

MATH 264B: Homework

Professor Rhoades

Ray Tsai

A16848188

Problem 1

Let $n, m \in \mathbb{Z}_{\geq 0}$. Give a *combinatorial* proof that

$$\sum_{i=0}^n \binom{m+i}{i} = \binom{m+n+1}{n}.$$

That is, interpret both sides as the cardinality of a set, and find a bijection between these sets.

Proof. It suffices to show that

$$\sum_{i=0}^n \binom{m+i}{m} = \binom{m+n+1}{m+1}.$$

Let C_i be the set of all m -element subsets of $[m+i]$, and let S be the set of all $(m+1)$ -element subsets of $[m+n+1]$. Consider the map $f : \bigsqcup_{i=1}^n C_i \rightarrow S$ by sending $A \in C_i$ to $A \cup \{m+i+1\} \in S$. This mapping is a bijection as we may recover A by removing the largest element of $f(A)$. Thus, $|\bigsqcup_{i=1}^n C_i| = |S|$, and the result now follows. \square

Problem 2

Let $\text{des} : \mathfrak{S}_n \rightarrow \mathbb{Z}_{\geq 0}$ be the descent statistic

$$\text{des}(w) := \#\{1 \leq i \leq n-1 : w(i) > w(i+1)\}$$

and consider the *Eulerian polynomial*

$$A_n(t) := \sum_{w \in \mathfrak{S}_n} t^{\text{des}(w)}.$$

Prove that $A_n(2) = [A_n(t)]_{t=2}$ is the number of ordered set partitions of $[n]$.

Proof. We say that a ordered partition is in canonical form if the elements of each block are in descending order. Let P_n be the set of all ordered set partitions of $[n]$. Define the operation $\phi : P_n \rightarrow \mathfrak{S}_n$ by erasing the brackets of an ordered partition in canonical form and interpreting the resulting string as a permutation. It is clear that ϕ is well-defined. Now consider the reverse operation $\psi : \mathfrak{S}_n \rightarrow 2^{P_n}$ by sending $w \in \mathfrak{S}_n$ to $\{p \in P_n : \phi(p) = w\}$, the set of all ordered partitions whose canonical form resembles w after erasing the brackets. Note that

$$|P_n| = \sum_{p \in P_n} |\phi(p)| = \sum_{w \in \mathfrak{S}_n} |\psi(w)|,$$

and so it suffices to show that $|\psi(w)| = 2^{\text{des}(w)}$. To see this, we start from the ordered singleton partition $p_0 \in \psi(w)$. Reading p_0 from left to right, we may choose to combine a block with its preceding block whenever a descent occurs, and the resulting partition will still be in $\psi(w)$. This gives us $2^{\text{des}(w)}$ ways to partition w into blocks. \square

Problem 3

How many (strong) compositions of n have an even number of even parts?

Proof. Let E_n be the set of all compositions of n with even number of even parts, and let O_n be the set of all compositions of n with odd number of even parts. We show $|E_n| = 2^{n-2}$ for $n \geq 2$ by proving that $|E_n| = |O_n|$. Consider the operation $\phi : E_n \rightarrow O_n$ by sending the composition $(\alpha_1, \dots, \alpha_k)$ to $(\alpha_1, \dots, \alpha_k - 1, 1)$ if $\alpha_k > 1$ and send $(\alpha_1, \dots, \alpha_k)$ to $(\alpha_1, \dots, \alpha_{k-1} + 1)$ if $\alpha_k = 1$. Notice that $\phi(\phi(\alpha_1, \dots, \alpha_k)) = (\alpha_1, \dots, \alpha_k)$, so ϕ is a bijection. But then there are 2^{n-1} compositions of n , so $E_n = 2^{n-2}$. \square

Problem 4

For $1 \leq i \leq n-1$, let s_i be the adjacent transposition $(i, i+1) \in \mathfrak{S}_n$. It is known that the set $S = \{s_1, \dots, s_{n-1}\}$ generates the group \mathfrak{S}_n . For $w \in \mathfrak{S}_n$, the *Coxeter length* $\ell_S(w)$ is the minimum number r so that $w = s_{i_1} \cdots s_{i_r}$ for some $1 \leq i_1, \dots, i_r \leq n$. Prove that $\ell_S(w) = \text{inv}(w)$ for all $w \in \mathfrak{S}_n$.

Proof. Let $w \in \mathfrak{S}_n$. Consider the bubble sorting algorithm that rearranges the identity permutation by swapping adjacent numbers. Let $w^{(i)}$ be the result after the i th iteration of the algorithm. Note $w^{(0)}$ is the identity permutation. Hence, in the i th iteration, we shift the i th number of w to the i th position. For all i , notice that $w_j^{(i)} = w_j$ for $1 \leq j \leq i$, and $w_j^{(i)} < w_k^{(i)}$ for $i < j < k \leq n$. Thus if $w_i = w_j^{(i-1)}$, then we know $j \geq i$ and the i th iteration of the algorithm would take $j-i$ adjacent transpositions to move w_i to the i th position. But then for $i < k < j$, we know $w_i = w_j^{(i-1)} > w_k^{(i-1)}$ and $w_k^{(i-1)} = w_m$ for some $m > i$. Additionally, $w_k = w_k^{(i-1)}$ for $1 \leq k < i$ so the numbers sorted before $w_j^{(i-1)}$ will not contribute to the number of inversions in w with respect to $w_i = w_j^{(i-1)}$. Hence, let $L(w)$ be the number of adjacent transpositions used to create w with this algorithm, then $\ell_S(w) \leq L(w) = \text{inv}(w)$. It remains to show that $\ell_S(w) \geq \text{inv}(w)$. Notice that the identity permutation is a product of 0 adjacent transpositions, and each transposition increases the number of inversions of a permutation by at most 1. Hence, we need at least $\text{inv}(w)$ adjacent transpositions to produce a permutation with $\text{inv}(w)$ inversions, and thus $\ell_S(w) \geq \text{inv}(w)$. \square

Problem 5

The set $T = \{(i\ j) : 1 \leq i < j \leq n\}$ of all transpositions generates the symmetric group \mathfrak{S}_n . For $w \in \mathfrak{S}_n$, the *absolute length* $\ell_T(w)$ is defined to be the minimum number r so that $w = t_1 t_2 \cdots t_r$ for some $t_1, t_2, \dots, t_r \in T$. Prove that $\ell_T(w) = n - \text{cyc}(w)$.

Proof. Let $w = c_1 \cdots c_k \in \mathfrak{S}_n$, where c_1, \dots, c_k are disjoint cycles and each c_i is a m_i -cycle. Note that c_i is a product of $m_i - 1$ transpositions, so w can be written as a product of $\sum_{i=1}^k (m_i - 1) = \left(\sum_{i=1}^k m_i\right) - k = n - k$ transpositions. Thus, $\ell_T(w) \leq n - k$. It remains to show that $\ell_T(w) \geq n - k$. Notice the identity permutation is a product of n disjoint 1-cycles, and each transposition decreases the number of disjoint cycles of a permutation by at most 1. It now follows that we need at least $n - k$ transpositions to produce a permutation with k cycles, and thus $\ell_T(w) \geq n - k$. \square

Problem 6

Prove the following identity of formal power series using the theory of partitions:

$$\prod_{i \geq 1} \frac{1}{1 - x^i y} = \sum_{k \geq 0} \frac{x^{k^2} y^k}{(1-x)(1-x^2) \cdots (1-x^k)(1-yx)(1-yx^2) \cdots (1-yx^k)}.$$

Proof. Note that the left-hand-side is the generating function for partitions, where the exponent of y represents the number of parts. Given $k \geq 0$, we show how to generate a partition with a $k \times k$ Durfee square. Start with a $k \times k$ Durfee square, this has generating function $x^{k^2} y$. We may choose two partitions with at most k parts to add to the right and bottom sides of the Durfee square. The generating function for partition with at most k parts is $\frac{1}{(1-x)(1-x^2) \cdots (1-x^k)}$. However, each part of the bottom partition contributes to an addition part to the whole partition. Hence, we need to use the generating function which records the number of parts, which is $\frac{1}{(1-yx)(1-yx^2) \cdots (1-yx^k)}$. For partitions with a $k \times k$ Durfee square, we now have the generating function $x^{k^2} y \cdot \frac{1}{(1-x)(1-x^2) \cdots (1-x^k)} \cdot \frac{1}{(1-yx)(1-yx^2) \cdots (1-yx^k)}$. This gives us the right-hand-side. \square

Problem 7

Prove the following identity of formal power series using the theory of partitions:

$$\prod_{i \geq 1} (1 + x^{2i-1}y) = \sum_{k \geq 0} \frac{x^{k^2} y^k}{(1-x^2)(1-x^4) \cdots (1-x^{2k})}.$$

Proof. Note that the left-hand-side equals the generating function for partitions into distinct odd parts where the exponent of y represents the number of parts. Let P_k be the set of such partitions of k parts. Given $\lambda \in P_k$, $|\lambda| \geq 1 + 3 + \cdots + (2k-1) = k^2$, as $\lambda_i \geq 2i-1$. Hence, for $\lambda \in P_k$, we may write $\lambda_i = 2i-1 + 2\mu_i$, where μ_i is even and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$. That is, we may generate P_k by starting with a partition of k^2 into k distinct odd parts, and we choose an non-decreasing sequence of even numbers (μ_1, \dots, μ_k) to add to the corresponding odd parts. This gives us the right-hand-side. \square

Problem 8

Let $\alpha = (\alpha_1, \dots, \alpha_r) \models n$ be a composition of a positive integer n and let \mathbb{F} be an arbitrary field. Let $P_\alpha \subseteq GL_n(\mathbb{F})$ be the parabolic subgroup of block upper-triangular invertible matrices whose diagonal blocks have sizes $\alpha_1 \times \alpha_1, \dots, \alpha_r \times \alpha_r$. Describe a bijection between cosets $GL_n(\mathbb{F})/P_\alpha$ and the family of flags $W_\bullet = (W_0 \subset W_1 \subset \dots \subset W_r)$ of subspaces of \mathbb{F}^n such that $\dim W_i = \alpha_1 + \dots + \alpha_i$.

Proof. Consider flag $0 = V_0 \subset V_1 \subset \dots \subset V_r = \mathbb{F}^n$, where V_i is the span of the first $\alpha_1 + \dots + \alpha_i$ standard basis vector. For $g \in GL_n(\mathbb{F})$, define the map by sending gP_α to $g(V_0) \subset g(V_1) \subset \dots \subset g(V_r)$. Notice $P_\alpha(V_i) = V_i$ for all i , so P_α is a stabilizer of the flag $V_0 \subset V_1 \subset \dots \subset V_r$. Hence, this map is well-defined as $gP_\alpha(V_i) = gV_i$, and it is obviously bijective. \square

Problem 9

Let $n > 1$ be an integer. Use a sign-reversing involution to prove the identity

$$\sum_{w \in \mathfrak{S}_n} (-1)^{\text{inv}(w)} = 0.$$

Your involution should also prove the identity

$$\sum_{w \in \mathfrak{S}_n} (-1)^{\text{cyc}(w)} = 0.$$

Proof. Consider the function $f : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ which swaps $w(1)$ and $w(2)$ if $w(1) = 1$, and swaps $w(1)$ and $w^{-1}(1)$ if $w(1) \neq 1$. Let $w \in \mathfrak{S}_n$. Observe that $f(f(w)) = 1$, so f is an involution. Suppose $w(1) = 1$. Then applying f to w increases the number of inversions by 1. Also, f merges the 1-cycle (1) with the cycle containing $w(2)$, so f reduces the number of cycles of w by 1. Now suppose $w(1) \neq 1$. Then applying f to w decreases the number of inversions by 1. Also, f splits the cycle which contains $w(1)$, so f increases the number of cycles of w by 1. Hence, f changes the parity of $\text{inv}(w)$ and $\text{cyc}(w)$. Thus,

$$\begin{aligned} \sum_{w \in \mathfrak{S}_n} (-1)^{\text{inv}(w)} &= \frac{1}{2} \left(\sum_{w \in \mathfrak{S}_n} (-1)^{\text{inv}(w)} + (-1)^{\text{inv}(f(w))} \right) = 0, \\ \sum_{w \in \mathfrak{S}_n} (-1)^{\text{cyc}(w)} &= \frac{1}{2} \left(\sum_{w \in \mathfrak{S}_n} (-1)^{\text{cyc}(w)} + (-1)^{\text{cyc}(f(w))} \right) = 0. \end{aligned}$$

□

Problem 10

Let n and m be positive integers and let $f(n, m)$ be the number of $0, 1$ -matrices of dimensions $n \times m$ which do not contain any zero rows or columns. Find a formula for $f(n, m)$.

Proof. For $S \subseteq [m]$, let $g(S)$ be the number of $0, 1$ -matrices of dimensions $n \times m$ that have no zero rows but have zero columns indexed by S , and let $h(S)$ be the number of $0, 1$ -matrices of dimensions $n \times m$ that have no zero rows and have zero columns exactly at the columns indexed by S . Note that for all $S \subseteq [m]$,

$$g(S) = \sum_{S \subseteq T \subseteq [m]} h(T).$$

Given $T \subseteq [m]$, there are $(2^{m-|T|} - 1)^n$ ways to choose n rows that is not completely zero in the columns not indexed by T and zero in the columns indexed by T , so $g(T) = (2^{m-|T|} - 1)^n$. Principle of Inclusion-Exclusion now yields

$$f(n, m) = h(\emptyset) = \sum_{T \subseteq [m]} (-1)^{|T|} g(T) = \sum_{k=1}^m (-1)^k \binom{m}{k} (2^{m-k} - 1)^n.$$

□

Problem 11

What is the average number of fixed points of a permutation $w \in \mathcal{S}_n$?

Proof. Let X be the expected number of fixed points in a random permutation $w \in \mathcal{S}_n$, and let X_i be the indicator for the event that $w(i) = i$. Then

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n \mathbb{P}(w(i) = i) = \sum_{i=1}^n \frac{1}{n} = 1.$$

□

Problem 12

Let q be a prime power and let \mathbb{F}_q be the finite field with q elements. For $k \leq n$, find a formula for the number of \mathbb{F}_q -linear surjections $\varphi : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^k$.

Proof. For each subspace U of \mathbb{F}_q^k , define $f(U)$ to be the number of linear maps $\mathbb{F}_q^n \rightarrow \mathbb{F}_q^k$ whose image lies in U , and define $g(U)$ to be the number of linear maps $\mathbb{F}_q^n \rightarrow \mathbb{F}_q^k$ whose image is U . Obviously,

$$f(U) = \sum_{V \text{ subspace of } U} g(V).$$

By Möbius inversion on the lattice of subspaces of \mathbb{F}_q^k ,

$$g(\mathbb{F}_q^k) = \sum_{V \text{ subspace of } \mathbb{F}_q^k} \mu(V, \mathbb{F}_q^k) f(V).$$

Since $\mu(V, \mathbb{F}_q^k) = (-1)^{k-\dim(V)} q^{\binom{k-\dim(V)}{2}}$ and $f(V) = q^{n \cdot \dim V}$,

$$g(\mathbb{F}_q^k) = \sum_{i=0}^k \sum_{\substack{V \subseteq \mathbb{F}_q^k \\ \dim V = i}} (-1)^{k-i} q^{\binom{k-i}{2}} q^{ni}.$$

□

Problem 13

Let P be a finite poset which has a minimum element $\hat{0}$. Prove that the order complex $\Delta(P)$ is contractible. (This means that $\Delta(P)$ can be continuously deformed to a point. For example, the 2-dimensional disc D^2 is contractible whereas its boundary circle S^1 is not.)

Proof. Note that $\hat{0}$ is minimal and any chain in P can be extended to another chain in P that includes $\hat{0}$ as a unique minimum element. Hence, all maximal simplices in $\Delta(P)$ are simplices of $\{\hat{0}\}$. Hence, given a maximal chain C in P , we continuously deform the corresponding simplex of C by iteratively removing the elements of starting from the end of the chain to $\hat{0}$. This deformation iteratively reduces the dimension of the corresponding simplex of C until it is a point at $\hat{0}$. Hence, $\Delta(P)$ is contractible.

(I have no idea what I wrote.)

□

Problem 14

Let Π_n be the lattice of set partitions of $[n]$, partially ordered by $\pi \leq \sigma$ if and only if π refines σ . These exercises outline an alternative proof that

$$\mu_{\Pi_n}[\hat{0}, \hat{1}] = (-1)^{n-1} \cdot (n-1)!.$$

- (a) Let $\pi = \{B_1 \mid \cdots \mid B_k\}$ be a set partition of $[n]$. Give a poset isomorphism

$$[\hat{0}, \pi]_{\Pi_n} \cong [\hat{0}, \hat{1}]_{\Pi_{|B_1|}} \times \cdots \times [\hat{0}, \hat{1}]_{\Pi_{|B_k|}}.$$

Proof. Let $\sigma \in [\hat{0}, \pi]_{\Pi_n}$. Note that each block of σ is a subset of some unique B_i . Hence, we may index the blocks of σ that refines B_i as $B_{i_1}, B_{i_2}, \dots, B_{i_{m_i}}$. Define $f : [\hat{0}, \pi]_{\Pi_n} \rightarrow [\hat{0}, \hat{1}]_{\Pi_{|B_1|}} \times \cdots \times [\hat{0}, \hat{1}]_{\Pi_{|B_k|}}$ by

$$f(\sigma) = (\sigma_{B_1}, \dots, \sigma_{B_k}),$$

where $\sigma_{B_i} = \{B_{i_1} \mid \cdots \mid B_{i_{m_i}}\} \in \Pi_{|B_i|}$. Since the operation of f can be reversed, f is well-defined and bijective. Suppose $x, y \in [\hat{0}, \pi]_{\Pi_n}$ such that $x \leq y$. Then $x_{B_i} \leq y_{B_i}$ for all i , so $f(x) \leq f(y)$. Hence, f is order-preserving. \square

- (b) We have a natural map $\varphi : \mathcal{S}_n \rightarrow \Pi_n$ where $\varphi(w)$ is the set partition of $[n]$ whose blocks are the cycles of the permutation $w \in \mathcal{S}_n$. If $\pi = \{B_1 \mid \cdots \mid B_k\}$ is a set partition of $[n]$, prove that the pre-image $\varphi^{-1}(\pi)$ has size

$$\#\varphi^{-1}(\pi) = (|B_1| - 1)! \cdots (|B_k| - 1)!.$$

Proof. Since there are $(r-1)!$ ways to arrange r distinct elements into a cycle, for each B_i there are $(|B_i| - 1)!$ possible cycles that would result in B_i as a block. The result now follows. \square

- (c) Use the results in Problems 1 and 2 to deduce that $\mu_{\Pi_n}[\hat{0}, \hat{1}]$. (Use induction on n . It may be helpful to remember a result about the alternating sum of Stirling numbers of the first kind.)

Proof. We proceed on induction on n . For $n = 1$, Π_1 only has one partition, so $\mu_{\Pi_1}[\hat{0}, \hat{1}] = 1$. Suppose $n \geq 2$. The Möbius function satisfies

$$\mu_{\Pi_n}[\hat{0}, \hat{1}] = - \sum_{\pi < \hat{1}} \mu_{\Pi_n}[\hat{0}, \pi].$$

For $\pi < \hat{1}$,

$$\mu_{\Pi_n}[\hat{0}, \pi] = \mu_{\Pi_{|B_1|}}[\hat{0}, \hat{1}] \cdots \mu_{\Pi_{|B_{k_\pi}|}}[\hat{0}, \hat{1}] = \prod_{i=1}^k (-1)^{|B_i|-1} (|B_i| - 1)! = (-1)^{n-k} (|B_1| - 1)! \cdots (|B_{k_\pi}| - 1)!,$$

by (a) and induction. But then by (b),

$$\mu_{\Pi_n}[\hat{0}, \hat{1}] = - \sum_{\pi < \hat{1}} (-1)^{n-k_\pi} \cdot \#\varphi^{-1}(\pi) = (-1)^{n-1} \sum_{k=2}^n (-1)^k c(n, k).$$

Since $\sum_{k=1}^n (-1)^k c(n, k) = 0$,

$$\mu_{\Pi_n}[\hat{0}, \hat{1}] = (-1)^{n-1} c(n, 1) = (-1)^{n-1} (n-1)!.$$

\square

Problem 15

Let \mathcal{A} be the arrangement in \mathbb{R}^n and let R be a region of \mathcal{A} . The *recession cone* of R is

$$\text{Rec}(R) := \{v \in \mathbb{R}^n \mid R + v \subseteq R\}.$$

Prove that $\text{Rec}(R)$ is a *cone* in \mathbb{R}^n . That is, prove that if $v_1, \dots, v_r \in \text{Rec}(R)$ and if $a_1, \dots, a_r > 0$ we have

$$a_1 v_1 + \dots + a_r v_r \in \text{Rec}(R).$$

Proof. Let $x \in R$. By definition of $\text{Rec}(R)$, we have $x + v_i \in R$ for all i . Since R is convex, $x + t v_i \in R$ for all $t \in [0, 1]$. But then given any $a_i > 0$, we may repeatedly apply the above argument to show that $x + a_i v_i \in R$. The result now follows from induction on r . \square

Problem 16

Let \mathcal{A} be the arrangement in \mathbb{R}^n with hyperplanes $x_i = x_j$ for $1 \leq i < j \leq n$ and

$$x_1 + x_2 + \cdots + x_n = 0.$$

Prove that \mathcal{A} has characteristic polynomial

$$\chi_{\mathcal{A}}(t) = (t-1)^2(t-2)(t-3) \cdots (t-n+1).$$

Proof. We now use the finite field method to compute $\chi_{\mathcal{A}}(t)$. Let q be a large prime power and let \mathbb{F}_q be the finite field with q elements. The finite field method yields

$$\begin{aligned} \chi_{\mathcal{A}}(q) &= \# \left(\mathbb{F}_q^n \setminus \bigcup_{H^{(p)} \in \mathcal{A}^{(p)}} H^{(p)} \right) \\ &= \#\{x \in \mathbb{F}_q^n \mid x_i \neq x_j \text{ for } i \neq j \text{ and } x_1 + \cdots + x_n \neq 0\} \\ &= \#\{x \in \mathbb{F}_q^n \mid x_i \neq x_j \text{ for } i \neq j\} - \#\{x \in \mathbb{F}_q^n \mid x_i \neq x_j \text{ for } i \neq j \text{ and } x_1 + \cdots + x_n = 0\}. \end{aligned}$$

For $a \in \mathbb{F}_q$, define

$$S_a = \#\{x \in \mathbb{F}_q^n \mid x_i \neq x_j \text{ for } i \neq j \text{ and } x_1 + \cdots + x_n = a\}.$$

Let $a, b \in \mathbb{F}_q$. Since q is prime, there exists unique $k \in \mathbb{F}_q$ such that $a + nk \equiv b \pmod{q}$. Define $f_{a,b} : S_a \rightarrow S_b$ by

$$f_{a,b}(x_1, \dots, x_n) = (x_1 + k, x_2 + k, \dots, x_n + k).$$

Define $f_{b,a}$ in the same fashion, and we get $f_{a,b} \circ f_{b,a} = f_{b,a} \circ f_{a,b}$ is the identity map. Hence, $|S_a| = |S_b|$ for all $a, b \in \mathbb{F}_q$, and so

$$|S_0| = \frac{1}{q} \cdot \#\{x \in \mathbb{F}_q^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Since

$$\#\{x \in \mathbb{F}_q^n \mid x_i \neq x_j \text{ for } i \neq j\} = q(q-1)(q-2) \cdots (q-n+1),$$

we have

$$\chi_{\mathcal{A}}(q) = (q-1)^2(q-2)(q-3) \cdots (q-n+1).$$

The result now follows. □