MATH 188: Homework #1

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We first give a proof for the general closed form of homogeneous linear recurrence relations for later use:

Proof. Given

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$$

for $n \geq d$, and the values of a_n, $n < d$, define $A(x) = \sum\limits_{n \geq 0} a_n x^n$. Then, we have

$$\begin{aligned} A(x) &= a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} + \sum_{n \geq d}(c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d})x^n \\ &= \sum_{n=0}^{d-1} a_n x^n + c_1 x \sum_{n \geq d-1} a_n x^n + c_2 x^2 \sum_{n \geq d-2} a_n x^n + \cdots + c_d x^d \sum_{n \geq 0} a_n x^n \\ &= \sum_{n=0}^{d-1} a_n x^n + c_1 x \left(A(x) - \sum_{n=0}^{d-2} a_n x^n \right) + c_2 x^2 \left(A(x) - \sum_{n=0}^{d-3} a_n x^n \right) + \cdots + c_d x^d A(x) \end{aligned}$$

Rearranged,

$$A(x) = \frac{p(x)}{1 - c_1 x - c_2 x^2 - \cdots - c_d x^d},$$

for some polynomial $p(x)$ of degree less than d. We may assume $1 - c_1 x - c_2 x^2 - \cdots - c_d x^d$ has factorization $(1 - r_1 x)^{w_1}(1 - r_2 x)^{w_2} \ldots (1 - r_s x)^{w_s}$. By applying partial fraction decomposition,

$$A(x) = \sum_{m=1}^{w_1} \frac{p_{1,m}(x)}{(1 - r_1 x)^m} + \sum_{m=1}^{w_2} \frac{p_{2,m}(x)}{(1 - r_2 x)^m} + \cdots + \sum_{m=1}^{w_s} \frac{p_{s,m}(x)}{(1 - r_s x)^m},$$

where $p_{i,m}(x)$ is some polynomial of degree less than m. Note that for $k < m$,

$$\frac{x^k}{(1 - r_i x)^m} = \sum_{n \geq 0} \binom{m+n-1}{n} r_i^n x^{n+k} = \sum_{n \geq k} \left(\binom{m+n-k-1}{n-k} r_i^{-k} \right) r_i^n x^n,$$

By the binomial theorem. But then

$$\binom{m+n-k-1}{n-k} r_i^{-k} = \frac{r_i^{-k}}{(m-1)!}(m+n-k-1)\cdots(n-k+2)(n-k+1),$$

so the each coefficient of r_i^n is really just a polynomial of n with degree less than $m-1$. It follows that for $n \geq d$, the recurrence relation has a closed form

$$a_n = \sum_{i=1}^{s} f_i(n) r_i^n,$$

for some polynomial $f_i(n)$ of degree less than w_i. $\square$

Problem 1

Find a closed formula for the following recurrence relation:

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 2,$$
$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \quad (n \geq 3).$$

Proof. The characteristic polynomial of this recurrence relation is defined to be

$$t^3 - 5t^2 + 8t - 4 = (t-1)(t-2)^2,$$

which has roots $t = 1, 2$. Note that 2 is a repeated root, and thus

$$a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 n 2^n.$$

Solving the system of equations

$$\begin{cases} 1 = \alpha_1 + \alpha_2 \\ 1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ 2 = \alpha_1 + 4\alpha_2 + 8\alpha_3 \end{cases},$$

we get

$$a_n = 2 - 2^n + n2^{n-1}.$$

□

Problem 2

Let $r_1, \dots, r_d$ be distinct numbers. Show that the determinant of the $d \times d$ matrix $(r_i^{j-1})_{i,j=1,\dots,d}$ is nonzero (interpret $0^0 = 1$). Explain why this implies that the sequences $(r_1^n)_{n\geq 0}, \dots, (r_d^n)_{n\geq 0}$ are linearly independent.

Proof. Given numbers $x_1, x_2, \dots, x_d$, define

$$M(x_1, x_2, \dots, x_d) = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{d-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_d & x_d^2 & \cdots & x_d^{d-1} \end{bmatrix}.$$

We first show by induction on d that,

$$\det M(x_1, x_2, \dots, x_d) = \prod_{1 \leq i < j \leq d} (x_j - x_i),$$

for all $d \geq 2$. We already know.

$$\det M(x_1, x_2) = \det \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} = x_2 - x_1.$$

Suppose $d > 2$. Note that the determinant remains the same after subtracting to each column the preceding column scaled by x_1. Hence,

$$\begin{aligned} \det M(x_1, x_2, \dots, x_d) &= \det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{d-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_d & x_d^2 & \cdots & x_d^{d-1} \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & x_2(x_2 - x_1) & \cdots & x_2^{d-2}(x_2 - x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_d - x_1 & x_d(x_d - x_1) & \cdots & x_d^{d-2}(x_d - x_1) \end{bmatrix} \\ &= \det \begin{bmatrix} x_2 - x_1 & x_2(x_2 - x_1) & \cdots & x_2^{d-2}(x_2 - x_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_d - x_1 & x_d(x_d - x_1) & \cdots & x_d^{d-2}(x_d - x_1) \end{bmatrix}. \end{aligned}$$

Since the entries of ith row share a common factor $(x_{i+1} - x_1)$, we may extract them from the determinant and get

$$\begin{aligned} \det M(x_1, x_2, \dots, x_d) &= \left(\prod_{1 \leq i \leq d-1} (x_{i+1} - x_1) \right) \det M(x_2, x_2, \dots, x_d) \\ &= \left(\prod_{1 \leq i \leq d-1} (x_{i+1} - x_1) \right) \left(\prod_{2 \leq i < j \leq d} (x_j - x_i) \right) = \prod_{1 \leq i < j \leq d} (x_j - x_i), \end{aligned}$$

by induction. Since all r_i's are distinct,

$$\det M(r_1, r_2, \dots, r_d) = \prod_{1 \leq i < j \leq d} (r_j - r_i) \neq 0.$$

But then $(r_1^n)_{0 \leq n < d}, \dots, (r_d^n)_{0 \leq n < d}$ are linearly independent, so $(r_1^n)_{n \geq 0}, \dots, (r_d^n)_{n \geq 0}$ are also linearly independent. (*Source cited: wikipedia.org/wiki/Vandermonde_matrix*) □

Problem 3

Let $(a_n)_{n\geq 0}$ be a sequence satisfying a linear recurrence relation whose characteristic polynomial is $(t^2-1)^d$. Show that there exist polynomials $p(n)$ and $q(n)$ of degree $\leq d-1$ such that

$$a_n = \begin{cases} p(n) & \text{if } n \text{ is even} \\ q(n) & \text{if } n \text{ is odd} \end{cases}.$$

Proof. Since $(t^2-1)^d = (t-1)^d(t+1)^d$,

$$\begin{aligned} a_n &= \alpha_0 + \alpha_1 n + \cdots + \alpha_{d-1}n^{d-1} + (-1)^n(\beta_0 + \beta_1 n + \cdots + \beta_{d-1}n^{d-1}) \\ &= \begin{cases} \sum_{k=0}^{d-1}(\alpha_k + \beta_k)n^k & \text{if } n \text{ is even} \\ \sum_{k=0}^{d-1}(\alpha_k - \beta_k)n^k & \text{if } n \text{ is odd} \end{cases}. \end{aligned}$$

The result follows by taking $p(n) = \sum_{0\leq k\leq d-1}(\alpha_k+\beta_k)n^k$ and $q(n) = \sum_{k=0}^{d-1}(\alpha_k - \beta_k)n^k$. □

Problem 4

(a) Suppose that $(a_n)_{n\geq 0}$ and $(a'_n)_{n\geq 0}$ both satisfy the same linear recurrence relation of order d and that they agree in d consecutive places, i.e., there exists k such that $a_k = a'_k$, $a_{k+1} = a'_{k+1}$, ..., $a_{k+d-1} = a'_{k+d-1}$. Show that these sequences are the same.

Proof. By assumption,

$$\begin{aligned} a_n &= c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d} \\ a'_n &= c_1 a'_{n-1} + c_2 a'_{n-2} + \cdots + c_d a'_{n-d}, \end{aligned}$$

for some $c_1, \ldots, c_d$, with $c_d \neq 0$. By induction, $a_n = a'_n$ for all $n \geq k$, so it remains to show the equality also holds true for $n < k$. Rearranging the equations, we get

$$\begin{aligned} a_n &= \frac{1}{c_d}(a_{n+d} - c_1 a_{n+d-1} - \cdots - c_{d-1} a_{n+1}) \\ a'_n &= \frac{1}{c_d}(a'_{n+d} - c_1 a'_{n+d-1} - \cdots - c_{d-1} a'_{n+1}), \end{aligned}$$

so by induction based on the k consecutive terms that both sequences agree we get $a_n = a'_n$ for all $n < k$, and this completes the proof. □

(b) Suppose that $(a_n)_{n\geq 0}$ satisfies the linear recurrence relation of order d

$$a_n = c_1 a_{n-1} + \ldots + c_d a_{n-d} \quad \text{for all } n \geq d$$

with $c_d \neq 0$. Show that there is a unique sequence $(b_n)_{n\in\mathbb{Z}}$ (indexed by *all* integers) such that $b_n = a_n$ for $n \geq 0$ and such that

$$b_n = c_1 b_{n-1} + \ldots + c_d b_{n-d} \quad \text{for all } n \in \mathbb{Z}. \tag{1}$$

Proof. Given $b_n = a_n$ for $n \geq 0$, define

$$b_n = \frac{1}{c_d}(b_{n+d} - c_1 b_{n+d-1} - \cdots - c_{d-1} b_{n+1}), \tag{2}$$

for $n < 0$. Rearranging (2), we know b_n follows (1) for $n \in \mathbb{Z}$. Hence, it remains to show the uniqueness of (b_n). Suppose there exists (b'_n) such that $b'_n = a_n$ for $n \geq 0$ and satisfies the recurrence relation for all $n \in \mathbb{Z}$. We already know (b_n) and (b'_n) agree for all nonnegative terms. But then by (2), (b_n) and (b'_n) agree with each negative term by backwards induction on negative n based on the first d nonnegative terms, so both sequences also agree on the negative terms. Hence, $(b_n) = (b'_n)$ and we are done. □

(c) Consider the Fibonacci sequence $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$. How does the negatively indexed Fibonacci sequence relate to the usual one?

Proof. For $n < 0$, f_n is defined as

$$f_n = -f_{n+1} + f_{n+2}.$$

Define a new sequence $(g_n)_{n\geq 0}$ as $g_n = f_{-n}$. The characteristic polynomial of (g_n) is $t^2 + t - 1$, which has roots $r'_1 = \frac{-1+\sqrt{5}}{2}$ and $r'_2 = \frac{-1-\sqrt{5}}{2}$. Notice that $r'_1 = -r_1$ and $r'_2 = -r_2$, where r_1, r_2 are the roots of the characteristic polynomial of the Fibonacci sequence. Since $g_0 = 0$ and $g_1 = 1$,

$$g_n = \frac{1}{\sqrt{5}}((r'_1)^n + (r'_2)^n) = \frac{(-1)^n}{\sqrt{5}}(r_1^n + r_2^n) = (-1)^n f_n,$$

so (g_n) is just the alternating Fibonacci sequence. □

Problem 5

Let $A_0(x), A_1(x), \dots$ and $B_0(x), B_1(x), \dots$ be sequences of formal power series. Assume that $\lim_{i\to\infty} A_i(x) = A(x)$ and $\lim_{i\to\infty} B_i(x) = B(x)$.

(a) Prove that $\lim_{i\to\infty}(A_i(x) + B_i(x)) = A(x) + B(x)$.

Proof. Note that for any n, there exists N_{a_n}, N_{b_n} such that $[x^n]A_i(x) = [x^n]A(x)$ and $[x^n]B_i(x) = [x^n]B(x)$, for all $i \geq N_n = \max(N_{a_n}, N_{b_n})$. Hence,

$$[x^n](A_i(x) + B_i(x)) = [x^n]A_i(x) + [x^n]B_i(x) = [x^n]A(x) + [x^n]B(x) = [x^n](A(x) + B(x)),$$

for $i \geq N_n$, and the result follows. □

(b) Prove that $\lim_{i\to\infty}(A_i(x)B_i(x)) = A(x)B(x)$.

Proof. Note that for any n, there exists N_{a_n}, N_{b_n} such that $[x^n]A_i(x) = [x^n]A(x)$ and $[x^n]B_i(x) = [x^n]B(x)$, for all $i \geq N_n = \max(N_{a_n}, N_{b_n})$. Given $m \geq 0$, take $N = \max(N_0, N_1, \dots, N_m)$. Then,

$$[x^m](A_i(x)B_i(x)) = \sum_{k=0}^{m}[x^k]A_i(x)[x^{m-k}]B_i(x) = \sum_{k=0}^{m}[x^k]A(x)[x^{m-k}]B(x) = [x^m](A(x)B(x)),$$

for $i \geq N$, and the result follows. □

Problem 6

Continuing from Problem 3, how does the statement generalize if the characteristic polynomial is $(t^k-1)^d$?

Proof. Notice $t^k - 1 = (t-1)(t-\omega)(t-\omega^2)\dots(t-\omega^k)$, where $\omega = e^{\frac{2\pi}{k}}$. Hence, for $m = 0, 1, \dots k-1$, take $p_m(n) = \sum_{i=1}^{k} \omega^{im} \sum_{j=0}^{d-1} \alpha_{i,j} n^j$, which are polynomials of degree at most $d-1$. Then,

$$
\begin{aligned}
a_n &= \sum_{i=1}^{k} \omega^{in} \sum_{j=0}^{d-1} \alpha_{i,j} n^j \\
&= \begin{cases} p_0(n) & \text{if } n \equiv 0 \pmod{k} \\ p_1(n) & \text{if } n \equiv 1 \pmod{k} \\ & \vdots \\ p_{k-1}(n) & \text{if } n \equiv k-1 \pmod{k} \end{cases}.
\end{aligned}
$$

□

Problem 7

Let p be a prime number and let $(a_n)_{n\geq 0}$ be a sequence such that $a_n \in \mathbb{Z}/p$ and which satisfies a homogeneous linear recurrence relation. Show that the sequence is in fact periodic.

Proof. By assumption,

$$a_n = c_1 a_{n-1} + c_2 a_{n-1} + \cdots + c_d a_{n-d},$$

for some $c_1, c_2, \ldots, c_d \in \mathbb{Z}/p$, $c_d \neq 0$. Since there are only p^d possible strings of length d, it is guaranteed that some length d string s_d repeats in the first dp^d terms. Suppose that s_d initially appeared at a_k and repeated at a_{k+l}, that is, $a_k = a_{k+l}, a_{k+1} = a_{k+1+l}, \ldots, a_{k+d-1} = a_{k+d-1+l}$. Note that $\mathbb{Z}/p$ is closed under taking multiplicative inverse. Hence, by problem 4(a), we have $(a_n)_{n\geq 0} = (a_{n+l})_{n\geq 0}$, and thus $(a_n)_{n\geq 0}$ is periodic. $\square$

Problem 8

Let $r_1, \dots, r_{d-1}$ be distinct numbers. Prove that the sequences $\alpha_1 = (r_1^n), \dots, \alpha_{d-1} = (r_{d-1}^n), \alpha_d = (nr_{d-1}^{n-1})$ are linearly independent by showing that the determinant of $(\alpha_{i,j-1})_{i,j=1,\dots,d}$ is nonzero (interpret $0^0 = 1$ and if $r_{d-1} = 0$, interpret $\alpha_{d,0} = 0$).

Proof. Given distinct d numbers $r_1, r_2, \dots, r_{d-1}$, define

$$M(r_1, r_2, \dots, r_{d-1}) = \begin{bmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^{d-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} & r_{d-1}^2 & \cdots & r_{d-1}^{d-1} \\ 0 & 1 & 2r_{d-1} & \cdots & dr_{d-1}^{d-1} \end{bmatrix}.$$

We first show by induction on d that,

$$\det M(r_1, r_2, \dots, r_{d-1}) \neq 0,$$

for any d distinct numbers, $d \geq 2$. We already know.

$$\det M(r_1) = \det \begin{bmatrix} 1 & r_1 \\ 0 & 1 \end{bmatrix} = 1.$$

Suppose we are given distinct $r_1, \dots, r_{d-1}$, for $d > 2$. Note that the determinant remains the same after subtracting to each column the preceding column scaled by r_1. Hence,

$$\begin{aligned}
\det M(r_1, r_2, \dots, r_{d-1}) &= \det \begin{bmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^{d-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} & r_{d-1}^2 & \cdots & r_{d-1}^{d-1} \\ 0 & 1 & 2r_{d-1} & \cdots & (d-1)r_{d-1}^{d-2} \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & r_2 - r_1 & r_2(r_2 - r_1) & \cdots & r_2^{d-2}(r_2 - r_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} - r_1 & r_{d-1}(r_{d-1} - r_1) & \cdots & r_{d-1}^{d-2}(r_{d-1} - r_1) \\ 0 & 1 & 2r_{d-1} - r_1 & \cdots & (d-1)r_{d-1}^{d-2} - (d-2)r_1 r_{d-1}^{d-3} \end{bmatrix} \\
&= \left(\prod_{1 \leq i \leq d-1} (r_{i+1} - r_1) \right) \det \begin{bmatrix} 1 & r_2 & \cdots & r_2^{d-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} & \cdots & r_{d-1}^{d-2} \\ 1 & 2r_{d-1} - r_1 & \cdots & (d-1)r_{d-1}^{d-2} - (d-2)r_1 r_{d-1}^{d-3} \end{bmatrix} \\
&= \left(\prod_{1 \leq i \leq d-1} (r_{i+1} - r_1) \right) \det \begin{bmatrix} 1 & r_2 & r_2^2 & \cdots & r_2^{d-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} & r_{d-1}^2 & \cdots & r_{d-1}^{d-2} \\ 0 & (r_{d-1} - r_1) & 2(r_{d-1} - r_1)r_{d-1} & \cdots & (d-2)(r_{d-1} - r_1)r_{d-1}^{d-3} \end{bmatrix} \\
&= (r_{d-1} - r_1) \left(\prod_{1 \leq i \leq d-1} (r_{i+1} - r_1) \right) \det M(r_2, r_3, \dots, r_{d-1}).
\end{aligned}$$

But then all r_i's are distinct, so $\det M(r_1, r_2, \dots, r_{d-1}) \neq 0$, by induction. The induction result implies that $(r_1^n)_{0 \leq n < d}, \dots, (r_{d-1}^n)_{0 \leq n < d}, (nr_{d-1}^{n-1})_{0 \leq n < d}$ are linearly independent, so $(r_1^n)_{n \geq 0}, \dots, (r_{d-1}^n)_{n \geq 0}, (nr_{d-1}^{n-1})_{n \geq 0}$ are also linearly independent. □