MATH 220A: Homework #6

Due on Nov 8, 2024 at 23:59pm

Professor Ebenfelt

Ray Tsai

A16848188

Problem 1

If $Tz = \frac{az+b}{cz+d}$, find z_2, z_3, z_4 (in terms of a, b, c, d) such that $Tz = (z, z_2, z_3, z_4)$.

Proof. Solving

$$Tz_2 = \frac{az_2 + b}{cz_2 + d} = 1, \quad Tz_3 = \frac{az_3 + b}{cz_3 + d} = 0, \quad Tz_4 = \frac{az_4 + b}{cz_4 + d} = \infty,$$

we get $z_2 = \frac{d-b}{a-c}$, $z_3 = \frac{-b}{a}$, and $z_4 = -\frac{d}{c}$. $\square$

Problem 2

If $Tz = \frac{az+b}{cz+d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$ where Γ is the unit circle $\{z : |z| = 1\}$.

Proof. Let $z \in \mathbb{C}$. Note that $|T(z)|^2 = (\frac{az+b}{cz+d})(\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}) = 1$ if and only if

$$|z|^2(|a|^2 - |c|^2) + (a\bar{b} - c\bar{d})z + (b\bar{a} - d\bar{c})\bar{z} + |b|^2 - |d|^2 = 0.$$

Suppose $|z| = 1$. Then

$$|a|^2 - |c|^2 = |d|^2 - |b|^2 = 1 \text{ and } \quad a\bar{b} - c\bar{d} = 0$$

is a sufficient condition so that $|T(z)| = 1$.

Now suppose $|T(z)|^2 = 1$. Then $|z|^2 = 1$ if

By Proposition 3.6, T is a composition of translations, rotations, dilations, and the inversion. Note also that T maps circles to circles. Thus, $T(\Gamma) = \Gamma$ if and only $Tz =$ □

Problem 3

Let $D = \{z : |z| < 1\}$ and find all Möbius transformations T such that $T(D) = D$.

Proof. $\square$

Problem 4

Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that f is constant.

Proof. Let $z_2, z_3, z_4 \in G$ such that $f(z_2), f(z_3), f(z_4)$ are distinct. Let T be a Möbius transformation such that $Tz = \frac{az+b}{cz+d} = (z, f(z_2), f(z_3), f(z_4))$. Since T is analytic and $T(f(z)) \in \mathbb{R}_\infty$ for all $z \in G$, $T(f(z))$ is constant by exercise 3.2.14. Thus,

$$(T(f(z)))' = T'(f(z))f'(z) = \frac{ad - bc}{(cf(z) + d)^2} f'(z) = 0.$$

Since $ad - bc \neq 0$, $f'(z) = 0$ for all $z \in G$ with $z \neq z_4$. But then G is connected and f' is continuous, so $\lim_{z \to z_4} f'(z) = f'(z_4) = 0$. Hence, $f' = 0$ and the result now follows. $\square$

Problem 5

Show that a Möbius transformation T satisfies $T(0) = \infty$ and $T(\infty) = 0$ iff $Tz = kz^{-1}$ for some k in $\mathbb{C}$.

Proof. Let $Tz = \frac{az+b}{cz+d}$. The converse is trivial. Suppose $T(0) = \infty$, and $T(\infty) = 0$. Then $Tz = (z, z_2, \infty, 0)$ for some k. By the first problem of this homework,

$$\frac{-b}{a} = \infty, \quad \frac{-d}{c} = 0,$$

which implies $a = d = 0$. Hence, $T = \frac{b}{c} z^{-1}$. $\square$