C8.3 Combinatorics: Sheet #1

Due on October 28, 2025 at 12:00pm

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Problem 1

Write down all antichains contained in $\mathcal{P}(1)$ and $\mathcal{P}(2)$. How many different antichains are there in $\mathcal{P}(3)$?

Proof. The antichains in $\mathcal{P}(1)$ are $\{\emptyset\}$ and $\{1\}$. The antichains in $\mathcal{P}(2)$ are $\{\emptyset\}$, $\{1\}$, $\{2\}$, $\{1, 2\}$, and $\{12\}$. There are 20 antichains in $\mathcal{P}(3)$. $\square$

Problem 2

(a) Look up Stirling's Formula. Use it to find an asymptotic estimate for $\binom{n}{n/2}$ of the form $(1+o(1))f(n)$ when n is even.

Proof. By Stirling's Formula,

$$\binom{n}{n/2} = \frac{n!}{(n/2)!(n/2)!} = \frac{(1+o(1))\sqrt{2\pi n}(n/e)^n}{(1+o(1))\pi n(n/2e)^n} = (1+o(1))2^n\sqrt{\frac{2}{\pi n}}.$$

□

(b) Now do the same for $\binom{n}{pn}$ where $p \in (0,1)$ is a constant and pn is an integer. Write your answer in terms of the binary entropy function

$$H(p) = -p\log p - (1-p)\log(1-p)$$

Proof. By Stirling's Formula,

$$\begin{aligned}
\binom{n}{pn} &= \frac{n!}{(pn)!((1-p)n)!} \\
&= \frac{(1+o(1))\sqrt{2\pi n}(n/e)^n}{(1+o(1))\sqrt{2\pi pn}(pn/e)^{pn}\sqrt{2\pi(1-p)n}((1-p)n/e)^{(1-p)n}} \\
&= (1+o(1))\frac{1}{\sqrt{2\pi p(1-p)n}} \cdot \frac{(n/e)^n}{(pn/e)^{pn}(n(1-p)/e)^{(1-p)n}} \\
&= (1+o(1))\frac{1}{\sqrt{2\pi p(1-p)n}} \cdot \frac{n^n}{(pn)^{pn}(n(1-p))^{(1-p)n}} \\
&= (1+o(1))\frac{1}{\sqrt{2\pi p(1-p)n}} \cdot \frac{2^{n\log n}}{2^{pn\log(pn)}2^{(1-p)n\log(n(1-p))}} \\
&= (1+o(1))\frac{2^{n\log n - pn(\log n+\log p)-(1-p)n(\log n+\log(1-p))}}{\sqrt{2\pi p(1-p)n}} \\
&= (1+o(1))\frac{2^{nH(p)}}{\sqrt{2\pi p(1-p)n}}.
\end{aligned}$$

□

Problem 3

Let $k \le n/2$, and suppose that $\mathcal{F}$ is an antichain in $\mathcal{P}[n]$ such that every $A \in \mathcal{F}$ has $|A| \le k$. Prove that $|\mathcal{F}| \le \binom{n}{k}$.

Proof. Let $\mathcal{P}_k[n]$ be the set of all subsets of $[n]$ of size $k \le n$. For $1 \le k \le n/2$, consider the bipartite subgraph G_k of the discrete cube Q_n induced by $[n]^{(k-1)} \sqcup [n]^{(k)}$. Note that there is edge between $A \in [n]^{(k-1)}$ and $B \in [n]^{(k)}$ if and only if $A \subseteq B$.

We now verify the conditions of Hall's Theorem to show that there is a matching saturating $[n]^{(k-1)}$. Let $S \subseteq [n]^{(k-1)}$ and let $T = \Gamma(S)$. Notice that each $A \in S$ has $n-k+1$ neighbors in T, whereas each $B \in T$ has $k-1$ neighbors in $[n]^{(k-1)}$. But then

$$|S| \cdot (n-k+1) = e(S,T) \le |T| \cdot k.$$

Since $k \le n/2$, we have $|S| \le |T| \cdot k/(n-k+1) \le |T|$. Hall's Theorem now furnishes a matching in G_k saturating $[n]^{(k-1)}$, for any $1 \le k \le n/2$. By connecting the matchings between G_k for $1 \le k \le n/2$, we get $\binom{n}{k}$ chains that partition $\mathcal{P}_k[n]$. It now follows that $\mathcal{F}$ intersects with any of these chains in at most one element, and so $|\mathcal{F}| \le \binom{n}{k}$. $\square$

Problem 4

Let $(P, \leq)$ be a poset. Suppose that every chain in P has at most k elements. Prove that P can be written as the union of k antichains.

Proof. For $x \in P$, define $h(x)$ as the length of the longest chain containing x as the maximal element. Notice that if $x > y$ then $h(x) > h(y)$, as we may append x to the end of any chain containing y. This implies x and y are incomparable if $h(x) = h(y)$. But then for any $x \in P$ we have $h(x) \leq k$. Thus for $1 \leq n \leq k$, $A_n = \{x \in P \mid h(x) = n\}$ is an antichain. The result now follows. $\square$

Problem 5

Suppose $\mathcal{F} \subset \mathcal{P}[n]$ is a set system containing no chain with $k+1$ sets.

(a) Prove that $\sum_{i=0}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}} \leq k$, where $\mathcal{F}_i = \mathcal{F} \cap [n]^{(i)}$ for each i.

Proof. Let C be a maximal chain in $\mathcal{P}[n]$. Then

$$k \geq \mathbb{E}[|F \cap C|] = \sum_{A \in \mathcal{F}} \mathbb{P}(A \in C) = \sum_{i=0}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}}.$$

□

(b) What is the maximum possible size of such a system?

Proof. Note that setting $\mathcal{F}$ to be the union of the center k layers of $\mathcal{P}[n]$ shows that

$$|\mathcal{F}| \geq \sum_{i=1}^{k} \binom{n}{\lfloor \frac{n-k}{2} \rfloor + i}.$$

By the LYM inequality, equality holds in (1) if and only if $A_j = [n]^{(i)}$ for some i. Thus, equality can be achieved when $\mathcal{F} = \bigsqcup_{i \in I} [n]^{(i)}$ for some $I \subseteq [n]$ of size k. But then

$$|\mathcal{F}| \leq \max_{I \in [n]^{(k)}} \sum_{i \in I} \binom{n}{i} = \sum_{i=1}^{k} \binom{n}{\lfloor \frac{n-k}{2} \rfloor + i}.$$

The result now follows. □

Problem 6

Let $\mathcal{A}$ be an antichain in $\mathcal{P}[n]$ that is not of the form $[n]^{(r)}$. Must there exist a maximal chain disjoint from $\mathcal{A}$?

Proof. For $A \in \mathcal{A}$, the fraction of maximal chains in $\mathcal{P}[n]$ that contain A is

$$\frac{|A|!(n-|A|)!}{n!} = \frac{1}{\binom{n}{|A|}}.$$

Let M be a random maximal chain in $\mathcal{P}[n]$. Since each maximal chain intersects with at most one element of $\mathcal{A}$,

$$\mathbb{P}(M \cap \mathcal{A} \neq \emptyset) = \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} = \sum_{i=0}^{n} \sum_{A \in \mathcal{A} \cap [n]^{(i)}} \frac{1}{\binom{n}{i}} = \sum_{i=0}^{n} \frac{|\mathcal{A} \cap [n]^{(i)}|}{\binom{n}{i}}.$$

But then $\mathcal{A}$ is not of the form $[n]^{(r)}$, so by the LYM inequality, the above sum is strictly less than 1. This completes the proof. $\square$

Problem 7

Let $(P, \leq)$ be an infinite poset. Must P contain an infinite chain or antichain?

Proof. Take any x_0 from P. Since P is infinite, at least one of

$$\{y \in P \mid y > x_0\}, \quad \{y \in P \mid y < x_0\}, \quad \{y \in P \mid y \text{ incomparable to } x_0\}$$

is infinite. Let S_1 be the set that is infinite, and pick any $x_1 \in S_1$. Iterate the above process on S_1 gives us a infinite sequence $x_0, x_1, x_2, \ldots$ in P. There exists a subsequence $x_{i_0}, x_{i_1}, x_{i_2}, \ldots$ where all elements are picked from the same choices. But then this subsequence is a chain or antichain. □