SC9 Probability on Graphs and Lattices: Sheet #2

Due on November 8, 2025 at 12:00pm

Professor C. Goldschmidt and Professor J. Jorritsma

Ray Tsai

Problem 1

Prove that $p_c(1) = 1$.

Proof. Let $p \in [0,1]$. If $0 \leftrightarrow \infty$, then for any $n \in \mathbb{Z}_{\geq 0}$ we have $0 \leftrightarrow n$ or $0 \leftrightarrow -n$. Notice

$$\mathbb{P}_p(0 \leftrightarrow n) = \mathbb{P}_p(0 \leftrightarrow -n) = p^n.$$

Thus,

$$\theta(p) = \lim_{n \to \infty} p^n = \begin{cases} 0 & \text{if } p < 1 \\ 1 & \text{if } p = 1 \end{cases}.$$

It now follows that $p_c(1) = \sup\{p \in [0,1] : \theta(p) = 0\} = 1$. $\qquad\square$

Problem 2

An event $A \subset \Omega_G$ is *decreasing* if whenever $\omega \leq \omega'$ and $\omega' \in A$, we also have $\omega \in A$. A function $f : \Omega_G \to \mathbb{R}$ is *decreasing* if $f(\omega) \geq f(\omega')$ whenever $\omega \leq \omega'$.

(a) Fix a vertex $v \in V(G)$. Are the following events and variables/functions increasing, decreasing, or neither?

   (i) $|C(v)|$

      Proof. Increasing, as including more edges does not decrease the size of the largest cluster. $\square$

  (ii) $\{\omega$ has no cycles$\}$

      Proof. Decreasing, as removing edges would not create a cycle. $\square$

  (iii) $\{v$ lies in the largest cluster or $|C(v)| = \infty\}$

      Proof. Neither, as adding or removing edges might cause another cluster that does not contain v to become the largest. $\square$

  (iv) $\{\omega$ forms a spanning tree of $G\}$

      Proof. Niether. Adding edges might create a cycle, and removing edges might disconnect the component. $\square$

(b) Prove that the Harris inequality holds for bounded decreasing functions $f, g : \Omega_G \to \mathbb{R}$, i.e.

$$\mathbb{E}_p[fg] \geq \mathbb{E}_p[f]\mathbb{E}_p[g].$$

Proof. Let $f' = -f$ and $g' = -g$. Then f' and g' are increasing, so by the Harris inequality for increasing functions,

$$\mathbb{E}_p[fg] = \mathbb{E}_p[f'g'] \geq \mathbb{E}_p[f']\mathbb{E}_p[g'] = \mathbb{E}_p[f]\mathbb{E}_p[g].$$

$\square$

(c) Suppose f is a bounded increasing function, and A a decreasing event, with $\mathbb{P}_p(A) > 0$. Show that

$$\mathbb{E}_p[f|A] \leq \mathbb{E}_p[f].$$

Proof. Let $f' = -f$ and $g = \mathbb{1}_A$. Then f' and g are decreasing. By (b) we have

$$\mathbb{E}_p[f'g] \geq \mathbb{E}_p[f']\mathbb{P}_p(A).$$

Rearranging and replacing f' with $-f$, we have

$$E_p[f|A] = \frac{\mathbb{E}_p[fg]}{\mathbb{P}_p(A)} \leq \mathbb{E}_p[f].$$

$\square$

Problem 3

Given $k \geq 2$ increasing events $A_1, \ldots, A_k \subset \Omega_G$, show that

$$\max_{1 \leq i \leq k} \mathbb{P}_p(A_i) \geq 1 - (1 - \mathbb{P}_p(A_1 \cup \cdots \cup A_k))^{1/k} \, .$$

Proof. By induction and the Harris inequality for decreasing events, we have

$$1 - \mathbb{P}_p(A_1 \cup \cdots \cup A_k) = \mathbb{P}_p(A_1^C \cap \cdots \cap A_k^C) \geq \mathbb{P}_p(A_1^C) \cdots \mathbb{P}_p(A_k^C) = \prod_{i=1}^{k}(1 - \mathbb{P}_p(A_i)) \geq \left(1 - \max_{1 \leq i \leq k} \mathbb{P}_p(A_i)\right)^k \, .$$

Rearranging now yields the result. $\qquad \square$

Problem 4

Let $\mathbb{T}_\infty$ be the infinite rooted binary tree, with root ρ, which is exhausted by the sequence $(\mathbb{T}_n)$, where $\mathbb{T}_n$ is the depth-n binary tree, consisting of all vertices in $\mathbb{T}_\infty$ at graph distance at most n from ρ. In lectures, we showed that the FUSF on $\mathbb{T}_\infty$ is $\mathbb{T}_\infty$ itself.

(a) Suppose that $(X_k)_{k \geq 0}$ is a SRW on $\mathbb{T}_\infty$. Let $D_k = d(\rho, X_k)$, where d denotes the graph distance. Describe the process $(D_k)_{k \geq 0}$, and find

$$\theta := \mathbb{P}(D_k > 1 \text{ for all } k \geq 1 \mid D_0 = 1).$$

What is $\mathbb{P}(D_k > m \text{ for all } k \geq 1 \mid D_0 = m)$ for $m > 1$?

Proof. Note that (D_n) is a simple random walk on $\mathbb{Z}_{\geq 0}$ with up probability $\frac{2}{3}$, down probability $\frac{1}{3}$, and reflection at 0. Put $q = \mathbb{P}(\text{SRW hits } 1 \mid D_0 = 2)$. Then,

$$\theta = \frac{2}{3}q, \quad \text{and} \quad q = \frac{1}{3} + \frac{2}{3}\mathbb{P}(\text{SRW hits } 1 \mid D_0 = 3).$$

But then by the strong Markov property and translation invariance,

$$\mathbb{P}(\text{SRW hits } 1 \mid D_0 = 3) = \mathbb{P}(\text{SRW hits } 2 \mid D_0 = 3) \cdot \mathbb{P}(\text{SRW hits } 1 \mid D_0 = 2) = q^2.$$

Thus, $q = \frac{1}{3} + \frac{2}{3}q^2$, which has the solution $\frac{1}{2}$. Thus by translation invariance,

$$\mathbb{P}(D_k > m \text{ for all } k \geq 1 \mid D_0 = m) = \theta = \frac{1}{3}.$$

$\square$

(b) Now write $\mathbb{T}_n^W$ for the wired version of $\mathbb{T}_n$, with wiring vertex w_n. Let one of the two neighbours of ρ be α. Prove that

$$\liminf_{n \to \infty} \mathbb{P}(\text{SRW on } \mathbb{T}_n^W \text{ started from } \rho \text{ hits } w_n \text{ before } \alpha) > 0. \tag{1}$$

Proof. Let β be the other child of ρ. Let $(X_k)_{k \geq 0}$ be the SRW on $\mathbb{T}_n$, and let $\tau_n = \inf\{k \geq 0 : D_k = n\}$. Then by the strong Markov property and translation invariance,

$$\mathbb{P}(\text{SRW hits } w_n \text{ before } \alpha \mid X_0 = \rho) = \mathbb{P}(X_1 = \beta \mid X_0 = \rho) \cdot \mathbb{P}(\text{SRW doesn't hit } \alpha \text{ before } \tau_{n+1} \mid X_0 = \beta) \geq \frac{1}{2}\theta.$$

The result now follows by taking lim inf as $n \to \infty$. $\square$

(c) By considering also a version of (1) for a SRW started from α, and using Wilson's algorithm on $\mathbb{T}_n^W$ with initial vertex w_n (or otherwise), prove that the edge $\{\rho, \alpha\}$ is not μ^W-almost surely present, and thus FUSF $\neq$ WUSF for $\mathbb{T}_\infty$.

Proof. By (a) we have

$$\mathbb{P}(\text{SRW on } \mathbb{T}_n^W \text{ started from } \alpha \text{ hits } w_n \text{ before } \rho) \geq \frac{1}{3}.$$

Now consider running Wilson's algorithm on $\mathbb{T}_n^W$ with an enumeration of the vertices $w_n, \rho, \alpha, \ldots$. Then the probability that the edge $\{\rho, \alpha\}$ is not present is

$$\mathbb{P}(\text{SRW on } \mathbb{T}_n^W \text{ started from } \rho \text{ hits } w_n \text{ before } \alpha) \cdot \mathbb{P}(\text{SRW on } \mathbb{T}_n^W \text{ started from } \alpha \text{ hits } w_n \text{ before } \rho).$$

By the previous results, taking the limit as $n \to \infty$ yields

$$\liminf_{n \to \infty} \mathbb{P}(\{\rho, \alpha\} \notin E(\mathbb{T}_n^W)) > 0.$$

The result now follows. $\square$

Problem 5

Let σ_n be the number of self-avoiding walks started from the origin in $\mathbb{Z}^d$ with length n.

(a) Prove that $\sigma_n \geq d^n$.

Proof. Consider any simple walks that only move in the positive direction. Since there are no cycles in such walks, they are self-avoiding walks. There are d^n such walks, and so $\sigma_n \geq d^n$.   ☐

(b) Explain why $\sigma_{m+n} \leq \sigma_m \sigma_n$.

Proof. Consider the set of all concatenations of two self-avoiding walks of length m and n. Then the set of self-avoiding walks of length $m + n$ is a subset of this set, and so $\sigma_{m+n} \leq \sigma_m \sigma_n$.   ☐

(c) (Fekete's lemma) Let (x_n) be a real sequence satisfying the subadditive property:

$$x_{n+m} \leq x_n + x_m \quad \text{for all } n, m \geq 1.$$

Prove that $\lim_{n \to \infty} \frac{x_n}{n} \in [-\infty, \infty)$ exists.

Proof. Let $A = \inf_n \frac{x_n}{n}$. By definition, $\liminf_{n \to \infty} \frac{x_n}{n} \geq A$, so it suffcies to show $\limsup_{n \to \infty} \frac{x_n}{n} \leq A$. Fix $\epsilon > 0$. Let (x_{n_k}) be a convergent subsequence of (x_n). Pick m such that $x_m / m < A + \epsilon$. Then for $n_k \geq m$, we have $n_k = qm + r$ where $q, r \in \mathbb{N}$ and $0 \leq r < m$. By the subadditive property,

$$\frac{x_{n_k}}{n_k} \leq \frac{q x_m + x_r}{n_k} < \frac{qm(A + \epsilon) + x_r}{n_k} \leq A + \epsilon + \frac{x_r}{n_k}.$$

But then

$$\lim_{k \to \infty} \frac{x_{n_k}}{n_k} \leq A + \epsilon.$$

Since ϵ is arbitrary, $\limsup_{n \to \infty} \frac{x_n}{n} \leq A$. This completes the proof.   ☐

(d) Hence, or otherwise, prove that there exists $\kappa \in [d, 2d - 1]$ such that for all $\varepsilon > 0$,

$$(\kappa - \varepsilon)^n \leq \sigma_n \leq (\kappa + \varepsilon)^n, \quad \text{for large enough } n.$$

(κ is known as the *connectivity constant* of $\mathbb{Z}^d$.)

Proof. Consider the sequence (x_n) where $x_n = \frac{\log \sigma_n}{n}$. By (b) we have

$$\frac{\log \sigma_{m+n}}{m + n} \leq \frac{\log \sigma_m \sigma_n}{m + n} = \frac{\log \sigma_m}{m + n} + \frac{\log \sigma_n}{m + n} \leq \frac{\log \sigma_m}{m} + \frac{\log \sigma_n}{n},$$

so (x_n) is subadditive. By (c) we have that $L = \lim_{n \to \infty} \frac{x_n}{n} \in [-\infty, \infty)$ exists. Thus,

$$\lim_{n \to \infty} \sigma_n^{\frac{1}{n}} = e^L.$$

Put $\kappa = e^L$. Since $d^n \leq \sigma_n \leq (2d - 1)^n$, we have $\kappa \in [d, 2d - 1]$.   ☐

Problem 6

Consider the one-dimensional slab $S_n := \mathbb{Z} \times \{1, 2, \ldots, n\}$ for any $n \in \mathbb{N}$, considered as a subgraph of the lattice $\mathbb{Z}^2$. Show that the critical probability p_c for percolation on S_n is 1.

Hint: you may wish to argue similarly to Question 1.

Proof. Let $p \in [0, 1]$. Let $A_k = \{(k, x) \in S_n : x \in [n]\}$. Let $\{0 \leftrightarrow A_k\} = \bigcup_{v \in A_k} \{0 \leftrightarrow v\}$ and note that $\mathbb{P}(0 \leftrightarrow A_k) = \mathbb{P}(0 \leftrightarrow A_{-k})$. If $0 \leftrightarrow \infty$, then for any $k \in \mathbb{Z}_{\geq 0}$ we have $0 \leftrightarrow A_k$ or $0 \leftrightarrow A_{-k}$. Notice that $0 \leftrightarrow A_k$ implies that for $1 \leq k \leq n$, there is an edge between A_k and A_{k-1}. Put $q = 1 - (1-p)^n$. We then have,

$$\mathbb{P}(0 \leftrightarrow A_k) \leq q^k.$$

Then

$$\lim_{k \to \infty} \mathbb{P}(0 \leftrightarrow A_k) \leq \lim_{k \to \infty} q^k = \begin{cases} 0 & \text{if } p < 1 \\ 1 & \text{if } p = 1 \end{cases}.$$

$\square$

Problem 7

Suppose that $\mathbb{T}_d$ is an infinite rooted tree in which each vertex has d children (so that all vertices except the root ρ have degree $d+1$). Suppose each edge of the tree is open independently with probability p. As usual, let

$$p_c := \sup\{p : \mathbb{P}_p(\rho \leftrightarrow \infty) = 0\}.$$

(a) Show that $p_c = 1/d$, and that $\mathbb{P}_{p_c}(\rho \leftrightarrow \infty) = 0$.

　　Hint: recall basic results about branching processes.

　　Proof.

$$p_c = \sup\{p : \mu \leq 1\} = \frac{1}{d},$$

　　and that $\mathbb{P}_{p_c}(\rho \leftrightarrow \infty) = 0$.　　　　□

(b) Prove that for $p \leq p_c$,

$$\mathbb{P}_p(\exists\, v \in V(\mathbb{T}_d) : v \leftrightarrow \infty) = 0.$$

　　Note that, unlike the case $\mathbb{Z}^d$ treated in the lemma in lectures, $\mathbb{T}_d$ is not vertex-transitive.

　　Proof. Suppose not. There exists $v \in V(\mathbb{T}_d)$ such that $\mathbb{P}_p(v \leftrightarrow \infty) > 0$. But then

$$\mathbb{P}_p(\rho \leftrightarrow \infty) \geq \mathbb{P}_p(\rho \leftrightarrow v) \cdot \mathbb{P}_p(v \leftrightarrow \infty) > 0,$$

　　contradiction.　　　　□

(c) Prove that for $p \in (p_c, 1)$,

$$\mathbb{P}_p\big(\exists\, v \in V(\mathbb{T}_d) : v \leftrightarrow \infty\big) = 1,$$

　　directly, without appeal to Kolmogorov's 0–1 law.

　　Proof. Let V_n be the set of vertices on layer n of the tree, and note that $|V_n| = d^n$. Let $\{V_n \leftrightarrow \infty\} = \bigcup_{v \in V_n}\{v \leftrightarrow \infty\}$. Since for any $v, v' \in V_n$, the events $\{v \leftrightarrow \infty\}$ and $\{v' \leftrightarrow \infty\}$ are independent and have the same probability,

$$\mathbb{P}(V_n \leftrightarrow \infty) = 1 - \prod_{v \in V_n}(1 - \mathbb{P}_p(v \leftrightarrow \infty)) = 1 - (1 - \mathbb{P}_p(v \leftrightarrow \infty))^{d^n}.$$

　　Since $p > p_c$, we must have $\mathbb{P}_p(v \leftrightarrow \infty) > 0$ for some $v \in V_n$. But then for any $\epsilon > 0$, there exists large enough n such that $\mathbb{P}_p\big(\exists\, v \in V(\mathbb{T}_d) : v \leftrightarrow \infty\big) \geq \mathbb{P}(V_n \leftrightarrow \infty) > 1 - \epsilon$. This completes the proof.　　□

(d) How many infinite clusters are there for $p \in (p_c, 1)$?

　　Proof. For $v \in V(\mathbb{T}_d)$, let $\mathbb{T}_d^{(v)}$ be the subtree of $\mathbb{T}_d$ rooted at v. By (c), there exists some vertex v_0 such that $v_0 \leftrightarrow \infty$ almost surely. Consider $\mathbb{T}_d^{(v_0)}$. Since $p < 1$, there exists some vertex $v_0' \in V(\mathbb{T}_d^{(v_0)})$ such that $v_0 \not\leftrightarrow v_0'$ almost surely. By (c), there exists some vertex $v_1 \in V(\mathbb{T}_d^{(v_0')})$ such that $v_1 \leftrightarrow \infty$ almost surely, and the infinite cluster containing v_1 is disjoint from the infinite cluster containing v_0. Recursively repeating this process shows that there are infinitely many infinite clusters.　　□

Problem 8

For any $d \geq 2$, the graph $\mathbb{Z}^d \oplus \mathbb{Z}^d$ is constructed by taking two disjoint copies of $\mathbb{Z}^d$, and for each labelled vertex $v \in \mathbb{Z}^d$, adding an edge between the two vertices with this label.

For $\mathbb{Z}^2 \oplus \mathbb{Z}^2$, prove that

$$1 - \frac{1}{\sqrt{2}} \leq p_c \leq \frac{1}{2}.$$

Hint: you may find it helpful to observe that $p = 1 - \frac{1}{\sqrt{2}}$ satisfies $2p - p^2 = \frac{1}{2}$.

Proof. By Kesten's theorem, it is obvious that $p_c \leq \frac{1}{2}$, otherwise there exists an infinite cluster in $\mathbb{Z}^2$ which is also an infinite cluster in $\mathbb{Z}^2 \oplus \mathbb{Z}^2$.

Now suppose $p < 1 - 1/\sqrt{2}$. Consider the dual graph L' of $\mathbb{Z}^2$, and define an edge $e \in E(L')$ as open if and only if the corresponding primal edges in at least one of the copies of $\mathbb{Z}^2$ are open. Then $e \in E(L')$ is open with probability $1 - (1 - p)^2 = 2p - p^2$. Suppose $2p - p^2 < 1/2$. By Kesten's theorem there does not exist an infinite cluster in L'. But then there are no infinite clusters in either copy of $\mathbb{Z}^2$. It now follows that $p_c \geq 1 - 1/\sqrt{2}$. $\qquad\square$

Problem 9

Let $\Lambda(n) = [-n, n]^2 \cap \mathbb{Z}^2$. Using the BK inequality, show that, for $\mathbb{Z}^2$,

$$\mathbb{P}_{1/2}\big(0 \leftrightarrow \partial\Lambda(n)\big) \geq \frac{1}{2\sqrt{n}}. \tag{2}$$

Hint: consider horizontal crossings of the box $[-n, n] \times [-n, n-1]$ and think about where such a crossing meets the y-axis in order to split it in two. Then relate the probabilities of each of those smaller crossings to $\mathbb{P}_{1/2}\big(0 \leftrightarrow \partial\Lambda(n)\big)$.

Proof. Consider the rectangle $J_n := ([-n, n] \cap \mathbb{Z}) \times ([-n, n-1] \times \mathbb{Z})$. Let L_n, R_n be the left and right boundaries of J_n. Let J'_n be the dual graph of J_n, and denote T_n, B_n the top and bottom boundaries of J'_n. Define $H_n := \{L_n \leftrightarrow R_n\}$ and $V_n := \{T_n \leftrightarrow B_n\}$. Also, let $Y = \{0\} \times ([-n, n-1] \cap \mathbb{Z})$.

Note that if $L_n \leftrightarrow R_n$, then J' must not contain a vertical crossing. On the other hand, if $L_n \not\leftrightarrow R_n$, then there must be a vertical crossing in J'. Thus $\mathbb{P}(H_n) + \mathbb{P}(V_n) = 1$. But then J and J' are isomorphic, and so by symmetry $\mathbb{P}_{1/2}(H_n) = \mathbb{P}_{1/2}(V_n) = \frac{1}{2}$.

Now note that by BK inequality,

$$\mathbb{P}_{1/2}(H_n) = \mathbb{P}_{1/2}\left(\bigcup_{y \in Y} \{L_n \leftrightarrow y\} \circ \{y \leftrightarrow R_n\}\right) \leq \sum_{y \in Y} \mathbb{P}_{1/2}(L_n \leftrightarrow y) \cdot \mathbb{P}_{1/2}(y \leftrightarrow R_n) \leq 2n \cdot \mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda(n))^2.$$

The result now follows from rearranging the above inequality. $\square$