

# **MATH 188: Homework #3**

Due on May 3, 2024 at 23:59pm

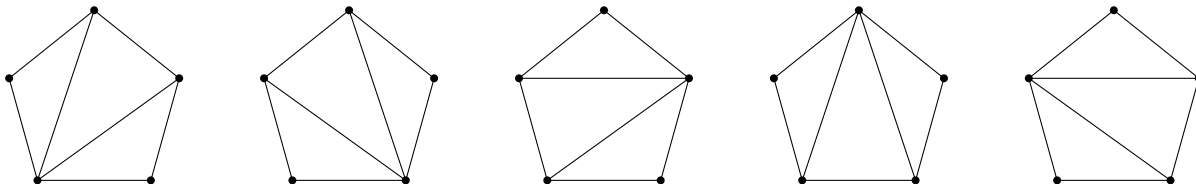
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## Problem 1

Let  $n$  be a positive integer. Show that the number of ways of triangulating (i.e., drawing diagonals between vertices that do not intersect except at vertices so that the regions are all triangles) a convex polygon with  $(n + 2)$  vertices is the  $n$ th Catalan number  $C_n$ . By convention, the “2-gon” and triangle both have exactly one triangulation and here are the 5 triangulations of a pentagon:



*Proof.* We proceed by induction on  $n$ . There is only  $C_1 = 1$  way to triangulate a triangle, so the base case is done. Suppose  $n > 1$ . Index the vertices in counter-clockwise order from 0 to  $n + 1$ , say  $v_0, v_1, \dots, v_{n+1}$ . We focus on  $v_0$ . The two clockwise most edges incident to  $v_0$  are  $\{v_0, v_1\}$  and  $\{v_0, v_k\}$ , for some  $2 \leq k \leq n + 1$ . Since there are no edges between  $\{v_0, v_1\}$  and  $\{v_0, v_k\}$ ,  $v_0 v_1 v_k$  form a triangle. Removing triangle  $v_0 v_1 v_k$ , we get an  $k$ -gon  $v_1 v_2 \dots v_k$  and an  $(n - k + 3)$ -gon  $v_k v_{k+1} \dots v_{n+1} v_0$ . By induction, there are  $C_{k-2} C_{n-k+1}$  ways to triangulate these two polygons, and thus there are  $C_{k-2} C_{n-k+1}$  triangulations of the  $(n + 2)$ -gon which contains the triangle  $v_0 v_1 v_k$ . Therefore, the total number of triangulations of an  $(n + 2)$ -gon is

$$\sum_{k=2}^{n+1} C_{k-2} C_{n-k+1} = \sum_{i=0}^{n-1} C_i C_{n-i-1} = C_n.$$

□

## Problem 2

Consider the following variation of counting balanced parentheses. We have a new symbol  $*$ . Let  $a_n$  be the number of length  $n$  strings consisting of left/right parentheses and  $*$  such that the result of deleting all of the  $*$ 's is a balanced set of parentheses ( $a_0 = 1$ ). Let  $A(x) = \sum_{n \geq 0} a_n x^n$ . Find polynomials  $a(x)$ ,  $b(x)$ ,  $c(x)$  in  $x$ , not all identically 0, such that

$$a(x)A(x)^2 + b(x)A(x) + c(x) = 0.$$

*Proof.*

□

### Problem 3

Let  $n$  be a positive integer. Consider the equation

$$x_1 + x_2 + \dots + x_8 = 2n.$$

For each of the following conditions, how many solutions are there? Give as simple of a formula as possible.

- (a) The  $x_i$  are non-negative even integers.

*Proof.* Let

$$C_{\text{even}} = \{(x_1, \dots, x_8) \mid x_1 + \dots + x_8 = 2n, x_i = 2k_i \text{ for some } k_i \in \mathbb{Z}_{\geq 0}\},$$

$$C_n = \{(y_1, \dots, y_8) \mid y_1 + \dots + y_8 = n, y_i \in \mathbb{Z}_{\geq 0}\}.$$

We show that  $C_n \simeq C_{\text{even}}$ . Define  $f : C_{\text{even}} \rightarrow C_n$  which sends  $(x_1, \dots, x_8)$  to  $(k_1, \dots, k_8)$  and  $g : C_n \rightarrow C_{\text{even}}$  which sends  $(y_1, \dots, y_8)$  to  $(2y_1, \dots, 2y_8)$ . Both  $f$  and  $g$  are obviously well-defined. Since

$$g(f(x_1, \dots, x_8)) = g(k_1, \dots, k_8) = (2k_1, \dots, 2k_8) = (x_1, \dots, x_8),$$

$$f(g(y_1, \dots, y_8)) = f(2y_1, \dots, 2y_8) = (y_1, \dots, y_8),$$

$f$  is a bijection, and thus  $C_n \simeq C_{\text{even}}$ . But then we know there are  $\binom{n+7}{7}$  weak compositions of  $n$  with 8 parts, and the result now follows.  $\square$

- (b) The  $x_i$  are positive odd integers.

*Proof.* Note that

$$\begin{aligned} \frac{x^8}{(1-x^2)^8} &= \left( x \sum_{a_1 \geq 0} x^{2a_1} \right) \cdots \left( x \sum_{a_8 \geq 0} x^{2a_8} \right) \\ &= \left( \sum_{a_1 \geq 0} x^{2a_1+1} \right) \cdots \left( \sum_{a_8 \geq 0} x^{2a_8+1} \right) \\ &= \sum_{\substack{(k_1, \dots, k_8) \in \mathbb{Z}_{\geq 1}^8, \\ k_i \text{ odd}}} x^{k_1 + \dots + k_8}, \end{aligned}$$

so the number of solutions where all  $x_i$ 's are positive odd integers are

$$[x^{2n}] \frac{x^8}{(1-x^2)^8} = [x^{2n-8}] \frac{1}{(1-x^2)^8} = [x^{n-4}] \frac{1}{(1-x)^8} = \binom{n+3}{7}.$$

$\square$

- (c) The  $x_i$  are non-negative integers and  $x_8 \leq 9$ .

*Proof.* Suppose  $x_8 = k$ , for some  $0 \leq k \leq 9$ . Then, there are  $\binom{2n-k+6}{6}$  solutions, as there are  $\binom{2n-k+6}{6}$  solutions to  $x_1 + \dots + x_7 = 2n - k$ . Hence, in total, there are  $\sum_{k=0}^9 \binom{2n-k+6}{6}$  solutions.  $\square$

## Problem 4

Let  $k, n$  be positive integers such that  $k \geq n$ .

(a) Show that

$$\sum_{(a_1, \dots, a_n)} a_1 a_2 \cdots a_n = \binom{n+k-1}{k-n},$$

where the sum is over all compositions of  $k$  into  $n$  parts.

*Proof.* Note that

$$\begin{aligned} \frac{x^n}{(1-x)^{-2n}} &= xD \left( \sum_{a_1 \geq 0} x^{a_1} \right) \cdots xD \left( \sum_{a_n \geq 0} x^{a_n} \right) \\ &= \left( x \sum_{a_1 \geq 1} a_1 x^{a_1-1} \right) \cdots \left( x \sum_{a_n \geq 1} a_n x^{a_n-1} \right) \\ &= \left( \sum_{a_1 \geq 1} a_1 x^{a_1} \right) \cdots \left( \sum_{a_n \geq 1} a_n x^{a_n} \right) \\ &= \sum_{(a_1, \dots, a_n) \in \mathbb{Z}_{\geq 1}^n} a_1 a_2 \cdots a_n x^{a_1 + \cdots + a_n}. \end{aligned}$$

Hence,

$$\sum_{\substack{(a_1, \dots, a_n) \in \mathbb{Z}_{\geq 1}^n \\ a_1 + \cdots + a_n = k}} a_1 a_2 \cdots a_n = [x^k] \frac{x^n}{(1-x)^{-2n}} = [x^{k-n}] \frac{1}{(1-x)^{-2n}} = \binom{n+k-1}{k-n}.$$

□

(b) Show that

$$\sum_{(a_1, \dots, a_n)} 2^{a_2-1} 3^{a_3-1} \cdots n^{a_n-1} = S(k, n),$$

where the sum is over all compositions of  $k$  into  $n$  parts.

*Proof.* Note that

$$\begin{aligned} F_n(x) &= \left( \frac{x}{1-x} \right) \left( \frac{x}{1-2x} \right) \cdots \left( \frac{x}{1-nx} \right) \\ &= \left( x \sum_{a_1 \geq 0} x^{a_1} \right) \left( x \sum_{a_2 \geq 0} (2x)^{a_2} \right) \cdots \left( x \sum_{a_n \geq 0} (nx)^{a_n} \right) \\ &= \left( x \sum_{a_1 \geq 1} x^{a_1-1} \right) \left( x \sum_{a_2 \geq 1} (2x)^{a_2-1} \right) \cdots \left( x \sum_{a_n \geq 1} (nx)^{a_n-1} \right) \\ &= \sum_{(a_1, \dots, a_n) \in \mathbb{Z}_{\geq 1}^n} 2^{a_2-1} \cdots n^{a_n-1} x^{a_1 + \cdots + a_n}. \end{aligned}$$

Hence,

$$\sum_{\substack{(a_1, \dots, a_n) \in \mathbb{Z}_{\geq 1}^n \\ a_1 + \cdots + a_n = k}} 2^{a_2-1} 3^{a_3-1} \cdots n^{a_n-1} = [x^k] F_n(x) = S(k, n).$$

□

## Problem 5

- (a) Give a closed formula for the number of pairs of subsets  $S, T$  of  $[n]$  such that  $S \subset T$  (i.e.,  $S \subseteq T$  and  $S \neq T$ ).

*Proof.* There are  $\binom{n}{k}$  ways to pick a subset of size  $k$ , and each subset of size  $k$  has  $2^k - 1$  strict subsets. Hence, the total number of  $S, T$  pairs is

$$\sum_{k=0}^n \binom{n}{k} (2^k - 1) = \sum_{k=0}^n \binom{n}{k} 2^k - \sum_{k=0}^n \binom{n}{k} = (1 + 2)^n - (1 + 1)^n = 3^n - 2^n,$$

by the binomial theorem. □

- (b) Give a closed formula for the number of  $k$ -tuples of subsets  $(S_1, \dots, S_k)$  of  $[n]$  such that  $\bigcup_{i=1}^k S_i = [n]$ .

*Proof.* Let  $a_n$  be the number of  $k$ -tuples of subsets  $(S_1, \dots, S_k)$  of  $[n]$  such that  $\bigcup_{i=1}^k S_i = [n]$ . Put  $a_0 = 1$ . We show that  $a_n = (2^k - 1)^n$  by induction on  $n$ . Given  $(S_1, \dots, S_k)$  such that  $\bigcup_{i=1}^k S_i = [n-1]$ , we have to add  $n$  to at least one of the  $S_i$ 's to ensure  $\bigcup_{i=1}^k S_i = [n]$ . Since for each such  $k$ -tuple there are  $2^k - 1$  ways to do so, we get

$$a_n = (2^k - 1)a_{n-1} = (2^k - 1)^n,$$

by induction. □

## Problem 6

Give a closed formula for the number of  $k$ -tuples of subsets  $(S_1, \dots, S_k)$  of  $[n]$  such that  $S_i \subseteq S_{i+1}$  for  $i = 1, \dots, k-1$ .

*Proof.* Notice that the first appearance of any  $j \in [n]$  in the tuple determines  $j$ 's existence in all  $S_i$ 's, as all subsequent sets in the tuple would also contain  $j$ . Since each  $j \in [n]$  can either first appear in one of the  $k$  sets or never appear, there are  $k+1$  choices for each element in  $[n]$ , and thus there are  $(k+1)^n$   $k$ -tuples of subsets  $(S_1, \dots, S_k)$  of  $[n]$  such that  $S_i \subseteq S_{i+1}$ .  $\square$

## Problem 7

What is the total number of parts of all compositions of  $k$ ?

*Proof.* The possible number of parts of a composition of  $k$  is anywhere between  $n = 1$  to  $n = k$ , so the total number of parts of all compositions is

$$\begin{aligned}\sum_{n=1}^k \binom{k-1}{n-1} n &= \sum_{n=0}^{k-1} \binom{k-1}{n} (n+1) \\ &= \sum_{n=1}^{k-1} \binom{k-1}{n} n + \sum_{n=0}^{k-1} \binom{k-1}{n}.\end{aligned}$$

Note that

$$(k-1)(x+1)^{k-2} = D(x+1)^{k-1} = \sum_{n=1}^{k-1} \binom{k-1}{n} n x^{n-1}.$$

Hence,

$$\sum_{n=1}^k \binom{k-1}{n-1} n = (k-1)(1+1)^{k-2} + (1+1)^{k-1} = (k+1)2^{k-2}.$$

□



## Problem 8

Fix an integer  $k \geq 2$ . Call a composition  $(a_1, \dots, a_n)$  of  $k$  doubly even if the number of  $a_i$  which are even is also even (i.e., there could be no even  $a_i$ , or 2 of them, or 4, etc.). Show that the number of doubly even compositions of  $k$  is  $2^{k-2}$ .

*Proof.* Let  $E$  be the set of doubly even compositions of  $k$ , and  $C$  be the set of compositions of  $k-1$ . We show that  $E \simeq C$ . Define  $f : E \rightarrow C$  as

$$f(a_1, \dots, a_n) = \begin{cases} (a_1, \dots, a_n - 1), & \text{if } a_n > 1 \\ (a_1, \dots, a_{n-1}), & \text{if } a_n = 1 \end{cases}.$$

On the other hand, define  $g : C \rightarrow E$  as

$$g(a_1, \dots, a_n) = \begin{cases} (a_1, \dots, a_n, 1), & \text{if } (a_1, \dots, a_n) \text{ is doubly even} \\ (a_1, \dots, a_n + 1), & \text{otherwise} \end{cases}.$$

Note that  $f$  is obviously well defined. Let  $(a_1, \dots, a_n) \in C$ . If  $(a_1, \dots, a_n)$  is doubly even, then  $(a_1, \dots, a_n, 1)$  is also doubly even. If  $(a_1, \dots, a_n)$  is not doubly even, then  $(a_1, \dots, a_n + 1)$  is doubly even, as incrementing  $a_n$  by 1 either increase or decrease the amount of even numbers in the tuple by 1. Hence,  $g$  is also well-defined.

Since

$$g(f(a_1, \dots, a_n)) = \begin{cases} (a_1, \dots, a_{n-1}, 1), & \text{if } a_n = 1 \\ (a_1, \dots, (a_n - 1) + 1), & \text{if } a_n > 1 \end{cases} = (a_1, \dots, a_n),$$

$$f(g(a_1, \dots, a_n)) = \begin{cases} (a_1, \dots, a_n), & \text{if } (a_1, \dots, a_n) \text{ is doubly even} \\ (a_1, \dots, (a_n + 1) - 1), & \text{otherwise} \end{cases} = (a_1, \dots, a_n),$$

$f$  and  $g$  are inverses of each other, and thus  $E \simeq C$ . Hence, the number of doubly even compositions of  $k$  is equal to the number of compositions of  $k-1$ , which is  $2^{k-2}$ .  $\square$

## Problem 9

Let  $F(n)$  be the number of set partitions of  $[n]$  such that every block has size  $\geq 2$ . Prove that

$$B(n) = F(n) + F(n+1),$$

where  $B(n)$  is the  $n$ th Bell number.

*Proof.* Let  $P$  be the set of all partitions of  $[n]$ ,  $A_k$  be the set partitions of  $[k]$  such that every block has size  $\geq 2$ , and let  $S$  be the set of partition of  $[n]$  which contains at least a singleton. It is obvious that  $P = A_n \sqcup S$  and  $|A_n| = F(n)$ . It remains to show that  $|S| = F(n+1)$ .

Define  $f : S \rightarrow A_{n+1}$  which puts all singletons of a partition into the same block as  $n+1$ . On the other hand, define  $g : A_{n+1} \rightarrow S$  which breaks the block containing  $n+1$  into singletons and removes  $n+1$ .

Let  $p, p' \in S$ , say  $p = p' = \{b_1, \dots, b_l, \{s_1\}, \dots, \{s_k\}\}$ , where  $|b_i| \geq 2$ . Then,

$$f(p) = f(p') = \{b_1, \dots, b_l, \{s_1, \dots, s_k, n+1\}\} \in A_{n+1},$$

so  $f$  is well-defined.

Now suppose  $q, q' \in A_{n+1}$ , say  $q = q' = \{b_1, \dots, b_l, \{s_1, \dots, s_k, n+1\}\}$ . Note that each block in  $q, q'$  has size at least 2. Then,

$$g(q) = g(q') = \{b_1, \dots, b_l, \{s_1\}, \dots, \{s_k\}\},$$

which contains at least one singleton, and thus  $g$  is well-defined.

Since

$$g(f(p)) = g(\{b_1, \dots, b_l, \{s_1, \dots, s_k, n+1\}\}) = \{b_1, \dots, b_l, \{s_1\}, \dots, \{s_k\}\} = p,$$

$$f(g(q)) = f(\{b_1, \dots, b_l, \{s_1\}, \dots, \{s_k\}\}) = \{b_1, \dots, b_l, \{s_1, \dots, s_k, n+1\}\} = q,$$

$f$  and  $g$  are inverses of each other, and so  $S \simeq A_{n+1}$ .

But then  $|S| = |A_{n+1}| = F(n+1)$ , and the result follows.  $\square$