MATH 220A: Homework #6

Due on Nov 8, 2024 at 23:59pm

Professor Ebenfelt

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Problem 1

If $Tz = \frac{az+b}{cz+d}$, find z_2, z_3, z_4 (in terms of a, b, c, d) such that $Tz = (z, z_2, z_3, z_4)$.

Proof. Solving

$$Tz_2 = \frac{az_2 + b}{cz_2 + d} = 1, \quad Tz_3 = \frac{az_3 + b}{cz_3 + d} = 0, \quad Tz_4 = \frac{az_4 + b}{cz_4 + d} = \infty,$$

we get $z_2 = \frac{d-b}{a-c}$, $z_3 = \frac{-b}{a}$, and $z_4 = -\frac{d}{c}$. $\square$

Problem 2

If $Tz = \frac{az+b}{cz+d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$ where Γ is the unit circle $\{z : |z| = 1\}$.

Proof. Note that $T(\Gamma) = \Gamma$ if and only if $T^{-1}(\Gamma) = \Gamma$, and thus T^{-1} perseves the symmetry between 0 and ∞. Let $k = T^{-1}(0) \notin \Gamma$. By the calculation in the textbook, k is symmetric to $\frac{1}{k}$. That is, $T(\frac{1}{k}) = \infty$. Solving

$$\begin{cases} \frac{ak+b}{ck+d} = 0, \\ \frac{\frac{a}{k}+b}{\frac{c}{k}+d} = \infty \end{cases},$$

we have $b = -ak, d = -\frac{c}{k}$. Put $m = \frac{a}{c}$ and T is of the form

$$Tz = \frac{m(z-k)}{z - \frac{1}{k}}.$$

If $k = 0$ or ∞, then $Tz = mz$ or $Tz = \frac{m}{z}$, so $T(\Gamma) = \Gamma$ if and only if $m \in \Gamma$. Suppose otherwise. Notice

$$\begin{aligned} |Tz|^2 &= |m\bar{k}|^2 \left(\frac{z-k}{\bar{k}z - 1} \right) \left(\frac{\bar{z} - \bar{k}}{k\bar{z} - 1} \right) \\ &= |m\bar{k}|^2 \left(\frac{|z|^2 + |k|^2 - 2Re(z\bar{k})}{|k|^2|z|^2 + 1 - 2Re(z\bar{k})} \right). \end{aligned}$$

But then when $z \in \Gamma$, $|Tz|^2 = |m\bar{k}|^2$. Thus in this case, $T(\Gamma) = \Gamma$ if and only if $k \notin \Gamma$ and $m\bar{k} \in \Gamma$.

Cominbing the two cases, we have $T(\Gamma) = \Gamma$ if and only if

$$Tz = \frac{m(z-k)}{z - \frac{1}{k}},$$

where $k \notin \Gamma$, $m \in \Gamma$ if $k = 0, \infty$ and $m\bar{k} \in \Gamma$ otherwise. □

Problem 3

Let $D = \{z : |z| < 1\}$ and find all Möbius transformations T such that $T(D) = D$.

Proof. We show that $T(D) = D$ if and only if $T^{-1}(0) \in D$. Obviously if $T(D) = D$ we have $T^{-1}(0) \in D$. Now suppose $k = T^{-1}(0) \in D$. Since T is continuous, $T(\partial D) = \partial D$, where $\partial D = \Gamma$ is the unit circle. By the previous problem, T is of the form

$$Tz = \frac{m(z-k)}{z - \frac{1}{\bar{k}}}.$$

Suppose $z \in D$. If $k = 0$ then $Tz = mz$ and $|Tz| = |mz| < 1$, so $Tz \in D$. Suppose $k \neq 0$. By the previous problem,

$$|Tz|^2 = |m\bar{k}|^2 \left(\frac{|z|^2 + |k|^2 - 2Re(z\bar{k})}{|k|^2|z|^2 + 1 - 2Re(z\bar{k})} \right).$$

Since $|k|^2(1 - |z|^2) < 1 - |z|^2$ for $k, z \in D$,

$$|k|^2 + |z|^2 < 1 + |z|^2,$$

and so $|Tz|^2 < 1$. $\square$

Problem 4

Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that f is constant.

Proof. Let $z_2, z_3, z_4 \in G$ such that $f(z_2), f(z_3), f(z_4)$ are distinct. Let T be a Möbius transformation such that $Tz = \frac{az+b}{cz+d} = (z, f(z_2), f(z_3), f(z_4))$. Since T is analytic and $T(f(z)) \in \mathbb{R}_\infty$ for all $z \in G$, $T(f(z))$ is constant by exercise 3.2.14. Thus,

$$(T(f(z)))' = T'(f(z))f'(z) = \frac{ad-bc}{(cf(z)+d)^2} f'(z) = 0.$$

Since $ad - bc \neq 0$, $f'(z) = 0$ for all $z \in G$ with $z \neq z_4$. But then G is connected and f' is continuous, so $\lim_{z \to z_4} f'(z) = f'(z_4) = 0$. Hence, $f' = 0$ and the result now follows. □

Problem 5

Show that a Möbius transformation T satisfies $T(0) = \infty$ and $T(\infty) = 0$ iff $Tz = kz^{-1}$ for some k in $\mathbb{C}$.

Proof. Let $Tz = \frac{az+b}{cz+d}$. The converse is trivial. Suppose $T(0) = \infty$, and $T(\infty) = 0$. Then $Tz = (z, z_2, \infty, 0)$ for some k. By the first problem of this homework,

$$\frac{-b}{a} = \infty, \quad \frac{-d}{c} = 0,$$

which implies $a = d = 0$. Hence, $T = \frac{b}{c}z^{-1}$. $\square$