C3.8 Combinatorics: Sheet #4

Due on Jan 14, 2026 at 12:00pm

Professor Goon

Ray Tsai

Problem 1

Show that, for some $c > 1$ and every $n \geq 5$, there is a family $\mathcal{F} \subset \mathcal{P}[n]$ of size at least c^n such that every set in $\mathcal{F}$ has odd size, and the intersection of any two distinct sets from $\mathcal{F}$ has odd size.

Proof. Put $m = \lceil n/2 \rceil - 1 \geq n/4$. Define

$$\mathcal{F} = \left\{ \{1\} \cup \bigcup_{i \in S} \{2i\} \cup \{2i+1\} : S \subseteq [m] \right\}.$$

Then $|\mathcal{F}| = 2^m \geq 2^{n/4}$ and for any A, B, there are correpsonding $S_A, S_B \subseteq [m]$ such that

$$A \cap B = \{1\} \cup \bigcup_{i \in S_A \cap S_B} \{2i\} \cup \{2i+1\}.$$

$\square$

Problem 2

Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}[n]$ be two set systems such that $|A \cap B|$ is even for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $|\mathcal{A}| \cdot |\mathcal{B}| \leq 2^n$. Can you describe the pairs $\mathcal{A}, \mathcal{B}$ for which we have equality?

[Hint: Show that if $A, A' \in \mathcal{A}$ then we may assume $A \Delta A' \in \mathcal{A}$.]

Proof. We first note that if $A, A' \in \mathcal{A}$, then $|A \Delta A'| = |A| + |A'| - 2|A \cap A'|$ is even. Thus we may assume $A \Delta A' \in \mathcal{A}$ for all $A, A' \in \mathcal{A}$. We work over $\mathbb{F}_2$. For $A \in \mathcal{A}$ and $B \in \mathcal{B}$, let $\chi_A, \chi_B \in \mathbb{F}_2^n$ and $\chi_A(i) = 1$ or $\chi_B(i) = 1$ if and only if $i \in A$ or $i \in B$, respectively. Let $V = \{\chi_A : A \in \mathcal{A}\}$ and $W = \{\chi_B : B \in \mathcal{B}\}$. Since $\chi_A + \chi_{A'} = \chi_{A \Delta A'}$ for all $A, A' \in \mathcal{A}$, we have that V is a linear subspace of $\mathbb{F}_2^n$. Similarly, W is a linear subspace of $\mathbb{F}_2^n$. Since $\langle \chi_A, \chi_B \rangle = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have that W is the orthogonal complement V. But then $\dim(V) + \dim(W) \leq n$, so $|\mathcal{A}| \cdot |\mathcal{B}| \leq 2^n$. Thus if $\mathcal{A}, \mathcal{B}$ achieves the bound, then V and W are linear subspaces of $\mathbb{F}_2^n$ of dimension $n/2$ and $W = V^{\perp}$. $\square$

Problem 3

Let P be a set of n points in the plane that do not all lie on a straight line. Prove that they determine at least n lines. [Hint: For each point, consider the set of lines that passes through it.]

Proof. Let $\mathcal{L} = \{L_1, \ldots, L_m\}$ be the set of lines determined by points in P. For $x \in P$, let $A_x = \{i \in [m] : x \in L_i\}$. Then $|A_x \cap A_{x'}| = 1$ for $x \neq x'$. Note that $\mathcal{A} = \{A_x : x \in P\} \subseteq [m]$. Thus, by Fisher's inequality, $n = |\mathcal{A}| \leq m$. This completes the proof. $\square$

Problem 4

Prove that a non-trivial decomposition of the edges of K_n into edge-disjoint complete subgraphs requires at least n subgraphs. Show how this bound can be achieved.

[Hint: Consider the set of cliques that contain a given vertex.]

Proof. Let $G_1, \ldots, G_m$ be a non-trivial decomposition of the edges of K_n into edge-disjoint complete subgraphs. For $x \in [n]$, let $A_x = \{i \in [m] : x \in A_i\}$. Note that $A_x \subseteq [m]$ and $|A_x \cap A_y| = 1$ for $x \neq y$. Since the decomposition is non-trivial, $|A_x| \geq 2$ for all x, and so each A_x is distinct. It now follows from Fisher's inequality that $n = |\{A_x\}_{x \in [n]}| \leq m$.

To see how this bound can be achieved, let G_1 be the clique induced by $[n] \setminus \{1\}$, and for $[n] \setminus \{1\}$ let G_i be the cliques of size 2 induced by 1 and i. $\square$

Problem 5

A set P in $\mathbb{R}^n$ is a *two-distance set* if there are positive real numbers α, β such that $||x-y||_2 \in \{\alpha, \beta\}$ for all distinct $x, y \in P$. Let $P = \{p_1, \dots, p_k\}$ be a two-distance set.

1. For each $i \in [k]$, let f_i be the polynomial in variables $x = (x_1, \dots, x_n)$ defined by

$$f_i(x) = (||x-p_i||_2^2 - \alpha^2)(||x-p_i||_2^2 - \beta^2).$$

   Show that the polynomials f_i are linearly independent. [Hint: Consider $f_i(p_j)$.]

   Proof. Suppose $f = \sum_{i=1}^k \lambda_i f_i = 0$ where $\lambda_i \in \mathbb{R}$. Notice $f_i(p_i) = \alpha^2\beta^2$ and $f_i(p_j) = 0$ for $i \neq j$. Thus $f(p_i) = \alpha^2\beta^2\lambda_i = 0$ for all i, and so $\lambda_i = 0$ for all i. Thus the polynomials f_i are linearly independent. □

2. Deduce that $k \leq \binom{n}{2} + 3n + 2$. [Hint: Find a basis for the space spanned by the polynomials f_i.]

   Proof. For $q \in \mathbb{R}^n$, write

$$\|x-q\|_2^2 - \alpha^2 = \|x\|_2^2 - 2\sum_{i=1}^k q_i x_i + \|q\|_2^2 - \alpha^2,$$

   and so $\|x-q\|_2^2 - \alpha^2$ is a linear combination of $\|x\|_2^2, \{x_i\}_{i\in[n]}$, and 1. Hence, each f_i can be written as a linear combination of

$$\|x\|_2^4, \{\|x\|_2^2 x_i\}_{i\in[n]}, \{x_i x_j\}_{i,j\in[n]}, \{x_i\}_{i\in[n]}, 1.$$

   Thus the span of $\{f_i\}_{i\in[k]}$ has dimension at most $1 + n + n^2 + n + 1 = \binom{n}{2} + 3n + 2$ above. That is,

$$k = |\{f_i\}_{i\in[k]}| \leq \binom{n}{2} + 3n + 2.$$

   □

Problem 6

Let $\mathcal{F}$ be a collection of functions from $[n]$ to $\mathbb{Z}$. Suppose that, for every pair of distinct functions $f, g \in \mathcal{F}$ we have $f(i) = g(i) + 1$ for some i. Prove that $|\mathcal{F}| \leq 2^n$.

[Hint: Look for a suitable collection of polynomials.]

Proof. In $\mathbb{Z}[x_1, \ldots, x_n]$, define

$$p_f(x_1, \ldots, x_n) = \prod_{i \in [n]} (x_i - f(i) - 1),$$

for $f \in \mathcal{F}$. We show that $\{\{p_f\}_{f \in \mathcal{F}}\}$ is linearly independent. Suppose $P = \sum_{f \in \mathcal{F}} \lambda_f p_f = 0$ where $\lambda_f \in \mathbb{Q}$. Let $g \in \mathcal{F}$. Then

$$P(g(1), \ldots, g(n)) = \lambda_g p_g(g(1), \ldots, g(n)) = 0.$$

But then $p_g(g(1), \ldots, g(n)) \neq 0$, so $\lambda_g = 0$ for all $g \in \mathcal{F}$. Thus the polynomials p_f are linearly independent in $\mathbb{Q}[x_1, \ldots, x_n]$. Since each p_f can be written is a linear combination of $\left\{\prod_{i \in S} x_i\right\}_{S \subseteq [n]}$,

$$|\mathcal{F}| = |\{p_f\}_{f \in \mathcal{F}}| \leq 2^n.$$

$\square$