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Math 180A **Homework 4** **Professor Carfagnini**

Question 1 Let X be a discrete random variable with possible values $\{1, 2, 3, 4, 5\}$ and probability mass function

x	1	2	3	4	5
$p(x)$	$\frac{1}{7}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{2}{7}$

(a) Compute $\mathbb{P}(X \leq 3)$

Solution. $\mathbb{P}(X \leq 3) = \frac{1}{7} + \frac{1}{14} + \frac{3}{14} = \frac{3}{7}$. □

(b) Compute $\mathbb{P}(X < 3)$

Solution. $\mathbb{P}(X < 3) = \frac{1}{7} + \frac{1}{14} = \frac{3}{14}$. □

(c) Compute $\mathbb{P}(X < 4.12 \,|\, X > 1.638)$.

Solution.

$$\begin{aligned}
\mathbb{P}(X < 4.12 \text{ and } X > 1.638) &= \frac{1}{14} + \frac{3}{14} + \frac{2}{7} = \frac{4}{7}, \\
\mathbb{P}(X > 1.638) &= \frac{1}{14} + \frac{3}{14} + \frac{2}{7} + \frac{2}{7} = \frac{6}{7}, \\
\mathbb{P}(X < 4.12 \,|\, X > 1.638) &= \frac{\mathbb{P}(X < 4.12 \text{ and } X > 1.638)}{\mathbb{P}(X > 1.638)} = \frac{2}{3}.
\end{aligned}$$

□

Question 2 Let X be a continuous random variable with density function

$$f(x) = \begin{cases} 3e^{-3x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that f is a density function.

Solution. f is a density function because

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_{0}^{\infty} 3e^{-3x}dx \\ &= -e^{-3x}\Big|_0^{\infty} \\ &= 1. \end{aligned}$$

□

(b) Compute $\mathbb{P}(-1 < X < 1)$.

Solution.

$$\begin{aligned} \mathbb{P}(-1 < X < 1) &= \int_{0}^{1} 3e^{-3x}dx \\ &= -e^{-3x}\Big|_0^{1} \\ &= 1 - e^{-3}. \end{aligned}$$

□

(c) Compute $\mathbb{P}(0 < X < 1)$.

Solution.

$$\begin{aligned} \mathbb{P}(0 < X < 1) &= \int_{0}^{1} 3e^{-3x}dx \\ &= -e^{-3x}\Big|_0^{1} \\ &= 1 - e^{-3}. \end{aligned}$$

□

(d) Compute $\mathbb{P}(X < 5)$.

Solution.

$$\begin{aligned} \mathbb{P}(X < 5) &= \int_{0}^{5} 3e^{-3x}dx \\ &= -e^{-3x}\Big|_0^{5} \\ &= 1 - e^{-15}. \end{aligned}$$

□

(e) Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Solution.

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_{0}^{\infty} 3xe^{-3x} dx \\
&= -xe^{-3x}\Big|_0^{\infty} + \int_0^{\infty} e^{-3x} dx \\
&= -e^{-3x}\left(x + \frac{1}{3}\right)\Big|_0^{\infty} \\
&= \frac{1}{3}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \int_0^{\infty} 3x^2 e^{-3x} dx - \frac{1}{9} \\
&= -x^2 e^{-3x}\Big|_0^{\infty} + \frac{2}{3}\mathbb{E}[X] - \frac{1}{9} = \frac{1}{9}.
\end{aligned}$$

$\square$

Question 3 Let X be a continuous random variable with a cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < \sqrt{2} \\ x^2 - 2 & \text{if } \sqrt{2} \leq x < \sqrt{3} \\ 1 & \text{if } \sqrt{3} \leq x \end{cases}$$

(a) Find $\mathbb{P}(X = 1.6)$.

Solution. $\mathbb{P}(X = 1.6) = F(1.6) - F(1.6) = 0$. $\square$

(b) Compute $\mathbb{P}(1 \leq X \leq \frac{3}{2})$.

Solution. $\mathbb{P}(1 \leq X \leq \frac{3}{2}) = F(\frac{3}{2}) - F(1) = \frac{1}{4}$. $\square$

(c) Compute $\mathbb{P}(1 < X \leq \frac{3}{2})$.

Solution. $\mathbb{P}(1 < X \leq \frac{3}{2}) = F(\frac{3}{2}) - F(1) = \frac{1}{4}$. $\square$

(d) Find the probability density function of X.

Solution. Let $p(x)$ be the probability density function of X.

$$\begin{aligned} p(x) &= \frac{d}{dx} F(x) \\ &= \begin{cases} 2x, & x \in [\sqrt{2}, \sqrt{3}) \\ 0, & x \notin [\sqrt{2}, \sqrt{3}). \end{cases} \end{aligned}$$

$\square$

(e) Find the smallest interval $[a, b]$ such that $\mathbb{P}(a \leq X \leq b) = 1$

Solution. We first notice that $F(x)$ is an increasing function and $0 \leq F(x) \leq 1$. Since $F(x) = 0$ if $x \leq \sqrt{2}$ and $F(x) = 1$ if $x \geq \sqrt{3}$, the smallest interval $[a, b]$ is $[\sqrt{2}, \sqrt{3}]$. $\square$

Question 4 Let X be a continuous random variable with a density function given by

$$f_X(x) = \begin{cases} \frac{1}{2}x^{-\frac{3}{2}} & \text{if } 1 < x < \infty \\ 0 & \text{otherwise} \end{cases}.$$

Compute $\mathbb{E}[X]$, $\text{Var}(X)$, and $\mathbb{E}[X^{\frac{1}{4}}]$.

Solution.

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_{1}^{\infty} \frac{1}{2} x^{-\frac{1}{2}} dx \\ &= \left[x^{\frac{1}{2}} \right]_1^{\infty} \to \infty. \end{aligned}$$

Thus, $\mathbb{E}[X]$ is divergent.

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= \int_{1}^{\infty} \frac{1}{2} x^{\frac{1}{2}} dx - \mathbb{E}[X]^2 \\ &= \left[\frac{1}{3} x^{\frac{3}{2}} \right]_1^{\infty} - \left(\left[x^{\frac{1}{2}} \right]_1^{\infty} \right)^2 \\ &= \lim_{x \to \infty} \frac{1}{3} x^{\frac{3}{2}} - x + 2x^{\frac{1}{2}} - \frac{4}{3} \to \infty \end{aligned}$$

Thus, $\text{Var}(X)$ is divergent.

$$\begin{aligned} \mathbb{E}[X^{\frac{1}{4}}] &= \int_{-\infty}^{\infty} x^{\frac{1}{4}} f_X(x) dx \\ &= \int_{1}^{\infty} \frac{1}{2} x^{-\frac{5}{4}} dx \\ &= \left[-2x^{-\frac{1}{4}} \right]_1^{\infty} = 2. \end{aligned}$$

□

Question 5 Let $X \sim \mathrm{Unif}([a,b])$. Compute its expectation and variance.

Solution.

$$\begin{aligned}
\mathbb{E}[X] &= \int_a^b \frac{x}{b-a} dx \\
&= \left. \frac{x^2}{2(b-a)} \right|_a^b \\
&= \frac{b+a}{2}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= -\left(\frac{b+a}{2}\right)^2 + \int_a^b \frac{x^2}{b-a} dx \\
&= \left. \frac{x^3}{3(b-a)} \right|_a^b - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{b^3-a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4} \\
&= \frac{(b-a)^2}{12}.
\end{aligned}$$

$\square$

Question 6 Show that $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

Proof. Let $p(x)$ be the density function of X, and let $\mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} xp(x)dx$. Thus,

$$\begin{aligned}
\text{Var}(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 p(x)dx \\
&= \int_{-\infty}^{\infty} x^2 p(x) - 2\mu x p(x) + \mu^2 p(x)dx \\
&= \int_{-\infty}^{\infty} x^2 p(x)dx - 2\mu \int_{-\infty}^{\infty} xp(x)dx + \mu^2 \int_{-\infty}^{\infty} p(x)dx \\
&= \mathbb{E}[X^2] - 2\mu^2 + \mu^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2.
\end{aligned}$$

$\square$

Question 7 Let X be a continuous random variable with a cumulative distribution function given by

$$F(x) = \begin{cases} \frac{x}{x+1} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

(a) Find the probability density function of X.

Solution. Let $f(x)$ be the probability density function of X.

$$\begin{aligned} f(x) &= \frac{d}{dx} F(x) \\ &= \begin{cases} \frac{1}{(x+1)^2} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}. \end{aligned}$$

□

(b) Calculate $\mathbb{E}[(1+X)^2 e^{-2X}]$.

Solution.

$$\begin{aligned} \mathbb{E}[(1+X)^2 e^{-2X}] &= \int_{-\infty}^{\infty} (1+x)^2 e^{-2x} f(x) dx \\ &= \int_0^{\infty} \frac{(1+x)^2 e^{-2x}}{(x+1)^2} dx \\ &= \int_0^{\infty} e^{-2x} dx \\ &= \left[-\frac{1}{2} e^{-2x} \right]_0^{\infty} = \frac{1}{2}. \end{aligned}$$

□

Question 8 Let $X \sim \text{Geom}(p)$ be a geometric random variable with parameter p. Compute its variance.

Solution. By the law of total expectation,

$$\begin{aligned}\mathbb{E}[X] &= p \cdot 1 + (1-p)(\mathbb{E}[X]+1) \\ &= \mathbb{E}[X](1-p) + 1 \\ &= \frac{1}{p}.\end{aligned}$$

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= \mathbb{E}[X(X-1)] + \frac{1}{p} - \frac{1}{p^2} \\ &= \frac{1}{p} - \frac{1}{p^2} + \sum_{n=1}^{\infty} n(n-1)p(1-p)^{n-1} \\ &= \frac{1}{p} - \frac{1}{p^2} + p(1-p)\sum_{n=0}^{\infty} n(n-1)(1-p)^{n-2} \\ &= \frac{1}{p} - \frac{1}{p^2} + p(1-p)\sum_{n=0}^{\infty} \frac{d^2}{d(1-p)^2}(1-p)^n \\ &= \frac{1}{p} - \frac{1}{p^2} + p(1-p)\left(\frac{d^2}{d(1-p)^2}\frac{1}{p}\right) \\ &= \frac{1}{p} - \frac{1}{p^2} + \frac{2(1-p)}{p^2} \\ &= \frac{1-p}{p^2}.\end{aligned}$$

$\square$