Question 1. The probability of getting a single pair in a poker hand of 5 cards is approximately 0.42. Find the approximate probability that out of 1000 poker hands there will be at least 450 with a single pair

Solution. Let X be the number of hands with a single pair out of 1000 poker hands. Then $X \sim Bin(1000, 0.42)$, and so

$$\mathbb{E}[X] = 420,$$
$$\text{Var}(X) = 243.6.$$

Let $Z = \frac{X-420}{\sqrt{243.6}}$. Thus,

$$\begin{aligned}\mathbb{P}(X \geq 450) &= \mathbb{P}\left(Z \geq \frac{30}{\sqrt{243.6}}\right) \\ &\approx 1 - \Phi\left(\frac{29}{\sqrt{243.6}}\right) \approx 0.0314.\end{aligned}$$

□

Question 2. Approximate the probability that out of 300 die rolls, we get exactly 100 numbers that are multiples of 3.

Solution. Let X be the number of multiples of 3 we get from 300 die rolls. We know $X \sim Bin(300, \frac{1}{3})$, and so

$$\mathbb{E}[X] = 100,$$
$$\mathrm{Var}(X) = \frac{200}{3}.$$

Let $Z = \frac{X-100}{\sqrt{\frac{200}{3}}}$.

$$\begin{aligned}
\mathbb{P}(99.5 \leq X \leq 100.5) &= \mathbb{P}\left(\frac{-0.5}{\sqrt{\frac{200}{3}}} \leq Z \leq \frac{0.5}{\sqrt{\frac{200}{3}}}\right) \\
&\approx 2\Phi\left(\sqrt{\frac{3}{800}}\right) - 1 \\
&\approx 0.0478.
\end{aligned}$$

$\square$

Question 3. We roll a pair of dice 10,000 times. Estimate the probability that the number of times we get snake eyes (two ones) is between 280 and 300.

Solution. Let X be the times we get snake eyes from rolling a pair of dice 10,000 times. The probability of getting a snake eye is $\frac{1}{36}$. Thus, $X \sim Bin(10000, \frac{1}{36})$, and so

$$\begin{aligned}\mathbb{E}[X] &= \frac{2500}{9}, \\ \operatorname{Var}(X) &= \frac{21875}{81}.\end{aligned}$$

Let $Z = \frac{X - \frac{2500}{9}}{\sqrt{\frac{21875}{81}}}$.

$$\begin{aligned}\mathbb{P}(280 \leq X \leq 300) &= \mathbb{P}\left(\frac{20}{\sqrt{21875}} \leq Z \leq \frac{200}{\sqrt{21875}}\right) \\ &\approx \Phi\left(\frac{200}{\sqrt{21875}}\right) - \Phi\left(\frac{20}{\sqrt{21875}}\right) \\ &\approx 0.3558.\end{aligned}$$

□

Question 4. On the first 300 pages of a book, you notice that there are, on average, 6 typos per page. What is the probability that there will be at least 4 typos on page 301?

Solution. Let X be the number of typos on a page. We assume that $X \sim \text{Possion}(6)$. Then,

$$\begin{aligned}
\mathbb{P}(X \geq 4) &= 1 - \mathbb{P}(X \leq 3) \\
&\approx 1 - e^{-6} \sum_{k=0}^{3} \frac{6^k}{k!} \\
&\approx 1 - e^{-6} (1 + 6 + 18 + 36) \\
&\approx 0.8488.
\end{aligned}$$

$\square$

Question 5. Let $T \sim Exp(1/3)$.

(a) Find $\mathbb{P}(T > 3)$.

Solution. $\mathbb{P}(T > 3) = e^{-1}$. □

(b) Find $\mathbb{P}(1 \leq T < 8)$.

Solution. $\mathbb{P}(1 \leq T < 8) = \mathbb{P}(1 \leq T) - \mathbb{P}(8 \leq T) = e^{-\frac{1}{3}} - e^{-\frac{8}{3}}$. □

(c) Find $\mathbb{P}(T > 4 \,|\, T > 1)$.

Solution. $\mathbb{P}(T > 4 \,|\, T > 1) = \mathbb{P}(T > 3) = e^{-1}$. □

Question 6. Over the course of 365 days, 1 million radioactive atoms of Cesium-137 decayed to 977,287 radioactive atoms. Use the Poisson distribution to estimate the probability that on a given day, 50 radioactive atoms decayed.

Solution. Let X be the number of radioactive atoms decayed in a day. Since 22,713 radioactive atoms of Celsium-137 decayed over the course of 365 days, we expect $\frac{22713}{365}$ to decay in a day, and so $X \sim \text{Poisson}\left(\frac{22713}{365}\right)$. Therefore, the probability of 50 radioactive atoms decaying is

$$\mathbb{P}(X = 50) = e^{-\frac{22713}{365}} \cdot \frac{\left(\frac{22713}{365}\right)^{50}}{50!} \approx 0.0155.$$

$\square$

Question 7. Telephone calls enter a college switchboard on an average of two every three minutes. What is the probability of 5 or more calls arriving in a 9-minute period?

Solution. Let X be the number of calls arriving in a 9-minute period. Since there are two calls arriving every three minutes on average, $\mathbb{E}[X] = 6$. We assume $X \sim \text{Poisson}(6)$. Then, the probability of 5 or more calls arriving in a 9-minute period is

$$\begin{aligned}
\mathbb{P}(X \geq 5) &= 1 - \mathbb{P}(X \leq 4) \\
&\approx 1 - \sum_{k=0}^{4} e^{-6} \cdot \frac{6^k}{k!} \\
&\approx 1 - e^{-6}\,(1 + 6 + 18 + 36 + 54) \\
&\approx 1 - 115e^{-6} = 0.7149.
\end{aligned}$$

□