MATH 140A: Homework #1

Due on Jan 19, 2023 at 23:59pm

Professor Seward

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Problem 1

If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Proof. Suppose for the sake of contradiction that $y = r + x$, $z = rx$ are rational. Since rational numbers are closed under addition, $x = y - r$ is also rational, contradiction. Similarly, since non-zero rational is closed under multiplication and taking multiplicative inverse, $x = \frac{z}{r}$ is rational, contradiction. Hence, $r + x$ and rx are irrational. $\square$

Problem 2

Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. Since $\beta \geq e$ and $e \geq \alpha$ for $e \in E$, $\beta \geq \alpha$. $\square$

Problem 3

Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Proof. Let $y = \inf A$. Since $y \leq x$ for $x \in A$, we know $-y \geq -x$ for $-x \in -A$, so $-y$ is an upper bound of $-A$. Since y is the greatest lower bound of A, there exists $a \in A$ such that $a < y + \epsilon$, for $\epsilon > 0$. This immediately follows that for $\epsilon > 0$, there exists $-a \in -A$ such that $-a > -y - \epsilon$, so $-y$ is the least upper bound of $-A$. In other words, $y = -\sup(-A)$. $\square$

Problem 4

Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = \frac{m}{n} = \frac{p}{q}$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

Proof. Let $x = (b^m)^{1/n}$, $y = (b^p)^{1/q}$. We know $x^n = b^m = b^{nr}$ and $y^q = b^p = b^{qr}$. Consider x^{nq}. Since $x^{nq} = b^{nrq} = y^{nq}$, we conclude that $x = y$, by Theorem 1.21. □

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

Proof. Let $r = \frac{m}{n}, s = \frac{p}{q}$, for $m, n, p, q \in \mathbb{Z}$ and $n, q > 0$. Consider $(b^{r+s})^{nq}$ and $(b^r b^s)^{nq}$. Since $(b^{r+s})^{nq} = b^{mq+np}$ and $(b^r b^s)^{nq} = (b^r)^{nq}(b^s)^{nq} = b^{mq}b^{np} = b^{mq+np}$, we know $(b^{r+s})^{nq} = (b^r b^s)^{nq}$. The result now follows by Theorem 1.21. □

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t < x$. Prove that

$$b^r = \sup B(r).$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x).$$

for every real x.

Proof. Let $b^t \in B(r)$. Suppose $r = \frac{m}{n}$, $t = \frac{p}{q}$, where $n, q > 0$. Since $r > t$, $mq > np$. Consider $(b^r)^{nq}$ and $(b^t)^{nq}$. We know $(b^r)^{nq} = b^{mq} > b^{np} = (b^t)^{nq}$. Let $y = b^r$, $x = b^t$. Since $y^{nq-1} + y^{nq-2}x + \cdots + x^{nq-1} > 0$, the identity $y^{nq} - x^{nq} = (y - x)(y^{nq-1} + y^{nq-2}x + \cdots + x^{nq-1}) > 0$ yields $y > x$, and thus $y = b^r$ is an upper bound of $B(r)$.

Let $y < b^r$ be a positive real number. We show that there exists $b^s \in B(r)$ such that $b^s > y$. Let $t = y^{-1}b^r$, and let $n \in \mathbb{Z}$ such that $n > \frac{b-1}{t-1}$. We know such n exists by the archimedean property and $t > 1$. Since $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \cdots + 1) > n(a-1)$ for $a > 1$, we know $b - 1 > n(b^{1/n} - 1) > \frac{b-1}{t-1}(b^{1/n} - 1)$. This immediately follows that $t = y^{-1}b^r > b^{1/n}$, so $b^{r-(1/n)} > y$. However, since $r - (1/n) \in \mathbb{Q}$, we have $b^{r-(1/n)} \in B(r)$. Therefore, y is not an upper bound of $B(r)$, which shows that $b^r = \sup B(r)$. □

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Proof. We show that $b^x b^y = \sup B(x+y)$. Note that $B(x+y) = \{b^t \,|\, t < x+y, t \in \mathbb{Q}\} = \{b^{g+h} \,|\, g < x, h < y, g, h \in \mathbb{Q}\} = \{b^g b^h \,|\, g < x, h < y, g, h \in \mathbb{Q}\}$. Let $b^g b^h \in B(x+y)$, for $g < x, h < y$. Since $b^g \in B(x)$ and $b^h \in B(y)$, we know $b^x > b^g$ and $b^y > b^h$. Thus $b^x b^y > b^g b^h$, so $b^x b^y$ is an upper bound of $B(x+y)$. Suppose that $k < b^x b^y$. Since $\frac{k}{b^y} < b^x$, there exists $b^l \in B(x)$ such that $\frac{k}{b^y} < b^l$. However, $\frac{k}{b^l} < b^y$, so there exists $b^s \in B(y)$ such that $\frac{k}{b^l} < b^s$. This immediately follows that there exists $b^l b^s \in B(x+y)$ such that $k < b^l b^s$, so $b^x b^y$ is the least upper bound of $B(x+y)$. □

Problem 5

Fix $b > 1, y > 0$, and prove that there is a unique real x such that $b^x = y$.

Proof. Since $b > 1$, we know $b^k > 1$ for positive integer k, so the identity $b^n - 1 = (b-1)(b^{n-1} + b^{n-2} + \cdots + 1)$ yields

$$b^n - 1 < n(b-1), \tag{1}$$

for positive integer n. Note that $b^{1/n} > 1$, otherwise $b = (b^{1/n})^n \leq 1$. Hence, we may apply (1) and get

$$b - 1 > n(b^{1/n} - 1). \tag{2}$$

Suppose that $b^w < y$. Let $t = yb^{-w}$, and let $n > \frac{b-1}{t-1}$, for $w \in \mathbb{R}$. We know such n exists by the Archimedean Property and $t = yb^{-w} > b^w b^{-w} = 1$. By (2), $b - 1 > n(b^{1/n} - 1) > \frac{b-1}{t-1}(b^{1/n} - 1)$, so $t > b^{1/n}$. Thus, $b^{w+(1/n)} < y$ for sufficiently large n. Similarly, when $b^w > y$, take $t = y^{-1}b^w$. Again, let $n > \frac{b-1}{t-1}$, and it follows by (2) that $t > b^{1/n}$. Thus, $y < b^{w-(1/n)}$ for sufficient large n.

Let A be the set of all w such that $b^w < y$. We show that $x = \sup A$ satisfies $b^x = y$. Suppose for the sake of contradiction that $b^x > y$. Then, we know there exists n such that $b^{x-(1/n)} > y$. However, since $x = \sup A$, $b^{x-(1/n)}$ must be in A, which contradicts that $b^{x-(1/n)} > y$. Suppose for the sake of contradiction that $b^x < y$. There exists n such that $b^{x+(1/n)} < y$. However, this means that there exists $b^{x+(1/n)} \in A$ such that $b^{x+(1/n)} > x = \sup A$, contradiction. Therefore, $b^x = y$.

Suppose that $b^x = b^{x'} = y$. x cannot be greater or lesser than x', otherwise $b^x \neq b^{x'}$, so x is unique. $\square$