

C3.8 Analytic Number Theory: Sheet #2

Due on November 4, 2025 at 12:00pm

Professor B. Green

Ray Tsai

Problem 1

Evaluate the sum $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$.

Proof. By Proposition 3.1(ii),

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\sum_{n=1}^{\infty} n^{-2}} = \frac{6}{\pi^2}.$$

□

Problem 2

Give a simple description of the function $\phi \star 1$.

Proof. Note that

$$\phi \star 1 = \sum_{d|n} \phi(n/d).$$

But then $\phi(n/d)$ is the number of $m \leq n$ such that $\gcd(m, n) = d$. Thus $\phi \star 1(n) = n$. □

Problem 3

Establish the following Dirichlet series:

$$(i) \sum_n \tau(n)n^{-s} = \zeta(s)^2 \text{ for } \operatorname{Re} s > 1;$$

Proof. Note that

$$1 \star 1 = \sum_{d|n} 1 = \tau(n).$$

The result now follows from Proposition 3.1(i). □

$$(ii) \sum_n \phi(n)n^{-s} = \frac{\zeta(s-1)}{\zeta(s)} \text{ for } \operatorname{Re} s > 2;$$

Proof. Since

$$\phi \star 1 = \sum_{d|n} \phi(n/d) = n,$$

by Proposition 3.1(i) we have

$$\zeta(s) \sum_n \phi(n)n^{-s} = \sum_n n \cdot n^{-s} = \zeta(s-1).$$

The result now follows. □

$$(iii) \sum_n \sigma(n)n^{-s} = \zeta(s)\zeta(s-1);$$

Proof. By (ii), $\zeta(s-1)$ is the Dirichlet series for $\phi \star 1 = n$. But then

$$n \star 1 = \sum_{d|n} d = \sigma(n).$$

The result now follows from Proposition 3.1(i). □

(iv) If $\lambda(n)$ is the Liouville function, that is to say the unique completely multiplicative function equal to -1 on the primes, then

$$\sum_n \lambda(n)n^{-s} = \frac{\zeta(2s)}{\zeta(s)} \text{ for } \operatorname{Re} s > 1.$$

Proof. Let $d \mid n$. Notice

$$\lambda(d) + \frac{\lambda(n)}{\lambda(d)} = \frac{\lambda(d)^2 + \lambda(n)}{\lambda(d)} = \frac{1 + \lambda(n)}{\lambda(d)} = \begin{cases} 2/\lambda(d) & \text{if } n \text{ is a square number} \\ 0 & \text{otherwise} \end{cases}$$

Thus $\lambda \star 1(n) = 0$ if n is not a square number. Suppose now n is a square number. Then n has an odd number of divisors d . That is if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then $|\alpha| = \alpha_1 + \cdots + \alpha_k$ is odd. But then $d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, $\lambda(d) = (-1)^{\sum_{i=1}^k \beta_i}$. Thus

$$\lambda \star 1(n) = \sum_{d|n} \lambda(d) = \sum_{0 \leq \beta_i \leq \alpha_i} (-1)^{\sum_{i=1}^k \beta_i} = \sum_{i=1}^{|\alpha|} (-1)^i = 1.$$

It now follows that

$$\zeta(s) \sum_n \lambda(n)n^{-s} = \sum_{\substack{n=1, \\ n=k^2}}^{\infty} n^{-s} = \sum_{n=1}^{\infty} n^{-2s} = \zeta(2s).$$

□

Problem 4

Obtain an asymptotic for $\sum_{n < X} \tau(n)$

Proof. Note that

$$\sum_{n < X} \tau(n) = \sum_{n < X} \sum_{ab=n} 1 = \sum_{ab < X} 1 = 2 \sum_{n \leq \sqrt{X}} \lfloor X/n \rfloor - \mathbb{1}_{X=k^2} \cdot (\sqrt{X})^2.$$

Since

$$\sum_{n \leq \sqrt{X}} \lfloor X/n \rfloor = (1 + o(1))X \sum_{n \leq \sqrt{X}} 1/n = (1 + o(1)) \left(\frac{1}{2} \log X + \gamma \right) X,$$

we have

$$\sum_{n < X} \tau(n) = (1 + o(1))X \log X.$$

□

Problem 5

True or false? There is a constant C such that $\tau(n) \leq \log^C n$ for all sufficiently large n . Justify your answer.

Proof. False. Suppose it is true and that n is large. Consider the product of the first k primes $n = p_1 p_2 \cdots p_k$.

$$\tau(n) = (1+1)(1+1) \cdots (1+1) = 2^k.$$

But then by the Prime Number Theorem, $p_k \sim k \log k$. Thus

$$\tau(n) = 2^k \leq \log^C n \leq \log^C p_k^k = (1 + o(1))(k^2 \log k)^C,$$

contradiction as $\lim_{k \rightarrow \infty} \frac{2^k}{(k^2 \log k)^C} = \infty$ for fixed C . □

Problem 6

Show that

$$\sum_n \Lambda(n) \left\lfloor \frac{Y}{n} \right\rfloor = \sum_{n \leq Y} \log n.$$

By considering $Y = X$ and $Y = X/2$, use this to prove that

$$\sum_{X/2 < n \leq X} \Lambda(n) \ll X.$$

Proof. By Legendre's formula, the exponent of the largest power of prime p that divides $n!$ is

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

Thus

$$\sum_n \Lambda(n) \left\lfloor \frac{Y}{n} \right\rfloor = \sum_{p \leq Y} \nu_p(Y!) \log p = \log \prod_{p \leq Y} p^{\nu_p(Y!)} = \log Y! = \sum_{n \leq Y} \log n.$$

Notice

$$\begin{aligned} \sum_n \Lambda(n) \left\lfloor \frac{X}{n} \right\rfloor - 2 \sum_n \Lambda(n) \left\lfloor \frac{X/2}{n} \right\rfloor &= \left(\sum_{n \leq X/2} \Lambda(n) \left\lfloor \frac{X}{n} \right\rfloor + \sum_{X/2 < n \leq X} \Lambda(n) \left\lfloor \frac{X}{n} \right\rfloor \right) - 2 \sum_{n \leq X/2} \Lambda(n) \left\lfloor \frac{X/2}{n} \right\rfloor \\ &= \sum_{n \leq X/2} \Lambda(n) \left(\left\lfloor \frac{X}{n} \right\rfloor - 2 \left\lfloor \frac{X/2}{n} \right\rfloor \right) + \sum_{X/2 < n \leq X} \Lambda(n) \left\lfloor \frac{X}{n} \right\rfloor \\ &\geq \sum_{X/2 < n \leq X} \Lambda(n), \end{aligned}$$

as $\left\lfloor \frac{X}{n} \right\rfloor - 2 \left\lfloor \frac{X/2}{n} \right\rfloor \geq 0$ for $n > X/2$ and $\left\lfloor \frac{X}{n} \right\rfloor = 1$ for $X/2 < n \leq X$. But then on the LHS, we have

$$\sum_n \Lambda(n) \left\lfloor \frac{X}{n} \right\rfloor - 2 \sum_n \Lambda(n) \left\lfloor \frac{X/2}{n} \right\rfloor = \log X! - 2 \log (X/2)! = \log \frac{X!}{(X/2)!(X/2)!}.$$

By the Stirling's Formula,

$$\frac{X!}{(X/2)!(X/2)!} = (1 + o(1)) 2^X \sqrt{\frac{2}{\pi X}}.$$

Thus

$$\log \frac{X!}{(X/2)!(X/2)!} = O(X).$$

The result now follows from combining all of the above. □

Problem 7

Write $L(X) := \sum_{n \leq X} \lambda(n)$ and $M(X) := \sum_{n \leq X} \mu(n)$. Establish the relations

$$L(X) = \sum_{d \leq \sqrt{X}} M\left(\frac{X}{d^2}\right) \quad \text{and} \quad M(X) = \sum_{d \leq \sqrt{X}} \mu(d) L\left(\frac{X}{d^2}\right),$$

and hence conclude that the statements $L(X) = o(X)$ and $M(X) = o(X)$ are equivalent.

Proof. By Problem 3(iv),

$$\lambda \star 1(n) = \begin{cases} 1 & \text{if } n \text{ is a square number} \\ 0 & \text{otherwise} \end{cases}.$$

Thus let $\text{sq}(n)$ denote the indicator function of the set of square numbers. The Möbius inversion formula then yields

$$\lambda(n) = \mu \star \text{sq}(n) = \sum_{d|n} \mu(n/d) \text{sq}(d).$$

Hence,

$$L(X) = \sum_{n \leq X} \sum_{d|n} \mu(n/d) \text{sq}(d) = \sum_{d \leq X} \text{sq}(d) \sum_{n \leq X/d} \mu(n) = \sum_{d \leq \sqrt{X}} \sum_{n \leq X/d^2} \mu(n) = \sum_{d \leq \sqrt{X}} M\left(\frac{X}{d^2}\right).$$

But then

$$\sum_{d \leq \sqrt{X}} \mu(d) L\left(\frac{X}{d^2}\right) = \sum_{d^2 k \leq X} \mu(d) \lambda(k) = \sum_{n \leq X} \sum_{d^2 | n} \mu(d) \lambda\left(\frac{n}{d^2}\right).$$

Note that

$$\begin{aligned} \sum_{d^2 | n} \mu(d) \lambda\left(\frac{n}{d^2}\right) &= \sum_{d^2 | n} \mu(d) \sum_{j | n/d^2} \mu(n/jd^2) \text{sq}(j) \\ &= \sum_{d^2 | n} \mu(d) \sum_{(jd)^2 | n} \mu(n/(jd)^2) \\ &= \sum_{m^2 | n} \mu(n/m^2) \sum_{d|m} \mu(d) \\ &= \sum_{m^2 | n} \mu(n/m^2) \cdot \delta(m) = \mu(n). \end{aligned}$$

Thus,

$$M(X) = \sum_{d \leq \sqrt{X}} \mu(d) L\left(\frac{X}{d^2}\right).$$

Suppose $L(X) = o(X)$ and that $M(X) = \Omega(X)$. Then

$$L(X) = \sum_{d \leq \sqrt{X}} M\left(\frac{X}{d^2}\right) = \Omega\left(X \sum_{d \leq \sqrt{X}} \frac{1}{d^2}\right) = \Omega(X),$$

contradiction. On the other hand, suppose $M(X) = o(X)$. Fix $\epsilon > 0$. Since $|M(Y)/Y| \leq 1$, we may pick N large enough such that

$$\left| \frac{1}{X} \sum_{N \leq d \leq \sqrt{X}} M\left(\frac{X}{d^2}\right) \right| = \left| \sum_{N \leq d \leq \sqrt{X}} \frac{1}{d^2} \cdot \frac{M\left(\frac{X}{d^2}\right)}{X/d^2} \right| \leq \left| \sum_{d \geq N} \frac{1}{d^2} \right| < \epsilon/2.$$

We also have

$$\lim_{X \rightarrow \infty} \frac{1}{X} \sum_{d \leq N} M\left(\frac{X}{d^2}\right) = \lim_{X \rightarrow \infty} \sum_{d \leq N} \frac{1}{d^2} \cdot \frac{M\left(\frac{X}{d^2}\right)}{X/d^2} = 0$$

But then

$$\frac{L(X)}{X} = \frac{1}{X} \sum_{d \leq N} M\left(\frac{X}{d^2}\right) + \frac{1}{X} \sum_{N \leq d \leq \sqrt{X}} M\left(\frac{X}{d^2}\right) < \epsilon,$$

as $X \rightarrow \infty$. Since ϵ was arbitrary, we have $L(X) = o(X)$. □

Problem 8

Give an asymptotic for $\sum_{n \leq X} \phi(n)$. (*Hint. Using the answer to Question 2, or otherwise, first establish that*

the expression to be estimated is $\sum_{d \leq X} \mu(d) \sum_{m \leq X/d} m$.)

Proof. Since $\phi \star 1 = n$, the Mobius inversion formula yields

$$\phi(n) = n \star \mu(n) = \sum_{d|n} \mu(d) \cdot \frac{n}{d}.$$

Thus we have

$$\sum_{n \leq X} \phi(n) = \sum_{n \leq X} \sum_{d|n} \mu(d) \cdot \frac{n}{d}.$$

Notice that for a fixed $d \leq X$, we will sum up n/d over all $n \leq X$ such that $d \mid n$, and times it by $\mu(d)$. In other words, we will sum up all integers $m \leq X/d$ and multiply it by $\mu(d)$. Thus the double sum can be written as

$$\begin{aligned} \sum_{n \leq X} \phi(n) &= \sum_{d \leq X} \mu(d) \sum_{m \leq X/d} m \\ &= \sum_{d \leq X} \mu(d) \cdot \frac{(1 + \lfloor X/d \rfloor) \lfloor X/d \rfloor}{2} \\ &= \frac{1}{2} \sum_{d \leq X} (1 + o(1)) \mu(d) \cdot (X/d) + \frac{1}{2} \sum_{d \leq X} (1 + o(1)) \mu(d) \cdot (X/d)^2 \\ &= \frac{X}{2} \sum_{d \leq X} (1 + o(1)) \frac{\mu(d)}{d} + \frac{X^2}{2} \sum_{d \leq X} (1 + o(1)) \frac{\mu(d)}{d^2}. \end{aligned}$$

By Problem 1, $\sum_{d \leq X} (1 + o(1)) \frac{\mu(d)}{d^2} \rightarrow \frac{6}{\pi^2}$ as $X \rightarrow \infty$. On the other hand,

$$\sum_{d \leq X} (1 + o(1)) \frac{\mu(d)}{d} \leq \sum_{d \leq X} (1 + o(1)) \frac{1}{d} = O(\log X).$$

Thus we may conclude that

$$\sum_{n \leq X} \phi(n) = \frac{3}{\pi^2} X^2 + O(X \log X).$$

□