MATH 180B: Homework #1

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Professor Carfagnini

Ray Tsai

A16848188

Problem 1

Let U, V, and W be independent random variables with equal variance σ^2. Define $X = U+W$ and $Y = V-W$. Find the covariance between X and Y.

Proof.

$$\begin{aligned}
Cov(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[(U+W)(V-W)] - \mathbb{E}[U+W]\mathbb{E}[V-W] \\
&= \mathbb{E}[UV + WV - UW - W^2] - (\mathbb{E}[U] + \mathbb{E}[W])(\mathbb{E}[V] - \mathbb{E}[W]) \\
&= \mathbb{E}[U]\mathbb{E}[V] + \mathbb{E}[W]\mathbb{E}[V] - \mathbb{E}[U]\mathbb{E}[W] - \mathbb{E}[W^2] - \mathbb{E}[U]\mathbb{E}[V] - \mathbb{E}[W]\mathbb{E}[V] + \mathbb{E}[U]\mathbb{E}[W] + \mathbb{E}[W^2] \\
&= 0.
\end{aligned}$$

$\square$

Problem 2

Let X and Y be independent binomial random variables having parameters (N, p) and (M, p), respectively. Let $Z = X + Y$.

(a) Argue that Z has a binomial distribution with parameters $(N+M, p)$ by writing X and Y as appropriate sums of Bernoulli random varaibles.

Proof. Since $\mathbb{P}(X = i) = \binom{N}{i} p^i (1-p)^{N-i}$ and $\mathbb{P}(Y = i) = \binom{M}{i} p^i (1-p)^{M-i}$, X is the sum of N indicators and Y is the sum of M indicators. Hence, we have $Z = X + Y$ as the sum of $M + N$ indicators. □

(b) Validate the results in (a) by evaluating the necessary convolution.

Proof. Since

$$\begin{aligned}
\mathbb{P}(Z = k) &= \sum_{i=0}^{k} \mathbb{P}(X = i)\mathbb{P}(Y = k - i) \\
&= \sum_{i=0}^{k} \binom{N}{i} p^i (1-p)^{N-i} \binom{M}{k-i} p^{k-i} (1-p)^{M-(k-i)} \\
&= p^k (1-p)^{(M+N)-k} \sum_{i=0}^{k} \binom{N}{i} \binom{M}{k-i} \\
&= \binom{M+N}{k} p^k (1-p)^{(M+N)-k},
\end{aligned}$$

Z has a binomial distribution with parameters $(N + M, p)$. □

Problem 3

Let X be a random variable. Recall that the moment generating function (or MGF for short) $M_X(t)$ of X is the function $M_X : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ defined by $t \mapsto \mathbb{E}[e^{tX}]$. Now suppose that $X \sim Gamma(\alpha, \lambda)$, where $\alpha, \lambda > 0$.

1. Prove that
$$M_X(t) = \begin{cases} \left(\frac{\lambda}{\lambda - t}\right)^\alpha & \text{if } t < \lambda; \\ \infty & \text{if } t \geq \lambda. \end{cases}$$

   Proof. Let $u = (\lambda - t)x$. We know $du = (\lambda - t)dx$. Then,
$$\begin{aligned} M_X(t) &= \int_0^\infty \frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}e^{-\lambda x}e^{tx}dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{(t-\lambda)x}dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty \left(\frac{u}{\lambda - t}\right)^{\alpha-1} e^{-u}\frac{du}{\lambda - t} \\ &= \left(\frac{\lambda}{\lambda - t}\right)^\alpha \frac{\int_0^\infty u^{\alpha-1}e^{-u}du}{\Gamma(\alpha)}. \end{aligned}$$

   If $t \geq \lambda$, we get $-u > 0$, so the integral $\int_0^\infty u^{\alpha-1}e^{-u}du$ would approach infinity. Otherwise, $\int_0^\infty u^{\alpha-1}e^{-u}du = \Gamma(\alpha)$, and we get $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha$. □

2. Recall that the MGF contains the information of the moments. In particular, if $m_l(X)$ is the l-th moment of X, then $M_X^{(l)}(0) = m_l(X)$, where $M_X^{(l)}$ denotes the l-th derivative of M_X. Use this to compute the mean and variance of X.

   Proof. Note that
$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{k=0}^\infty \mathbb{E}\left[\frac{(tX)^k}{k!}\right] = \sum_{k=0}^\infty \frac{t^k}{k!}\mathbb{E}[X^k].$$

   Since all the terms after the first one in $M_X^{(l)}$ is multiplied by a power of t, only the first term remains when t is set to 0, and thus $m_l(X) = M_X^{(l)}(0) = \mathbb{E}[X^l]$. To calculate the mean μ and variance σ^2 of X, we only need to calculate $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$, namely $m_1(X)$ and $m_2(X)$. Since $t < \lambda$,
$$m_1(X) = \left.\frac{\alpha\lambda^\alpha}{(\lambda - t)^{\alpha+1}}\right|_{t=0} = \frac{\alpha}{\lambda}$$
$$m_2(X) = \left.\frac{\alpha(\alpha+1)\lambda^\alpha}{(\lambda - t)^{\alpha+2}}\right|_{t=0} = \frac{\alpha(\alpha+1)}{\lambda^2},$$

   and thus $\mu = m_1(X) = \frac{\alpha}{\lambda}$ and $\sigma^2 = m_2(X) - m_1(X)^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha}{\lambda^2}$. □

Problem 4

Suppose that (X_1, X_2) has the bivariate normal distribution with marginals $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ and correlation $Corr(X_1, X_2) = \rho$. Let $Y_1 = 2X_1 + X_2$ and $Y_2 = X_1 - X_2$. Determine the distribution of the random vector (Y_1, Y_2).

Proof. Let $X = (X_1, X_2)^T$, $Y = (Y_1, Y_2)^T$. Note that $Corr(X_1, X_2) = \frac{Cov(X_1,X_2)}{\sigma_1\sigma_2}$, so $Cov(X_1, X_2) = Cov(X_2, X_1) = \rho\sigma_1\sigma_2$. Thus, we get the covariance matrix of X, which is

$$\Sigma_X = \mathbb{E}[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^T] = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$. Since $Y = A^T X$ and X is a bivariate Gaussgian random variable, we get

$$\mu_Y = \mathbb{E}[Y] = A^T\mathbb{E}[X] = A^T(\mu_1, \mu_2)^T = (2\mu_1 + \mu_2, \mu_1 - \mu_2)^T,$$

and

$$\begin{aligned} \Sigma_Y &= \mathbb{E}[(Y - \mathbb{E}[Y])(Y - \mathbb{E}[Y])^T] \\ &= \mathbb{E}[(A^T X - A^T\mathbb{E}[X])(A^T X - A^T\mathbb{E}[X])^T] \\ &= \mathbb{E}[A^T(X - E[X])(X - \mathbb{E}[X])A] \\ &= A^T\mathbb{E}[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^T]A \\ &= A^T\Sigma_X A \\ &= \begin{bmatrix} 4\sigma_1^2 + 5\rho\sigma_1\sigma_2 + \sigma_2^2 & 2\sigma_1^2 - \rho\sigma_1\sigma_2 - \sigma_2^2 \\ 2\sigma_1^2 - 3\rho\sigma_1\sigma_2 - \sigma_2^2 & \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 \end{bmatrix}. \end{aligned}$$

Therefore, $Y \sim \mathcal{N}(\mu_Y, \Sigma_Y)$. □

Problem 5

Let $X \sim \text{Unif}[-1,1]$. Consider the functions $g, h : [-1,1] \to [-1,1]$ given by

$$g(x) = \begin{cases} 1-x & \text{if } x \in [0,1]; \\ x & \text{if } x \in [-1,0), \end{cases}$$

and

$$h(x) = \begin{cases} x & \text{if } x \in [0,1]; \\ -(x+1) & \text{if } x \in [-1,0). \end{cases}$$

(a) Prove that $Y = g(X)$ and $Z = h(X)$ are both uniform $Y, Z \sim \text{Unif}[-1,1]$.

Proof. Let $k \in [-1,1]$, and let $\alpha = \mathbb{P}(X=0)$. Note that $\mathbb{P}(X=x) = \alpha$, for all $x \in [-1,1]$. Suppose that $k \geq 0$. Then, $\mathbb{P}(Y=k) = \mathbb{P}(X = 1-k) = \alpha$ and $\mathbb{P}(Z=k) = \mathbb{P}(X=k) = \alpha$. Suppose that $k < 0$. Then, $\mathbb{P}(Y=k) = \mathbb{P}(X=k) = \alpha$ and $\mathbb{P}(Z=k) = \mathbb{P}(X = -(k+1)) = \alpha$. Since $\mathbb{P}(Y=k) = \mathbb{P}(Z=k) = \alpha$ for all $l \in [-1,1]$, $Y, Z \sim \text{Unif}[-1,1]$. $\square$

(b) Prove that $Cov(X,Y) = Cov(X,Z)$.

Proof. Since

$$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[XY] = \alpha\left(\int_{-1}^{0} x^2 dx + \int_{0}^{1} x(1-x) dx\right) = \frac{\alpha}{2}$$

and

$$Cov(X,Z) = \mathbb{E}[XZ] - \mathbb{E}[X]\mathbb{E}[Z] = \mathbb{E}[XZ] = \alpha\left(\int_{-1}^{0} -(x+1)x dx + \int_{0}^{1} x^2 dx\right) = \frac{\alpha}{2},$$

we get $Cov(X,Y) = Cov(X,Z)$. $\square$

(c) Prove that the random vectors (X,Y) and (X,Z) do not have the same joint distribution. This can be done by finding a subset $B \subset \mathbb{R}^2$ such that

$$\mathbb{P}((X,Y) \in B) \neq \mathbb{P}((X,Z) \in B).$$

Proof. Consider $B = \{(x,x) \mid x \in [0,1]\}$. Since $\mathbb{P}((X,Y) \in B) = 0 \neq \frac{1}{2} = \mathbb{P}((X,Z) \in B)$, (X,Y) and (X,Z) do not have the same joint distribution. $\square$