# MATH 190A: Homework #2

Due on Jan 22, 2025 at 12:00pm

*Professor McKernan*

Section A02 8:00AM - 8:50AM
Section Leader: Zhiyuan Jiang

Source Consulted: Textbook, Lecture, Discussion

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# Problem 1

Let $X$ be a set. Give $X$ the topology $\mathcal{T}$ where every finite set is closed, plus $X$. If $Y$ is a subset of $X$ then determine

(i) The interior of $Y$.

*Proof.* If $X \backslash Y$ is finite, then $Y \in \mathcal{T}$ and so $\text{int}(Y) = Y$. Otherwise, $X \backslash Y$ is infinite and so $Y$ does not contain a set whose complement is finite. Thus, $\text{int}(Y) = \emptyset$. $\qquad\square$

(ii) The closure of $Y$.

*Proof.* If $Y$ is finite, then $Y$ is closed so $Y = \overline{Y}$. If $Y$ is infinite, then there is no finite set that contains $Y$ so $\overline{Y} = X$. $\qquad\square$

(iii) The boundary of $Y$.

*Proof.* By (i) and (ii),

$$\partial Y = \begin{cases} \emptyset & \text{if } Y \text{ is finite and } X \backslash Y \text{ is finite} \\ Y & \text{if } Y \text{ is finite and } X \backslash Y \text{ is infinite} \\ X \backslash Y & \text{if } Y \text{ is infinite and } X \backslash Y \text{ is finite} \\ X & \text{if } Y \text{ is infinite and } X \backslash Y \text{ is infinite} \end{cases}$$

$\qquad\square$

# Problem 2

Let $a < b \in \mathbb{R}$ and let $Y = [a, b) \subset \mathbb{R}$. What is

(i) The interior of $Y$?

*Proof.* Obviously, the largest open set contained in $Y$ is $(a, b)$, so $\text{int}(Y) = (a, b)$. $\qquad\square$

(ii) The closure of $Y$?

*Proof.* Obviously, the smalled closed set contained in $Y$ is $[a, b]$, so $\overline{Y} = [a, b]$. $\qquad\square$

(iii) The boundary of $Y$?

*Proof.*
$$\partial Y = \overline{Y} \backslash \text{int}(Y) = \{a, b\}.$$

$\qquad\square$

# Problem 3

Let $(X, d)$ be a metric space, let $a \in X$ be a point of $X$ and let $r > 0$ be a positive real. Let

$$B = \overline{B_r}(a) = \{x \in X \mid d(a, x) \leq r\}.$$

Show that $B$ is closed in $X$. $\overline{B_r}(a)$ is called the *closed ball of radius $r$ centred about $a$.*

*Proof.* Let $x \in X \backslash B$. Let $\epsilon = d(a, x) - r > 0$. Then $B_\epsilon(x) \subset X \backslash B$, so $X \backslash B$ is covered by a collection of open sets. Thus, $X \backslash B$ is closed and the result follows. $\square$

# Problem 4

True or false? If true then give a proof and if false then give a counterexample.

(i) Let $(X, d)$ be a metric space, let $a \in X$ be a point of $X$ and let $r > 0$ be a positive real. Let $Y = B_r(a)$ be the *open ball of radius $r$ centred about $a$*. Then the closure of $Y$ is the closed ball of radius $r$ centred about $a$.

*Proof.* True. By problem 3, $\overline{B}_r(a)$ is closed, so it suffices to show $\overline{B}_r(a)$ is the smallest closed set. Suppose not. Let $C$ be a closed set such that $B_r(a) \subset C \subset \overline{B}_r(a)$. There exists some $x$ such that $d(a, x) = r$ and $x \notin C$. Since $B_s(x) \cap B_r(a) \neq \emptyset$ for any $s > 0$, there are no open balls in $X \backslash C$ that contain $x$. Thus $C$ is not closed, contradiction. $\qquad\square$

(ii) If $(X, \mathcal{T})$ is a topological space and $Y \subset X$ is a subset then

$$\overline{X \setminus Y} = X \setminus \operatorname{int}(Y).$$

*Proof.* True. Since $C = \overline{X \setminus Y}$ is the smalled closed set that contains $X \setminus Y$, we know $X \backslash C$ is the largest open set that contains $Y$. Thus, $X \backslash C = \operatorname{int}(Y)$ and the result follows. $\qquad\square$

(iii) If $(X, \mathcal{T})$ is a topological space and $Y \subset X$ and $Z \subset X$ are two subsets then

$$\operatorname{int}(Y \cup Z) = \operatorname{int}(Y) \cup \operatorname{int}(Z).$$

*Proof.* False. Let $X = \mathbb{R}$ and $Y = [0, 1]$ and $Z = [1, 2]$. Then $\operatorname{int}(Y \cup Z) = (0, 2)$ but $\operatorname{int}(Y) \cup \operatorname{int}(Z) = (0, 1) \cup (1, 2) = (0, 1) \cup [1, 2] = (0, 2)$. $\qquad\square$

(iv) The integers $\mathbb{Z}$ are dense in the reals $\mathbb{R}$.

*Proof.* False. Let $x \in \mathbb{R} \backslash \mathbb{Z}$. There exists some $n \in \mathbb{Z}$ such that $n < x < n + 1$. Let $r = \min(x - n, n - x + 1)$. Then $B_r(x) \cap \mathbb{Z} = \emptyset$, so $\mathbb{Z}$ is not dense in $\mathbb{R}$ by lemma 4.12. $\qquad\square$

(v) The rationals $\mathbb{Q}$ are dense in the reals $\mathbb{R}$.

*Proof.* True. Let $x \in \mathbb{R}$ and $r > 0$. Fix $q$ such that $\frac{1}{q} < r$. There exists some $p$ such that $\frac{p-1}{q} \leq x \leq \frac{p}{q} < x + r$. Thus $\frac{p}{q} \in B_r(x) \cap \mathbb{Q}$, so $\mathbb{Q}$ is dense. $\qquad\square$

# Problem 5

Show that every open subset of $\mathbb{R}$ is a disjoint union of open intervals.

*Proof.* Let $U$ be an open set of $\mathbb{R}$. Then $U$ is a union of a collection of open intervals. Suppose $I_1 = (a, b)$ and $I_2 = (c, d)$ are two open intervals in $U$ such that $I_1 \cap I_2 \neq \emptyset$. Then $I_1 \cup I_2 = (\min(a, c), \max(b, d))$ is an open interval contained in $U$. Thus, $U$ is a disjoint union of open intervals. $\qquad\square$

# Problem 6

Let $(X, \mathcal{T})$ be a topological space. Starting with any subset $Y \subset X$ (and any $X$) what is the maximum number of distinct subsets one can obtain by taking the closure and the complement (as many times as you please, in whatever order you please)?

*Proof.* 14? $\square$