MATH 140B: Homework #9

Due on Jun 7, 2024 at 23:59pm

Professor Seward

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Problem 1

Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x + 2\pi) = f(x)$ for all x.

(a) Compute the Fourier coefficients of f.

Proof. Let c_n denote the nth fourier coefficient of f. We first note that

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{\delta}{\pi}.$$

For $n \neq 0$,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx}\, dx = \frac{1}{2in\pi}(e^{in\delta} - e^{-in\delta}) = \frac{\sin(n\delta)}{n\pi}.$$

□

(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

Proof. Since $f(t) = 1$ for all $t \in (-\delta, \delta)$, it follows from Theorem 8.14 that

$$\sum_{-\infty}^{\infty} c_n = f(0) = 1.$$

Since $\frac{\sin(-n\delta)}{-n\pi} = \frac{\sin(n\delta)}{n\pi}$,

$$\pi = \delta + \sum_{n \neq 0} \frac{\sin(n\delta)}{n} = \delta + 2 \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n},$$

and the result now follows from rearranging the equation. □

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

Proof. Note that $\frac{\sin^2(n\delta)}{(n\pi)^2}$ is an even function with respect to n. By Parseval's theorem

$$\frac{\delta^2}{\pi^2} + 2 \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{(n\pi)^2} = \sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{\delta}{\pi}.$$

The result now follows from rearranging the equation. □

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(d) Let $\delta \to 0$ and prove that
$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

Proof. We first show that the improper integral exists. Pick $\epsilon > 0$. By L'Hopital's rule,
$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x \to 0} \cos x = 1,$$
and thus there exists $\nu > 0$ such that $\left|(\frac{\sin x}{x})^2 - 1\right| < \epsilon$ whenever $|x| < \nu$. Hence,
$$\nu(1-\epsilon) \leq \int_0^\nu \left(\frac{\sin x}{x}\right)^2 dx \leq \nu(1+\epsilon),$$
and so the the improper integral $\int_0^A \left(\frac{\sin x}{x}\right)^2 dx$ exists. On the other hand,
$$\left|\int_A^n \left(\frac{\sin x}{x}\right)^2 dx\right| \leq \int_A^n \frac{1}{x^2}\, dx = \frac{1}{A} - \frac{1}{n},$$
and thus $\int_A^n \left(\frac{\sin x}{x}\right)^2 dx \to \frac{1}{A}$ as $n \to \infty$.
Since the improper integral exists, there exists large enough A, such that
$$\left|\int_A^\infty \left(\frac{\sin x}{x}\right)^2 dx\right| < \epsilon/3.$$
There exists small enough $\delta' \in (0, \delta)$ such that $A/\delta' \in \mathbb{Z}^+$ and the partition $P = \{n\delta' \mid n \in \mathbb{Z}, 0 \leq n \leq A/\delta'\}$, yields
By (c), given $\delta > 0$, there exists large enough N_1 such that for all $N \geq N_1$,
$$\left|\sum_{n=1}^N \frac{\sin^2(n\delta)}{n^2\delta} - \frac{\pi - \delta}{2}\right| < \epsilon/3.$$
Since the improper integral exists, there exists large enough N_2, such that
$$\left|\int_{N_2}^\infty \left(\frac{\sin x}{x}\right)^2 dx\right| < \epsilon/3.$$
Since the improper integral exists, there exists large enough N_3 such that for all $N \geq N_3$, the paritition $P = \{n\delta \mid n \in \mathbb{Z}, 0 \leq n \leq N\}$ on $[0, N_2]$ yields
$$\left|\int_0^A \left(\frac{\sin x}{x}\right)^2 dx - \sum_{n=1}^N \frac{\sin^2(n\delta)}{n^2\delta}\right| < \epsilon/3.$$
Put $N = \max(N_1, N_2)$, and

□

(e) Put $\delta = \frac{\pi}{2}$ in (c). What do you get?

Proof.
$$\sum_{n=1}^\infty \frac{\sin^2(n\pi/2)}{n^2\pi/2} = \frac{2}{\pi}\sum_{k=1}^\infty \frac{1}{(2k+1)^2} = \frac{\pi}{4},$$
and thus
$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

□

Problem 2

Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. For $x \in \mathbb{R}$, define $f(x + 2\pi) = f(x)$.

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = \pi$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} \, dx = -\frac{1}{in} e^{-2\pi in} - \frac{1}{2\pi (in)^2} (e^{-in2\pi} - 1) = \frac{i}{n}.$$

By Parseval's theorem,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

On the left-hand-side, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx = \frac{4\pi^2}{3}.$$

On the right-hand-side, since $|c_n|^2 = \frac{1}{n^2} = |c_{-n}|^2$,

$$\sum_{-\infty}^{\infty} |c_n|^2 = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hence, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6} \left(\frac{4\pi^2}{3} - \pi^2 \right) = \frac{\pi^2}{6}.$$

□