

## Math 158 HW2

**Question 2.5.2.** A tournament is an orientation of a complete graph. Prove that every tournament contains a directed path containing all of its vertices.

*Proof.* Let  $T_n$  be a  $n$ -vertex tournament. We will prove by induction on  $n$  to show that  $T_n$  is traceable for all  $n$ .  $T_1$  is obviously traceable as it only contains one vertex.  $T_2$  is traceable, as it contains only one directed edge that connects all the vertices in the graph. Suppose that a  $T_k$  contains a directed hamiltonian  $uv$ -path  $P$ , for some  $k \geq 2$ . We denote the vertex after  $x$  in  $P$  as  $x^+$ , for some  $x \in V(P)$ . By adding a vertex  $w$  and  $k$  directed edges to  $T_k$ , we get a  $T_{k+1}$ . If  $e = (w, u)$  or  $(v, w) \in E(T_{k+1})$ , we can connect  $e$  with  $P$  to obtain a hamiltonian path in  $T_{k+1}$ . If  $(w, u), (v, w) \notin E(T_{k+1})$ , we know  $(u, w), (w, v) \in E(T_{k+1})$  because  $N_{T_{k+1}}(w) = V(P)$ , which ensures  $d_{T_{k+1}}^+(w), d_{T_{k+1}}^-(w) \geq 1$ . Hence, there exist  $x \in V(P)$  such that  $(x, w), (w, x^+) \in E(T_{k+1})$ . We can then add  $w$  and  $(x, w), (w, x^+)$  to  $P - (w, w^+)$  to get a directed hamiltonian path in  $T_{k+1}$ . Thus, if  $T_k$  is traceable, then  $T_{k+1}$  is also traceable. Therefore, all tournaments are traceable.  $\square$

**Question 2.5.9.** The closure of an  $n$ -vertex graph  $G$ , denoted  $C(G)$ , consists in adding edges between any two non-adjacent vertices  $u$  and  $v$  such that  $d_G(u) + d_G(v) \geq n$ . Prove that a graph  $G$  is hamiltonian if and only if  $C(G)$  is hamiltonian.

*Proof.* If  $G$  is hamiltonian,  $G$  contains a hamiltonian cycle  $H \subseteq G$ . Since  $C(G)$  contains  $G$  and  $V(C(G)) = V(G)$ , we have  $H \subseteq G \subseteq C(G)$ , and thus  $C(G)$  is hamiltonian.

Suppose that  $C(G)$  has a hamiltonian cycle  $F$ . If  $F$  does not contain any edges that are not in  $G$ , then  $G$  is hamiltonian. Otherwise, there exists  $\{u, v\} \in E(F)$  such that  $\{u, v\} \notin E(G)$ , which implies  $d_G(u) + d_G(v) \geq n$ . Let  $P = F - \{u, v\}$  be a hamiltonian  $uv$ -path of  $C(G)$ , say  $v_1v_2 \dots v_n$ , and  $N(v)^+ = \{v_{i+1} : v_i \in N_G(v)\}$ . We then have  $N(v)^+ \cup N(u) \subseteq V(P) \setminus \{u\}$ , which shows that  $|N(v)^+ \cup N(u)| \leq n-1$ . Since  $|N(v)^+| + |N(u)| = d_G(u) + d_G(v) \geq n$ , we have

$$|N(v)^+ \cap N(u)| = |N(v)^+| + |N(u)| - |N(v)^+ \cup N(u)| \quad (1)$$

$$\geq n - (n-1) = 1. \quad (2)$$

Hence,  $N(v)^+ \cap N(u) \neq \emptyset$ . Let  $v_k \in N(v)^+ \cap N(u)$ , we can then get a new hamiltonian cycle  $P - \{v_k, v_{k+1}\} + \{u, v_{k+1}\} + \{v_k, v\}$ . This shows that all  $e \in E(F) \setminus E(G)$  can be removed from  $F$  to obtain a hamiltonian cycle that only consists of edges in  $G$ , which shows that  $G$  is hamiltonian. Therefore,  $C(G)$  is hamiltonian if and only if  $G$  is hamiltonian.  $\square$

**Question 2.5.11.** Let  $G$  be a hamiltonian bipartite graph of a minimum degree of at least three. Prove that  $G$  contains at least two hamiltonian cycles.

*Proof.* Let  $C$  be a hamiltonian cycle in bipartite graph  $G(A, B)$ , and let  $u, v, w \in A$  such that  $N_C(u) = \{v, w\}$ . Consider the hamiltonian  $uv$ -path  $P = C - \{u, v\}$ . Since  $G$  is bipartite and  $v \in B$ , we know  $N(v)^+ \in B$ , and thus the vertices of  $P$  obtained by all possible rotations are all in  $B$ . Let  $G'(A', B') \subseteq G$  such that  $C \subseteq G'$  and  $d_{G'}(b) = 3$  for all  $b \in B'$ . We know  $G'$  exists because  $\delta(G) \geq 3$ . Let  $H$  be a graph whose vertices are hamiltonian paths of  $G'$  starting with the edge  $\{u, w\}$ , where two hamiltonian paths in  $G'$  form an edge of  $H$  if they are obtained from one another by rotation. If  $Q \in H$  is a hamiltonian path that ends at a vertex  $x$ , then  $Q$  has  $3 - 1 = 2$  possible rotations in  $G'$  unless  $\{u, x\} \in E(G')$ , in which case would have  $3 - 2 = 1$  rotations instead. In the latter case,  $Q$  together with  $\{u, w\}$  would form a hamiltonian cycle in  $G'$ . By the Handshake Theorem, since the number of vertices with odd degrees is even, there is an even number of paths  $Q$  in  $G'$  which ends at a neighbor of  $u$ . Therefore,  $G'$  has an even amount of hamiltonian cycles. Since  $G'$  already contains a hamiltonian cycle  $C$ , it must contain some other hamiltonian cycle  $C'$ . Since  $C, C' \subseteq G' \subseteq G$ ,  $G$  has at least two hamiltonian cycles.  $\square$

**Question 3.8.2.** A tiling of an  $m \times n$  chess board is a set of dominoes that cover all the squares on the chess board exactly once (each domino covers two adjacent squares).

- (a) For which  $m \geq 1$  and  $n \geq 1$  does an  $m \times n$  chess board having a tiling?

*Solution.* Since the number of squares on the chess board must be even to have a perfect matching,  $m$  or  $n$  is even. Assume, without loss of generality, that  $m$  is even. We will prove by induction on  $m$ . If  $m = 2$ , then we can match each square in one column to one in the adjacent column that is adjacent to it, and this is a perfect matching  $M_2$ . Suppose that there is a perfect matching  $M_k$  for each  $k \times n$  chessboard, where  $2 \leq k \leq m$  and is even. We can then split a  $(m+2) \times n$  chessboard into a  $2 \times n$  and  $m \times n$  board. We can then find a perfect matching  $M_2 \cup M_m$ . This also shows true for even  $n$ . Therefore, for all  $(m, n) \in \{(a, b) \in \mathbb{N}^2 : ab \text{ is even}\}$ , an  $m \times n$  chessboard has tiling.  $\square$

- (b) If we remove two squares from an  $m \times n$  chessboard, when do the remaining squares have a tiling?

*Solution.* There must be an even amount of squares to have tiling, so if a  $m \times n$  chessboard has tiling after two squares removed, then the  $m \times n$  chessboard also has an even amount of squares. Thus,  $m$  or  $n$  needs to be even.

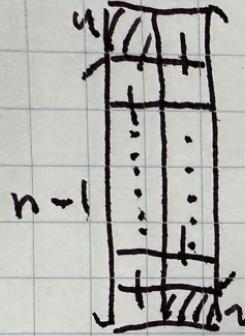
Assume, without loss of generality, that  $m$  is even. Let  $G$  be a grid graph whose vertex set contains all squares on a  $m \times n$  chessboard, and each pair of vertices forms an edge if they are adjacent to each other on the board. Since  $m$  is even,  $G$  has a perfect matching. Let  $v_{xy}$  correspond to the square in the  $x$ th row and  $y$ th column, for some  $1 \leq x \leq n$ ,  $1 \leq y \leq m$ . Let  $\{c_1, c_2\}$  be a set of two colors. We color  $v_{xy}$  with  $c_1$  if  $x+y$  is even and  $c_2$  if  $x+y$  is odd. This shows that every square can be colored with no same-colored squares being adjacent. Since each domino covers a  $c_1$  square and a  $c_2$  square, each color must have the same number of squares to have tiling. Therefore, if we remove two squares with the same color, then  $G$  does not have a tiling.

Suppose that we remove two squares  $u, v$  with different colors. If  $n = 1$ , then  $G - \{u\} - \{v\}$  has a tiling if there are only even components.

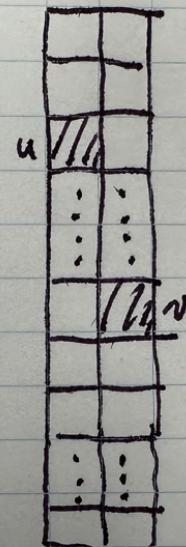
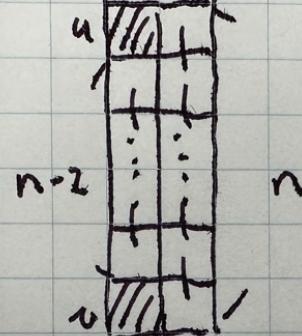
**Claim 1.** *If  $m = 2$ , then  $G - \{u\} - \{v\}$  has a tiling.*

Suppose that  $u, v$  are each in the first and last rows and  $n$  is some natural number. Consider the case where  $n$  is odd. Since  $u, v$  have different colors,  $u, v$  are in different columns. This means that the two columns in  $G - \{u\} - \{v\}$  both have  $n-1$  number of squares, which is even. Consider the case where  $n$  is even. Since  $u, v$  has different colors,  $u, v$  are in the same column. This means that the two columns in  $G - \{u\} - \{v\}$  have  $n$  and  $n-2$  numbers of squares respectively, which are also even. Thus, in both cases, we can tile along the columns and cover all squares, which is a tiling. Suppose that  $u, v$  are in the  $i$ th and  $j$ th rows, for some  $1 < i \leq j < n$ . We can then remove the first  $i-1$  rows and the last  $n-j$  rows, as there are two columns so we can find a tiling of them by putting a domino in each row. What is left is a  $2 \times (j-i+1)$  board with two corners on both sides removed, which we just proved to have tiling in the first case. Therefore,  $G - \{u\} - \{v\}$  has a tiling if  $m = 2$ .

$n$  is odd

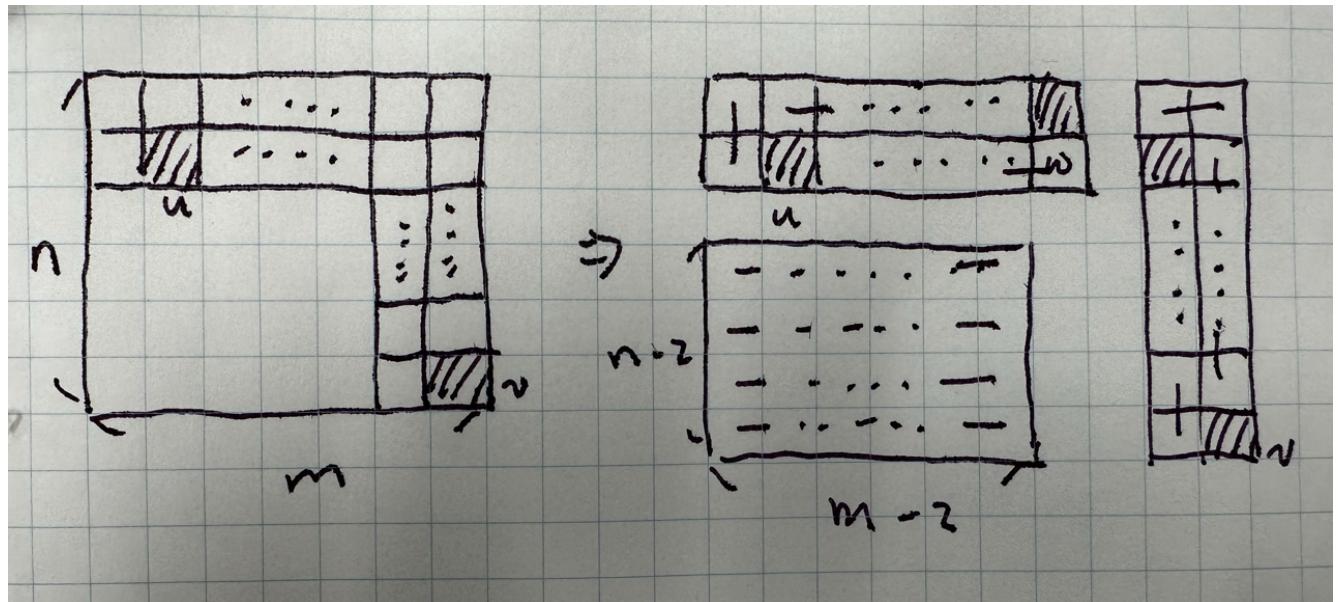


$n$  is even



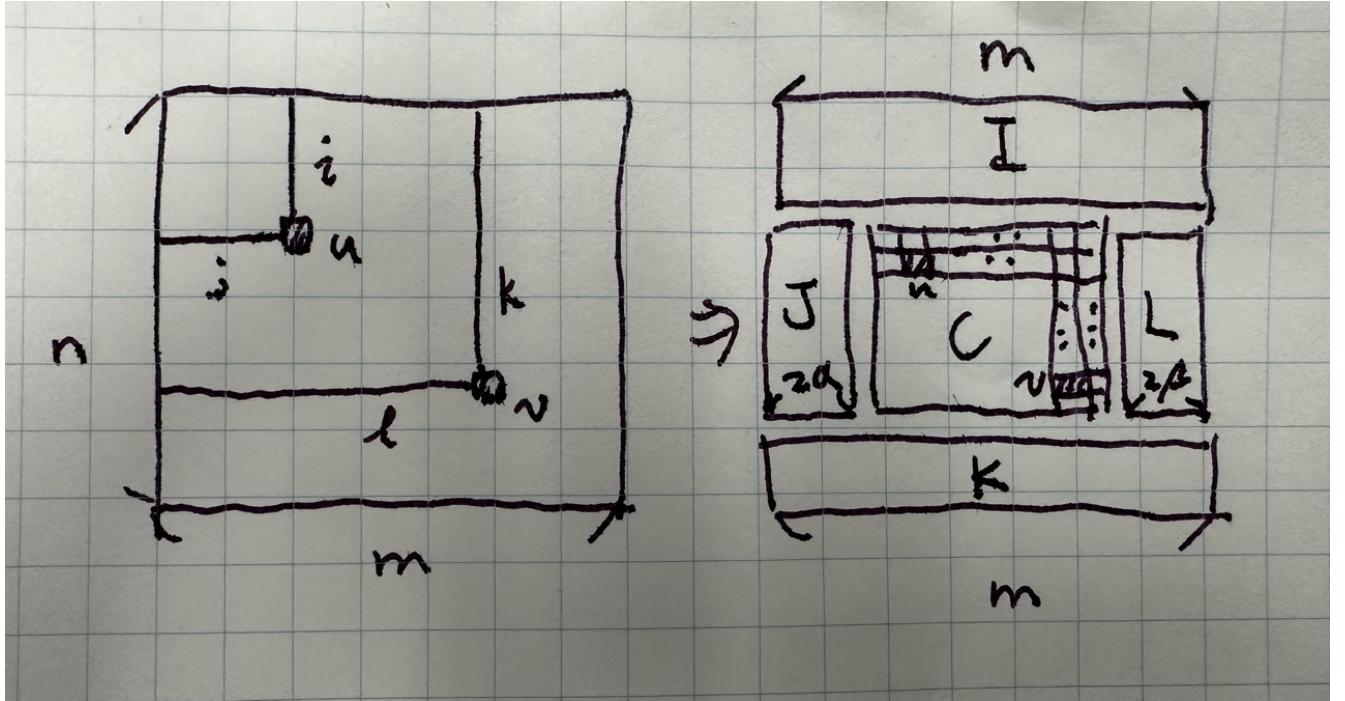
**Claim 2.** Suppose that  $n \geq 2$ . If  $u, v$  are each in a  $2 \times 2$  corner on the opposite side, then  $G - \{u\} - \{v\}$  has a perfect matching.

Assume, without loss of generality, that  $u$  is in the top left  $2 \times 2$  corner and  $v$  is in the bottom right  $2 \times 2$  corner. Since the  $(m-2) \times (n-2)$  squares on the bottom left have a perfect matching  $M_1$ , as it has an even side, we can first take it out. What is left are the first two rows and last two columns of  $G$ , so we can split it into two parts, a  $(m-2) \times 2$  board  $A$  that contains  $u$  and a  $(2 \times n)$  board  $B$  that contains  $v$ . Let  $w \in V(B) \cap N_G(V(A))$  such that  $w$  shares the same color with  $u$ . We remove  $w$  from  $B$  to  $A$ , and we then have two parts,  $A + \{w\}$  and  $B - \{w\}$ . We can view  $A + \{w\}$  as a  $2 \times (m-1)$  board missing two different color squares  $u$  and a square next to  $w$ . By Claim 1, since both  $A + \{w\}$  and  $B - \{w\}$  have exactly two columns and are missing two different-colored squares,  $A + \{w\}$  and  $B - \{w\}$  each has a perfect matching  $M_2$  and  $M_3$  respectively. Therefore,  $G - \{u\} - \{v\}$  has a perfect matching  $M_1 \cup M_2 \cup M_3$ .



Finally, we will show that  $G - \{u\} - \{v\}$  has a perfect matching for all  $m, n \geq 2$ ,  $m$  is even. Suppose that  $u$  is in the  $i$ th row  $j$ th column and  $v$  is in the  $k$ th row  $l$ th column of  $G$ . Assume, without loss of generality, that  $k \geq i$  and  $l \geq j$ . We can first take out a  $m \times (i-1)$  board  $I$  that contains the first  $(i-1)$  rows of  $G$  and a  $m \times (n-k)$  board  $K$  that contains the last  $(n-k)$  rows of  $G$ . Since they both have an even side  $m$ , they have a perfect matching  $M_i$  and  $M_k$  respectively. What's left is a  $m \times (k-i+1)$  board  $G'$ . We can then take

out the left-most  $2\alpha$  and right-most  $2\beta$  columns of  $G'$ , where  $\alpha$  is the greatest integer such that  $2\alpha < j$  and  $\beta$  is the greatest integer such that  $m - 2\beta > l$  and obtain a  $2\alpha \times (k - i + 1)$  board  $J$  and a  $2\beta \times (k - i + 1)$  board  $L$ . Since  $J$  and  $L$  each have an even side  $2\alpha$  and  $2\beta$ , they have perfect matching  $M_J$  and  $M_L$  respectively. What is left is a  $(m - 2(\alpha + \beta)) \times (k - i + 1)$  board  $C$  with  $u, v$  in opposite side  $2 \times 2$  corners. By Claim 2,  $C$  contains a perfect matching  $M_C$ . We now found a perfect matching  $M_I \cup M_J \cup M_K \cup M_L \cup M_C$  of  $G - \{u\} - \{v\}$ . Therefore, if we remove two squares from a  $m \times n$  chessboard, it has tiling if and only if  $mn$  is even and the two removed squares have different colors and all boards have an even number of squares.



□

**Question 3.8.8.**

- (a) Let  $G$  be an  $n$  by  $n$  bipartite graph of minimum degree more than  $n/2$ . Prove that  $G$  has a perfect matching.

*Solution.* Suppose that there is a non-hamiltonian  $n$  by  $n$  bipartite graph of minimum degree at least  $n/2$ . Amongst all such graphs, let  $H(A, B)$  be one with parts  $A$  and  $B$  and a maximum number of edges. If we add an edge  $e = \{v_1, v_{2n}\}$  between non-adjacent vertices in  $H$ , we would have a graph with a hamiltonian cycle  $C$ , and so  $C - e$  is a hamiltonian path in  $H$ , say  $v_1 v_2 \dots v_{2n}$ . Assume, without loss of generality, that  $v_1 \in A$  and  $v_{2n} \in B$ . Let  $N(v_1)^+ = \{v_{i+1} : v_i \in N(v_1)\}$ . Since  $N(v_1)^+ \cup N(v_{2n}) \subseteq A$ , we have  $|N(v_1)^+ \cup N(v_{2n})| \leq n$ . Since  $\delta(H) > n/2$ , we have  $|N(v_1)^+| + |N(v_{2n})| \geq n + 1$ . Thus, we have

$$|N(v_1)^+ \cap N(v_{2n})| = |N(v_1)^+| + |N(v_{2n})| - |N(v_1)^+ \cup N(v_{2n})| \quad (3)$$

$$\geq n + 1 - n = 1. \quad (4)$$

This shows that  $N(v_1)^+ \cap N(v_{2n}) \neq \emptyset$ , which proves that  $H$  contains a hamiltonian cycle, a contradiction. Therefore, there exists a hamiltonian path  $P$  in  $G$  such, say  $u_1 u_2 \dots u_{2n}$ . Let  $f = \{(u_i, u_{i+1}) : i \text{ is even}\}$ . We can then find a perfect matching  $M = P - f$  of  $G$ . Hence,  $G$  has a perfect matching.  $\square$

- (b) Let  $G$  be a  $2n$ -vertex graph of minimum degree at least  $n$ . Prove that  $G$  has a perfect matching.

*Solution.* If  $n = 1$ ,  $G$  itself is a perfect matching for  $G$ . Suppose that  $n \geq 2$ . By Dirac's Theorem, since  $\delta(G) \geq |V(G)|/2$ ,  $G$  contains a hamiltonian path  $P$ , say  $v_1 v_2 \dots v_{2n}$ . Let  $e = \{(v_i, v_{i+1}) \in E(P) : i \text{ is even}\}$ , then  $M = P - e$  is a perfect matching in  $G$ . Therefore,  $G$  has a perfect matching.  $\square$