C8.4 Probabilistic Combinatorics: Sheet #0

Due on January 26, 2026 at 12:00pm

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Problem 1

Prove the following inequalities:

(a) $1 + x \leqslant e^x$ for all real x.

Proof. Note that $y = x + 1$ is the tangent line to $y = e^x$ at $x = 0$. Since e^x is convex, $e^x \geq x + 1$ for all real x. $\square$

(b) $e^{nx/(1+x)} \leqslant (1 + x)^n \leqslant e^{nx}$ for $x > -1, n \geqslant 0$.

Proof. Since $x > -1$, by (a) we have $e^x \geq 1 + x > 0$. Hence, $(1 + x)^n \leq e^{nx}$. Let $z = -x/(1 + x)$. By (a),

$$\frac{1}{1+x} = 1 + z \leq e^z = \frac{1}{e^{x/(1+x)}} \implies 0 < e^{x/(1+x)} \leq 1 + x.$$

The result now follows. $\square$

(c) $k! \geqslant (k/e)^k$ for $k \geqslant 1$.

Proof. Note that

$$e^k = \sum_{i=0}^{\infty} \frac{k^i}{i!} \geq 1 + k + \frac{k^2}{2!} + \cdots + \frac{k^k}{k!} \geq \frac{k^k}{k!}.$$

The result now follows. $\square$

(d) $\left(\frac{n}{k}\right)^k \leqslant \binom{n}{k} \leqslant \frac{n^k}{k!} \leqslant \left(\frac{en}{k}\right)^k$ for $1 \leqslant k \leqslant n$.

Proof. It is obvious that $\binom{n}{k} \leq \frac{n^k}{k!}$. By (c), $\frac{n^k}{k!} \leq \left(\frac{ne}{k}\right)^k$. Since $n \geq k$, we have $n/k \geq (n - i)/(k - i)$ for all $1 \leq i \leq k$. Hence,

$$\binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-k+1}{1} \geq \left(\frac{n}{k}\right)^k.$$

$\square$

Problem 2

For the following functions $f(n)$ and $g(n)$, decide whether $f = o(g)$ or $g = o(f)$ or $f = \Theta(g)$ as $n \to \infty$:

(a) $f(n) = \binom{n}{k}, g(n) = n^k$, first for k fixed and then for the case where $k = k(n) \to \infty$ as $n \to \infty$;

Proof. Since $\left(\frac{n}{k}\right)^k \leqslant \binom{n}{k} \leqslant \frac{n^k}{k!}$,

$$\frac{1}{k^k} \leq \frac{f(n)}{g(n)} \leq \frac{1}{k!}.$$

Thus if k is fixed, then $f(n) = \Theta(g)$. If $k = k(n) \to \infty$ as $n \to \infty$, then $f(n) = o(g)$. □

(b) $f(n) = (\log n)^{1000}, g(n) = n^{1/1000}$.

Proof. Put $x = \log n$. By applying L'Hopital's rule 1000 times,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{x\to\infty} \frac{x^{1000}}{e^{x/1000}} = \lim_{x\to\infty} \frac{1000!}{(0.001)^{1000} e^{x/1000}} = 0.$$

□

Problem 3

Find the simplest function $f(n)$ you can such that $(n-2)^{n+2}/n^n \sim f(n)$ as $n \to \infty$.

Proof. Note that $(n-2)^{n+2}/n^n = (n-2)^2 \cdot (1-2/n)^n$. But then

$$\lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^n = e^{-2}.$$

Thus,

$$(n-2)^{n+2}/n^n \sim n^2 e^{-2}.$$

$\square$

Problem 4

Show that if $n, k, \ell \geqslant 1$ are integers and $0 < p < 1$, then

$$R(k,\ell) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}.$$

Proof. Let $G \sim G(n,p)$. Let X be the number of cliques of size k and independent sets of size ℓ in G. Then $\mu = \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}$ is the expected number of k-cliques and ℓ-independent sets in G. Since $\mathbb{P}(X \leq \mu) > 0$, there exists a configuration such that G has at most μ k-cliques and ℓ-independent sets. Removing a vertex from each k-clique and ℓ-independent set in G yields a ramsey graph with at least $n - \mu$ vertices. The result now follows. $\square$

Problem 5

Let H be an r-uniform hypergraph with fewer than $\frac{3^{r-1}}{2^r}$ edges. Prove that the vertices of H can be coloured using three colours in such a way that in each edge, all three colours are represented.

Proof. Randomly the vertices of H with 3 colors uniformly and independently. Let A_e be the event that not all three colors are represented in edge e. Then,

$$\mathbb{P}(A_e) = \frac{2^r - 1}{3^{r-1}}.$$

Then the probability that there exists an edge without all three colors is

$$\mathbb{P}\left(\bigcup_{e \in E(H)} A_e\right) \leq \sum_{e \in E(H)} \mathbb{P}(A_e) < \frac{3^{r-1}}{2^r} \cdot \frac{2^r - 1}{3^{r-1}} < 1.$$

Thus, there exists a coloring of H such that no edge contains all three colors. $\square$

Problem 6

Let F be a collection of binary strings ("codewords") of finite length, where the ith codeword has length c_i. Suppose that no member of F is an initial segment of another member (so you can decode any string made up by concatenating codewords as you go along, without looking ahead). Show that $\sum_i 2^{-c_i} \leqslant 1$ (the *Kraft inequality* for prefix-free codes).

Proof. Let s be a random string of infinite length. Let X be the number of codewords in F that are prefixes of s. Since no member of F is an initial segment of another member, $X \leq 1$. But then

$$1 \geq \mathbb{E}[X] = \sum_{s_i \in F} \mathbb{P}(s_i \text{ is a prefix of } s) = \sum_{s_i \in F} 2^{-c_i}.$$

$\square$