MATH 140A: Homework #8

Due on Mar 8, 2024 at 23:59pm

Professor Seward

Ray Tsai

A16848188

Problem 1

Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{(1+a_n)}$ diverges.

Proof. Note that if $a_n > 1$, then $\frac{a_n}{a_n+1} = 1 - \frac{1}{a_n+1} > \frac{1}{2}$. On the other hand, if $a_n \leq 1$, we have $\frac{a_n}{a_n+1} \geq \frac{a_n}{2}$. If there are infinitely many n such that $a_n > 1$, then the series obviously diverges, as it would be greater than the sum of infinitely many $\frac{1}{2}$. Hence, we may assume there exists $N \geq 0$ such that $a_n \leq 1$ for all $n \geq N$. But then

$$\sum \frac{a_n}{(1+a_n)} \geq \sum_{n=1}^{N-1} \frac{a_n}{(1+a_n)} + \frac{1}{2} \sum_{n=N}^{\infty} a_n.$$

Since $\sum a_n$ diverges, $\frac{1}{2} \sum_{n=N}^{\infty} a_n$ diverges, by comparison test. The result now follows. □

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

Proof. We first note that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.$$

Fix $\epsilon \in (0,1)$. Since S_n is increasing and unbounded, $\frac{s_N}{s_{N+k}} \to 0$. Hence, we may find large enough k such that $\frac{s_N}{s_{N+k}} < 1 - \epsilon$. But then $\sum_{n=N+1}^{N+k} \frac{a_n}{s_n} \geq \epsilon$, which fails to meet the Cauchy criterion. □

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

Proof. Since

$$\frac{a_n}{s_n^2} \leq \frac{a_n}{s_{n-1}s_n} = \frac{1}{s_{n-1}} - \frac{1}{s_n},$$

the consecutive terms cancel out, and we get $\sum_{n=1}^{N} \frac{a_n}{s_n^2} \leq \sum_{n=2}^{N} \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{a_1} - \frac{1}{s_N}$. But then s_n is increasing and unbounded, and thus

$$\frac{1}{a_1} \leq \lim_{N \to \infty} \sum \frac{a_n}{s_n^2} \leq \lim_{N \to \infty} \frac{1}{a_1} - \frac{1}{s_N} = \frac{1}{a_1}.$$

Hence, the series converges to $\frac{1}{a_1}$. □

Problem 2

Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

Proof. Let $A = \sum_{n=1}^{\infty} a_n$. We know $r_n = A - s_n$, where s_n is the sum of the first $n-1$ terms of a_n. Note that for $n > m$, we have $r_n < r_m$, as $s_n > s_m$. Hence,

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m + \cdots + a_{n-1}}{r_m} = \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m}.$$

Fix

$\lim_{n\to\infty} \sum_{k=m}^{n} \frac{a_n}{r_n}$ □

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.