Superimposed Extremal Graphs

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1 Introduction

Given graph G with n vertices, let $G_1, \dots, G_m$ be subgraphs of G. Let F be a graph with at least one edge. Our goal is to determine the maximum sum of the number of edges in each G_i, i.e. $\sum_{i=1}^{m} e(G_i)$, with the constraint of $E(G_i) \cap E(G_j)$ not including F for all distinct i, j.

2 Content

- Examine the case where $G_1, \dots, G_m$ are induced
  - The case $F = K_3$.
  - Generalize to any F.
- Examine the non-induced case
  - The case $F = K_3$.

3 Induced Case

In this section, we assume that $G_1, \dots, G_m$ are induced subgraphs of G.

3.1 Triangle-Free Case

Theorem 3.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j. For $m \geq 2$,*

$$\sum_{i=1}^{m} e(G_i) \leq m \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = \cdots = G_m = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

We claim that it suffices to show for the case $m = 2$. Suppose the theorem holds for $m = 2$. Put $G_{m+1} = G_1$ and we have

$$\sum_{i=1}^{m} e(G_i) = \frac{1}{2} \sum_{i=1}^{m} (e(G_i) + e(G_{i+1})) \leq \frac{1}{2} \sum_{i=1}^{m} 2 \left\lfloor \frac{n^2}{4} \right\rfloor = m \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality only if $G_i = G_{i+1} = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ for all i. That is, $G_1 = G_2 = \cdots = G_m = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

Proof for $m = 2$. Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both G_1 and G_2. Let $A = V(G_1)\backslash C$, and let $B = V(G_2)\backslash C$. For simplicity, put $a = |A|$, $b = |B|$, and $c = |C|$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to a, b, c. Since G_1, G_2 are induced graphs, we have $\{u, v\} \in E(G_1)$ if and only if $\{u, v\} \in E(G_2)$, for $u, v \in C$. This implies the subgraph of G_1 induced by C is identical to the subgraph of G_2 induced by C. In other words, $E(G_1[C]) = E(G_2[C]) = E(G_i) \cap E(G_j)$, which is triangle-free. By Mantel's Theorem, $e(G_1[C]) \leq \left\lfloor \frac{c^2}{4} \right\rfloor$, with equality if and only if $G_1[C] = K_{\lceil \frac{c}{2} \rceil, \lfloor \frac{c}{2} \rfloor}$. Hence, we may write

$$\begin{aligned} e(G_1) + e(G_2) &\leq \binom{|V(G_1)|}{2} + \binom{|V(G_2)|}{2} - 2\left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor\right] \\ &= \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor\right]. \end{aligned}$$

Define $f(a, b, c)$ as the function on the right-hand-side. We show that $f(a, b, c)$ attains its maximum at $a = b = 0$ and $c = n$. Note that

$$\begin{aligned} f(a, b-1, c+1) - f(a, b, c) &= (a + c) - 2\left(c - \left\lfloor \frac{(c+1)^2}{4} \right\rfloor + \left\lfloor \frac{c^2}{4} \right\rfloor\right) \\ &= (a + c) - 2\left\lfloor \frac{c}{2} \right\rfloor \\ &= \begin{cases} a & c \text{ is even} \\ a + 1 & c \text{ is odd} \end{cases}. \end{aligned}$$

Do the extremal graphs really have to be $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$'s when n is odd? Consider $n = 3$, $G_1 = K_2$, and $G_2 = K_3$. $e(G_1) + e(G_2) = 4 = 2\left\lfloor \frac{3^2}{4} \right\rfloor$. □

3.2 Generalize to any F

Theorem 3.2. *Suppose that $E(G_i) \cap E(G_j)$ does not include F for distinct i, j. For $m \geq 2$,*

$$\sum_{i=1}^{m} e(G_i) \leq m \cdot \mathrm{ex}(n, F),$$

with equality if and only if $G_1 = G_2 = \cdots = G_m$ are equal to an extremal F-free graph.

By the same argument as in Theorem 3.1, it suffices to show the statement holds for $m = 2$.

Proof for $m = 2$. Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both G_1 and G_2. Let $A = V(G_1)\backslash C$, and let $B = V(G_2)\backslash C$. For simplicity, put $a = |A|$, $b = |B|$, and $c = |C|$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to a, b, c. Since G_1, G_2 are induced graphs, we have $E(G_1[C]) = E(G_2[C]) = E(G[C]) = E(G_i) \cap E(G_j)$, which is F-free. Hence, we may write

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \mathrm{ex}(c, F)\right].$$

Define $f(a, b, c)$ as the function on the right-hand-side. We show that $f(a, b, c)$ attains its maximum at $a = b = 0$ and $c = n$. By a theorem of Simonovits, if F is r-colorable, then $\text{ex}(c, F) = \text{ex}(c, K_r) + \text{ex}(c, \tilde{F})$, where $\tilde{F}$ is the family of residue subgraphs of F after F is embedded into K_r. Hence, we may write

$$\begin{aligned} f(a, b-1, c+1) - f(a, b, c) &= a - c + 2[\text{ex}(c+1, F) - \text{ex}(c, F)] \\ &= a - c + 2\left[(c+1) - \left\lceil \frac{c+1}{r-1} \right\rceil \right. \\ &\qquad \left. + \text{ex}(c+1, \tilde{F}) - \text{ex}(c, \tilde{F})\right] \\ &\geq a - c + 2\left\lfloor \frac{c+1}{2} \right\rfloor \geq a. \end{aligned}$$

Same problem as in Theorem 3.1. TODO: show $\text{ex}(c+1, \tilde{F}) > \text{ex}(c, \tilde{F})$ unless $\tilde{F} = ?$ □

4 Non-induced Case

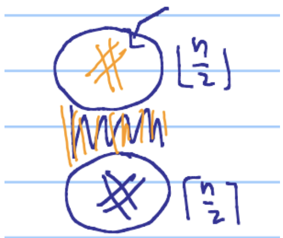
We now remove the assumption that $G_1, \ldots, G_m$ are induced subgraphs. Again, we first consider the triangle-free case.

4.1 Triangle-Free Case

Theorem 4.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j. Then,*

$$\sum_{i=1}^{m} e(G_i) \leq \binom{n}{2} + (m-1)\left\lfloor \frac{n^2}{4} \right\rfloor.$$

The natural extremal construction is to simply put $G_1 = K_n$ and the rest as $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. However, even for $m = 2$ there are multiple extremal constructions. For example, put G_1 as $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and connect all possible pairs of vertices on the left part. On the other hand, put G_2 as $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and connect all possible pairs of vertices on the right part.



Then, $E(G_1) \cap E(G_2)$ is triangle-free and

$$\begin{aligned} e(G_1) + e(G_2) &= 2e(G_1 \cap G_2) + e(G_1 \Delta G_2) \\ &= 2\left\lfloor \frac{n^2}{4} \right\rfloor + \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor = \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

Here we introduce the notation of *compression* of $G_1, \ldots, G_m$, which is the graph obtained by moving all edges in only one G_i to G_1. Performing compression for the case $m = 2$, we get

$$e(G_1) + e(G_2) = e(G_1) + e(G_1 \cap G_2) \leq \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = K_n$ and $G_2 = G_1 \cap G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. That is, the extremal graphs for $m = 2$ are isomorphic, up to compression.

We use the notion of compression to solve for $m = 3, 4$:

Theorem 4.2. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j. Then,*

$$e(G_1) + e(G_2) + e(G_3) \leq \binom{n}{2} + 2\left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = K_n$ and $G_2, G_3 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ after compression.

Proof. Compressing G_1, G_2, G_3 yields

$$\begin{aligned} e(G_1) + e(G_2) + e(G_3) &= e(G_1) + e(G_1 \cap G_2) + e(G_1 \cap G_3) \\ &\qquad + 2[e(G_2 \cap G_3) - e(G_1 \cap G_2 \cap G_3)] \\ &\leq e(G_1) + e(G_1 \cap G_2) + e(G_1 \cap G_3) \quad \text{(should be lowerbound)} \\ &\leq \binom{n}{2} + 2\left\lfloor \frac{n^2}{4} \right\rfloor, \end{aligned}$$

with equality if and only if $G_1 = K_n$ and $G_1 \cap G_2, G_1 \cap G_3 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. The result now follows. □

TODO: solve $m = 4$.