

# MATH 188: Homework #2

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## Problem 1

Let  $F(x)$  be a formal power series with  $F(0) = 0$ .

- (a) Show that there exists a formal power series  $G(x)$  with  $G(0) = 0$  such that  $F(G(x)) = x$  if and only if  $[x^1]F(x) \neq 0$ .

*Proof.* Let  $F(x) = \sum_{n=0}^{\infty} a_n x^n$ , for some nonzero  $a_1$  and  $a_0 = 0$ . We look for a formal power series  $G(x) = \sum_{n=0}^{\infty} b_n x^n$  such that  $F(G(x)) = x$  and  $b_0 = 0$ . That is,

$$\begin{aligned} F(G(x)) &= \sum_{i=1}^{\infty} a_i G(x)^i \\ &= \sum_{n=1}^{\infty} x^n \sum_{i=1}^n a_i \sum_{m_1+m_2+\dots+m_i=n} b_{m_1} b_{m_2} \cdots b_{m_i} = x. \end{aligned}$$

Note that the inner summation terminates at  $n$ , as we are enumerating through compositions of  $n$ , which could not exceed  $n$  terms. By comparing coefficients, we have

$$b_0 = 0, \quad b_1 = \frac{1}{a_1},$$

and for  $n \geq 2$ ,

$$\sum_{i=1}^n a_i \sum_{m_1+m_2+\dots+m_i=n} b_{m_1} b_{m_2} \cdots b_{m_i} = 0. \quad (1)$$

Here, we already know that  $G(x)$  exists only if  $[x^1]F(x) \neq 0$ , it remains to show the converse. Suppose  $[x^1]F(x) \neq 0$ . We already determined the unique existence of  $b_1$ . For  $n \geq 2$ , rearranging (1) gives an expression of  $b_n$  uniquely determined by  $a_1, \dots, a_n, b_1, \dots, b_{n-1}$ . But then the existence of  $b_1, \dots, b_{n-1}$  are shown by induction, and this ensures the unique existence of  $b_n$ .  $\square$

- (b) Assuming  $[x^1]F(x) \neq 0$ , show that  $G(x)$  is unique and also satisfies  $G(F(x)) = x$ . You may use without proof that composition of formal power series is associative.

*Proof.* Uniqueness of  $G(x)$  is shown in (a). We know  $[x^1]G(x) \neq 0$ . By (a), there exists a formal power series  $H(x)$  with  $H(0) = 0$  such that  $G(H(x)) = x$ . But then  $F(x) = F(G(H(x))) = H(x)$ .  $\square$

## Problem 2

Evaluate the following sums:

(a)

$$\sum_{i=0}^n \binom{n}{i} \frac{1}{2^i}$$

*Proof.* By the binomial theorem,

$$\sum_{i=0}^n \binom{n}{i} \frac{1}{2^i} = \left(1 + \frac{1}{2}\right)^n = \frac{3^n}{2^n}.$$

□

(b)

$$\sum_{i=0}^n i^2 \binom{n}{i} 3^i$$

*Proof.* By the binomial theorem,

$$\begin{aligned} \sum_{n \geq 1} i \binom{n}{i} x^{i-1} &= \left( \sum_{n \geq 0} \binom{n}{i} x^i \right)' = ((1+x)^n)' = n(1+x)^{n-1}, \\ \sum_{n \geq 2} i(i-1) \binom{n}{i} x^{i-2} &= \left( \sum_{n \geq 0} \binom{n}{i} x^i \right)'' = ((1+x)^n)'' = n(n-1)(1+x)^{n-2}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=0}^n i^2 \binom{n}{i} 3^i &= \sum_{i=0}^n i(i-1) \binom{n}{i} 3^i + \sum_{i=0}^n i \binom{n}{i} 3^i \\ &= 9 \sum_{i=2}^n i(i-1) \binom{n}{i} 3^{i-2} + 3 \sum_{i=1}^n i \binom{n}{i} 3^{i-1} \\ &= 9 \left( \sum_{i=0}^n \binom{n}{i} 3^i \right)'' + 3 \left( \sum_{i=0}^n \binom{n}{i} 3^i \right)' \\ &= 9n(n-1)(1+3)^{n-2} + 3n(1+3)^{n-1} \\ &= \frac{9}{16} n(n-1)4^n + \frac{3}{4} n4^n = 3n(3n+1)4^{n-2}. \end{aligned}$$

□

### Problem 3

Let  $a, b$  be non-negative integers.

- (a) By comparing coefficients in  $(1+x)^{a+b} = (1+x)^a(1+x)^b$ , prove that for any non-negative integer  $n$ , we have

$$\binom{a+b}{n} = \sum_{i=0}^n \binom{a}{i} \binom{b}{n-i}.$$

*Proof.* By the binomial theorem,

$$\begin{aligned} \binom{a+b}{n} &= [x^n](1+x)^{a+b} \\ &= [x^n](1+x)^a(1+x)^b \\ &= \sum_{i=0}^n ([x^i](1+x)^a) ([x^{n-i}](1+x)^b) \\ &= \sum_{i=0}^n \binom{a}{i} \binom{b}{n-i}. \end{aligned}$$

□

- (b) Now prove this identity using a counting argument.

*Proof.* Consider choosing  $n$  animals from  $a$  dogs and  $b$  cats. Suppose that we picked  $i$  dogs. There are  $\binom{a}{i}$  ways of choosing them. In order to have  $n$  animals in total, we then have to pick  $n-i$  cats, which has  $\binom{b}{n-i}$  ways. The possible values for  $i$  are between 0 and  $n$ , and thus we get the identity

$$\binom{a+b}{n} = \sum_{i=0}^n \binom{a}{i} \binom{b}{n-i}.$$

□

## Problem 4

How many ways can we arrange the letters of: MISSISSIPPI?

*Proof.* There are one M, four I's, two P's, and four S's, and we have 11 slots in total. We first choose a slot for the M, which has  $\binom{11}{1}$  ways. Then, we choose 4 slots from the remaining 10 slots for the I's, which has  $\binom{10}{4}$  ways. Then, we choose 2 slots from the remaining 6 slots for the P's, which has  $\binom{6}{2}$  ways. Finally, we choose 4 slots from the remaining 4 slots for the S's, which has  $\binom{4}{4}$  ways. In total, there are

$$\binom{11}{1} \binom{10}{4} \binom{6}{2} \binom{4}{4} = \frac{11!}{4!2!4!}$$

ways of arranging the letters of MISSISSIPPI.

□

## Problem 5

Let  $f(t) = \sum_{k=0}^d f_k t^k$  be a degree  $d$  polynomial with rational coefficients. From lecture, we know that there exist unique rational numbers  $g_0, \dots, g_d$  such that

$$\sum_{n \geq 0} f(n) x^n = \frac{g_0 + g_1 x + \dots + g_d x^d}{(1-x)^{d+1}}. \quad (2)$$

Now assume that  $f(a)$  is an integer for  $a = 0, \dots, d$ . (The  $f_k$  don't have to be integers for this to be true, for example  $f(n) = n(n-1)/2$  has this property.) Prove that this implies that the  $g_k$  are all integers and that  $f(a)$  is an integer whenever  $a$  is an integer.

*Proof.* From (2), for  $k = 0, 1, \dots, d$ ,

$$\begin{aligned} g_k &= [x^k](1-x)^{d+1} \sum_{n \geq 0} f(n) x^n \\ &= \sum_{i=0}^k (-1)^{k-i} \binom{d+1}{k-i} f(i), \end{aligned}$$

which is an integer as  $f(i)$  and  $\binom{d+1}{k-i}$  are both integers, for  $i = 0, \dots, d$ . But then for  $n \in \mathbb{Z}_{\geq 0}$ ,

$$\begin{aligned} f(n) &= [x^n](1-x)^{-(d+1)}(g_0 + g_1 x + \dots + g_d x^d) \\ &= \sum_{k=0}^d \binom{d+n-k}{n-k} g_k = \sum_{k=0}^d \binom{d+n-k}{d} g_k. \end{aligned}$$

Note that  $h(n) = \sum_{k=0}^d \binom{d+n-k}{d} g_k$  is a polynomial of degree  $d$ . Since  $f(n) - h(n) = 0$  for all  $n \in \mathbb{Z}_{\geq 0}$ , it follows from the Fundamental Theorem of Algebra that  $f(n) = h(n)$ . Since  $g_k \in \mathbb{Z}$  and  $\binom{d+n-k}{n-k} \in \mathbb{Z}$  whenever  $n \in \mathbb{Z}$ , we know  $f(n)$  is an integer whenever  $n \in \mathbb{Z}$ .  $\square$

## Problem 6

Let  $n \geq 2$  be an integer.

(a) Prove that

$$\sum_{i=0}^n i \binom{n}{i} (-1)^{i-1} = 0.$$

*Proof.* By the binomial theorem,

$$\sum_{n \geq 1} i \binom{n}{i} x^{i-1} = \left( \sum_{n \geq 0} \binom{n}{i} x^i \right)' = ((1+x)^n)' = n(1+x)^{n-1},$$

and thus

$$\sum_{i=0}^n i \binom{n}{i} (-1)^{i-1} = n(1+(-1))^{n-1} = 0$$

□

(b) Compute

$$\sum_{\substack{0 \leq i \leq n \\ i \text{ even}}} i \binom{n}{i}.$$

*Proof.*

$$\begin{aligned} \sum_{\substack{0 \leq i \leq n \\ i \text{ even}}} i \binom{n}{i} &= \frac{1}{2} \left( \sum_{i=0}^n i \binom{n}{i} - \sum_{i=0}^n i \binom{n}{i} (-1)^{i-1} \right) \\ &= \frac{1}{2} (n(1+1)^{n-1}) = n2^{n-2}. \end{aligned}$$

□

## Problem 7

- (a) Let  $a, b$  be rational numbers. Show that for any formal power series  $A(x)$  with  $A(0) = 1$ , we have

$$A(x)^a A(x)^b = A(x)^{a+b}.$$

[Remember that we defined rational powers in a very specific way, so your proof needs to use this definition.]

*Proof.* Let  $a = m/n$ ,  $b = p/q$ , for some  $m, n, p, q \in \mathbb{Z}$ . Then,

$$\begin{aligned} A(x)^a A(x)^b &= (A(x)^{1/nq})^{mq} (A(x)^{1/nq})^{np} \\ &= A(x)^{(mq+np)/nq} \\ &= A(x)^{a+b}. \end{aligned}$$

□

- (b) Deduce from (a) that

$$\binom{a+b}{n} = \sum_{i=0}^n \binom{a}{i} \binom{b}{n-i}$$

for all non-negative integers  $n$ .

*Proof.* Put  $A(x) = (1+x)$ . Since  $(1+x)^a (1+x)^b = (1+x)^{a+b}$ ,

$$\begin{aligned} \binom{a+b}{n} &= [x^n](1+x)^{a+b} \\ &= [x^n](1+x)^a (1+x)^b \\ &= \sum_{i=0}^n ([x^i](1+x)^a) ([x^{n-i}](1+x)^b) \\ &= \sum_{i=0}^n \binom{a}{i} \binom{b}{n-i}. \end{aligned}$$

□



## Problem 8

Assume now that we deal with complex-coefficient formal power series. Define the following sets of formal power series:

$$V = \{F(x) \mid F(0) = 0\}, \quad W = \{G(x) \mid G(0) = 1\}.$$

- (a) Given  $F \in V$ , show that  $\mathbf{E}(F) = \sum_{n \geq 0} \frac{F^n(x)}{n!}$  is the *unique* formal power series  $G \in W$  such that  $DG = DF \cdot G$ . This defines a function  $\mathbf{E}: V \rightarrow W$ . [Convention:  $F^0(x) = 1$  even if  $F(x) = 0$ .]

*Proof.* It is easy to see that

$$DG = \sum_{n \geq 0} \frac{D(F^n(x))}{n!} = \sum_{n \geq 1} DF \cdot \frac{F^{n-1}(x)}{(n-1)!} = DF \sum_{n \geq 0} \frac{F^n(x)}{n!} = DF \cdot G,$$

and  $G(0) = F^0(0) = 1$ . It remains to show that  $D$  is unique. Suppose there exists  $G = \sum_{n \geq 0} b_n x^n$ ,  $G' = \sum_{n \geq 0} b'_n x^n \in W$  such that  $\mathbf{E}(F) = G$  and  $\mathbf{E}(F) = G'$ . Suppose  $DF = \sum_{n \geq 0} a_n x^n$ . We know  $DG = DF \cdot G$  and  $DG' = DF \cdot G'$ . By comparing coefficients, for  $k \geq 1$ ,

$$\begin{aligned} \frac{b_k}{k+1} &= [x^k]DG = [x^k](DF \cdot G) = \sum_{i=0}^k a_i b_{k-i}, \\ \frac{b'_k}{k+1} &= [x^k]DG = [x^k](DF \cdot G) = \sum_{i=0}^k a_i b'_{k-i}. \end{aligned}$$

In particular, for  $k \geq 1$ ,

$$b_k = \frac{k+1}{-a_0 k - a_0 + 1} \sum_{i=1}^k a_i b_{k-i}, \quad b'_k = \frac{k+1}{-a_0 k - a_0 + 1} \sum_{i=1}^k a_i b'_{k-i},$$

so  $b_k, b'_k$  are uniquely determined by the corresponding previous coefficients, and thus  $G = G'$  if and only if they agree with the constant term. But then  $G(0) = G'(0) = 1$ , and the result follows.  $\square$

- (b) Given  $G \in W$ , show that there is a *unique* formal power series  $F \in V$  such that  $DF(x) = DG(x)/G(x)$ . We define the function  $\mathbf{L}: W \rightarrow V$  by  $\mathbf{L}(G) = F$ . [For the rest, it is unnecessary to use explicit formulas for  $\mathbf{L}$  and  $\mathbf{E}$  and in fact it may be easier to only use the uniqueness properties above.]

*Proof.* Since  $G(0) = 1$ , there exists  $G^{-1}(x)$  such that  $G(x)G^{-1}(x) = G(x)^{-1}G(x) = 1$ , so  $DG(x)/G(x)$  is unique given  $G$ . Suppose  $DG(x)/G(x) = \sum_{n \geq 0} a_n x^n$ . There exists  $F = \sum_{n \geq 1} \frac{1}{n} a_{n-1} x^n \in V$  such that

$$DF = \sum_{n \geq 1} a_{n-1} x^{n-1} = \sum_{n \geq 0} a_n x^n = DG(x)/G(x).$$

That is, all coefficients  $a_n$  of  $DF$  are uniquely determined by  $DG(x)/G(x)$ . But then all coefficients of  $F$  are uniquely determined, as  $F$  has no constant term.  $\square$

- (c) Show that  $\mathbf{E}$  and  $\mathbf{L}$  are inverses of each other.

*Proof.* Let  $F \in V$ .  $\mathbf{E}$  maps  $F$  to some unique  $G' \in W$  such that  $DG' = DF \cdot G'$ , that is,  $DF = DG'/G'$ . Then,  $\mathbf{L}$  maps  $G'$  back to some unique  $F'$  such that  $DF' = DG'/G' = DF$ . But then both  $F$  and  $F'$  have no constant terms, so  $F$  and  $F'$  actually agree with all coefficients. Hence,  $\mathbf{L}(\mathbf{E}(F)) = F$ .

Let  $G \in W$ .  $\mathbf{L}$  maps  $G$  to some unique  $F'' \in V$  such that  $DG/G = DF''$ , and  $\mathbf{E}$  maps  $F''$  back to some unique  $G''$  such that  $DG'' = DF'' \cdot G'' = DG/G \cdot G''$ . But then  $DG''/G'' = DG/G$ . By comparing coefficients, for all  $k \geq 0$  we get

$$\sum_{i=0}^k b''_{k-i}(i+1)b_{i+1} = [x^k]DG'' \cdot G = [x^k]DG \cdot G'' = \sum_{i=0}^k b_{k-i}(i+1)b''_{i+1}.$$

Since  $b_0 = b'_0 = 1$ , it follows from induction that  $b_k = b'_k$  for all  $k \in \mathbb{Z}_{\geq 0}$ , and so  $G = G''$ . Hence,  $\mathbf{E}(\mathbf{L}(G)) = G$ .  $\square$

- (d) Show that  $\mathbf{E}(F_1 + F_2) = \mathbf{E}(F_1)\mathbf{E}(F_2)$  for all  $F_1, F_2 \in V$ .

*Proof.* Let  $G_1 = \mathbf{E}(F_1)$ ,  $G_2 = \mathbf{E}(F_2)$ , and  $G = \mathbf{E}(F_1 + F_2)$ . Since

$$\begin{aligned} D(G_1G_2) &= DG_1 \cdot G_2 + DG_2 \cdot G_1 \\ &= (DF_1 \cdot G_1)G_2 + (DF_2 \cdot G_2)G_1 \\ &= (DF_1 + DF_2)(G_1G_2) \\ &= D(F_1 + F_2)(G_1G_2). \end{aligned}$$

Note that  $G_1G_2 \in W$ . But then  $G$  is the unique element in  $W$  such that  $DG = D(F_1 + F_2)G$ , and so  $\mathbf{E}(F_1 + F_2) = G = G_1G_2 = \mathbf{E}(F_1)\mathbf{E}(F_2)$ .  $\square$

- (e) Show that  $\mathbf{L}(G_1G_2) = \mathbf{L}(G_1) + \mathbf{L}(G_2)$  for all  $G_1, G_2 \in W$ .

*Proof.* Let  $F_1 = \mathbf{L}(G_1)$ ,  $F_2 = \mathbf{L}(G_2)$ . Since

$$\begin{aligned} D(F_1 + F_2) &= DF_1 + DF_2 \\ &= DG_1/G_1 + DG_2/G_2, \end{aligned}$$

$$G_1G_2D(F_1F_2) = DG_1 \cdot G_2 + DG_2 \cdot G_1 = D(G_1G_2), \quad (3)$$

that is,  $D(F_1F_2) = D(G_1G_2)/G_1G_2$ . But then  $F_1F_2 \in F$ , so  $F_1F_2$  is the unique element in  $W$  that satisfies (3), and thus  $\mathbf{L}(G_1G_2) = F_1F_2 = \mathbf{L}(G_1) + \mathbf{L}(G_2)$ .  $\square$

- (f) If  $m$  is a positive integer and  $G \in W$ , show that  $\mathbf{E}(\frac{\mathbf{L}(G)}{m})$  is an  $m$ th root of  $G$ . [This gives an alternative proof for the existence of  $m$ th roots and in fact we can now define powers for any complex number  $m$ :  $F^m = \mathbf{E}(m\mathbf{L}(F))$ .]

*Proof.* By (e),

$$\mathbf{L} \left[ \left( \mathbf{E} \left( \frac{\mathbf{L}(G)}{m} \right) \right)^m \right] = m\mathbf{L} \left[ \mathbf{E} \left( \frac{\mathbf{L}(G)}{m} \right) \right] = m \cdot \frac{\mathbf{L}(G)}{m} = \mathbf{L}(G).$$

But then  $\mathbf{L}$  is bijective, and the result follows.  $\square$

(g) Show that if  $\sum_{i \geq 0} F_i(x)$  converges to  $F(x)$ , then  $\prod_{i \geq 0} \mathbf{E}(F_i)$  converges to  $\mathbf{E}(F)$ .

*Proof.* By (d),

$$\mathbf{E}(F(x)) = \mathbf{E}\left(\sum_{i \geq 0} F_i(x)\right) = \prod_{i \geq 0} \mathbf{E}(F_i).$$

□

(h) Show that if  $\prod_{i \geq 0} G_i(x)$  converges to  $G(x)$ , then  $\sum_{i \geq 0} \mathbf{L}(G_i)$  converges to  $\mathbf{L}(G)$ .

*Proof.* By (e),

$$\mathbf{L}(G(x)) = \mathbf{L}\left(\prod_{i \geq 0} G_i(x)\right) = \sum_{i \geq 0} \mathbf{L}(G_i).$$

□