# MATH 220A: Homework #4

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# Problem 1

Prove that $\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$ and $\liminf(a_n + b_n) \ge \liminf a_n + \liminf b_n$ for $\{a_n\}$ and $\{b_n\}$ sequences of real numbers.

*Proof.* Let $A = \limsup a_n$ and $B = \limsup b_n$. Pick $\epsilon > 0$. Then there exists $N_1, N_2$ such that $a_n \le A + \epsilon/2$ for all $n \ge N_1$, and $b_n \le B + \epsilon/2$ for all $n \ge N_2$. Put $N = \max(N_1, N_2)$. Then for all $n \ge N$, we have $a_n + b_n \le A + B + \epsilon$. But then $\epsilon$ is arbitrary, and thus $\limsup(a_n + b_n) \le A + B$.

Let $A = \liminf a_n$ and $B = \liminf b_n$. Pick $\epsilon > 0$. Then there exists $N_1, N_2$ such that $a_n \ge A - \epsilon/2$ for all $n \ge N_1$, and $b_n \ge B - \epsilon/2$ for all $n \ge N_2$. Put $N = \max(N_1, N_2)$. Then for all $n \ge N$, we have $a_n + b_n \ge A + B - \epsilon$. But then $\epsilon$ is arbitrary, and thus $\liminf(a_n + b_n) \ge A + B$. $\square$

# Problem 2

Find the radius of convergence for each of the following power series:

(a) $\sum_{n=0}^{\infty} a^n z^n$, $a \in \mathbb{C}$

*Proof.* By the comparison test, the radius of convergence is $R = \lim |a^n/a^{n+1}| = \frac{1}{|a|}$ when $a \neq 0$, and $R = \infty$ when $a = 0$. □

(b) $\sum_{n=0}^{\infty} a^{n^2} z^n$, $a \in \mathbb{C}$

*Proof.* By the comparison test, the radius of convergence is

$$R = \lim |a^{n^2}/a^{(n+1)^2}| = \lim |a^{-2n-1}| = \begin{cases} 0 & \text{if } |a| > 1 \\ 1 & \text{if } |a| = 1 \\ \infty & \text{if } |a| < 1 \end{cases}.$$

□

(c) $\sum_{n=0}^{\infty} k^n z^n$, $k$ an integer $\neq 0$

*Proof.* By the comparison test, the radius of convergence is $R = \lim |k^n/k^{n+1}| = \frac{1}{|k|}$. □

(d) $\sum_{n=0}^{\infty} z^{n!}$

*Proof.* Note that

$$\sum_{n=0}^{\infty} z^{n!} = \sum_{k=0}^{\infty} a_k z^k,$$

where $a_1 = 2$, $a_k = 1$ if $k = n!$ for $n \in \mathbb{Z}_{\geq 2}$, and $a_k = 0$ otherwise. Then by the root test, the radius of convergence is

$$R = \frac{1}{\limsup |a_k|^{1/k}} = 1.$$

□

# Problem 3

Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

is 1, and discuss convergence for $z = 1$, $-1$, and $i$. (Hint: The $n$th coefficient of this series is not $(-1)^n/n$.)

*Proof.* Note that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)} = \sum_{k=0}^{\infty} a_k z^k,$$

where $a_k = \frac{(-1)^n}{n}$ if there exists $n$ such that $k = n(n+1)$, otherwise $a_k = 0$. Then by the root test,

$$\frac{1}{R} = \limsup |a_k|^{1/k} = \limsup \left| \frac{(-1)^n}{n} \right|^{1/n(n+1)} = \limsup n^{-1/n(n+1)} = \limsup e^{-\ln n/n(n+1)} = 1,$$

as $\lim \frac{\ln n}{n(n+1)} = 0$. Thus the radius of convergence is $R = 1$.

When $z = 1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the alternating test.

When $z = -1$, since $n(n+1)$ is even,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+n(n+1)}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

so the series again converges by the alternating test.

When $z = i$, since $n(n+1)$ is even

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+\frac{n(n+1)}{2}}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n+3)}{2}}}{n} = \sum_{n=1}^{\infty} a_n.$$

where

$$a_n = \frac{(-1)^{\frac{n(n+3)}{2}}}{n} = \begin{cases} \frac{1}{n} & \text{if } n \equiv 0, 1 \pmod 4 \\ -\frac{1}{n} & \text{if } n \equiv 2, 3 \pmod 4 \end{cases}.$$

Put $b_0 = a_1$, $b_k = a_{2k} + a_{2k+1} = (-1)^k \left( \frac{1}{2k} + \frac{1}{2k+1} \right)$ for $k \geq 1$. Then $\sum_{n=1}^{\infty} a_n = \sum_{k=0}^{\infty} b_k$. But then $|b_k|$ decreases monotonically and $\lim b_k = 0$, so the series converges by the alternating test. $\square$

# Problem 4

Show that $f(z) = |z|^2 = x^2 + y^2$ has a derivative only at the origin.

*Proof.* Suppose that $f'(z)$ exists for some $z \in \mathbb{C}$. Then

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(h)}{h} = \lim_{h \to 0} \frac{(z+h)(\bar{z} + \bar{h}) - z\bar{z}}{h} = \lim_{h \to 0} \frac{z\bar{h} + \overline{z\bar{h}} + h\bar{h}}{h} = \lim_{h \to 0} \frac{2Re(z\bar{h})}{h} + \bar{h}.$$

Suppose $\{h_n\} \to 0$. If $\{h_n\} \subseteq \mathbb{R}$, then

$$f'(z) = \lim_{n \to \infty} \frac{2Re(zh_n)}{h_n} + h_n = \lim_{n \to \infty} \frac{2h_n x}{h_n} = 2x.$$

If $\{h_n\} \subseteq i\mathbb{R}$, then

$$f'(z) = \lim_{n \to \infty} \frac{2Re(-zh_n)}{h_n} - h_n = \lim_{n \to \infty} \frac{2h_n y}{ih_n} = -2yi.$$

Since $f'(z) = 2x = 2yi$, we must have $x = y = 0$, so $z = 0$. Thus $f'(z)$ only exists at the origin. $\square$

# Problem 5

Describe the following sets:

(a) $\{z : e^z = i\}$

*Proof.* Put $z = x + iy$, where $x, y \in \mathbb{R}$. We have $e^z = e^{x+iy} = e^x e^{iy} = i$. Then $e^x = 1$ and $e^{iy} = \cos y + i \sin y = i$. Hence $x = 0$ and $y = \frac{\pi}{2} + 2\pi k$ for some $k \in \mathbb{Z}$, which yields

$$\{z : e^z = i\} = \left\{ \frac{(4k+1)i\pi}{2} \mid k \in \mathbb{Z} \right\}.$$

□

(b) $\{z : e^z = -1\}$

*Proof.* Put $z = x + iy$, where $x, y \in \mathbb{R}$. We have $e^z = e^{x+iy} = e^x e^{iy} = -1$. Then $e^x = 1$ and $e^{iy} = \cos y + i \sin y = -1$. Hence $x = 0$ and $y = \pi + 2\pi k$ for some $k \in \mathbb{Z}$, which yields

$$\{z : e^z = -1\} = \{(2k+1)i\pi \mid k \in \mathbb{Z}\}.$$

□

(c) $\{z : e^z = -i\}$

*Proof.* Put $z = x + iy$, where $x, y \in \mathbb{R}$. We have $e^z = e^{x+iy} = e^x e^{iy} = -i$. Then $e^x = 1$ and $e^{iy} = \cos y + i \sin y = -i$. Hence $x = 0$ and $y = -\frac{\pi}{2} + 2\pi k$ for some $k \in \mathbb{Z}$, which yields

$$\{z : e^z = -i\} = \left\{ \frac{(4k-1)i\pi}{2} \mid k \in \mathbb{Z} \right\}.$$

□

(d) $\{z : \cos z = 0\}$

*Proof.* Put $z = x + iy$, where $x, y \in \mathbb{R}$. Since $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = 0$, we have $e^{2iz} = e^{-2y} e^{2ix} = -1$. Hence, $y = 0$ and $x = \frac{\pi}{2} + \pi k$. Thus,

$$\{z : \cos z = 0\} = \left\{ \frac{(2k+1)\pi}{2} \mid k \in \mathbb{Z} \right\}.$$

□

(e) $\{z : \sin z = 0\}$

*Proof.* Put $z = x + iy$, with $x, y \in \mathbb{R}$. Since $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = 0$, we have $e^{2iz} = e^{-2y} e^{2ix} = 1$. Hence, $y = 0$ and $x = \pi k$ for some $k \in \mathbb{Z}$. Thus,

$$\{z : \cos z = 0\} = \{k\pi \mid k \in \mathbb{Z}\}.$$

□

# Problem 6

Prove the following generalization of Proposition 2.20. Let $G$ and $\Omega$ be open in $\mathbb{C}$ and suppose $f$ and $h$ are functions defined on $G$, $g : \Omega \to \mathbb{C}$ and suppose that $f(G) \subseteq \Omega$. Suppose that $g$ and $h$ are analytic, $g'(\omega) \neq 0$ for any $\omega$, that $f$ is continuous, $h$ is one-to-one, and that they satisfy $h(z) = g(f(z))$ for $z$ in $G$. Show that $f$ is analytic. Give a formula for $f'(z)$.

*Proof.* Let $z \in \mathbb{C}$. Since $h$ is injective, $g(f(z+k)) = h(z+k) \neq h(z) = g(f(z))$ for all $k \neq 0$, and so $f(z+k) \neq f(z)$ for all $k \neq 0$. Since $h$ is analytic,

$$h'(z) = \lim_{k\to 0} \frac{h(z+k) - h(z)}{k} = \lim_{k\to 0} \frac{g(f(z+k)) - g(f(z))}{k} = \lim_{k\to 0} \frac{g(f(z+k)) - g(f(z))}{f(z+k) - f(z)} \cdot \frac{f(z+k) - f(z)}{k}.$$

But then $f$ is continuous, so $f(z+k) \to f(z)$ as $k \to 0$, and thus

$$\lim_{k\to 0} \frac{g(f(z+k)) - g(f(z))}{f(z+k) - f(z)} = g'(f(z)).$$

Hence,

$$h'(z) = g'(f(z)) \lim_{k\to 0} \frac{f(z+k) - f(z)}{k},$$

so $f'(z) = \lim_{k\to 0} \frac{f(z+k)-f(z)}{k} = \frac{h'(z)}{g'(f(z))}$ exists. □