

# MATH 188: Homework #1

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We first give a proof for the general closed form of homogeneous linear recurrence relations for later use:

*Proof.* Given

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$$

for  $n \geq d$ , and the values of  $a_n$ ,  $n < d$ , define  $A(x) = \sum_{n \geq 0} a_n x^n$ . Then, we have

$$\begin{aligned} A(x) &= a_0 + a_1 x + \cdots + a_{d-1} x^{d-1} + \sum_{n \geq d} (c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}) x^n \\ &= \sum_{n=0}^{d-1} a_n x^n + c_1 x \sum_{n \geq d-1} a_n x^n + c_2 x^2 \sum_{n \geq d-2} a_n x^n + \cdots + c_d x^d \sum_{n \geq 0} a_n x^n \\ &= \sum_{n=0}^{d-1} a_n x^n + c_1 x \left( A(x) - \sum_{n=0}^{d-2} a_n x^n \right) + c_2 x^2 \left( A(x) - \sum_{n=0}^{d-3} a_n x^n \right) + \cdots + c_d x^d A(x) \end{aligned}$$

Rearranged,

$$A(x) = \frac{p(x)}{1 - c_1 x - c_2 x^2 - \cdots - c_d x^d},$$

for some polynomial  $p(x)$  of degree less than  $d$ . We may assume  $1 - c_1 x - c_2 x^2 - \cdots - c_d x^d$  has factorization  $(1 - r_1 x)^{w_1} (1 - r_2 x)^{w_2} \cdots (1 - r_s x)^{w_s}$ . By applying partial fraction decomposition,

$$A(x) = \sum_{m=1}^{w_1} \frac{p_{1,m}(x)}{(1 - r_1 x)^m} + \sum_{m=1}^{w_2} \frac{p_{2,m}(x)}{(1 - r_2 x)^m} + \cdots + \sum_{m=1}^{w_s} \frac{p_{s,m}(x)}{(1 - r_s x)^m},$$

where  $p_{i,m}(x)$  is some polynomial of degree less than  $m$ . Note that for  $k < m$ ,

$$\frac{x^k}{(1 - r_i x)^m} = \sum_{n \geq 0} \binom{m+n-1}{n} r_i^n x^{n+k} = \sum_{n \geq k} \left( \binom{m+n-k-1}{n-k} r_i^{-k} \right) r_i^n x^n,$$

By the binomial theorem. But then

$$\binom{m+n-k-1}{n-k} r_i^{-k} = \frac{r_i^{-k}}{(m-1)!} (m+n-k-1) \cdots (n-k+2)(n-k+1),$$

so the each coefficient of  $r_i^n$  is really just a polynomial of  $n$  with degree less than  $m-1$ . It follows that for  $n \geq d$ , the recurrence relation has a closed form

$$a_n = \sum_{i=1}^s f_i(n) r_i^n,$$

for some polynomial  $f_i(n)$  of degree less than  $w_i$ . □

## Problem 1

Find a closed formula for the following recurrence relation:

$$\begin{aligned} a_0 &= 1, & a_1 &= 1, & a_2 &= 2, \\ a_n &= 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \quad (n \geq 3). \end{aligned}$$

*Proof.* The characteristic polynomial of this recurrence relation is defined to be

$$t^3 - 5t^2 + 8t - 4 = (t - 1)(t - 2)^2,$$

which has roots  $t = 1, 2$ . Note that 2 is a repeated root, and thus

$$a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 n 2^n.$$

Solving the system of equations

$$\begin{cases} 1 = \alpha_1 + \alpha_2 \\ 1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ 2 = \alpha_1 + 4\alpha_2 + 8\alpha_3 \end{cases},$$

we get

$$a_n = 2 - 2^n + n2^{n-1}.$$

□

## Problem 2

Let  $r_1, \dots, r_d$  be distinct numbers. Show that the determinant of the  $d \times d$  matrix  $(r_i^{j-1})_{i,j=1,\dots,d}$  is nonzero (interpret  $0^0 = 1$ ). Explain why this implies that the sequences  $(r_1^n)_{n \geq 0}, \dots, (r_d^n)_{n \geq 0}$  are linearly independent.

*Proof.* Given numbers  $x_1, x_2, \dots, x_d$ , define

$$M(x_1, x_2, \dots, x_d) = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{d-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_d & x_d^2 & \cdots & x_d^{d-1} \end{bmatrix}.$$

We first show by induction on  $d$  that,

$$\det M(x_1, x_2, \dots, x_d) = \prod_{1 \leq i < j \leq d} (x_j - x_i),$$

for all  $d \geq 2$ . We already know.

$$\det M(x_1, x_2) = \det \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} = x_2 - x_1.$$

Suppose  $d > 2$ . Note that the determinant remains the same after subtracting to each column the preceding column scaled by  $x_1$ . Hence,

$$\begin{aligned} \det M(x_1, x_2, \dots, x_d) &= \det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{d-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_d & x_d^2 & \cdots & x_d^{d-1} \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & x_2(x_2 - x_1) & \cdots & x_2^{d-2}(x_2 - x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_d - x_1 & x_d(x_d - x_1) & \cdots & x_d^{d-2}(x_d - x_1) \end{bmatrix} \\ &= \det \begin{bmatrix} x_2 - x_1 & x_2(x_2 - x_1) & \cdots & x_2^{d-2}(x_2 - x_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_d - x_1 & x_d(x_d - x_1) & \cdots & x_d^{d-2}(x_d - x_1) \end{bmatrix}. \end{aligned}$$

Since the entries of  $i$ th row share a common factor  $(x_{i+1} - x_1)$ , we may extract them from the determinant and get

$$\begin{aligned} \det M(x_1, x_2, \dots, x_d) &= \left( \prod_{i=1}^{d-1} (x_{i+1} - x_1) \right) \det M(x_2, x_2, \dots, x_d) \\ &= \left( \prod_{i=1}^{d-1} (x_{i+1} - x_1) \right) \left( \prod_{2 \leq i < j \leq d} (x_j - x_i) \right) = \prod_{1 \leq i < j \leq d} (x_j - x_i), \end{aligned}$$

by induction. Since all  $r_i$ 's are distinct,

$$\det M(r_1, r_2, \dots, r_d) = \prod_{1 \leq i < j \leq d} (r_j - r_i) \neq 0.$$

But then  $(r_1^n)_{0 \leq n < d}, \dots, (r_d^n)_{0 \leq n < d}$  are linearly independent, so  $(r_1^n)_{n \geq 0}, \dots, (r_d^n)_{n \geq 0}$  are also linearly independent. (Source cited: [wikipedia.org/wiki/Vandermonde\\_matrix](https://en.wikipedia.org/wiki/Vandermonde_matrix))  $\square$

### Problem 3

Let  $(a_n)_{n \geq 0}$  be a sequence satisfying a linear recurrence relation whose characteristic polynomial is  $(t^2 - 1)^d$ . Show that there exist polynomials  $p(n)$  and  $q(n)$  of degree  $\leq d - 1$  such that

$$a_n = \begin{cases} p(n) & \text{if } n \text{ is even} \\ q(n) & \text{if } n \text{ is odd} \end{cases}.$$

*Proof.* Since  $(t^2 - 1)^d = (t - 1)^d(t + 1)^d$ ,

$$\begin{aligned} a_n &= \alpha_0 + \alpha_1 n + \cdots + \alpha_{d-1} n^{d-1} + (-1)^n (\beta_0 + \beta_1 n + \cdots + \beta_{d-1} n^{d-1}) \\ &= \begin{cases} \sum_{k=0}^{d-1} (\alpha_k + \beta_k) n^k & \text{if } n \text{ is even} \\ \sum_{k=0}^{d-1} (\alpha_k - \beta_k) n^k & \text{if } n \text{ is odd} \end{cases}. \end{aligned}$$

The result follows by taking  $p(n) = \sum_{0 \leq k \leq d-1} (\alpha_k + \beta_k) n^k$  and  $q(n) = \sum_{k=0}^{d-1} (\alpha_k - \beta_k) n^k$ . □

## Problem 4

- (a) Suppose that  $(a_n)_{n \geq 0}$  and  $(a'_n)_{n \geq 0}$  both satisfy the same linear recurrence relation of order  $d$  and that they agree in  $d$  consecutive places, i.e., there exists  $k$  such that  $a_k = a'_k$ ,  $a_{k+1} = a'_{k+1}$ , ...,  $a_{k+d-1} = a'_{k+d-1}$ . Show that these sequences are the same.

*Proof.* By assumption,

$$\begin{aligned} a_n &= c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d} \\ a'_n &= c_1 a'_{n-1} + c_2 a'_{n-2} + \cdots + c_d a'_{n-d}, \end{aligned}$$

for some  $c_1, \dots, c_d$ , with  $c_d \neq 0$ . By induction,  $a_n = a'_n$  for all  $n \geq k$ , so it remains to show the equality also holds true for  $n < k$ . Rearranging the equations, we get

$$\begin{aligned} a_n &= \frac{1}{c_d} (a_{n+d} - c_1 a_{n+d-1} - \cdots - c_{d-1} a_{n+1}) \\ a'_n &= \frac{1}{c_d} (a'_{n+d} - c_1 a'_{n+d-1} - \cdots - c_{d-1} a'_{n+1}), \end{aligned}$$

so by induction based on the  $k$  consecutive terms that both sequences agree we get  $a_n = a'_n$  for all  $n < k$ , and this completes the proof.  $\square$

- (b) Suppose that  $(a_n)_{n \geq 0}$  satisfies the linear recurrence relation of order  $d$

$$a_n = c_1 a_{n-1} + \cdots + c_d a_{n-d} \quad \text{for all } n \geq d$$

with  $c_d \neq 0$ . Show that there is a unique sequence  $(b_n)_{n \in \mathbb{Z}}$  (indexed by *all* integers) such that  $b_n = a_n$  for  $n \geq 0$  and such that

$$b_n = c_1 b_{n-1} + \cdots + c_d b_{n-d} \quad \text{for all } n \in \mathbb{Z}. \quad (1)$$

*Proof.* Given  $b_n = a_n$  for  $n \geq 0$ , define

$$b_n = \frac{1}{c_d} (b_{n+d} - c_1 b_{n+d-1} - \cdots - c_{d-1} b_{n+1}), \quad (2)$$

for  $n < 0$ . Rearranging (2), we know  $b_n$  follows (1) for  $n \in \mathbb{Z}$ . Hence, it remains to show the uniqueness of  $(b_n)$ . Suppose there exists  $(b'_n)$  such that  $b'_n = a_n$  for  $n \geq 0$  and satisfies the recurrence relation for all  $n \in \mathbb{Z}$ . We already know  $(b_n)$  and  $(b'_n)$  agree for all nonnegative terms. But then by (2),  $(b_n)$  and  $(b'_n)$  agree with each negative term by backwards induction on negative  $n$  based on the first  $d$  nonnegative terms, so both sequences also agree on the negative terms. Hence,  $(b_n) = (b'_n)$  and we are done.  $\square$

- (c) Consider the Fibonacci sequence  $f_0 = 0$ ,  $f_1 = 1$ , and  $f_n = f_{n-1} + f_{n-2}$ . How does the negatively indexed Fibonacci sequence relate to the usual one?

*Proof.* For  $n < 0$ ,  $f_n$  is defined as

$$f_n = -f_{n+1} + f_{n+2}.$$

Define a new sequence  $(g_n)_{n \geq 0}$  as  $g_n = f_{-n}$ . The characteristic polynomial of  $(g_n)$  is  $t^2 + t - 1$ , which has roots  $r'_1 = \frac{-1+\sqrt{5}}{2}$  and  $r'_2 = \frac{-1-\sqrt{5}}{2}$ . Notice that  $r'_1 = -r_1$  and  $r'_2 = -r_2$ , where  $r_1, r_2$  are the roots of the characteristic polynomial of the Fibonacci sequence. Since  $g_0 = 0$  and  $g_1 = 1$ ,

$$g_n = \frac{1}{\sqrt{5}} ((r'_1)^n + (r'_2)^n) = \frac{(-1)^n}{\sqrt{5}} (r_1^n + r_2^n) = (-1)^n f_n,$$

so  $(g_n)$  is just the alternating Fibonacci sequence.  $\square$

## Problem 5

Let  $A_0(x), A_1(x), \dots$  and  $B_0(x), B_1(x), \dots$  be sequences of formal power series. Assume that  $\lim_{i \rightarrow \infty} A_i(x) = A(x)$  and  $\lim_{i \rightarrow \infty} B_i(x) = B(x)$ .

(a) Prove that  $\lim_{i \rightarrow \infty} (A_i(x) + B_i(x)) = A(x) + B(x)$ .

*Proof.* Note that for any  $n$ , there exists  $N_{a_n}, N_{b_n}$  such that  $[x^n]A_i(x) = [x^n]A(x)$  and  $[x^n]B_i(x) = [x^n]B(x)$ , for all  $i \geq N_n = \max(N_{a_n}, N_{b_n})$ . Hence,

$$[x^n](A_i(x) + B_i(x)) = [x^n]A_i(x) + [x^n]B_i(x) = [x^n]A(x) + [x^n]B(x) = [x^n](A(x) + B(x)),$$

for  $i \geq N_n$ , and the result follows.  $\square$

(b) Prove that  $\lim_{i \rightarrow \infty} (A_i(x)B_i(x)) = A(x)B(x)$ .

*Proof.* Note that for any  $n$ , there exists  $N_{a_n}, N_{b_n}$  such that  $[x^n]A_i(x) = [x^n]A(x)$  and  $[x^n]B_i(x) = [x^n]B(x)$ , for all  $i \geq N_n = \max(N_{a_n}, N_{b_n})$ . Given  $m \geq 0$ , take  $N = \max(N_0, N_1, \dots, N_m)$ . Then,

$$[x^m](A_i(x)B_i(x)) = \sum_{k=0}^m [x^k]A_i(x)[x^{m-k}]B_i(x) = \sum_{k=0}^m [x^k]A(x)[x^{m-k}]B(x) = [x^m](A(x)B(x)),$$

for  $i \geq N$ , and the result follows.  $\square$

## Problem 6

Continuing from Problem 3, how does the statement generalize if the characteristic polynomial is  $(t^k - 1)^d$ ?

*Proof.* Notice  $t^k - 1 = (t - 1)(t - \omega)(t - \omega^2) \dots (t - \omega^{k-1})$ , where  $\omega = e^{\frac{2\pi}{k}}$ . Hence, for  $m = 0, 1, \dots, k-1$ , take  $p_m(n) = \sum_{i=1}^k \omega^{im} \sum_{j=0}^{d-1} \alpha_{i,j} n^j$ , which are polynomials of degree at most  $d-1$ . Then,

$$\begin{aligned} a_n &= \sum_{i=1}^k \omega^{in} \sum_{j=0}^{d-1} \alpha_{i,j} n^j \\ &= \begin{cases} p_0(n) & \text{if } n \equiv 0 \pmod{k} \\ p_1(n) & \text{if } n \equiv 1 \pmod{k} \\ \vdots & \\ p_{k-1}(n) & \text{if } n \equiv k-1 \pmod{k} \end{cases}. \end{aligned}$$

□



## Problem 7

Let  $p$  be a prime number and let  $(a_n)_{n \geq 0}$  be a sequence such that  $a_n \in \mathbb{Z}/p$  and which satisfies a homogeneous linear recurrence relation. Show that the sequence is in fact periodic.

*Proof.* By assumption,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d},$$

for some  $c_1, c_2, \dots, c_d \in \mathbb{Z}/p$ ,  $c_d \neq 0$ . Since there are only  $p^d$  possible strings of length  $d$ , it is guaranteed that some length  $d$  string  $s_d$  repeats in the first  $dp^d$  terms. Suppose that  $s_d$  initially appeared at  $a_k$  and repeated at  $a_{k+l}$ , that is,  $a_k = a_{k+l}, a_{k+1} = a_{k+1+l}, \dots, a_{k+d-1} = a_{k+d-1+l}$ . Note that  $\mathbb{Z}/p$  is closed under taking multiplicative inverse. Hence, by problem 4(a), we have  $(a_n)_{n \geq 0} = (a_{n+l})_{n \geq 0}$ , and thus  $(a_n)_{n \geq 0}$  is periodic.  $\square$

## Problem 8

Let  $r_1, \dots, r_{d-1}$  be distinct numbers. Prove that the sequences  $\alpha_1 = (r_1^n), \dots, \alpha_{d-1} = (r_{d-1}^n), \alpha_d = (nr_{d-1}^{n-1})$  are linearly independent by showing that the determinant of  $(\alpha_{i,j-1})_{i,j=1,\dots,d}$  is nonzero (interpret  $0^0 = 1$  and if  $r_{d-1} = 0$ , interpret  $\alpha_{d,0} = 0$ ).

*Proof.* Given distinct  $d$  numbers  $r_1, r_2, \dots, r_{d-1}$ , define

$$M(r_1, r_2, \dots, r_{d-1}) = \begin{bmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^{d-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} & r_{d-1}^2 & \cdots & r_{d-1}^{d-1} \\ 0 & 1 & 2r_{d-1} & \cdots & dr_{d-1}^{d-1} \end{bmatrix}.$$

We first show by induction on  $d$  that,

$$\det M(r_1, r_2, \dots, r_{d-1}) \neq 0,$$

for any  $d$  distinct numbers,  $d \geq 2$ . We already know.

$$\det M(r_1) = \det \begin{bmatrix} 1 & r_1 \\ 0 & 1 \end{bmatrix} = 1.$$

Suppose we are given distinct  $r_1, \dots, r_{d-1}$ , for  $d > 2$ . Note that the determinant remains the same after subtracting to each column the preceding column scaled by  $r_1$ . Hence,

$$\begin{aligned} \det M(r_1, r_2, \dots, r_{d-1}) &= \det \begin{bmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^{d-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} & r_{d-1}^2 & \cdots & r_{d-1}^{d-1} \\ 0 & 1 & 2r_{d-1} & \cdots & (d-1)r_{d-1}^{d-2} \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & r_2 - r_1 & r_2(r_2 - r_1) & \cdots & r_2^{d-2}(r_2 - r_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} - r_1 & r_{d-1}(r_{d-1} - r_1) & \cdots & r_{d-1}^{d-2}(r_{d-1} - r_1) \\ 0 & 1 & 2r_{d-1} - r_1 & \cdots & (d-1)r_{d-1}^{d-2} - (d-2)r_1r_{d-1}^{d-3} \end{bmatrix} \\ &= \left( \prod_{i=2}^{d-1} (r_i - r_1) \right) \det \begin{bmatrix} 1 & r_2 & \cdots & r_2^{d-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} & \cdots & r_{d-1}^{d-2} \\ 1 & 2r_{d-1} - r_1 & \cdots & (d-1)r_{d-1}^{d-2} - (d-2)r_1r_{d-1}^{d-3} \end{bmatrix} \\ &= \left( \prod_{i=2}^{d-1} (r_i - r_1) \right) \det \begin{bmatrix} 1 & r_2 & r_2^2 & \cdots & r_2^{d-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{d-1} & r_{d-1}^2 & \cdots & r_{d-1}^{d-2} \\ 0 & (r_{d-1} - r_1) & 2(r_{d-1} - r_1)r_{d-1} & \cdots & (d-2)(r_{d-1} - r_1)r_{d-1}^{d-3} \end{bmatrix} \\ &= (r_{d-1} - r_1) \left( \prod_{i=2}^{d-1} (r_i - r_1) \right) \det M(r_2, r_3, \dots, r_{d-1}). \end{aligned}$$

But then all  $r_i$ 's are distinct, so  $\det M(r_1, r_2, \dots, r_{d-1}) \neq 0$ , by induction. The induction result implies that  $(r_1^n)_{0 \leq n < d}, \dots, (r_{d-1}^n)_{0 \leq n < d}, (nr_{d-1}^{n-1})_{0 \leq n < d}$  are linearly independent, so  $(r_1^n)_{n \geq 0}, \dots, (r_{d-1}^n)_{n \geq 0}, (nr_{d-1}^{n-1})_{n \geq 0}$  are also linearly independent.  $\square$