Note Title 6/21/2024

$G_1, G_2, \ldots, G_m \subseteq G$

- G has n vertices
- G_i induced subgraph
- $E(G_i) \cap E(G_j)$ is triangle-free

maximize $\displaystyle\sum_{i=1}^{m} e(G_i)$

<u>Theorem</u> $\displaystyle\sum_{i=1}^{m} e(G_i) \leq m \lfloor n^2/4 \rfloor$ if $m \geq 2$

with equality iff $G_1 = G_2 = \cdots = G_m = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

<u>Claim</u> Only need it for G_1 and G_2 (i.e. $m=2$).

since

$$\sum_{i=1}^{m} e(G_i) = \frac{1}{2}\sum_{i=1}^{m}\left(e(G_i) + e(G_{i+1})\right)$$

$e(G_{m+1}) = e(G_1)$

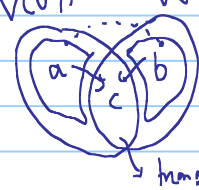
$$\leq \frac{1}{2}\sum_{i=1}^{m} 2\left\lfloor \frac{n^2}{4} \right\rfloor = m\left\lfloor \frac{n^2}{4} \right\rfloor$$

with equality only if $G_i = G_{i+1} = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ for all i.

Implies $G_1 = G_2 = \cdots = G_m = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

<u>Proof for $m=2$</u>

$V(G_1)$ $V(G_2)$



triangle free $\leq \left\lfloor \frac{c^2}{4} \right\rfloor$

max $\boxed{e(G_1) + e(G_2) \leq \binom{a}{2} + \binom{b}{2} + ac + bc + \lfloor c^2/4 \rfloor}$

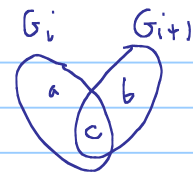
subject to $\boxed{a + b + c = n}$

a maximum occurs when $a = b = 0$ and $G_1 = G_2 = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ (and $c = n$).

Prove the same for any non-bipartite F i.e. $E(G_i) \cap E(G_j)$ are F-free.

<u>Theorem</u> $\displaystyle\sum_{i=1}^{m} e(G_i) \leq m \cdot ex(n, F)$ with equality

if and only if $G_1 = G_2 = \cdots = G_m =$ an extremal F-free graph.



Proof Need only $m=2$.

$$ex(n,F) = \left(1-\frac{1}{\chi(F)-1}\right)\binom{n}{2} + n^{\gamma} \qquad (\gamma<2)$$

$$\binom{a+c}{2}+\binom{b+c}{2} - 2\left(\binom{c}{2} - ex(c,F)\right) - ex(c,F)$$

$$= \underline{\binom{a+c}{2}+\binom{b+c}{2} - 2\binom{c}{2}} + ex(c,F)$$

$2\binom{c}{2} - ex(c,F)$ strictly decreasing as a function of c

$$2\binom{c}{2} - ex(c,F) < 2\binom{c+1}{2} - ex(c.$$