

MATH 180B: Homework #6

Due on Mar 08, 2024 at 23:59pm

Professor Carfagnini

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Problem 1

Which states are transient and which are recurrent in the Markov chain whose transition probability matrix is

$$\begin{array}{c}
 \\
 0 \\
 1 \\
 2 \\
 3 \\
 4 \\
 5
 \end{array}
 \begin{array}{c}
 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\
 \left\| \begin{array}{cccccc}
 \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\
 \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{array} \right\|
 \end{array}
 ?$$

Proof. The communicating classes of the matrix are $\{0\}$, $\{1\}$, $\{3\}$, $\{5\}$, $\{2, 4\}$. Note that $\{5\}$ and $\{2, 4\}$ are closed classes, so states 2, 4, 5 are recurrent. In addition since 0, 1, 3 are connected all to closed classes, they will eventually get stuck in those closed classes and never return to the original state, and thus they are transient. \square

Problem 2

Determine the communicating classes and period for each state of the Markov chain whose transition probability matrix is

$$\begin{array}{c}
 \\
 0 \\
 1 \\
 2 \\
 3 \\
 4 \\
 5
 \end{array}
 \begin{array}{c}
 \\
 \left\| \begin{array}{cccccc}
 0 & 1 & 2 & 3 & 4 & 5 \\
 \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}
 \end{array} \right\| \\
 \\
 \end{array}
 ?$$

Proof. Since states each of 0, 1 is not accessible to any other classes, they each form their own classes. Since there exists a cycle $2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 2$ which passes through all the rest of the states, they all belong to a class. Hence, the communicating classes are $\{0\}$, $\{1\}$, $\{2, 3, 4, 5\}$. Since 0 and 5 are both connected to itself, $\{0\}$ and $\{2, 3, 4, 5\}$ have period 1. Since 1 is not accessible from any states including itself, $d(1) = 0$. \square

Problem 3

Recall the first return distribution

$$f_{ii}^{(n)} = \Pr\{X_1 \neq i, X_2 \neq j, \dots, X_{n-1} \neq i, X_n = i \mid X_0 = i\} \text{ for } n = 1, 2, \dots,$$

with $f_{ii}^{(0)} = 0$ by convention. Using equation (4.16), determine $f_{00}^{(n)}$, $n = 1, 2, 3, 4$, for the Markov chain whose transition probability matrix is

$$\begin{array}{c} \begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left\| \begin{array}{ccccc} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array} \right\| \end{array} \end{array}.$$

Proof. Call that matrix P . Note that

$$P^2 = \begin{array}{c} \begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left\| \begin{array}{ccccc} \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{array} \right\| \end{array}, \quad P^3 = \begin{array}{c} \begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left\| \begin{array}{ccccc} \frac{1}{8} & \frac{1}{8} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{array} \right\| \end{array}, \quad P^4 = \begin{array}{c} \begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left\| \begin{array}{ccccc} \frac{3}{8} & \frac{1}{16} & \frac{1}{8} & \frac{7}{16} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{16} & \frac{1}{8} & \frac{1}{8} & \frac{9}{16} \end{array} \right\| \end{array}.$$

Hence, we get

$$\begin{aligned} P_{00}^1 &= 0 = f_{00}^{(0)} P_{00}^1 + f_{00}^{(1)} P_{00}^0 = f_{00}^{(1)} \\ P_{00}^2 &= \frac{1}{4} = f_{00}^{(0)} P_{00}^2 + f_{00}^{(1)} P_{00}^1 + f_{00}^{(2)} P_{00}^0 = f_{00}^{(2)} \\ P_{00}^3 &= \frac{1}{8} = f_{00}^{(0)} P_{00}^3 + f_{00}^{(1)} P_{00}^2 + f_{00}^{(2)} P_{00}^1 + f_{00}^{(3)} P_{00}^0 = f_{00}^{(3)} \\ P_{00}^4 &= \frac{3}{8} = f_{00}^{(0)} P_{00}^4 + f_{00}^{(1)} P_{00}^3 + f_{00}^{(2)} P_{00}^2 + f_{00}^{(3)} P_{00}^1 + f_{00}^{(4)} P_{00}^0 = \frac{1}{4} \cdot \frac{1}{4} + f_{00}^{(4)}, \end{aligned}$$

and thus $f_{00}^{(1)} = 0, f_{00}^{(2)} = \frac{1}{4}, f_{00}^{(3)} = \frac{1}{8}, f_{00}^{(4)} = \frac{5}{16}$. □

Problem 4

Let $\{\alpha_i : i = 1, 2, \dots\}$ be a probability distribution, and consider the Markov chain whose transition probability matrix is

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & \dots \\ \hline 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 4 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

What condition on the probability distribution $\{\alpha_i : i = 1, 2, \dots\}$ is necessary and sufficient in order that a limiting distribution exist, and what is this limiting distribution? Assume $\alpha_1 > 0$ and $\alpha_2 > 0$ so that the chain is aperiodic.

Proof. Call that matrix P . We show that $\sum_{k=1}^{\infty} k\alpha_k < \infty$ is the necessary and sufficient condition to the existence of the limiting distribution.

P is obviously aperiodic and irreducible. Hence, if state 0 is recurrent, every state in P is recurrent. Let $R = \inf\{n \geq 1; X_n = 0\}$. Note that $P(R \leq k \mid X_0 = 0) = \sum_{i=1}^k f_{00}^{(i)} = \sum_{i=1}^k \alpha_i$. Clearly, $\lim_{k \rightarrow \infty} \sum_{i=1}^k \alpha_i = 1$. It follows that $f_{00} = \lim_{k \rightarrow \infty} \sum_{i=1}^k f_{00}^{(i)} = 1$, so P is indeed recurrent.

Suppose that $\sum_{k=1}^{\infty} k\alpha_k < \infty$. Since $m_0 = E[R \mid X_0 = 0] = \sum_{k=1}^{\infty} k\alpha_k < \infty$, the state 0 is positively recurrent, with

$$\begin{aligned} \pi_0 &= \lim_{n \rightarrow \infty} P_{00}^{(n)} = \frac{1}{m_0} = \frac{1}{\sum_{k=1}^{\infty} k\alpha_k} \\ \pi_1 &= (1 - \alpha_1)\pi_0 \\ \pi_2 &= (1 - \alpha_1 - \alpha_2)\pi_0 \\ &\vdots \\ \pi_n &= \left(1 - \sum_{i=1}^n \alpha_i\right)\pi_0 \\ &\vdots \end{aligned}$$

Hence, the limiting distribution is $\pi_k = \frac{1 - \sum_{i=1}^k \alpha_i}{m_0}$.

Suppose that $\sum_{k=1}^{\infty} k\alpha_k = \infty$. Then, $m_0 = \infty$. It follows that $\pi_0 = \frac{1}{m_0} = 0$, so the limiting distribution does not exist. \square

Problem 5

Determine the period of state 0 in the Markov chain whose transition probability matrix is

$$P = \begin{array}{c} \begin{array}{cccccccc} & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 \\ \begin{array}{c} 3 \\ 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ -3 \\ -4 \end{array} & \left\| \begin{array}{cccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right\| \end{array} \end{array}$$

Proof. There are two communicating classes in P , namely $\{1, 2, 3\}$ and $\{0, -1, -2, -3, -4\}$. Note that 0 is accessible to both classes. But since $\{1, 2, 3\}$ is a closed class, there are no path to return to 0 after entering that class. Hence, we may focus on the class $\{0, -1, -2, -3, -4\}$. Since the only path to return to 0 is via the cycle $0 \rightarrow -1 \rightarrow -2 \rightarrow -3 \rightarrow -4 \rightarrow 0$, the state 0 is of period 5. \square

Problem 6

A Markov chain on states $0, 1, \dots$ has transition probabilities

$$P_{ij} = \frac{1}{i+2} \quad \text{for } j = 0, 1, \dots, i, i+1.$$

Find the stationary distribution.

Proof. We solve for $\pi = \pi P$ and get $\pi_0 = \sum_{i=0}^{\infty} \frac{\pi_i}{i+2}$ and $\pi_k = \sum_{i=k}^{\infty} \frac{\pi_{i-1}}{i+1}$, for $k > 0$. Hence, we get $\pi_k = \frac{\pi_0}{k!}$. Since $\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{k=0}^{\infty} \frac{1}{k!} = e\pi_0 = 1$, we get $\pi_0 = \frac{1}{e}$, and so $\pi_k = \frac{1}{k!e}$. \square