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**Question 1.** Suppose that a class of students is star-gazing on top of the local mathematics building from the hours of 11 PM through 3 AM. Suppose further that meteors arrive (i.e. they are seen) according to a Poisson process with intensity $\lambda = 4$ per hour. Find the following.

(a) The probability that the students see more than 2 meteors in the first hour.

*Solution.* Let $X$ be the number of meteors seen in the first hour, so $X \sim \text{Poisson}(4)$. Then,

$$\begin{aligned}\mathbb{P}(X > 2) &= 1 - \mathbb{P}(X \leq 2) \\ &= 1 - e^{-4}(1 + 4 + 8) \\ &= 1 - \frac{13}{e^4}.\end{aligned}$$

□

(b) The probability that they see zero meteors in the first hour, but at least ten meteors in the final three hours.

*Solution.* Let $X_1$ be the number of meteors seen in the first hour and $X_2$ be the number of meteors seen in the last 3 hours, so $X_1 \sim \text{Poisson}(4)$ and $X_1 \sim \text{Poisson}(12)$. We know,

$$\begin{aligned}\mathbb{P}(X_1 = 0) &= e^{-4}, \\ \mathbb{P}(X_2 \geq 10) &= 1 - \mathbb{P}(X \leq 9) \\ &= 1 - e^{-12}\sum_{k=0}^{9}\frac{12^k}{k!},\end{aligned}$$

and $X_1, X_2$ are independent to each other. Thus,

$$\mathbb{P}(X_1 = 0, X_2 \geq 10) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 \geq 10) = e^{-4} - e^{-16}\sum_{k=0}^{9}\frac{12^k}{k!}.$$

□

(c) Given that there were 13 meteors seen all night, what is the probability there were no meteors seen in the first hour?

*Solution.* Let $X_1$ be the number of meteors seen in the first hour, $X_2$ be the number of meteors seen in the last 3 hours, and $X$ be the number of meteors seen all night, so $X_1 \sim \text{Poisson}(4)$, $X_2 \sim \text{Poisson}(12)$, and $X \sim \text{Poisson}(16)$. Then,

$$\begin{aligned}\mathbb{P}(X_1 = 0 | X = 13) &= \frac{\mathbb{P}(X_1 = 0, X = 13)}{\mathbb{P}(X = 13)} \\ &= \frac{\mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 13)}{\mathbb{P}(X = 13)} \\ &= \frac{e^{-4}e^{-12} \cdot \frac{12^{13}}{13!}}{e^{-16}\frac{16^{13}}{13!}} \\ &= \left(\frac{3}{4}\right)^{13}.\end{aligned}$$

□

**Question 2.** Let $\{N_t\}_{t>0}$ be a Poisson process with rate $\lambda$, that is, for each $t > 0$, $N_t = N((0,t]) \sim Pois(t\lambda)$. Let $X_1$ be the first arrival time of $N_t$, that is the first time a customer arrives (a car passes by etc). Show that

$$\mathbb{P}(X_1 \leq x \,|\, N(t) = 1) = \frac{x}{t},$$

for $0 \leq x \leq t$. That is, show that given $N(t) = 1$, then $X_1$ is uniformly distributed in $(0,t]$.

*Proof.*

$$\begin{aligned}
\mathbb{P}(X_1 \leq x \,|\, N(t) = 1) &= \frac{\mathbb{P}(X_1 \leq x, N(t) = 1)}{\mathbb{P}(N(t) = 1)} \\
&= \frac{\mathbb{P}(N([0,x]) = 1)\mathbb{P}(N((x,t]) = 0)}{\mathbb{P}(N(t) = 1)} \\
&= \frac{\lambda x e^{-\lambda x} e^{\lambda(x-t)}}{\lambda t e^{-\lambda t}} \\
&= \frac{x e^{\lambda(-x+x-t)}}{t e^{-\lambda t}} \\
&= \frac{x}{t}.
\end{aligned}$$

$\square$

**Question 3.** Suppose that the random variable X has a density function

$$f(x) \begin{cases} \frac{1}{2}x^2e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Find the moment generating function $M(t)$ of $X$.

*Solution.*

$$\begin{aligned} M(t) &= \mathbb{E}[e^{tX}] \\ &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ &= \int_0^{\infty} \frac{1}{2}x^2 e^{(t-1)x} dx. \end{aligned}$$

For $t = 1$,

$$\int_0^{\infty} \frac{1}{2}x^2 e^{(t-1)x} dx = \int_0^{\infty} \frac{1}{2}x^2 dx \to \infty.$$

For $t \neq 1$,

$$\int_0^{\infty} \frac{1}{2}x^2 e^{(t-1)x} dx = \left( \frac{x^2}{2(t-1)} - \frac{x}{(t-1)^2} + \frac{1}{(t-1)^3} \right) e^{(t-1)x} \Big|_0^{\infty}$$

Since

$$\begin{aligned} \lim_{x\to\infty} \left( \frac{x^2}{2(t-1)} - \frac{x}{(t-1)^2} + \frac{1}{(t-1)^3} \right) e^{(t-1)x} &= \lim_{x\to\infty} \frac{\frac{x^2}{2(t-1)} - \frac{x}{(t-1)^2} + \frac{1}{(t-1)^3}}{e^{(1-t)x}} \\ &= \lim_{x\to\infty} \frac{\frac{x}{(t-1)} - \frac{1}{(t-1)^2}}{(1-t)e^{(1-t)x}} \\ &= \lim_{x\to\infty} \frac{e^{(t-1)x}}{(t-1)^3}, \end{aligned}$$

we have $\lim_{x\to\infty} \frac{e^{(t-1)x}}{(t-1)^3} \to \infty$, for $t > 1$, and $\lim_{x\to\infty} \frac{e^{(t-1)x}}{(t-1)^3} = 0$, for $t < 1$. Therefore,

$$M(t) = \begin{cases} \infty & t \geq 1 \\ \frac{1}{(1-t)^3} & t < 1 \end{cases}.$$

□

**Question 4.** Let $X$ be a random variable with moment generating function $M_X(t)$. Let us consider a new random variable $Y = aX + b$, for some real numbers $a, b$. Write the moment-generating function $M_Y(t)$ of $Y$ in terms of $M_X(t)$.

*Solution.*

$$\begin{aligned} M_Y(t) &= \mathbb{E}[e^{tY}] \\ &= \mathbb{E}[e^{atX+bt}] \\ &= e^{bt}\mathbb{E}[e^{atX}] \\ &= e^{bt}M_X(at). \end{aligned}$$

$\square$

**Question 5.** Suppose that $U \sim Unif[0,1]$. Let $Y = e^{\frac{U}{1-U}}$. Find the probability density function of $Y$.

*Solution.* Let $F_X$ be the cumulative distribution function of $U$, and let $f_Y$ be the probability density function of $Y$. Since $U \sim Unif[0,1]$, $F_X(x) = \mathbb{P}(U \leq x) = x$, for $x \in [0,1]$. Since $e^{\frac{U}{1-U}} > 0$, $Y$ only takes positive values, and so $F_Y(t) = 0$ for $t \leq 0$. Therefore, for $t > 0$, the cumulative distribution function of $Y$ is

$$\begin{aligned} F_Y(t) &= \mathbb{P}(Y \leq t) \\ &= \mathbb{P}(e^{\frac{U}{1-U}} \leq t) \\ &= \mathbb{P}\left(\frac{U}{1-U} \leq \ln t\right) \\ &= \mathbb{P}\left(\frac{1}{U} - 1 \geq \frac{1}{\ln t}\right) \\ &= \mathbb{P}\left(U \leq \frac{\ln t}{\ln t + 1}\right). \end{aligned}$$

For $t < 1$, $\frac{\ln t}{\ln t + 1} \notin [0,1]$, and so $f_Y(t) = 0$. For $t \geq 1$, $F_Y(t) = \mathbb{P}\left(U \leq \frac{\ln t}{\ln t+1}\right) = \frac{\ln t}{\ln t+1}$, and so $f_Y(t) = F_Y'(t) = \frac{1}{t(\ln t+1)^2}$. Therefore,

$$f_Y(t) = \begin{cases} \frac{1}{t(\ln t+1)^2} & t \geq 1 \\ 0 & t < 1 \end{cases}.$$

$\square$

**Question 6.** Let $X$ be a random variable with moment generating

$$M_X(t) = e^{2(e^{2t}-1)}.$$

Compute $\mathbb{E}[X^3]$.

*Solution.*

$$\begin{aligned}
M_X(t) &= e^{2(e^{2t}-1)},\\
M_X'(t) &= 4e^{2(e^{2t}+t-1)},\\
M_X''(t) &= 8(2e^{2t}+1)e^{2(e^{2t}+t-1)},\\
M_X'''(t) &= 16(4e^{6t}+6e^{4t}+e^{2t})e^{2e^{2t}-2}.
\end{aligned}$$

Thus,

$$\mathbb{E}[X^3] = M_X'''(0) = 176.$$

□

**Question 7.** Suppose that $X$ is uniform on $[-2,3]$ and let $Y = |X-1|$. Find the density function of $Y$.

*Solution.* We first note that $Y$ only takes positive values. Thus, for $t < 0$, $\mathbb{P}(Y < t) = 0$. For $t > 3$, since $[-2,3] \subset [1-t, 1+t]$, $\mathbb{P}(Y \leq t) = \mathbb{P}(X \in [1-t,1+t]) = 1$. For $t \leq 2$, $\mathbb{P}(Y \leq t) = \mathbb{P}(X \in [1-t,1+t]) = \frac{2t}{5}$. For $2 < t \leq 3$, $\mathbb{P}(Y \leq t) = \mathbb{P}(X \in [1-t,1+t]) = \frac{t+2}{5}$. Thus, the cumulative distribution function of $Y$ is

$$F_Y(t) = \begin{cases} 0 & t < 0 \\ \frac{2t}{5} & 0 \leq t \leq 2 \\ \frac{t+2}{5} & 2 < t \leq 3 \\ 1 & t > 3 \end{cases}.$$

Therefore, the density function of $Y$ is

$$f_Y(t) = F_Y'(t) = \begin{cases} \frac{2}{5} & 0 \leq t \leq 2 \\ \frac{1}{5} & 2 < t \leq 3 \\ 0 & \text{otherwise} \end{cases}.$$

□