# SC9 Probability on Graphs and Lattices: Sheet #3

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# Problem 1

**Obtaining magnetisation from pressure.**

Let $m_G(\omega) = \frac{1}{|V(G)|}\sum_{i\in V(G)} \sigma_i(\omega)$ be the magnetisation per site. Let $\Psi_{G;\beta,h} = \log(Z_{G;\beta,h})/|V(G)|$ be the pressure. Show that

$$\langle m_G\rangle_{G;\beta,h} = \frac{\partial \Psi_{G;\beta,h}}{\partial h}.$$

*Proof.*

$$\begin{aligned}
\frac{\partial \Psi_{G;\beta,h}}{\partial h} &= \frac{1}{|V(G)|Z_{G;\beta,h}} \cdot \sum_{\omega\in\Omega_G} \frac{\partial e^{-\mathcal{H}_{G;\beta,h}(\omega)}}{\partial h} \\
&= \frac{1}{|V(G)|Z_{G;\beta,h}} \cdot \sum_{\omega\in\Omega_G} e^{-\mathcal{H}_{G;\beta,h}(w)} \cdot \frac{\partial(-\mathcal{H}_{G;\beta,h}(\omega))}{\partial h} \\
&= \frac{1}{|V(G)|Z_{G;\beta,h}} \cdot \sum_{\omega\in\Omega_G} \sum_{i\in V(G)} e^{-\mathcal{H}_{G;\beta,h}(w)} \cdot \sigma_i(\omega) \\
&= \frac{1}{|V(G)|} \cdot \sum_{i\in V(G)} \sigma_i(\omega) \cdot \sum_{\omega\in\Omega_G} \frac{e^{-\mathcal{H}_{G;\beta,h}(w)}}{Z_{G;\beta,h}} \\
&= m_G(\omega).
\end{aligned}$$

□

# Problem 2

**Spatial Markov property.**

Fix $(\beta, h) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, $\eta \in \{\pm 1\}^{\mathbb{Z}^d}$ and let $G' \subseteq G$ be finite subgraphs of $\mathbb{Z}^d$. Show that we have

$$\mu^{\eta}_{G;\beta,h}(\cdot \mid \sigma_i = \eta_i, \forall i \in V(G) \setminus V(G')) = \mu^{\eta}_{G';\beta,h}(\cdot).$$

*Proof.*

$$\begin{aligned}\mathcal{H}^{\eta}_{G;\beta,h}(\omega \mid \sigma_i = \eta_i, \forall i \in V(G) \setminus V(G')) &= -\beta \sum_{(i,j) \in E(G) \cup \partial G} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i \in V(G)} \sigma_i(\omega) \\ &= \mathcal{H}^{\eta}_{G';\beta,h}(\omega) - \beta \sum_{(i,j) \in E(G \setminus G') \cup \partial G} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i \in V(G \setminus G')} \sigma_i(\omega).\end{aligned}$$

Since all vertices in $V(G) \setminus V(G')$ are conditioned on $\eta$, we may write the last two sums as a constant $C = C(\eta)$. Thus,

$$\mathcal{H}^{\eta}_{G;\beta,h}(\omega \mid \sigma_i = \eta_i, \forall i \in V(G) \setminus V(G')) = \mathcal{H}^{\eta}_{G';\beta,h}(\omega) + C.$$

Let

$$Z' = \sum_{\substack{\omega \in \Omega_G \\ \sigma_i = \eta_i, \forall i \in V(G) \setminus V(G')}} e^{-\mathcal{H}^{\eta}_{G';\beta,h}(\omega) - C} = e^{-C} \cdot Z_{G';\beta,h}.$$

It now follows that

$$\mu^{\eta}_{G;\beta,h}(\cdot \mid \sigma_i = \eta_i, \forall i \in V(G) \setminus V(G')) = \frac{e^{-\mathcal{H}^{\eta}_{G';\beta,h}(\cdot) - C}}{e^{-C} \cdot Z_{G';\beta,h}} = \frac{e^{-\mathcal{H}^{\eta}_{G';\beta,h}(\cdot)}}{Z_{G';\beta,h}} = \mu^{\eta}_{G';\beta,h}(\cdot).$$

$\square$

# Problem 3

**The measures $\mu^{+}_{G;\beta,h}$ and $\mu^{-}_{G;\beta,h}$ are extremal.**

Let $f : \{\pm 1\}^{\mathbb{Z}^d} \to \mathbb{R}$ be an increasing function. Show that for all configurations $\eta \leq \omega$ in $\{\pm 1\}^{\mathbb{Z}^d}$ we have

$$\langle f \rangle^{\eta}_{G;\beta,h} \leq \langle f \rangle^{\omega}_{G;\beta,h}.$$

*Proof.* Let

$$R^{\omega,\eta}_{G;\beta,h}(\sigma) = \exp\Big(\mathcal{H}^{\eta}_{G;\beta,h}(\sigma) - \mathcal{H}^{\omega}_{G;\beta,h}(\sigma)\Big) = \exp\left(\beta \sum_{(i,j)\in\partial G} \sigma_i \cdot (\omega_j - \eta_j)\right).$$

Then

$$\begin{aligned}
\langle f \rangle^{\omega}_{G;\beta,h} &= \sum_{\Omega^{\omega}_G} f(\sigma) \cdot \frac{e^{-\mathcal{H}^{\omega}_{G;\beta,h}(\sigma)}}{Z^{\omega}_{G;\beta,h}} \\
&= \sum_{\Omega^{\omega}_G} f(\sigma) \cdot \frac{e^{-\mathcal{H}^{\eta}_{G;\beta,h}(\sigma)} \cdot R^{\omega,\eta}_{G;\beta,h}(\sigma)}{Z^{\omega}_{G;\beta,h}} \\
&= \frac{Z^{\eta}_{G;\beta,h}}{Z^{\omega}_{G;\beta,h}} \sum_{\Omega^{\eta}_G} f(\sigma) \cdot R^{\omega,\eta}_{G;\beta,h}(\sigma) \cdot \frac{e^{-\mathcal{H}^{\eta}_{G;\beta,h}(\sigma)}}{Z^{\eta}_{G;\beta,h}} \\
&= \frac{Z^{\eta}_{G;\beta,h}}{Z^{\omega}_{G;\beta,h}} \langle f R^{\omega,\eta}_{G;\beta,h} \rangle^{\eta}_{G;\beta,h}.
\end{aligned}$$

Note that

$$\sum_{\Sigma^{\omega}_G} \mu^{\omega}_{G;\beta,h}(\sigma) = \frac{Z^{\eta}_{G;\beta,h}}{Z^{\omega}_{G;\beta,h}} \sum_{\Sigma^{\eta}_G} \mu^{\eta}_{G;\beta,h}(\sigma) \cdot R^{\omega,\eta}_{G;\beta,h}(\sigma) = \frac{Z^{\eta}_{G;\beta,h}}{Z^{\omega}_{G;\beta,h}} \langle R^{\omega,\eta}_{G;\beta,h} \rangle^{\eta}_{G;\beta,h} = 1,$$

so

$$\frac{Z^{\eta}_{G;\beta,h}}{Z^{\omega}_{G;\beta,h}} = \frac{1}{\langle R^{\omega,\eta}_{G;\beta,h} \rangle^{\eta}_{G;\beta,h}}.$$

It now follows from the FKG inequality that

$$\langle f \rangle^{\omega}_{G;\beta,h} = \frac{\langle f R^{\omega,\eta}_{G;\beta,h} \rangle^{\eta}_{G;\beta,h}}{\langle R^{\omega,\eta}_{G;\beta,h} \rangle^{\eta}_{G;\beta,h}} \geq \frac{\langle f \rangle^{\eta}_{G;\beta,h} \cdot \langle R^{\omega,\eta}_{G;\beta,h} \rangle^{\eta}_{G;\beta,h}}{\langle R^{\omega,\eta}_{G;\beta,h} \rangle^{\eta}_{G;\beta,h}} = \langle f \rangle^{\eta}_{G;\beta,h}.$$

□

# Problem 4

**Spin at 0 and average magnetisation.**

Define $m^+(\beta, h)$ and $m^-(\beta, h)$ as

$$\lim_{n\to\infty} \frac{\langle \sum_{i\in\Lambda(n)} \sigma_i \rangle^+_{\Lambda(n);\beta,h}}{|\Lambda(n)|}, \qquad \lim_{n\to\infty} \frac{\langle \sum_{i\in\Lambda(n)} \sigma_i \rangle^-_{\Lambda(n);\beta,h}}{|\Lambda(n)|},$$

respectively. Show that $\langle \sigma_0 \rangle^{\pm}_{\beta,h} = m^{\pm}(\beta, h)$.

*Hint: Show an upper and lower bound. Argue (and use) that for an increasing function $f$ and subgraphs $G' \subseteq G \subseteq \mathbb{Z}^d$, $\langle f \rangle^+_{G',\beta,h} \geq \langle f \rangle^+_{G,\beta,h}$, then establish bounds on the* limsup *and* liminf *using appropriate choices of $G', G$ and $f$.*

*Proof.* We first note that

$$\langle \sigma_0 \rangle^+_{\beta,h} = \langle \sigma_i \rangle^+_{\beta,h}.$$

for all $i \in \mathbb{Z}^d$. By Problem 3 and the strong Markov property, for increasing function $f$,

$$\langle f \rangle^+_{G',\beta,h} \geq \mathbb{E}_\alpha[\langle f \rangle^\alpha_{G',\beta,h}] = \langle f \rangle^+_{G,\beta,h}.$$

But then $\sigma_i$ is increasing, so

$$\langle \sigma_0 \rangle^+_{\beta,h} = \langle \sigma_i \rangle^+_{\beta,h} \leq \langle \sigma_i \rangle^+_{\Lambda(n);\beta,h},$$

for any $n \in \mathbb{N}$. Let

$$m_n^+(\beta, h) = \frac{\sum_{i\in\Lambda(n)} \langle \sigma_i \rangle^+_{\Lambda(n);\beta,h}}{|\Lambda(n)|}.$$

Then

$$\liminf_{n\to\infty} m_n^+(\beta, h) \geq \langle \sigma_0 \rangle^+_{\beta,h}.$$

On the other hand, fix an integer $k$. Let $\Lambda_i(k)$ denote the length $k$ box centred at $i$. Assume $n$ is sufficiently large and consider the sites $i \in \Lambda(n)$ such that $\Lambda_i(k) \subset \Lambda(n)$. Then

$$\langle \sigma_i \rangle^+_{\Lambda(n);\beta,h} \leq \langle \sigma_i \rangle^+_{\Lambda_i(k);\beta,h} = \langle \sigma_0 \rangle^+_{\Lambda(k);\beta,h}.$$

Thus

$$\limsup_{n\to\infty} m_n^+(\beta, h) \leq \langle \sigma_0 \rangle^+_{\Lambda(k);\beta,h}.$$

But then

$$\langle \sigma_0 \rangle^+_{\beta,h} \leq \liminf_{n\to\infty} m_n^+(\beta, h) \leq \limsup_{n\to\infty} m_n^+(\beta, h) \leq \lim_{k\to\infty} \langle \sigma_0 \rangle^+_{\Lambda(k);\beta,h} = \langle \sigma_0 \rangle^+_{\beta,h}.$$

The result for the negative case follows from symmetry. □

# Problem 5

**Ising model on the complete graph.**

The *Curie-Weiss* model is the Ising model on the complete graph $K_N$ with $N$ vertices. Despite the lack of geometry, there are still interesting phase transitions. Let $h = 0$ throughout.

(i) Let $R(\omega) = \#\{i : \sigma_i(\omega) = 1\}$ be the number of $+$ spins (which is $N$ minus the number of $-$ spins). Show that the probability that $R(\omega) = k$ is given by

$$\mu_{K_N;\beta,0}(R(\omega) = k) = \frac{1}{\tilde{Z}_{N,\beta}} \binom{N}{k} \exp\left(-2\beta k(N-k)\right)$$

for $0 \le k \le N$ (where $\tilde{Z}_{N,\beta}$ is a normalising constant).

*Proof.* Let $\omega \in \Omega^+_{K_N}$ such that $R(\omega) = k$. Let $C = e^{-\beta\binom{N}{2} - 2k(N-k)}$. Then

$$\mathcal{H}_{K_N;\beta,0}(\omega) = -\beta \sum_{(i,j)\in E(K_N)} \sigma_i(\omega)\sigma_j(\omega) = -\beta\left(\binom{k}{2} + \binom{N-k}{2} - k(N-k)\right) = -\beta\left(\binom{N}{2} - 2k(N-k)\right).$$

But then

$$\begin{aligned}
\mu_{K_N;\beta,0}(R(\omega) = k) &= \frac{\binom{N}{i}\exp\left(-\beta\left(\binom{N}{2} - 2k(N-k)\right)\right)}{\sum_{i=0}^N \binom{N}{i}\exp\left(-\beta\left(\binom{N}{2} - 2i(N-i)\right)\right)} \\
&= \frac{C\binom{N}{k}\exp\left(-\beta\left(2k(N-k)\right)\right)}{C\sum_{i=0}^N \binom{N}{i}\exp\left(-\beta\left(2i(N-i)\right)\right)} \\
&= \frac{1}{\tilde{Z}_{N,\beta}}\binom{N}{k}\exp\left(-2\beta k(N-k)\right).
\end{aligned}$$

□

(ii) Consider taking the limit $N \to \infty$, with $\beta = \theta/N$ for some fixed $\theta > 0$. Investigate which $k$ (approximately) maximises the probability in (i). Interpret your findings.

*[Perhaps useful: from Stirling's formula, if $k = (\alpha + o(1))N$ as $N \to \infty$, then $\binom{N}{k} \approx \exp\left[N(f(\alpha) + o(1))\right]$, where $f(\alpha) = -\alpha\log\alpha - (1-\alpha)\log(1-\alpha)$.]*

*Proof.*

$$\mu_{K_N;\theta/N,0}(R(\omega) = k) = \frac{1}{\tilde{Z}_{N,\theta/N}}\binom{N}{k}\exp\left(\frac{-2\theta k(N-k)}{N}\right).$$

Write $k = (\alpha + o(1))N$ as $N \to \infty$. Let $f(\alpha) = -\alpha\log\alpha - (1-\alpha)\log(1-\alpha)$. By Stirling's formula,

$$\binom{N}{k} \approx \exp\left[N(f(\alpha) + o(1))\right].$$

Put $\phi(\alpha) = f(\alpha) - 2\theta(\alpha(1-\alpha))$ and we have

$$\mu_{K_N;\theta/N,0}(R(\omega) = k) = \frac{1}{\tilde{Z}_{N,\theta/N}}\exp(\phi(\alpha) + o(1)).$$

Note that the exponential function is increasing, so the maximum is achieved when $\phi(\alpha)$ is maximized.

□

# Problem 6

*FKG and BK in the context of the random-cluster model.*

- One can show that the FKG inequality holds for the random-cluster model for all $G$ when $q \geq 1$. Does it also hold for $q < 1$?

  *Proof.* No. Consider graph $G$ with two vertices and two parallel edges $x, y$. Let $A_x$ and $A_y$ be the event where edge $x$ and $y$ are open, respectively. Then

  $$\mathbb{P}(\emptyset) = \frac{q^2(1-p)^2}{Z}, \quad \mathbb{P}(A_x) = \mathbb{P}(A_y) = \frac{qp}{Z}, \quad \mathbb{P}(A_x \cap A_y) = \frac{qp^2}{Z}, \quad Z = q^2(1-p)^2 + 2qp(1-p) + qp^2.$$

  If the FKG inequality held, we would have

  $$\frac{qp^2}{Z} \geq \left(\frac{qp}{Z}\right)^2 \implies Z \geq q.$$

  But then

  $$q \leq q^2(1-p)^2 + 2qp(1-p) + qp^2 \implies q \geq 1,$$

  contradiction. □

- Fix $q > 1$. Does the BK inequality hold for the random-cluster model on $G$ with parameters $p, q$ for all $G, p \in (0, 1)$?

  *Proof.* No. Using the same events as in the previous proof, note that

  $$\mathbb{P}(A_x \circ A_y) = \mathbb{P}(A_x \cap A_y) = \frac{qp^2}{Z}.$$

  Since $\frac{q}{Z} < 1$ for $q > 1$,

  $$\mathbb{P}(A_x \circ A_y) = \frac{qp^2}{Z} > \left(\frac{qp}{Z}\right)^2 = \mathbb{P}(A_x)\mathbb{P}(A_y),$$

  contradicting the BK inequality. □

# Problem 7

**Uniqueness of Gibbs measure for Ising model at high temperature.**

(i) Let $G$ be a finite subgraph of $\mathbb{Z}^d$. Let $\beta > 0$ and $p \in (0,1)$ be connected by $p = 1 - e^{-\beta}$. Fix $q \geq 2$.

Let $\phi^{\text{free}}_{G;p,q}$ and $\phi^{\text{wired}}_{G;p,q}$ be free and wired random-cluster measures on $G$ respectively, and let $\mu_{G;\beta,q}$ and $\mu^b_{G;\beta,q}$ be Gibbs measures for the Potts model on $G$ with free boundary conditions, and with all-$b$ boundary conditions for some fixed colour $b$, respectively.

Show that for any $i, j \in V(G)$,

$$\begin{aligned}\mu_{G;\beta,q}(\sigma_i = \sigma_j) &= \frac{1}{q} + \frac{q-1}{q}\phi^{\text{free}}_{G;p,q}(i \leftrightarrow j)\\ \mu^b_{G;\beta,q}(\sigma_i = \sigma_b) &= \frac{1}{q} + \frac{q-1}{q}\phi^{\text{wired}}_{G;p,q}(i \leftrightarrow \mathbb{Z}^d \setminus V(G)).\end{aligned}$$

*Proof.* By Theorem 3.12, $\sigma_i = \sigma_j$ in the free random-cluster model with probability $1/q$ if $i$ and $j$ are not in the same cluster and probability 1 otherwise. Thus,

$$\mu_{G;\beta,q}(\sigma_i = \sigma_j) = \frac{1}{q}(1 - \phi^{\text{free}}_{G;p,q}(i \leftrightarrow j)) + \phi^{\text{free}}_{G;p,q}(i \leftrightarrow j) = \frac{1}{q} + \frac{q-1}{q}\phi^{\text{free}}_{G;p,q}(i \leftrightarrow j).$$

Similarly, $\sigma_i = \sigma_b$ in the wired random-cluster model with probability $1/q$ if $i$ is disconnected from the boundary and probability 1 otherwise. Thus,

$$\mu^b_{G;\beta,q}(\sigma_i = \sigma_b) = \frac{1}{q}(1-\phi^{\text{wired}}_{G;p,q}(i \leftrightarrow \mathbb{Z}^d \setminus V(G))) + \phi^{\text{wired}}_{G;p,q}(i \leftrightarrow \mathbb{Z}^d \setminus V(G)) = \frac{1}{q} + \frac{q-1}{q}\phi^{\text{wired}}_{G;p,q}(i \leftrightarrow \mathbb{Z}^d \setminus V(G)).$$

□

(ii) By applying the second identity in (i) to the + boundary and − boundary Ising models, show that at sufficiently high temperature (i.e. for $\beta$ sufficiently small), there is a unique Gibbs measure for the Ising model in $\mathbb{Z}^d$.

*[Hint: you may like to use the FKG inequality to compare connectivity probabilities for percolation with those for random-cluster with $q = 2$.]*

*Proof.* Note that $-2^{k(\omega)}$ is an increasing function. By the FKG inequality,

$$-\phi^{\text{wired}}_{G;p,2}(i \leftrightarrow \mathbb{Z}^d \setminus V(G)) = \frac{\langle -2^{k(\omega)}\mathbb{1}_{i \leftrightarrow \mathbb{Z}^d \setminus V(G)}\rangle^{\text{wired}}_{G;p,1}}{\langle 2^{k(\omega)}\rangle^{\text{wired}}_{G;p,1}} \geq -\phi^{\text{wired}}_{G;p,1}(i \leftrightarrow \mathbb{Z}^d \setminus V(G)).$$

Thus,

$$\phi^{\text{wired}}_{G;p,2}(i \leftrightarrow \mathbb{Z}^d \setminus V(G)) \leq \phi^{\text{wired}}_{G;p,1}(i \leftrightarrow \mathbb{Z}^d \setminus V(G)).$$

This implies that vertex $i$ is more likely to be connected to the boundary in the percolation model than in the random-cluster model with $q = 2$. Let $(G_n)$ be an exhaustion of $\mathbb{Z}^d$ by finite subgraphs. Then when $p < p_c$,

$$\lim_{n\to\infty} \phi^{\text{wired}}_{G;p,2}(i \leftrightarrow \mathbb{Z}^d \setminus V(G)) \leq \lim_{n\to\infty} \phi^{\text{wired}}_{G;p,1}(i \leftrightarrow \mathbb{Z}^d \setminus V(G)) = 0.$$

It now follows from $(i)$ that

$$\lim_{n\to\infty} \mu^+_{G_n;\beta,q}(\sigma_i = +) = \lim_{n\to\infty} \mu^-_{G_n;\beta,q}(\sigma_i = -) = \frac{1}{q}.$$

□