MATH 180B: Homework #2

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Professor Carfagnini

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Problem 1

Suppose U and V are independent and follow the geometric distribution

$$p(k) = p(1-p)^k \quad \text{for } k = 0, 1, \ldots.$$

Define the random variable $Z = U + V$.

(a) Determine the joint probability mass function $p_{U,Z}(u, z) = \mathbb{P}\{U = u, Z = z\}$.

Proof. Since $p_{U,Z}(u, z) = \mathbb{P}\{U = u, Z = z\} = \mathbb{P}\{U = u, V = z - u\}$, we have

$$p_{U,Z}(u, z) = p(1-p)^u p(1-p)^{z-u} = p^2(1-p)^z.$$

$\square$

(b) Determine the conditional probability mass function for U given that $Z = n$.

Proof. Note that $p_Z(z) = \sum_{u=0}^{z} p_U(u)p_U(z-u) = \sum_{u=0}^{z} p_U(u)p_v(v) = \sum_{u=0}^{z} p^2(1-p)^z = (z+1)p^2(1-p)^z$. Thus,

$$\begin{aligned} p_{U|Z}(u|n) &= \frac{p_{U,Z}(u,n)}{p_Z(n)} \\ &= \frac{p^2(1-p)^n}{(n+1)p^2(1-p)^n} \\ &= \frac{1}{n+1}. \end{aligned}$$

$\square$

Problem 2

Let N have a Poisson distribution with parameter $\lambda = 1$. Conditioned on $N = n$, let X have a uniform distribution over the integers $0, 1, \ldots, n+1$. What is the marginal distribution for X?

Proof.

$$\begin{aligned}
p_X(x) &= \sum_{n=0}^{\infty} p_{X|N}(x|n)p_N(n) \\
&= \sum_{n=x-1}^{\infty} \frac{1}{n+2} \cdot \frac{1}{n!}e^{-1} \\
&= e^{-1} \sum_{n=x-1}^{\infty} \frac{1}{(n+2)n!} \\
&= e^{-1} \sum_{n=x-1}^{\infty} \frac{1}{(n+1)!} - \frac{1}{(n+2)!} \\
&= e^{-1} \left(\sum_{n=x-2}^{\infty} \frac{1}{(n+2)!} - \sum_{n=x-1}^{\infty} \frac{1}{(n+2)!} \right) \\
&= e^{-1} \left(\frac{1}{x!} + \sum_{n=x-1}^{\infty} \frac{1}{(n+2)!} - \sum_{n=x-1}^{\infty} \frac{1}{(n+2)!} \right) \\
&= \frac{1}{e(x!)}.
\end{aligned}$$

□

Problem 3

Suppose that upon striking a plate a single electron is transformed into a number N of electrons, where N is a random variable with mean μ and standard deviation σ. Suppose that each of these electrons strikes a second plate and releases further electrons, independently of each other and each with the same probability distribution as N. Let Z be the total number of electrons emitted from the second plate. Determine the mean and variance of Z.

Proof. Note that $Z = \xi_1 + \cdots + \xi_N$, where ξ_k is the number of electrons struck by the kth electron from the first plate. ξ_k shares the same distribution as N, for all k. Thus,

$$\mathbb{E}[Z] = \mathbb{E}[N]\mathbb{E}[N] = \mu^2,$$

$$Var(Z) = \mathbb{E}[N]Var(N) + \mathbb{E}[N]^2 Var(N) = \mu\sigma^2 + \mu^2\sigma^2 = \mu(\mu + 1)\sigma^2.$$

□

Problem 4

A six-sided die is rolled, and the number N on the uppermost face is recorded. From a jar containing 10 tags numbered $1, 2, \ldots, 10$ we then select N tags at random without replacement. Let X be the smallest number on the drawn tags. Determine $\mathbb{P}(X = 2)$ and $\mathbb{E}[X]$.

Proof. Note that $\mathbb{P}(X = x|N = n) = \frac{\binom{10-x}{n-1}}{\binom{10}{n}}$. Thus,

$$\begin{aligned}
\mathbb{P}(X = 2) &= \sum_{n=1}^{6} \mathbb{P}(X = 2|N = n)\mathbb{P}(N = n) \\
&= \frac{1}{6}\sum_{n=1}^{6} \mathbb{P}(X = 2|N = n) \\
&= \frac{1}{6}\sum_{n=1}^{6} \frac{\binom{8}{n-1}}{\binom{10}{n}} \\
&= \frac{1}{6}\sum_{n=1}^{6} \frac{n}{9} - \frac{n^2}{90} \\
&= \frac{119}{540},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X|N]] \\
&= \mathbb{E}\left[\sum_{x=1}^{11-N} x\mathbb{P}(X = x|N)\right] \\
&= \mathbb{E}\left[\sum_{x=1}^{11-N} \frac{x\binom{10-x}{N-1}}{\binom{10}{N}}\right] \\
&= \mathbb{E}\left[\frac{1}{\binom{10}{N}} \sum_{x=1}^{11-N} \binom{x}{1}\binom{10-x}{N-1}\right] \\
&= \mathbb{E}\left[\frac{11}{N+1}\right] \approx 2.92024
\end{aligned}$$

□

Problem 5

Suppose that $\xi_1, \xi_2, \dots$ are independent and identically distributed with $\mathbb{P}\{\xi_k = \pm 1\} = \frac{1}{2}$. Let N be independent of $\xi_1, \xi_2, \dots$ and follow the geometric probability mass function

$$p_N(k) = \alpha(1-\alpha)^k \quad \text{for } k = 0, 1, \dots,$$

where $0 < \alpha < 1$. Form the random sum $Z = \xi_1 + \dots + \xi_N$.

(a) Determine the mean and variance of Z.

Proof. Since the $\mathbb{E}[N] = \frac{1-\alpha}{\alpha}$ and $Var(N) = \frac{1-\alpha}{\alpha^2}$,

$$\mathbb{E}[Z] = \mathbb{E}[N]\mathbb{E}[\xi_1] = 0, \quad Var(Z) = \mathbb{E}[N]Var(\xi_1) + \mathbb{E}[\xi_1]^2 Var(N) = \frac{1}{\alpha}.$$

□

(b) Evaluate the higher moments $m_3 = \mathbb{E}[Z^3]$ and $m_4 = \mathbb{E}[Z^4]$.

Proof. Note that $\xi_i^2 = 1$ and $\mathbb{E}[\xi_i] = \mathbb{E}[\xi_i\xi_j] = 0$. Hence,

$$\mathbb{E}[\xi_i\xi_j\xi_k] = \mathbb{E}[\xi_i\xi_j\xi_k\xi_m] = \mathbb{E}[\xi_i^2\xi_j\xi_k] = \mathbb{E}[\xi_i^3\xi_j] = 0,$$

so we only need to care about $\mathbb{E}[\xi_i^2\xi_j^2]$ and $\mathbb{E}[\xi_i^4]$. We thus get

$$\begin{aligned}
\mathbb{E}[Z^3] &= \mathbb{E}\left[\mathbb{E}\left[\sum_i \sum_j \sum_k \xi_i\xi_j\xi_k | N\right]\right] \\
&= \mathbb{E}\left[N(N-1)(N-2)\mathbb{E}[\xi_i\xi_j\xi_k] + N(3N-2)\mathbb{E}[\xi_i]\right] = 0, \\
\mathbb{E}[Z^4] &= \mathbb{E}\left[\sum_i \sum_j \sum_k \sum_m \mathbb{E}[\xi_i\xi_j\xi_k\xi_m] | N\right] \\
&= \mathbb{E}\left[3N(N-1)\mathbb{E}[\xi_i^2\xi_j^2] + N\mathbb{E}[\xi_i^4]\right] \\
&= \mathbb{E}\left[(3N^2 - 2N)\mathbb{E}[1]\right] \\
&= \mathbb{E}\left[3N^2 - 2N\right] = 3(Var(N) + \mathbb{E}[N]^2) - 2\mathbb{E}[N] = \frac{6-5\alpha}{\alpha^2}.
\end{aligned}$$

□

Problem 6

To form a slightly different random sum, let $\xi_0, \xi_1, \ldots$ be independent identically distributed random variables and let N be a nonnegative integer-valued random variable, independent of $\xi_0, \xi_1, \ldots$. The first two moments are

$$\mathbb{E}[\xi_k] = \mu, \quad Var[\xi_k] = \sigma^2,$$
$$\mathbb{E}[N] = \nu, \quad Var[N] = \tau^2.$$

Determine the mean and variance of the random sum $Z = \xi_0 + \ldots + \xi_N$.

Proof.

$$\mathbb{E}[Z] = \mathbb{E}[\xi_k](\mathbb{E}[N] + 1) = \mu(\nu + 1),$$
$$Var(Z) = (\mathbb{E}[N] + 1)Var(\xi_k) + \mathbb{E}[\xi_k]^2 Var(N) = (\nu + 1)\sigma^2 + \mu^2\tau^2.$$

□