MATH 140A: Homework #3

Due on Feb 5, 2024 at 23:59pm

Professor Seward

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Problem 1

A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint*: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Proof. Let p be a n-degree polynomial of integer coefficients. By the Fundamental Theorem of Algebra, p has n complex roots. Notice that since $\mathbb{Z}^i$ is countable for all $i > 0$, $S = \bigcup_{i=1}^{\infty} \{i\} \times \mathbb{Z}^i$ is countable, by Theorem 2.12. This follows that for $m \in \mathbb{N}$, each $(m, a_0, a_1, \ldots, a_m) \in S$, gives m algebraic numbers and S contains all possible tuples of integer coefficients, so the set

$$\bigcup_{(n, a_0, a_1, \ldots, a_n) \in S} \{z \mid a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0\}$$

contains all algebraic numbers and it is countable. $\square$

Problem 2

Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. Do E and E' always have the same limit points?

Proof. Let p be a limit point of E'. It suffices to show that there exists some $k \in E$ such that $d(p,k) < r$, for all $r > 0$. Since p is a limit point, there exists $q \in E'$ such that $d(p,q) < \frac{r}{2}$. However, as $q \in E'$, q is a limit point of E, so there exists $k \in E$ such that $d(q,k) < \frac{r}{2}$. Hence, $d(p,k) < d(p,q) + d(q,k) < r$, so p is a limit point of E. It follows that $p \in E'$ so E' is closed.

We prove that E and $\overline{E}$ have the same limit points. E' is obviously contained in the set of limit points of $\overline{E}$, so it suffices to show the converse. Let x be a limit point of $\overline{E}$. We show that $x \in E'$. Since $\overline{E}$ is closed, $x \in \overline{E} = E \cup E'$. We may assume that $x \in E$, otherwise we are done. For $r > 0$, we know that there exists $y \in \overline{E}$ such that $d(x,y) < \frac{r}{2}$. If $y \notin E$, then y is a limit point of E, so there exists $z \in E$ such that $d(y,z) < \frac{r}{2}$. But then $d(x,z) < d(x,y) + d(y,z) < r$. Hence, there exists some elements in E such that its in $N_r(x)$, for any $r > 0$. Thus, x is a limit point of E, so $x \in E'$.

To see that E and E' do not always share the same limit points, consider $E = \{0, 1, \frac{1}{2}, \ldots\}$. Since $E' = \{0\}$, E' does not have any limit points. $\square$

Problem 3

Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\overline{B}_n = \bigcup_{i=1}^{n} \overline{A}_i$ for $n = 1, 2, 3, \ldots$.

Proof. We first show that $M' \cup N' = (M \cup N)'$, for subsets M, N. Since $x \in M' \cup N'$ is a limit point of M or N, we get $x \in (M \cup N)'$. Hence, it just need to show that $(M \cup N)' \subseteq M' \cup N'$. Suppose $y \notin M' \cup N'$. Then, there exists $r, s > 0$ such that $N_r(y)$ does not contain any points in M and $N_s(y)$ does not contain any points in N. Hence, $N_{\min(r,s)}(y)$ does not contain any points in $M \cup N$, and thus $y \notin (M \cup N)'$. By the contrapositive of the statement, we get $(M \cup N)' \subseteq M' \cup N'$. Now that we have shown $M' \cup N' = (M \cup N)'$, we get $\overline{M} \cup \overline{N} = \overline{M \cup N}$.

We may now prove $\overline{B}_n = \bigcup_{i=1}^{n} \overline{A}_i$ by induction on n. The base case is trivial. For $n > 1$,

$$\begin{aligned}
\overline{B}_n &= \overline{\left(\bigcup_{i=1}^{n} A_i\right)} \\
&= \overline{\left(A_n \cup \bigcup_{i=1}^{n-1} A_i\right)} \\
&= \overline{A_n} \cup \overline{\left(\bigcup_{i=1}^{n-1} A_i\right)}.
\end{aligned}$$

Hence, $\overline{B}_n = \overline{A_n} \cup \overline{\left(\bigcup_{i=1}^{n-1} A_i\right)} = \overline{A_n} \cup \bigcup_{i=1}^{n-1} \overline{A}_i = \bigcup_{i=1}^{n} \overline{A}_i$, by induction. $\square$

(b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supset \bigcup_{i=1}^{\infty} \overline{A}_i$. Show, by an example, that this inclusion can be proper.

Proof. Let $x \in \bigcup_{i=1}^{\infty} \overline{A}_i$. Then, $x \in A_i \cup A_i'$, for some $i \in \mathbb{N}$. Hence, we may assume that x is the limit point of some A_i, otherwise $x \in A_i \subset B \subset \overline{B}$ and we are done. However, $N_r(x)$ contains a point in $A_i \subset B$ for $r > 0$, so x is also a limit point of B, and thus $x \in \overline{B}$.

Let $A_i = \{\frac{1}{i}\}$, for $i \in \mathbb{N}$. Note that A_i does not have a limit point. But then $B = \{\frac{1}{k} \mid k \in \mathbb{N}\}$ has a limit point 0. Therefore, $0 \in \overline{B} \setminus \bigcup_{i=1}^{\infty} \overline{A}_i$. $\square$

Problem 4

Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Proof. This is true. Let $x = (x_1, x_2) \in E$. Since x is an interior point in E, there exists $r > 0$ such that $N_r(x) \subseteq E$. Since $x \in \mathbb{R}^2$, there exists $k = (x_1 - \frac{r}{2}, x_2 - \frac{r}{2}) \in \mathbb{R}^2$ such that $d(x,k) = \sqrt{(x_1 - (x_1 - \frac{r}{2}))^2 + (x_2 - (x_2 - \frac{r}{2}))^2} = \frac{r}{\sqrt{2}} < r$, so $N_r(x)$ is not empty. Hence, for any $t > 0$, if $t > r$ we can find $k \in N_r(x)$ such that $d(x,y) < r < t$. Otherwise, since $x \in \mathbb{R}^2$, there exists $s = (x_1 - \frac{t}{2}, x_2 - \frac{t}{2}) \in \mathbb{R}^2$ such that $d(x,s) = \frac{t}{\sqrt{2}} < t \leq r$. But then $s \in N_r(x)$. Therefore, x is a limit point in E.

However, this does not hold true for closed sets. Consider any non-empty finite set S in $\mathbb{R}^2$. S does not have any limit points. $\square$

Problem 5

Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q. \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed?

Proof. We first check that d is a valid metric. By definition, we already know $d(p,q) = d(q,p)$ is positive than $p \neq q$, otherwise it is 0. Let $r \in X$. We show that $d(p,q) \leq d(p,r) + d(r,q)$ holds. Since d is nonnegative, we may assume that $p \neq q$, otherwise we are done. Then, r canot be equal to both p and q, so at least one of $d(p,r)$, $d(r,q)$ is 1. Therefore, $d(p,q) \leq 1 \leq d(p,r) + d(r,q)$, and thus d is a metric.

Let $E \subset X$ be finite and non-empty. Since for $e \in E$, $N_{\frac{1}{\pi}}(e) = \{e\} \subset E$, so every point in E is an interior point, which makes E an open set. Since any set in X is an union of finite sets, all sets in X is thus an open set. However, any set in X is also the complement of a set, so any set in X is also closed. $\square$

Problem 6

For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= |x^2-y^2|, \\ d_4(x,y) &= |x-2y|, \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Proof. We first note that $d_i(x,x) = 0$ and $d_i(x,y) = d_i(y,x)$, for $i \in \{1,2,5\}$. d_3 is not a metric as $d(1,-1) = 0$. d_4 is not a metric as $d_4(1,1) \neq 0$. Hence, we only need to check the triangle inequality for each d_i. Let $z \in \mathbb{R}$.

For d_1, choose $x = 1$, $y = 0$, and $z = \frac{1}{2}$. Since $(x-y)^2 = 1 \geq \frac{1}{4} = (x-z)^2 + (z-y)^2$, d_1 is not a matric.

For d_2, since $|x-y| \leq |x-z| + |z-y|$ and $2\sqrt{|x-z||z-y|} \geq 0$, we get

$$|x-y| \leq |x-z| + |z-y| + 2\sqrt{|x-z||z-y|} = (\sqrt{|x-y|} + \sqrt{|y-z|})^2,$$

and thus the triangle equality is met by taking the square roots of both sides. Hence, d_2 is a metric.

For d_5, we show that $\frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|y-z|}{1+|y-z|}$. By multiplying both sides by the denominators and clearing the repeated terms on both sides, we get

$$|x-y| \leq |x-z| + |z-y| + 2|x-z||z-y| + 2|x-y||x-z||z-y|.$$

Since $|x-y| \leq |x-z| + |z-y|$, the above inequality holds, and thus d_5 is a metric. $\square$

Problem 7

Prove that the set of all injections from the set of natural numbers to itself is uncountable.

Proof. Let S be a countable set of injections from $\mathbb{N}$ to $\mathbb{N}$, and we index each function in S, say $s_1, s_2, \ldots$. Note that we may view an injection from $\mathbb{N}$ to $\mathbb{N}$ as an infinite sequence that does not have repeated numbers. We wish to construct an injection not already in S. We start with some injection $f : \mathbb{N} \to \mathbb{N}$. Whenever $f(2k) = s_k(2k)$, we update f by swapping $f(2k)$ with $f(2k+1)$, as $f(2k) \neq f(2k+1)$. Note that we merely changed the ordering of f, so f remains to be an injection. Hence, $f(2k) \neq s_k(2k)$ for all $s_k \in S$, so f is an injection not in S. The result then follows. $\square$