MATH 140A: Homework #4

Due on Feb 9, 2024 at 23:59pm

Professor Seward

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Problem 1

Let E° denote the set of all interior points of a set E.

(a) Prove that E° is always open.

Proof. Let $p \in E^\circ$. Since p is an interior point of E, there exists a neighborhood N of p such that $N \subset E$. Let $q \in N$. Since N is an open set, there exists a neighborhood V of q such that $V \subset N \subset E$, and thus q is also an interior point of E. Hence, $N \subset E^\circ$, so E° is open. $\square$

(b) Prove that E is open if and only if $E^\circ = E$.

Proof. Suppose that E is open. By definition, we know $E \subset E^\circ$. For $p \in E^\circ$, we know there exists a neighborhood N of p such that $N \subset E$. However, $p \in N \subset E$, and thus $E^\circ \subset E$. Therefore, $E^\circ = E$.

We now assume the converse. For $p \in E$, since $p \in E^\circ$, p is an interior point of E, and thus E is open. $\square$

(c) If $G \subseteq E$ and G is open, prove that $G \subseteq E^\circ$.

Proof. Let $g \in G$. Since g is an interior point of G, there exists a neighborhood N of g such that $N \subset G \subset E$. Thus, g is also an interior point of G, and the result now follows. $\square$

(d) Prove that the complement of E° is the closure of the complement of E.

Proof. Let $p \in (E^\circ)^c$. Then, for all neighborhood N of p, N is not a subset of E, and thus N contains a point in E^c, which makes p is limit point of E^c. Therefore, $p \in \overline{E^c} = (E^c)' \cup E^c$. $\square$

(e) Do E and $\overline{E}$ always have the same interiors?

Proof. No. Consider $E = \mathbb{Q}$ in $\mathbb{R}$. By the density of $\mathbb{Q}$, any real number is a limit point of $\mathbb{Q}$, so $\overline{E} = \mathbb{R}$. However, $\mathbb{Q}$ has no interiors points, but $\mathbb{R}$ is open. Hence, $\mathbb{Q}$ and $\mathbb{R}$ do not have the same interiors. $\square$

(f) Do E and E° always have the same closures?

Proof. No. Consider the set $E = \{\pi\}$ in $\mathbb{R}$. We know that E is closed so $\overline{E} = E$. However, E does not contain any interior points, so $E^\circ = \emptyset$. It immediately follows that $\emptyset$ is closed, and thus $\overline{E^\circ} = \emptyset \neq \overline{E}$. $\square$

Problem 2

Let $K \subseteq \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$ for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Proof. Let $\{G_\alpha\}$ be an open cover of K. Then, 0 must be in some G_{α_0}. Since G_{α_0} is an open set, there exists $N_\epsilon(0) = (-\epsilon, \epsilon) \subset G_{\alpha_0}$. Hence, $(-\epsilon, \epsilon)$ covers all $\frac{1}{k} \in K$ such that $k > \frac{1}{\epsilon}$. By the archimedean property, we may find an integer $m > \frac{1}{\epsilon}$. For natural number $n < m$, we may find a G_{α_n} that covers $\frac{1}{n}$. Thus, with the choice of m indices $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$,

$$K \subset G_{\alpha_0} \cup G_{\alpha_0} \cup \ldots \cup G_{\alpha_{m-1}},$$

which proves the compactness of K. $\square$

Problem 3

Give an example of an open cover of the segment $(0,1)$ which has no finite subcover.

Proof. Consider $\{G_\alpha\}$, where $G_\alpha = (0,\alpha)$, for $\alpha \in (0,1)$. $\{G_\alpha\}$ is clearly an infinite collection. Let $p \in (0,1)$ and $q \in (p,1)$. Since there exists G_q such that $p \in G_q$, $\{G_\alpha\}$ is an infinite open cover of $(0,1)$. Let $A \subset (0,1)$ be non-empty and finite. Then, there exists $k = \max A$. Since $k < 1$, there exists $h \in (k,1)$ such that $h \notin G_a = (0,a)$, for all $a \in A$, as $a < k < h < 1$. Hence, $\{G_\alpha\}$ has no finite subcover. $\square$

Problem 4

A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. *Hint:* Consider the set of points which have only rational coordinates.

Proof. Consider $\mathbb{Q}^k \subset \mathbb{R}^k$. Note that $\mathbb{Q}$ is countable, so $\mathbb{Q}^k$ is countable, by Theorem 2.12. Let S be a non-empty open subset of $\mathbb{R}^k$ and let $x = (x_1, x_2, \ldots, x_k) \in S$. Then, we may find a ball $B_\epsilon(x) \subset S$, for some $\epsilon > 0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r_i \in \mathbb{Q}$ such that $|x_i - r_i| < \frac{\epsilon}{\sqrt{k}}$, for all $1 \le i \le k$. Let $r = (r_1, r_2, \ldots, r_k)$. Then, we may find a rational coordinate r such that

$$\|x - r\| = \left(\sum_{i=1}^{k}(x_i - r_i)^2\right)^{\frac{1}{2}} < \left(\sum_{i=1}^{k}\frac{\epsilon^2}{k}\right)^{\frac{1}{2}} = \epsilon.$$

Hence, $r \in S$, and thus $\mathbb{R}^k$ contains a dense set $\mathbb{Q}^k$. $\square$

Problem 5

A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $x \in V_\alpha \subseteq G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$. Prove that every separable metric space has a *countable base*. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

Proof. Suppose that X is a separable metric space. Let $S \subset X$ be a countable dense subset, and let $\{V_\alpha\}$ be the collection of neighborhoods $N_r(s)$, for all $s \in S$ and $r \in \mathbb{Q}^+$. Since $\{V_\alpha\}$ contains neighborhoods of countably many points with countably many radii, $\{V_\alpha\}$ is countable. Let $x \in X$ and let G be an open set in X such that $x \in G$. Then, there exists an open neighborhood $N_\delta(x) \subset G$, for some $\delta > 0$. Let $\epsilon \in \mathbb{Q}^+$ such that $\epsilon < \delta/2$. Since S is dense in X, there exists $k \in S$ such that $d(x,k) < \epsilon$. Consider $N_\epsilon(k)$. Since $d(x,k) < \epsilon$, we already know $x \in N_\epsilon(k)$. However, notice that for all $b \in N_\epsilon(k)$, $d(x,b) \leq d(x,k) + d(k,b) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \delta$, and thus $x \in N_\epsilon(k) \subset N_\delta(x) \subset G$. It immediately follows that $N_\epsilon(k) = V_\alpha$, for some α, so $\{V_\alpha\}$ is a countable base for X. $\square$

Problem 6

Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ $(n = 1, 2, 3, \ldots)$, and consider the centers of the corresponding neighborhoods.

Proof. Suppose that the process does not terminate after a finite number of steps. Then, $\{x_i\}_{i=1}^{\infty}$ is an infinite subset of X, which means that there exists a limit point s of $\{x_i\}_{i=1}^{\infty}$. But then $N_{\frac{\delta}{2}}(s)$ contains at most 1 point in $\{x_i\}_{i=1}^{\infty}$, which contradicts the fact that any neighborhoods of a limit point contain infinitly many points. Hence, for all $p \in X$, there exists x_i such that $p \in N_\delta(x_i)$, and thus $\{N_\delta(x_i)\}$ is a finite open cover of X. Let S be the set of all x_i we pick via the process, with $\delta = \frac{1}{n}(n = 1, 2, 3, \ldots)$. Since S is a union of countably many finite sets, S is countable. Let G be a non-empty open subset of X, and let $g \in G$. There exists $\epsilon > 0$ such that $N_\epsilon(g) \subset G$. By the archimedean property, there exists $k \in \mathbb{N}$ such that $k\epsilon > 1$, and thus $\frac{1}{k} < \epsilon$. We know there exists $x_i \in S$ such that $g \in N_{\frac{1}{k}}(x_i)$. But then $d(x_i, g) < \frac{1}{k} < \epsilon$, so $x_i \in G$. Hence, S is a countable dense subset of X, and the result now follows. $\square$

Problem 7

Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments. *Hint:* Use Exercise 2.22.

Proof. From Exercise 2.22, we know $\mathbb{R}^1$ is separable. The proof of Exercise 2.23 gives us a countable base $\{V_{\alpha,\beta}\}$ of $\mathbb{R}_1$, where $V_{\alpha,\beta} = (\alpha - \beta, \alpha + \beta)$, $\alpha, \beta \in \mathbb{Q}$, $\beta > 0$. In other words, every open set in $\mathbb{R}^1$ is the union of a subcollection of $\{V_{\alpha,\beta}\}$, which is a union of an at most countable collection of segments. Let S be an open set in $\mathbb{R}^1$, and let K be the subcollection of $\{V_{\alpha,\beta}\}$ whose union of all elements is S. For each pair of segments $u, v \in K$, if $u \cap v \neq \emptyset$, then we replace u and v with segment $u \cup v$. We repeat this process until all segments in K are disjoint. We call this new collection K'. Since the union of every segment in K' remains to be S and $|K'| \leq |K|$, K' is an at most countable collection of disjoint segments such that $S = \bigcup_{k \in K'} k$. $\square$