MATH 188: Homework #2

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Professor Kunnawalkam Elayavalli

Ray Tsai

A16848188

Problem 1

Let $F(x)$ be a formal power series with $F(0) = 0$.

(a) Show that there exists a formal power series $G(x)$ with $G(0) = 0$ such that $F(G(x)) = x$ if and only if $[x^1]F(x) \neq 0$.

Proof. Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$, for some nonzero a_1 and $a_0 = 0$. We look for a formal power series $G(x) = \sum_{n=0}^{\infty} b_n x^n$ such that $F(G(x)) = x$ and $b_0 = 0$. That is,

$$\begin{aligned} F(G(x)) &= \sum_{i=1}^{\infty} a_i G(x)^i \\ &= \sum_{n=1}^{\infty} x^n \sum_{i=1}^{n} a_i \sum_{m_1+m_2+\cdots+m_i=n} b_{m_1} b_{m_2} \cdots b_{m_i} = x. \end{aligned}$$

Note that the inner summation terminates at n, as we are enumerating through compositions of n, which could not exceed n terms. By comparing coefficients, we have

$$b_0 = 0, \quad b_1 = \frac{1}{a_1},$$

and for $n \geq 2$,

$$\sum_{i=1}^{n} a_i \sum_{m_1+m_2+\cdots+m_i=n} b_{m_1} b_{m_2} \cdots b_{m_i} = 0. \tag{1}$$

Here, we already know that $G(x)$ exists only if $[x^1]F(x) \neq 0$, it remains to show the converse. Suppose $[x^1]F(x) \neq 0$. We already determined the unique existence of b_1. For $n \geq 2$, rearranging (1) gives an expression of b_n uniquely determined by $a_1, \ldots a_n, b_1, \ldots b_{n-1}$. But then the existence of $b_1, \ldots b_{n-1}$ are shown by induction, and this ensures the unique existence of b_n. □

(b) Assuming $[x^1]F(x) \neq 0$, show that $G(x)$ is unique and also satisfies $G(F(x)) = x$. You may use without proof that composition of formal power series is associative.

Proof. Uniqueness of $G(x)$ is shown in (a). We know $[x^1]G(x) \neq 0$. By (a), there exists a formal power series $H(x)$ with $H(0) = 0$ such that $G(H(x)) = x$. But then $F(x) = F(G(H(x))) = H(x)$. □

Problem 2

Evaluate the following sums:

(a)

$$\sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^i}$$

Proof. By the binomial theorem,

$$\sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^i} = \left(1 + \frac{1}{2}\right)^n = \frac{3^n}{2^n}.$$

□

(b)

$$\sum_{i=0}^{n} i^2 \binom{n}{i} 3^i$$

Proof. By the binomial theorem,

$$\sum_{n \geq 1} i \binom{n}{i} x^{i-1} = \left(\sum_{n \geq 0} \binom{n}{i} x^i\right)' = ((1+x)^n)' = n(1+x)^{n-1},$$

$$\sum_{n \geq 2} i(i-1) \binom{n}{i} x^{i-2} = \left(\sum_{n \geq 0} \binom{n}{i} x^i\right)'' = ((1+x)^n)'' = n(n-1)(1+x)^{n-2}.$$

Hence,

$$\begin{aligned}
\sum_{i=0}^{n} i^2 \binom{n}{i} 3^i &= \sum_{i=0}^{n} i(i-1) \binom{n}{i} 3^i + \sum_{i=0}^{n} i \binom{n}{i} 3^i \\
&= 9 \sum_{i=2}^{n} i(i-1) \binom{n}{i} 3^{i-2} + 3 \sum_{i=1}^{n} i \binom{n}{i} 3^{i-1} \\
&= 9 \left(\sum_{i=0}^{n} \binom{n}{i} 3^i\right)'' + 3 \left(\sum_{i=0}^{n} \binom{n}{i} 3^i\right)' \\
&= 9n(n-1)(1+3)^{n-2} + 3n(1+3)^{n-1} \\
&= \frac{9}{16} n(n-1)4^n + \frac{3}{4} n4^n = 3n(3n+1)4^{n-2}.
\end{aligned}$$

□

Problem 3

Let a, b be non-negative integers.

(a) By comparing coefficients in $(1+x)^{a+b} = (1+x)^a(1+x)^b$, prove that for any non-negative integer n, we have

$$\binom{a+b}{n} = \sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}.$$

Proof. By the binomial theorem,

$$\begin{aligned}\binom{a+b}{n} &= [x^n](1+x)^{a+b} \\ &= [x^n](1+x)^a(1+x)^b \\ &= \sum_{i=0}^{n} \left([x^i](1+x)^a\right)\left([x^{n-i}](1+x)^b\right) \\ &= \sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}.\end{aligned}$$

□

(b) Now prove this identity using a counting argument.

Proof. Consider choosing n animals from a dogs and b cats. Suppose that we picked i dogs. There are $\binom{a}{i}$ ways of choosing them. In order to have n animals in total, we then have to pick $n-i$ cats, which has $\binom{b}{n-i}$ ways. The possible values for i are between 0 and n, and thus we get the identity

$$\binom{a+b}{n} = \sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}.$$

□

Problem 4

How many ways can we arrange the letters of: MISSISSIPPI?

Proof. There are one M, four I's, two P's, and four S', and we have 11 slots in total. We first choose a slot for the M, which has $\binom{11}{1}$ ways. Then, we choose 4 slots from the remaining 10 slots for the I's, which has $\binom{10}{4}$ ways. Then, we choose 2 slots from the remaining 6 slots for the P's, which has $\binom{6}{2}$ ways. Finally, we choose 4 slots from the remaining 4 slots for the S's, which has $\binom{4}{4}$ ways. In total, there are

$$\binom{11}{1}\binom{10}{4}\binom{6}{2}\binom{4}{4} = \frac{11!}{4!2!4!}$$

ways of arranging the letters of MISSISSIPPI. $\square$

Problem 5

Let $f(t) = \sum_{k=0}^{d} f_k t^k$ be a degree d polynomial with rational coefficients. From lecture, we know that there exist unique rational numbers $g_0, \dots, g_d$ such that

$$\sum_{n \geq 0} f(n)x^n = \frac{g_0 + g_1 x + \ldots + g_d x^d}{(1-x)^{d+1}}. \tag{2}$$

Now assume that $f(a)$ is an integer for $a = 0, \dots, d$. (The f_k don't have to be integers for this to be true, for example $f(n) = n(n-1)/2$ has this property.) Prove that this implies that the g_k are all integers and that $f(a)$ is an integer whenever a is an integer.

Proof. From (2), for $k = 0, 1, \dots, d$,

$$\begin{aligned} g_k &= [x^k](1-x)^{d+1} \sum_{n \geq 0} f(n)x^n \\ &= \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1}{k-i} f(i), \end{aligned}$$

which is an integer as $f(i)$ and $\binom{d+1}{k-i}$ are both integers, for $i = 0, \dots, d$. But then for $n \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} f(n) &= [x^n](1-x)^{-(d+1)}(g_0 + g_1 x + \ldots + g_d x^d) \\ &= \sum_{k=0}^{d} \binom{d+n-k}{n-k} g_k = \sum_{k=0}^{d} \binom{d+n-k}{d} g_k. \end{aligned}$$

Note that $h(n) = \sum_{k=0}^{d} \binom{d+n-k}{d} g_k$ is a polynomial of degree d. Since $f(n) - h(n) = 0$ for all $n \in \mathbb{Z}_{\geq 0}$, it follows from the Fundamental Theorem of Algebra that $f(n) = h(n)$. Since $g_k \in \mathbb{Z}$ and $\binom{d+n-k}{n-k} \in \mathbb{Z}$ whenever $n \in \mathbb{Z}$, we know $f(n)$ is an integer whenever $n \in \mathbb{Z}$. □

Problem 6

Let $n \geq 2$ be an integer.

(a) Prove that

$$\sum_{i=0}^{n} i\binom{n}{i}(-1)^{i-1} = 0.$$

Proof. By the binomial theorem,

$$\sum_{n\geq 1} i\binom{n}{i}x^{i-1} = \left(\sum_{n\geq 0}\binom{n}{i}x^i\right)' = ((1+x)^n)' = n(1+x)^{n-1},$$

and thus

$$\sum_{i=0}^{n} i\binom{n}{i}(-1)^{i-1} = n(1+(-1))^{n-1} = 0$$

$\square$

(b) Compute

$$\sum_{\substack{0\leq i\leq n \\ i \text{ even}}} i\binom{n}{i}.$$

Proof.

$$\begin{aligned}\sum_{\substack{0\leq i\leq n \\ i \text{ even}}} i\binom{n}{i} &= \frac{1}{2}\left(\sum_{i=0}^{n} i\binom{n}{i} - \sum_{i=0}^{n} i\binom{n}{i}(-1)^{i-1}\right) \\ &= \frac{1}{2}(n(1+1)^{n-1}) = n2^{n-2}.\end{aligned}$$

$\square$

Problem 7

(a) Let a, b be rational numbers. Show that for any formal power series $A(x)$ with $A(0) = 1$, we have

$$A(x)^a A(x)^b = A(x)^{a+b}.$$

[Remember that we defined rational powers in a very specific way, so your proof needs to use this definition.]

Proof. By definition, $A(x)^{m/n} = (A(x)^{1/n})^m = (A(x)^m)^{1/n}$. Let $a = m/n$, $b = p/q$, for some $m, n, p, q \in \mathbb{Z}$. Then,

$$\begin{aligned} A(x)^a A(x)^b &= (A(x)^{1/nq})^{mq} (A(x)^{1/nq})^{np} \\ &= (A(x)^{1/nq})^{mq+np} \\ &= A(x)^{a+b}. \end{aligned}$$

$\square$

(b) Deduce from (a) that

$$\binom{a+b}{n} = \sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i}$$

for all non-negative integers n.

Proof. Put $A(x) = (1 + x)$. Since $(1 + x)^a (1 + x)^b = (1 + x)^{a+b}$,

$$\begin{aligned} \binom{a+b}{n} &= [x^n](1+x)^{a+b} \\ &= [x^n](1+x)^a (1+x)^b \\ &= \sum_{i=0}^{n} \left([x^i](1+x)^a\right) \left([x^{n-i}](1+x)^b\right) \\ &= \sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i}. \end{aligned}$$

$\square$

Problem 8

Assume now that we deal with complex-coefficient formal power series. Define the following sets of formal power series:

$$V = \{F(x) \mid F(0) = 0\}, \quad W = \{G(x) \mid G(0) = 1\}.$$

(a) Given $F \in V$, show that $\mathbf{E}(F) = \sum_{n\geq 0} \frac{F^n(x)}{n!}$ is the *unique* formal power series $G \in W$ such that $DG = DF \cdot G$. This defines a function $\mathbf{E}\colon V \to W$. [Convention: $F^0(x) = 1$ even if $F(x) = 0$.]

Proof. It is easy to see that

$$DG = \sum_{n\geq 0} \frac{D(F^n(x))}{n!} = \sum_{n\geq 1} DF \cdot \frac{F^{n-1}(x)}{(n-1)!} = DF \sum_{n\geq 0} \frac{F^n(x)}{n!} = DF \cdot G,$$

and $G(0) = F^0(0) = 1$. It remains to show that G is unique. Suppose there exists $G = \sum_{n\geq 0} b_n x_n, G' = \sum_{n\geq 0} b'_n x_n \in W$ such that $\mathbf{E}(F) = G$ and $\mathbf{E}(F) = G'$. Suppose $DF = \sum_{n\geq 0} a_n x_n$ We know $DG = DF \cdot G$ and $DG' = DF \cdot G'$. By comparing coefficients, for $k \geq 1$,

$$\frac{b_k}{k+1} = [x^k]DG = [x^k](DF \cdot G) = \sum_{i=0}^{k} a_i b_{k-i},$$
$$\frac{b'_k}{k+1} = [x^k]DG = [x^k](DF \cdot G) = \sum_{i=0}^{k} a_i b'_{k-i}.$$

In particular, for $k \geq 1$,

$$b_k = \frac{k+1}{-a_0 k - a_0 + 1} \sum_{i=1}^{k} a_i b_{k-i}, \quad b_k = \frac{k+1}{-a_0 k - a_0 + 1} \sum_{i=1}^{k} a_i b_{k-i},$$

so b_k, b'_k are uniquely determined by the corrresponding previous coefficients, and thus $G = G'$ if and only if $b_0 = b'_0$. But then $G(0) = G'(0) = 1$, and the result follows. □

(b) Given $G \in W$, show that there is a *unique* formal power series $F \in V$ such that $DF(x) = DG(x)/G(x)$. We define the function $\mathbf{L}\colon W \to V$ by $\mathbf{L}(G) = F$. [For the rest, it is unnecessary to use explicit formulas for $\mathbf{L}$ and $\mathbf{E}$ and in fact it may be easier to only use the uniqueness properties above.]

Proof. Since $G(0) = 1$, there exists $G^{-1}(x)$ such that $G(x)G^{-1}(x) = G(x)^{-1}G(x) = 1$, so $DG(x)/G(x)$ is unique given G. Suppose $DG(x)/G(x) = \sum_{n\geq 0} a_n x^n$. There exists $F = \sum_{n\geq 1} \frac{1}{n} a_{n-1} x^n \in V$ such that

$$DF = \sum_{n\geq 1} a_{n-1} x^{n-1} = \sum_{n\geq 0} a_n x^n = DG(x)/G(x).$$

That is, all coefficients a_n of DF are uniquely determined by $DG(x)/G(x)$. But then all coefficients of F are uniquely determined, as F has no constant term. □

(c) Show that $\mathbf{E}$ and $\mathbf{L}$ are inverses of each other.

Proof. Let $F \in V$. $\mathbf{E}$ maps F to some unique $G' \in W$ such that $DG' = DF \cdot G'$, that is, $DF = DG'/G'$. Then, $\mathbf{L}$ maps G' back to some unique F' such that $DF' = DG'/G' = DF$. But then both F and F' have no constant terms, so F and F' actually agree with all coefficients. Hence, $\mathbf{L}(\mathbf{E}(F)) = F$.

Let $G \in W$. $\mathbf{L}$ maps G to some unique $F'' \in V$ such that $DG/G = DF''$, and $\mathbf{E}$ maps F'' back to some unique G'' such that $DG'' = DF'' \cdot G'' = DG/G \cdot G''$. But then $DG''/G'' = DG/G$. By comparing coefficients, for all $k \geq 0$ we get

$$\sum_{i=0}^{k} b''_{k-i}(i+1)b_{i+1} = [x^k]DG'' \cdot G = [x^k]DG \cdot G'' = \sum_{i=0}^{k} b_{k-i}(i+1)b''_{i+1}.$$

Since $b_0 = b''_0 = 1$, it follows from induction that $b_k = b''_k$ for all $k \in \mathbb{Z}_{\geq 0}$, and so $G = G''$. Hence, $\mathbf{E}(\mathbf{L}(G)) = G$. ☐

(d) Show that $\mathbf{E}(F_1 + F_2) = \mathbf{E}(F_1)\mathbf{E}(F_2)$ for all $F_1, F_2 \in V$.

Proof. Let $G_1 = \mathbf{E}(F_1)$, $G_2 = \mathbf{E}(F_2)$, and $G = \mathbf{E}(F_1 + F_2)$. Since

$$\begin{aligned}
D(G_1G_2) &= DG_1 \cdot G_2 + DG_2 \cdot G_1 \\
&= (DF_1 \cdot G_1)G_2 + (DF_2 \cdot G_2)G_1 \\
&= (DF_1 + DF_2)(G_1G_2) \\
&= D(F_1 + F_2)(G_1G_2).
\end{aligned}$$

Note that $G_1G_2 \in W$. But then G is the unique element in W such that $DG = D(F_1 + F_2)G$, and so $\mathbf{E}(F_1 + F_2) = G = G_1G_2 = \mathbf{E}(F_1)\mathbf{E}(F_2)$. ☐

(e) Show that $\mathbf{L}(G_1G_2) = \mathbf{L}(G_1) + \mathbf{L}(G_2)$ for all $G_1, G_2 \in W$.

Proof. Let $F_1 = \mathbf{L}(G_1)$, $F_2 = \mathbf{L}(G_2)$. Since

$$\begin{aligned}
D(F_1 + F_2) &= DF_1 + DF_2 \\
&= DG_1/G_1 + DG_2/G_2,
\end{aligned}$$

$$G_1G_2D(F_1 + F_2) = DG_1 \cdot G_2 + DG_2 \cdot G_1 = D(G_1G_2), \tag{3}$$

that is, $D(F_1 + F_2) = D(G_1G_2)/G_1G_2$. But then $F_1 + F_2 \in F$, so $G_!$ is the unique element in W that satisfies (3), and thus $\mathbf{L}(G_1G_2) = F_1 + F_2 = \mathbf{L}(G_1) + \mathbf{L}(G_2)$. ☐

(f) If m is a positive integer and $G \in W$, show that $\mathbf{E}(\frac{\mathbf{L}(G)}{m})$ is an mth root of G. [This gives an alternative proof for the existence of mth roots and in fact we can now define powers for any complex number m: $F^m = \mathbf{E}(m\mathbf{L}(F))$.]

Proof. By (e),

$$\mathbf{L}\left[\left(\mathbf{E}\left(\frac{\mathbf{L}(G)}{m}\right)\right)^m\right] = m\mathbf{L}\left[\mathbf{E}\left(\frac{\mathbf{L}(G)}{m}\right)\right] = m \cdot \frac{\mathbf{L}(G)}{m} = \mathbf{L}(G).$$

But then $\mathbf{L}$ is bijective, and the result follows. ☐

(g) Show that if $\sum_{i\geq 0} F_i(x)$ converges to $F(x)$, then $\prod_{i\geq 0} \mathbf{E}(F_i)$ converges to $\mathbf{E}(F)$.

Proof. By (d),

$$\mathbf{E}(F(x)) = \mathbf{E}\left(\sum_{i\geq 0} F_i(x)\right) = \prod_{i\geq 0} \mathbf{E}(F_i).$$

$\square$

(h) Show that if $\prod_{i\geq 0} G_i(x)$ converges to $G(x)$, then $\sum_{i\geq 0} \mathbf{L}(G_i)$ converges to $\mathbf{L}(G)$.

Proof. By (e),

$$\mathbf{L}(G(x)) = \mathbf{L}\left(\prod_{i\geq 0} G_i(x)\right) = \sum_{i\geq 0} \mathbf{L}(G_i).$$

$\square$