MATH 140A: Homework #6

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Professor Seward

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Problem 1

Calculate $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$.

Proof. We show that the limit is $\frac{1}{2}$. Since $\lim_{n\to\infty}\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}=1$, we have

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right) &= \lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)\left(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right)\\ &= \lim_{n\to\infty}\left(\frac{n}{\sqrt{n^2+n}+n}\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right),\end{aligned}$$

by Theorem 3.3. Note that

$$\frac{1}{1+\frac{1}{n}+1}=\frac{1}{\frac{1}{n}+2}<\frac{1}{\sqrt{1+\frac{1}{n}}+1}<\frac{1}{1+1}=\frac{1}{2}.$$

Since $\frac{1}{\frac{1}{n}+2}\to\frac{1}{2}$, the result follows from Theorem 3.19. ☐

Problem 2

Find the upper and lower limits of the sequence (s_n) defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}, \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Proof. We first show that $s_{2m+1} = 1 - 2^{-m}$ by induction on m. If $m = 0$, $s_1 = 1 - 2^0 = 0$. Suppose $m > 0$. We know $s_{2m+1} = s_{2m} + \frac{1}{2} = \frac{s_{2(m-1)+1}}{2} + \frac{1}{2}$. It follows that

$$\frac{s_{2(m-1)+1}}{2} + \frac{1}{2} = \frac{1 - 2^{-(m-1)}}{2} + \frac{1}{2} = 1 - 2^{-m},$$

by induction. Hence $s_{2m+1} = 1 - 2^{-m}$, and thus $s_{2m} = s_{2m+1} - \frac{1}{2} = \frac{1}{2} - 2^{-m}$. By Theorem 3.20,

$$\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} (1 - 2^{-m}) = 1,$$

$$\lim_{m\to\infty} s_{2m} = \lim_{m\to\infty} \left(\frac{1}{2} - 2^{-m}\right) = \frac{1}{2}.$$

Since subsequences of s_n contains either a subsequence of s_{2m} or a subsequence of s_{2m+1}, any convergence sequence converges to either 1 or $\frac{1}{2}$. Therefore, the upper limit and lower limit of (s_n) are 1 and $\frac{1}{2}$, respectively. $\square$

Problem 3

For any two real sequences $(a_n), (b_n)$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Proof. The inequality obviously holds for the case $\limsup_{n\to\infty} a_n = \infty$ and $\limsup_{n\to\infty} b_n > -\infty$.

Suppose $\limsup_{n\to\infty} a_n = -\infty$ and $\limsup_{n\to\infty} b_n < \infty$. Then, there are no subsequential limits for a_n and b_n is bounded above by some b. Consider subequence $(a_{n_k} + b_{n_k})$. Suppose for the sake of contradiction that $(a_{n_k} + b_{n_k})$ converges at some point p. Let $r > 0$. Since a_n has no subsequential limits, there are only at most finitely many values of n such that $a_n > p - r - b$. It follows that the neighborhood $N_r(p)$ only contains at most finitely many values of n such that $a_n + b_n \in N_r(p)$, contradiction. Hence,

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = -\infty,$$

and the inequality holds.

It remains to show the case for $\limsup_{n\to\infty} a_n = p$ and $\limsup_{n\to\infty} b_n = q$, for some $p, q \in \mathbb{R}$. Since both a_n and b_n have subsequential limits, a_n and b_n are bounded. It follows that $(a_n + b_n)$ are also bounded, so $\limsup_{n\to\infty}(a_n + b_n) = r$, for some $r \in \mathbb{R}$, by Theorem 3.6. Theorem 3.7 shows that there exists subsequence $(a_{n_k} + b_{n_k})$ such that $a_{n_k} + b_{n_k} \to r$. Since a_{n_k} is bounded, there exists subsequence $a_{n_{k_p}}$ of a_{n_k} such that $a_{n_{k_p}} \to \limsup_{k\to\infty} a_{n_k}$. The subsequence $(a_{n_{k_p}} + b_{n_{k_p}})$ of $(a_{n_k} + b_{n_k})$ also converges to r. By Theorem 3.3, $\lim_{p\to\infty} a_{n_{k_p}} + \lim_{p\to\infty} b_{n_{k_p}} = \lim_{p\to\infty}(a_{n_{k_p}} + b_{n_{k_p}})$, and so $b_{n_{k_p}}$ is also a convergence sequence. Hence, we have shown the existence of convergence subsequences $a_{n_{k_p}}$ and $b_{n_{k_p}}$. It immediately follows that

$$r = \lim_{n\to\infty}(a_{n_k} + b_{n_k}) = \lim_{p\to\infty}(a_{n_{k_p}} + b_{n_{k_p}}) = \lim_{p\to\infty} a_{n_{k_p}} + \lim_{p\to\infty} b_{n_{k_p}} \le p + q,$$

and this completes the proof. $\square$

Problem 4

If (s_n) is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1} \quad (n = 0, 1, 2, \ldots).$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

Proof. Fix $\epsilon > 0$. There exsits N such that for all $n \geq N$, $|s - s_n| < \frac{\epsilon}{2}$. Pick integer N' such that $\frac{\epsilon}{2} N' > \sum_{i=0}^{N-1} |s - s_i|$. Then for $n \geq \max(N, N')$,

$$\begin{aligned}
|s - \sigma_n| &= \left| s - \frac{1}{n+1} \sum_{i=0}^{n} s_i \right| \\
&\leq \frac{1}{n+1} \sum_{i=0}^{n} |s - s_i| \\
&= \frac{1}{n+1} \left(\sum_{i=0}^{N-1} |s - s_i| + \sum_{i=N}^{n} |s - s_i| \right) \\
&< \frac{1}{n+1} \left(\sum_{i=0}^{N-1} |s - s_i| + (n - N + 1) \frac{\epsilon}{2} \right) \\
&= \frac{\sum_{i=0}^{N-1} |s - s_i|}{n+1} + \frac{n - N + 1}{n+1} \cdot \frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence, $\sigma_n \to s$. □

(b) Construct a sequence (s_n) which does not converge, although $\lim \sigma_n = 0$.

Proof. Consider (s_n), with $s_1 = 1$, $s_{2k} = -1$, and $s_{2k+1} = 1$. s_n obviously does not converge. Since

$$\begin{aligned}
\sigma_n &= \begin{cases} \frac{1}{n} \left(\sum_{i=1}^{k} 1 + \sum_{i=1}^{k} -1 \right) & n = 2k, \text{ for some } k \in \mathbb{N} \\ \frac{1}{n} \left(1 + \sum_{i=1}^{k} 1 + \sum_{i=1}^{k} -1 \right) & n = 2k+1, \text{ for some } k \in \mathbb{N} \end{cases} \\
&= \begin{cases} 0 & n = 2k, \text{ for some } k \in \mathbb{N} \\ \frac{1}{n} & n = 2k+1, \text{ for some } k \in \mathbb{N} \end{cases},
\end{aligned}$$

we get $\sigma_n \to 0$. □

Problem 5

Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \dots$ by the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

Prove that (x_n) decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

Proof. We show that $x_n > \sqrt{\alpha}$ by induction on n. $x_1 > \sqrt{\alpha}$, obviously. Suppose $n > 1$. By induction, $x_{n-1} > \sqrt{\alpha}$, and the induction result then follows from

$$\frac{(x_{n-1} - \sqrt{\alpha})^2}{2x_{n-1}} = \frac{1}{2}\left(x_{n-1} + \frac{\alpha}{x_{n-1}}\right) - \sqrt{\alpha} = x_n - \sqrt{\alpha} > 0.$$

Notice that since $x_n^2 > \alpha$, we substitute α from the recursion formula and get $x_{n+1} < x_n$, and thus x_n is monotonically decreasing. It remains to show $x_n \to \sqrt{\alpha}$. Note that $\lim x_n = \lim x_{n+1} = a$, for some $a \geq \sqrt{\alpha}$. But then $a = \frac{1}{2}\left(a + \frac{\alpha}{a}\right)$, the solving the equation gives us $a = \sqrt{\alpha}$, and we are done. $\square$

Problem 6

Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$ and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(a) Prove that $x_1 > x_3 > x_5 > \dots$.

Proof. We first note that

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = \frac{\alpha + \left(\frac{\alpha + x_{n-1}}{1 + x_{n-1}}\right)}{1 + \left(\frac{\alpha + x_{n-1}}{1 + x_{n-1}}\right)} = \frac{2\alpha + (1+\alpha)x_{n-1}}{(1+\alpha) + 2x_{n-1}} = x_{n-1} + \delta_n, \tag{1}$$

where $\delta_n = \frac{\alpha - x_{n-1}^2}{\frac{1}{2}(1+\alpha) + x_{n-1}}$. Hence, if $x_{n-1} > \sqrt{\alpha}$, then $\delta_n < 0$ and thus $x_{n+1} < x_{n-1}$. Otherwise, we have $\delta_n > 0$, and so $x_{n+1} > x_{n-1}$.

Let $a_m = x_{2m-1}$, for $m \geq 1$. We now show that $a_m > \sqrt{\alpha}$ by induction on m. The base case is clear. Suppose $m > 1$. By induction, $a_{m-1} > \sqrt{\alpha}$, and so $a_m - a_{m-1} = \delta_m < 0$. Hence, a_m is monotonically decreasing. $\square$

(b) Prove that $x_2 < x_4 < x_6 < \dots$.

Proof. Similar to (a), we show that $b_m = x_{2m} < \sqrt{\alpha}$ by induction on m. We first prove the base case $m = 1$. Let $\epsilon = x_1 - \sqrt{\alpha} > 0$. Then,

$$x_2 = x_1 + \frac{\alpha - x_1^2}{1 + x_1} = x_1 + \frac{(\sqrt{\alpha} - x_1)(\sqrt{\alpha} + x_1)}{1 + x_1} = x_1 - \frac{\sqrt{\alpha} + x_1}{1 + x_1} \cdot \epsilon.$$

It follows that $\frac{\sqrt{\alpha}+x_1}{1+x_1} > 1$, so $x_2 < x_1 - \epsilon = \sqrt{\alpha}$. Suppose $m > 1$. Define δ_m the way we did in (1). By induction, $b_{m-1} < \sqrt{\alpha}$, and so $b_m - b_{m-1} = \delta_m > 0$. Hence, b_m is monotonically increasing. $\square$

(c) Prove that $\lim x_n = \sqrt{\alpha}$.

Proof. We show that both subsequences a_n and b_n converge to $\sqrt{\alpha}$. Since both a_n and b_n are bounded and monotomic, by Theorem 3.14, $a_n \to a$ and $b_n \to b$, where $a \geq \sqrt{\alpha} \geq b$. Notice that $\lim a_n = \lim a_{n+1} = a$ and $\lim b_n = \lim b_{n+1} = b$. By (1),

$$a = a + \lim \delta_m = a + \frac{\alpha - a^2}{\frac{1}{2}(1+\alpha) + a},$$

$$b = b + \lim \delta_m = b + \frac{\alpha - b^2}{\frac{1}{2}(1+\alpha) + b},$$

and solving the equations gives us $a = b = \sqrt{\alpha}$. Take $\gamma > 0$. There exists m_a and m_b such that $|a_k - \sqrt{\alpha}|, |b_l - \sqrt{\alpha}| < \gamma$, for all $k > m_a$ and $l > m_b$. Hence, for all $n \geq \max(m_a, m_b)$, we have $|x_n - \sqrt{\alpha}| < \gamma$, and the result follows. $\square$

Problem 7

Suppose (p_n) is a Cauchy sequence in a metric space X, and some subsequence (p_{n_i}) converges to a point $p \in X$. Prove that the full sequence (p_n) converges to p.

Proof. Fix $\epsilon > 0$. There exists integer N such that $d(p_n, p_m) < \frac{\epsilon}{2}$, for all $m, n \geq N$. Since (p_{n_i}) converges, there exists N' such that $d(p_{n_i}, p) < \frac{\epsilon}{2}$, for all $i \geq N'$. Hence, for all $n \geq N$, pick $i > N'$ such that $n_i \geq N$ and we have

$$d(p_n, p) \leq d(p_n, p_{n_1}) + d(p_{n_i}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and the result follows. □