

MATH 100: Homework #3

Due on October 19, 2023 at 12:00pm

Professor McKernan

Section A02 5:00PM - 5:50PM

Section Leader: Castellano

Source Consulted: Textbook, Lecture, Discussion, Office Hour

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Problem 1

“The union of two subgroups of a group G is a subgroup of G .” True or False? If true then give a proof and if false then give a counterexample.

Proof. The statement is false. Consider the D_3 , the groups of symmetries of a triangle, and its subgroups $\{I, F_1\}$, $\{I, F_2\}$, two cyclic subgroups of distinct flips. Since $F_1 F_2 = R$, their union $\{I, F_1, F_2\}$ is not closed under the operation of G , and thus it's not a subgroup. \square

Problem 2

Verify that the relation \sim is an equivalence relation on the set S given.

- (b) $S = \mathbb{C}$, the complex numbers, $a \sim b$ if $|a| = |b|$.

Proof. We check each property of a equivalence relation.

Reflexivity: $|a| = |a|$, and so $a \sim a$, trivial.

Symmetry: Suppose that $|a| = |b|$, then $|b| = |a|$. Thus, $a \sim b$ implies $b \sim a$.

Transitivity: Suppose that $a \sim b$ and $b \sim c$. Then $|a| = |b| = |c|$, and so $a \sim c$.

Thus, \sim is an equivalence relation. Its equivalence classes are sets of complex numbers of the same distance to the origin, namely, circles of different radius centering the origin on the complex plane. \square

- (c) $S =$ straight lines in the plane, $a \sim b$ if a, b are parallel.

Proof. We again check each property of a equivalence relation.

Reflexivity: a is parallel to itself so $a \sim a$.

Symmetry: Suppose that $a \sim b$. Then a, b are parallel to each other, and so $b \sim a$.

Transitivity: Suppose that $a \sim b$ and $b \sim c$. Let s be the slope of b . Since $a \sim b$ and $b \sim c$, the slope of a, b, c are all s , and so $a \sim c$.

Thus, \sim is an equivalence relation. Its equivalence classes are sets of straight lines with the same slope. \square

Problem 3

For each subgroup of D_4 , list all the left and right cosets. (Since D_4 has many subgroups, it is only necessary to do this up to the obvious symmetries)

Proof. The left and right cosets of $\{I\}$ are all the sets that only contain a non-identity element in D_4 .

The left and right cosets of D_4 is D_4 itself.

Since $\{I, R_1, R_2, R_3\}$ contains 4 elements, by the Lagrange Theorem, the only possible left/right cosets of it is $\{I, R_1, R_2, R_3\}$ itself and the rest of the elements $\{F_1, F_2, F_3, F_4\}$, namely, all of the flips.

For $\{I, F_1\}$, its left cosets are $\{I, F_1\}, \{F_2, R^2\}, \{F_3, R\}, \{F_4, R^3\}$, while the right cosets are $\{I, F_1\}, \{F_2, R^2\}, \{F_3, R^3\}, \{F_4, R\}$.

For $\{I, F_2\}$, its left cosets are $\{I, F_2\}, \{F_1, R^2\}, \{F_3, R^3\}, \{F_4, R\}$, while the right cosets are $\{I, F_2\}, \{F_1, R^2\}, \{F_3, R\}, \{F_4, R^3\}$.

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For $\{I, R^2\}$, its left and right cosets are both $\{I, R^2\}, \{R, R^3\}, \{F_1, F_2\}, \{F_3, F_4\}$. □

Problem 4

In \mathbb{Z}_{16} , write down all the cosets of the subgroup $H = \{[0], [4], [8], [12]\}$.

Proof.

$$[0] + H = H$$

$$[1] + H = \{[1], [5], [9], [13]\}$$

$$[2] + H = \{[2], [6], [10], [14]\}$$

$$[3] + H = \{[3], [7], [11], [15]\}$$

Since $[4] + H = [0] + H = H$, $[a] + H$ repeats the above listed cosets, for all $a \geq 4$.

Thus, we have obtained all cosets of H . □

Problem 5

In problem 4, what is the index of H in \mathbb{Z}_{16} ?

Proof. As listed in above question, there are 4 left/right cosets of H , and thus $[\mathbb{Z}_{16}; H] = 4$. □

Problem 6

If aH and bH are distinct left cosets of H in G , are Ha and Hb distinct right cosets of H in G ?

Proof. No. Consider D_4 's subgroup $H = \{I, F_1\}$. From problem 3, we know $F_3H = \{F_3, R\}$ and $R^3H = \{F_4, R_3\}$ are distinct cosets. However $HF_3 = \{F_3, R^3\} = HR^3$ are the same. Thus the statement is disproved. \square

Problem 7

If G is a finite abelian group and a_1, \dots, a_n are all elements, show that $x = a_1 a_2 \dots a_n$ must satisfy $x^2 = e$.

Proof. We first prove that for all $k \geq 1$, $\prod_{1 \leq j \leq k} a_j = a_k \prod_{1 \leq j < k} a_j$. Let $y = \prod_{1 \leq j \leq k} a_j \in G$. Since G is abelian,

$$\prod_{1 \leq j \leq k} a_j = y a_k = a_k y = a_k \prod_{1 \leq j < k} a_j. \quad (1)$$

We now prove that we can rearrange $x = a_1 a_2 \dots a_n$ into any ordering by induction on n . The base case is trivial. For $n > 1$, suppose we aim to rearrange $x = a_1 a_2 \dots a_n$ into some ordering such that a_n is the l -th element in the order. We can first take the last $n - l + 1$ elements and apply (1) to move a_n to the l -th position. Then, by induction, we can rearrange the first $l - 1$ elements and the last $n - l$ elements into the desired ordering, and thus the statement is proven.

Since each element has one unique inverse, we can rearrange $x = a_1 a_2 \dots a_n$ into $x = a_{m_n} a_{m_{n-1}} \dots a_{m_1}$, such that $a_i a_{m_i} = e$ for all $1 \leq i \leq n$. Therefore,

$$\begin{aligned} x^2 &= a_1 a_2 \dots a_{n-1} (a_n a_{m_n}) a_{m_{n-1}} \dots a_{m_1} \\ &= a_1 a_2 \dots a_{n-2} (a_{n-1} a_{m_{n-1}}) a_{m_{n-2}} \dots a_{m_1} \\ &= a_1 a_{m_1} \\ &= e. \end{aligned}$$

□

Problem 8

If G is of odd order, what can you say about the x in problem 16?

Proof. Since G is of odd order, G cannot have subgroups of order 2, and thus for all non identity $a \in G$, $a^2 \neq e$, otherwise $\{e, a\}$ would be a subgroup of order 2 in G . This implies that each non-identity element can be paired with a unique inverse distinct to itself. By the result we obtained in the previous question, we can rearrange x such that each non-identity element in the sequence is next to its inverse. By associativity, each non-identity element in the new ordering would pair up with its neighboring inverse and resolve to e , and thus we get $x = e$. \square

Problem 9

Let G be a group, H a subgroup of G , and let S be the set of all distinct right cosets of H in G , T the set of all left cosets of H in G . Prove that there is a 1-1 mapping of S onto T .

Proof. Consider the function $f : S \rightarrow T$, $f(Hx) = x^{-1}H$, for $x \in G$. Let $Ha, Ha' \in S$, such that $Ha = Ha'$. This implies that $Haa'^{-1} = H$, and so $aa'^{-1} \in H$. Let $h = aa'^{-1}$. We know $a^{-1}h = a^{-1}aa'^{-1} = a'^{-1} \in a^{-1}H$, and thus $f(Ha) = a^{-1}H = a'^{-1}H = f(Ha')$.

We first show f is injective. Let $a, b \in G$, such that $f(Ha) = f(Hb)$. Then, we know $a^{-1}H = b^{-1}H$, and so $ba^{-1}H = H$, which implies $ba^{-1} \in H$. Let $h = ba^{-1} \in H$. We then get $ha = b \in Ha$, and thus $Ha = Hb$.

We now show f is surjective. For all $y = bH \in T$, we have $x = Hb^{-1} \in S$, so that $f(x) = bH$.

Therefore, f is a 1-1 mapping of S onto T . □

Problem 10

If $aH = bH$ forces $Ha = Hb$ in G , show that $aHa^{-1} = H$ for every $a \in G$.

Proof. Let $b \in aH$. Then, $aH = bH$, which forces $b \in Hb = Ha$. Thus, $aH \subseteq Ha$, so $aHa^{-1} \subseteq H$. We now show that $|aHa^{-1}| \geq |H|$. Define $f : aHa^{-1} \rightarrow H$ as $f(x) = a^{-1}xa$.

Let $x = aha^{-1}, x' = ah'a^{-1}$, for some h, h' , such that $x = x'$. By cancellation, we know $h = h'$. Then, $f(x) = a^{-1}xa = h = h' = a^{-1}x'a = f(x')$, and so f is well defined.

For each $y \in H$, we have $x = aya^{-1}$, such that $f(x) = a^{-1}(aya^{-1})a = y$. Thus, f is surjective, and so $|aHa^{-1}| \geq |H|$. Since $aHa^{-1} \subseteq H$ and $|aHa^{-1}| \geq |H|$, we have $aHa^{-1} = H$. \square

Problem 11

If in a group G , $aba^{-1} = b^i$, show that $a^rba^{-r} = b^{i^r}$ for all positive integers r .

Proof. We proceed by induction on r . The base case $aba^{-1} = b^i$ is already given. For $r > 1$, we get $a^rba^{-r} = a \cdot a^{r-1}ba^{-(r-1)} \cdot a^{-1}$. By induction,

$$\begin{aligned} a \cdot a^{r-1}ba^{-(r-1)} \cdot a^{-1} &= ab^{i^{r-1}}a^{-1} \\ &= \underbrace{aba^{-1}aba^{-1} \dots aba^{-1}}_{i^{r-1} \text{ times}} \\ &= (b^i)^{i^{r-1}} \\ &= b^{i^r}, \end{aligned}$$

and we are done. □

Problem 12

If in G , $a^5 = e$ and $aba^{-1} = b^2$, find $o(b)$ if $b \neq e$.

Proof. Since $aba^{-1} = b^2$, by the result we obtained from the previous question, we know $a^5ba^{-5} = b = b^{2^5}$, and thus we get $b^{2^5-1} = e$. Since $2^5 - 1 = 31$ is a prime number and $b \neq e$, there are no positive $r < 31$ such that $b^r = e$, and so $o(b) = 31$. \square

Challenge Problems

Problem 13

Let G be an abelian group of order n , and a_1, \dots, a_n its elements. Let $x = a_1 a_2 \dots a_n$. Show that:

- (a) If G has exactly one element $b \neq e$ such that $b^2 = e$, then $x = b$.

Proof. In problem 7, we proved that we can rearrange $x = a_1 a_2 \dots a_n$ into any ordering. For all $a_i^2 \neq e$, we rearrange $a_1 a_2 \dots a_n$ such that a_i is next to its inverse, which allows each a_i to pair up with its neighboring inverse and resolve to e . Thus, the sequence becomes $x = be = e$, and we are done. \square

- (b) If G has more than one element $b \neq e$ such that $b^2 = e$, then $x = e$.

Proof. idk bro. \square

- (c) If n is odd, then $x = e$.

Proof. Proved in problem 8. \square

Problem 14

“Every countable group is finitely generated.” True or False? If true then give a proof and if false then give a counterexample.

Proof. Consider \mathbb{Q} under addition. We know \mathbb{Q} is countable. Suppose for sake of contradiction that there exists a finite set $S = \{s_1, s_2, \dots, s_n\} \subset \mathbb{Q}$, $s_i = \frac{a_i}{b_i}$ for $a_i, b_i \in \mathbb{Z} - \{0\}$, such that $\langle S \rangle = \mathbb{Q}$. Let $b = \prod_{s_i \in S} b_i$. Then, all elements in $\langle S \rangle$ can be represented in the form of $\frac{c_i}{b}$, $c_i \in \mathbb{Z}$. However, we can find $\frac{p}{q} \in \mathbb{Q}$, such that $p, q \in \mathbb{N}$, $\gcd(q, b) = 1$, so that $\frac{p}{q}$ cannot be represented in the form of $\frac{c_i}{b}$, contradiction. Therefore, \mathbb{Q} under addition is a countable group that cannot be finitely generated. \square