(a_1, b_1) = h(a_2, b_2)$, then $a_1 = a_2$ and $b_1 = b_2$.

For all $(x, y) \in \mathbb{R}^2$ and $xy = 1$, $x, y \neq 0$ because if $x = 0$ or $y = 0$ then $xy = 0 \neq 1$.

Suppose that $h(a_1, b_1) = h(a_2, b_2)$. Since $a_1b_1 = a_2b_2 = 1$ and $a_1, a_2 \neq 0$, we can assume that $b_1 = \frac{1}{a_1}, b_2 = \frac{1}{a_2}$. Since $h(a_1, b_1) = h(a_2, b_2)$, we know that $a_1 = a_2$. Since $a_1 = a_2$, we have $b_1 = \frac{1}{a_1} = \frac{1}{a_2} = b_2$.

Therefore, h is injective by definition. $\square$

Proposition 11. *h is surjective.*

Proof. Let $c \in \mathbb{R}$. We will prove that there exist $(a, b) \in \mathbb{R}^2$ such that $h(a, b) = c$.

Let $c = a$. Since $h(a, b) = a$, we know that $h(a, b) = c$

Therefore, h is surjective by definition. $\square$

(d)

Proposition 12. *k is injective.*

Proof. First, let a be some non-negative even number $2m$, b be some non-negative odd number $2n + 1$, $a, b, m, n \in \mathbb{Z}$, $a, b, m, n \geq 0$. We will use contradiction to prove that if a, b are not both even or both odd, then $k(a) \neq k(b)$. Suppose for the sake of contradiction that $k(a) = k(b)$ and

$$k(a) = \frac{2m}{2} = m \tag{1}$$
$$k(b) = -\frac{(2n+1)+1}{2} = -n-1 \tag{2}$$
$$\tag{3}$$

Since $k(a) = k(b)$, we know that

$$m = -n-1 \tag{4}$$
$$m+n = -1 \tag{5}$$

This contradicts our original assumption that $m \geq 0, n \geq 0$. Therefore, if $k(a) = k(b)$, then a, b are both even or both odd.

Let $a_1, a_2 \in \mathbb{Z}$, $a_1, a_2 \geq 0$. Now we will show that if $k(a_1) = k(a_2)$, then $a_1 = a_2$.

Suppose that $k(a_1) = k(a_2)$. We can separate it into 2 cases, a_1, a_2 are both even, a_1, a_2 are both odd.

If a_1, a_2 are both even, since $k(a_1) = k(a_2)$, we know

$$\frac{a_1}{2} = \frac{a_2}{2} \tag{6}$$
$$a_1 = a_2 \tag{7}$$

If a_1, a_2 are both odd, since $k(a_1) = k(a_2)$, we know

$$-\frac{a_1+1}{2} = -\frac{a_2+1}{2} \tag{8}$$
$$a_1 + 1 = a_2 + 1 \tag{9}$$
$$a_1 = a_2 \tag{10}$$

Therefore, k is injective by definition. □

Proposition 13. *k is surjective.*

Proof. Let $b \in \mathbb{Z}$. We will prove that there exist some non-negative integer a such that $k(a) = b$. We can separate it into 2 cases, $b \geq 0$ and $b < 0$.

If $b \geq 0$, let $b = \frac{a}{2}$, then $k(a) = \frac{a}{2} = b$.

If $b < 0$, let $b = -\frac{a+1}{2}$, then $k(a) = -\frac{a+1}{2} = b$.

Therefore, k is surjective by definition. □

6. (a)

Proposition 14. *α is not a well defined function.*

Proof. Let $x \in \mathbb{R}$. We will show that $\alpha(x) \notin \mathbb{Z}$. Consider the case $x = \frac{1}{2}$. $\alpha(x) = \frac{1}{2}$, which is not an integer. Therefore, α is not a well defined function. $\square$

(b)

Proposition 15. *β is a not well defined function.*

Proof. Let $x \in \mathbb{Z}$. We will show that there exists $y_1, y_2 \in \{-1, 0, 1\}$, such that $\beta(x) = y_1$, $\beta(x) = y_2$, and $y_1 \neq y_2$. Consider the case $x = 2, y_1 = 1, y_2 = -1$. $\beta(x) = 1 = y_1$, and $\beta(x) = -1 = y_2$. This shows that there exists $x \in \mathbb{Z}$ such that $\beta(x)$ has multiple possible values. Therefore, β is not a well defined function. $\square$

(c)

Proposition 16. *$|-|$ is a well defined function.*

Proof. For existence: let $x \in \mathbb{R}$. We will show that there exists $y \in \mathbb{R}_{\geq 0}$ such that $|x| = y$. We can separate it into 2 cases, $x \geq 0$ and $x \leq 0$.

If $x \geq 0$, let $y = x$. Since $y = x \geq 0$, we have $y \in \mathbb{R}_{\geq 0}$. We then have $|x| = x = y$.
If $x \leq 0$, let $y = -x$. Since $x \leq 0$, we have $-x = y \geq 0$. Thus, $y \in \mathbb{R}_{\geq 0}$. We then have $|x| = -x = y$.

Therefore, for each $x \in \mathbb{R}$, there exists $y \in \mathbb{R}_{\geq 0}$ such that $|x| = y$.

For uniqueness: let $a \in \mathbb{R}, b_1 = |a|, b_2 = |a|, b_1, b_2 \in \mathbb{R}_{\geq 0}$. We will show that $b_1 = b_2$. We can separate it into 2 cases, $a \geq 0$ and $a \leq 0$.

If $a \geq 0$, $b_1 = |a| = a$ and $b_2 = |a| = a$. Thus, $y_1 = y_2$.
If $a \leq 0$, $b_1 = |a| = -a$ and $b_2 = |a| = -a$. Thus, $b_1 = b_2$.

Therefore, for each $a \in \mathbb{R}$, $|a|$ is unique.

Therefore, $|-|$ is a well defined function. $\square$

(d)

Proposition 17. *γ is a not well defined function.*

Proof. Let $x \in \mathbb{R}$. Consider the case $x = 0$. We will show that $\gamma(x) \notin \mathbb{R}$. $\gamma(x) = \frac{0}{|0|} = \frac{0}{0}$, which is undefined. Therfore, γ is not a well defined function. $\square$

7. (a)

Proposition 18. *If $g \circ f$ is injective, then f is injective.*

Proof. We will prove by contradiction. Suppose for the sake of contradiction that $c = f(a) = f(b)$, $a \neq b$, $a, b \in A$, $c \in B$.

$g(f(a)) = g(f(b)) = g(c)$. Since $g \circ f$ is injective, we know that $g(f(a)) = g(f(b))$ implies $a = b$. However, it contradicts our original assumption that $a \neq b$.

Therefore, f is injective. $\square$

(b)

Proposition 19. *If $g \circ f$ is injective, then g does not have to be injective.*

Proof. Consider the case

$$f : \mathbb{R}_{\geq 0} \to \mathbb{R} \tag{11}$$
$$f(x) = x \tag{12}$$
$$g : \mathbb{R} \to \mathbb{R}_{\geq 0} \tag{13}$$
$$g(x) = x^2 \tag{14}$$

Combining f and g, we get

$$g \circ f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \tag{15}$$
$$g \circ f(x) = x^2 \tag{16}$$

We will first show that $g \circ f$ is injective.
Let a, b be some non-negative real numbers. Suppose that $g \circ f(a) = g \circ f(b)$. We will show that $a = b$.

Since $g \circ f(a) = a^2$, $g \circ f(b) = b^2$,

$$a, b \geq 0 \tag{17}$$
$$a^2 = b^2 \tag{18}$$
$$a = b \tag{19}$$

Therefore, $g \circ f(b)$ is injective. However, g is not injective, as both $g(-1)$ and $g(1)$ equals to 1, and $-1, 1 \in \mathbb{R}$.

Therefore, g does not have to be injective. $\square$

(c)

Proposition 20. *If $g \circ f$ is surjective, then f does not have to be surjective.*

Proof. Consider the case

$$f : \{0\} \to \mathbb{R} \tag{20}$$
$$f(x) = 0 \tag{21}$$
$$g : \mathbb{R} \to \{0\} \tag{22}$$
$$g(x) = 0 \tag{23}$$

Combining f and g, we get

$$g \circ f : \{0\} \to \{0\} \tag{24}$$
$$g \circ f(x) = 0 \tag{25}$$

$g \circ f$ in surjective because $g \circ f(0) = 0$ and there are no elements other than 0 in $\{0\}$. However, f is not surjective, since there are no $a \in \{0\}$ such that $f(a) = 1$.
Therefore, f does not have to be surjective. □

(d)

Proposition 21. *If $g \circ f$ is surjective, then g is surjective.*

Proof. We will prove by contradiction. Suppose for the sake of contradiction that there exists some $z \in C$ such that for all $k \in B$, $g(k) \neq z$.

Since $g \circ f$ is surjective, we know that there exist some $x \in A$ such that $g \circ f(x) = z$. Let $y = f(x)$, $y \in B$. We then have

$$g \circ f(x) = g(f(x)) \tag{26}$$
$$= g(y) \tag{27}$$
$$= z \tag{28}$$

This shows that there exists some $y \in B$ such that $g(y) = z$. However, this contradicts our assumption that for all $k \in B$, $g(k) \neq z$.

Therefore, g is surjective. □

8.

Proposition 22. *For all $n \in \mathbb{Z}^+$, we have*

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{29}$$

Proof. We proceed by induction on n.

If $n = 1$, then $1^2 = 1$, and $\frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$. Thus, the equation is correct when $n = 1$.

If $n = 2$, then $1^2 + 2^2 = 5$, and $\frac{2(2+1)(2\cdot 2+1)}{6} = \frac{2(3)(5)}{6} = 5$. Thus, the equation is correct when $n = 2$.

Suppose $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for some $n \in \mathbb{Z}^+$.
We then have

$$\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6} \tag{30}$$
$$= \frac{n(n+1)(2n+3)}{6} + \frac{2(n+1)(2n+3)}{6} \tag{31}$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{6} + \frac{2(n+1)(2n+3)}{6} \tag{32}$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{6n^2 + 12n + 6}{6} \tag{33}$$
$$= \frac{n(n+1)(2n+1)}{6} + n^2 + 2n + 1 \tag{34}$$
$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \tag{35}$$
$$= 1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 \tag{36}$$

Thus, the equation also work for $n + 1$ when n works.
Therefore, for all $n \in \mathbb{Z}^+$,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{37}$$

□

9.

Proposition 23. *Define a sequence* $\{a_n\}$ *by* $a_1 = 1$, $a_2 = 3$, *and* $a_{n+2} = a_{n+1} + a_n$. *For all* $n \geq 1$, *we have*

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \tag{38}$$

Proof. We proceed by induction on n.

If $n = 1$, then

$$\begin{aligned}
1 &= a_1 && (39)\\
&= \left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right) && (40)\\
&= \frac{1}{2} + \frac{1}{2} && (41)\\
&= 1. && (42)
\end{aligned}$$

Thus, the equation is correct when $n = 1$.
If $n = 2$, then

$$\begin{aligned}
3 &= a_2 && (43)\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^2 + \left(\frac{1-\sqrt{5}}{2}\right)^2 && (44)\\
&= \left(\frac{6+2\sqrt{5}}{4}\right) + \left(\frac{6-2\sqrt{5}}{4}\right) && (45)\\
&= \frac{6}{4} + \frac{6}{4} && (46)\\
&= 3. && (47)
\end{aligned}$$

Thus, the equation is correct when $n = 2$.
Suppose that for some $n \geq 1$, we have

$$\begin{aligned}
a_n &= \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n && (48)\\
a_{n+1} &= \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} && (49)
\end{aligned}$$

We then have

$$
\begin{aligned}
a_{n+2} &= \left(\frac{1+\sqrt{5}}{2}\right)^{n+2} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+2} && (50)\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{1-\sqrt{5}}{2}\right)^{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n} && (51)\\
&= \left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n} && (52)\\
&= \left(1+\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(1+\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n} && (53)\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{1-\sqrt{5}}{2}\right)^{n} + \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n} && (54)\\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{1-\sqrt{5}}{2}\right)^{n} + \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} && (55)\\
&= a_n + a_{n+1} && (56)
\end{aligned}
$$

Thus, the equation is correct for $n+2$ if the equation is correct for $n+1$ and n. Since the equation is correct when $n = 1$ and $n = 2$, the equation is correct for all $n \geq 1$.

Therefore, for all $n \geq 1$,

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{1-\sqrt{5}}{2}\right)^{n} \tag{57}$$

□