MATH 264B: Homework

Professor Rhodes

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Problem 1

Let $n, m \in \mathbb{Z}_{\geq 0}$. Give a *combinatorial* proof that

$$\sum_{i=0}^{n} \binom{m+i}{i} = \binom{m+n+1}{n}.$$

That is, interpret both sides as the cardinality of a set, and find a bijection between these sets.

Proof. It suffices to show that

$$\sum_{i=0}^{n} \binom{m+i}{m} = \binom{m+n+1}{m+1}.$$

Let C_i be the set of all m-element subsets of $[m+i]$, and let S be the set of all $(m+1)$-element subsets of $[m+n+1]$. Consider the map $f : \bigsqcup_{i=1}^{n} C_i \to S$ by sending $A \in C_i$ to $A \cup \{m+i+1\} \in S$. This mapping is a bijection as we may recover A by removing the largest element of $f(A)$. Thus, $|\bigsqcup_{i=1}^{n} C_i| = |S|$, and the result now follows. $\square$

Problem 2

Let des : $\mathfrak{S}_n \to \mathbb{Z}_{\geq 0}$ be the descent statistic

$$\mathrm{des}(w) := \#\{1 \leq i \leq n-1 : w(i) > w(i+1)\}$$

and consider the *Eulerian polynomial*

$$A_n(t) := \sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(w)}.$$

Prove that $A_n(2) = [A_n(t)]_{t=2}$ is the number of ordered set partitions of $[n]$.

Proof. We say that a ordered partition is in canonical form if the elements of each block are in descending order. Let P_n be the set of all ordered set partitions of $[n]$. Define the operation $\phi : P_n \to \mathfrak{S}_n$ by erasing the brackets of an ordered partition in canonical form and interpreting the resulting string as a permutation. It is clear that ϕ is well-defined. Now consider the reverse operation $\psi : \mathfrak{S}_n \to 2^{P_n}$ by sending $w \in \mathfrak{S}_n$ to $\{p \in P_n : \phi(p) = w\}$, the set of all ordered partitions whose canonical form resembles w after erasing the brackets. Note that

$$|P_n| = \sum_{p \in P_n} |\phi(p)| = \sum_{w \in \mathfrak{S}_n} |\psi(w)|,$$

and so it suffices to show that $|\psi(w)| = 2^{\mathrm{des}(w)}$. To see this, we start from the ordered singleton partition $p_0 \in \psi(w)$. Reading p_0 from left to right, we may choose to combine a block with its preceding block whenever a descend occurs, and the resulting partition will still be in $\psi(w)$. This gives us $2^{\mathrm{des}(w)}$ ways to partition w into blocks. $\square$

Problem 3

How many (strong) compositions of n have an even number of even parts?

Proof. Let E_n be the set of all compositions of n with even number of even parts, and let O_n be the set of all compositions of n with odd number of even parts. We show $|E_n| = 2^{n-2}$ for $n \geq 2$ by proving that $|E_n| = |O_n|$. Consider the operation $\phi : E_n \to O_n$ by sending the composition $(\alpha_1, \dots, \alpha_k)$ to $(\alpha_1, \dots, \alpha_k - 1, 1)$ if $\alpha_k > 1$ and send $(\alpha_1, \dots, \alpha_k)$ to $(\alpha_1, \dots, \alpha_{k-1} + 1)$ if $\alpha_k = 1$. Notice that $\phi(\phi(\alpha_1, \dots, \alpha_k)) = (\alpha_1, \dots, \alpha_k)$, so ϕ is an inversion. $\square$