MATH 140B: Homework #7

Due on May 29, 2024 at 23:59pm

Professor Seward

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Problem 1

Let K be the unit circle in the complex plane (i.e., the set of all z with $|z| = 1$), and let $\mathscr{A}$ be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta} \quad (\theta \text{ real}).$$

Then $\mathscr{A}$ separates points on K and $\mathscr{A}$ vanishes at no point of K, but nevertheless there are continuous functions on K which are not in the uniform closure of $\mathscr{A}$.

Hint: For every $f \in \mathscr{A}$

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta}\, d\theta = 0,$$

and this is also true for every f in the closure of $\mathscr{A}$.

Proof. Since $\mathscr{A}$ contains the identity function, $\mathscr{A}$ seperate points and vanishes at no points of K. We now show that there exists functions not in the uniform closure of $\mathscr{A}$. Suppose

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}.$$

Then,

$$\begin{aligned}
\int_0^{2\pi} f(e^{i\theta}) e^{i\theta}\, d\theta &= \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{in\theta} e^{i\theta}\, d\theta \\
&= \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{(n+1)i\theta}\, d\theta \\
&= \sum_{n=0}^{N} c_n \int_0^{2\pi} \cos((n+1)\theta)\, d\theta + \sum_{n=0}^{N} ic_n \int_0^{2\pi} \sin((n+1)\theta)\, d\theta \\
&= \sum_{n=0}^{N} \frac{c_n}{n+1} \int_0^{2\pi(n+1)} \cos(u)\, du + \sum_{n=0}^{N} \frac{ic_n}{n+1} \int_0^{2\pi(n+1)} \sin(u)\, du = 0.
\end{aligned}$$

Now suppose g is a limit point of $\mathscr{A}$. There exists a sequence $\{f_m\}$ of functions from $\mathscr{A}$ which converges to g uniformly. Then,

$$\int_0^{2\pi} g(e^{i\theta}) e^{i\theta}\, d\theta = \lim_{m\to\infty} \int_0^{2\pi} f_m(e^{i\theta}) e^{i\theta}\, d\theta = 0.$$

Now consider $g(e^{i\theta}) = e^{-i\theta}$. We have

$$\int_0^{2\pi} g(e^{i\theta}) e^{i\theta}\, d\theta = \int_0^{2\pi} e^{-i\theta} e^{i\theta}\, d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

But then g is not in the uniform closure of $\mathscr{A}$. $\square$

Problem 2

Define

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$.

Proof. We proceed by induction on n to show that

$$f^{(n)}(x) = \begin{cases} p_n(1/x)e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $p_n(1/x)$ is some polynomial function on $\frac{1}{x}$. Suppose $n = 1$. By Theorem 8.6(f),

$$f'(0) = \lim_{h \to 0} h^{-1} e^{-1/h^2} = 0.$$

On the other hand,

$$f'(x) = 2x^{-3}e^{-1/x^2}$$

if $x \neq 0$, and the base case is done. Now suppose $n \geq 2$. If $x = 0$,

$$\begin{aligned} f^{(n)}(0) &= \lim_{h \to 0} h^{-1}(f^{(n-1)}(h) - f^{(n-1)}(0)) \\ &= \lim_{h \to 0} (h^{-1} p_{n-1}(1/h))e^{-1/h^2} = 0, \end{aligned}$$

by induction and Theorem 8.6(f). If $n \neq 0$, by induction,

$$\begin{aligned} f^{(n)}(x) &= (p_{n-1}(1/x)e^{-1/x^2})' \\ &= (p_{n-1}(1/x))'e^{-1/x^2} + p_{n-1}(1/x)(e^{-1/x^2})' \\ &= \frac{1}{x}p'_{n-1}(1/x)e^{-1/x^2} + 2x^{-3}p_{n-1}(1/x)e^{-1/x^2} \\ &= p_n(1/x)e^{-1/x^2}, \end{aligned}$$

for some polynomial $p_n(1/x)$.

$\square$

Problem 3

Prove the following limit relations:

(a) $\lim_{x\to 0} \frac{b^x-1}{x} = \log b \quad (b > 0)$.

Proof. By L'Hopital's rule,

$$\lim_{x\to 0} \frac{b^x-1}{x} = \lim_{x\to 0} \frac{e^{x\log b}-1}{x} = \lim_{x\to 0} e^{x\log b}\log b = \log b.$$

$\square$

(b) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$.

Proof. By L'Hopital's rule,

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1.$$

$\square$

(c) $\lim_{x\to 0}(1+x)^{1/x} = e$.

Proof. By (b),

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0} e^{\frac{\log(1+x)}{x}} = e.$$

$\square$

(d) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$.

Proof. Fix x. Put $y = \left(1+\frac{x}{n}\right)^n$. Then,

$$\lim_{n\to\infty} \log y = \lim_{n\to\infty} n\log\left(1+\frac{x}{n}\right).$$

By L'Hopital's rule,

$$\begin{aligned} \lim_{n\to\infty} n\log\left(1+\frac{x}{n}\right) &= \lim_{n\to\infty} \frac{\log\left(1+\frac{x}{n}\right)}{\frac{1}{n}} \\ &= \lim_{a\to 0} \frac{\log(1+ax)}{a} \\ &= \lim_{a\to 0} \frac{x}{1+ax} = x. \end{aligned}$$

It now follows that

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^{\log y} = e^x.$$

$\square$

Problem 4

(a) $\lim_{x\to 0} \frac{e-(1+x)^{1/x}}{x}$.

Proof. Exercise 4(c) shows that $\lim_{x\to 0}(1+x)^{1/x} = e$, and so we may apply L'Hopital's rule, and get

$$\lim_{x\to 0} \frac{e-(1+x)^{1/x}}{x} = \lim_{x\to 0} -\left(\frac{1}{x}\log(1+x)\right)'(1+x)^{1/x} = e\lim_{x\to 0}\frac{\log(1+x)-\frac{x}{1+x}}{x^2}.$$

Again by L'Hopital's rule

$$e\lim_{x\to 0}\frac{\log(1+x)-\frac{x}{1+x}}{x^2} = e\lim_{x\to 0}\frac{1}{2(1+x)^2} = e/2.$$

□

(b) $\lim_{n\to\infty} \frac{n}{\log n}\left(n^{1/n}-1\right)$.

Proof. We first note that $\frac{\log n}{n} \to 0$ as $n\to\infty$. Hence,

$$\lim_{n\to\infty}\frac{n}{\log n}\left(n^{1/n}-1\right) = \lim_{n\to\infty}\frac{e^{\frac{\log n}{n}}-1}{\frac{\log n}{n}} = \lim_{a\to 0}\frac{e^a-1}{a}.$$

But then this is just the derivative of e^x at $x=0$, which is 1. □

(c) $\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)}$.

Proof. We apply L'Hopital's rule three times and get,

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)} &= \lim_{x\to 0}\frac{\sin x - x\cos x}{x\cos x(1-\cos x)}\\
&= \lim_{x\to 0}\frac{x\sin x}{(\cos x - x\sin x)(1-\cos x)+x\cos x\sin x}\\
&= \lim_{x\to 0}\frac{\sin x + x\cos x}{-2x\sin^2 x+(4\cos x-2)\sin x+2x\cos^2 x - x\cos x}\\
&= \lim_{x\to 0}\frac{2\cos x - x\sin x}{-6\sin^2 x+(x-8x\cos x)\sin x+6\cos^2 x-3\cos x}\\
&= \frac{2}{3}.
\end{aligned}$$

□

(d) $\lim_{x\to 0}\frac{x-\sin x}{\tan x - x}$.

Proof. We apply L'Hopital's rule three times and get,

$$\begin{aligned}
\lim_{x\to 0}\frac{x-\sin x}{\tan x - x} &= \lim_{x\to 0}\frac{x\cos x - \cos x\sin x}{\sin x - x\cos x}\\
&= \lim_{x\to 0}\frac{\sin^2 x - x\sin x - \cos^2 x + \cos x}{x\sin x}\\
&= \lim_{x\to 0}\frac{(4\cos x - 2)\sin x - x\cos x}{\sin x + x\cos x}\\
&= \lim_{x\to 0}\frac{-4\sin^2 x + x\sin x + 4\cos^2 x - 3\cos x}{2\cos x - x\sin x} = \frac{1}{2}.
\end{aligned}$$

□

Problem 5

Suppose $f(x)f(y) = f(x+y)$ for all real x and y.

(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx},$$

where c is a constant.

Proof. It is obvious that for $x \neq 0$, $f(x) = f(x)f(0)$, so $f(0) = 1$. We also note that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} = f'(0)f(x).$$

Let $c = f'(0)$ and consider $g(x) = e^{cx}f(-x)$. Since $g(0) = 1$ and

$$g'(x) = ce^{cx}f(-x) - e^{cx}f'(0)f(-x) = e^{cx}f(-x)(c - f'(0)) = 0,$$

we have $g(x) = 1$ for all x. But then

$$e^{cx}f(-x) = f(0) = f(x)f(-x).$$

Since f is non zero,

$$e^{-cx} = f(x).$$

□

(b) Prove the same thing, assuming only that f is continuous.

Proof. Given $r = p/q \in \mathbb{Q}$, we have

$$f(r) = f\left(p \cdot \frac{1}{q}\right) = (f(1/q))^n = (f(1)^{1/q})^p = (f(1))^r.$$

Hence,

$$\log f(r) = r \log f(1).$$

Put $c = \log f(1)$. Then,

$$f(r) = e^{\log f(r)} = e^{cr},$$

so $f(r) = e^{cr}$ for $r \in \mathbb{Q}$. Now for $x \in \mathbb{R}$, we have

$$e^{cx} = \sup e^{cr} = \sup f(r) = f(x) \quad (r < x, r \in \mathbb{Q})$$

as $\mathbb{Q}$ is dense in $\mathbb{R}$ and f is continuous. □