MATH 188: Homework #4

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Problem 1

(a) Let r be a fixed nonnegative integer. Show that both $S(n+r, n)$ and $c(n+r, n)$ are polynomial functions of n of degree $2r$ for $n \geq 0$.

Proof. We first prove the case for $S(n+r, n)$. Consider the number k of non-singleton blocks in a partition of $[n+r]$ with n blocks. To count the number of partitions with exactly k non-singleton blocks, we first pick the $n-k$ elements from $[n+r]$ that are in singletons, and then we calculate the number $a_{k,r}$ of possible orientations of the remaining $k+r$ elements. Note that $a_{k,r}$ is not dependent on n. Hence, summing over all possible k, we have

$$S(n+r, n) = \sum_{k=1}^{r} \binom{n+r}{n-k} a_{k,r} = \sum_{k=1}^{r} \binom{n+r}{r+k} a_{k,r}.$$

But then $\binom{n+r}{r+k} a_{k,r}$ is a polynomial of n of degree $r+k$. Since $r+k$ goes up to $2r$ exactly once, $S(n+r, n)$ is a polynomial of n of degree $2r$.

The similar argument works for $c(n+r, n)$. Consider the number k of non-trivial cycles in a permutation of size $n+r$ with n disjoint cycles. To count the number of permutation with exactly k non-trivial cycles, we first pick the $n-k$ elements from $[n+r]$ such that each of them are cycles on its own, and then we calculate the number $a_{k,r}$ of possible cycle formations of the remaining $k+r$ elements. Note that $b_{k,r}$ is not dependent on n. Hence, summing over all possible k, we have

$$c(n+r, n) = \sum_{k=1}^{r} \binom{n+r}{n-k} b_{k,r} = \sum_{k=1}^{r} \binom{n+r}{r+k} b_{k,r}.$$

But then $\binom{n+r}{r+k} b_{k,r}$ is a polynomial of n of degree $r+k$. Since $r+k$ goes up to $2r$ exactly once, $c(n+r, n)$ is a polynomial of n of degree $2r$. $\square$

(b) Compute these polynomials for $r = 2, 3$.

Proof. We first compute $S(n+r, n)$ for $r = 2, 3$. When $r = 2$, there are either 1 or 2 non-singleton blocks. If there is only one non-singleton block, then 3 elements are in a block and the remaining elements each form a singleton, which has $\binom{n+2}{3}$ possibilities. If there are 2 non-singleton blocks, then there are 2 blocks of size 2 and $n-2$ singletons, which has $3\binom{n+2}{4}$ possibilities. Hence, $S(n+2, n) = \binom{n+2}{3} + 3\binom{n+2}{4}$.When $r = 3$, the number of non-singleton blocks ranges from 1 to 3. If there is only one non-singleton block, then 4 elements are in a block and the remaining elements each form a singleton, which has $\binom{n+3}{4}$ possibilities. If there are 2 non-singleton blocks, then there is a block of size 2, a block of size 3, and $n-2$ singletons, which has $\binom{5}{2}\binom{n+3}{5} = 10\binom{n+3}{5}$ possibilities. If there are 3 non-singleton blocks, then there are 3 blocks of size 2 and all singletons for the rest, which has $\frac{1}{3!}\binom{6}{2}\binom{4}{2}\binom{n+3}{6} = 15\binom{n+3}{6}$ possibilities. Hence, $S(n+3, n) = \binom{n+3}{4} + 10\binom{n+3}{5} + 15\binom{n+3}{6}$.

We now compute $c(n+r, n)$ for $r = 2, 3$. When $r = 2$, there are either 1 or 2 non-trivial cycles. If there is only one non-trivial cycle, then there is a 3-cycle and $n-1$ singletons, which has $2\binom{n+2}{3}$ possibilities. If there are 2 non-trivial cycles, then there are 2 transpositions and $n-2$ singletons, which has $3\binom{n+2}{4}$ possibilities. Hence, $c(n+2, n) = 2\binom{n+2}{3} + 3\binom{n+2}{4}$. When $r = 3$, the number of non-trivial cycles ranges from 1 to 3. If there is only one non-trivial cycle, then there is a 4-cycle and $n-1$ singletons, which has $3!\binom{n+3}{4}$ possibilities. If there are 2 non-trivial cycles, then there is a transposition, a 3-cycle, and $n-2$ singletons, which has $2\binom{5}{2}\binom{n+3}{5} = 20\binom{n+3}{5}$ possibilities. If there are 3 non-trivial cycles, then there are 3 transpositions and all singletons for the rest, which has $\frac{1}{3!}\binom{6}{2}\binom{4}{2}\binom{n+3}{6} = 15\binom{n+3}{6}$ possibilities. Hence, $c(n+3, n) = 6\binom{n+3}{4} + 20\binom{n+3}{5} + 15\binom{n+3}{6}$. $\square$

Problem 2

For $n > 0$, let a_n be the number of partitions of n such that every part appears at most twice, and let b_n be the number of partitions of n such that no part is divisible by 3. Set $a_0 = b_0 = 1$. Show that $a_n = b_n$ for all n.

Proof. Let $A(x)$ be the generating function of a_n and $B(x)$ be the generating function of b_n. Since a_n is the number of partitions of n such that every part appears at most twice,

$$A(x) = \sum_{n\geq 0} a_n x^n = \prod_{i\geq 1}(1 + x^i + x^{2i}),$$

as we either choose 1, x^i, or x^{2i} from the ith term, when multiplying out the right side. What we get then is x^N where N where N is the sum of the i where we chose x^i or x^{2i}. But we get x^N one time for every partition of N into parts which repeat at most once, so the coefficient is a_N.

On the other hand, since b_n is the number of partitions of n such that no part is divisible by 3,

$$B(x) = \sum_{n\geq 0} b_n x^n = \prod_{i\geq 1, 3\nmid i} \frac{1}{1-x^i} = \frac{\prod_{i\geq 1} \frac{1}{1-x^i}}{\prod_{i\geq 1} \frac{1}{1-x^{3i}}} = \prod_{i\geq 1} \frac{1-x^{3i}}{1-x^i}.$$

But then notice that $1 + x^i + x^{2i} = \frac{1-x^{3i}}{1-x^i}$ for all i. Hence,

$$A(x) = \prod_{i\geq 1} \frac{1-x^{3i}}{1-x^i} = B(x),$$

and the result now follows. □

Problem 3

Let y be a variable. Prove the following generalization of Example 3.27:

$$\prod_{i\geq 0}(1+x^{2i+1}y) = \sum_{r\geq 0}\frac{x^{r^2}y^r}{(1-x^2)(1-x^4)\cdots(1-x^{2r})}$$

Proof. Notice that $[y^k x^n]\prod_{i\geq 0}(1+x^{2i+1}y)$ is counting the number of partitions of n with k distinct odd parts, as the exponent of the y term indicates the number of times we picked the $x^{2i+1}y$ term when expanding the multiplication. On the other hand, from Example 3.27 we know

$$[y^k x^n]\sum_{r\geq 0}\frac{x^{r^2}y^r}{(1-x^2)(1-x^4)\cdots(1-x^{2r})} = [y^k x^n]\sum_{r\geq 0} y^r\left(x^{r^2}\sum_{n\geq 0}p_{\leq r}(n)x^{2n}\right)$$

is counting the number of self-conjugate partitions of n with a Durfee square of size k. We now show that there is a bijection between the set of self-conjugate partitions with Durfee square of size r and the set of partition with r distinct odd parts. Given a self-conjugate partition of n which has a Durfee square of size r, we may use the reversible transformation described in Theorem 3.26 to obtain a new partition of n with r distinct odd parts, and thus the bijection. The result now follows from the bijection. $\square$

Problem 4

(a) Use the following q-analogue of Pascal's identity (you don't need to prove it)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \quad \text{for } n \geq k > 0$$

to show that if d is a non-negative integer, then

$$\sum_{n\geq 0} \begin{bmatrix} n+d \\ n \end{bmatrix}_q x^n = \prod_{i=0}^{d} (1-q^i x)^{-1} = \frac{1}{(1-x)(1-qx)\cdots(1-q^d x)}$$

Proof. We proceed by induction on d. If $d = 0$, then $\sum_{n\geq 0} \begin{bmatrix} n \\ n \end{bmatrix}_q x^n = \sum_{n\geq 0} x^n = \frac{1}{1-x}$, and the base case is done. Suppose $d \geq 1$. Then,

$$\begin{aligned}
\sum_{n\geq 0} \begin{bmatrix} n+d \\ n \end{bmatrix}_q x^n &= 1 + \sum_{n\geq 1} q^n \begin{bmatrix} n+(d-1) \\ n \end{bmatrix}_q x^n + \sum_{n\geq 1} \begin{bmatrix} (n-1)+d \\ n-1 \end{bmatrix}_q x^n \\
&= \sum_{n\geq 0} \begin{bmatrix} n+(d-1) \\ n \end{bmatrix}_q (qx)^n + x \sum_{n\geq 0} \begin{bmatrix} n+d \\ n \end{bmatrix}_q x^n \\
&= \frac{1}{1-x} \sum_{n\geq 0} \begin{bmatrix} n+(d-1) \\ n \end{bmatrix}_q (qx)^n \\
&= \frac{1}{(1-x)(1-qx)\cdots(1-q^d x)},
\end{aligned}$$

where the last equality follows from induction. □

(b) Give a direct explanation (i.e., independent of the Schubert decomposition explanation from lecture) for why the coefficient of x^n of the right side is the sum $\sum q^{|\lambda|}$ over all integer partitions λ whose Young diagram fits in the $n \times d$ rectangle.

Proof. Note that

$$[x^n] \prod_{i=0}^{d} (1-q^i x)^{-1} = \sum_{\substack{(a_0,\ldots,a_d)\in\mathbb{Z}^d \\ a_0+\cdots+a_d=n}} q^{a_1+\cdots+da_d}.$$

We now show the bijection between the weak compositions of n with $d+1$ parts and the integer partitions λ whose Young diagram fits in the $n \times d$ rectangle.

Given an integer partitions λ whose Young diagram fits in the $n \times d$ rectangle, let a_i be the number of parts of λ that are equal to $i \geq 1$ and put $a_0 = n - a_1 - \cdots - a_d$. Then, $(a_0, a_1, \ldots, a_d)$ is a weak composition of n.

On the other hand, given a weak compositions $(a_0, \ldots, a_d)$ of n, there is an integer partition λ with a_i number of i's for all $i \geq 1$. Since each part of λ is at most d and $\ell(\lambda) \leq n$, the Young diagram of λ fits in the $n \times d$ rectangle.

But then $a_1 + \cdots + da_d = |\lambda|$, and thus

$$\sum_{\substack{(a_0,\ldots,a_d)\in\mathbb{Z}^d \\ a_0+\cdots+a_d=n}} q^{a_1+\cdots+da_d} = \sum q^{|\lambda|}.$$

□

Problem 5

Let V, W be $\mathbf{F}_q$-vector spaces with $\dim V = n$ and $\dim W = m$.

(a) How many linear maps $V \to W$ are there?

Proof. Consider the number of ways we can map the canonical basis vectors $e_1, \ldots, e_n$ of V to some vectors in W. Since there are q^m choices of vectors for each e_i to be sent to, there are q^{mn} choices in total. Hence, there are q^{mn} linear maps $V \to W$. $\square$

(b) Suppose $n \geq m$. How many surjective linear maps $V \to W$ are there?

Proof. By the universal property of a quotient and the First Isomorphism Theorem, any surjective linear map $\phi : V \to W$ corresponds to a unique induced isomorphism $u : V/\mathrm{Ker}\ \phi \to W$. Note that Ker ϕ is of $(n-m)$-dimension and the number of isomorphisms $V/\mathrm{Ker}\ \phi \to W$ is equal to $|\mathbf{GL}_m(\mathbf{F}_q)|$. Hence, there is a bijection between the set of surjective linear maps $V \to W$ and $\mathbf{Gr}_{n-m}(\mathbf{F}_q^n) \times \mathbf{GL}_m(\mathbf{F}_q)$. But then by Theorem 3.34 and 3.35,

$$|\mathbf{Gr}_{n-m}(\mathbf{F}_q^n)| = \begin{bmatrix} n \\ m \end{bmatrix}_q, \quad |\mathbf{GL}_m(\mathbf{F}_q)| = \prod_{i=0}^{m-1} (q^m - q^i),$$

and thus there are $\begin{bmatrix} n \\ m \end{bmatrix}_q \prod_{i=0}^{m-1} (q^m - q^i) = \prod_{i=0}^{m-1} (q^n - q^i)$ surjective linear maps $V \to W$. $\square$

(c) Pick $k \leq \min(m, n)$. How many rank k linear maps $V \to W$ are there?

Proof. By the universal property of a quotient and the First Isomorphism Theorem, any linear map $\phi : V \to W$ of rank k corresponds to a unique induced isomorphism from V/K to some k-dimensional subspace U of W, where K is the kernel of ϕ. Note that K is of $(n-k)$-dimension and the number of isomorphisms $V/K \to W$ is equal to $|\mathbf{GL}_k(\mathbf{F}_q)|$. Since there are $|\mathbf{Gr}_{n-k}(\mathbf{F}_q^n)|$ choices for K, $|\mathbf{Gr}_k(\mathbf{F}_q^m)|$ choices for U, and $|\mathbf{GL}_k(\mathbf{F}_q)|$ choices for isomorphisms $V/K \to W$, there are

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} (q^k - q^i) = \begin{bmatrix} m \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} (q^n - q^i)$$

rank k linear maps $V \to W$, by Theorem 3.34 and 3.35. $\square$

Problem 6

Prove
$$\sum_{n\geq 1} x^{n(n-1)/2} = \prod_{n\geq 1} \frac{1-x^{2n}}{1-x^{2n-1}}.$$

Proof. By the Jacobi triple product,
$$\prod_{n\geq 1}(1-x^{2n})(1+x^{2n-1}y^2)(1+x^{2n-1}y^{-2}) = \sum_{n=-\infty}^{\infty} x^{n^2}y^{2n}.$$

Hence, substituting x as $\sqrt{x}$ and y as $\sqrt[4]{x}$, we have
$$\begin{aligned}\prod_{n\geq 1}(1-x^n)(1+x^n)(1+x^{n-1}) &= \sum_{n=-\infty}^{\infty} x^{n(n+1)/2} \\ &= 1 + \sum_{n\geq 1} x^{n(n+1)/2} + x^{n(n-1)/2} \\ &= \sum_{n\geq 0} x^{n(n+1)/2} + \sum_{n\geq 1} x^{n(n-1)/2} \\ &= 2\sum_{n\geq 0} x^{n(n+1)/2}.\end{aligned}$$

It now follows that
$$\begin{aligned}\sum_{n\geq 0} x^{n(n+1)/2} &= \frac{1}{2}\prod_{n\geq 1}(1-x^n)(1+x^n)(1+x^{n-1}) \\ &= \frac{1}{2}\prod_{n\geq 1}(1-x^{2n})(1+x^{n-1}) \\ &= \left(\prod_{n\geq 1}(1-x^{2n})\right)\left(\frac{1}{2}\prod_{n\geq 0}(1+x^n)\right) \\ &= \left(\prod_{n\geq 1}(1-x^{2n})\right)\left(\prod_{n\geq 1}(1+x^n)\right) \\ &= \prod_{n\geq 1}\frac{1-x^{2n}}{1-x^{2n-1}},\end{aligned}$$

where the last step follows from Theorem 3.25. □

Problem 7

Pick integers satisfying $1 \leq k_1 < k_2 < \cdots < k_r \leq n$. Let X be the set of subspaces $W_1, \dots, W_r$ of F_q^n such that $\dim W_i = k_i$ for all i and $W_i \subset W_{i+1}$ for $i < r$.

(a) Find a formula for $|X|$ by generalizing Example 3.39, i.e., use a q-analogue of a multinomial coefficient.

Proof. For any $n \geq k_r > \cdots > k_1 \geq 1$, we show that there are $\begin{bmatrix} n \\ k_1, k_2 - k_1, \dots, n - k_r \end{bmatrix}_q$ ways of picking subspaces $W_1, \dots, W_r$ of $\mathbf{F}_q^n$ by induction on r. If $r = 1$, it is obvious that there are $\begin{bmatrix} n \\ k_1 \end{bmatrix}_q$ ways of picking W_1. Suppose $r \geq 2$. There are $\begin{bmatrix} n \\ k_r \end{bmatrix}_q$ ways of picking W_r. But then by induction, there are $\begin{bmatrix} k_r \\ k_1, k_2 - k_1, \dots, k_r - k_{r-1} \end{bmatrix}_q$ ways of picking $W_1, \dots, W_{r-1}$ which are contained in W_r. It now follows that there are

$$\begin{bmatrix} n \\ k_r \end{bmatrix}_q \begin{bmatrix} k_r \\ k_1, k_2 - k_1, \dots, k_r - k_{r-1} \end{bmatrix}_q = \begin{bmatrix} n \\ k_1, k_2 - k_1, \dots, n - k_r \end{bmatrix}_q$$

ways of picking $W_1, \dots, W_r$ of $\mathbf{F}_q^n$. □

(b) $|X|$ is also a polynomial in q; find an explicit description of this polynomial using a generalization of the Schubert decomposition of the Grassmannian.

Proof. By the Schubert decomposition of $\mathbf{Gr}_k(\mathbf{F}_q^n)$, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = |\mathbf{Gr}_k(\mathbf{F}_q^n)| = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|},$$

where λ is any integer partition whose Young diagram fits into the $k \times (n-k)$ box. It now follows that,

$$\begin{aligned} |X| &= \begin{bmatrix} n \\ k_1, k_2 - k_1, \dots, n - k_r \end{bmatrix}_q \\ &= \begin{bmatrix} n \\ k_r \end{bmatrix}_q \begin{bmatrix} k_r \\ k_{r-1} \end{bmatrix}_q \cdots \begin{bmatrix} k_2 \\ k_1 \end{bmatrix}_q \\ &= \left(\sum_{\lambda \subseteq k_r \times (n-k_r)} q^{|\lambda|} \right) \left(\sum_{\lambda \subseteq k_{r-1} \times (k_r - k_{r-1})} q^{|\lambda|} \right) \cdots \left(\sum_{\lambda \subseteq k_1 \times (k_2 - k_1)} q^{|\lambda|} \right). \end{aligned}$$

□