MATH 220A: Homework #1

Due on Oct 4, 2024 at 23:59pm

Professor Ebenfelt

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Problem 1

Let Λ be a circle lying in S. Then there is a unique plane P in $\mathbb{R}^3$ such that $P \cap S = \Lambda$. Recall from analytic geometry that

$$P = \{(x_1, x_2, x_3) : x_1\beta_1 + x_2\beta_2 + x_3\beta_3 = l\}$$

where $(\beta_1, \beta_2, \beta_3)$ is a vector orthogonal to P and l is some real number. It can be assumed that $\beta_1^2+\beta_2^2+\beta_3^2 = 1$. Use this information to show that if Λ contains the point N, then its projection on $\mathbb{C}$ is a straight line. Otherwise, Λ projects onto a circle in $\mathbb{C}$.

Proof. Suppose $N \in \Lambda$. Then the stereographic projection line for every $x \in \Lambda$ is in P, and thus the projection of all $x \in \Lambda$ is in $P \cap \mathbb{C}$. On the other hand, for all $z \in P \cap \mathbb{C}$, stereographic projection line for z contains some point in $\Lambda \backslash \{N\}$. Therefore, the projection of Λ on $\mathbb{C}$ is $P \cap \mathbb{C}$, which is a straight line.

Suppose $N \notin \Lambda$, namely $\beta \neq l$. Let $x \in \Lambda$. By representing $x = (x_1, x_2, x_3)$ in terms of its complex plane projection z, we get

$$x_1\beta_1 + x_2\beta_2 + x_3\beta_3 = \frac{(z+\bar{z})\beta_1}{|z|^2+1} + \frac{-i(z-\bar{z})\beta_2}{|z|^2+1} + \frac{(|z|^2-1)\beta_3}{|z|^2+1} = l.$$

Rearranged,

$$\frac{l+\beta_3}{\beta_3 - l} = |z|^2 - \frac{\beta_1 - i\beta_2}{l-\beta_3}z - \frac{\beta_1 + i\beta_2}{l-\beta_3}\bar{z}.$$

But then circles in $\mathbb{C}$ are of the form

$$r^2 = (z-a)\overline{(z-a)} = |z|^2 - \bar{a}z - a\bar{z} + |a|^2,$$

for $r \in \mathbb{R}$ and $z, a \in \mathbb{C}$. Hence we are done. ☐

Problem 2

Prove that a set $G \subseteq X$ is open if and only if $X - G$ is closed.

Proof. If G is open, then its complement $X - G$ is closed, by definition. If $X - G$ is closed, it's complement $X - (X - G) = G$ is open, by definition. □

Problem 3

Let (X, d) be a metric space and $Y \subseteq X$. Suppose $G \subseteq X$ is open; show that $G \cap Y$ is open in (Y, d). Conversely, show that if $G_1 \subseteq Y$ is open in (Y, d), there is an open set $G \subseteq X$ such that $G_1 = G \cap Y$.

Proof. Let $B_X(x; \epsilon)$ denote the open ball in metric space (X, d) centered at x with radius ϵ.

Let $x \in G \cap Y$. If G is open, then there exists $\epsilon > 0$ such that $B_X(x; \epsilon) \subseteq G$. But then $B_Y(x; \epsilon) = B_X(x; \epsilon) \cap Y \subseteq G \cap Y$, so $G \cap Y$ is open in (Y, d).

Suppose $G_1 \subseteq Y$ is open in (Y, d). For all $x \in G_1$, there exists $\epsilon_x > 0$ such that $B_Y(x; \epsilon) \subseteq G_1$. Put $G = \bigcup_{x \in G_1} B_X(x; \epsilon)$. G is open in X, as it is an union of open sets. Since $x \in B_X(x; \epsilon)$ for all $x \in G_1$, $G_1 \subseteq G \cap Y$. Since $B_X(x; \epsilon) \cap Y = B_Y(x; \epsilon) \subseteq G_1$ for all $x \in G_1$, $G \cap Y = \bigcup_{x \in G_1} Y \cap B_X(x; \epsilon) \subseteq G_1$, and hence the equality. $\square$

Problem 4

The purpose of this exercise is to show that a connected subset of $\mathbb{R}$ is an interval.

(a) Show that a set $A \subseteq \mathbb{R}$ is an interval if and only if for any two points a and b in A with $a < b$, the interval $[a, b] \subseteq A$.

Proof. Suppose $A \subset \mathbb{R}$ is an interval. Let $a, b \in A$, with $b > a$. By definition of an interval, $x \in A$ for all $a < x < b$, and thus $[a, b] \subseteq A$.

Suppose that $[a, b] \subseteq A$ for all $a, b \in A$ with $b > a$. We may assume that $A \neq \mathbb{R}$, otherwise we are done. If A is bounded both above and below, then $m = \inf A$ and $M = \sup A$ exist. Pick $\epsilon > 0$. Since $M - \epsilon, m + \epsilon \in A$, $[m + \epsilon, M - \epsilon]$ is contained in A. Hence, for all $\epsilon > 0$, $x \in A$ for all $m + \epsilon \leq x \leq M - \epsilon$, that is, $(m, M) \subseteq A$. But then $x \notin A$ for all $x > M$ or $x < m$, so A is either $[m, M], [m, M), (m, M]$, or (m, M). Suppose WLOG that A is not bounded below. Then $M = \sup A$ exists as $A \neq \mathbb{R}$. Let $x < M$. Since the interval $[x, (x + M)/2] \subseteq A$, we know $x \in A$, and thus $(-\infty, M) \subseteq A$. But then $x \notin A$ for all $x > M$, so A is either $(-\infty, M]$ or $(-\infty, M)$. $\square$

(b) Use part (a) to show that if a set $A \subseteq \mathbb{R}$ is connected then it is an interval.

Proof. Let $a, b \in A$, with $b > a$. Suppose for the sake of contradiction that $[a, b]$ is not a subset of A. There exists $x \notin A$ such that $a < x < b$. But then by the last exercise, $(-\infty, x) \cap A$ and $(x, \infty) \cap A$ are open sets in (A, d), and their union is A. Hence, A is disconnected, contradiction. The result now follows from (a). $\square$

Problem 5

Prove the following generalization of Lemma 2.6. If $\{D_j : j \in J\}$ is a collection of connected subsets of X and if for each j and k in J we have $D_j \cap D_k \neq \emptyset$ then $\mathcal{D} = \bigcup\{D_j : j \in J\}$ is connected.

Proof. Let A be a nonempty subset of the metric space $(\mathcal{D}, d)$ which is both open and closed. Then $A \cap D_j$ is both open and closed in (D_j, d) for all j. Since $A \neq \emptyset$ and D_j is connected for all j, $A \cap D_j = D_j$ for some j. But then $D_j \cap D_k = A \cap D_k \neq \emptyset$ for all k, so $A \cap D_k = D_k$, as D_k is connected. Therefore, $\mathcal{D} = A$. $\square$