MATH 190A: Homework #9

Due on Mar 12, 2025 at 12:00pm

Professor McKernan

Section A02 8:00AM - 8:50AM
Section Leader: Zhiyuan Jiang

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

Show that an arbitrary product of Hausdorff spaces is Hausdorff.

Proof. Let $\{X\alpha\}$ be a collection of Hausdorff spaces and consider

$$X = \prod_\alpha X_\alpha.$$

Let $x, y \in X$ such that $x \neq y$. Then there exists some index β such that $x_\beta \neq y_\beta$. Since X_β is Hausdorff, there exists disjoint open sets $U, V \subset X_\beta$ such that $x_\beta \in U$ and $y_\beta \in V$. Let $p_\beta : X \to X_\beta$ be the natural projection map. Then $p_\beta^{-1}(U)$ and $p_\beta^{-1}(V)$ are open in X and $x \in p_\beta^{-1}(U)$. $y \in p_\beta^{-1}(V)$. Moreover, $p_\beta^{-1}(U)$ and $p_\beta^{-1}(V)$ are disjoint and U, V are joint, so X is Hausdorff. $\square$

Problem 2

Show that an arbitrary product of connected sets is connected.

Proof. Let $\{X_\alpha\}$ be a collection of connected spaces and consider

$$X = \prod_{\alpha \in \Lambda} X_\alpha.$$

First note that since X_α is connected, $\prod_{\alpha \in F} X_\alpha$ is connected for all finite $F \subseteq \Lambda$. Suppose for sake of contradiction that X may be partitioned into two disjoint nonempty open sets U, V. Fix $a \in X$. For $\alpha \in \Lambda$, let

$$Z_\alpha = \{x \in X \mid x_\beta = a_\beta, \beta \neq \alpha\},$$

the set of points in X that agree with a in all coordinates except α. Then $f_\alpha : X_\alpha \to Z_\alpha$ which sends k to (x_β) where $x_\beta = a_\beta$ for $\beta \neq \alpha$ and $x_\alpha = k$ is a homeomorphism. Since X_α is connected, Z_α is connected. Hence, for each $\alpha \in \Lambda$, Z_α is contained in either U or V. Since U, V are nonempty, we may find $Z_{\alpha_1}, Z_{\alpha_2}$ such that $Z_{\alpha_1} \subseteq U$ and $Z_{\alpha_2} \subseteq V$. Now consider $W = Z_{\alpha_1} \cup Z_{\alpha_2} \subset X$. Then the function $g : X_{\alpha_1} \times X_{\alpha_2} \to W$ which sends (k, m) to (x_β) where $x_\beta = a_\beta$ for $\beta \neq \alpha_1, \alpha_2$ and $(x_{\alpha_1}, x_{\alpha_2}) = (k, m)$ is a homeomorphism. But then this implies that $W = (W \cap U) \cup (W \cap V)$ is connected, contradiction. $\square$

Problem 3

Let $f : X \to Y$ be a continuous and surjective function.

(i) Show that f is a quotient map if and only if Y has the finest topology such that f is continuous (that is, $V \subset Y$ is open if and only if $U = f^{-1}(V)$ is open).

Proof. Suppose f is a quotient map. □

(ii) Show that if f is open then f is a quotient map.

(iii) Show that if f is closed then f is a quotient map.

(iv) Show that if X is compact and Y is Hausdorff then f is a quotient map.

(v) Show that if f has a right inverse, that is, there is a continuous function $g : Y \to X$ such that $f \circ g : Y \to Y$ is the identity, then f is a quotient map.

(vi) Show that if $p : X \times Y \to Y$ is the projection then p is a quotient map.

(vii) Let X be a topological space and let $Y \subset X$ be a subspace. We say that r is a *retraction* of X onto Y if $r : X \to Y$ is a continuous map whose restriction to Y is the identity, $r|_Y = i_Y$.
If $r : X \to Y$ is a retraction then show that r is a quotient map.

(viii) Show that the composition of quotient maps is a quotient map.

Problem 4

Let

$$X = \{(x, y) \mid \text{either } x \geq 0 \text{ or } y = 0\} \subset \mathbb{R}^2.$$

Let $p : \mathbb{R}^2 \to \mathbb{R}$ denote projection onto the first factor and let

$$f : X \to \mathbb{R}$$

be the restriction of p to X.

Show that f is a quotient map, even though f is neither open nor closed and X.

Problem 5

Consider the Möbius strip X_1 and the closed unit ball X_2 in $\mathbb{R}^2$. The boundary of both is homeomorphic to S^1 (in fact the boundary of the closed unit ball is S^1). Define a topological space X by taking the quotient of the disjoint union of X_1 and X_2, modulo an equivalence relation that identifies the two boundaries (in other words, pick a homeomorphism between the two boundaries and identify corresponding points).

Show that X is homeomorphic to the real projective plane.