# MATH 180B: Homework #5

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# Problem 1

A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array} \right\| \end{array}$$

Every period that the process spends in state 0 incurs a cost of \$2. Every period that the process spends in state 1 incurs a cost of \$5. Every period that the process spends in state 2 incurs a cost of \$3. What is the long run cost per period associated with this Markov chain?

*Proof.* Since $P$ is regular, $P$ has a limiting distribution $\pi = (\pi_0, \pi_1, \pi_2)^T$. We solve for

$$\begin{cases} (I - P^T)\pi = 0 \\ \sum_i \pi_i = 1 \end{cases}$$

and get $\pi = (\frac{5}{12}, \frac{2}{11}, \frac{53}{132})$. Since $\pi_i$ can also be interpreted as the long run mean fraction of time the process spent in state $i$, long run cost per period is $\frac{5}{12} \cdot \$2 + \frac{2}{11} \cdot \$5 + \frac{53}{132} \cdot \$3 = \$\frac{389}{132} \approx \$2.94697$. $\square$

# Problem 2

Five balls are distributed between two urns, labeled A and B. Each period, an urn is selected at random, and if it is not empty, a ball from that urn is removed and placed into the other urn. In the long run what fraction of time is urn A empty?

*Proof.* Let $\{X_n\}$ denote the number of balls in urn $A$ at step $n$. The transition probability matrix for this process is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{cccccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \right\| \end{array}.$$

Since $P_{00} > 0$ and every pair of states $i, j$ obviously communicates, $P$ is doubly stochasatic and regular, and thus the long run mean fraction of time urn $A$ is empty is $\frac{1}{6}$. $\square$

# Problem 3

A Markov chain has the transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right\| \end{array}.$$

where $\alpha_i \geq 0$, $i = 1, \ldots, 6$, and $\alpha_1 + \ldots + \alpha_6 = 1$. Determine the limiting probability of being in state 0.

*Proof.* Suppose $\alpha_6 = 0$, then we treast $P$ as a transition matrix for states 0 to 4. Then, since 0 communicates with every state, every pair of states $i, j$ are accessible. With $P_{00} > 0$, we know $P$ has a limiting distribution $\pi = (\pi_0, \ldots, \pi_5)$. We then get the system of equations

$$\begin{cases} \pi_0 = \alpha_1\pi_0 + \pi_1 \\ \pi_1 = \alpha_2\pi_0 + \pi_2 \\ \pi_2 = \alpha_3\pi_0 + \pi_3 \\ \pi_3 = \alpha_4\pi_0 + \pi_4 \\ \pi_4 = \alpha_5\pi_0 + \pi_5 \\ \pi_5 = \alpha_6\pi_0 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1 \end{cases}.$$

Solving for it, we get

$$\begin{cases} \pi_1 = (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\pi_0 \\ \pi_2 = (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\pi_0 \\ \pi_3 = (\alpha_4 + \alpha_5 + \alpha_6)\pi_0 \\ \pi_4 = (\alpha_5 + \alpha_6)\pi_0 \\ \pi_5 = \alpha_6\pi_0 \end{cases},$$

and thus the limiting probability of being in state 0 is $\pi_0 = (1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6)^{-1}$. $\square$

# Problem 4

Consider a Markov chain with transition probability matrix

$$P = \begin{Vmatrix} p_0 & p_1 & p_2 & \cdots & p_N \\ p_N & p_0 & p_1 & \cdots & p_{N-1} \\ p_{N-1} & p_N & p_0 & \cdots & p_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ p_1 & p_2 & p_3 & \cdots & p_0 \end{Vmatrix}$$

where $0 < p_0 < 1$ and $p_0 + p_1 + \cdots + p_N = 1$. Determine the limiting distribution.

*Proof.* We already know that $P$ is aperiodic, as $P_{ii} = p_0 > 0$. Since $p_0 + p_1 + \cdots + p_N = 1$ and $p_0 < 1$, there exsits $p_i > 0$ for some $i \neq 0$, there exists a directed hamiltonian cycle in the state transition diagram, and thus $P$ is an irreducible stochastic matrix. It follows that the limiting distribution of $P$ exists. Since $P$ is doubly stochastic, the limiting distribution is $\left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$. $\square$

# Problem 5

A component of a computer has an active life, measured in discrete units, that is a random variable $\xi$, where

| $k$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\Pr\{\xi = k\}$ | 0.1 | 0.3 | 0.2 | 0.4 |

Suppose that one starts with a fresh component, and each component is replaced by a new component upon failure. Let $X_n$ be the *remaining life* of the component in service at the *end* of period $n$. When $X_n = 0$, a new item is placed into service at the *start* of the next period.

(a) Set up the transition probability matrix for $\{X_n\}$.

*Proof.* The transition probability matrix is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0.1 & 0.3 & 0.2 & 0.4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right\| \end{array}$$

□

(b) By showing that the chain is regular and solving for the limiting distribution, determine the long run probability that the item in service at the end of a period has no remaining life and therefore will be replaced.

*Proof.* $P$ is aperiodic, as $P_{00} > 0$. Notice that for all state $i > 0$, 0 is accessible from state $i$ by following the path $i \to i-1 \to \cdots \to 0$. Since we also know that every positive state is accessible from 0, $P$ is irreducible. It follows that $P$ is regular, so the limiting distribution $\pi = (\pi_0, \ldots, \pi_4)$ exists. We then get the system of equations

$$\begin{cases} \pi_0 = 0.1\pi_0 + \pi_1 \\ \pi_1 = 0.3\pi_0 + \pi_2 \\ \pi_2 = 0.2\pi_0 + \pi_3 \\ \pi_3 = 0.4\pi_0 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \end{cases},$$

and solving it gives $\pi = (\frac{10}{29}, \frac{9}{29}, \frac{6}{29}, \frac{4}{29})$. □

(c) Relate this to the mean life of a component.

*Proof.* Notice that $E[\xi]\pi_0 = (1 \cdot 0.1 + 2 \cdot 0.3 + 3 \cdot 0.2 + 4 \cdot 0.4) \cdot \frac{10}{29} = 1$. This implies that the number of periods that we make replacements of component multiplied by the mean life of a component approaches the number of periods of the process in the long run. □

# Problem 6

Consider a computer system that fails on a given day with probability $p$ and remains "up" with probability $q = 1 - p$. Suppose the repair time is a random variable $N$ having the probability mass function $p(k) = \beta(1-\beta)^{k-1}$ for $k = 1, 2, \ldots$, where $0 < \beta < 1$. Let $X_n = 1$ if the computer is operating on day $n$ and $X_1 = 0$ if not. Show that $\{X_n\}$ is a Markov chain with transition matrix

$$\begin{array}{c|cc|} & 0 & 1 \\ 0 & \alpha & \beta \\ 1 & p & q \end{array}$$

and $\alpha = 1 - \beta$. Determine the long run probability that the computer is operating in terms of $\alpha, \beta, p$, and $q$.

*Proof.* We already know $\mathbb{P}(X_n = i \mid X_{n-1} = 1, \ldots, X_0 = i_0) = \mathbb{P}(X_n = i \mid X_{n-1} = 1)$. Notice that $p(k)$ is a geometric distribution. Hence, on any given day that our computer is broken, the probability of it being repaired on that day is $\beta$ regardless of when the repairment was incurred. Hence, $\{X_n\}$ is a Markov chain and has the above transition matrix. Name that matrix $P$. Since $p, \beta > 0$ and $\beta < 1$, $P$ is irreducible and aperiodic, and thus the limiting distribution $\pi = (\pi_0, \pi_1)^T$ exists. We then solve for $(I - P^T)\pi = 0$ and get $\pi_0 = \frac{p}{\beta+p}$ and $\pi_1 = \frac{\beta}{\beta+p}$. □