MATH 180B: Homework #7

Due on Mar 15, 2024 at 23:59pm

Professor Carfagnini

Ray Tsai

A16848188

Problem 1

Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $X(t)$ be the number of customers that have arrived up to time t. Determine the following probabilities and conditional probabilities:

(a) $\Pr\{X(1) = 2\}$.

Proof.

$$\Pr\{X_1 = 2\} = \frac{\lambda^2 e^{-\lambda}}{2!} = 2e^{-2}.$$

□

(b) $\Pr\{X(1) = 2 \text{ and } X(3) = 6\}$.

Proof.

$$\begin{aligned}\Pr\{X_1 = 2 \text{ and } X_3 = 6\} &= \Pr\{X_1 = 2 \text{ and } X_3 - X_1 = 4\} \\ &= \Pr\{X_1 = 2\} \Pr\{X_3 - X_1 = 4\} \\ &= 2e^{-2} \cdot \frac{(2\lambda)^4 e^{-2\lambda}}{4!} \\ &= 2e^{-2} \cdot \frac{32e^{-4}}{3} \\ &= \frac{64e^{-6}}{3}.\end{aligned}$$

□

(c) $\Pr\{X(1) = 2 | X(3) = 6\}$.

Proof. By Theorem 5.6,

$$\begin{aligned}\Pr\{X_1 = 2 | X_3 = 6\} &= \binom{6}{2} \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{3}\right)^4 \\ &= 15 \cdot \frac{1}{9} \cdot \frac{16}{81} \\ &= \frac{80}{243}.\end{aligned}$$

□

(d) $\Pr\{X(3) = 6 | X(1) = 2\}$.

Proof.

$$\begin{aligned}\Pr\{X_3 = 6 | X_1 = 2\} &= \frac{\Pr\{X_3 - X_1 = 4\} \Pr\{X_1 = 2\}}{\Pr\{X_1 = 2\}} \\ &= \Pr\{X_3 - X_1 = 4\} \\ &= \frac{32e^{-4}}{3}.\end{aligned}$$

□

Problem 2

Shocks occur to a system according to a Poisson process of rate λ. Suppose that the system survives each shock with probability α, independently of other shocks, so that its probability of surviving k shocks is α^k. What is the probability that the system is surviving at time t?

Proof. Let X_t denote the number of shocks at time t and S_t be the event that the system survives at time t. Then,

$$\begin{aligned}
\Pr(S_t) &= \sum_{k=0}^{\infty} \Pr\{S_t \mid X_t = k\} \Pr\{X_t = k\} \\
&= \sum_{k=0}^{\infty} \alpha^k \cdot \frac{(t\lambda)^k e^{-t\lambda}}{k!} \\
&= e^{-t\lambda} \sum_{k=0}^{\infty} \frac{(\alpha t\lambda)^k}{k!} \\
&= e^{-t\lambda} e^{-\alpha t\lambda} = e^{(\alpha-1)t\lambda}.
\end{aligned}$$

□

Problem 3

Determine numerical values to three decimal places for $\Pr\{X = k\}$, $k = 0, 1, 2$, when

(a) X has a binomial distribution with parameters $n = 20$ and $p = 0.06$.

Proof.

$$\Pr\{X = k\} = \binom{20}{k}(0.06)^k(0.94)^{20-k}.$$

Hence,

$$\Pr\{X = 0\} = 0.290, \quad \Pr\{X = 1\} = 0.370, \quad \Pr\{X = 2\} = 0.225.$$

□

(b) X has a binomial distribution with parameters $n = 40$ and $p = 0.03$.

Proof.

$$\Pr\{X = k\} = \binom{40}{k}(0.03)^k(0.97)^{40-k}.$$

Hence,

$$\Pr\{X = 0\} = 0.296, \quad \Pr\{X = 1\} = 0.366, \quad \Pr\{X = 2\} = 0.221.$$

□

(c) X has a Poisson distribution with parameter $\lambda = 1.2$.

Proof.

$$\Pr\{X = k\} = \frac{(1.2)^k e^{-1.2}}{k!}.$$

Hence,

$$\Pr\{X = 0\} = 0.301, \quad \Pr\{X = 1\} = 0.361, \quad \Pr\{X = 2\} = 0.217.$$

□

Problem 4

For $i = 1, \ldots, n$, let $\{X_i(t); t \geq 0\}$ be independent Poisson processes, each with the same parameter λ. Find the distribution of the first time that at least one event has occurred in every process.

Proof. Let T denote the first time that at least one event has occurred in every process. Then, the CDF of the desired distribution is

$$\begin{aligned}
\Pr\{T < t\} &= \prod_{i=1}^{n} \Pr\{X_t \geq 1\} \\
&= \prod_{i=1}^{n} 1 - (t\lambda)^0 e^{-t\lambda} \\
&= \left(1 - e^{-t\lambda}\right)^n
\end{aligned}$$

□

Problem 5

Customers arrive at a certain facility according to a Poisson process of rate λ. Suppose that it is known that five customers arrived in the first hour. Determine the mean total waiting time $E[W_1 + W_2 + \ldots + W_5]$.

Proof. We give each customer a label. Let U_k denote the time customer k arrive at the facility. By Theorem 5.7, U_k is of a uniform distribution. Hence,

$$E[W_1 + W_2 + \ldots + W_5] = E[U_1 + U_2 + \ldots + U_5] = 5E[U_1] = \frac{5}{2}.$$

$\square$

Problem 6

Let $W_1, W_2, \ldots$ be the event times in a Poisson process $\{X(t); t \geq 0\}$ of rate λ. Suppose it is known that $X(1) = n$. For $k < n$, what is the conditional density function of $W_1, \ldots, W_{k-1}, W_{k+1}, \ldots, W_n$, given that $W_k = w$?

Proof.

$$
\begin{aligned}
&f_{W_1,\ldots,W_{k-1},W_{k+1},\ldots,W_n|W_k,X_1=n}(w_1,\ldots,w_n|w) \\
&\qquad = f_{W_1,\ldots,W_{k-1}|X_w=k-1}(w_1,\ldots,w_{k-1}) f_{W_1,\ldots,W_{n-k}|X_{1-w}=n-k}(w_1,\ldots,w_{n-k}) \\
&\qquad = \frac{(k-1)!}{w^{1-k}} \cdot \frac{(n-k)!}{(1-w)^{k-n}}
\end{aligned}
$$

□

Problem 7

Let $W_1, W_2, \ldots$ be the event times in a Poisson process $\{X(t); t \geq 0\}$ of rate λ, and let $f(w)$ be an arbitrary function. Verify that

$$E\left[\sum_{i=1}^{X(t)} f(W_i)\right] = \lambda \int_0^t f(w)dw.$$

Proof. Let $U_1, \ldots, U_n$ denote independent random variables that are uniformly distributed in $(0, t]$. By Theorem 5.7,

$$\begin{aligned}
E\left[\sum_{i=1}^{X(t)} f(W_i)\right] &= \sum_{n=1}^{\infty} E\left[\sum_{i=1}^{n} f(W_i) | X(t) = n\right] \Pr\{X(t) = n\} \\
&= \sum_{n=1}^{\infty} E\left[\sum_{i=1}^{n} f(U_i)\right] \Pr\{X(t) = n\} \\
&= E\left[f(U_1)\right] \sum_{n=1}^{\infty} n \Pr\{X(t) = n\} \\
&= t\lambda E\left[f(U_1)\right] \\
&= t\lambda \int_0^t f(w) \Pr\{U_1 = w\} dw \\
&= \lambda \int_0^t f(w)dw.
\end{aligned}$$

□