MATH 190A: Homework #3

Due on Jan 29, 2025 at 12:00pm

Professor McKernan

Section A02 8:00AM - 8:50AM
Section Leader: Zhiyuan Jiang

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

Let $(X, \mathcal{T})$ be a topological space. If $A \subset X$ is any subset then we say that $x \in X$ is an **accumulation point** if the closure of $A \setminus \{x\}$ contains x. Show that the closure of A is the union of A and all of its accumulation points.

Proof. It suffices to show that the set of accumulation points of A that are not in A equals $\overline{A} \backslash A$. If x is an accumulation point of A and $x \notin A$, then x contained in the closure of $A \backslash \{x\} = A$. Now suppose $x \in \overline{A} \backslash A$. Then x is in the closure of $A \backslash \{x\} = A$, so x is an accumulation point of A. $\square$

Problem 2

Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$ and let $(Y, \mathcal{S})$ be a topological space with basis $\mathcal{C}$. Show that

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof. Note that

$$X \times Y = \bigcup_{B \in \mathcal{B}} B \times \bigcup_{C \in \mathcal{C}} C = \bigcup_{B \in \mathcal{B}, C \in \mathcal{C}} B \times C = \bigcup_{D \in \mathcal{D}} D,$$

so $\mathcal{D}$ covers $X \times Y$.

Suppose $D_1, D_2 \in \mathcal{D}$. Then $D_1 = B_1 \times C_1$ and $D_2 = B_2 \times C_2$, and thus

$$D_1 \cap D_2 = (B_1 \cap B_2) \times (C_1 \cap C_2) \in \mathcal{D}.$$

$\square$

Problem 3

Let $(X, \mathcal{T})$ be a topological space. We say that $(X, \mathcal{T})$ is **Hausdorff** if for any two points $x \neq y \in X$ we may find two disjoint neighborhoods F and G of x and y. Show that the following are equivalent:

(i) $(X, \mathcal{T})$ is Hausdorff.

(ii) For any two points $x \neq y$ we can find two disjoint open subsets U and V such that $x \in U$ and $y \in V$.

(iii) For any two points $x \neq y$ we can find a closed neighborhood A of x not containing y (that is, $y \notin A$).

(iv) The diagonal

$$\Delta = \{(x, x) \mid X \times X\}$$

is closed in the product topology.

Proof. (i) to (ii): If F and G are disjoint neighborhoods of x and y, then $\text{int}(F)$, $\text{int}(G)$ are disjoint open sets containing x and y.

(ii) to (iii): If U and V are disjoint open sets such that $x \in U$ and $y \in V$, then $A = V^c$ is a closed neighborhood of x not containing y.

(iii) to (iv): Suppose $x, y \in X$ such that $x \neq y$. Then there exists a closed neighborhood A of x that does not contain y. But then $\text{int}(A) \times A^c$ is an open neighborhood of (x, y) that does not intersect with Δ. Hence, Δ^c is open.

(iv) to (i): Since Δ^c is open, for each $x \neq y$ there exists an open set $U \times V \subseteq \Delta^c$ containing (x, y), where $U, V \subseteq X$. For $(a, b) \in U \times V$, since $U \times V \cap \Delta = \emptyset$, $a \neq b$. Thus, U and V are disjoint neighborhoods of x and y. $\square$

Problem 4

True or false? If true then give a proof and if false then give a counterexample.

(i) If $(X, \mathcal{T})$ is a topological space and $Y \subset X$ is a subset and $U \subset Y$ is open in the subspace topology then U is open in X.

Proof. False. Consider $X = \mathbb{R}$, and $\mathcal{T}$ is the Eclidean topology. If $Y = [0, 1]$, then $U = (0, 1]$ is open in Y but not in X. $\square$

(ii) If $(X, \mathcal{T})$ is a Hausdorff topological space then every singleton subset $\{x\}$ is closed.

Proof. True. Let $x \in X$. Then for any $y \in X$ with $y \neq x$, there exist closed neighborhood U_y of x that does not contain y. But then

$$\bigcup_{y \in X, x \neq y} U_y^c = X \backslash \{x\}$$

is open. $\square$

(iii) If $(X, \mathcal{T})$ is a topological space and every singleton subset is closed then $(X, \mathcal{T})$ is Hausdorff.

Proof. False. Consider the topology given in homework 2 problem 1 and let X be infinite. Every singleton is closed, but it is not Hausdorff, as any two non-empty open sets intersect. $\square$

(iv) If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are Hausdorff topological spaces then the product $(X \times Y, \mathcal{R})$ is Hausdorff.

Proof. True. For distinct points $(x_1, y_y), (x_2, y_2)$, there exists U_1, U_2, V_1, V_2 such that $x_1 \in U_1$, $x_2 \in U_2$, $y_1 \in V_1$, $y_2 \in V_2$ and $U_1 \cap U_2 = \emptyset$, $V_1 \cap V_2 = \emptyset$. Then $U_1 \times V_1$ and $U_2 \times V_2$ are disjoint neighborhoods of (x_1, y_1) and (x_2, y_2). $\square$

(v) If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are two topological spaces and $A \subset X, B \subset Y$ then

$$\overline{A \times B} = \overline{A} \times \overline{B}$$

in the product topology on $X \times Y$.

Proof. True. Let $(x, y) \in \overline{A \times B}$. Then any open neighborhoods $U \times V$ of (x, y) intersects with $A \times B$. This implies any open neighborhoods U of x intersects with A and any open neighborhood V of y intersects with B. Thus, $x \in \overline{A}$ and $y \in \overline{B}$.

On the other hand, Let $(x, y) \in \overline{A} \times \overline{B}$. Then any open neighborhoods U of x intersects with A and any open neighborhood V of y intersects with B. But then any open neighborhoods $U \times V$ of (x, y) intersects with $A \times B$. $\square$

(vi) Every subspace of a Hausdorff topological space is Hausdorff.

Proof. True. Let x, y be distinct points in the subspace Y of X. Then there exist disjoint neighborhoods U, V of x and y in X. But then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y in Y. $\square$

Problem 5

If (X, d) is a metric space then the induced topological space $(X, \mathcal{T})$ is Hausdorff.

Proof. Let $x, y \in X$ such that $x \neq y$, and let $r = d(x, y)/2$. Then the open balls $B(x, r)$ and $B(y, r)$ are disjoint neighborhoods of x and y. $\square$