MATH 264A: Homework

Due on Nov 2, 2024 at 23:59pm

Professor Warnke

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Problem 1

(a) Let $G = (V, E)$ be an n-vertex graph and suppose that each vertex $v \in V$ is associated with a list $S(v)$ of colors of size at least $4r$, where r is a positive integer. Suppose also that for each $v \in V$ and each $c \in S(v)$, there are at most r neighbors u of v such that $c \in S(u)$. Using induction, prove that there are at least $(2r)^n$ proper colorings of G under which each vertex v receives a color from its list $S(v)$.

Remark: a proof using the classical Lovász Local Lemma (LLL) requires $2er \approx 5.44r$ instead of $4r$.

Proof. For $S \subseteq V$, define N_S as the number of proper colorings of G under which v receives a color from its list $S(v)$ for all $v \in S$. It suffices to show that for all $T \subseteq V$ and $x \in T$, $N_T / N_{T \setminus \{x\}} \geq 2r$. We proceed by induction on $|T|$. When $T = \{x\}$, $N_T = S(x)$ and $N_{T\setminus\{x\}} = N_\emptyset = 1$, we have $N_T / N_{T\setminus\{x\}} \geq 4r \geq 2r$. Suppose $|T| > 2$. Then,

$$|S(x)| \cdot N_{T\setminus\{x\}} = N_T + B,$$

where B is the number of improper colorings of T which becomes proper if we ignore the color of x. Notice that any element counted by B contains a vertex u which is a neighbor of v that shares the same color c as x and a proper coloring of $T\setminus\{u, x\}$. This yields the upper bound

$$B \leq |S(x)| \cdot (\#\text{choice of vertex } u) \cdot N_{T\setminus\{u,x\}} \leq 4r \cdot r \cdot N_{T\setminus\{u,x\}}.$$

By induction, $N_{T\setminus\{u,x\}} \leq \frac{1}{2r} \cdot N_{T\setminus\{x\}}$, and thus

$$N_T = |S(x)| \cdot N_{T\setminus\{x\}} - B \geq 4rN_{T\setminus\{x\}} - 4r \cdot r \cdot \frac{1}{2r} \cdot N_{T\setminus\{x\}} = 2rN_{T\setminus\{x\}},$$

and this completes the induction. □

(b) A k-SAT formula is an expression such as

$$(x_1 \text{ OR } x_4 \text{ OR } \overline{x_6}) \text{ AND } (x_1 \text{ OR } \overline{x_2} \text{ OR } x_5),$$

where the variables x_i take values true or false, $\overline{x_i}$ means not x_i, and k distinct variables or their negations are OR-ed together in each clause. A formula is called satisfiable if there is an assignment of values to the variables making the expression true. Suppose that in a given k-SAT formula Φ no variable lies in more than $2^k/(ek)$ clauses. Using induction, prove that Φ has at least $(2 - 2/k)^n$ many satisfying assignments (which in fact remains true if we relax the assumed $2^k/(ek)$ upper bound to $2^k/k \cdot (1 - 1/k)^{k-1}$).

Proof. Let Φ_i denote the k-SAT formula which AND's together all the clauses of Φ that involve only the first i variables. Define N_i as the number of satisfying assignments of Φ_i. We show that $N_i \geq (2-2/k)^i$ by induction on $i \geq k$. And then I'm stuck. □

(c) Let $\mathcal{A}$ be an alphabet, and let $\mathcal{F}$ be a set of forbidden strings. Assume that there exists $\beta > 0$ such that

$$|\mathcal{A}| - \sum_{f \in \mathcal{F}} \beta^{1-|f|} \geq \beta.$$

Using induction, prove that there exists at least β^n words of length n over alphabet $\mathcal{A}$ that avoid all the substrings in $\mathcal{F}$.

Proof. Define N_k to be the set of words of length n over alphabet $\mathcal{A}$ that avoid all the substrings in $\mathcal{F}$. We show that $|N_k|/|N_{k-1}| \geq \beta$ by induciton on $k \geq 1$. Since $|N_0| = 1$,

$$\frac{|N_1|}{|N_0|} = |N_1| \geq |\mathcal{A}| - |\{f \in F \mid |f| = 1\}| \geq \beta,$$

as $|\{f \in F \mid |f| = 1\}| \leq \sum_{f \in \mathcal{F}} \beta^{1-|f|}$. Hence, the base case holds. Suppose $k > 1$. Let B denote the set of words over $\mathcal{A}$ of the form $a_1 \ldots a_k$ that contains some substring in F, with $a_1 \ldots a_{k-1} \in N_{k-1}$. Then

$$|\mathcal{A}| \cdot |N_{k-1}| = |N_k| + |B|.$$

By construction, any word in B consists of a forbidden string f at the end and some word in $N_{k-|f|}$ at the beginning. Summing over all f, we have the bound

$$|B| \leq \sum_{f \in \mathcal{F}} N_{k-|f|}.$$

By induction, $N_{k-|f|} \leq \beta^{1-|f|} \cdot |N_{k-1}|$, and thus

$$|B| \leq |N_{k-1}| \sum_{f \in \mathcal{F}} \beta^{1-|f|}.$$

Therefore,

$$|N_k| = |\mathcal{A}| \cdot |N_{k-1}| - B \geq |N_{k-1}| \left(|\mathcal{A}| - \sum_{f \in \mathcal{F}} \beta^{1-|f|} \right) \geq \beta |N_{k-1}|,$$

and this completes the induction. It now follows that

$$|N_n| = \frac{|N_n|}{|N_{n-1}|} \cdot \frac{|N_{n-1}|}{|N_{n-2}|} \cdots \frac{|N_1|}{|N_0|} \geq \beta^n.$$

$\square$

Problem 2

In the inductive proof of the 'almost all triangle-free graphs are 2-colorable' result, we defined the following sets (using $\Gamma(v)$ and $\Gamma(S)$ to denote the set of neighbors of a vertex v or set of vertices S):

- $\mathcal{C}ol_2(n)$ is the set of all 2-colorable graphs on n vertices.
- $\mathcal{T}(n)$ is the set of all triangle-free graphs on n vertices.
- $\mathcal{A}(n) \subseteq \mathcal{T}(n)$ is the subset of graphs containing a vertex v such that $|\Gamma(v)| \leq \log_2 n$.
- $\mathcal{B}(n) \subseteq \mathcal{T}(n)$ is the subset of graphs containing a vertex set Q of size $|Q| = \log_2 n$, such that $|\Gamma(Q)| \leq (1/2 - 1/1000)n$.
- $\mathcal{D}(n) \subseteq \mathcal{T}(n) \setminus (\mathcal{A}(n) \cup \mathcal{B}(n))$ is the subset of graphs containing an edge $\{x, y\}$ and vertex sets $Q_x \subseteq \Gamma(x)$ and $Q_y \subseteq \Gamma(y)$, such that $|Q_x| = \log_2 n$, $|Q_y| = \log_2 n$, and $|\Gamma(Q_x) \cap \Gamma(Q_y)| \geq n/100$.

(a) Prove that $|\mathcal{D}(n)|/|\mathcal{T}(n-2)| \leq 2^{(1-1/2000)n}$ for all sufficiently large $n \geq n_0$.

Proof. To generate a graph in $\mathcal{D}(n)$, we first pick two vertices x, y to be adjacent and then place a triangle free graph on the remaining $n-2$ vertices. Lastly, we pick two subsets from the $n-2$ vertices to be x and y's neighbors respectively. Since the graph is not in $\mathcal{A}(n)$, $|\Gamma(x)|, |\Gamma(y)| > \log_2 n$. But then the graph is also not in $\mathcal{B}(n)$, so $\Gamma(\Gamma(x)), \Gamma(\Gamma(y))$ each have size greater than $(1/2 - 1/1000)n$. Since the graph is triangle-free, x cannot be adjacent to any vertex in $\Gamma(\Gamma(x))$ and similarly for y, and thus $|\Gamma(x)|, |\Gamma(y)| \leq (1/2 + 1/1000)n$. This yields the bound

$$\mathcal{D}(n) \leq \binom{n}{2} \cdot |\mathcal{T}(n-2)| \cdot 2^{(1/2-1/1000)n} \cdot 2^{(1/2-1/1000)n} \leq 2^{(1-1/2000)n} \cdot |\mathcal{T}(n-2)|,$$

for large enough n. $\square$

(b) Prove that $|\mathcal{C}ol_2(n)|/|\mathcal{C}ol_2(n-1)| \geq 2^{\frac{1}{2}(n-1)}$ for all sufficiently large $n \geq n_0$.

Proof. Each graph in $\mathcal{C}ol_2(n)$ consists of vertex n, a graph $H \in \mathcal{C}ol_2(n-1)$, and edges between n and H. Since H is bipartite, H contains an independent set I_H of size at least $2^{\frac{1}{2}(n-1)}$. By picking a graph H from $\mathcal{C}ol_2(n-1)$ and adding some edges between n and I_H, we may uniquely generate a graph in $\mathcal{C}ol_2(n)$. Therefore,

$$|\mathcal{C}ol_2(n)| \geq |\mathcal{C}ol_2(n-1)| \cdot 2^{\frac{1}{2}(n-1)},$$

and the result now follows. $\square$

Problem 3

In this problem we discuss in more detail one calculation in the proof of the Bregman's Theorem from class. Given a bipartite graph $G = (L \cup R, E)$ with $|L| = |R| = n$, fix a perfect matching M of G. For each vertex $i \in L$, there thus is a unique vertex X_i such that $\{i, X_i\} \in M$ is a matching edge. Now write R_i for the set of $j \in L$ for which $\{j, X_j\} \in M$ and $\{i, X_j\} \in E$, i.e., the set of vertices $j \in L$ for which there is a matching edge in M that contains j and a neighbor of i. By construction, we have $|R_i| = \deg_G(i)$. Using a (permutation) counting argument, show that for each vertex $i \in L$ and $1 \leq j \leq \deg_G(i)$ we have

$$\mathbb{P}(\text{vertex } i \text{ appears in } \pi \text{ in the } j\text{th position among the vertices in } R_i) = \frac{1}{\deg_G(i)}.$$

Proof.

$$\begin{aligned}
&\mathbb{P}(\text{vertex } i \text{ in the } j\text{th position among the vertices in } R_i \text{ in } \pi) \\
&= \frac{\#\{\text{permutations of } R_i \text{ with } i \text{ appearing in the } j\text{th position}\}}{\#\{\text{permutations of } R_i\}} \\
&= \frac{\#\{\text{permutations of } R_i \backslash \{i\}\}}{\#\{\text{permutations of } R_i\}} \\
&= \frac{(|R_i| - 1)!}{|R_i|!} = \frac{1}{\deg_G(i)}.
\end{aligned}$$

□

Problem 4

(a) An order n Latin square is an $n \times n$ matrix in which each row and column is a permutation of $[n]$. Using the entropy method, prove that the number of order n Latin squares is at most

$$L(n) \leq \left((1 + o(1))\frac{n}{e^2}\right)^{n^2}$$

as $n \to \infty$.

Proof. Choose X uniformly at random from all order n Latin squares. Let X_i denote the ith row and let X_{ij} denote the ith row jth column of X. Independently and uniformly take a random a_{ij} from $[0, 1]$ for all $i, j \in [n]$, and denote $A = (a_{ij})$. Let $R_{ij}(A) = \{X_{il} : a_{il} < a_{ij}\}$ and $C_{ij}(A) = \{X_{kj} : a_{kj} < a_{ij}\}$. Note that $\mathbb{P}(a_{ij} = a_{kl}) = 0$ for all $(i, j) \neq (k, l)$. By the Chain rule,

$$\log_2 L(n) = H(X) = \mathbb{E}_A \sum_{i,j\in[n]} H(X_{ij} \mid X_{kl} : a_{kl} < a_{ij}).$$

Let $N_{ij}(A, X) = |[n] - R_{ij}(A) \cup C_{ij}(A)|$. Rewriting the entropy in expectation form,

$$H(X_{ij} \mid X_{kl} : a_{kl} < a_{ij}) = \mathbb{E}_X[\log_2 N_{ij}(A, X)],$$

and thus

$$H(X) = \mathbb{E}_A \sum_{i,j\in[n]} \mathbb{E}_X[\log_2 N_{ij}(A, X)] = \mathbb{E}_X \sum_{i,j\in[n]} \mathbb{E}_A[\log_2 N_{ij}(A, X) \mid X].$$

By Jensen's inequality,

$$\mathbb{E}_A[\log_2 N_{ij}(A, X) \mid X] \leq \mathbb{E}_{a_{ij}}[\log_2 \mathbb{E}_{A\setminus\{a_{ij}\}}[N_{ij}(A, X) \mid X, a_{ij}]].$$

For $m \in [n]$, notice

$$\mathbb{P}[m \notin (R_{ij}(A) \cup C_{ij}(A))] = \mathbb{P}(m \notin R_{ij}(A))\mathbb{P}(m \notin C_{ij}(A)) = (1 - a_{ij})^2,$$

and thus

$$\mathbb{E}_{A\setminus\{a_{ij}\}}[N_{ij}(A, X) \mid X, a_{ij}] = \sum_{m\in[n]} \mathbb{P}[m \notin (R_{ij}(A) \cup C_{ij}(A))] = n(1 - a_{ij})^2.$$

Hence,

$$\mathbb{E}_{a_{ij}}[\log_2 \mathbb{E}_{A\setminus\{a_{ij}\}}[N_{ij}(A, X) \mid X, a_{ij}]] = \int_0^1 \log_2 n(1 - x^2)\, dx = \log_2(n) - 2.$$

It now follows that

$$H(X) \leq \mathbb{E}_X \sum_{i,j\in[n]} \mathbb{E}_{a_{ij}}[\log_2 n(1 - a_{ij}^2)] \leq n^2(\log_2(n) - 2),$$

and thus the result. (I don't know where I lost the $o(1)$ term.) $\square$

(b) Sudoku puzzles are order 9 Latin squares, divided into 9 smaller 3×3 blocks, with the additional constraint that each block must contain all the symbols $\{1, \ldots, 9\}$. Sudoku squares of order N can be defined similarly for any square number $N = n^2$ (to clarify: normal Sudoku puzzles are simply order 9 Sudoku squares). Using the entropy method, prove that the number of order N Sudoku squares is at most

$$S(n) \leq \left((1 + o(1))\frac{N}{e^3}\right)^{N^2}$$

as $N \to \infty$.

Proof. Choose X uniformly at random from all order n Latin squares. Let X_i denote the ith row and let X_{ij} denote the ith row jth column of X. Independently and uniformly take a random a_{ij} from $[0, 1]$ for all $i, j \in [n]$, and denote $A = (a_{ij})$. Define

- $R_{ij}(A) = \{X_{il} : a_{il} < a_{ij}\}$
- $C_{ij}(A) = \{X_{kj} : a_{kj} < a_{ij}\}$
- $S_{ij}(A) = \{X_{kl} : a_{kl} < a_{ij}$ and X_{kl} in the same square as $X_{ij}\}$.

Note that $\mathbb{P}(a_{ij} = a_{kl}) = 0$ for all $(i, j) \neq (k, l)$. By the Chain rule,

$$\log_2 L(n) = H(X) = \mathbb{E}_A \sum_{i,j \in [n]} H(X_{ij} \mid X_{kl} : a_{kl} < a_{ij}).$$

Let $N_{ij}(A, X) = |[N] - R_{ij}(A) \cup C_{ij}(A) \cup S_{ij}(A)|$. Rewriting the entropy in expectation form,

$$H(X_{ij} \mid X_{kl} : a_{kl} < a_{ij}) = \mathbb{E}_X[\log_2 N_{ij}(A, X)],$$

and thus

$$H(X) = \mathbb{E}_A \sum_{i,j \in [n]} \mathbb{E}_X[\log_2 N_{ij}(A, X)] = \mathbb{E}_X \sum_{i,j \in [n]} \mathbb{E}_A[\log_2 N_{ij}(A, X) \mid X].$$

By Jensen's inequality,

$$\mathbb{E}_A[\log_2 N_{ij}(A, X) \mid X] \leq \mathbb{E}_{a_{ij}}[\log_2 \mathbb{E}_{A \setminus \{a_{ij}\}}[N_{ij}(A, X) \mid X, a_{ij}]].$$

For $m \in [N]$, notice

$$\mathbb{P}[m \notin (R_{ij}(A) \cup C_{ij}(A) \cup S_{ij}(A))] = \mathbb{P}(m \notin R_{ij}(A))\mathbb{P}(m \notin C_{ij}(A))\mathbb{P}(m \notin S_{ij}(A)) = (1 - a_{ij})^3,$$

and thus

$$\mathbb{E}_{A \setminus \{a_{ij}\}}[N_{ij}(A, X) \mid X, a_{ij}] = \sum_{m \in [N]} \mathbb{P}[m \notin (R_{ij}(A) \cup C_{ij}(A) \cup S_{ij}(A))] = N(1 - a_{ij})^3.$$

Hence,

$$\mathbb{E}_{a_{ij}}[\log_2 \mathbb{E}_{A \setminus \{a_{ij}\}}[N_{ij}(A, X) \mid X, a_{ij}]] = \int_0^1 \log_2 N(1 - x^2)\, dx = \log_2(N) - 3.$$

It now follows that

$$H(X) \leq \mathbb{E}_X \sum_{i,j \in [n]} \mathbb{E}_{a_{ij}}[\log_2 n(1 - a_{ij}^2)] \leq N^2(\log_2(N) - 3),$$

and thus the result. (I don't know where I lost the $o(1)$ term.) $\square$