MATH 220A: Homework #2

Due on Oct 11, 2024 at 23:59pm

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Problem 1

Show that if $F \subset X$ is closed and connected, then for every pair of points a, b in F and each $\epsilon > 0$, there are points $z_0, z_1, \ldots, z_n$ in F with $z_0 = a$, $z_n = b$, and $d(z_{k-1}, z_k) < \epsilon$ for $1 \leq k \leq n$. Is the hypothesis that F be closed needed? If F is a set which satisfies this property, then F is not necessarily connected, even if F is closed. Give an example to illustrate this.

Proof. We give a proof without assuming that F is closed. Suppose there exists $a, b \in F$ and $\epsilon > 0$ such that there do not exist $z_0, z_1, \ldots, z_n \in F$ with $z_0 = a$, $z_n = b$, and $d(z_{k-1}, z_k) < \epsilon$ for $1 \leq k \leq n$. Define

$$A \coloneqq \{z \in F \mid \exists z_0 = a, z_1, \ldots, z_n = z, d(z_{k-1}, z_k) < \epsilon, \forall 1 \leq k \leq n\},$$
$$B \coloneqq \{z \in F \mid \exists z_0 = b, z_1, \ldots, z_n = z, d(z_{k-1}, z_k) < \epsilon, \forall 1 \leq k \leq n\}.$$

Then $A \cap B = \emptyset$ and $A, B \neq \emptyset$, as $a \in A$ and $b \in B$. Let $x \in A$. There exists $z_0 = a, z_1, \ldots, z_n = x$ such that $d(z_{k-1}, z_k) < \epsilon$ for all $1 \leq k \leq n$. For any point $y \in B(x, \epsilon)$, putting $z_{n+1} \in y$ shows that $y \in A$. Same argument applies to B, and so A and B are open sets. Hence, we may assume that $C = F \backslash (A \cup B)$ is nonempty, otherwise F is disconnected. Let $x \in C$. If there exists $z_0 = a, z_1, \ldots, z_n = y, d(z_{k-1}, z_k) < \epsilon, \forall 1 \leq k \leq n$ for some $y \in B(x, \epsilon)$, then putting $z_{n+1} \in x$ shows that $x \in A$, contradiction. Same argument works for B. Thus, $B(x, \epsilon) \subset C$, C is open. But then C and $A \cup B$ are open sets that separates F, so F is disconnected, contradiction. The result now follows.

We now give an counter example to the converse of the statement. Counterexample?? □

Problem 2

Let z_n, z be points in $\mathbb{C}$ and let d be the metric on $\mathbb{C}_\infty$. Show that $|z_n - z| \to 0$ if and only if $d(z_n, z) \to 0$. Also show that if $|z_n| \to \infty$ then $\{z_n\}$ is Cauchy in $\mathbb{C}_\infty$. (Must $\{z_n\}$ converge in $\mathbb{C}_\infty$?)

Proof. For $z_n, z \in \mathbb{C}$, the distance function on $\mathbb{C}_\infty$ is defined as

$$d(z_n, z) \coloneqq \frac{2|z - z_n|}{\sqrt{(1+|z|^2)(1+|z_n|^2)}}.$$

Since $|z_n - z'| \to 0$, we have $d(z_n, z) \to 0$ as the numerator goes to 0 and the demominator is at least 1. Conversely, suppose for sake of contradiction that $|z_n - z|$ does not converge to 0 as $d(z_n, n) \to 0$. Since the numerator of $d(z_n, z)$ is not 0, $d(z_n, n) \to 0$ converges to 0 only if the demominator $\sqrt{(1+|z|^2)(1+|z_n|^2)}$ has approaches ∞. But then $|z_n| \to \infty$, contradiction.

Fix $\epsilon > 0$. Note that $d(z_n, \infty) = \frac{2}{\sqrt{(1+|z_n|^2)}} \to 0$ as $|z_n| \to \infty$. Hence, there exists large enough n_0 such that for all $n, m > n_0$, $d(z_n, \infty), d(z_m, \infty) < \epsilon/2$. The result now follows that

$$d(z_n, z_m) \leq d(z_n, \infty) + d(z_m, \infty) < \epsilon,$$

for all $n, m > n_0$. □

Problem 3

Put a metric d on $\mathbb{R}$ such that $|x_n - x| \to 0$ if and only if $d(x_n, x) \to 0$, but that $\{x_n\}$ is a Cauchy sequence in $(\mathbb{R}, d)$ when $|x_n| \to \infty$. (Hint: Take inspiration from $\mathbb{C}_\infty$.)

Proof. Define

$$d(x,y) \coloneqq \frac{2|x-y|}{\sqrt{(1+x^2)(1+y^2)}},$$

for real x, y. Since d is merely the real number case of the metric on $\mathbb{C}_\infty$, d is a metric on $\mathbb{R}$ and the statement "$|x_n - x| \to 0$ if and only if $d(x_n, x) \to 0$" follows from the same argument as the previous problem. Now, suppose $|x_n| \to \infty$. Fix $\epsilon > 0$. Pick $N > 4/\epsilon$. There exists n_0 such that for all $n > n_0$, $|x_n| > N$. But then

$$d(x_n, x_m) = \frac{2|x_n - x_m|}{\sqrt{(1+x_n^2)(1+x_m^2)}} \le \frac{2|x_n| + |x_m|}{\sqrt{x_n^2 x_m^2}} = 2\left(\frac{1}{|x_n|} + \frac{1}{|x_m|}\right) < \frac{4}{N} < \epsilon,$$

for all $n, m > n_0$. The result now follows. $\square$

Problem 4

Prove the converse of proposition 4.4: A set $K \subset X$ is compact if every collection $\mathscr{F}$ of closed subsets of K with the finite intersection property has nonempty intersection.

Proof. We prove the contrapositive. Suppose K is not compact. There exists an open cover $\{U_\alpha\}$ of K such that no finite subcover exists. Let $\mathscr{F}$ be the collection of closed subsets $\{K \backslash U_\alpha\}$. Given any finite subcollection $\{K \backslash U_{\alpha_i}\}_{i=1}^n$, the intersection $\bigcap_{i=1}^n K \backslash U_i = K \backslash \left(\bigcup_{i=1}^n U_{\alpha_i}\right) \neq \emptyset$, and so $\mathscr{F}$ has the finite interseciton property. But then $\bigcap_\alpha K \backslash U_\alpha = K \backslash \left(\bigcup_\alpha U_\alpha\right) = \emptyset$. $\square$

Problem 5

Show that the union of a finite number of compact sets is compact.

Proof. Let $K_1, K_2, \ldots, K_n$ be compact sets. Let $\{U_\alpha\}$ be an open cover of $K_1 \cup K_2 \cup \ldots \cup K_n$. Since K_1 is compact, there exists a finite subcover $\{U_{\alpha_1}, \ldots, U_{\alpha_{n_1}}\}$ of K_1. Similarly, there exists a finite subcover $\{U_{\alpha_{n_1+1}}, \ldots, U_{\alpha_{n_2}}\}$ of K_2, and so on. The union of these finite subcovers is a finite subcover of $K_1 \cup K_2 \cup \ldots \cup K_n$, and so $K_1 \cup K_2 \cup \ldots \cup K_n$ is compact. $\square$