MATH 190A: Homework #4

Due on Feb 3, 2025 at 12:00pm

Professor McKernan

Section A02 8:00AM - 8:50AM
Section Leader: Zhiyuan Jiang

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

Find all topologies on the set

$$X = \{a, b, c, d\}$$

up to homeomorphism. (This means, give a list of topologies on X, such that every other topology on X is homeomorphic to exactly one topology in your list).

Proof. bruh.

1. $\{\emptyset, X\}$
2. $\{\emptyset, X, \{a, b\}\}$
3. $\{\emptyset, X, \{a, b, c\}\}$
4. $\{\emptyset, X, \{a, b\}, \{c, d\}\}$
5. $\{\emptyset, X, \{a, b\}, \{a, b, c\}\}$
6. $\{\emptyset, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$
7. $\{\emptyset, X, \{a\}\}$
8. $\{\emptyset, X, \{a\}, \{a, b\}\}$
9. $\{\emptyset, X, \{a\}, \{a, b, c\}\}$
10. $\{\emptyset, X, \{a\}, \{b, c, d\}\}$
11. $\{\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}\}$
12. $\{\emptyset, X, \{a\}, \{a, d\}, \{a, b, c\}\}$
13. $\{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$
14. $\{\emptyset, X, \{a\}, \{a, b, c\}, \{a, c, d\}\}$
15. $\{\emptyset, X, \{a\}, \{b, c\}, \{a, d\}, \{a, b, c\}\}$
16. $\{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$
17. $\{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$
18. $\{\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$
19. $\{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$
20. $\{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$
21. $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$
22. $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$
23. $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$
24. $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$
25. $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$

Problem 1 continued on next page...

26. $\{\emptyset, X, \{a\}, \{b\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}\}$
27. $\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}\}$
28. $\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}\}$
29. $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$
30. $\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}\}$
31. $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}\}$
32. $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{c,d\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}$
33. $\wp(X)$

$\square$

Problem 2

Let X and Y be two sets. Give both sets the topology where the closed sets are the finite sets, plus the whole set. Under what conditions are X and Y homeomorphic?

Proof. X and Y are homeomorphic if X and Y have the same cardinality. When X, Y are finite, the given topology is just the discrete topology, so they are homeomorphic if and only if $|X| = |Y|$. If X, Y are infinite, let $f : X \to Y$ be a bijection. Then f is continuous since the preimage of any closed set in Y is closed in X. The inverse function $f^{-1} : Y \to X$ is also continuous since the preimage of any closed set in X is closed in Y. Thus, f is a homeomorphism if $|X| = |Y|$. $\square$

Problem 3

Show that any two closed and bounded intervals in $\mathbb{R}$ are homeomorphic.

Proof. By lemma 7.3, it suffices to show that any any closed interval $[a,b]$ is homeomorphic to $[0,1]$. Let $f : [0,1] \to [a,b]$ be defined as

$$f(x) = a + (b-a)x.$$

A basis for the subspace topology is given by intervals $(\alpha,\beta), (\alpha,1]$, $[0,\beta)$, and $[0,1]$, for $\alpha,\beta \in (0,1)$ and $\alpha < \beta$, and f sends these to intervals $(a+(b-a)\alpha, a+(b-a)\beta), (a+(b-a)\alpha, b]$, $[a, a+(b-a)\beta)$, and $[a,b]$, respectively. The inverse of f is the function $g : [a,b] \to [0,1]$ which is defined as

$$g(x) = \frac{x-a}{b-a}.$$

A basis for the subspace topology for $[a,b]$ is given by intervals $(r,s), (r,d]$, $[c,s)$, and $[c,d]$, for $r,s \in (a,b)$, and g sends these to intervals $(\frac{r-a}{b-a}, \frac{s-a}{b-a}), (\frac{r-a}{b-a}, 1]$, $[0, \frac{s-a}{b-a})$, and $[0,1]$, respectively. Thus, f and g are homeomorphisms by lemma 7.4. $\square$

Problem 4

Complete the proof of Theorem 7.2.

Proof. Let $a \in \mathbb{R}$.

Define $f : (0, \infty) \to (a, \infty)$ to be $f(x) = a + x$. f sends interval $(\alpha, \beta) \subseteq (0, \infty)$ to $(\alpha + a, \beta + a) \subseteq (a, \infty)$. f's inverse $f^{-1} : (a, \infty) \to (0, \infty)$ defined as $f^{-1}(x) = x - a$ sends $(\gamma, \eta) \subseteq (a, \infty)$ to $(\gamma - a, \eta - a) \subseteq (0, \infty)$. Thus, f and f^{-1} are continuous by lemma 7.4, and thus $(0, \infty), (a, \infty)$ are homeomorphic.

Define $g : (a, \infty) \to (-\infty, a)$ to be $g(x) = -x$. g sends interval $(\alpha, \beta) \subseteq (a, \infty)$ to $(-\beta, -\alpha) \subseteq (-\infty, a)$. g's inverse $g^{-1} : (-\infty, a) \to (a, \infty)$ defined as $g^{-1}(x) = -x$ sends $(\gamma, \eta) \subseteq (-\infty, a)$ to $(-\eta, \gamma) \subseteq (a, \infty)$. Thus, g and g^{-1} are continuous by lemma 7.4, and thus $(a, \infty), (-\infty, a)$ are homeomorphic.

Define $h : (0, \infty) \to \mathbb{R}$ to be $h(x) = \ln x$. h sends interval $(\alpha, \beta) \subseteq (0, \infty)$ to $(\ln \alpha, \ln \beta) \subseteq \mathbb{R}$. h's inverse $h^{-1} : \mathbb{R} \to (0, \infty)$ defined as $h^{-1}(x) = e^x$ sends $(\gamma, \eta) \subseteq \mathbb{R}$ to $(e^\gamma, e^\eta) \subseteq (0, \infty)$. Thus, h and h^{-1} are continuous by lemma 7.4, and thus $(0, \infty), \mathbb{R}$ are homeomorphic. $\square$

Problem 5

True or false? If true then give a proof and if false then give a counterexample.

(i) If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are topological spaces then $X \times Y$ and $Y \times X$, both with the product topology, are homeomorphic.

Proof. True. Consider the map $f : X \times Y \to Y \times X$ that sends (x, y) to (y, x). This map is a bijection. Given any open set $U \times V \subseteq X \times Y$, we have $f(U \times V) = V \times U$ is open in $Y \times X$. Similarly, given any open set $V \times U \subseteq Y \times X$, we have $f^{-1}(V \times U) = U \times V$ is open in $X \times Y$. Thus, $X \times Y$ and $Y \times X$ are homeomorphic. □

(ii) Let $\mathcal{T}$ be the Euclidean topology on $\mathbb{R}$ and let $\mathcal{S}$ be the topology where the open sets are half open intervals of the form (a, ∞). Then $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{S})$ are homeomorphic.

Proof. False. Since $\mathcal{S} \subseteq \mathcal{T}$ but $(0, 1) \in \mathcal{T} \backslash \mathcal{S}$, $\mathcal{T}$ is strictly finer than $\mathcal{S}$. □

(iii) Let $\mathcal{T}$ be the topology where the open sets are half open intervals of the form (a, ∞) and let $\mathcal{S}$ be the topology where the open sets are half open intervals of the form $(-\infty, a)$. Then $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{S})$ are homeomorphic.

Proof. True. Consider the bijection $f : \mathbb{R} \to \mathbb{R}$ that sends x to $-x$. Then the induced map $F : \mathcal{T} \to \mathcal{S}$ is a bijection that sends (a, ∞) to $(-\infty, a)$. Thus, F is a homeomorphism. □

(iv) If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are homeomorphic topological spaces then $(X, \mathcal{T})$ is Hausdorff if and only if $(Y, \mathcal{S})$ is Hausdorff.

Proof. True. Let $f : X \to Y$ be a homeomorphism. Let $x_1, x_2 \in X$ be distinct points. Then $f(x_1), f(x_2) \in Y$ are distinct points. Since $(X, \mathcal{T})$ is Hausdorff, there exists disjoint open sets $U_1, U_2 \in \mathcal{T}$ such that $x_1 \in U_1$ and $x_2 \in U_2$. Then $f(U_1), f(U_2) \in \mathcal{S}$ are open sets such that $f(x_1) \in f(U_1)$, $f(x_2) \in f(U_2)$. Note that $f(U_1), f(U_2)$ are disjoint, otherwise $f^{-1}(f(U_1) \cap f(U_2)) = U_1 \cap U_2$ is nonempty. By syymmetry, the converse is also true. □

(v) If we are given four topological spaces, X, Y, Z and W and $X \times Y$ is homeomorphic to $Z \times W$ then X is homeomorphic to Z or X is homeomorphic to W.

Proof. False. Consider $X = Y = \mathbb{R}$, $Z = \mathbb{R}^2$, and $W = \{0\}$. Obviously $X \times Y = \mathbb{R}^2$ is homeomorphic to $Z \times W = \mathbb{R}^2 \times \{0\}$. However, X is not homeomorphic to Z or W. □