MATH 190A: Homework #8

Due on Mar 5, 2025 at 12:00pm

Professor McKernan

Section A02 8:00AM - 8:50AM
Section Leader: Zhiyuan Jiang

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

Show that if X is a Hausdorff topological space and A and B are compact, then $A \cap B$ is compact.

Proof. Since A, B are compact in a Hausdorff space, they are closed and Hausdorff. But then $A \cap B$ is a closed compact subset of A, so $A \cap B$ is compact. $\square$

Problem 2

Let

$$f : X \to Y$$

be a function with graph

$$\Gamma_f = \{(x, f(x)) \mid x \in X\}.$$

Suppose that X and Y are topological spaces.

(i) If Y is Hausdorff and f is continuous, then show that Γ_f is closed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) \backslash \Gamma_f$. Since $y \neq f(x)$, there exists disjoint open sets $U, V \subseteq Y$ such that $y \in U$ and $f(x) \in V$. But then f is continuous, so $f^{-1}(V)$ is open and $f(f^{-1}(V)) \subseteq V$. Hence, $f^{-1}(V) \times U$ is an open neighborhood of (x, y) that does not intersect Γ_f. The result now follows. □

(ii) Show that if Y is compact, then

$$p : X \times Y \to X$$

is a closed map (that is, the image of a closed set is closed).

Proof. Let $A \subseteq X \times Y$ be closed. Pick $x \in X \backslash p(A)$. Then $\{x\} \times Y$ is contained in the open set $(X \times Y) \backslash A$. Since Y is compact, the Tube Lemma yields an open neighborhood U_x of x such that $U_x \times Y \subseteq (X \times Y) \backslash A$. But then $U_x \times Y$ is an open neighborhood of x that does not intersect $p(A)$, so $p(A)$ is closed.

□

(iii) Show that if Y is compact and Γ_f is closed in $X \times Y$, then f is continuous.

Proof. Let $V \subseteq Y$ be open. Let $B = \Gamma_f \cap (X \times (Y \backslash V))$. Since B is closed, the projection $p : X \times Y \to X$ is a closed map, and so $p(B) = X \backslash f^{-1}(V)$ is closed in X. It now follows that $f^{-1}(V)$ is open, so f is continuous. □

Problem 3

Let A and B be subspaces of the topological spaces X and Y. Suppose that

$$A \times B \subset W \subset X \times Y.$$

If A and B are compact and W is open, then show that we can find U and V open in X and Y such that

$$A \times B \subset U \times V \subset W.$$

Proof. Since W is open, for each $(a, b) \in A \times B$ there exists open sets $U_{a,b} \subseteq X, V_{a,b} \subseteq Y$ such that $U_{a,b} \times V_{a,b} \subseteq W$, $a \in U_{a,b}$ and $b \in V_{a,b}$. Fix $a \in A$. Since B is compact, there exists a finite subcover $\{V_{a,b_i}\}_{i=1}^n$ of B. Let $U_a = \bigcap_{i=1}^n U_{a,b_i}$ and $V_a = \bigcup_{i=1}^n V_{a,b_i}$. Note that U_a is an open neighborhood of a and $U_a \times V_a \subseteq W$. Since $\{U_a\}_{a \in A}$ is an open cover of A and A is compact, there exists a finite subcover $\{U_{a_i}\}_{i=1}^n$ of A. Let $U = \cup_{i=1}^n U_{a_i}$ and $V = \cap_{i=1}^n V_{a_i}$. Since $B \subseteq V_{a_i}$ for all i, $B \subseteq V$. But then $A \times B \subseteq U \times V \subseteq W$ and U, V are open. $\square$

Problem 4

Let f and g be two functions

$$f : X \to Y \quad \text{and} \quad g : X \to Y.$$

Let

$$h : X \to Y \times Y$$

be the function

$$h(x) = (f(x), g(x)).$$

(i) Show that $f = g$ if and only if the image of h lands in the diagonal

$$\Delta = \{(y, y) \mid y \in Y\}.$$

Proof. $f = g$ if and only if $(f(x), g(x)) \in \Delta$ for all $x \in X$ if and only if $h(X) \subseteq \Delta$. □

(ii) Now suppose that f and g are continuous functions and Y is Hausdorff. If $A \subset X$ is dense and $f|_A = g|_A$, then show that $f = g$.

Proof. By homework 3 question 3, Δ is closed in $Y \times Y$. Hence, $h^{-1}(\Delta)$ is closed in X and $A \subseteq h^{-1}(\Delta)$. But then $X = \overline{A} \subseteq h^{-1}(\Delta)$, so $f = g$. □

Problem 5

Let X be a Hausdorff topological space.

(i) If A and B are disjoint compact subspaces of X, then show that we can find disjoint open sets U and V such that

$$A \subset U \quad \text{and} \quad B \subset V.$$

Proof. For $a \in A$ and $b \in B$, there exists disjoint open sets $U_{a,b}$ and $V_{a,b}$ such that $a \in U_{a,b}$ and $b \in V_{a,b}$. Fix $a \in A$. Then $\{V_{a,b}\}_{b \in B}$ is an open cover of B. Since B is compact, there exists a finite subcover $\{V_{a,b_i}\}_{i=1}^n$ of B. Let $U_a = \bigcap_{i=1}^n U_{a,b_i}$ and $V_a = \bigcup_{i=1}^n V_{a,b_i}$. Note that U_a is an open neighborhood of a and $U_a \cap B = \emptyset$. Since $\{U_a\}_{a \in A}$ is an open cover of A and A is compact, there exists a finite subcover $\{U_{a_i}\}_{i=1}^n$ of A. Let $U = \bigcup_{i=1}^n U_{a_i}$ and $V = \bigcap_{i=1}^n V_{a_i}$. Then $A \subseteq U$ and $B \subseteq V$ and $U \cap V = \emptyset$. $\square$

(ii) Suppose in addition that X is compact. Suppose we are given a collection of connected closed subsets

$$\{F_\alpha \mid \alpha \in \Lambda\}.$$

which are totally ordered by inclusion, so that given α and $\beta \in \Lambda$, either

$$F_\alpha \subset F_\beta \quad \text{or} \quad F_\beta \subset F_\alpha.$$

Show that

$$\bigcap_\alpha F_\alpha$$

is closed and connected.

Proof. Let $F = \bigcap_\alpha F_\alpha$. Since F is the intersection of closed sets, F is closed. Suppose $f : F \to \{0,1\}$ is continuous and not constant. Since F_α is connected for all α, $f(F_\alpha)$ is a singleton. Hence $f^{-1}(0) \supseteq F_\alpha$ and $f^{-1}(1) \supseteq F_\beta$ for some $\alpha, \beta \in \Lambda$. But then $f^{-1}(0) \cap f^{-1}(1) = \emptyset$, so F_α and F_β are not ordered, contradiction. $\square$