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Math 180A **Homework 3** **Professor Carfagnini**

Question 1 Let $c > 0$ and $X \sim \text{Unif}[0, c]$. Show that the random variable $Y = c - X$ has the same cumulative distribution function as X and hence also the same density function.

Solution. Let F be the cumulative distribution function. For $x \in [0, c]$,

$$F_X(x) = \mathbb{P}(X \leq x) = \frac{x}{c}, \tag{1}$$
$$F_Y(x) = \mathbb{P}(Y \leq x) \tag{2}$$
$$= \mathbb{P}(X \geq c - x) = \frac{x}{c}. \tag{3}$$

Thus, X and Y have the same cumulative distribution.

Since the density function $p_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}F_Y(x) = p_Y(x) = \frac{1}{c}$, the density functions of X, Y are the same. $\square$

Question 2 Parts (a) and (b) ask for an example of a random variable X whose cumulative distribution function $F(x)$ satisfies $F(1) = \frac{1}{3}$, $F(2) = \frac{3}{4}$, and $F(3) = 1$.

(a) Make X discrete and give its probability mass function.

Solution. An urn contains 12 balls with 4 of them numbered 1, 5 of them numbered 2, and 3 of them numbered 3. Let X be the number of the selected ball. The mass function

$$p(x) = \begin{cases} \frac{1}{3}, & x = 1 \\ \frac{5}{12}, & x = 2 \\ \frac{1}{4}, & x = 3 \end{cases}.$$

Since $F(1) = p(1) = \frac{1}{3}$, $F(2) = p(1) + p(2) = \frac{1}{3} + \frac{5}{12} = \frac{3}{4}$, and $F(3) = 1$, X satisfies the given condition. □

(b) Make X continuous and give its probability density function.

Solution. Let $a = 4 - \sqrt{13}$. Define distribution

$$F(x) = \begin{cases} 0, & x < a \\ -\frac{1}{12}x^2 + \frac{2}{3}x - \frac{1}{4}, & x \in [a, 3] \\ 1, & x > 3. \end{cases}$$

Let $X \sim F$. Since $F(1) = \frac{1}{3}$, $F(2) = \frac{3}{4}$, and $F(3) = 1$, X satisfies the given condition. The density function of X is $p(x) = F'(x) = \begin{cases} -\frac{1}{6} + \frac{2}{3}, & x \in [a, 3] \\ 0, & x \notin [a, 3] \end{cases}$. □

Question 3 Let A and B be two disjoint events. Under what conditions are they independent?

Solution. Suppose A, B are independent. Then $\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B) = 0$. Thus, A, B are independent if $\mathbb{P}(A)$ or $\mathbb{P}(B)$ is 0. $\square$

Question 4 Suppose that the events A, B, and C are mutually independent with

$$\mathbb{P}(A) = \frac{1}{2}, \quad \mathbb{P}(B) = \frac{1}{3}, \quad \mathbb{P}(C) = \frac{1}{4}.$$

Compute $\mathbb{P}(AB \cup C)$.

Solution.

$$\begin{aligned}
\mathbb{P}(AB \cup C) &= \mathbb{P}(AB) + \mathbb{P}(C) - \mathbb{P}(ABC) && (4) \\
&= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} && (5) \\
&= \frac{3}{8}. && (6)
\end{aligned}$$

□

Question 5 Let n be a fixed integer and $c \in R$. Let us consider the function

$$p(x) = \begin{cases} cx, & x \in [n] \\ 0, & \text{otherwise} \end{cases}.$$

Find the value of c so that $p(x)$ is the probability mass function of a random variable X.

Solution.

$$\sum_{x \in [n]} p(x) = c \sum_{x \in [n]} x = \frac{cn(n+1)}{2} = 1, \tag{7}$$

$$c = \frac{2}{n^2 + n}. \tag{8}$$

□

Question 6 Let us consider the function

$$f(x) = \begin{cases} cxe^x, & x \in (0,3) \\ 0, & \text{otherwise} \end{cases},$$

for some $c \in \mathbb{R}$. Is it possible to find a value for c so that $f(x)$ is the probability density function of a random variable X?

Solution. Suppose that $\int_{-\infty}^{\infty} f(x)dx = c\int_0^3 xe^x dx = c[(x-1)e^x]_0^3 = c(2e^3+1) \leq 1$, $f(x)$ can be a probability density function for $0 \leq c \leq \frac{1}{2e^3+1}$. $\square$

Question 7 A fair coin is flipped twice. Let X be the number of heads observed.

(a) Give the possible values and probability mass function for X.

Solution. The possible values are $0, 1, 2$. The probability mass function $p(x) = \frac{\binom{2}{x}}{4}$, for $x \in \{0, 1, 2\}$. $\square$

(b) Find $\mathbb{P}(X \geq 1)$ and $\mathbb{P}(X > 1)$.

Solution.

$$\mathbb{P}(X \geq 1) = p(1) + p(2) = \frac{3}{4} \tag{9}$$

$$\mathbb{P}(X > 1) = p(2) = \frac{1}{4}. \tag{10}$$

$\square$

Question 8 Suppose that X is a discrete random variable with possible values $\mathbb{N} = \{1, 2, 3, \dots\}$, and probability mass function

$$p_X(k) = \frac{c}{k(k+1)},$$

for some constant $c > 0$. What is the value of c?

Solution.

$$\sum_{k \in \mathbb{N}} p_X(k) = c \sum_{k \in \mathbb{N}} \frac{1}{k(k+1)} \tag{11}$$

$$= c \sum_{k \in \mathbb{N}} \frac{1}{k} - \frac{1}{k+1} \tag{12}$$

$$= c \cdot \left(1 - \frac{1}{\infty}\right) = 1. \tag{13}$$

Therefore, $c = 1$. □

Question 9 We choose a number from the set $\{10, 11, 12, ..., 99\}$ uniformly at random.

(a) Let X be the first digit and Y be the second digit of the chosen number. Show that X and Y are independent random variables.

Solution. The possible values for X are $1, \ldots, 9$, Y are $0, 1, \ldots, 9$. Let d_1 be a possible value of X and d_2 be a possible value of Y. There are 9 numbers that start with d_1 and 10 numbers that end with d_2, and there is only one number that starts with d_1 and ends with d_2. Since the number is chosen uniformly at random, $\mathbb{P}(X = d_1) = \frac{1}{9}$ and $\mathbb{P}(Y = d_2) = \frac{1}{10}$. Since

$$\mathbb{P}(X = d_1 \text{ and } Y = d_2) = \frac{1}{90} = \mathbb{P}(X = d_1)\mathbb{P}(Y = d_2),$$

X, Y are independent variables. $\square$

(b) Let X be the first digit of the chosen number and Z be the sum of the two digits. Show that X and Z are not independent.

Solution. The possible values for X are $1, \ldots, 9$, and Z are $1, \ldots, 18$. Since it's impossible for the sum of the digits to be 18 when the first digit is 1, $\mathbb{P}(X = 1 \text{ and } Z = 18) = 0$. Since 99 is the only number whose sum of digits is 18, $\mathbb{P}(Z = 18) = \frac{1}{90}$. Since

$$\mathbb{P}(X = 1)\mathbb{P}(Z = 18) = \frac{1}{9} \cdot \frac{1}{90}, \tag{14}$$

$$\mathbb{P}(X = 1 \text{ and } Z = 18) = 0, \tag{15}$$

X, Z are not independent.

$\square$