MATH 140B: Homework #2

Due on Apr 19, 2024 at 23:59pm

Professor Seward

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Problem 1

Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x).$$

Show by example that the limit may exist even if $f''(x)$ does not.

Proof. Put $g(h) = f(x+h)+f(x-h)-2f(x)$. Since g is differentiable in a neighborhood of x and $g(h) \to 0$ as $h \to 0$, we may apply the L'Hospotal's Rule and get

$$\begin{aligned}
\lim_{h\to 0} \frac{g(h)}{h^2} &= \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} \\
&= \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{2h} - \lim_{h\to 0} \frac{f'(x-h)-f'(x)}{2h} \\
&= \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{2h} - \lim_{k\to 0} \frac{f'(x+k)-f'(x)}{-2k} \\
&= \frac{f''(x)}{2} + \frac{f''(x)}{2} = f''(x).
\end{aligned}$$

Consider $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$. f is not continuous at 0, so $f''(0)$ does not exist. But then $f(h)+f(-h)-2f(0) = 0$ for all $h > 0$, so $\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$ exists. ☐

Problem 2

Suppose $a \in \mathbb{R}^1$, f is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a, ∞). Prove that

$$M_1^2 \leq 4M_0M_2. \tag{1}$$

Does $M_1^2 \leq 4M_0M_2$ hold for vector-valued functions too?

Proof. Let $x \in (a, \infty)$. Put $h > 0$. By Taylor's Theorem, there eixsts $t \in (x, x + 2h)$ such that

$$f(x + 2h) = f(x) + 2hf'(x) + 2h^2 f''(t),$$

that is,

$$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] + hf''(t).$$

But then

$$-\frac{M_0}{h} - hM_2 \leq f'(x) \leq \frac{M_0}{h} + hM_2.$$

It follows that

$$M_1^2 \leq \left(\frac{M_0}{h} + hM_2\right)^2 = \left(\frac{M_0^2}{h^2} + h^2M_2^2\right) + 2M_0M_2 \leq 4M_0M_2,$$

as $\frac{M_0^2}{h^2} + h^2M_2^2 \geq 2M_0M_2$ by AM-GM.

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1 & x \in (-1, 0) \\ \frac{x^2-1}{x^2+1} & x \in [0, \infty) \end{cases}.$$

we know

$$f'(x) = \begin{cases} 4x & x \in (-1, 0) \\ \frac{4x}{(x^2+1)^2} & x \in [0, \infty) \end{cases}, \; f''(x) = \begin{cases} 4 & x \in (-1, 0) \\ \frac{4(-3x^2+1)}{(x^2+1)^3} & x \in [0, \infty) \end{cases}, \; f'''(x) = \begin{cases} 0 & x \in (-1, 0) \\ \frac{48x(x^2-1)}{(x^2+1)^4} & x \in [0, \infty) \end{cases}$$

Since $f' < 0$ when $x < 0$ but $f' > 0$ when $x > 0$, $f(x)$ monotonically decreases from 1 to -1 then monotonically approaches 1, and thus $M_0 = 1$.

When $x < 0$, since $f'' > 0$, f' monotonically increases from -4 to 0. Notice that $\frac{4(-3x^2+1)}{(x^2+1)^3} = 0$ has a single positive root at $x = \frac{1}{\sqrt{3}}$. Since $f'(0) = 0$, $f'(1/\sqrt{3}) = \frac{3\sqrt{3}}{4}$, and $\lim_{x\to\infty} f'(x) = 0$, $|f'(x)| \leq \frac{3\sqrt{3}}{4} < 4$ for nonnegative x. Hence, $M_1 = 4$.

Notice that $f'''(x) = 0$ has a single positive root at $x = 1$. But then $f''(0) = 4, f''(1) = -1, \lim_{x\to\infty} f''(x) = 0$, so $M_2 = 4$.

Therefore, the equality of (1) holds for this example.

We now show that (1) also holds for vector valued functions. Let $f'(x) = (f_1(x), \ldots, f_n(x))$ be a twice differentiable vector valued function on (a, ∞). Let M_0^f, M_1^f, M_2^f be the least upper bounds of $\|f(x)\|$, $\|f'(x)\|$, $\|f''(x)\|$, respectively. Pick $\epsilon > 0$. There exists $c \in \mathbb{R}$ such that $\|f'(c)\| \geq M_1^f - \epsilon$. Let $u = \frac{f'(c)}{\|f'(c)\|}$ and define $g(x) = u \cdot f(x)$. Let M_0^g, M_1^g, M_2^g be the least upper bounds of $|g(x)|$, $|g'(x)|$, $|g''(x)|$, respectively. We know

$$M_1^g \geq g'(c) = u \cdot f'(c) = \|f'(c)\| \geq M_1^f - \epsilon,$$

for arbitrary ϵ, and thus $M_1^g \geq M_1^f$. But then by Cauchy-Schwarz inequality,

$$g(x)^2 \leq \|u\|\|f(x)\|^2 \leq M_0, \quad g'(x)^2 \leq \|u\|\|f''(x)\|^2 \leq M_2,$$

as $\|u\| = 1$. Hence, applying (1) on g, we get $M_1^f \leq M_1^g \leq 2\sqrt{M_0^g M_2^g} \leq 2\sqrt{M_0^f M_2^f}$. ∎

Problem 3

Suppose f is a real function on $(-\infty, \infty)$. Call x a fixed point of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

Proof. Suppose for contradiction that f has multiple fixed points, say x, y, $x < y$. By MVT, there exists $t \in (x, y)$ such that

$$f(y) - f(x) = x - y = (x - y)f'(t).$$

But then $f'(t) = 1$, contradiction. □

(b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

Proof. We can easily see that

$$f'(t) = 1 + \frac{-e^t}{(1 + e^t)^2}.$$

Since $e^t, (1 + e^t)^2 > 0$ and $e^t < (1 + e^t)^2$, we have $0 < \frac{e^t}{(1+e^t)^2} < 1$, and so $0 < f'(t) < 1$.

Suppose t is a fixed point of f, which implies $t + (1+e^t)^{-1} = t$. But then $(1+e^t)^{-1} = 0$, contradiction. □

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point of f exists, and that $x = \lim_{n\to\infty} x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \ldots$.

Proof. Since $x_{n+1} = f(x_n)$ and $x_n = f(x_{n-1})$, by MVT,

$$|f(x_n) - f(x_{n-1})| = |x_{n+1} - x_n| = |f'(t)(x_n - x_{n-1})| \leq |f'(t)||(x_n - x_{n-1})| \leq A|x_n - x_{n-1}|,$$

for some t, and thus $|x_{n+1} - x_n| \leq A^{n-1}|x_2 - x_1|$. But then for $m, n \geq N$,

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \\ &= (x_2 - x_1) \sum_{k=n-1}^{m-2} A_k \\ &\leq (x_2 - x_1) \sum_{k=N}^{\infty} A_k \leq \frac{|x_2 - x_1|A^N}{1 - A}. \end{aligned}$$

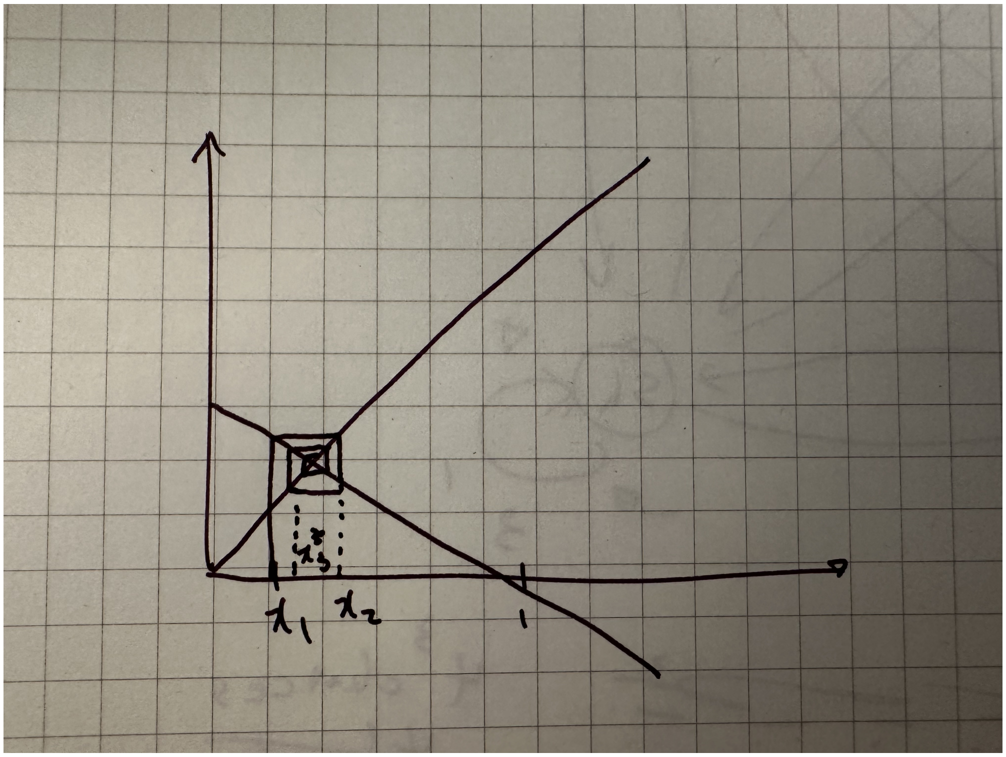
As $A < 1$, $|x_m - x_n| \to 0$ as $N \to \infty$. Therefore, (x_n) is a Cauchy sequence in the reals, which converges to some x. But then $f(x) = \lim\limits_{n\to\infty} f(x_n) = x_{n+1} = x$, so x is a fixed point. □

Problem 3 continued on next page. . .

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \ldots.$$

Proof. Take $f(x) = \frac{1-x}{2}$ and consider the following diagram:



$\square$

Problem 4

Suppose α increases on $[a,b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f\, d\alpha = 0$.

Proof. Pick arbitrary $\epsilon > 0$. We first note that the infimum of $f(x)$ over any interval in $[a,b]$ is 0, so $L(P,f,\alpha) = 0$. Since α is continuous at x_0, there exists $\delta > 0$ such that $|\alpha(x) - \alpha(x_0)| < \epsilon/2$ whenever $|x - x_0| < \delta$. Consider the partition $P = \{a, x_0 - \delta', x_0 + \delta', b\}$, where $0 < \delta' < \min\{\delta, x_0 - a, b - x_0\}$. We then have

$$\begin{aligned} U(P,f,\alpha) &= \alpha(x_0 + \delta') - \alpha(x_0 - \delta') \\ &= (\alpha(x_0 + \delta') - \alpha(x_0)) + (\alpha(x_0) - \alpha(x_0 - \delta')) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, $U(P,f,\alpha) - L(P,f,\alpha) = \epsilon$, and so $f \in \mathscr{R}(\alpha)$ by Theorem 6.6. Since $L(P,f,\alpha) \leq \int f d\alpha \leq U(P,f,\alpha)$, we have $\int f d\alpha = 0$. $\square$

Problem 5

Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof. Suppose for the sake of contradicion that there exists some $x_0 \in [a, b]$ with $f(x_0) = \epsilon$, for some $\epsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all $|x - x_0| < \delta$. Consider the partition $P = \{a, x_0 - \delta', x_0 + \delta', b\}$, where $0 < \delta' < \min\{\delta, x_0 - a, b - x_0\}$. We know $m = \inf f(x) > 0$, for $x \in (x_0 - \delta', x_0 + \delta')$. But then $L(P, f) \geq 2\delta' m > 0$, which forces $\underline{\int_a^b} f(x)dx > 0$, contradiction. □

Problem 6

Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = 1/2$. Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+)$ equals $f(0)$ and that then

$$\int f \, d\beta_1 = f(0).$$

Proof. Suppose $f \in \mathscr{R}(\beta_1)$. Pick $\epsilon > 0$. There exists partition P such that

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon.$$

Let P^* be a refinement which contains 0. Let $\delta \in P^*$ such that $[0, \delta]$ is an interval given by the partition P. Then, $U(P^*, f, \beta_1) - L(P^*, f, \beta_1) = \sup f(x) - \inf f(x) < \epsilon$, $x \in [0, \delta]$. But then, $|f(t) - f(0)| < \epsilon$ whenever $t \in (0, \delta)$. Hence, $f(0+) = f(0)$.

We now suppose $f(0+) = f(0)$. Pick $\epsilon > 0$. There exists $\delta > 0$ such that $|f(t) - f(0)| < \epsilon/2$ whenever $t \in (0, \delta)$. Let $\delta' < \min(1, \delta)$ be positive. Consider the partition $P = \{-1, 0, \delta', 1\}$. Then,

$$U(P, f, \beta_1) - L(P, f, \beta_1) = f(s) - f(t) \le |f(s) - f(0)| + |f(t) - f(0)| < \epsilon,$$

for some $s, t \in [0, \delta']$. Hence, by Theorem 6.6, $f \in \mathscr{R}(\beta_1)$. Note that for any P which contains 0, we have $U(P, f, \beta_1) = M$ and $L(P, f, \beta_1) = m$, where $M = \sup_{x \in (0, \delta')} f(x)$ and $m = \inf_{x \in (0, \delta')} f(x)$. But then $M < f(0) + \epsilon$ and $m > f(0) - \epsilon$. Hence,

$$f(0) - \epsilon < L(P, f, \beta_1) \le \int f \, d\beta_1 \le U(P, f, \beta_1) < f(0) + \epsilon,$$

for arbitrary ϵ, and the result follows. $\square$

(b) State and prove a similar result for β_2.

Proof. We show that $f \in \mathscr{R}(\beta_2)$ if and only if $f(0-)$ equals $f(0)$ and that then $\int f \, d\beta_2 = f(0)$.

Suppose $f \in \mathscr{R}(\beta_2)$. Pick $\epsilon > 0$. There exists partition P such that

$$U(P, f, \beta_2) - L(P, f, \beta_2) < \epsilon.$$

Let P^* be a refinement which contains 0. Let $-\delta \in P^*$ such that $[-\delta, 0]$ is an interval given by the partition P. Then, $U(P^*, f, \beta_2) - L(P^*, f, \beta_2) = \sup f(x) - \inf f(x) < \epsilon$, $x \in [-\delta, 0]$. But then, $|f(t) - f(0)| < \epsilon$ whenever $t \in (-\delta, 0)$. Hence, $f(0-) = f(0)$.

We now suppose $f(0-) = f(0)$. Pick $\epsilon > 0$. There exists $\delta > 0$ such that $|f(t) - f(0)| < \epsilon/2$ whenever $t \in (-\delta, 0)$. Let $\delta' < \min(1, \delta)$ be positive. Consider the partition $P = \{-1, -\delta', 0, 1\}$. Then,

$$U(P, f, \beta_2) - L(P, f, \beta_2) = f(s) - f(t) \le |f(s) - f(0)| + |f(t) - f(0)| < \epsilon,$$

for some $s, t \in [0, \delta']$. Hence, by Theorem 6.6, $f \in \mathscr{R}(\beta_2)$. Note that for any P which contains 0, we have $U(P, f, \beta_2) = M$ and $L(P, f, \beta_2) = m$, where $M = \sup_{x \in (\delta', 0)} f(x)$ and $m = \inf_{x \in (\delta', 0)} f(x)$. But then $M < f(0) + \epsilon$ and $m > f(0) - \epsilon$. Hence,

$$f(0) - \epsilon < L(P, f, \beta_2) \le \int f \, d\beta_2 \le U(P, f, \beta_2) < f(0) + \epsilon,$$

for arbitrary ϵ, and the result follows. $\square$

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.

Proof. Suppose $f \in \mathscr{R}(\beta_3)$. Pick $\epsilon > 0$. There exists partition P such that

$$U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon.$$

Let P^* be a refinement which contains 0. Let $[x_i, 0]$, $[0, x_{i+1}]$ be the intervals given by P^* which contains 0. Then,

$$U(P^*, f, \beta_3) - L(P^*, f, \beta_3) = \frac{1}{2}\left(\sup_{x \in [x_i, 0]} f(x) - \inf_{x \in [x_i, 0]} f(x) + \sup_{x \in [0, x_{i+1}]} f(x) - \inf_{x \in [0, x_{i+1}]} f(x)\right) < \epsilon/2$$

But then, $|f(t) - f(0)| < \epsilon$ whenever $t \in (-\delta, \delta)$, where $\delta = \min(|x_i|, |x_{i+1}|)$. Hence, f is continuous at 0.

We now suppose f is continuous at 0. Pick $\epsilon > 0$. There exists $\delta > 0$ such that $|f(t) - f(0)| < \epsilon/2$ whenever $t \in (-\delta, \delta)$. Let $\delta' < \min(1, \delta)$ be positive. Consider the partition $P = \{-1, -\delta', \delta', 1\}$. Then,

$$U(P, f, \beta_3) - L(P, f, \beta_3) = f(s) - f(t) \leq |f(s) - f(0)| + |f(t) - f(0)| < \epsilon,$$

for some $s, t \in [-\delta', \delta']$. Hence, by Theorem 6.6, $f \in \mathscr{R}(\beta_2)$. □

(d) If f is continuous at 0 prove that

$$\int f \, d\beta_1 = \int f \, d\beta_2 = \int f \, d\beta_3 = f(0).$$

Proof. We have already shown $\int f \, d\beta_1 = \int f \, d\beta_2 = f(0)$, from (a), (b). It remains show $\int f \, d\beta_3 = f(0)$. Pick $\epsilon > 0$. There exists $\delta > 0$ such that $|f(t) - f(0)| < \epsilon/2$ whenever $|t| < \delta$. But then for any P which contains $-\delta, 0, \delta$, we have $U(P, f, \beta_3) < f(0) + \epsilon$ and $L(P, f, \beta_3) > f(0) - \epsilon$. Hence,

$$f(0) - \epsilon < L(P, f, \beta_3) \leq \int f \, d\beta_3 \leq U(P, f, \beta_3) < f(0) + \epsilon,$$

for arbitrary ϵ, and the result follows. □

Problem 7

If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$.

Proof. Take any partition $P = \{x_0 = a, \ldots, x_n = b\}$. Notice that there exists an irrational in any interval, so $L(P, f) = 0$. But then $\mathbb{Q}$ is dense in $\mathbb{R}$, so for any distinct x_i, x_{i+1}, there exists $q \in \mathbb{Q}$ such that $x_i < q < x_{i+1}$. But then $U(P, f) = \sum_{i=1}^{n}(x_i - x_{i-1}) = b - a > 0$. Hence, $\inf U(P, f) = b - a \neq 0 = \sup L(P, f)$, and the result now follows. $\square$