MATH 190A: Homework #6

Due on Feb 19, 2025 at 12:00pm

Professor McKernan

Section A02 8:00AM - 8:50AM
Section Leader: Zhiyuan Jiang

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

Let X be the topological space whose closed sets are the finite sets plus the whole of X. When is X connected?

Proof. If X is finite, then X has the discrete topology, and is disconnected. Suppose X is infinite. Let $Y \subset X$ be open and Y is not empty or X. Then $X \backslash Y$ is finite, so Y is infinite. But then Y is not closed. Hence, X is connected if and only if X is infinite. $\square$

Problem 2

Let X be a topological space and let Y be a connected subspace. If

$$Y \subset Z \subset \overline{Y}$$

then prove that Z is connected.

Proof. Let A be the connected component of Z that contains Y. Since A is closed and contains Y, $Z \subseteq \overline{Y} \subseteq A \subseteq Z$. Hence, Z is connected. $\square$

Problem 3

Let $Y \subset \mathbb{R}^n$ be a subset.

(i) We say that Y is **convex** if the line between any two points p and q of Y is contained in Y,

$$\{tp + (1-t)q \mid t \in [0,1]\} \subset Y.$$

Show that if Y is convex then it is path-connected.

Proof. Let $p, q \in Y$. By definition, the line between p and q is contained in Y, and thus Y is path-connected. $\square$

(ii) We say that Y is **star-shaped** about y_0 if for any point $y \in Y$ the line connecting y_0 to y is contained in Y. Show that if Y is star-shaped then it is path-connected.

Proof. Let $p, q \in Y$. There is a path from p to y_0 and a path from y_0 to q. By connecting the two paths, we have a path from p to q. Thus, Y is path-connected. $\square$

Problem 4

True or false? If true then give a proof and if false then give a counterexample.

(i) The set

$$\mathbb{Q}^2 = \{(a, b) \mid a, b \in \mathbb{Q}\}$$

is connected.

Proof. False. Let $A = \{a \in \mathbb{Q}, a > \sqrt{2}\}$ and $B = \{b \in \mathbb{Q}, b < \sqrt{2}\}$. Then $(A \times \mathbb{Q}) \cap (B \times \mathbb{Q}) = \emptyset$ and $\mathbb{Q}^2 = (A \times \mathbb{Q}) \cup (B \times \mathbb{Q})$. But then $A = (\sqrt{2}, \infty) \cap \mathbb{Q}$ and $B = (-\infty, \sqrt{2}) \cap \mathbb{Q}$ are open. Thus, $\mathbb{Q}^2$ is disconnected. $\square$

(ii) The set

$$\mathbb{Q}^2 = \{(a, b) \mid a, b \in \mathbb{Q}\}$$

is path-connected.

Proof. False. Since

$$\mathbb{Q}^2$$

is disconnected, it cannot be path-connected. $\square$

(iii) The set

$$\mathbb{R}^2 \setminus \mathbb{Q}^2$$

is path-connected.

Proof. True. For $a \in \mathbb{R}\backslash\mathbb{Q}$, the lines $\{(a, r) \mid r \in \mathbb{R}\}$ and $\{(r, a) \mid r \in \mathbb{R}\}$ are in $\mathbb{R}^2\backslash\mathbb{Q}^2$. Let $p, q \in \mathbb{R}^2\backslash\mathbb{Q}^2$. Assume that $p_1, q_1 \notin \mathbb{Q}$. Then for some $k \in (\mathbb{R}\backslash\mathbb{Q}) \cap (p_2, q_2)$, the path

$$\{(p_1, r) \mid p_2 \leq r \leq k\} \cup \{(r, k) \mid p_1 \leq r \leq q_1\} \cup \{(q_1, r) \mid k \leq r \leq q_2\}$$

is a path from p to q and is contained in $\mathbb{R}^2\backslash\mathbb{Q}^2$. We may also find a path from p to q with a similar appraoch if $p_1, q_2 \notin \mathbb{Q}$. The result now follows. $\square$

(iv) The set

$$\mathbb{R}^2 \setminus \mathbb{Q}^2$$

is connected.

Proof. True. Since $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path-connected, it is also connected. $\square$

(v) If $f : X \to Y$ is continuous and surjective and X is path-connected then Y is path-connected.

Proof. True. Let $y_1, y_2 \in Y$. Since $f : X \to Y$ is surjective, there exist points $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. As X is path-connected, there is a continuous path $\gamma : [0, 1] \to X$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Then the composition $f \circ \gamma : [0, 1] \to Y$ is continuous, with $(f \circ \gamma)(0) = y_1$ and $(f \circ \gamma)(1) = y_2$. Hence, Y is path-connected. $\square$

(vi) If X and Y are path-connected topological spaces then $X \times Y$ is path-connected.

Problem 4 continued on next page. . .

Proof. True. Let (x_1, y_1) and (x_2, y_2) be any two points in $X \times Y$. Since X is path-connected, there exists a continuous path $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Similarly, there exists a continuous path $\sigma : [0, 1] \to Y$ with $\sigma(0) = y_1$ and $\sigma(1) = y_2$.

Define a path $\eta : [0, 1] \to X \times Y$ by

$$\eta(t) = \begin{cases} \big(\gamma(2t),\, y_1\big) & \text{if } 0 \le t \le \frac{1}{2}, \\ \big(x_2,\, \sigma(2t-1)\big) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then $\eta(0) = (x_1, y_1)$ and $\eta(1) = (x_2, y_2)$. Since the concatenation of continuous functions is continuous, η is a continuous path connecting (x_1, y_1) to (x_2, y_2). Therefore, $X \times Y$ is path-connected. $\square$

(vii) The path components of a topological space are closed.

Proof. False. Let $B = \{(x, \sin(1/x)) \in x \in (0, 1]\}$ and $A = \{(0, y) \in y \in [-1, 1]\}$. Consider $X = A \cup B \subset \mathbb{R}^2$. We know $X = \overline{B}$, so B is not closed. But then B is a path component of X. $\square$

(viii) The connected components of a topological space are open.

Proof. True. let C be a connected component of X. Let $x \in C$. For any connected subsets U containing x, we have $U \subseteq C$. Let $\{U_\alpha\}$ be the collection of connected subsets in C. Then $C = \bigcup_\alpha U_\alpha$. But then C is a union of open sets. $\square$

Problem 5

Let X be a connected topological space. We say that a point x is a **cut point** if $X - \{x\}$ is disconnected.

(i) Let $Y \subset \mathbb{R}^2$ be the union of two closed disks that intersect at one point (so that the boundary circles are tangent). Identify the cut points.

Proof. Let D_1 and D_2 be the two disks. We may assume that $D_1 = \overline{B}_1(1,0)$ and $D_2 = \overline{B}_1(-1,0)$. Let x be the point of tangency, i.e $x = (0,0)$. If x is removed, then D_1 and D_2 are disjoint. But then $D_1 \backslash \{0\} = Y \cap ((-2,0) \times \mathbb{R})$ and $D_2 \backslash \{0\} = Y \cap ((0,1) \times \mathbb{R})$ are open sets. Hence, x is a cut point of Y. $\square$

(ii) If $f : X \to Y$ is a homeomorphism then show that $x \in X$ is a cut point if and only if $y = f(x)$ is a cut point.

Proof. Suppose $x \in X$ is a cut point. Then $X \backslash \{x\}$ is disconnected. Let U and V be disjoint open sets such that $X - \{x\} = U \cup V$. Then $f(U), f(V)$ are disjoint open sets in Y such that $f(X - \{x\}) = Y - \{y\} = f(U) \cup f(V)$. But then $Y = f(X)$ is connected, and so $y = f(x)$ is a cut point. $\square$

(iii) Show that $[0,1]$ and $(0,1)$ are not homeomorphic.

Proof. Note that every point in $(0,1)$ is a cut point. But 0 and 1 are not cut points in $[0,1]$. Since homeomorphisms perserve cut points, $[0,1]$ and $(0,1)$ are not homeomorphic. $\square$

(iv) Give a complete list of all intervals in $\mathbb{R}$, up to homeomorphism.

Proof. $(0,1), [0,1), [0,1]$. $\square$

(v) Show that if $\mathbb{R}$ is homeomorphic to $\mathbb{R}^n$ then $n = 1$.

Proof. Note that every point in $(0,1)$ is a cut point. But $\mathbb{R}^n$ have no cut points for $n \geq 2$. Since homeomorphisms perserve cut points, R is homeomorphic to $\mathbb{R}^n$ only if $n = 1$. $\square$