MATH 100B: Homework #7

Due on Feb 29, 2024 at 12:00pm

Professor McKernan

Section A02 6:00PM - 6:50PM
Section Leader: Castellano-Macías

Source Consulted: Textbook, Lecture, Discussion, Office Hour

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Problem 1

Are the following true or false? If true prove them; if false give counterexamples.

(i) Every prime ideal is maximal.

Proof. False. Consider $\langle x \rangle$ in $\mathbb{Z}[x]$. $\mathbb{Z}[x]/\langle x \rangle \simeq \mathbb{Z}$, so $\langle x \rangle$ is a prime ideal but not maximal. □

(ii) Every ring is a PID.

Proof. False. Consider $\mathbb{Z}[x^2, x^3]$. Notice that since $x \notin \mathbb{Z}[x^2, x^3]$, x^2 and x^3 are irreducible. But then $x^6 = (x^2)(x^2)(x^2) = (x^3)(x^3)$, so the factorization is not unique in $\mathbb{Z}[x^2, x^3]$. It follows that $\mathbb{Z}[x^2, x^3]$ is not a UFD, and thus not a PID. □

(iii) Every UFD is a PID.

Proof. False. Consider $\mathbb{Z}[x]$. Since $\mathbb{Z}$ is a UFD, $\mathbb{Z}[x]$ is a UFD, by Gauss' Lemma. Let $I = \langle x^2, 4 \rangle$. Note that $x \notin I$ so $I \neq \mathbb{Z}[x]$. Suppose for the sake of contradiction that $I = \langle p(x) \rangle$ for some $p(x) \in \mathbb{Z}[x]$. We know $p(x) = a$ for some $a \in \mathbb{Z}$, otherwise $4 \notin I$. But then $a = 1$, otherwise $x^2 \notin I$. The result now follows from the contradiction that $I = \mathbb{Z}[x]$. □

Problem 2

If

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$$

is an ascending chain of submodules of an R-module M then show that the union N is a submodule.

Proof. Let $m, n \in N$. We know $m \in N_i$, $n \in N_j$, for some i, j. Let $k = \max(i, j)$. Since $m, n \in N_k$, we have $m + n, m^{-1}, 0 \in N_k$ and $m + n = n + m$. Hence, N is an abelian group under addition. Since $m, n \in N_k$, m, n follows all 4 multiplicative conditions of a module, and the result follows $\square$