MATH 264C: Homework

Professor Rhoades

Ray Tsai

A16848188

Problem 1

Let $c_n = \frac{1}{n+1}\binom{2n}{n}$ be the n^{th} Catalan number. Prove that

$$c_n = \sum_{i=1}^{n} c_{i-1} \cdot c_{n-i}$$

for all $n > 0$.

Proof. Note that c_n is the number of Dyck paths of size n. Observe that a Dyck path of size n consists of a prime Dyck path of size i starting at the origin and a Dyck path of size $n-i$, for some $i \in [n]$. Let p_n denote the number of prime Dyck paths of size n. Then

$$c_n = \sum_{i=1}^{n} p_i \cdot c_{n-i}.$$

Since we may bijectively map prime Dyck paths of size i to Dyck paths of size $i-1$ by removing the first and last steps, we have $p_i = c_{i-1}$. The result now follows. $\square$

Problem 2

Let $\tau \in \mathfrak{S}_k$ be a permutation (thought of as a 'pattern'). A permutation $w \in \mathfrak{S}_n$ is called τ-*avoiding* if its one-line notation $[w(1), \ldots, w(n)]$ does not contain a subsequence

$$w(i_1), \ldots, w(i_k) \quad \text{with } 1 \leq i_1 < \cdots < i_k \leq n$$

order-isomorphic to τ.

Prove that the number of 312-avoiding permutations $w \in \mathfrak{S}_n$ equals the Catalan number c_n. (Hint: Consider $w^{-1}(1)$.) Explain why c_n also counts τ-avoiding permutations in $\mathfrak{S}_n$ for the patterns $\tau = 132, 213, 231$.

Proof. Let A_n be the set of 312-avoiding permutations in $\mathfrak{S}_n$. For $i \in [n]$, we show a bijection between $\{w \in A_n : w^{-1}(1) = i\}$ and $A_{i-1} \times A_{n-i}$. Let $w \in A_n$ with $w^{-1}(1) = i$. Write $L = [w(1), \ldots, w(i-1)]$ and $R = [w(i+1), \ldots, w(n)]$. Notice that if $w(k) > w(m)$ for some $1 \leq k < i < m \leq n$, then $w(k), w(i), w(m)$ is a 312-pattern, contradiction. Hence, $\max L < \min R$. Now let $w_L = [w(1) - 1, \ldots, w(i-1) - 1] \in A_{i-1}$ and $w_R = [w(i+1) - i, \ldots, w(n) - i] \in A_{n-i}$ and map w to (w_L, w_R), and the above operation is reversible. This gives the desired bijection. Therefore, A_n has the recursion formula

$$|A_n| = \sum_{i=1}^{n} |A_{i-1}| \cdot |A_{n-i}|,$$

which is the same as the recursion for the Catalan numbers.

Since permutations 312 and 231 are isomorphic, we only need to consider 132 and 213. Consider the involutions $\text{inv}(w)$ and $\text{rev}(w)$, which maps w to w^{-1} and the reverse of w, respectively. Since $\text{inv}(312) = 132$ and $\text{rev}(312) = 213$, inv maps 312-avoiding permutations to 132-avoiding permutations, and rev maps 312-avoiding permutations to 213-avoiding permutations. Hence, we may the composition of the above bijection and inv or rev gives the desired bijection. Therefore, c_n also counts 132-avoiding and 213-avoiding permutations. $\square$

Problem 3

Let $S_{n,k}$ be the Stirling number of the second kind counting set partitions of $[n]$ into k blocks. Find an expression for the exponential generating function

$$\sum_{n \geq 0} S_{n,k} \cdot \frac{x^n}{n!}$$

where k is fixed.

Proof. Since $e^x - 1 = \sum_{n \geq 1} x^n/n!$ is the exponential generating function for non-empty sets, $(e^x - 1)^k$ counts the number of ways to partition number of ways to $[n]$ into k ordered blocks. Hence, $(e^x - 1)^k/k!$ counts the number of ways to partition $[n]$ into k unordered blocks, which is just $S_{n,k}$. Therefore,

$$\sum_{n \geq 0} S_{n,k} \cdot \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}.$$

$\square$

Problem 4

Each of n distinguishable telephone poles is painted red, white, blue, or yellow. An odd number are painted blue and an even number are painted yellow. In how many ways can this be done?

Proof. Write $E(x) = \sum_{n \text{ even}} x^n/n!$ and $O(x) = \sum_{n \text{ odd}} x^n/n!$. Then

$$E(x) = \frac{e^x + 1}{2} \quad \text{and} \quad O(x) = \frac{e^x - 1}{2}.$$

Note that $E(x)$ and $O(x)$ are the exponential generating function for even and odd sets, respectively. Then the exponential generating funtion for the ways to paint the poles is

$$e^x \cdot e^x \cdot E(x) \cdot O(x) = e^{2x} \cdot \frac{e^x - 1}{2} \cdot \frac{e^x + 1}{2} = \frac{e^{4x} - e^{2x}}{4}.$$

$\square$

Problem 5

Let G be a finitely generated group and write

$$\mathrm{Hom}(G, \mathfrak{S}_n) := \{\text{all homomorphisms } G \to \mathfrak{S}_n\}.$$

For any $n \geq 0$, the set $\mathrm{Hom}(G, \mathfrak{S}_n)$ is finite (why?). We consider the exponential generating function

$$F(x) := \sum_{n \geq 0} \#\mathrm{Hom}(G, \mathfrak{S}_n) \frac{x^n}{n!}.$$

If g_n denotes the number of *transitive* actions of G on $[n]$, prove that

$$F(x) = \exp\left(\sum_{d \geq 1} g_d \cdot \frac{x^d}{d!}\right).$$

Proof. Since a homomorphism $G \to \mathfrak{S}_n$ is equivalent to a G-action on $[n]$, we have $\#\mathrm{Hom}(G, \mathfrak{S}_n) = \#\{G\text{-actions on } [n]\}$. Given an G-action α on $[n]$, α induces a partition of $[n]$ into orbits $O_1 \sqcup \cdots \sqcup O_k$, and the restriction of α to O_i is a transitive action of G on O_i. In particular, α may be determined by k transitive actions on each orbit. Applying the exponential generating function, we get

$$F(x) = \exp\left(\sum_{d \geq 1} g_d \cdot \frac{x^d}{d!}\right).$$

$\square$

Problem 6

In the context of the previous problem, if $j_d(G)$ is the number of subgroups of G having index d, show that

$$g_d = j_d(G) \cdot (d-1)!.$$

Proof. Let T_d be the set of transitive G-actions on $[d]$, and let D be the set of subgroups of G with index d. Consider the map $f : T_d \to D$ which maps action α to $\mathrm{Stab}_\alpha(1)$. We show that f is surjective. Let $H \in D$. Let G act on G/H by left multiplication and note that it is transitive. Then each $g \in G$ yields a permutation $\sigma_g : G/H \to G/H$ by sending aH to $(ga)H$. Since H has index d, there is a bijection $\phi : G/H \to [d]$ such that $\phi(H) = 1$. Let $\alpha : G \to \mathfrak{S}_d$ by sending g to $\phi \circ \sigma_g \circ \phi^{-1}$. Then α is a transitive action of G on $[d]$ with $\mathrm{Stab}_\alpha(1)$. This shows that f is surjective. Since $f^{-1}(H)$ is the set of transitive actions on $[d]$ with H as the stabilizer of 1, there are $(d-1)!$ choices to map the remaining $d-1$ cosets of H to $[d]\backslash\{1\}$. Thus,

$$g_d = \sum_{H \subseteq G, [G:H]=d} |f^{-1}(H)| = j_d(G)(d-1)!.$$

$\square$

Problem 7

For each of the following sets of graphs, let $f(n)$ be the number of graphs on the labeled vertex set $[n]$ such that every connected component is isomorphic to some graph in the set. In each case, find the exponential generating function of $f(n)$, where $f(0) := 1$.

(a) Cycles C_i of length $i \geq k$ where $k \geq 3$ is fixed.

Proof. Given n vertices, the number of ways to order them into an undirected cycle is $(n-1)!/2$. Hence, let

$$C(x) = \sum_{n \geq k} \frac{(n-1)!}{2} \cdot \frac{x^n}{n!} = \frac{1}{2} \sum_{n \geq k} \frac{x^n}{n} = -\frac{1}{2} \ln(1-x) - \frac{1}{2} \sum_{i=1}^{k-1} \frac{x^i}{i}$$

be the exponential generating function that counts cycles of length k or more. Since the connected components of a graph on $[n]$ may be viewed as a partition of $[n]$,

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \exp(C(x)) = (1-x)^{-1/2} \exp\left(-\frac{1}{2} \sum_{i=1}^{k-1} \frac{x^i}{i}\right).$$

□

(b) Stars $K_{i,1}$ for some $i \geq 1$. (Here $K_{r,s}$ is the complete bipartite graph.)

Proof. Given n vertices, the number of ways to order them into a $K_{n-1,1}$ is n. Hence, let

$$S(x) = \sum_{n \geq 1} n \cdot \frac{x^n}{n!} = \sum_{n \geq 1} \frac{x^n}{(n-1)!} = x(e^x - 1)$$

be the exponential generating function that counts stars. Since the connected components of a graph on $[n]$ may be viewed as a partition of $[n]$,

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \exp(S(x)) = \exp[x(e^x - 1)].$$

□

(c) Wheels W_i for $i \geq 4$. (Here W_i is obtained from C_{i-1} by adding a new vertex connected to every other vertex.)

Proof. Given n vertices, the number of ways to order them into a W_n is $n \cdot (n-2)!/2$. Hence, let

$$W(x) = \sum_{n \geq 4} \frac{n(n-2)!}{2} \cdot \frac{x^n}{n!} = \sum_{n \geq 4} \frac{x^n}{2(n-1)} = \frac{x}{2} \sum_{n \geq 3} \frac{x^n}{n} = \frac{x}{2}\left(-\ln(1-x) - x - \frac{x^2}{2}\right)$$

be the exponential generating function that counts wheels. Since the connected components of a graph on $[n]$ may be viewed as a partition of $[n]$,

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \exp(W(x)) = \exp\left[\frac{x}{2}\left(-\ln(1-x) - x - \frac{x^2}{2}\right)\right].$$

□

(d) Paths P_i with $i \geq 1$ vertices. (So that P_1 is a single vertex and P_2 is a single edge.)

Proof. Given $n \geq 2$ vertices, the number of ways to order them into an undirected path is $n!/2$. Hence, let

$$P(x) = x + \sum_{n \geq 2} \frac{n!}{2} \cdot \frac{x^n}{n!} = x + \frac{1}{2} \sum_{n \geq 2} x^n = \frac{x^2 - 2x + 2}{2 - 2x}$$

be the exponential generating function that counts paths. Since the connected components of a graph on $[n]$ may be viewed as a partition of $[n]$,

$$\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \exp(P(x)) = \exp(x^2 - 2x + 2) - \exp(2 - 2x).$$

□

Problem 8

Let a and b be positive integers. Give combinatorial interpretations of the numbers

$$e_a(1, 2, \dots, b) \quad \text{and} \quad h_a(1, 2, \dots, b).$$

(Here we set all remaining variables in these symmetric functions to 0.)

Proof. We first note that

$$e_a(1, 2, \dots, b) = \sum_{S \subseteq [b], |S| = a} \prod_{i \in S} i.$$

But then this is just $(-1)^a$ times the $(b+1-a)$-th coefficient of the falling factorial

$$(x)_{b+1} = x(x-1)\cdots(x-b) = \sum_{k \geq 0} (-1)^{b+1-k} c(b+1, k) x^k$$

where $c(n, k)$ is the unsigned Stirling number of the first kind. In other words,

$$e_a(1, 2, \dots, b) = c(b+1, b+1-a),$$

which is the number of permutations of $[b+1]$ with $b+1-a$ disjoint cycles.

For h_a, we note that

$$h_a(1, 2, \dots, b) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_a \leq b} i_1 i_2 \cdots i_a.$$

But then this is the coefficient of x^a in the generating function

$$(x + x^2 + \cdots)(2x + (2x)^2 + \cdots)\cdots(bx + (bx)^2 + \cdots) = \prod_{k=1}^{b} \frac{1}{1 - kx}.$$

□

Problem 9

Describe transition matrices between the e-basis $\{e_\lambda\}$ and the h-basis $\{h_\lambda\}$ of the ring Λ of symmetric functions.

Proof. We first show by induction on d that

$$h_d = \sum_{\alpha \vDash d} (-1)^{d-\ell(\alpha)} e_\alpha.$$

When $d = 1$, $h_1 = e_1$. Suppose $d \geq 2$. Since $h_d = (-1)^{d-1} e_d + \sum_{k=1}^{d-1} (-1)^{k-1} h_k e_{d-k}$, by induction we have

$$\begin{aligned}
h_d &= (-1)^{d-1} e_d + \sum_{k=1}^{d-1} (-1)^{d-k-1} \left(\sum_{\alpha \vDash k} (-1)^{k-\ell(\alpha)} e_\alpha \right) e_{d-k} \\
&= (-1)^{d-1} e_d + \sum_{k=1}^{d-1} \sum_{\alpha \vDash k} (-1)^{d-\ell(\alpha)-1} e_{\alpha \cup \{d-k\}} \\
&= \sum_{\alpha \vDash d} (-1)^{d-\ell(\alpha)} e_\alpha
\end{aligned}$$

For partition λ, let $m_i(\lambda)$ be the number of parts of λ that are equal to i. Then

$$h_d = \sum_{\alpha \vDash d} (-1)^{d-\ell(\alpha)} e_\alpha = \sum_{\lambda \vdash d} (-1)^{d-\ell(\lambda)} \left(\frac{\ell(\lambda)!}{\prod_{i \geq 1} m_i(\lambda)} \right) e_\lambda.$$

Hence, for partition μ,

$$h_\mu = \prod_{i=1}^{\ell(\mu)} \sum_{\lambda \vdash \mu_i} (-1)^{\mu_i - \ell(\lambda)} \left(\frac{\ell(\lambda)!}{\prod_{i \geq 1} m_i(\lambda)} \right) e_\lambda.$$

$\square$

Problem 10

Let $\mathbb{Y}$ be Young's lattice and let $\mathbb{C}[\mathbb{Y}]$ be the complex vector space with basis $\mathbb{Y}$. Define linear functions

$$U : \mathbb{C}[\mathbb{Y}] \to \mathbb{C}[\mathbb{Y}] \quad \text{and} \quad D : \mathbb{C}[\mathbb{Y}] \to \mathbb{C}[\mathbb{Y}]$$

by

$$U(\lambda) = \sum_{\lambda \prec \nu} \nu \quad \text{and} \quad D(\lambda) = \sum_{\mu \prec \lambda} \mu.$$

Prove that

$$DU - UD = I$$

where $I : \mathbb{C}[\mathbb{Y}] \to \mathbb{C}[\mathbb{Y}]$ is the identity operator. Use this relation to deduce

$$D^n U^n(\varnothing) = n! \cdot \varnothing$$

where $\varnothing \in \mathbb{Y}$ is the empty partition. Explain why this proves

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

Proof. Note that U, D are linear functions which sends λ to the sum of all Young's tableaux that are greater than λ and less than λ, respectively. Suppose λ has d removable boxes. Let ν be a Young's tableau obtained by adding a box a to λ then removing a box b. If $a \neq b$, then we may also get ν by first removing b then adding a. Hence, if $\lambda \neq \nu$, then the number of ways to obtain ν from λ by first adding then removing a box is equal to the number of ways to obtain ν from λ by first removing then adding a box. Now suppose $a = b$. There are $d+1$ ways to obtain $\nu = \lambda$ from λ by first adding then removing the same box. But then there are only d ways to obtain $\nu = \lambda$ from λ by first removing then adding the same box. Hence, we have $(DU - UD)\lambda = \lambda$ for all $\lambda \in \mathbb{Y}$, which gives us the desired relation.

Now that $DU = UD + I$, we can apply this relation repeatedly to get

$$DU^n = (UD + I)U^{n-1} = U(DU^{n-1}) + U^{n-1} = U^2(DU^{n-2}) + 2U^{n-1} = \cdots = U^n D + nU^{n-1}.$$

It now follows that

$$D^n U^n(\varnothing) = D^{n-1}(DU^n)(\varnothing) = D^{n-1}U^n D(\varnothing) + nD^{n-1}U^{n-1}(\varnothing).$$

But then $D^{n-1}U^n D(\varnothing) = 0$. Hence,

$$D^n U^n(\varnothing) = nD^{n-1}U^{n-1}(\varnothing) = n(n-1)D^{n-2}U^{n-2}(\varnothing) = \cdots = n! \cdot \varnothing.$$

Finally, note that f^λ is the number of standard Young's tableaux of shape λ. In other words, f^λ counts the paths from $\varnothing$ to λ in Young's lattice, so $U^n(\varnothing) = \sum_{\lambda \vdash n} f^\lambda \cdot \lambda$. But then for each $\lambda \vdash n$, the number of paths from $\varnothing$ to λ is also f^λ, so $D^n(\lambda) = f^\lambda \cdot \varnothing$. It now follows that

$$n! \cdot \varnothing = D^n U^n(\varnothing) = \sum_{\lambda \vdash n} f^\lambda D^n(\lambda) = \left(\sum_{\lambda \vdash n} (f^\lambda)^2 \right) \cdot \varnothing.$$

$\square$

Problem 11

Let G be a finite group acting on a finite set X. Prove that the multiplicity of the trivial representation in $\mathbb{C}[X]$ is the number of orbits in the action of G on X.

Proof. For $g \in G$, note that $\chi(g) = \operatorname{tr}(\rho(g)) = |X^g|$, the number of fixed points of g in X. By Burnside's lemma, the multiplicity of the trivial representation in $\mathbb{C}[X]$ is

$$\frac{1}{|G|}\sum_{g\in G}\chi(g) = \frac{1}{|G|}\sum_{g\in G}|X^g| = |G/X|,$$

the number of orbits in the action of G on X. $\square$