Math 158 Textbook Solutions

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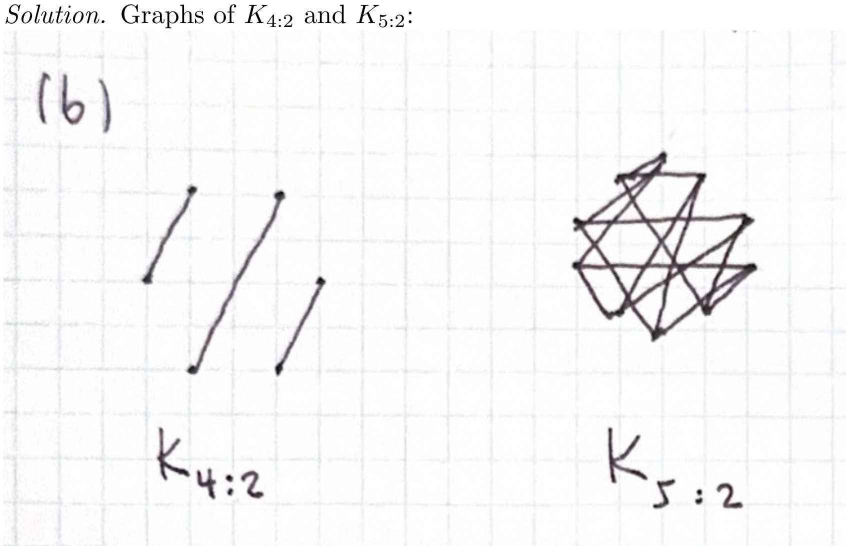
Problem 1.7.2. Let $K_{n:r}$ denote the Kneser graph, whose vertex set is the set of r-element subsets of an n-element set, and where two vertices form an edge if the corresponding sets are disjoint.

(a) Describe $K_{n:1}$ for $n \geq 1$.

Solution. Since $\forall v, u \in V(K_{n:1})$, $v \cap u = \emptyset$. Thus, $\forall v, u \in V(K_{n:1})$, $\{v, u\} \in E(K_{n:1})$, which makes $K_{n:1}$ a K_n complete graph. $\square$

(b) Draw $K_{4:2}$ and $K_{5:2}$.

Solution. Graphs of $K_{4:2}$ and $K_{5:2}$:



$\square$

(c) Determine $|E(K_{n:r})|$ for $n \geq 2r \geq 1$.

Solution. For each $v \in V(K_{n:r}$, v forms edges with other vertices whose vertex set is r of the other $n - r$ elements that are not in the vertex set of v, which implies that $d_{K_{n:r}}(v) = \binom{n-r}{r}$. Since there are $\binom{n}{r}$ vertices in $K_{n:r}$, by the Handshake Theorem, we have $|E(K_{n:r})| = \binom{n}{r}\binom{n-r}{r}/2$. $\square$

Problem 1.7.4. Let G be a digraph such that every vertex has a positive in-degree. Prove that G contains a directed cycle.

Proof. We will prove this by contradiction. Let $v \in V(G)$. Suppose for the sake of contradiction that G does not contain any directed cycle. Starting from v, we can find a path P by tracing back to a vertex with an edge directed to the current vertex we're on. We then add the vertex to P and go to that vertex, and we repeat the previous actions. Since every vertex in G has a positive in-degree, we can always find another vertex that has a directed edge to the current vertex we're on and not in P. However, this makes G have infinitely many vertices, which is a contradiction. Therefore, G contains a directed cycle. $\square$

Problem 1.7.12. Let g be an n-vertex graph with $n \geq 2$ and $\delta(G) \geq (n-1)/2$. Prove that G is connected and the diameter of G is at most two.

Proof. We will first prove that G is connected by contradiction. Suppose for the sake of contradiction that G is disconnected. Let $n = |V(G)|$, $v \in V(G)$, H be the component of G that contains v. Since $d_G(v) \geq \delta(G) \geq (n-1)/2$, we have $|V(H)| \geq (n-1)/2 + 1 = (n+1)/2$, which implies that other components in G contain at most $n - (n+1)/2 = (n-1)/2$ vertices. However, this shows that $\Delta(G - V(H)) \leq (n-1)/2 - 1 < (n-1)/2$, which contradicts $\delta(G) \geq (n-1)/2$ because H is disconnected to $G - V(H)$. Therefore, G is connected.

We will now prove that the diameter of G is at most two. Let $u, w \in V(G)$. If $u \in N(w)$, then $d_G(u,w) = 1$. If $u \notin N(w)$, then $N(u), N(w) \in V(G) \backslash \{u, w\}$. Since $|N(u)|, |N(w)| \geq \delta(G) \geq (n-1)/2$, we have $|N(u)| + |N(w)| > n - 2 = |V(G) \backslash \{u, w\}|$. Hence, $|N(u)| \cap |N(w)| \neq \emptyset$, which means that $d_G(u,w) = 2$. Therefore, the diameter of G is at most two. $\square$

Problem 1.7.14.a. Let P and Q be the longest paths in a connected graph G. Prove that

$$V(P) \cap V(Q) \neq \emptyset.$$

Proof. We will prove this by contradiction. Let P, Q be the longest paths in a connected graph G, with $\{p_1, p_2, \ldots, p_{n+1}\}$ and $\{q_1, q_2, \ldots, q_{n+1}\}$ as their vertex sets respectively, and $n = |E(P)| = |E(Q)|$. Suppose for the sake of contradiction that $V(P) \cap V(Q) = \emptyset$. Since G is connected, there must be a path R that starts from p_i and ends at q_j, for some $1 \leq i, j \leq n+1$. Let $m = d_G(p_i, q_j)$. Since $p_i \neq q_j$, we have $m \geq 1$. Let P' be the longer path between $p_1 p_2 \ldots p_i$ and $p_i p_{i+1} \ldots p_{n+1}$, Q' be the longer path between $q_1 q_2 \ldots q_j$ and $q_j q_{j+1} \ldots q_{n+1}$. By connecting P', Q', and R, we get a new path S. Since $|E(P')|, |E(Q')| \geq n/2$, $|E(R)| = m \geq 1$, we have $|E(S)| \geq n+1$, which contradicts that P, Q are the longest paths on G. Therefore, if P, Q are the longest paths in a connected graph, then $V(P) \cap V(Q) \neq \emptyset$. $\square$

Problem 2.5.2. A tournament is an orientation of a complete graph. Prove that every tournament contains a directed path containing all of its vertices.

Proof. Let T_n be a n-vertex tournament. We will prove by induction on n to show that T_n is traceable for all n. T_1 is obviously traceable as it only contains one vertex. T_2 is traceable, as it contains only one directed edge that connects all the vertices in the graph. Suppose that a T_k contains a directed hamiltonian uv-path P, for some $k \geq 2$. We denote the vertex after x in P as x^+, for some $x \in V(P)$. By adding a vertex w and k directed edges to T_k, we get a T_{k+1}. If $e = (w, u)$ or $(v, w) \in E(T_{k+1})$, we can connect e with P to obtain a hamiltonian path in T_{k+1}. If $(w, u), (v, w) \notin E(T_{k+1})$, we know $(u, w), (w, v) \in E(T_{k+1})$ because $N_{T_{k+1}}(w) = V(P)$, which ensures $d^+_{T_{k+1}}(w), d^-_{T_{k+1}}(w) \geq 1$. Hence, there exist $x \in V(P)$ such that $(x, w), (w, x^+) \in E(T_{k+1})$. We can then add w and $(x, w), (w, x^+)$ to $P - (w, w^+)$ to get a directed hamiltonian path in T_{k+1}. Thus, if T_k is traceable, then T_{k+1} is also traceable. Therefore, all tournaments are traceable. □

Problem 2.5.7. Prove that a graph G of minimum degree at least $k \geq 2$ containing no triangles contains a cycle of length at least $2k$.

Proof. Let P be the longest path in G, say $v_1 v_2 \dots v_t$. Then $N(v_t) \subseteq V(P)$. Since G does not contain any triangles, if $v_p, v_q \in N(v_t)$ for some $p > q$, then $p - q \geq 2$. Since $|N(v_t)| \geq \delta(G) \geq k$ and $d_P(v_p, v_q) \geq 2$ for some $v_p, v_q \in N(v_t)$, we have $t \geq 2k + 1$ and v_t has a neighbor v_i for some $i \leq t - 2k$, which proves that the cycle $v_i v_{i+1} \dots v_t v_i$ has a length of at least $2k$. □

Problem 2.5.9. The closure of an n-vertex graph G, denoted $C(G)$, consists in adding edges between any two non-adjacent vertices u and v such that $d_G(u) + d_G(v) \geq n$. Prove that a graph G is hamiltonian if and only if $C(G)$ is hamiltonian.

Proof. If G is hamiltonian, G contains a hamiltonian cycle $H \subseteq G$. Since $C(G)$ contains G and $V(C(G)) = V(G)$, we have $H \subseteq G \subseteq C(G)$, and thus $C(G)$ is hamiltonian.

Suppose that $C(G)$ has a hamiltonian cycle F. If F does not contain any edges that are not in G, then G is hamiltonian. Otherwise, there exists $\{u, v\} \in E(F)$ such that $\{u, v\} \notin E(G)$, which implies $d_G(u) + d_G(v) \geq n$. Let $P = F - \{u, v\}$ be a hamiltonian uv-path of $C(G)$, say $v_1 v_2 \dots v_n$, and $N(v)^+ = \{v_{i+1} : v_i \in N_G(v)\}$. We then have $N(v)^+ \cup N(u) \subseteq V(P) \backslash \{u\}$, which shows that $|N(v)^+ \cup N(u)| \leq n - 1$. Since $|N(v)^+| + |N(u)| = d_G(u) + d_G(v) \geq n$, we have

$$\begin{aligned} |N(v)^+ \cap N(u)| &= |N(v)^+| + |N(u)| - |N(v)^+ \cup N(u)| \qquad (1) \\ &\geq n - (n - 1) = 1. \qquad (2) \end{aligned}$$

Hence, $N(v)^+ \cap N(u) \neq \emptyset$. Let $v_k \in N(v)^+ \cap N(u)$, we can then get a new hamiltonian cycle $P - \{v_k, v_{k+1}\} + \{u, v_{k+1}\} + \{v_k, v\}$. This shows that all $e \in E(F) \backslash E(G)$ can be removed from F to obtain a hamiltonian cycle that only consists of edges in G, which shows that G is hamiltonian. Therefore, $C(G)$ is hamiltonian if and only if G is hamiltonian. □

Problem 2.5.11. Let G be a hamiltonian bipartite graph of a minimum degree of at least three. Prove that G contains at least two hamiltonian cycles.

Proof. Let C be a hamiltonian cycle in bipartite graph $G(A, B)$, and let $u, v, w \in A$ such that $N_C(u) = \{v, w\}$. Consider the hamiltonian uv-path $P = C - \{u, v\}$. Since G is bipartite and $v \in B$, we know $N(v)^+ \in B$, and thus the vertices of P obtained by all possible rotations are all in B. Let $G'(A', B') \subseteq G$ such that $C \subseteq G'$ and $d_{G'}(b) = 3$ for all $b \in B'$. We know G' exists because $\delta(G) \geq 3$. Let H be a graph whose vertices are hamiltonian paths of G' starting with the edge $\{u, w\}$, where two hamiltonian paths in G' form an edge of H if they are obtained from one another by rotation. If $Q \in H$ is a hamiltonian path that ends at a vertex x, then Q has $3 - 1 = 2$ possible rotations in G' unless $\{u, x\} \in E(G')$, in which case would have $3 - 2 = 1$ rotations instead. In the latter case, Q together with $\{u, w\}$ would form a hamiltonian cycle in G'. By the Handshake Theorem, since the number of vertices with odd degrees is even, there is an even number of paths Q in G' which ends at a neighbor of u. Therefore, G' has an even amount of hamiltonian cycles containing $\{u, w\}$. Since G' already contains a hamiltonian cycle C, it must contain some other hamiltonian cycle C'. Since $C, C' \subseteq G' \subseteq G$, G has at least two hamiltonian cycles. □

Problem 3.8.1. A school with 20 professors forms 10 committees, each containing 6 professors, such that every professor is on exactly 3 committees. Prove that it is possible to select a distinct representative from each committee.

Proof. Let G be a bipartite graph with the set of all professors and committees, where each professor and committee form an edge if the professor is in the committee. Let C be a subset of the set of all committees. Since each committee has an edge with 6 professors, we know that C is incident to $6|C|$ edges. Since each professor has an edge with 3 committees, we know that $N(C)$ is incident to $3|N(C)|$ edges. Since the edges $N(C)$ is incident to include the edges C is incident to, $3|N(C)| \geq 6|C|$, and thus $|N(C)| \geq |C|$. By Hall's Theorem, there is a matching saturating the set of all committees, so it is possible to select a distinct representative from each committee. □

Problem 3.8.2. A tiling of an $m \times n$ chess board is a set of dominoes that cover all the squares on the chess board exactly once (each domino covers two adjacent squares).

(a) For which $m \geq 1$ and $n \geq 1$ does an $m \times n$ chess board having a tiling?

Solution. Since the number of squares on the chess board must be even to have a perfect matching, m or n is even. Assume, without loss of generality, that m is even. We will prove by induction on m. If $m = 2$, then we can match each square in one column to one in the adjacent column that is adjacent to it, and this is a perfect matching M_2. Suppose that there is a perfect matching M_k for each $k \times n$ chessboard, where $2 \leq k \leq m$ and is even. We can then split a $(m + 2) \times n$ chessboard into a $2 \times n$ and

$m \times n$ board. We can then find a perfect matching $M_2 \cup M_m$. This also shows true for even n. Therefore, for all $(m, n) \in \{(a, b) \in \mathbb{N}^2 : ab \text{ is even}\}$, an $m \times n$ chessboard has tiling. □

(b) If we remove two squares from an $m \times n$ chessboard, when do the remaining squares have a tiling?

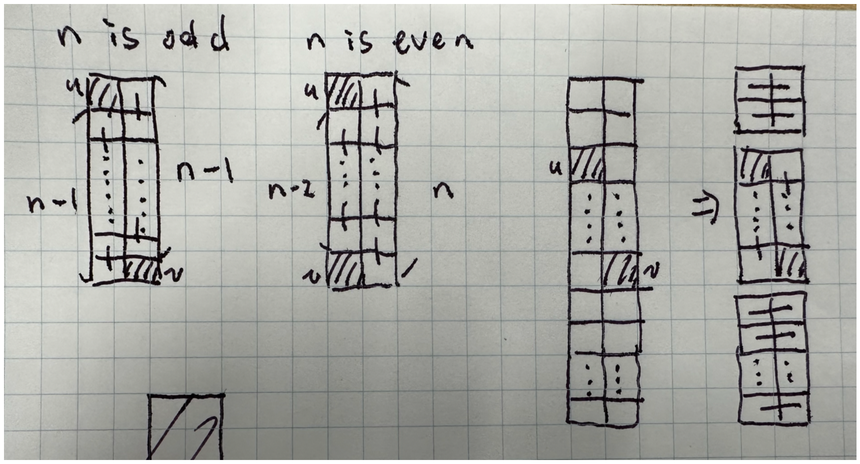
Solution. There must be an even amount of squares to have tiling, so if a $m \times n$ chessboard has tiling after two squares removed, then the $m \times n$ chessboard also has an even amount of squares. Thus, m or n needs to be even.

Assume, without loss of generality, that m is even. Let G be a grid graph whose vertex set contains all squares on a $m \times n$ chessboard, and each pair of vertices forms an edge if they are adjacent to each other on the board. Since m is even, G has a perfect matching. Let v_{xy} correspond to the square in the xth row and yth column, for some $1 \leq x \leq n$, $1 \leq y \leq m$. Let $\{c_1, c_2\}$ be a set of two colors. We color v_{xy} with c_1 if $x + y$ is even and c_2 if $x + y$ is odd. This shows that every square can be colored with no same-colored squares being adjacent. Since each domino covers a c_1 square and a c_2 square, each color must have the same number of squares to have tiling. Therefore, if we remove two squares with the same color, then G does not have a tiling.

Suppose that we remove two squares u, v with different colors. If $n = 1$, then $G - \{u\} - \{v\}$ has a tiling if there are only even components.

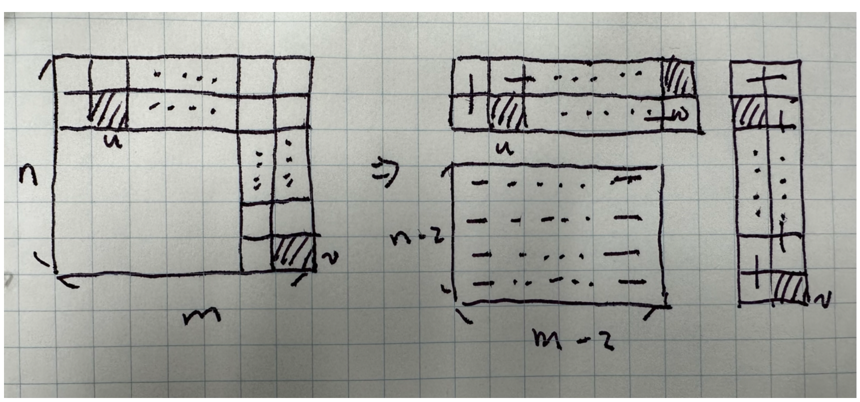
Claim 1. *If $m = 2$, then $G - \{u\} - \{v\}$ has a tiling.*

Suppose that u, v are each in the first and last rows and n is some natural number. Consider the case where n is odd. Since u, v have different colors, u, v are in different columns. This means that the two columns in $G - \{u\} - \{v\}$ both have $n - 1$ number of squares, which is even. Consider the case where n is even. Since u, v has different colors, u, v are in the same column. This means that the two columns in $G - \{u\} - \{v\}$ have n and $n - 2$ numbers of squares respectively, which are also even. Thus, in both cases, we can tile along the columns and cover all squares, which is a tiling. Suppose that u, v are in the ith and jth rows, for some $1 < i \leq j < n$. We can then remove the first $i - 1$ rows and the last $n - j$ rows, as there are two columns so we can find a tiling of them by putting a domino in each row. What is left is a $2 \times (j - i + 1)$ board with two corners on both sides removed, which we just proved to have tiling in the first case. Therefore, $G - \{u\} - \{v\}$ has a tiling if $m = 2$.

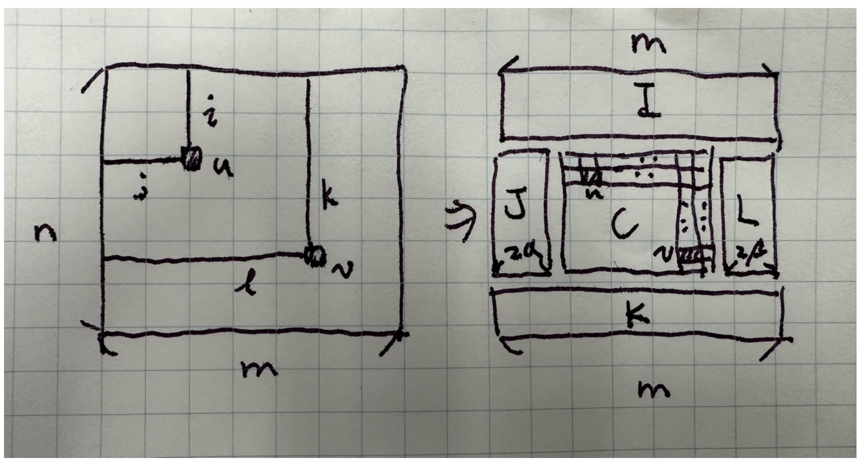


Claim 2. *Suppose that $n \geq 2$. If u, v are each in a 2×2 corner on the opposite side, then $G - \{u\} - \{v\}$ has a perfect matching.*

Assume, without loss of generality, that u is in the top left 2×2 corner and v is in the bottom right 2×2 corner. Since the $(m-2) \times (n-2)$ squares on the bottom left have a perfect matching M_1, as it has an even side, we can first take it out. What is left are the first two rows and last two columns of G, so we can split it into two parts, a $(m-2) \times 2$ board A that contains u and a $(2 \times n)$ board B that contains v. Let $w \in V(B) \cap N_G(V(A))$ such that w shares the same color with u. We remove w from B to A, and we then have two parts, $A + \{w\}$ and $B - \{w\}$. We can view $A + \{w\}$ as a $2 \times (m-1)$ board missing two different color squares u and a square next to w. By Claim 1, since both $A + \{w\}$ and $B - \{w\}$ have exactly two columns and are missing two different-colored squares, $A + \{w\}$ and $B - \{w\}$ each has a perfect matching M_2 and M_3 respectively. Therefore, $G - \{u\} - \{v\}$ has a perfect matching $M_1 \cup M_2 \cup M_3$.



Finally, we will show that $G - \{u\} - \{v\}$ has a perfect matching for all $m, n \geq 2$, m is even. Suppose that u is in the ith row jth column and v is in the kth row lth column of G. Assume, without loss of generality, that $k \geq i$ and $l \geq j$. We can first take out a $m \times (i-1)$ board I that contains the first $(i-1)$ rows of G and a $m \times (n-k)$ board K that contains the last $(n-k)$ rows of G. Since they both have an even side m, they have a perfect matching M_i and M_k respectively. What's left is a $m \times (k-i+1)$ board G'. We can then take out the left-most 2α and right-most 2β columns of G', where α is the greatest integer such that $2\alpha < j$ and β is the greatest integer such that $m - 2\beta > l$ and obtain a $2\alpha \times (k-i+1)$ board J and a $2\beta \times (k-i+1)$ board L. Since J and L each have an even side 2α and 2β, they have perfect matching M_J and M_L respectively. What is left is a $(m - 2(\alpha+\beta)) \times (k-i+1)$ board C with u, v in opposite side 2×2 corners. By Claim 2, C contains a perfect matching M_C. We now found a perfect matching $M_I \cup M_J \cup M_K \cup M_L \cup M_C$ of $G - \{u\} - \{v\}$. Therefore, if we remove two squares from a $m \times n$ chessboard, it has tiling if and only if mn is even and the two removed squares have different colors and all boards have an even number of squares.



□

Problem 3.8.3. Let e be an edge of a connected cubic graph such that $G - e$ is disconnected. Prove that every perfect matching of G contains e.

Proof. Suppose for sake of contradiction that there exists a perfect matching M of G such that $e \notin M$. Let G_1 and G_2 be the two components in $G - e$ respectively. Since $e \notin M$, $G - e$ has a perfect matching, and thus G_1 and G_2 both have a perfect matching. Since each vertex in G is degree three and we only removed one edge, G_1 and G_2 each have an even number of vertices with degree 3 and a single vertex with degree 2. This makes G_1 and G_2 odd components, which contradicts that they both have a perfect matching. Therefore, e must be in the perfect matching of G. □

Problem 3.8.5. Determine $\chi'(G)$ when G is the Petersen graph.

Proof. Since the maximum degree of the Petersen graph is 3, its edge chromatic number is 3 or 4 by Vizing's Theorem. Suppose the Petersen graph can be 3 edge colored with $\{1, 2, 3\}$. Since Petersen's graph is cubic, each vertex is incident with edges of all colors. Take an edge $\{u, v\}$ on the pentagon and color it with 1. Let x be u's neighbor on the pentagram and y be that of v's. Since $\{u, x\}$ and $\{v, y\}$ cannot be colored with 1, there are 2 edges with color 1 on the pentagram. Since a pentagon cannot be 2 edge-colored, at least three colors appear on the edges of the pentagon. This means that all three colors appear at least twice on the 5 edges of the pentagram, contradiction. Therefore, the edge chromatic number of the Petersen graph is 4. □

Problem 3.8.8.

(a) Let G be an n by n bipartite graph of minimum degree more than $n/2$. Prove that G has a perfect matching.

Solution. Suppose that there is a non-hamiltonian n by n bipartite graph of minimum degree at least $n/2$. Amongst all such graphs, let $H(A, B)$ be one with parts A and B and a maximum number of edges. If we add an edge $e = \{v_1, v_{2n}\}$ between non-adjacent vertices in H, we would have a graph with a hamiltonian cycle C, and so $C - e$ is a hamiltonian path in H, say $v_1 v_2 \dots v_{2n}$. Assume, without loss of generality, that $v_1 \in A$ and $v_{2n} \in B$. Let $N(v_1)^+ = \{v_{i+1} : v_i \in N(v_1)\}$. Since $N(v_1)^+ \cup N(v_{2n}) \subseteq A$, we have $|N(v_1)^+ \cup N(v_{2n})| \leq n$. Since $\delta(H) > n/2$, we have $|N(v_1)^+| + |N(v_{2n})| \geq n + 1$. Thus, we have

$$|N(v_1)^+ \cap N(v_{2n})| = |N(v_1)^+| + |N(v_{2n})| - |N(v_1)^+ \cup N(v_{2n})| \tag{3}$$
$$\geq n + 1 - n = 1. \tag{4}$$

This shows that $N(v_1)^+ \cap N(v_{2n}) \neq \emptyset$, which proves that H contains a hamiltonian cycle, a contradiction. Therefore, there exists a hamiltonian path P in G such, say $u_1 u_2 \dots u_{2n}$. Let $f = \{(u_i, u_{i+1}) : i \text{ is even}\}$. We can then find a perfect matching $M = P - f$ of G. Hence, G has a perfect matching. □

(b) Let G be a $2n$-vertex graph of minimum degree at least n. Prove that G has a perfect matching.

Solution. If $n = 1$, G itself is a perfect matching for G. Suppose that $n \geq 2$. By Dirac's Theorem, since $\delta(G) \geq |V(G)|/2$, G contains a hamiltonian path P, say $v_1 v_2 \dots v_{2n}$. Let $e = \{(v_i, v_{i+1}) \in E(P) : i \text{ is even}\}$, then $M = P - e$ is a perfect matching in G. Therefore, G has a perfect matching. □

Problem 3.8.9. Let A_k be the set of subsets of $\{1, 2, \dots, n\}$ of size k. Prove that for $k < n/2$, there is an injective function $f : A_k \to A_{k+1}$ such that $a \subseteq f(a)$ for all $a \in A_k$. For instance, if $k = 1$ and $n = 3$ then the function

$$f(\{1\}) = \{1, 2\} \quad f(\{2\}) = \{2, 3\} \quad f(\{3\}) = \{1, 3\}$$

is an example of such a function $f : A_1 \to A_2$.

Proof. Let G be a bipartite graph with parts A_k and A_{k+1}, for some $k < n/2$, and each $a_k \in A_k$ forms an edge with $a_{k+1} \in A_{k+1}$ if $a_k \subset a_{k+1}$. Since $k < n/2$, $|A_k| = \binom{n}{k} \leq \binom{n}{k+1} = |A_{k+1}|$. For each $a_k \in A_k$, there are $n-k$ number of $a_{k+1} \in A_{k+1}$ such that $a_k \subset a_{k+1}$, so each a_k has $n-k$ neighbors. For each $a_{k+1} \in A_{k+1}$, there are $k+1$ number of $a_k \in A_k$ such that $a_k \subset a_{k+1}$, so each a_{k+1} has $k+1$ neighbors. Since both sides are incident to the same number of edges, we know $(n-k)|A_k| = (k+1)|A_{k+1}|$, and so $n-k \geq k+1$ because $|A_k| \leq |A_{k+1}|$. Let $S \subseteq A_k$. We know there are $(n-k)|S|$ edges that are incident with S, and there are $(k+1)|N(S)|$ edges that are incident with $N(S)$. Since the edges that are incident with $N(S)$ contain the ones that are incident with S, we have $(k+1)|N(S)| \geq (n-k)|S| \geq (k+1)|S|$, and so $|N(S)| \geq |S|$. Therefore, by Hall's Theorem, there is a matching saturating A_k in G, which shows that there is an injection from A_k to A_{k+1}. $\square$

Problem 3.8.11. Let A be an n by n matrix of zeros and ones. Suppose every row and every column of A has exactly k ones. Prove that we can pick n ones from A, no two in the same row or column.

Proof. Let G be a n by n bipartite graph with the set of rows R and columns C of A as its two parts, where $r \in R$ and $c \in C$ form an edge if the rth row cth column of A is one. We know that G is k-regular. Let $S \subseteq R$. We know that S is incident to $k|S|$ edges and $N(S)$ is incident to $k|N(S)|$ edges. Since the edges $N(S)$ is incident to include the edges S is incident to, $k|N(S)| \geq k|S|$, and thus $|N(S)| \geq |S|$, Hall's condition met. Therefore, there exists a perfect matching of G, and thus we can pick n ones from A such that no two are in the same row or column. $\square$

Problem 3.8.17. An independent set in a graph G is a set $X \subseteq V(G)$ such that $e(X) = 0$, and the independence number $\alpha(G)$ is the largest size of an independent set in G. A vertex cover of G is a set of vertices $X \subset V(G)$ such that $e \cap X \neq \emptyset$ for every edge $e \in E(G)$. The minimum size of a vertex cover of G, the vertex cover number, is denoted $\beta(G)$.

(a) Prove that for any graph G, $\alpha(G) + \beta(G) = |V(G)|$.

Proof. Let C be the smallest vertex cover of G, and let I be the largest independent set in G. First, Suppose for the sake of contradiction that $|C \cup I| < |V(G)|$. Let $L = V(G) \backslash (C \cup I) \neq \emptyset$. Since there does not exist $e \in E(G)$ such that $e \subseteq V(G) \backslash C$, we know $e(L) = 0$, and thus $e(L \cup I) = 0$, which makes $L \cup I$ a larger independent set, contradiction. Therefore, $|C \cup I| = |V(G)|$. Suppose for the sake of contradiction that $C \cap I \neq \emptyset$. Let $v \in C \cap I$. Since $v \in I$, we know $N(v) \subseteq V(G) \backslash I \subseteq C$. Since all neighbors of v are in C, we can remove v to get a smaller vertex cover $C \backslash \{v\}$, contradiction. Therefore, $C \cap I = \emptyset$, and thus $|V(G)| = |I \cup C| = |I| + |C| = \alpha(G) + \beta(G)$. $\square$

(b) Prove that $\mu(G) \le \beta(G) \le 2\mu(G)$.

Proof. Let M be the maximum matching of G, and C be the minimum vertex cover. Since no two exposed vertices form an edge, the neighbors of exposed vertices are all saturated vertices, so the set of all saturated vertices is a vertex cover. Thus, $\beta(G) \le 2\mu(G)$. For each edge $e \in M$, we know $e \cap C \ne \emptyset$, so $\mu(G) \le \beta(G)$. Therefore, $\mu(G) \le \beta(G) \le 2\mu(G)$. □

Problem 4.7.2. Determine $\chi'(G)$ and $\chi(G)$ for each of the graphs shown below.

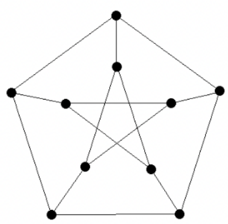


FIGURE 4.7. The Petersen graph

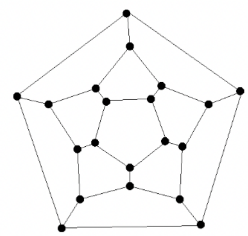


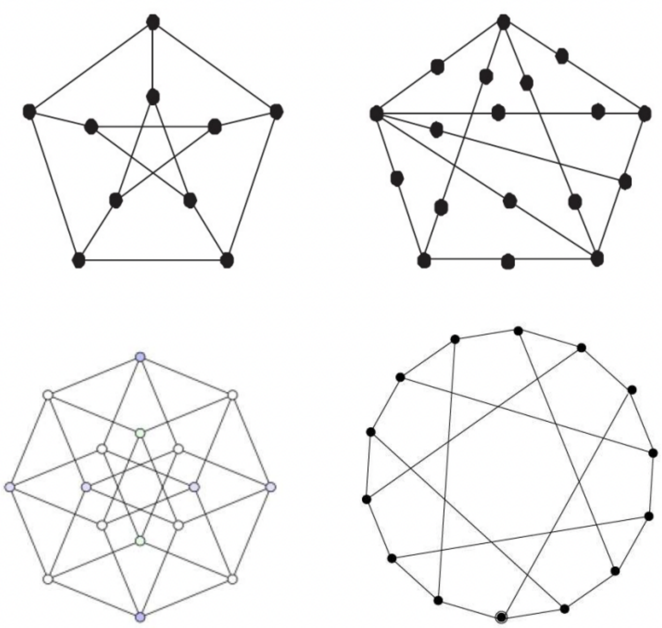
FIGURE 4.8. The dodecahedron graph

Proof. Since the maximum degree of the Petersen graph is 3, its edge chromatic number is 3 or 4 by Vizing's Theorem. Suppose the Petersen graph can be 3 edge colored with $\{1, 2, 3\}$. Since Petersen's graph is cubic, each vertex is incident with edges of all colors. Take an edge $\{u, v\}$ on the pentagon and color it with 1. Let x be u's neighbor on the pentagram and y be that of v's. Since $\{u, x\}$ and $\{v, y\}$ cannot be colored with 1, there are 2 edges with color 1 on the pentagram. Since a pentagon cannot be 2 edge-colored, at least three colors appear on the edges of the pentagon. This means that all three colors appear at least twice on the 5 edges of the pentagram, contradiction. Therefore, the edge chromatic number of the Petersen graph is 4.

Since the Petersen graph contains an odd cycle, it cannot be 2-colored. By Brook's Theorem, since the Petersen graph is not a complete graph nor an odd cycle, it can be 3-colored because the maximum degree is 3.

Let D be a dodecahedron graph. Since D contains an odd cycle, $\chi'(D), \chi(D) \ge 3$. By Theorem 9 in 4.4, since D is a cubic planar graph and all planar graphs are 4-colorable, $\chi'(D) = 3$. By Brook's Theorem, since D is cubic and not a complete graph nor an odd cycle, $\chi(D) = 3$. □

Problem 4.7.5. Determine which of the graphs in the figure below is planar. Justify your answers.



Solution. Since all four graphs have cycles, we can check by using the equation

$$|E(G)| \leq \frac{g}{g-2}(|V(G)|-2),$$

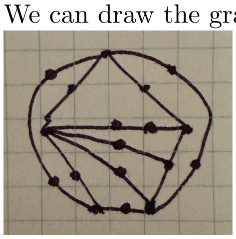
where g is the length of the shortest cycle in the graph.

The graph on the top left has 15 edges and 10 vertices, and the shortest cycle has a length of 5.

$$15 > \frac{40}{3} = \frac{5}{5-2}(10-2),$$

and thus it is not planar.

We can draw the graph on the top right in the following form:



Therefore, the graph is planar.

The graph on the bottom left has 32 edges and 16 vertices, and the shortest cycle has a length of 4.

$$32 > 28 = \frac{4}{4-2}(16-2),$$

and thus it is not planar.

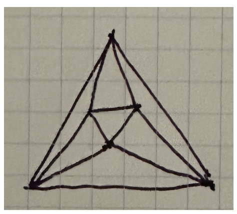
The graph on the bottom right has 20 edges and 14 vertices, and the shortest cycle has a length of 6.

$$20 > 18 = \frac{6}{6-2}(14-2),$$

and thus it is not planar. $\square$

Problem 4.7.7. A maximal plane graph is a plane graph $G = (V, E)$ with $n \geq 3$ vertices such that if we join any two non-adjacent vertices in G, we obtain a non-plane graph

(a) Draw a maximal plane graph on six vertices.



(b) Show that a maximal plane graph on n points has $3n-6$ edges and $2n-4$ faces.

Proof. A maximal plane graph G only contains triangular faces. G is connected because if it's not, we can add an edge to connect two components and still get a plane graph, which contradicts G's maximality. Let $n = |V(G)|$. By theorem, we know that $E(G) \leq 3n-6$. Since G is a maximal plane graph, we have $E(G) = 3n-6$. By Euler's Formula, we have $n-(3n-6)+|F(G)| = 2$. Therefore, rearranged, we have $|F(G)| = 2n-4$ and $|E(G) = 3n-6$. $\square$

(c) A triangulation of an n-gon is a plane graph whose vertex set is the vertex set of a convex n-gon in the plane, whose infinite face boundary is a convex n-gon, and all of whose other faces are triangles. How many edges does a triangulation of an n-gon have?

Solution. To triangulate a n-gon, we can pick a vertex v from the n-gon and connect it with all other vertices in the graph. Since v is already connected to two vertices, we only need to add $n-1-2 = n-3$ edges. Therefore, including the original n edges, a triangulation of an n-gon has $2n-3$ edges. $\square$

Problem 4.7.8. Show that every triangle-free planar graph is 4-colorable.

Proof. Let G be an n-vertex triangle-free planar graph. We will first check with Theorem 3 in chapter 4. Since the smallest possible cycle in G is at least length 4, we have

$$|E(G)| \leq 2(n-2).$$

Let d be the sum of all degrees in G. By the Handshake Theorem, we know

$$d = 2|E(G)| \leq 4n - 8.$$

If $\delta(G) \geq 4$, then $d \geq 4n$, contradiction. Therefore, we can always find a vertex with a degree lesser or equal to 3 in every triangle-free planar graph. We will then proceed by induction on n. We already know all paths are 4-colorable. If $n = 4$, G is a length 4 cycle, which is 4 colorable. Suppose $n > 4$. Pick a vertex v from G such that $d(v) \leq 3$. Since removing v from G does not create any triangles and the graph would remain planar, $G - \{v\}$ is 4-colorable by induction. Since v is connected to vertices of at most 3 colors, we can assign the 4th color to v. Therefore, every triangle-free planar graph is 4-colorable. □

Problem 4.7.14. Let $\omega(G)$ – the clique number of G – be the maximum number of vertices in a complete subgraph of a graph G.

(a) Prove that for every graph G, $\chi(G) \geq \omega(G)$.

Proof. We know that $\chi(K_n) = n$. Since $K_{\omega(G)} \subseteq G$, we have $\chi(G) \geq \omega(G)$. □

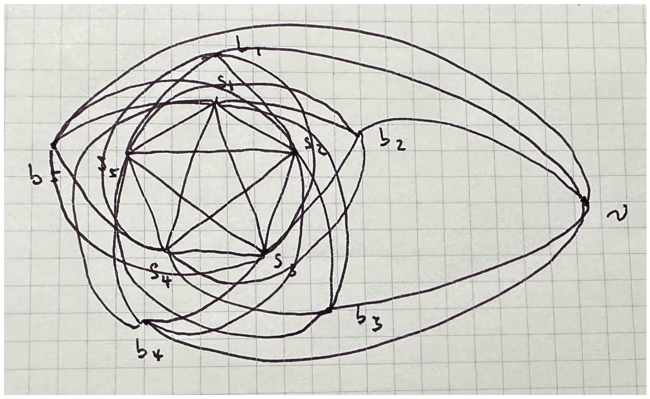
(b) Prove that for every graph G, $\chi(G) \geq |V(G)|/\alpha(G)$.

Proof. Suppose that we color G with $\chi(G)$ colors. Let C be a set of vertices with the same color. Since $e(C) = 0$, C is an independent set. Thus, each set of vertices with the same color is an independent set and has a size less than $\alpha(G)$, and there are $\chi(G)$ of them. Therefore, $\alpha(G)\chi(G) \geq |V(G)|$, and, rearranged, we get $\chi(G) \geq |V(G)|/\alpha(G)$ □

(c) For each $k \geq 2$, find a graph G such that $\chi(G) = k + 1$ and $\omega(G) = k$.

Solution. For $k = 2$, a length 5 cycle has $\omega(G) = k$ and $\chi(G) = k + 1$. For $k > 2$, we can start from a k-complete graph F with a vertex set $\{s_1, s_2, \ldots, s_k\}$, each colored differently with the set of colors $C = \{c_1, c_2, \ldots, c_k\}$. Let $H = (V(F) \cup V, E(F) \cup E)$, where $V = \{b_1, b_2, \ldots, b_k\}$ and $E = \{\{b_i, s_j\} : b_i \in V, s_j \in V(F), i \neq j\}$. Since each $b_i \in V(H)$ is connected to $k - 1$ vertices of different colors, there is still one color available for b_i, so we color b_i with that color. Now all $b_i \in H$ are colored differently. We then add a vertex v to H and let v form an edge with each $b_i \in V$, and we call this graph G. Since v has k neighbors with k colors, it must be colored by a new color. Therefore, we get a graph G where $\chi(G) = k + 1$ and $\omega(G) = k$. □

Below is an illustration of what G looks like when $k = 5$.



Problem 5.9.2. Let $k \geq 1$. Prove that an n-vertex bipartite graph containing no matching of size k has at most $(k-1)(n-k+1)$ edges for $n \geq 2k$. For each $k \geq 1$ and $n \geq 2k$, give an example of a graph with exactly $(k-1)(n-k+1)$ edges and no matching of size k.

Proof. Let G be a n-vertex bipartite graph. For $n \geq 2k$, we prove by induction on n that if G has no matching of size k and has at least $(k-1)(n-k+1)$ edges, then $G = K_{k-1,n-k+1}$. For $n = 2k$, G has at least $k^2 - 1$ edges. Suppose G has parts with sizes $k + \gamma$ and $k - \gamma$, then $e(G) \leq (k+\gamma)(k-\gamma) = k^2 - \gamma^2$. Since $k^2 - \gamma^2 \geq e(G) \geq k^2 - 1$, γ can only be 0 or 1. Suppose $\gamma = 0$. $G \neq K_{k,k}$ because it has no matching of size k. Suppose $G = K_{k,k} - \{u, v\}$, for some $u, v \in V(K_{k,k})$. Since G contains a $K_{k-1,k-1}$ that does not contain v and some vertex $w \neq u$, G has a matching M of size $k-1$ such that $\{u, w\} \notin M$. Since u, w forms an edge in G, $M \cup \{u, w\}$ is a matching of G with size k. Thus, for $n = 2k$, $G = K_{k-1,k+1}$.

For $n \geq 2k+1$, let G be an n-vertex graph with no matching of size k and $e(G) \geq (k-1)(n-k+1)$. Let H be a subgraph with $(k-1)(n-k+1)$ edges. Suppose for the sake of contradiction that $\delta(H) \geq k$. Let P be the longest path in H, say $v_1 v_2 \ldots v_m$. We know $N(v_1) \subseteq V(P)$. Since H is bipartite, H does not contain any triangles, so there exists $v_i \in N(v_1)$ for some $2k \leq i \leq m$. Thus, $v_1 v_2 \cdots v_i v_1$ is a cycle of length at least $2k$ in H, and the cycle contains a matching of size k, contradiction. Thus, $\delta(H) \leq k - 1 = \delta(K_{k-1,n-k+1})$. If v is a vertex of minimum degree in H, then

$$
\begin{aligned}
e(H - \{v\}) &\geq e(K_{k-1,n-k+1}) - \delta(K_{k-1,n-k+1}) && (5)\\
&= (k-1)(n-k+1) - (k-1) = e(K_{k-1,n-k}). && (6)
\end{aligned}
$$

By induction, $H - \{v\} = K_{k-1,n-k}$, and so $d_H(v) = (k-1)(n-k+1) - e(K_{k-1,n-k}) = k-1$. Let A, B be parts of $H - \{v\}$ such that $|A| = k-1$ and $|B| = n-k$. Suppose for sake of contradiction that $A \cup \{v\}$ is a part of H. Let $S \subset A \cup \{v\}$ such that $S \neq \emptyset$. If $S = \{v\}$, then $|N(S)| = k-1 \geq |S|$. If $S \neq \{v\}$, then $S \cap A \neq \emptyset$. Since each vertex in A is connected to all vertices in B, $|N(S)| = |B| = n - k \geq |S|$. By Hall's Theorem, there is a matching saturating $A \cup \{v\}$, which has a size of k, contradiction. Therefore, $B \cup \{v\}$ is part of H, so $H = K_{k-1,n-k+1}$. Since $K_{k-1,n-k+1}$ is a maximal graph that has no matching of size k, $G = H = K_{k-1,n-k+1}$, and thus G is an example of the required graph. □

Problem 5.9.3. Determine for all $n \geq 1$ the value of $\mathrm{ex}(n, P_3)$.

Proof. By the Erdös-Gallai Theorem, we know $\mathrm{ex}(n, P_3) \leq n$, with equality if and only if $3|n$ and every component of the graph is K_3. Thus, if $3|n$, a graph that consists of a union of K_3 has n edges and is a maximal graph that does not contain any P_3, so $\mathrm{ex}(n, P_3) \geq n$. If $3 \nmid n$, we have $\mathrm{ex}(n, P_3) \leq n - 1$. Since $K_{n-1,1}$ is a maximal graph that has no P_3 and $e(K_{n-1,1}) = n - 1$, $\mathrm{ex}(n, P_3) \geq n - 1$. Therefore,

$$\mathrm{ex}(n, P_3) = \begin{cases} n, & \text{if } n|3 \\ n-1, & \text{otherwise.} \end{cases}$$

□

Problem 5.9.8. Let G be a graph. Prove that there exists a partition (A, B) of $V(G)$ such that $e(A, B) \geq \frac{1}{2}e(G)$ and $|A| \leq |B| \leq |A| + 1$.

Proof. We will first prove by induction on n to show that there exists a partition (A, B) of $V(G)$ such that $e(A, B) \geq \frac{1}{2}e(G)$ and $|A| = |B|$, for $n = |V(G)|$ is even. The case $n = 2$ is true since $e(A, B) = e(G)$. For $n > 2$, we remove two vertices u, v from G and obtain G'. By induction, there exists a partition (A', B') of $V(G')$ such that $e(A', B') \geq \frac{1}{2}(e(G) - d(u) - d(v))$ and $|A'| = |B'|$. Since $d(u) + d(v) = e(u, A') + e(u, B') + e(v, A') + e(u, B')$, we know $\max(e(u, A') + e(v, B'), e(u, B') + e(v, A')) \geq \frac{1}{2}(d(u) + d(v))$. Suppose without loss of generality that $e(u, A') + e(v, B') \geq \frac{1}{2}(d(u) + d(v))$. Let $A = A' \cup \{v\}$, $B = B' \cup \{u\}$. Then (A, B) is a partition of $V(G)$ such that $e(A, B) \geq \frac{1}{2}e(G)$.

Suppose that n is odd. Let $v \in G$. We know there exists a partition (A', B') of $V(G) \backslash \{v\}$ such that $e(A', B') \geq \frac{1}{2}(e(G) - d(v))$ and $|A'| = |B'|$. Since $d(v) = e(v, A') + e(v, B')$, $\max(e(v, A'), e(v, B')) \geq \frac{1}{2}d(v)$. Suppose, without loss of generality, that $e(v, A') \geq \frac{1}{2}d(v)$. Let $A = A'$, $B = B' \cup \{v\}$. Then (A, B) is a partition of $V(G)$ such that $e(A, B) \geq \frac{1}{2}e(G)$. □

Problem 5.9.10. A bowtie is a graph B consisting of two triangles sharing exactly one vertex. Determine $\mathrm{ex}(n, B)$ for all $n \geq 1$.

Proof. Let G be a graph with at least $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ egdes. G does not contain B for $n < 5$, so we can assume $n \geq 5$. We will prove by induction on n to show that if G does not contain B then G is a balanced complete bipartite graph plus an edge. If $n = 5$ □

Problem 5.9.12. Let G be a bipartite graph with parts of sizes m and n, not containing a 4-cycle. Prove that

$$|E(G)| \leq m\sqrt{n} + m + n$$

Proof. Let M, N be parts of G such that $|M| = m$, $|N| = n$. We count the number of $K_{1,2}$. Since no set of 2 vertices have more than 1 common neighbor, we get

$$\sum_{v \in N} \binom{d(v)}{2} \leq \binom{m}{2} \leq \frac{m^2}{2}.$$

Let d be the average degree of the vertices in N. If $d \leq 1$, then we are done as $|E(G)| = nd \leq n$. Suppose $d \geq 2$. Define $f : \mathbb{R} \to \mathbb{R}$ to be $f(x) = \begin{cases} \binom{x}{2} & , x \geq 2 \\ 0 & , x < 2 \end{cases}$. Since f is convex, Jensen's inequality gives

$$\sum_{v \in N} \binom{d(v)}{2} \geq n\binom{d}{2} \geq \frac{n(d-1)^2}{2}.$$

Thus, we get

$$n(d-1)^2 \leq m^2 \tag{7}$$

$$d \leq \frac{m}{\sqrt{n}} + 1 \tag{8}$$

Therefore, $|E(G)| = nd \leq m\sqrt{n} + n \leq m\sqrt{n} + m + n$. ☐

Problem 6.3.9. Prove that for $n > 2^k$, every k-coloring of $E(K_n)$ gives a monochromatic odd cycle

Proof. Suppose for sake of contradiction that G is a k-edge-colored K_n with no monochromatic odd cycle, for $n \geq 2^k + 1$. G contains a subgraph k-colored K_{2^k+1} with no monochromatic odd cycle, we name it G_k. We obtain $H \subseteq G_k$ by picking a color from G_k and removing all edges that are not that color. Since G_k contains no monochromatic odd cycles, H is bipartite, say with parts A, B. Assume, without loss of generality, that $|A| \geq 2^{k-1} + 1$. Let $H' = G[A]$. Then H' contains a $(k-1)$-edge-coloring of a $K_{2^{k-1}+1}$ with no monochromatic odd cycle, we name it G_{k-1}. By recursively finding a smaller G_r, we can find G_3, a 1-edge-colored K_3 with no monochromatic odd cycle, contradiction. Therefore, G contains a monochromatic odd cycle. ☐