MATH 220A: Homework #8

Due on Nov 22, 2024 at 23:59pm

Professor Ebenfelt

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Problem 1

(a) Prove Abel's Theorem: Let $\sum a_n(z-a)^n$ have radius of convergence 1 and suppose that $\sum a_n$ converges to A. Prove that

$$\lim_{r\to 1^-} \sum a_n r^n = A.$$

(Hint: Find a summation formula which is the analogue of integration by parts.)

Proof. We may assume that $a = 0$ and $\sum a_n = A = 0$, as we can always adjust the value of a_0. Let $S_n = \sum_{k=0}^n a_k$. Then for $r \in (0,1)$,

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n r^n &= a_0 + \sum_{n=1}^{\infty}(S_n - S_{n-1})r^n && (1)\\
&= a_0 + \sum_{n=1}^{\infty} S_n r^n - \sum_{n=1}^{\infty} S_{n-1} r^n && (2)\\
&= \sum_{n=0}^{\infty} S_n r^n - r\sum_{n=0}^{\infty} S_n r^n && (3)\\
&= (1-r)\sum_{n=0}^{\infty} S_n r^n. && (4)
\end{aligned}$$

Pick $\epsilon > 0$. Since $\sum a_n \to 0$, there exists integer N such that for all $k \geq N$, $|S_k| < \epsilon/2$. Then

$$\left|(1-r)\sum_{n=N}^{\infty} S_n r^n\right| \leq (1-r)\sum_{n=N}^{\infty}|S_n|r^n \leq \frac{\epsilon}{2}(1-r)\sum_{n=N}^{\infty} r^n = \frac{\epsilon}{2}(1-r)\frac{r^N}{1-r} < \epsilon/2,$$

for all $r \in (0,1)$. Since $\sum_{n=N}^{N} S_n r^n = M$ for some constant M, pick $r < (1-\epsilon/2M, 1)$ and we have

$$\left|(1-r)\sum_{n=0}^{N-1} S_n r^n\right| = (1-r)M < \epsilon/2.$$

Therefore,

$$\sum a_n r^n < \epsilon$$

for r sufficiently close to 1, and the result now follows. ☐

(b) Use Abel's Theorem to prove that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$.

Proof. Consider $\log(1+z)$. The power series expansion of $\log(1+z)$ is $\sum a_n z^n = \sum \frac{(-1)^{n+1}}{n} z^n$, which has radius of convergence 1, as $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$. By Abel's Theorem,

$$\log 2 = \lim_{r\to 1^-}\sum \frac{(-1)^{n+1}}{n} r^n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots.$$

☐

Problem 2

Give the power series expansion of $\log z$ about $z = i$ and find its radius of convergence.

Proof. Note that the nth derivative of $\log z$ is $(-1)^{n+1}(n-1)!z^{-n}$. Let $a_n = \frac{(-1)^{n+1}}{n} i^{-n}$. Then the power series expansion of $\log z$ about $z = i$ is

$$\sum a_n (z-i)^n = \sum \frac{(-1)^{n+1}}{n} i^{-n} (z-i)^n.$$

The radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{ni} \right| = 1.$$

□

Problem 3

Evaluate
$$\int_\gamma \frac{z^2+1}{z(z^2+4)}\,dz$$
where $\gamma(s) = re^{is}$, $0 \le s \le 2\pi$, for all possible values of r, $0 < r < 2$ and $2 < r < \infty$.

Proof. Define $\phi(s,t) = \frac{i(r^2e^{2is}+t)}{r^2e^{2is}+4t}$ for $t \in [0,1]$ and $s \in [0,2\pi]$. Note that ϕ is continuously differentiable, and thus $g(t) = \int_0^{2\pi} \phi(s,t)\,ds$ is continuously differentiable. Notice
$$g(1) = \int_0^{2\pi} \frac{i(r^2e^{2is}+1)}{r^2e^{2is}+4}\,ds = \int_0^{2\pi} \frac{r^2e^{2is}+1}{re^{is}(r^2e^{2is}+4)} ire^{is}\,ds = \int_\gamma \frac{z^2+1}{z(z^2+4)}\,dz.$$
By Leibniz's Rule,
$$g'(t) = \int_0^{2\pi} \frac{\partial}{\partial t}\phi(s,t)\,ds = \int_0^{2\pi} \frac{-3ir^2e^{2is}}{(r^2e^{2is}+4t)^2}\,ds.$$
Define $\Phi(s) = \frac{3}{2}(r^2e^{2is}+4t)^{-1}$. Since $\Phi'(s) = \frac{-3ir^2e^{2is}}{(r^2e^{2is}+4t)^2}$, we have $g'(t) = \Phi(2\pi) - \Phi(0) = 0$. That is, $g(t)$ is constant for $t \in [0,1]$. It now follows that
$$\int_\gamma \frac{z^2+1}{z(z^2+4)}\,dz = g(1) = g(0) = \int_0^{2\pi} \frac{i(r^2e^{2is})}{r^2e^{2is}}\,ds = 2\pi i.$$
□

Problem 4

Let f be an entire function and suppose there is a constant M, an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.

Proof. Since f is entire, f jas a power series expansion about $z = 0$ of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

Let $\gamma(t) = re^{it}$, with $r > R$ and $t \in [0, 2\pi]$. By Corollary 2.13,

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{w^{k+1}}\, dw.$$

For $r > R$, $|f(z)| \leq M|z|^n$ and thus

$$\begin{aligned}
|f^{(k)}(0)| &= \left| \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{w^{k+1}}\, dw \right| \\
&\leq \frac{k!}{2\pi} \int_\gamma \frac{|f(w)|}{|w|^{k+1}}\, |dw| \\
&\leq \frac{k!M}{2\pi} \int_\gamma |w|^{n-k-1}\, |dw| \\
&= \frac{k!M}{2\pi} r^{n-k-1} \int_\gamma |dw| = k!Mr^{n-k}.
\end{aligned}$$

But then r can be arbitrarily large, and thus $f^{(k)}(0) = 0$ for all $k > n$. Therefore,

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$$

is a polynomial of degree $\leq n$. □

Problem 5

Find all entire functions f such that $f(x) = e^x$ for $x \in \mathbb{R}$.

Proof. Let $g(x) = f(x) - e^x$. Then $g(x) = 0$ for $x \in \mathbb{R}$. Since g is entire and $\{z \in \mathbb{C} : g(z) = 0\}$ has a limit point at 0, $g(z) = 0$ for all $z \in \mathbb{C}$ by Theorem 3.7. Therefore, $f(x) = e^x$ for all $x \in \mathbb{C}$. $\square$