MATH 140A: Homework #5

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Professor Seward

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Problem 1

Construct a compact set of real numbers whose limit points form a countable set.

Proof. Consider

$$S = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \bigcup_{k \in \mathbb{N}} S_k,$$

where $S_k = \left\{\frac{1}{k} + \frac{1}{n} \mid n > k(k-1), n \in \mathbb{N}\right\}$. Note that S_k is bounded below by $\frac{1}{k}$ and bounded above by $\frac{1}{k-1}$, as $\sup S_k < \frac{1}{k} + \frac{1}{k(k-1)} = \frac{1}{k-1}$. Thus, S_i and S_j are disjoint if $i \neq j$. We claim $S' = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Since 0 is a limit point of $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ and $\frac{1}{k}$ is the limit point of S_k, we only need to prove that $S' \subseteq \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$. It is obvious that S is bounded above by 2 and below by 0, so we only need to consider points in $[0, 2]$. Let $x \in [0, 2]$ be a limit point of S. Suppose for sake of contradiction that $x \notin \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$. If $x > 1$, then $x = \epsilon + 1$, for some positive ϵ. But then $B_{\frac{\epsilon}{2}}(x) \cap S = B_{\frac{\epsilon}{2}}(x) \cap S_1 = \left\{1 + \frac{1}{n} \mid n < \frac{1}{\epsilon}, n \in \mathbb{N}\right\}$ is finite. Hence, we may assume $x < 1$. Then, $\frac{1}{p} < x < \frac{1}{p-1}$ for some $p \in \mathbb{N}$, by the archimedean property. This means that $B_\delta(x) \cap S = B_\delta(x) \cap S_p$, for $\delta < \frac{1}{2p(p-1)}$. But then x is the limit point of S_p, which is $\frac{1}{p}$, contradiction. Hence, $x \in \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \subset S$, and thus $S = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$, which is countable. Since $S' \in S$ and $S \subset [0, 2]$, S is compact, by Theorem 2.41. $\square$

Problem 2

Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

Proof. E is obviously bounded above by 3 and below by -3, otherwise there exists $p \in E$ such that $p^2 > 3^2 > 3$.

We show that E is closed. Let $x \in E^c$. Then, either $x^2 \leq 2$ or $x^2 \geq 3$. Suppose that $x^2 \leq 2$. Then, $-\sqrt{2} \leq x \leq \sqrt{2}$. Pick $\epsilon < \min(\sqrt{2}-x, x+\sqrt{2})$. Since $x+\epsilon < x+(\sqrt{2}-x) = \sqrt{2}$ and $x-\epsilon > x-(x+\sqrt{2}) = -\sqrt{2}$, $B_\epsilon(x)$ is bounded above by $\sqrt{2}$ and below by $-\sqrt{2}$. Thus, $B_\epsilon(x) \subset [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q} \subset E^c$, x is an interior point of E^c. Suppose that $x^2 \geq 3$. Then, either $x \geq \sqrt{3}$ or $x \leq -\sqrt{3}$. If $x \geq \sqrt{3}$, then $B_{x-\sqrt{3}}(x)$ is bounded below by $\sqrt{3}$, so $B_{x-\sqrt{3}}(x) \subset E^c$. Otherwise, $B_{-\sqrt{3}-x}(x)$ is bounded above by $-\sqrt{3}$, and thus $B_{-\sqrt{3}-x}(x) \subset E^c$. Hence, x is an interior point of E^c. It follows that E^c is open, and thus E is closed.

We now show that E is not compact. Consider the set $S = \{2 < r^2 < 3\}$ under $\mathbb{R}$. Let $r \in S$. Since $2 < r^2 < 3$, either $\sqrt{2} < r < \sqrt{3}$ or $-\sqrt{3} < r < -\sqrt{2}$. Hence, $S = (-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})$, which is open by Theorem 2.24. By Theorem 2.34, S is not compact in $\mathbb{R}$. By Theorem 2.33, S is not compact relative to $\mathbb{Q} \subset \mathbb{R}$. But then $E = S \cap \mathbb{Q}$, so E is not compact.

By Theorem 2.30, we also know $E = S \cap \mathbb{Q}$ is open in $\mathbb{Q}$. $\square$

Problem 3

Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

Proof. We show by Cantor's diagonalization argument that E is uncountable. Let C be a countable set of E. We associate each number in C a unique index, say $a_1, a_2, \ldots \in C$. Define $f : [0,1] \times \mathbb{N} \to \mathbb{Z}_{\geq 0}$ that maps (a, n) to the nth decimal digit of a. Let $k \in [0,1]$ such that $f(k,n) = 4$ if $f(a_n, n) = 7$ and $f(k,n) = 7$ otherwise, for $n \in \mathbb{N}$. Since the decimal expansion is of k contains only 4 and 7, k is in E. However, $k \neq a_i$, for all $a_i \in C$. Hence, E is uncountable.

Note that $E \subset [0.4, 0.8]$. Since there does not exist $a \in E$ such that $0.8 < a < 1$, E is not dense in $[0,1]$.

For compactness, we already know E is bounded, so it suffices to show that E is closed, by Theorem 2.41.

Let $x \in E^c$. x contains a decimal digit other than 4 and 7. Consider the first such digit, say the nth digit. Let $\delta > 0$ such that $\delta < 10^{-n}$. Then, $N_\delta(x)$ does not contain any point in E, and thus x cannot be a limit point of E. Therefore, $E' \subset E$, so E is closed.

However, E is not perfect. Consider $0.4 \in E$. $N_{0.01}(0.4) \cap E = \emptyset$ and thus 0.4 is not a limit point of E. $\square$

Problem 4

Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?

Proof. Yes. Let $E_0 = [r, s]$, where r, s are two irrational numbers. We inductively remove all rational numbers from E_0. Since $\mathbb{Q}$ is countable, associate an index to each rational number, say $a_1, a_2, \ldots \in E_0$. We construct E_n by removing the segment (r_n, s_n) from E_{n-1}, for some irrational r_n, s_n such that $r_n < a_n < s_n$, and make it E_n. Note that E_n union of intervals and thus $E_0 \supset E_1 \supset \cdots$ is a chain of compact sets, by Theorem 2.41. Let $E = \bigcap_{i=0}^{\infty} E_i$. E is closed and nonempty, by Theorem 2.24 and 2.36. For rational $q \in [r, s]$, $q \notin E_n$, and thus $q \notin E$. Hence, E does not contain any rational numbers. It remains to show that every point in E is a limit point. Notice that E does contain any segments, as $\mathbb{Q}$ is dense in $\mathbb{R}$ so any segment $(a, b) \subset \mathbb{R}$ contains a rational number. Let $x \in E$ and let $\epsilon > 0$. Let I_n be an interval of E_n which contains x. Since the open set $N_\epsilon(x)$ is not contained in E, there exists large enough n such that $I_n \subset N_\epsilon(x)$, and thus $N_\epsilon(x)$ contains the end point of I_n. It follows that x is a limit point of E, as $(N_\epsilon(x) \backslash \{x\}) \cap E \neq \emptyset$, and this completes the proof. $\square$

Problem 5

(a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

Proof. Since A, B are closed, $A = \overline{A}$ and $B = \overline{B}$. Since $\overline{A} \cap B = A \cap \overline{B} = A \cap B = \emptyset$, the result follows. □

(b) Prove the same for disjoint open sets.

Proof. It suffices to show that $A' \cap B = A \cap B' = \emptyset$. Let $a \in A$. Since there exists a neighborhood N of a such that $N \subset A$, N contains not point of B, and thus $A \cap B' = \emptyset$. By symmetry, we also know $B' \cap A = \emptyset$, and this completes to proof. □

(c) Fix $p \in X, \delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

Proof. Since $A = N_\delta(p)$, A is obviously an open set. Moreover, B is the complement of $\overline{A}$, so B is an open set disjoint to A. The result now follows from (b). □

(d) Prove that every connected metric space with at least two points is uncountable. *Hint:* Use (c).

Proof. Let X be a metric space and $p, q \in X$ such that $p < q$. Then, $d(p, q) > 0$. Since X is connected, there exists $m \in X$ such that $d(p, m) = \delta$, for every $\delta \in [0, d(p, q)]$, otherwise X is separated by (c). Hence, there exists an surjective mapping $X \to [0, d(p, q)]$ that maps r to ϵ, for some $d(p, r) = \epsilon$. However, $[0, d(p, q)]$ is uncountable, and thus X is uncountable. □

Problem 6

Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)

Proof. Closures of connected sets are connected. Let X be a nonempty connected set. Suppose for the sake of contradiction that $\overline{X}$ is not connected. Then, $\overline{X} = A \cup B$, where $\overline{A} \cap B = \overline{B} \cap A = \emptyset$. We know $X \not\subset A$, otherwise B contains limit points of A, which forces $\overline{A} \cap B \neq \emptyset$. Similarly, $X \not\subset B$, so $X \cap A$ and $X \cap B$ are both nonempty. Hence, X is the union of disjoint sets $X \cap A$ and $X \cap B$. However, since $\overline{A} \cap B = \overline{B} \cap A = \emptyset$, we have $(\overline{X \cap A}) \cap (X \cap B) = (\overline{X \cap B}) \cap (X \cap A) = \emptyset$, contradicting that X is connected.

However, interiors of connected sets are not always connected. Let $p, q \in \mathbb{R}^2$, $p \neq q$. Let $A = \{a \in \mathbb{R}^2 \mid d(p, a) \leq \frac{1}{2} d(p, q)\}$ and $B = \{b \in \mathbb{R}^2 \mid d(q, b) \leq \frac{1}{2} d(p, q)\}$, and let $E = A \cup B$. Note that $A \cap B \neq \emptyset$. Suppose for the sake of contradiction tha E is not connected. Then, E can be partitioned into two nonempty sets G, H, such that $\overline{G} \cap H = \overline{H} \cap G = \emptyset$. Let $x \in A \cap B$. Suppose WLOG that $x \in G$. Let $y \in H$. Since $y \in E$, we know y is in A or B. Say that $y \in A$. Then, $x \in A \cap G$ and $y \in A \cap H$. However, since G, H are disjoint, A can be separated into two disjoint sets $A \cap G$ and $A \cap H$, contradiction. Hence, E is connected. The interior points of E is $N_{\frac{1}{2} d(p,q)}(p) \cup N_{\frac{1}{2} d(p,q)}(q)$, which is the union of two disjoint open sets. The result now follows from Problem 5 (b). $\square$

Problem 7

Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $a \in A$, $b \in B$, and define

$$p(t) = (1-t)a + tb$$

for $t \in \mathbb{R}^1$. Put $A_0 = p^{-1}(A)$, $B_0 = p^{-1}(B)$. [Thus $t \in A_0$ if and only if $p(t) \in A$.]

(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}^1$.

Proof. A_0 and B_0 are disjoint, otherwise there exists $x \in A_0 \cap B_0$ such that $p(x) \in A \cap B$. Let k be a limit point of A_0. Suppose for the sake of contradiction that $k \in B_0$. Then, for $\epsilon > 0$, there exists $m \in N_{\frac{\epsilon}{|b-a|}}(k) \cap A_0$. Hence, $d(p(k), p(m)) = |[(1-k)a + kb] - [(1-m)a + mb]| = (k-m)|b-a| < \epsilon$. Since ϵ is arbitrary and $p(m) \in A$, $p(k)$ is a limit point of A. But then $p(k) \in B_0 \cap \overline{A_0}$, contradiction. Hence, $k \notin B_0$. By symmetry, we also know that $A_0 \cap \overline{B_0} = \emptyset$, and thus A_0 and B_0 are separated. □

(b) Prove that there exists $t_0 \in (0,1)$ such that $p(t_0) \notin A \cup B$.

Proof. Note that $0 \in A_0$ and $1 \in B_0$. Let $t_0 = \sup(A \cap [0,1])$. By Theorem 2.28, $t_0 \in \overline{A_0}$, and thus $t_0 \notin B_0$, as A_0, B_0 are separated. In particular, $0 \leq t_0 < 1$. If $t_0 \notin A_0$, it follows that $0 < t_0 < 1$ and $t_0 \notin A_0 \cup B_0$. If $t_0 \in A_0$, then $t_0 \notin \overline{B_0}$, and thus there exists $t_0' \notin B_0$ such that $t_0 < t_0' < 1$. Hence, $0 < t_0' < 1$ and $t_0 \notin A_0 \cup B_0$. Since there is a point $t_0 \in (0,1) \ni (A_0 \cup B_0)$, $p(t_0) \notin A \cup B$. □

(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.

Proof. Let $S \subset \mathbb{R}^k$ be not connected. Then, $S = A \cup B$, for separated A, B. But then there exists $t \in (0,1)$ such that $(1-t)a + tb \notin A \cup B$, by (b), and thus S is not convex. The result now follows from the contrapositive. □