MATH 140B: Homework #6

Due on May 17, 2024 at 23:59pm

Professor Seward

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Problem 1

Suppose g and f_n $(n = 1, 2, 3, \dots)$ are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^\infty g(x)\,dx < \infty.$$

Prove that

$$\lim_{n \to \infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx.$$

Proof. Since $|f_n| \leq g$ and $\int_t^\infty g(x)\,dx < \infty$, we know $\int_t^\infty f_n(x)\,dx$ exists. Define $h_k = \int_{1/k}^\infty f(x)\,dx$. Pick $\epsilon > 0$. Since $\int_0^\infty g(x)\,dx$ exists, there exists N such that $\int_0^{1/n} g(x)\,dx < \epsilon/2$ for all $n \geq N$. Let $\beta > \alpha \geq N$. There exists large enough n such that $|f(x) - f_n(x)| < \epsilon(\beta - \alpha)/2\alpha\beta$ for all $x \in [1/\beta, 1/\alpha]$. But then,

$$
\begin{aligned}
|h_\beta - h_\alpha| &= \left| \int_{1/\beta}^{1/\alpha} f(x)\,dx \right| \\
&\leq \int_{1/\beta}^{1/\alpha} |f(x) - f_n(x)| + |f_n(x)|\,dx \\
&\leq \int_{1/\beta}^{1/\alpha} |f(x) - f_n(x)|\,dx + \int_{1/\beta}^{1/\alpha} g(x)\,dx \\
&< \epsilon/2 + \epsilon/2 = \epsilon,
\end{aligned}
$$

Thus, h_k converges uniformly by Cauchy criterion, and so $\int_t^\infty f(x)\,dx$ exists for $t \in (0, \infty)$.

Let $I_n(t) = \int_t^\infty f_n(x)\,dx$ and let $I(t) = \int_t^\infty f(x)\,dx$. We show that $I_n \to I$ uniformly on $(0, \infty)$. Again, pick $\epsilon > 0$. There exists $t_1, t_2 \in (0, \infty)$, $t_2 > t_1$, such that $\int_0^{t_1} g(x)\,dx < \epsilon/6$ and $\int_{t_2}^\infty g(x)\,dx < \epsilon/6$. Since f_n converges to f uniformly, there exists N such that $|f_n(x) - f(x)| < \epsilon/3(t_2 - t_1)$ for all $n \geq N$ and $x \in [t_1, t_2]$. Hence, for all $n \geq N$ and $t \in (0, \infty)$, let $t' \in (0, \min(t, t_1))$ and we have

$$
\begin{aligned}
|I(t) - I_n(t)| &\leq \int_{t'}^\infty |f(x) - f_n(x)|\,dx \\
&= \int_{t'}^{t_1} |f(x) - f_n(x)|\,dx + \int_{t_1}^{t_2} |f(x) - f_n(x)|\,dx + \int_{t_2}^\infty |f(x) - f_n(x)|\,dx \\
&\leq 2\int_0^{t_1} g(x)\,dx + \int_{t_1}^{t_2} |f(x) - f_n(x)|\,dx + 2\int_{t_2}^\infty g(x)\,dx \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}
$$

Hence, $I_n \to I$ uniformly on $(0, \infty)$. By Theorem 7.11,

$$\lim_{n \to \infty} \lim_{t \to 0} I_n(t) = \lim_{t \to 0} I(t).$$

$\square$

Problem 2

Assume that (f_n) is a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$ for all x and all n.

Prove that there is a function f and a sequence (n_k) such that

$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$

for every $x \in \mathbb{R}^1$. (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem*.)

Proof. $\square$

Problem 3

Suppose f is a real continuous function on $\mathbb{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$, and (f_n) is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

Proof. Pick $\epsilon > 0$. Since (f_n) is equicontinuous on $[0, 1]$, there exists $\delta > 0$ such that

$$|f(nx) - f(ny)| < \epsilon,$$

whenever $|x - y| < \delta$, for all n and $x, y \in [0, 1]$. Let $s, t \in [0, \infty]$. Put integer $n > \max(|s - t|/\delta, s, t)$. Since $|\frac{s}{n} - \frac{t}{n}| = |s - t|/n < \delta$ and $\frac{s}{n}, \frac{t}{n} \in [0, 1]$, we have

$$|f(s) - f(t)| < \epsilon.$$

But then ϵ is arbitrary, so f is constant on $[0, \infty]$. $\qquad\qquad\square$

Problem 4

Let (f_n) be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)\, dt \quad (a \le x \le b).$$

Prove that there exists a subsequence (F_{n_k}) which converges uniformly on $[a, b]$.

Proof. We show that (F_n) is point-wise bounded and equicontinuous on $[a, b]$.

Since (f_n) is uniformly bounded, there exists K such that $|f_n(x)| < K$ for all x and n. Hence,

$$|F_n(x)| \le \int_a^x |f_n(t)|\, dt < K(x - a),$$

and so F_n is point-wise bounded.

Pick $\epsilon > 0$. Let $\delta = \epsilon / K(b - a)$. Then,

$$|F_n(x) - F_n(y)| \le \int_x^y |f_n(t)|\, dt \le K(y - x) < \epsilon,$$

for all $n \in \mathbb{N}$ and $x, y \in [a, b]$ such that $\delta > y - x > 0$. Hence, (F_n) is equicontinuous.

The result now follows from Theorem 7.25. $\square$

Problem 5

Let K be a compact metric space, let S be a subset of $\mathscr{C}(K)$. Prove that S is compact (with respect to the metric defined in Definition 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on K.)

Proof. Suppose S is compact. S is closed so S is uniformly closed by definition.

Pick $\epsilon > 0$ and pick $\nu \in (0, \epsilon/2)$. Consider the open cover $\{B_\nu(f)\}_{f \in S}$. There exists $\{f_i\}_{i=1}^n$ such that $\bigcup_{i=1}^n B_\nu(f_i) \supset S$. Since K is compact, f is uniformly continuous for all $f \in \mathscr{C}(K)$. Hence, for each f_i, there exists δ_i such that $|f_i(x) - f_i(y)| < \epsilon - 2\nu$ for all $x, y \in K$ such that $d(x, y) < \delta_i$. Put $\delta = \min_{1 \le i \le n} \delta_i$. Let $g \in S$. Then $\sup_{x \in K} |g(x) - f_i(x)| < \nu$ for some f_i. But then

$$|g(x) - g(y)| \le |g(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - g(y)|$$
$$< \nu + \epsilon - 2\nu + \nu = \epsilon.$$

Define $\Phi(x) = \nu + \max_{1 \le i \le n} |f_i(x)|$. Then, $f(x) < \Phi(x)$ for all $f \in S$, so S is pointwise bounded.

Now suppose that S is uniformly closed, pointwise bounded, and equicontinuous. Let $T \subset S$ be an infinite subset of S. By Theorem 7.25, T contains a uniformly convergent sequence, which converges to some $f \in S$ as S is uniformly closed. Hence, every infinite subset of S has a limit point in S, and thus S is compact. $\square$

Problem 6

If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)x^n \, dx = 0 \quad (n = 0, 1, 2, \ldots),$$

prove that $f(x) = 0$ on $[0, 1]$.

Proof. By the Weierstrass Theorem, there exists a sequence of polynomials P_n which converges to f uniformly on $[0, 1]$. By the given identity,

$$\int_0^1 f(x)P_n(x) \, dx = 0.$$

But then

$$\int_0^1 f^2(x) \, dx = \lim_{n \to \infty} \int_0^1 f(x)P_n(x) \, dx = 0,$$

and the result now follows. $\square$