

# MATH 140A: Homework #2

Due on October 13, 2023 at 11:00pm

*Professor Mohammadi*

Section A02

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## Problem 1

Let  $A \subset \mathbb{R}$  be a non-empty subset which satisfies the following two properties

1.  $A = A + A$
2. For every  $\epsilon > 0$ , there exists some  $a \in A$  so that  $0 < a < \epsilon$ .

Prove that for every  $x \in \mathbb{R}^{>0}$ , there exists some  $a \in A$  so that

$$0 < x - a < \epsilon.$$

*Proof.* Let  $a \in A$ . We first show that for  $n \in \mathbb{N}$ ,  $na \in A$  by induction on  $n$ . We already know  $a \in A$ . For  $n > 1$ , since  $(n-1)a \in A$ , we know  $na = a + (n-1)a \in A$  by rule 1. Thus,  $na \in A$ , for all  $n \in \mathbb{N}$ .

Since  $\epsilon > 0$ , there exists  $a \in A$  such that  $0 < a < \epsilon$ , by rule 2. We assume that  $\epsilon < x$ , otherwise we are done. Now we show that there exists  $n \in \mathbb{N}$  such that  $0 < x - na < \epsilon$ . Let  $0 < \frac{x-\epsilon}{a} < n < \frac{x}{a}$ . By the Archimedean Property, we know there exists  $n > \frac{x-\epsilon}{a}$ . Since  $\epsilon > a$ , the gap  $\frac{x}{a} - \frac{x-\epsilon}{a} = \frac{\epsilon}{a} > 1$ , and so there exists such natural number  $n$  within the interval. Thus, we get

$$0 = x - a \cdot \frac{x}{a} < x - na < x - a \cdot \frac{x-\epsilon}{a} = \epsilon.$$

□

## Problem 2

Let  $a, b, c, d \in \mathbb{R}$  and assume  $a < b$  and  $c < d$ . Give an explicit one-to-one correspondence between

1. The points of the two open intervals  $(a, b)$  and  $(c, d)$ .

*Proof.* Define  $f : (a, b) \rightarrow (c, d)$  to be  $f(x) = \frac{(d-c)x + (cb-ad)}{b-a}$ . Let  $l, m \in (a, b)$ . Since  $a < l < b$ ,

$$\begin{aligned} \frac{d-c}{b-a}a &< \frac{d-c}{b-a}l < \frac{d-c}{b-a}b \\ \frac{(d-c)a + (cb-ad)}{b-a} &< \frac{(d-c)l + (cb-ad)}{b-a} < \frac{(d-c)b + (cb-ad)}{b-a} \\ c &< f(l) < d. \end{aligned}$$

Suppose that  $l = m$ . Then

$$\frac{(d-c)l + (cb-ad)}{b-a} = f(l) = f(m) = \frac{(d-c)m + (cb-ad)}{b-a},$$

and so  $f$  is well defined.

Suppose  $f(l) = f(m)$ . Then,

$$\begin{aligned} \frac{(d-c)l + (cb-ad)}{b-a} &= \frac{(d-c)m + (cb-ad)}{b-a} \\ (d-c)l + (cb-ad) &= (d-c)m + (cb-ad) \\ (d-c)l &= (d-c)m \\ l &= m. \end{aligned}$$

Thus,  $f$  is injective.

Let  $y \in (c, d)$ . There exists  $x = \frac{(b-a)y - (cb-ad)}{d-c} \in (a, b)$  such that  $f(x) = y$ , and so  $f$  is surjective.

Thus,  $f$  is an one-to-one correspondence. □

2. The points of the two closed intervals  $[a, b]$  and  $[c, d]$ .

*Proof.* Define  $f : [a, b] \rightarrow [c, d]$  to be  $f(x) = \frac{(d-c)x + (cb-ad)}{b-a}$ . Let  $l, m \in [a, b]$ . Since  $a \leq l \leq b$ ,

$$\begin{aligned} \frac{d-c}{b-a}a &\leq \frac{d-c}{b-a}l \leq \frac{d-c}{b-a}b \\ \frac{(d-c)a + (cb-ad)}{b-a} &\leq \frac{(d-c)l + (cb-ad)}{b-a} \leq \frac{(d-c)b + (cb-ad)}{b-a} \\ c &\leq f(l) \leq d. \end{aligned}$$

Suppose that  $l = m$ . Then

$$\frac{(d-c)l + (cb-ad)}{b-a} = f(l) = f(m) = \frac{(d-c)m + (cb-ad)}{b-a},$$

and so  $f$  is well defined.

Suppose  $f(l) = f(m)$ . Then,

$$\begin{aligned} \frac{(d-c)l + (cb-ad)}{b-a} &= \frac{(d-c)m + (cb-ad)}{b-a} \\ (d-c)l + (cb-ad) &= (d-c)m + (cb-ad) \\ (d-c)l &= (d-c)m \\ l &= m. \end{aligned}$$

Thus,  $f$  is injective.

Let  $y \in [c, d]$ . There exists  $x = \frac{(b-a)y-(cb-ad)}{d-c} \in [a, b]$  such that  $f(x) = y$ , and so  $f$  is surjective.

Thus,  $f$  is an one-to-one correspondence.  $\square$

3. The points of the closed interval  $[a, b]$  and the open interval  $(c, d)$ .

*Proof.* Define  $f : [a, b] \rightarrow (c, d)$  to be

$$f(x) = \begin{cases} c + \frac{d-c}{n+2}, & x = a + \frac{b-a}{n}, n \in \mathbb{N} \\ \frac{c+d}{2}, & x = a \\ \frac{(d-c)x+(cb-ad)}{b-a}, & \text{otherwise.} \end{cases}$$

Note that the product of  $f$  of different cases would not be equal.

Obviously,  $f(x) \in (c, d)$  for all  $x \in [a, b]$ . Let  $k, m \in [a, b]$ . If  $k = m = a$ , then  $f(k) = f(m) = \frac{c+d}{2}$ . If  $k = m = a + \frac{b-a}{n}$ , for some  $n \in \mathbb{N}$ , then  $f(k) = f(m) = c + \frac{d-c}{n+2}$ . Otherwise,  $\frac{(d-c)k+(cb-ad)}{b-a} = \frac{(d-c)m+(cb-ad)}{b-a}$ , which implies that  $f(k) = f(m)$ . Therefore,  $f$  is well defined.

Suppose that  $f(k) = f(m)$ . If  $f(k) = f(m) = \frac{c+d}{2}$ , then  $k = m = a$ . If  $f(k) = f(m) = c + \frac{d-c}{n+2}$  for some  $n \in \mathbb{N}$ , then  $k = m = a + \frac{b-a}{n}$ . If  $f(k) = \frac{(d-c)k+(cb-ad)}{b-a} = \frac{(d-c)m+(cb-ad)}{b-a} = f(m)$ , Then  $k = m$ , by the results we obtained from previous parts. Thus,  $f$  is injective.

Let  $y \in (c, d)$ . There exists

$$x = \begin{cases} a + \frac{b-a}{n}, & y = c + \frac{d-c}{n+2}, n \in \mathbb{N} \\ a, & y = \frac{c+d}{2} \\ \frac{(b-a)x+(ad-cb)}{d-c}, & \text{otherwise,} \end{cases}$$

such that  $f(x) = y$ . Thus,  $f$  is surjective.

Therefore,  $f$  is bijective.  $\square$

4. The points of the closed interval  $[a, b]$  and  $\mathbb{R}$

*Proof.* Consider  $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$  and  $\tan^{-1} : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ . Since  $\tan$  and  $\tan^{-1}$  are inverses of each other, they are bijective. We can then use the function  $f$  we defined in part 3 to get a bijective mapping from  $[a, b]$  to  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . Thus, we get a bijection  $(\tan \circ f) : [a, b] \rightarrow \mathbb{R}$ ,

$$(\tan \circ f)(x) = \begin{cases} \tan(-\frac{\pi}{2} + \frac{\pi}{n+2}), & x = a + \frac{b-a}{n}, n \in \mathbb{N} \\ 0, & x = a \\ \tan(\frac{2\pi x - \pi(b+a)}{2(b-a)}), & \text{otherwise.} \end{cases}$$

$\square$

### Problem 3

Fix  $b > 1, y > 0$ , and prove that there is a unique real  $x$  such that  $b^x = y$ .

*Proof.* We first show that for any positive integer  $n$ ,  $b^n - 1 \geq n(b - 1)$ . We show that  $b^n > 1$  by induction on  $n$ . We already know  $b > 1$ . For  $n > 1$ ,  $b^n = b \cdot b^{n-1} > 1$ , since  $b^{n-1} > 1$  by induction. Thus,

$$b^n - 1 = (b - 1)(b^{n-1} + \cdots + b + 1) \geq (b - 1)n. \quad (1)$$

By Theorem 1.21, we know that there exists a unique  $a \in \mathbb{R}^+$  such that  $a^n = b$ . Suppose that  $a \leq 1$ . We show that  $a^n \leq 1$  by induction on  $n$ . For  $n > 1$ , we know that  $a^n = a \cdot a^{n-1} \leq 1$ , since  $a^{n-1} \leq 1$  by induction. Thus,  $a$  must be greater than 1. Then, by (1), we know that  $b - 1 = a^n - 1 \geq (a - 1)n = (b^{\frac{1}{n}} - 1)n$ .

Let  $t > 1$ . Suppose that  $n > \frac{b-1}{t-1}$ , then  $nt - n > b - 1$ . Note that we know there exists  $n > \frac{b-1}{t-1}$  by the Archimedean Property. Since  $n \geq 1$ , we know  $t > b$ . Note that since  $a^n > 1$  for all  $n \in \mathbb{N}$ ,  $b = b^{\frac{1}{n}} \cdot a^{n-1} \geq b^{\frac{1}{n}}$ . Thus, we get

$$t > b \geq b^{\frac{1}{n}}. \quad (2)$$

Let  $w \in \mathbb{R}$ . Suppose that  $b^w < y$ . Let  $t = y \cdot b^{-w} > b^w \cdot b^{-w} = 1$ . By (2), there exists  $n > \frac{b-1}{t-1}$ , such that  $t = y \cdot b^{-w} > b^{\frac{1}{n}}$ , and so  $y > b^{w+\frac{1}{n}}$ . Suppose that  $b^w < y$ . Let  $t = b^w y^{-1}$ . Similarly, there exists  $n > \frac{b-1}{t-1}$ , such that  $t = b^w y^{-1} > b^{\frac{1}{n}}$ , and so  $b^{w-\frac{1}{n}} > y$ .

Let  $A$  be the set of all  $w$  such that  $b^w < y$ . We will show that  $x = \sup A$  satisfies  $b^x = y$ . Suppose for the sake of contradiction that  $b^x < y$ . Then, by the result we obtained above, we know there exists a large enough  $n \in \mathbb{N}$ , such that  $b^x < b^{x+\frac{1}{n}} < y$ . This implies that there exists  $x + \frac{1}{n} \in A$ , which contradicts that  $x = \sup A$ . Suppose for the sake of contradiction that  $b^x > y$ . Then, by the result we obtained above, there exists a large enough  $n \in \mathbb{N}$ , such that  $b^x > b^{x-\frac{1}{n}} > y$ , contradicting the fact that  $x = \sup A$ . Thus,  $b^x = y$ .

Suppose that  $b^z = b^x = y$ .  $x \not\prec z$ , otherwise  $b^z < b^x$ , contradiction. Similarly, we also know  $x \not\succ z$ . Therefore,  $x$  is unique.  $\square$

## Problem 4

If  $x, y$  are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

*Proof.* We square both sides. On the right-hand-side, we have

$$\begin{aligned} |x - y|^2 &= (x - y)\overline{(x - y)} \\ &= |x|^2 + |y|^2 - y\bar{x} - x\bar{y} \end{aligned}$$

Note that  $\overline{x\bar{y}} = y\bar{x}$ , so  $y\bar{x} + x\bar{y} = 2\operatorname{Re}(x\bar{y})$ . On the left-hand-side, we have

$$\begin{aligned} (|x| - |y|)^2 &= |x|^2 + |y|^2 - 2|x||y| \\ &= |x|^2 + |y|^2 - 2|x||\bar{y}| \\ &= |x|^2 + |y|^2 - 2|x\bar{y}| \end{aligned}$$

Since  $\operatorname{Re}(x\bar{y}) \leq |x\bar{y}|$ ,

$$\begin{aligned} |x|^2 + |y|^2 - 2|x\bar{y}| &\leq |x|^2 + |y|^2 - 2\operatorname{Re}(x\bar{y}) \\ &= |x|^2 + |y|^2 - y\bar{x} - x\bar{y}, \end{aligned}$$

and thus  $||x| - |y|| \leq |x - y|$ . □

## Problem 5

If  $z$  is a complex number such that  $|z| = 1$ , that is, such that  $z\bar{z} = 1$ , compute

$$|1 + z|^2 + |1 - z|^2$$

*Proof.*

$$\begin{aligned} |1 + z|^2 + |1 - z|^2 &= (1 + z)\overline{(1 + z)} + (1 - z)\overline{(1 - z)} \\ &= 1 + z + \bar{z} + z\bar{z} + 1 - z - \bar{z} + z\bar{z} \\ &= 4. \end{aligned}$$

□

## Problem 6

Prove that

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

if  $x \in \mathbb{R}^k$  and  $y \in \mathbb{R}^k$ . Interpret this geometrically, as a statement about parallelograms.

*Proof.*

$$\begin{aligned} |x + y|^2 + |x - y|^2 &= |x|^2 + |y|^2 + 2x \cdot y + |x|^2 + |y|^2 - 2x \cdot y \\ &= 2|x|^2 + 2|y|^2. \end{aligned}$$

Interpreting geometrically, if  $x, y$  were the neighboring sides of a parallelogram, then  $x + y$  and  $x - y$  are its diagonals. Thus, the equation suggests that the sum of the squares of the sides is equal to the sum of the square of the diagonals.  $\square$