

# MATH 140B: Homework #4

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*Professor Seward*

**Ray Tsai**

A16848188

## Problem 1

Show that integration by parts can sometimes be applied to the “improper” integrals defined in Exercises 6.7 and 6.8. (State the appropriate hypotheses, formulate a theorem, and prove it.) For instance, show that

$$\int_0^\infty \frac{\cos x}{1+x} dx = \int_0^\infty \frac{\sin x}{(1+x)^2} dx.$$

Show that one of these integrals converges absolutely, but that the other does not.

**Theorem** Let  $F, G$  be differentiable functions on  $[a, \infty)$ , where  $F' = f \in \mathcal{R}$  and  $G' = g \in \mathcal{R}$ . Suppose both  $\lim_{x \rightarrow \infty} F(x)G(x)$  and  $\int_a^\infty f(x)G(x) dx$  exist. Then

$$\int_a^\infty F(x)g(x) dx = \lim_{x \rightarrow \infty} F(x)G(x) - F(a)G(a) - \int_a^\infty f(x)G(x) dx.$$

*Proof.* Put  $H(x) = F(x)G(x)$ . By Theorem 6.13, we know  $H' \in \mathcal{R}$ . For finite  $b > a$ , applying Theorem 6.21 to  $H$  and its derivative yields

$$H(b) - H(a) = \int_a^b F(x)g(x) + f(x)G(x) dx,$$

that is,

$$\int_a^b F(x)g(x) dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x) dx.$$

But then by assumption,  $\lim_{x \rightarrow \infty} F(x)G(x)$  and  $\int_a^\infty f(x)G(x) dx$  exist, and thus  $\int_a^\infty F(x)g(x) dx$  also converges.  $\square$

Put  $F(x) = \frac{1}{1+x}$  and  $G(x) = \sin x$ . We know  $f(x) = -\frac{1}{(1+x)^2} \in \mathcal{R}$ ,  $g(x) = \cos x$ . Note that

$$\lim_{x \rightarrow \infty} |F(x)G(x)| = \lim_{x \rightarrow \infty} \left| \frac{\sin x}{1+x} \right| \leq \lim_{x \rightarrow \infty} \left| \frac{1}{1+x} \right| = 0 = F(0)G(0).$$

By exercise 6.8, we know that  $\int_0^\infty \left| \frac{\sin x}{(1+x)^2} \right| dx$  converges as  $\sum_{n=0}^\infty \frac{|\sin x|}{(1+n)^2}$  converges by comparison test with  $\sum_{n=0}^\infty \frac{1}{(1+n)^2}$ . Hence, again by exercise 6.8,  $\int_0^\infty \frac{\sin x}{(1+x)^2} dx$  also converges, as  $\sum_{n=0}^\infty \frac{\sin x}{(1+n)^2}$  converges absolutely. Since the hypothesis holds, we may apply our theorem stated above and get

$$\int_0^\infty \frac{\cos x}{1+x} dx = \lim_{x \rightarrow \infty} \frac{\sin x}{1+x} - \frac{\sin 0}{1} + \int_0^\infty \frac{\sin x}{(1+x)^2} dx = \int_0^\infty \frac{\sin x}{(1+x)^2} dx.$$

To see that  $\int_0^\infty \frac{\cos x}{1+x} dx$  does not converge absolutely, we again apply exercise 6.8. Since

$$\sum_{n \geq 0} \left| \frac{\cos x}{1+x} \right| \geq \sum_{n \geq 0} \frac{1}{1+x}$$

diverges,  $\int_0^\infty \left| \frac{\cos x}{1+x} \right| dx$  also diverges.

## Problem 2

Let  $\alpha$  be a fixed increasing function on  $[a, b]$ . For  $u \in \mathcal{R}(\alpha)$ , define

$$\|u\|_2 = \left( \int_a^b |u|^2 \right)^{1/2}.$$

Suppose  $f, g, h \in \mathcal{R}(\alpha)$ , and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

*Proof.*

$$\begin{aligned} \|f - h\|_2 &= \|f - g + g - h\|_2 \\ &= \left( \int_a^b |f - g + g - h|^2 \right)^{1/2} \\ &= \left( \int_a^b |f - g|^2 + 2 \int_a^b |(f - g)(g - h)| + \int_a^b |g - h|^2 \right)^{1/2} \\ &\leq \left( \int_a^b |f - g|^2 + 2 \int_a^b |f - g| \int_a^b |g - h| + \int_a^b |g - h|^2 \right)^{1/2} \\ &= \left( \int_a^b |f - g|^2 \right)^{1/2} + \left( \int_a^b |g - h|^2 \right)^{1/2} \\ &= \|f - g\|_2 + \|g - h\|_2. \end{aligned}$$

□

### Problem 3

With the notations of Exercise 6.11, suppose  $f \in \mathcal{R}(\alpha)$  and  $\epsilon > 0$ . Prove that there exists a continuous function  $g$  on  $[a, b]$  such that  $\|f - g\|_2 < \epsilon$ .

*Proof.* Pick  $\epsilon > 0$ . Since  $f \in \mathcal{R}(\alpha)$ , there exists a partition  $P = \{x_0, \dots, x_n\}$  on  $[a, b]$  such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon^2/2M.$$

Suppose  $|f| < M$ . Define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i).$$

if  $x_{i-1} \leq t \leq x_i$ . Note that  $g$  is defined to be linear on every interval  $[x_i, x_{i+1}]$ , and  $g$  remains continuous between neighboring intervals. Hence,  $g$  is continuous on  $[a, b]$ . In addition, on every interval  $[x_i, x_{i+1}]$ , since  $g(t)$  is between  $f(x_i)$  and  $f(x_{i+1})$ , we have  $m_i \leq g(t) \leq M_i$  for all  $t \in [x_i, x_{i+1}]$ . But then

$$\begin{aligned} \|f - g\|_2^2 &= \int_a^b |f - g|^2 \\ &\leq U(P, |f - g|^2, \alpha) \\ &= \sum_{i=1}^n \sup_{x \in [x_i, x_{i+1}]} (f(x) - g(x))^2 \Delta \alpha_i \\ &\leq \sum_{i=1}^n (M_i - m_i)^2 \Delta \alpha_i \\ &\leq 2M[U(P, f, \alpha) - L(P, f, \alpha)] < \epsilon^2, \end{aligned}$$

and thus  $\|f - g\|_2 < \epsilon$ . □

## Problem 4

Define

$$f(x) = \int_x^{x+1} \sin(t^2) dt.$$

(a) Prove that  $|f(x)| < \frac{1}{x}$  if  $x > 0$ .

*Proof.* By Theorem 6.17 and 6.19, we may substitute  $t^2$  by  $u$  and get

$$f(x) = \int_{x^2}^{(x+1)^2} \sin(u) du^{1/2} = \int_{x^2}^{(x+1)^2} \frac{\sin(u)}{2u^{1/2}} du.$$

Put  $F(x) = \frac{1}{2u^{1/2}}$  and  $G(x) = -\cos(x)$ . Applying Theorem 6.22 then yields

$$f(x) = \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \frac{1}{2} \int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du.$$

But then notice

$$\left| \int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du \right| \leq \int_{x^2}^{(x+1)^2} \left| \frac{\cos u}{2u^{3/2}} \right| du < \int_{x^2}^{(x+1)^2} \frac{1}{2u^{3/2}} du = \frac{1}{x} - \frac{1}{x+1}.$$

Hence,

$$\begin{aligned} |f(x)| &\leq \left| \frac{\cos(x^2)}{2x} \right| + \left| \frac{\cos[(x+1)^2]}{2(x+1)} \right| + \left| \frac{1}{2} \int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du \right| \\ &< \frac{1}{2x} + \frac{1}{2(x+1)} + \frac{1}{2} \left( \frac{1}{x} - \frac{1}{x+1} \right) = \frac{1}{x}. \end{aligned}$$

□

(b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where  $|r(x)| < \frac{c}{x}$  and  $c$  is a constant.

*Proof.* By (a),

$$\begin{aligned} 2xf(x) &= \left( \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \frac{1}{2} \int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du \right) \\ &= \cos(x^2) - \frac{x}{(x+1)} \cdot \cos[(x+1)^2] - x \int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du \\ &< \cos(x^2) - \cos[(x+1)^2] + \frac{\cos[(x+1)^2]}{x+1} + \frac{1}{x+1}, \end{aligned}$$

and thus

$$|r(x)| < \left| \frac{\cos[(x+1)^2] + 1}{x+1} \right| < \frac{2}{x}.$$

□

(c) Does  $\int_0^\infty \sin(t^2) dt$  converge?

*Proof.*

$$\begin{aligned}
 \int_0^\infty \sin(t^2) dt &= \sum_{x=0}^\infty f(x) \\
 &= f(0) + \sum_{x=1}^\infty \frac{\cos(x^2)}{2x} - \sum_{x=1}^\infty \frac{\cos[(x+1)^2]}{2x} + \sum_{x=1}^\infty \frac{r(x)}{2x} \\
 &= f(0) + \sum_{x=1}^\infty \frac{\cos(x^2)}{2x} - \sum_{x=1}^\infty \frac{x+1}{x} \cdot \frac{\cos[(x+1)^2]}{2(x+1)} + \sum_{x=1}^\infty \frac{r(x)}{2x} \\
 &= f(0) + \sum_{x=1}^\infty \frac{\cos(x^2)}{2x} - \sum_{x=2}^\infty \frac{x}{x-1} \cdot \frac{\cos(x^2)}{2x} + \sum_{x=1}^\infty \frac{r(x)}{2x} \\
 &= f(0) + \frac{\cos 1}{2} + \sum_{x=2}^\infty \frac{\cos(x^2)}{2x(1-x)} + \sum_{x=1}^\infty \frac{r(x)}{2x}.
 \end{aligned}$$

But then

$$\begin{aligned}
 \sum_{x=1}^\infty \frac{|r(x)|}{2x} &< \sum_{x=1}^\infty \frac{1}{x^2}, \\
 \sum_{x=2}^\infty \left| \frac{\cos(x^2)}{2x(1-x)} \right| &< \sum_{x=2}^\infty \left| \frac{1}{2x(1-x)} \right| \\
 &= \frac{1}{2} \sum_{x=2}^\infty \frac{1}{x(x-1)} \\
 &< \frac{1}{2} \sum_{x=1}^\infty \frac{1}{x^2},
 \end{aligned}$$

and thus both series converge by comparison test. Since all terms of  $f(0) + \frac{\cos 1}{2} + \sum_{x=2}^\infty \frac{\cos(x^2)}{2x(1-x)} + \sum_{x=1}^\infty \frac{r(x)}{2x}$  converge,  $\int_0^\infty \sin(t^2) dt$  converges.  $\square$

## Problem 5

Suppose  $f$  is a real, continuously differentiable function on  $[a, b]$ ,  $f(a) = f(b) = 0$ , and

$$\int_a^b f^2(x) dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x) dx \geq \frac{1}{4}.$$

*Proof.* By Theorem 6.22,

$$\begin{aligned} \int_a^b x f(x) f'(x) dx &= b f(b) - a f(a) - \int_a^b f(x) (f(x) + x f'(x)) \\ &= b f(b) - a f(a) - \int_a^b f^2(x) dx - \int_a^b x f(x) f'(x) dx. \end{aligned}$$

But then

$$2 \int_a^b x f(x) f'(x) dx = b f(b) - a f(a) - 1 = -1,$$

and the result follows.

It now follows from Hölder's inequality that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x) dx \geq \left( \int_a^b x f(x) f'(x) dx \right)^2 = \frac{1}{4}.$$

□

## Problem 6

Suppose  $\alpha$  increases monotonically on  $[a, b]$ ,  $g$  is continuous, and  $g(x) = G'(x)$  for  $a \leq x \leq b$ . Prove that

$$\int_a^b \alpha(x)g(x) dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

*Proof.* Let  $P = \{x_0, x_1, \dots, x_n\}$  be any partition on  $[a, b]$ . For each segment  $(x_{i-1}, x_i)$ , the mean value theorem furnishes some  $t_i \in (x_{i-1}, x_i)$  such that  $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$ . Since  $\alpha$  increases monotonically,

$$\begin{aligned} \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i &= \sum_{i=1}^n \alpha(x_i)(G(x_i) - G(x_{i-1})) \\ &= G(b)\alpha(b) - G(a)\alpha(a) + \sum_{i=1}^n \alpha(x_{i-1})G(x_{i-1}) - \sum_{i=1}^n \alpha(x_i)G(x_{i-1}) \\ &= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})\Delta \alpha_i, \end{aligned}$$

for any partition  $P$ . Pick  $\epsilon > 0$ . There exist partitions  $P_1, P_2$  such that

$$U(P_1, \alpha(x_i)g(t_i)) - L(P_1, \alpha(x_i)g(t_i)) < \epsilon/2,$$

$$U(P_2, G, \alpha) - L(P_2, G, \alpha) < \epsilon/2.$$

Let  $P = \{x_0, \dots, x_n\} = P_1 \cup P_2$ . By Theorem 6.7,

$$\begin{aligned} &\left| G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha - \int_a^b \alpha(x)g(x) dx \right| \\ &\leq \left| \sum_{i=1}^n G(x_{i-1})\Delta \alpha_i - \int_a^b G d\alpha \right| + \left| \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i - \int_a^b \alpha(x)g(x) dx \right| < \epsilon. \end{aligned}$$

□



## Problem 7

Let  $\gamma_1$  be a curve in  $\mathbb{R}^k$  defined on  $[a, b]$ , let  $\phi$  be a continuous 1-1 mapping of  $[c, d]$  into  $[a, b]$  such that  $\phi(c) = a$ ; and define  $\gamma_2(s) = \gamma_1(\phi(s))$ . Prove that  $\gamma_2$  is an arc, a closed curve, or a rectifiable curve if and only if the same is true of  $\gamma_1$ . Prove that  $\gamma_2$  and  $\gamma_1$  have the same length.

*Proof.* Since  $\phi$  is a continuous 1-1 mapping on a compact space, its inverse mapping  $\psi$  from  $[a, b]$  into  $[c, d]$  is also a continuous mapping, by Theorem 4.17. But then  $\gamma_2(s) = \gamma_1(\phi(s))$  and  $\gamma_2(\psi(t)) = \gamma_1(t)$ , so  $\gamma_1$  is 1-1 if and only if  $\gamma_2$  is. Additionally, since  $\phi$  is a continuous bijection with  $\phi(a) = c$ , we know  $\phi(d) = b$ , and thus  $\gamma_2(c) = \gamma_1(\phi(c)) = \gamma_1(\phi(d)) = \gamma_2(d)$  if and only if  $\gamma_1(a) = \gamma_1(b)$ . Given a partition  $P = \{x_0, \dots, x_n\}$  on  $[c, d]$ ,  $\gamma_1$  yields a partition  $P' = \{y_0, \dots, y_n\}$  on  $[a, b]$ , with  $[x_i, x_{i+1}]$  corresponding to  $[y_i, y_{i+1}]$  for all  $i$ . But then,

$$\Lambda(P, \gamma_1) = \sum_{i=1}^n |\gamma_1(x_i) - \gamma_1(x_{i-1})| = \sum_{i=1}^n |\gamma_2(\psi(x_i)) - \gamma_2(\psi(x_{i-1}))| = \sum_{i=1}^n |\gamma_2(y_i) - \gamma_2(y_{i-1})| = \Lambda(P', \gamma_2).$$

Therefore,  $\gamma_1$  is a rectifiable curve if and only if  $\gamma_2$  is, and

$$\Lambda(\gamma_1) = \sup \Lambda(P, \gamma_1) = \sup \Lambda(P', \gamma_2) = \Lambda(\gamma_2),$$

and the result now follows. □