MATH 140B: Homework #8

Due on Jun 3, 2024 at 23:59pm

Professor Seward

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Problem 1

If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

Proof. Consider the function $f(x) = x - \sin x$. Since $\cos x < 1 = (x)'$ in $(0, \pi/2)$, $f'(x) = x - \cos x > 0$ in $(0, \pi/2)$, so f is strictly increasing in $(0, \pi/2)$. But then $f(x) > f(0) = 0$ for all $x \in (0, \pi/2)$. It now follows that $\frac{\sin x}{x} < 1$.

Now consider $g(x) = \frac{\sin x}{x}$. $g'(x) = \frac{x \cos x - \sin x}{x^2}$. We now show that $x < \tan x = \frac{\sin x}{\cos x}$ in $(0, \pi/2)$. Put $h(x) = \tan x - x$. Since $|\cos x| < 1$ in $(0, \pi/2)$, $h'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} - 1 = \frac{1}{\cos^2 x} - 1 > 1$. But then $h(0) = 0$ and h is strictly increasing, so $\tan x - x > 0$ in $(0, \pi/2)$. It now follows that $g'(x) < \frac{\tan x \cos x - \sin x}{x^2} = 0$ and $g(\pi/2) = \frac{2}{\pi}$, and thus $\frac{2}{\pi} < \frac{\sin x}{x}$ for all $0 < x < \frac{\pi}{2}$. $\square$

Problem 2

For $n = 0, 1, 2, \ldots$ and x real, prove that

$$|\sin nx| \leq n|\sin x|.$$

Proof. We proceed by indcution on n. The base case $n = 0$ is trivial. Suppose $n \geq 1$.

$$\begin{aligned}
|\sin nx| &= \left|\frac{1}{2i}(e^{nix} - e^{-nix})\right| \\
&= \left|\frac{1}{2i}[(e^{(n-1)ix} - e^{-(n-1)ix})(e^{ix} + e^{-ix}) + (e^{(n-1)ix} + e^{-(n-1)ix})(e^{ix} - e^{-ix})]\right| \\
&= [\sin(n-1)x \cdot \cos x + \cos(n-1)x \cdot \sin x] \\
&\leq |\sin(n-1)x \cdot \cos x| + |\cos(n-1)x \cdot \sin x| \\
&\leq |\sin(n-1)x| + |\sin x|
\end{aligned}$$

By induction,

$$|\sin nx| = |\sin(n-1)x| + |\sin x| \leq (n-1)|\sin x| + |\sin x| = n|\sin x|.$$

□

Problem 3

Put $s_N = 1 + \left(\frac{1}{2}\right) + \cdots + \left(\frac{1}{N}\right)$. Prove that

$$\lim_{N \to \infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ, is called Euler's constant. Its numerical value is 0.5772.... It is not known whether γ is rational or not.)

Proof. Let $f_n = s_n - \log n$. Since $\frac{1}{x}$ is a decreasing function, $\int_n^{n+1} \frac{1}{x}\,\mathrm{d}x \geq \frac{1}{n+1}$. Thus,

$$f_{n+1} - f_n = \frac{1}{n+1} - (\log(n+1) - \log n) = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x}\,\mathrm{d}x \leq 0,$$

and so $\{f_n\}$ is a monotonically decreasing sequence. But then $\int_1^n \frac{1}{x}\,dx \leq \sum_{k=1}^{n-1} \frac{1}{k}$. Hence,

$$f_n = \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\,dx \geq \frac{1}{n} > 0,$$

so f_n is bounded below. The result now follows from Theorem 3.14. $\square$

Problem 4

Prove that $\sum 1/p$ diverges; the sum extends over all primes.

Proof. Given N, let $p_1, \dots, p_k$ be those primes that divide at least one integer at most N. Each $n \leq N$ is a product of powers of p_j's. Since $\prod_{j=1}^k \left(1 + \frac{1}{p_j} + \frac{1}{p^2} + \cdots\right)$ is the sum of all inverses of numbers whose factorization consists of only powers of p_j's,

$$\begin{aligned}\sum_{n=1}^N \frac{1}{n} &= \sum_{n=1}^N \frac{1}{p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}} \\ &\leq \prod_{j=1}^k \left(1 + \frac{1}{p_j} + \frac{1}{p^2} + \cdots\right) \\ &= \prod_{j=1}^k \left(1 - \frac{1}{p_j}\right)^{-1}.\end{aligned}$$

We now show that $e^{2x} \geq (1-x)^{-1}$ for $x \in (0, 1/2)$. Put $f(x) = (1-x)e^{2x}$. Since $f'(x) = (1-2x)e^{2x} > 0$ for $x \in (0, 1/2)$ and $f(0) = 1$, we have $f(x) \geq 1$ in $(0, 1/2)$, and thus $e^{2x} \geq (1-x)^{-1}$. Hence, we have

$$\prod_{j=1}^k \left(1 - \frac{1}{p_j}\right)^{-1} \leq \exp \sum_{j=1}^k \frac{2}{p_j}.$$

The logarithmic function is monotonically increasing, so we get

$$\frac{1}{2} \log \left(\sum_{n=1}^N \frac{1}{n}\right) \leq \sum_{j=1}^k \frac{1}{p_j}.$$

Since $k \to \infty$ as $N \to \infty$ and $\sum_{n=1}^N \frac{1}{n}$ diverges, $\sum_{j=1}^k \frac{1}{p_j}$ diverges, by comparison test. $\square$

Problem 5

Suppose $f \in \mathscr{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to \infty$. Prove that

$$\lim_{t \to 0} t \int_0^\infty e^{-tx} f(x)\, dx = 1 \quad (t > 0).$$

Proof. Pick $\epsilon > 0$. There exists A such that $|f(x) - 1| < \epsilon$ for all $x \geq A$. Since $|e^{-tx}| < 1$ for all $t > 0$,

$$\lim_{t \to 0^+} t \left| \int_0^A e^{-tx} f(x)\, dx \right| \leq \lim_{t \to 0^+} t \int_0^A |f(x)|\, dx = 0.$$

On the other hand, for $t > 0$,

$$e^{-At}(1-\epsilon) \leq \left| \int_A^\infty te^{-tx}(1-\epsilon)\, dx \right| \leq t \left| \int_A^\infty e^{-tx} f(x)\, dx \right| \leq \left| \int_A^\infty te^{-tx}(1+\epsilon)\, dx \right| \leq e^{-At}(1+\epsilon).$$

Thus, $t \left| \int_A^\infty e^{-tx} f(x)\, dx \right| = e^{-At}$, as ϵ is arbitrary. It now follows that

$$\begin{aligned}
\lim_{t \to 0^+} t \left| \int_0^\infty e^{-tx} f(x)\, dx \right| &= \lim_{t \to 0^+} t \left| \int_0^A e^{-tx} f(x)\, dx + \int_0^\infty e^{-tx} f(x)\, dx \right| \\
&\leq \lim_{t \to 0^+} t \left| \int_0^A e^{-tx} f(x)\, dx \right| + \lim_{t \to 0^+} t \left| \int_A^\infty e^{-tx} f(x)\, dx \right| \\
&= \lim_{t \to 0^+} t \left| \int_A^\infty e^{-tx} f(x)\, dx \right| \\
&= \lim_{t \to 0^+} e^{-At} \\
&= 1.
\end{aligned}$$

□

Problem 6

If α is real and $-1 < x < 1$, prove Newton's binomial theorem

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

Proof. Since

$$\lim_{n\to\infty} \left| \frac{\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}x^{n+1}}{\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n} \right| = \lim_{n\to\infty} \left| \frac{n-\alpha}{n+1} \right| |x| < 1,$$

the series on the right converges in $(-1,1)$ by the ratio test. Let $f(x)$ denote the function on the right-hand side. By Theorem 8.1, $f(x)$ is differentiable. Note that

$$f'(x) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^n.$$

Hence, we have

$$\begin{aligned}
(1+x)f'(x) &= \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^n + \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^{n+1} \\
&= \alpha + \sum_{n=1}^{\infty} \left(\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} \right) x^n \\
&= \alpha + \sum_{n=1}^{\infty} (n+\alpha-n) \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n \\
&= \alpha + \alpha \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n \\
&= \alpha f(x).
\end{aligned}$$

Since $f(0) = 1$ and f is continuous, there exists $R \in (0,1)$ such that $f(x) > 0$ in $(-R,R)$. Hence, $(\log f(x))' = \frac{f'(x)}{f(x)} = \frac{\alpha}{1+x}$ in $(-R,R)$, which shares the same derivative with $\log(1+x)^\alpha$. But then for $x \in (-R,R)$,

$$\log f(x) = \log f(x) - \log f(0) = \int_0^x \frac{\alpha}{1+t}\, dt = \alpha \log(1+x) = \log(1+x)^\alpha,$$

and so $f(x) = \exp(\log f(x)) = \exp(\log(1+x)^\alpha) = (1+x)^\alpha$. Now let $S = \{K \in (0,1) \mid f(x) > 0 \text{ if } x \in [-K,K]\}$. Suppose for contradiction that $A = \sup S < 1$. We know $f(x) = (1+x)^\alpha$ in $(-A,A)$. But then

$$\lim_{x\to A} f(x) = (1+A)^\alpha > 0 \text{ and } \lim_{x\to -A} f(x) = (1+-A)^\alpha > 0.$$

By continuity, there exists δ such that $f(x) > 0$ in $(-A-\delta, A+\delta)$, contradiction. Hence, $f(x) = (1+x)^\alpha$ in $(-1,1)$. $\square$