MATH 140B: Homework #2

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Professor Seward

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Problem 1

Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x).$$

Show by example that the limit may exist even if $f''(x)$ does not.

Proof. Put $g(h) = f(x+h)+f(x-h)-2f(x)$. Since g is differentiable in a neighborhood of x and $g(h) \to 0$ as $h \to 0$, we may apply the L'Hospotal's Rule and get

$$\begin{aligned}
\lim_{h\to 0} \frac{g(h)}{h^2} &= \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} \\
&= \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{2h} - \lim_{h\to 0} \frac{f'(x-h)-f'(x)}{2h} \\
&= \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{2h} - \lim_{k\to 0} \frac{f'(x+k)-f'(x)}{-2k} \\
&= \frac{f''(x)}{2} + \frac{f''(x)}{2} = f''(x).
\end{aligned}$$

Consider $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$. f is not continuous at 0, so $f''(0)$ does not exist. But then $f(h)+f(-h)-2f(0) = 0$ for all $h > 0$, so $\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$ exists. $\square$

Problem 2

Suppose $a \in \mathbb{R}^1$, f is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a, ∞). Prove that

$$M_1^2 \leq 4M_0M_2.$$

Hint: If $h > 0$, Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$$

for some ξ in $(x, x+2h)$. Hence

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.$$

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty), \end{cases}$$

and show that $M_0 = 1$, $M_1 = 4$, and $M_2 = 4$. Does $M_1^2 \leq 4M_0M_2$ hold for vector-valued functions too?

Proof. □

Problem 3

Suppose f is a real function on $(-\infty, \infty)$. Call x a fixed point of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

(b) Show that the function f defined by

$$f'(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point of f exists, and that $x = \lim_{n\to\infty} x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \ldots$.

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \ldots.$$

Problem 4

Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{C}(\alpha)$ and that $\int f\, d\alpha = 0$.

Proof. $\square$