MATH 220B: Homework #3

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Professor Xiao

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Problem 1

Prove Lemma 1.5: If (S, d) is a metric space then

$$\mu(s,t) = \frac{d(s,t)}{1+d(s,t)}$$

is also a metric on S. A set is open in (S, d) iff it is open in (S, μ); a sequence is a Cauchy sequence in (S, d) iff it is a Cauchy sequence in (S, μ).

Proof. We first show that μ is a metric. Let $s, t, u \in S$. Then $\mu(s,s) = 0$, $\mu(s,t) > 0$ if $s \neq t$, $\mu(s,t) = \mu(t,s)$. We now prove the triangle inequality. Note that

$$\frac{d(s,u)}{1+d(s,u)} \leq \frac{d(s,t)+d(t,u)}{1+d(s,t)+d(t,u)},$$

Hence, it suffices to show that for $a, b \geq 0$,

$$\frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

Notice

$$\frac{a}{1+a} + \frac{b}{1+b} = 2 - \left(\frac{1}{1+a} + \frac{1}{1+b}\right)$$

and

$$\frac{a+b}{1+a+b} = 1 - \frac{1}{1+a+b}.$$

Since

$$\frac{1}{1+a} + \frac{1}{1+b} - 1 = \frac{1-ab}{1+a+b+ab} \leq \frac{1}{1+a+b},$$

we have

$$\frac{a}{1+a} + \frac{b}{1+b} = 2 - \left(\frac{1}{1+a} + \frac{1}{1+b}\right) \geq 1 - \frac{1}{1+a+b} = \frac{a+b}{1+a+b}.$$

Since $\frac{t}{1+t}$ is continuous and strictly increasing on $[0, \infty)$, for $\delta > 0$ there exists $\epsilon > 0$ such that $d(s,t) < \delta$ if and only if $\mu(s,t) < \epsilon$. Hence, a set $U \subseteq S$ is open in (S, d) if and only if U is open in (S, μ). Similarly, a sequence $\{s_n\}$ is a Cauchy sequence in (S, μ) if and only if for $\epsilon > 0$ there exists N such that for all $m, n \geq N$,

$$\mu(s_n, s_m) < \epsilon \iff d(s_n, s_m) < \delta,$$

where the δ corresponds to ϵ as above. $\square$

Problem 2

Suppose $\{f_n\}$ is a sequence in $C(G, \Omega)$ which converges to f and $\{z_n\}$ is a sequence in G which converges to a point z in G. Show $\lim f_n(z_n) = f(z)$.

Proof. Let $\epsilon > 0$. Since $f_n \to f$ on G, there exists N such that for all $n \geq N$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2},$$

for all $x \in G$. Since $z_n \to z$ and f is continuous, there exists M such that for all $n \geq M$,

$$d|f(z_n) - f(z)| < \frac{\epsilon}{2},$$

Hence, for all $n \geq \max(N, M)$,

$$|f_n(z_n) - f(z)| \leq |f_n(z_n) - f_n(z)| + |f_n(z) - f(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

Problem 3

(Dini's Theorem) Consider $C(G,\mathbb{R})$ and suppose that $\{f_n\}$ is a sequence in $C(G,\mathbb{R})$ which is monotonically increasing (i.e., $f_n(z) \leq f_{n+1}(z)$ for all z in G) and $\lim f_n(z) = f(z)$ for all z in G, where $f \in C(G,\mathbb{R})$. Show that $f_n \to f$.

Proof. Let $K \subseteq G$ be compact. Fix $\epsilon > 0$. Let $g_n = f - f_n$. Let $K_n = \{x \in K \mid g_n(x) \geq \epsilon\} = g^{-1}([\epsilon, \infty))$. Since g_n is continuous and $[\epsilon, \infty)$ is closed, K_n is closed. But then K_n is a closed subset of a compact set, so K_n is compact. Since $g_{n+1}(z) \geq g_n(z)$, we have $K_{n+1} \subseteq K_n$. Let $z \in K$. Since $\lim_{n\to\infty} g_n(z) = 0$, we know $z \notin K_n$ for large enough n, and so $\bigcap_{n\geq 1} K_n = \emptyset$. But then K_N is empty for some N. Hence, $0 \leq g_n(z) < \epsilon$ for all $z \in K$, $n \geq N$. The result now follows. $\square$

Problem 4

(a) Let f be analytic on $B(0; R)$ and let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for } |z| < R.$$

If

$$f_n(z) = \sum_{k=0}^{n} a_k z^k,$$

show that $f_n \to f$ in $C(G; \mathbb{C})$.

Proof. Let $r \in (0, R)$. Since f converges on $B(0; R)$, the series $\sum_{n=0}^{\infty} a_n r^n$ converges. But then by the Weierstrass M-test, f_n converges to f uniformly on $\overline{B}_r(0)$. The result now follows. $\square$

(b) Let $G = \text{ann}(0; 0, R)$ and let f be analytic on G with Laurent series development

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Put

$$f_n(z) = \sum_{k=-\infty}^{n} a_k z^k$$

and show that $f_n \to f$ in $C(G; \mathbb{C})$.

Proof. Write $f(z) = f^-(z) + f^+(z)$, with $f^-(z) = \sum_{n=-\infty}^{-1} a_n z^n$ and $f^+(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $f_n^- = \sum_{k=1}^{n} a_{-k} z^{-k}$ and $f_n^+ = \sum_{k=0}^{n} a_k z^k$. Let $0 < r_1 < r_2 < R$. Since f converges on $\text{ann}(0; 0, R)$, the series $\sum_{n=-\infty}^{-1} a_n r_1^n$ and $\sum_{n=0}^{\infty} a_n r_2^n$ converges. By the Weierstrass M-test, f_n^- converges to f^- uniformly on $\overline{\text{ann}(0; r_1, r_2)}$ and f_n^+ converges to f^+ uniformly on $\overline{\text{ann}(0; r_1, r_2)}$. Since $f_n(z) = f^-(z) + f_n^+(z)$, the result follows. $\square$

Problem 5

Prove Vitali's Theorem: If G is a region and $\{f_n\} \subset H(G)$ is locally bounded and $f \in H(G)$ that has the property that

$$A = \{z \in G : \lim f_n(z) = f(z)\}$$

has a limit point in G, then $f_n \to f$.

Proof. Define $g_n = f_n - f$. Since $\{f_n\}$ is locally bounded, $\{g_n\}$ is locally bounded. By Montel's Theorem, theres is a converging subsequence $\{g_{n_k}\}$, say $g_{n_k} \to g$. But then $g(z) = 0$ on A and A has a limit point, so $g(z) = 0$ on G. This implies every converging subsequence of $\{g_n\}$ converges to 0 on G, which forces $g_n \to 0$. Therefore, $f_n = f + g_n \to f$. $\square$

Problem 6

Let $D = B(0;1)$ and for $0 < r < 1$ let $\gamma_r(t) = re^{2\pi it}$, $0 \leq t \leq 1$. Show that a sequence $\{f_n\}$ in $H(D)$ converges to f iff

$$\int_{\gamma_r} |f(z) - f_n(z)|\,|dz| \to 0 \quad \text{as } n \to \infty$$

for each r, $0 < r < 1$.

Proof. Suppose that $f_n \to f$. Pick $\epsilon > 0$. Then there exists N such that for all $n \geq N$, $|f(z) - f_n(z)| < \epsilon$. Hence,

$$\int_{\gamma_r} |f(z) - f_n(z)|\,|dz| < \epsilon \int_{\gamma_r} |dz| = \epsilon \cdot 2\pi r \to 0,$$

as $\epsilon \to 0$.

We now show the converse. Fix $r \in (0,1), \epsilon > 0$. Let $g_n = f(z) - f_n(z)$. Since g_n is analytic,

$$|g_n(z)| = \frac{1}{2\pi} \int_{\gamma_r} \frac{g_n(w)}{w-z}\,|dw| \leq \frac{1}{2\pi r} \int_{\gamma_r} |g_n(w)|\,|dz|$$

on $\overline{B}_r(0)$. Hence, $g_n(z) \to 0$ on any closed disk $B_0(r)$, $0 < r < 1$, and the result now follows. □

Problem 7

Let $\{f_n\} \subset H(G)$ be a sequence of one-one functions which converge to f. Show that either f is one-one or f is a constant function.

Proof. Suppose f is not one-one or constant. There exists $z_1, z_2 \in G$ such that $f(z_1) = f(z_2)$. Consider sequence $g_n(z) = f_n(z) - f_n(z_1)$. Let $g = f - f(z_1)$. Note that $g_n \to g$ and g_n has at most one zero. Since g is analytic, its zeros are isolated, so we may find a closed disk D such that g does not vanish on ∂D and $z_1, z_2 \in K$. By Hurwitz's Theorem, for large enough n, g_n and g have the same number of zeros in K. But then g has zeros z_1 and z_2 in K while g_n has at most one zero in K, contradiction. $\square$

Problem 8

Suppose that $\{f_n\}$ is a sequence in $H(G)$, f is a non-constant function, and $f_n \to f$ in $H(G)$. Let $a \in G$ and $\alpha = f(a)$; show that there is a sequence $\{a_n\}$ in G such that:

(i) $a = \lim a_n$;

(ii) $f_n(a_n) = \alpha$ for sufficiently large n.

Proof. Define $g(z) = f(z) - \alpha$. Since g is analytic and non-constant, the zeros of g are isolated. Hence, we may find a sequence $\{r_n\}$ such that $r_n \to 0$ and g does not vanish on $\partial B_{r_n}(a)$. Since $f_n \to f$ uniformly on closed balls, there exists N such that for $n \geq N$ we have

$$\max_{|z-a|=r_n} |f_n(z) - f(z)| < \min_{|z-a|=r_n} |g(z)|.$$

Put $g_n(z) = f_n(z) - \alpha$. Since for $n \geq N$

$$|g_n(z) - g(z)| = |f_n(z) - f(z)| < |g(z)|$$

on $\partial B_{r_n}(a)$, $g_n(z)$ and $g(z)$ have the same number of zeros in $B_{r_n}(a)$, which is at least one. Let a_n be a zero of $g_n(z)$ in $B_{r_n}(a)$. Then we have $f_n(a_n) = \alpha$ for all $n \geq N$. Since $r_n \to 0$, $a_n \to 0$. □

Problem 9

Let f be analytic on $G = \{z : \operatorname{Re} z > 0\}$, one-one, with $\operatorname{Re} f(z) > 0$ for all $z \in G$, and $f(a) = a$ for some real number a. Show that $|f'(a)| \leq 1$.

Proof. Since G is a simply connected region and $G \neq \mathbb{C}$, there is a unique analytic one-one function $g : G \to D$ such that $g(a) = 0$. Consider $h = g \circ f \circ g^{-1}$. Note that h maps D to D and $h(0) = 0$. By Schwarz's Lemma,

$$|h'(0)| = |g'(a)f'(a)(g^{-1})'(0)| \leq 1$$

But then $(g^{-1})'(0)g'(a) = (g^{-1})'(0)g'(g^{-1}(0)) = 1$, and the result now follows. $\square$