C8.3 Combinatorics: Sheet #3

Due on November 27, 2025 at 12:00pm

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Problem 1

Let $\mathcal{A} \subset \mathcal{P}[n]$ be an upset and $\mathcal{B} \subset \mathcal{P}[n]$ be a downset. Prove that $|\mathcal{A} \cap \mathcal{B}| \leq 2^{-n}|\mathcal{A}| \cdot |\mathcal{B}|$.

Proof. Note that $\mathcal{A}^C$ is a downset. By Kleitman's Theorem,

$$|\mathcal{B}| - |\mathcal{A} \cap \mathcal{B}| = |\mathcal{A}^C \cap \mathcal{B}| \geq \frac{|\mathcal{A}^C||\mathcal{B}|}{2^n} = \frac{(2^n - |\mathcal{A}|)|\mathcal{B}|}{2^n} = |\mathcal{B}| - \frac{|\mathcal{A}||\mathcal{B}|}{2^n}.$$

The result now follows. $\square$

Problem 2

The i-compression operator π_i is defined by $\pi_i(A) = A \setminus \{i\}$ and, for a set system $\mathcal{A}$,

$$\pi_i(\mathcal{A}) = \{\pi_i(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : \pi_i(A) \in \mathcal{A}\}.$$

Let $\mathcal{F} \subset \mathcal{P}[n]$ be a set system and $\mathcal{A} = \pi_i(\mathcal{F})$ for some $i \in [n]$. Show that $\mathrm{tr}_{\mathcal{A}}(S) \leq \mathrm{tr}_{\mathcal{F}}(S)$ for every $S \subset [n]$.

Proof. Suppose $B \subseteq S$ such that $B = A \cap S$ for some $A \in \mathcal{A}$. Let $F \in \mathcal{F}$ such that $\pi_i(F) = A$. We may assume $F = A \cup \{i\}$ otherwise $F \cap S = F \cap A = B$ and we are done. If $i \notin S$, then $F \cap S = A \cap S = B$. If $i \in S$, then $F \cap S = B \cup \{i\} \notin \mathcal{A} \mid S$. Thus, regardless of whether $B \in \mathcal{F} \mid S$ or not,

$$\mathrm{tr}_{\mathcal{A}}(S) \leq \mathrm{tr}_{\mathcal{F}}(S).$$

□

Problem 3

(a) Let $X = \mathbb{R}$ and let $\mathcal{F} = \{[a, b] : a < b\}$. What is the VC-dimension of $\mathcal{F}$?

Proof. Note that the VC-dimension of $\mathcal{F}$ is at least 2: the set $\{0, 2\}$ is shattered by the intervals $[3, 4], [0, 1], [1, 2], [0, 2]$.

It has VC-dimension less than 3: consider any set of three points $a < b < c$. Then there are no intervals that contain both a and c while excluding b.

Hence, the VC-dimension of $\mathcal{F}$ is 2. $\square$

(b) What if $X = \mathbb{R}^2$ and $\mathcal{F} = \{[a, b] \times [c, d] : a < b \text{ and } c < d\}$?

Proof. The VC-dimension of $\mathcal{F}$ is at least 4, as it can shatter the set $\{(-1, -1), (1, 1), (-1, 1), (1, -1)\}$.

It has VC-dimension less than 4: Let S be a set of 5 elements. Let x_M, x_m, y_M, y_m be the maximum and minimum x- and y-coordinates of the points in S, respectively. Then the box that contains $\{x_M, x_m, y_M, y_m\}$ must contain the rest of S. $\square$

Problem 4

Let $\mathcal{F}$ be the collection of all convex sets in $\mathbb{R}^2$. Show that $\mathcal{F}$ does not have bounded VC-dimension.

Proof. For any $n \in \mathbb{N}$, consider a set S of n points lying on the unit circle. Then for any subset $T \subseteq S$, the polygon formed by the points in T is convex and only contains points in T. Hence, the VC-dimension of $\mathcal{F}$ is at least n. This completes the proof. $\square$

Problem 5

A *sunflower* is a sequence $F_1, \dots, F_k$ of sets such that for some set S, and all $i < j$,

$$F_i \cap F_j = S.$$

Let $r, s \geq 1$. Prove that there is $m = m(r, s)$ such that every sequence of m sets from $\mathbb{N}^{(r)}$ has a subsequence of length s that forms a sunflower.

[Bonus question: explain the term *sunflower* by means of a nice picture.]

Proof. Fix $s \geq 1$. We proceed by induction on r to show that $m(r, s)$ is bounded. Note that if $m(1, s) \geq s^2+1$, then either there are s distinct singletons or there exists subsequence $F_{i_1} = F_{i_2} = \cdots = F_{i_s}$, by the pigeonhole principle. But then either case yields a sunflower, so the base case is done. Suppose $r \geq 2$. By induction, $m(r-1, s) < \infty$. Let $F_1, \dots, F_k$ be a sequence of $k \geq (m(r-1, s)+1)^2 + 1$ sets from $\mathbb{N}^{(r)}$. Just as the base case, there exists a subsequence $F_{i_1}, \dots F_{i_{m(r-1,s)+1}}$, such that either there exists $f_i \in F_i$ with $f_i \neq f_j$ for $i \neq j$, or there exists $f \in \mathbb{N}$ with $f \in \bigcap_{i=1}^{m(r-1,s)+1} F_i$. In either case, consider the sequence $F_1 \setminus \{f_1\}, \dots, F_{m(r-1,s)+1} \setminus \{f_{m(r-1,s)+1}\}$ or $F_1 \setminus \{f\}, \dots, F_{m(r-1,s)+1} \setminus \{f\}$. Since it is a sequence from $\mathbb{N}^{(r-1)}$ of length $> m(r-1, s)$, there is a subsequence $F_{j_1}, \dots, F_{j_s}$ such that $F_{j_1} \setminus \{f_{j_1}\}, \dots, F_{j_s} \setminus \{f_{j_s}\}$ or $F_{j_1} \setminus \{f\}, \dots, F_{j_s} \setminus \{f\}$ is a sunflower. But then $F_{j_1}, \dots, F_{j_s}$ is a sunflower of length s in either case. This completes the proof. $\square$

Problem 6

Let $\mathcal{F} \subset \mathcal{P}[n]$ be a set system. The *dual set system* $\mathcal{F}^*$ has vertex set $\mathcal{F}$, and for each $i \in [n]$, there is an edge $\{F \in \mathcal{F} : i \in F\}$ (we ignore duplicate edges). Prove that for every positive integer d there is a constant $f(d)$ such that if $\mathcal{F}$ has VC-dimension at most d then $\mathcal{F}^*$ has VC-dimension at most $f(d)$.

Proof. Suppose $\mathcal{F}$ has VC-dimension d. Suppose for the sake of contradiction that $\mathcal{F}^*$ has VC-dimension 2^{d+1}. Then there exists $\mathcal{S} \subseteq \mathcal{F}$ of size 2^{d+1} that is shattered by $\mathcal{F}^*$. Consider the indicent matrix M, whose rows are indexed by the elements of $[n]$, columns are indexed by the elements of $\mathcal{S}$, and $M_{i,F} = \mathbb{1}_{i \in F}$ for $F \in \mathcal{S}, i \in [n]$. Since $\mathcal{S}$ is shattered by $\mathcal{F}^*$, there are $2^{2^{d+1}}$ unique binary vectors of length 2^{d+1} among the rows of M. Omit all duplicate rows of M so that M turns into a $2^{2^{d+1}} \times 2^{d+1}$ matrix with unique rows. Let M' be the $(d+1) \times 2^{d+1}$ matrix whose columns are the binary expansions of the numbers $0, \dots, 2^{d+1} - 1$ in order. Note that the columns of M' are distinct. Since the rows of M contain all possible binary vectors of length 2^{d+1}, each row of M' corresponds to a unique row of M. Let $X \subseteq [n]$ be the set of $d+1$ rows in M that M' corresponds to. Then X is shattered by $\mathcal{F}$, contradicting that $\mathcal{F}$ has VC-dimension d. Thus F^* has VC-dimension at most $f(d) = 2^{d+1} - 1$. □

Problem 7

Suppose that $\mathcal{F}_1, \dots, \mathcal{F}_s \subset \mathcal{P}(n)$ are intersecting families. Prove that $|\mathcal{F}_1 \cup \dots \cup \mathcal{F}_s| \leq 2^n - 2^{n-s}$.

Proof. For $1 \leq i \leq s$ and $\mathcal{S} \subseteq \mathcal{F}_i$, Define

$$\mathcal{D}_i = \{D \in \mathcal{P}(n) : F \subseteq D \text{ for some } F \in \mathcal{F}_i\}.$$

Note that $\mathcal{D}_i$ is intersecting and $|\mathcal{F}_i| = |\mathcal{D}_i|$. Since

$$|\mathcal{F}_1 \cup \dots \cup \mathcal{F}_s| \leq |\mathcal{D}_1 \cup \dots \cup \mathcal{D}_s| = 2^n - |\mathcal{D}_1^C \cap \dots \cap \mathcal{D}_s^C|,$$

it suffices to show that $|\mathcal{D}_1^C \cap \dots \cap \mathcal{D}_s^C| \geq 2^{n-s}$. Since $|\mathcal{D}_i| \leq 2^{n-1}$, we have $|\mathcal{D}_i^C| \geq 2^{n-1}$. For $D \in \mathcal{D}_i$, notice that if $D' \supseteq D$ then $D' \in \mathcal{D}_i$, so $\mathcal{D}_i$ is an upset. But then $\mathcal{D}_i^C$ is a downset. It now follows from the Kleitman's Theorem that

$$|\mathcal{D}_1^C \cap \dots \cap \mathcal{D}_s^C| \geq \frac{|\mathcal{D}_1^C||\mathcal{D}_2^C| \cdots |\mathcal{D}_s^C|}{(2^n)^{s-1}} \geq \frac{2^{s(n-1)}}{2^{n(s-1)}} = 2^{n-s}.$$

□