# MATH 188: Homework #6

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# Problem 1

The following exercise gives another proof of Cayley's formula, and at the same time provides new information that our proof doesn't give.

Let $n \geq 1$ and let $x_1, \dots, x_n$ be variables. Given a labeled tree $T$ with vertices $1, \dots, n$, define the monomial $x(T) = x_1^{d_1} \cdots x_n^{d_n}$ where $d_i$ is the degree of vertex $i$, i.e., the number of edges containing $i$. Define $\mathbf{C}_n = \sum_T x(T)$ where the sum is over all labeled trees $T$ with vertices $1, \dots, n$. Also define

$$\mathbf{D}_n = x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}.$$

(a) Given a polynomial $p(x_1, \dots, x_n)$, let $p^{(i)}$ be the result of plugging in $x_i = 0$ into the partial derivative $\frac{\partial p}{\partial x_i}$, i.e., the coefficient of $x_i$ if you think of the other variables as constants. If $n \geq 2$, show that

$$\mathbf{C}_n^{(n)} = (x_1 + x_2 + \cdots + x_{n-1})\mathbf{C}_{n-1},$$

$$\mathbf{D}_n^{(n)} = (x_1 + x_2 + \cdots + x_{n-1})\mathbf{D}_{n-1}.$$

*Proof.* Since a tree is connected, all vertices has positive degree. Hence, we have

$$\mathbf{C}_n^{(n)} = \frac{\partial}{\partial x_n} \sum_T x(T) \bigg|_{x_n=0} = \sum_T d_n x_1^{d_1} \cdots x_n^{d_n - 1} \Big|_{x_n=0} = \sum_{T; d_n=1} x_1^{d_1} \cdots x_{n-1}^{d_{n-1}}. \tag{1}$$

A tree must contains a vertex of degree 1. Given a $T$ with $d_n = 1$, suppose $j$ is the only neighbor of $n$. Then, $x(T) = x_n x_j x(T_{n-1})$, where $T_{n-1} = T - \{n\}$ a labeled tree with vertex set $[n-1]$. On the other hand, given a labeled tree $T_{n-1}$ with vertex set $[n-1]$, we may choose a vertex which connects to $n$ and get $T$ with $d_n = 1$, with $x(T) = x_n x_j x(T_{n-1})$. It now follows that

$$\begin{aligned}
\mathbf{C}_n^{(n)} &= \frac{\partial}{\partial x_n} \sum_T x(T) \bigg|_{x_n=0} \\
&= \frac{\partial}{\partial x_n} \sum_{T; d_n=1} x(T) \bigg|_{x_n=0} + \frac{\partial}{\partial x_n} \sum_{T; d_n \neq 1} x(T) \bigg|_{x_n=0} \\
&= \sum_{j=1}^{n-1} \sum_{\substack{T; d_n=1, \\ \{j,n\} \in e(T)}} \frac{\partial}{\partial x_n} x_n x_j x(T_{n-1}) \bigg|_{x_n=0} \\
&= \sum_{j=1}^{n-1} x_j \sum_{T_{n-1}} x(T_{n-1}) = \sum_{j=1}^{n-1} x_j \mathbf{C}_{n-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbf{D}_n^{(n)} = x_1 x_2 \cdots x_{n-1} (x_1 + x_2 + \cdots + x_n)^{n-2} \\
&+ (n-2) x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-3} |_{x_n=0} \\
&= x_1 x_2 \cdots x_{n-1} (x_1 + x_2 + \cdots + x_{n-1})^{n-2} \\
&= (x_1 + x_2 + \cdots + x_{n-1}) \mathbf{D}_{n-1}.
\end{aligned}$$

□

Problem 1 continued on next page. . .

(b) Assuming that $\mathbf{C}_{n-1} = \mathbf{D}_{n-1}$ show that $\mathbf{C}_n^{(i)} = \mathbf{D}_n^{(i)}$ for all $i = 1, \ldots, n$.

*Proof.* Define $\mathbf{C}_{[n]-\{i\}} = \sum_{T_{[n]-\{i\}}} x(T_{[n]-\{i\}})$, where the sum is over all labeled trees $T_{[n]-\{i\}}$ with vertices $[n] - \{i\}$. Also define $\mathbf{D}_{[n]-\{i\}} = x_1 x_2 \cdots x_n x_i^{-1}(x_1 + x_2 + \cdots + x_n - x_i)^{n-3}$.

Using the same argument in (a), we may show that

$$\mathbf{C}_n^{(i)} = \sum_{T;d_i=1} \prod_{j \neq i} x_j^{d_j} = \sum_{j=1, j\neq i}^{n} x_j \mathbf{C}_{[n]-\{i\}},$$

for all $i$. On the other hand, for all $i$,

$$\begin{aligned}\mathbf{D}_n^{(i)} = x_i^{-1} x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2} \\ &+ (n-2) x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-3}|_{x_i=0} \\ &= x_1 x_2 \cdots x_n x_i^{-1}(x_1 + x_2 + \cdots + x_n - x_i)^{n-2} \\ &= (x_1 + x_2 + \cdots + x_n - x_i)\mathbf{D}_{[n]-\{i\}}.\end{aligned}$$

Note that the only differences between $\mathbf{C}_{n-1}, \mathbf{C}_{[n]-\{i\}}$ and between $\mathbf{D}_{n-1}, \mathbf{D}_{[n]-\{i\}}$ are the indexing of the variables. Hence, $\mathbf{C}_{n-1} = \mathbf{D}_{n-1}$ also implies that $\mathbf{C}_{[n]-\{i\}} = \mathbf{D}_{[n]-\{i\}}$. It now follows that

$$\mathbf{C}_n^{(i)} = \sum_{j=1, j\neq i}^{n} x_j \mathbf{C}_{[n]-\{i\}} = \sum_{j=1, j\neq i}^{n} x_j \mathbf{D}_{[n]-\{i\}} = \mathbf{D}_n^{(i)},$$

for all $i$. □

(c) Conclude that $\mathbf{C}_n = \mathbf{D}_n$ for all $n \geq 1$.

*Proof.* We proceed by induction on $n$. When $n = 1$, there are only one label tree, which is a singleton. Hence, $\mathbf{C}_1 = 1 = x_1 x_1^{-1} = \mathbf{D}_1$. Suppose $n \geq 2$. Since each tree has at least a leaf, each term in $\mathbf{C}_n$ has some $x_i$ with power 1. On the other hand, note that each term in the expansion of $(x_1 + x_2 + \cdots + x_n)^{n-2}$ misses at least one $x_i$, and thus each term in $\mathbf{D}_n$ has some $x_i$ with degree 1. By induction and (b), we have $\mathbf{C}_n^{(i)} = \mathbf{D}_n^{(i)}$ for all $i \in [n]$. It follows that the terms with some single degree $x_i$ are equal in $\mathbf{C}_n$ and $\mathbf{D}_n$, which entail every term. Hence, $\mathbf{C}_n = \mathbf{D}_n$. □

# Problem 2

How many ways are there to list the letters of the word MATHEMATICS so that no two consecutive letters are the same?

*Proof.* The repeated characters in the word MATHEMATICS are A, M, and T. Let $S_A, S_M, S_T$ each be the set of ways to list MATHEMATICS with consecutive A, M, T, respectively. By inclusion-exclusion, the number of arrangements of MATHEMATICS with consecutive same characters is

$$|S_A \cup S_M \cup S_T| = |S_A| + |S_M| + |S_T| - |S_A \cap S_M| - |S_A \cap S_T| - |S_T \cap S_T| + |S_A \cap S_M \cap S_T|.$$

Since A, M, and T each appears exactly twice in MATHEMATICS,

$$|S_A \cup S_M \cup S_T| = 3|S_A| - 3|S_A \cap S_M| + |S_A \cap S_M \cap S_T|,$$

by symmetry. Notice that to count elements in $S_A$, we may view $AA$ as a single character, and

$$|S_A| = \frac{10!}{2!2!}.$$

Similarily, to count elements in $S_A \cap S_M$, we may $AA$ and $MM$ as characters and get

$$|S_A \cap S_M| = \frac{9!}{2!}.$$

Using the same idea, we get

$$|S_A \cap S_M \cap S_T| = 8!.$$

Hence,

$$|S_A \cup S_M \cup S_T| = 3 \cdot \frac{10!}{2!2!} - 3 \cdot \frac{9!}{2!} + 8!.$$

In total there are $\frac{11!}{2!2!2!}$ arrangements of MATHEMATICS, so the number of arrangements of MATHEMATICS with no consecutive repeated characters is

$$\frac{11!}{2!2!2!} - 3 \cdot \frac{10!}{2!2!} + 3 \cdot \frac{9!}{2!} - 8! = 2772000.$$

□

# Problem 3

Let $n \geq 2$ be an integer. We have $n$ married couples ($2n$ people in total).

(a) How many ways can we have the $2n$ people stand in a line so that no person is standing next to their spouse?

*Proof.* Let $L$ be the set of all arrangements of $2n$ people in a line. Put $A = A_1 \cup \cdots \cup A_n$, where $A_i$ is set of ways to line up $n$ couples with the $i$th couple standing next to each other. Let $S \subseteq [n]$. Define

$$f(S) = |\{x \in A \mid x \in A_i \text{ if and only if } i \in S\}|,$$
$$g(S) = |\{x \in A \mid x \in A_i \text{ if } i \in S\}|.$$

Note that $g(\emptyset) = |A|$ and $f(\emptyset) = 0$. To calculate $g(S) = |\cap_{i \in S} A_i|$, we may view each couple in $S$ as a unit of people and account the ordering of each unit. Then, we would have $2n - |S|$ unit of people, with $|S|$ units each having 2 arrangements, so

$$g(S) = (2n - |S|)!2^{|S|}.$$

By inclusion-exclusion,

$$|A| = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} (-1)^{|S|-1}(2n - |S|)!2^{|S|} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (2n-k)!2^k$$

Since we are calculating the case where no person stands next to their spouse, we have

$$|L \backslash A| = |L| - |A| = (2n)! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (2n-k)!2^k = \sum_{k=0}^{n} (-2)^k \binom{n}{k} (2n-k)!.$$

□

(b) Same as (a), but replace "line" by "circle".

*Proof.* Let $L$ be the set of all arrangements of $2n$ people in a line. Put $A = A_1 \cup \cdots \cup A_n$, where $A_i$ is set of ways to line up $n$ couples with the $i$th couple standing next to each other. Let $S \subseteq [n]$. Define

$$f(S) = |\{x \in A \mid x \in A_i \text{ if and only if } i \in S\}|,$$
$$g(S) = |\{x \in A \mid x \in A_i \text{ if } i \in S\}|.$$

Note that $g(\emptyset) = |A|$ and $f(\emptyset) = 0$. To calculate $g(S) = |\cap_{i \in S} A_i|$, we may view each couple in $S$ as a unit of people and account the ordering of each unit. Then, we would have $2n - |S|$ unit of people, with $|S|$ units each having 2 arrangements, so

$$g(S) = \frac{1}{2n - |S|} \cdot (2n - |S|)!2^{|S|} = (2n - |S| - 1)!2^{|S|}.$$

Note that we divide by $2n - |S|$ to disregard shifting the circle. By inclusion-exclusion,

$$|A| = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} (-1)^{|S|-1}(2n - |S| - 1)!2^{|S|} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (2n-k-1)!2^k$$

Hence, the number of ways to have $n$ couples stand in a circle with no person standign next to their spouse is

$$|L \backslash A| = \frac{1}{2n} \cdot (2n)! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (2n-k-1)!2^k = \sum_{k=0}^{n} (-2)^k \binom{n}{k} (2n-k-1)!.$$

□

# Problem 4

Let $q$ be a prime power and $n$ a positive integer. Let $V$ be an $n$-dimensional $\mathbf{F}_q$-vector space and let $P$ be the poset whose elements are linear subspaces of $V$ with the ordering $X \le Y$ if $X$ is contained in $Y$. Show that the Möbius function of $P$ is given by

$$\mu(X,Y) = (-1)^d q^{\binom{d}{2}}$$

where $d = \dim Y - \dim X$.

*Proof.* For $X \le Y$, it suffices to show that

$$\delta_{X,Y} = \sum_{U \in [X,Y]} (-1)^{d_u} q^{\binom{d_u}{2}},$$

where $d_u = \dim U - \dim X$. Put $x = \dim X$ and $y = \dim Y$. The number of vector spaces of dimension $k$ in the interval $[X,Y]$ is $\begin{bmatrix} d \\ k-x \end{bmatrix}_q$. It now follows that

$$\begin{aligned}\sum_{U \in [X,Y]} (-1)^{d_u} q^{\binom{d_u}{2}} &= \sum_{k=x}^{y} \begin{bmatrix} d \\ k-x \end{bmatrix}_q (-1)^{k-x} q^{\binom{k-x}{2}} \\ &= \sum_{i=0}^{d} \begin{bmatrix} d \\ i \end{bmatrix}_q (-1)^i q^{\binom{i}{2}}.\end{aligned}$$

By Theorem 3.2.4 from Sagen,

$$\sum_{i=0}^{d} \begin{bmatrix} d \\ i \end{bmatrix}_q (-1)^i q^{\binom{i}{2}} = \begin{cases} \prod_{i=0}^{d-1}(1-q^i) & d > 0 \\ 1 & d = 0 \end{cases} = \delta_{X,Y}.$$

□

# Problem 5

Let $\Pi_n$ be the poset of set partitions of $[n]$ and let $\mu$ be its Möbius function. Write a formula for the number of connected labeled graphs with vertex set $[n]$ using $\mu$.

*Proof.* Let $\mathcal{G}$ be the set of all labeled graphs with vertex set $[n]$, and let $P = \{S_1, \dots, S_m\} \in \Pi_n$. Define

$$f(P) = |\{G \in \mathcal{G} \mid i, j \text{ connected in } G \text{ if and only if } i, j \in S_k \text{ for some } k\}|,$$

$$g(P) = |\{G \in \mathcal{G} \mid i, j \text{ connected in } G \text{ only if } i, j \in S_k \text{ for some } k\}| = \prod_{k=1}^{m} 2^{\binom{|S_k|}{2}} = 2^{\sum_{k=1}^{m} \binom{|S_k|}{2}}.$$

Note that $f(\{[n]\})$ is the number of connected labeled graphs with vertex set $[n]$ and $g(\{[n]\}) = 2^{\binom{n}{2}}$. By definition, $g(P) = \sum_{Q \leq P} f(Q)$. It now follows by the Möbius inversion that

$$f(\{[n]\}) = \sum_{P \in \Pi_n} \mu(P, \{[n]\}) g(P) = \sum_{P \in \Pi_n} \mu(P, \{[n]\}) 2^{\sum_{S \in P} \binom{|S|}{2}}.$$

□

# Problem 6

$F(x) = \sum_{n \geq 0} f_n x^n$ is a formal power series that satisfies the following identity:

$$F(x) = \exp\left(\frac{x}{2}(F(x)+1)\right).$$

Find a formula for $f_n$.

*Proof.* We first note that $f_0 = F(0) = 1$. Adding 1 then multiplying $\frac{x}{2}$ on both sides of the given identity yields

$$\frac{x}{2}(F(x)+1) = \frac{x}{2}\left[\exp\left(\frac{x}{2}(F(x)+1)\right)+1\right].$$

Take $G(x) = \frac{1}{2}(e^x+1)$ and $A(x) = \frac{x}{2}(F(x)+1)$. Since $A(0) = 0$ and $G(0) \neq 0$, the Lagrange inversion formula gives

$$\begin{aligned}
\frac{n+1}{2}[x^n]F(x) &= (n+1)[x^{n+1}]A(x) \\
&= [x^n](G(x)^{n+1}) \\
&= [x^n]\frac{1}{2^{n+1}}(e^x+1)^{n+1} \\
&= [x^n]\frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}e^{kx} \\
&= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{k^n}{n!}
\end{aligned}$$

That is, for $n \geq 1$,

$$f_n = \frac{1}{2^n(n+1)!}\sum_{k=0}^{n+1}\binom{n+1}{k}k^n.$$

□

# Problem 7

Reminder: Lagrange's version of the Taylor remainder theorem says this: if $f(x)$ is an infinitely differentiable function whose Taylor series at 0 converges at $x = r$, then there exists $\xi$ between 0 and $r$ such that

$$f(r) - \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} r^i = \frac{f^{(n+1)}(\xi)}{(n+1)!} r^{n+1}.$$

Use the Taylor remainder theorem to show that

$$\left| \frac{1}{e} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} \right| \leq \frac{1}{(n+1)!}$$

and conclude from this that the number of derangements of $n$ objects is inside the closed interval

$$\left[ \frac{n!}{e} - \frac{1}{n+1}, \frac{n!}{e} + \frac{1}{n+1} \right].$$

In particular, show that it is the closest integer to $n!/e$.

*Proof.* Consider $f(r) = e^{-r}$. When $r = 1$, Taylor remainder theorem yields

$$e^{-1} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \frac{(-1)^{n+1} e^{-\xi}}{(n+1)!},$$

for some $\xi$ between 0 and 1. But then $|e^{-\xi}| \leq 1$, so

$$\left| \frac{1}{e} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} \right| = \left| \frac{(-1)^{n+1} e^{-\xi}}{(n+1)!} \right| \leq \frac{1}{(n+1)!}.$$

Let $D(n)$ be the number of derangements of size $n$. It now follows from Theorem 6.14 that

$$\frac{n!}{e} - \frac{1}{n+1} \leq D(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \leq \frac{n!}{e} + \frac{1}{n+1}.$$

Obviously $\frac{n!}{e}$ is not an integer. Since $D(n)$ is an integer and $\frac{1}{n+1} \leq \frac{1}{2}$ for all $n \geq 1$, $D(n)$ is the closest integer to $\frac{n!}{e}$. $\square$

# Problem 8

Let $d_n$ be the number of derangements of $[n]$, and let

$$D(x) = \sum_{n \geq 0} \frac{d_n}{n!} x^n.$$

(a) Using the structure interpretation for products of EGF, show that

$$D(x)e^x = \frac{1}{1-x}.$$

*Proof.* Let $D(S)$ denote the set of derangements of $S$. Define structures $\alpha(S) = D(S)$ and $\beta(S) = \{0\}$. Hte product of two structures is

$$(\alpha \cdot \beta)(S) = \bigsqcup_{T \subseteq S} D(T) \times \{0\} \simeq \bigsqcup_{T \subseteq S} D(T).$$

But then given a derangement of some subset $T \subseteq S$, we get a permutation $\sigma$ of $S$ with $\sigma(i) = i$ if and only if $i \in S \backslash T$. On the other hand, given a permutation $\sigma$ of $S$, we get a derangement of $T = \{i \in S \mid \sigma(i) \neq i\} \subseteq S$. Hence, $\bigsqcup_{T \subseteq S} D(T) \simeq S_n$, the symmetry group of degree $n$. It now follows that $(\alpha \cdot \beta)(S) = |S|!$, and so

$$D(x)e^x = E_{\alpha \cdot \beta}(x) = \sum_{n \geq 0} x^n = \frac{1}{1-x}.$$

□

(b) Show how this implies the formula we previously obtained:

$$d_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}.$$

*Proof.* Rearranging the result of (a), we get

$$D(x) = \frac{1}{1-x} e^{-x},$$

and so

$$d_n = n![x^n] \frac{1}{1-x} e^{-x} = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}.$$

□

# Problem 9

For a positive integer $n$, define

$$f(n) = |\{i \in \mathbb{Z} \mid 1 \leq i \leq n, \gcd(n,i) = 1\}|.$$

(a) Show that

$$n = \sum_{d|n} f(d)$$

where the sum is over all positive integers $d$ that divide $n$.

*Proof.*

$$\begin{aligned}
\sum_{d|n} f(d) &= \sum_{d|n} f(n/d) \\
&= \sum_{d|n} |\{i \in \mathbb{Z} \mid 1 \leq i \leq n/d, \gcd(n/d, i) = 1\}| \\
&= \sum_{d|n} |\{i \in \mathbb{Z} \mid 1 \leq i \leq n, \gcd(n, i) = d\}| \\
&= n.
\end{aligned}$$

□

(b) Use Möbius inversion to show that

$$f(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

where the product is over the primes $p$ that divide $n$.

*Proof.* Define $g(n) = n$, for $n \in \mathbb{N}$. In (a), we already established that $g(n) = \sum_{d|n} f(d)$. By the Möbius inversion formula,

$$\begin{aligned}
f(n) &= \sum_{d|n} g(d)\mu(d,n) \\
&= \sum_{d|n} d\mu(d,n) \\
&= \sum_{d|n} \frac{n}{d}\mu\left(\frac{n}{d}, n\right) \\
&= n\sum_{d|n} \frac{1}{d}\mu\left(\frac{n}{d}, n\right) \\
&= n \sum_{\substack{d=p_1\cdots p_k \\ p_i|n \text{ and distinct}}} \frac{(-1)^k}{d} \\
&= n \sum_{\substack{d=p_1\cdots p_k \\ p_i|n \text{ and distinct}}} \left(-\frac{1}{p_1}\right)\cdots\left(-\frac{1}{p_k}\right) \\
&= n\prod_{p|n}\left(1 - \frac{1}{p}\right).
\end{aligned}$$

□

# Problem 10

There are $n$ people sitting at a circular table. How many ways can they rearrange seats so that no one sits next to someone they were sitting next to before?

*Proof.* idk bro. $\square$

# Problem 11

Let $q$ be a prime power and let $N_n$ be the number of monic irreducible polynomials of degree $n$ with coefficients in $\mathbf{F}_q$:

(a) Using that polynomials over a field satisfy unique factorization, show that

$$(1-qx)^{-1} = \prod_{d\geq 1}(1-x^d)^{-N_d}$$

*Proof.* By the binomial theorem,

$$(1-x^d)^{-N_d} = \sum_{k\geq 0}\binom{-N_d}{k}(x^d)^k.$$

Note that $\binom{-N_d}{k}$ is the number of ways to pick a multiset of size $k$ from $N_d$ elements. Given a monic polynomial, we may view its factorization as a multiset of irreducible polynomials. Hence, $[x^{dk}](1-x^d)^{-N_d}$ is the number of ways to pick a monic polynomial whose factorization is $k$ irreducible polynomials of degree $d$. But then

$$[x^n]\prod_{d\geq 1}(1-x^d)^{-N_d} = [x^n]\prod_{d\geq 1}\sum_{k\geq 0}\binom{-N_d}{k}(x^d)^k$$

is just the number monic polynomials of degree $n$. Since there are $n$ undetermined coefficients in a monic polynomial of degree $n$, there are $q^n$ monic polynomials of degree $n$. In other words,

$$\prod_{d\geq 1}(1-x^d)^{-N_d} = (1+qx+q^2x^2+\cdots) = (1-qx)^{-1}.$$

□

(b) Take the logarithmic derivative of (a) and compare the coefficient of $x^{n-1}$ to get

$$q^n = \sum_{d|n} dN_d.$$

*Proof.*

$$\mathcal{L}((1-qx)^{-1}) = q(1-qx)(1-qx)^{-2} = q(1-qx)^{-1}.$$

$$\begin{aligned}
\mathcal{L}\left(\prod_{d\geq 1}(1-x^d)^{-N_d}\right) &= \sum_{d\geq 1} N_d\mathcal{L}((1-x^d)^{-1}) \\
&= \sum_{d\geq 1} dN_d x^{d-1}(1-x^d)^{-1} \\
&= \sum_{d\geq 1} dN_d(x^{d-1}+x^{2d-1}+x^{3d-1}+\cdots).
\end{aligned}$$

Hence,

$$\begin{aligned}
q^n &= [x^{n-1}]q(1-qx)^{-1} \\
&= [x^{n-1}]\sum_{d\geq 1} dN_d x^{d-1}(1-x^d)^{-1} \\
&= [x^n]\sum_{d\geq 1} dN_d(x^d+x^{2d}+x^{3d}+\cdots) \\
&= \sum_{d|n} dN_d.
\end{aligned}$$

□

Problem 11 continued on next page... 

(c) Use Möbius inversion to get a formula for $N_n$.

*Proof.* Since $q^n = \sum_{d|n} dN_d$, by Möbius inversion,

$$nN_d = \sum_{d|n} q^d \mu(d, n),$$

and so

$$N_d = \frac{1}{n} \sum_{d|n} q^d \mu(d, n).$$

□