MATH 264A Lecture Notes

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This note is for the graduate combinatorics course MATH 264A at UC San Diego, taught by Professor Lutz Warnke in 2024 Fall. The proofs below are merely my attempts of recreating the contents in lectures, which might not be accurate representations of what was actually taught.

Some Basic Tools

Here we introduce some simple but powerful tools.

Inductive Approaches

This method is done by changing the size of the problem, e.g. adding an vertex or an edge in a graph.

Example 0.1. *Every n-vertex graph with maximum degree Δ has $\geq \beta^n$ valid vertex colorings with $\leq \lceil \Delta + \beta \rceil$ colors.*

Proof. Color the vertices $v_1, v_2, \ldots, v_n$ sequentially. Since v_i has $\leq \Delta$ neighbors already colored, there are $\geq \lceil \Delta + \beta \rceil - \Delta \geq \beta$ choices to color v_i. Define N_i as # valid colorings of $v_1, \ldots, v_i$. Then, the *Telescoping Product* now yields

$$N_n = \frac{N_n}{N_{n-1}} \cdot \frac{N_{n-2}}{N_{n-1}} \cdots \frac{N_1}{N_0} \cdot N_0 \geq \beta^n,$$

as $N_0 = 1$. $\square$

Despite being an extremely basic technique, induction can prove several advanced theorems if used artfully. The following are some exciting theorems which can be proven by induction:

1. Strengthen Lovász Local Lemma (LLL)
2. Chromatic number of triangle-free graph with max-degree Δ is $\leq (1 + o(1)) \frac{\Delta}{\log \Delta}$ as $\Delta \to \infty$.
3. Almost all triangle-free graphs are bipartite.

Double Counting/Switching

Also known as the Pertubation method, e.g. change of location of edges.

Example 0.2. *Find the $\#\Pi \in S_n$ without fix-points, i.e. $\Pi(i) \neq i$ for all i.*

Proof. We prove this by a basic approach which consists of several steps:

Step 1: Define the "Switching Operation." Let $S_{n,k}$ be the set of permutations with k fix-points. Define the switching operation to transform $\pi \in S$ to $\pi' \in S_{n,1}$.

Step 2: Consider the auxiliary bipartite graph. Let $S_{n,0}, S_{n,1}$ be parts of the bipartite graph. Connect $\pi \in S_{n,0}$ with $\pi' \in S_{n,1}$ if π' results from π through the switching operation.

Step 3: Double count the degrees.

$$\sum_{\pi \in S_{n,0}} \deg \pi = \sum_{\pi' \in S_{n,1}} \deg \pi'$$

Step 4: Degree essentially transfers to ratio. Suppose $\deg \pi \approx a$ and $\deg \pi' \approx b$, for all $\pi \in S_{n,0}$ and $\pi' \in S_{n,1}$. Then,

$$\frac{|S_{n,0}|}{|S_{n,1}|} \approx \frac{b}{a}.$$

□

This method can be applied to count d-regular graphs with certain properties, i.e. random model without independence.

Asymptotic Methods

Rather than finding the close form of a discrete function, sometimes it is significantly easier to approximate the function in asymptotic settings.

Bootstrapping

Suppose we have an equation $w(z)e^{w(z)} = z$ and we try to extract $w(z)$. By bootstrapping, $w(z) = \ln z - \ln \ln z + o(1)$.

Integral-Approximation

As the title suggests, this method estimates a summation $\sum_{k \in I} f(k)$ with its integral counterpart $\int_I f(x)\, dx$. For example, the the summation derived from the Fibonacci Tiliing Problem can be estimated by the Laplace-Method, i.e.

$$\sum_{0 \leq k \leq \frac{n}{2}} \binom{n-k}{k} \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} \qquad n \to \infty.$$