MATH 220B: Homework #2

Due on Feb 8, 2025 at 23:59pm

Professor Xiao

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Problem 1

Suppose f is analytic on $\overline{B}(0;1)$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Find the number of solutions (counting multiplicities) of the equation $f(z) = z^n$ where n is an integer larger than or equal to 1.

Proof. Let $g(z) = z^n$, $h(z) = f(z) - g(z)$. Since

$$|h(z) + g(z)| = |f(z)| < 1 = |g(z)|$$

for $|z| = 1$, by Rouche's theorem, $h(z)$ has the same number of zeros as $g(z)$ in $B(0;1)$, that is, n zeros. $\square$

Problem 2

Prove the following Minimum Principle. If f is a non-constant analytic function on a bounded open set G and is continuous on $\overline{G}$, then either f has a zero in G or $|f|$ assumes its minimum value on ∂G. (See Exercise IV. 3.6.)

Proof. If there exists $a \in G$ such that $|f(a)| \leq |f(z)|$ for all $z \in G$, then $f(a) = 0$ by Exercise IV.3.6. Otherwise, $|f|$ assumes its minimum value on ∂G as it is continuous on $\overline{G}$. $\square$

Problem 3

Let G be a bounded region and suppose f is continuous on $\overline{G}$ and analytic on G. Show that if there is a constant $c \geq 0$ such that $|f(z)| = c$ for all z on the boundary of G then either f is a constant function or f has a zero in G.

Proof. Suppose f is not constant. By the Maximum Modulus Principle, $|f(z)| \leq c$ for all $z \in G$ otherwise $|f|$ would assume its maximum value in G. But then by the Minimum Principle we just proved, f has a zero in G. □

Problem 4

(a) Let f be entire and non-constant. For any positive real number c show that the closure of $\{z : |f(z)| < c\}$ is the set $\{z : |f(z)| \leq c\}$.

Proof. Since f is continuous, it suffices to show that any z with $|f(z)| = c$ are in the closure of $\{z : |f(z)| < c\}$. Suppose there exists z_0 such that $|f(z_0)| = c$ and z_0 is not in the closure of $\{z : |f(z)| < c\}$. Then there exists $r > 0$ such that $B_r(z_0) \cap \{z : |f(z)| < c\} = \emptyset$. That is, $|f(z)| \geq c$ for all $z \in B_r(z_0)$. But then $f(B_r(z_0))$ is open by the Open Mapping Theorem, so $f(B_r(z_0))$ contains an open neighborhood U of $f(z_0)$. This implies $|f(z_0)| < c$ for some $z \in B_r(z_0)$, contradiction. $\square$

(b) Let p be a polynomial and show that each component of $\{z : |p(z)| < c\}$ contains a zero of p.

Proof. We may assume p is not constant. Note that each component of $\{z : |p(z)| < c\}$ is bounded, otherwise p is constant by the Louiville's Theorem. By (a), the closure of $\{z : |p(z)| < c\}$ is $\{z : |p(z)| \leq c\}$. Since each component G of $\{z : |p(z)| \leq c\}$ is bounded and $|p(z)| = c$ for all $z \in \partial G$, p has a zero in G by the previous problem. $\square$

Problem 5

Suppose that both f and g are analytic on $\overline{B}(0;R)$ with $|f(z)| = |g(z)|$ for $|z| = R$. Show that if neither f nor g vanishes in $B(0;R)$ then there is a constant λ, $|\lambda| = 1$, such that $f = \lambda g$.

Proof. We first show that the multiplicities of the zeros of f and g on the boundary are the same. Suppose f has a zero of order n at z_0 and g has a zero of order m at z_0 with $|z_0| = R$ and $n \geq m$. Then $f(z) = (z - z_0)^n F(z)$ and $g(z) = (z - z_0)^m G(z)$ for some analytic functions F and G with $F(z_0), G(z_0) \neq 0$. Since $|f(z)| = |g(z)|$ for $|z| = R$, we have $|z - z_0|^{n-m} = \left|\frac{G(z)}{F(z)}\right|$. But then $n = m$ and $F(z_0) = G(z_0)$, otherwise $G(z_0) = 0$. Hence, we may define $h(z) = \frac{g(z)}{f(z)}$ on $\overline{B}_R(0)$. Since $|h(z)| = 1$ for $|z| = R$ and h has not zeros in $B_R(0)$, $|h(z)| = 1$ for all $z \in \overline{B}_R(0)$ by Exercise VI.1.2. The result now follows. □

Problem 6

Let f be analytic in the disk $B(0;R)$ and for $0 \leq r < R$ define $A(r) = \max\{\text{Re} f(z) : |z| = r\}$. Show that unless f is a constant, $A(r)$ is a strictly increasing function of r.

Proof. Assume that f is not a constant. Let $0 \leq r_1 < r_2 < R$. Consider $g(z) = e^{f(z)}$ over $\overline{B}_{r_2}(0)$. Note that $|g(z)| = e^{\text{Re} f(z)}$ attains the maximum at the same point as $\text{Re} f(z)$. Suppose $A(r_1) \geq A(r_2)$. Then $|g(z)|$ attains a maximum in $B_{r_2}(0)$, which makes $g(z)$ constant by the Maximum Modulus Principle, contradiction. $\square$

Problem 7

Does there exist an analytic function $f : D \to D$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$?

Proof. By the Schwarz-Pick Lemma,

$$|f'(\frac{1}{2})| \leq \frac{1 - |f(\frac{1}{2})|^2}{1 - |\frac{1}{2}|^2} = \frac{1 - \frac{9}{16}}{1 - \frac{1}{4}} = \frac{7}{12} < \frac{2}{3},$$

and thus such analytic function does not exist. □

Problem 8

Suppose $f : D \to \mathbb{C}$ satisfies $\operatorname{Re} f(z) \geq 0$ for all z in D and suppose that f is analytic.

(a) Show that $\operatorname{Re} f(z) > 0$ for all z in D.

Proof. Suppose $z \in D$ such that $\operatorname{Re} f(z) = 0$. By the Open Mapping Theorem, $f(D)$ is open, so there exists $r > 0$ such that $B_r(f(z)) \subset f(D)$. But then $B_r(f(z))$ contains points with negative real part, contradiction. $\square$

(b) By using an appropriate Möbius transformation, apply Schwarz's Lemma to prove that if $f(0) = 1$ then

$$|f(z)| \leq \frac{1+|z|}{1-|z|}$$

for $|z| < 1$. What can be said if $f(0) \neq 1$?

Proof. Let $\phi(z) = \frac{z-1}{z+1}$ and consider $g(z) = \phi \circ f(z)$. Note that $g(0) = \phi(f(0)) = \phi(1) = 0$. Since ϕ maps $\{z : \operatorname{Re}(z) > 0\}$ to D, g maps D to D. By Schwarz's Lemma, $|g(z)| \leq |z|$ for $z \in D$. That is,

$$|z| \geq \frac{|f(z)-1|}{|f(z)+1|} \geq \frac{|f(z)|-1}{|f(z)|+1}.$$

The result now follows from rearranging the inequality. If $f(0) = \alpha$ for some $\alpha \neq 1$, apply the transformation $\phi(z) = \frac{z-a}{z+a}$ instead. $\square$

(c) Show that f also satisfies

$$|f(z)| \geq \frac{1-|z|}{1+|z|}.$$

Proof. Note that $\operatorname{Re} \frac{1}{f(z)} > 0$ for all $z \in D$. Hence, consider $h(z) = \phi \circ (1/f)(z)$. $h(0) = 0$ and h maps D to D. By Schwarz's Lemma, $|h(z)| \leq |z|$ for $z \in D$. That is,

$$|z| \geq \frac{|1/f(z)-1|}{|1/f(z)+1|} \geq \frac{1-|f(z)|}{1+|f(z)|}.$$

The result now follows. $\square$

Problem 9

Suppose f is analytic in some region containing $\overline{B}(0;1)$ and $|f(z)| = 1$ where $|z| = 1$. Find a formula for f. (Hint: First consider the case where f has no zeros in $\overline{B}(0;1)$.)

Proof. Suppose f has no zeros in $\overline{B}_1(0)$. Then by the exercise in the start of this assignment, $f = c$ with $|c| = 1$. Suppose f has zeros $a_1, \ldots, a_m$ in $\overline{B}_1(0)$. Since $|f(z)| = 1$ for $|z| = 1$, we know $a_1, \ldots, a_m \in B_1(0)$. Then $f(z) = g(z)\prod_{i=1}^m z - a_i$ with $g(z) \neq 0$ for all $z \in \overline{B}_1(0)$. Consider $\frac{f(z)}{\prod_{i=i}^m \phi_{a_i}(z)}$. Since $|\phi_{a_i}(z)| = 1$ for $|z| = 1$, $\frac{f(z)}{\prod_{i=i}^m \phi_{a_i}(z)} = 1$ for $|z| = 1$. But then for all a_i,

$$\lim_{z \to a_i} \frac{f(z)}{\prod_{i=i}^m \phi_{a_i}(z)} = \lim_{z \to a_i} g(z) \prod_{i=i}^m (1 - \overline{a_i} z) \neq 0$$

for all $z \in B_1(0)$. Hence, $\frac{f(z)}{\prod_{i=i}^m \phi_{a_i}(z)} \neq 0$ on $\overline{B}_1(0)$. By the exercise in the start of this assignment, $\frac{f(z)}{\prod_{i=i}^m \phi_{a_i}(z)} = 1$. It now follows that $f(z) = \prod_{i=1}^m \phi_{a_i}(z)$. ☐

Problem 10

Is there an analytic function f on $B(0;1)$ such that $|f(z)| < 1$ for $|z| < 1$, $f(0) = \frac{1}{2}$, and $f'(0) = \frac{3}{4}$. If so, find such an f. Is it unique?

Proof. Let $\phi_{\frac{1}{2}}(z) = \frac{z-1/2}{1-z/2}$ be defined as in the textbook, and let $g = \phi_{\frac{1}{2}} \circ f$. Since g maps $B(0;1)$ to $B(0;1)$, $g(0) = 0$, and $|g'(0)| = |\phi'_{\frac{1}{2}} \circ f(0) \cdot f'(0)| = 1$, by Schwarz's Lemma, $g(z) = cz$ for some $|c| = 1$. Hence, $f(z) = \frac{\frac{1}{2}+cz}{1+\frac{c}{2}z}$. $\square$