# MATH 140A: Homework #2

Due on October 13, 2023 at 11:00pm

*Professor Mohammadi*

Section A02


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# Problem 1

Let $A \subset R$ be a non-empty subset which satisfies the following two properties

1. $A = A + A$
2. For every $\epsilon > 0$, there exists some $a \in A$ so that $0 < a < \epsilon$.

Prove that for every $x \in R^{>0}$, there exists some $a \in A$ so that

$$0 < x - a < \epsilon.$$

*Proof.* Let $a \in A$. We first show that for $n \in \mathbb{N}$, $na \in A$ by induction on $n$. We already know $a \in A$. For $n > 1$, since $(n-1)a \in A$, we know $na = a + (n-1)a \in A$ by rule 1. Thus, $na \in A$, for all $n \in \mathbb{N}$.

Since $\epsilon > 0$, there exists $a \in A$ such that $0 < a < \epsilon$, by rule 2. We assume that $\epsilon < x$, otherwise we are done. Now we show that there exists $n \in \mathbb{N}$ such that $0 < x - an < \epsilon$. Let $0 < \frac{x-\epsilon}{a} < n < \frac{x}{a}$. Since $\epsilon > a$, we know such $n$ exists because $\frac{x}{a} - \frac{x-\epsilon}{a} = \frac{\epsilon}{a} > 1$, and so there must exists a natural number within the interval. Thus, we get

$$0 = x - a \cdot \frac{x}{a} < x - an < x - a \cdot \frac{x-\epsilon}{a} = \epsilon.$$

$\square$

# Problem 2

Let $a, b, c, d \in \mathbb{R}$ and assume $a < b$ and $c < d$. Give an explicit one-to-one correspondence between

1. The points of the two open intervals $(a, b)$ and $(c, d)$.

   *Proof.* Define $f : (a, b) \to (c, d)$ to be $f(x) = \frac{(d-c)x+(cb-ad)}{b-a}$. Let $l, m \in (a, b)$. Since $a < l < b$,

   $$\frac{d-c}{b-a}a < \frac{d-c}{b-a}l < \frac{d-c}{b-a}b$$
   $$\frac{(d-c)a+(cb-ad)}{b-a} < \frac{(d-c)l+(cb-ad)}{b-a} < \frac{(d-c)b+(cb-ad)}{b-a}$$
   $$c < f(l) < d.$$

   Suppose that $l = m$. Then

   $$\frac{(d-c)l+(cb-ad)}{b-a} = f(l) = f(m) = \frac{(d-c)m+(cb-ad)}{b-a},$$

   and so $f$ is well defined.

   Suppose $f(l) = f(m)$. Then,

   $$\frac{(d-c)l+(cb-ad)}{b-a} = \frac{(d-c)m+(cb-ad)}{b-a}$$
   $$(d-c)l+(db-ad) = (d-c)m+(db-ad)$$
   $$(d-c)l = (d-c)m$$
   $$l = m.$$

   Thus, $f$ is injective.

   Let $y \in (c, d)$. There exists $x = \frac{(b-a)y-(cb-ad)}{d-c} \in (a, b)$ such that $f(x) = y$, and so $f$ is surjective.

   Thus, $f$ is an one-to-one correspondence. □

2. The points of the two closed intervals $[a, b]$ and $[c, d]$.

   *Proof.* Define $f : [a, b] \to [c, d]$ to be $f(x) = \frac{(d-c)x+(cb-ad)}{b-a}$. Let $l, m \in [a, b]$. Since $a \le l \le b$,

   $$\frac{d-c}{b-a}a \le \frac{d-c}{b-a}l \le \frac{d-c}{b-a}b$$
   $$\frac{(d-c)a+(cb-ad)}{b-a} \le \frac{(d-c)l+(cb-ad)}{b-a} \le \frac{(d-c)b+(cb-ad)}{b-a}$$
   $$c \le f(l) \le d.$$

   Suppose that $l = m$. Then

   $$\frac{(d-c)l+(cb-ad)}{b-a} = f(l) = f(m) = \frac{(d-c)m+(cb-ad)}{b-a},$$

   and so $f$ is well defined.

   Suppose $f(l) = f(m)$. Then,

   $$\frac{(d-c)l+(cb-ad)}{b-a} = \frac{(d-c)m+(cb-ad)}{b-a}$$
   $$(d-c)l+(db-ad) = (d-c)m+(db-ad)$$
   $$(d-c)l = (d-c)m$$
   $$l = m.$$

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Thus, $f$ is injective.

Let $y \in [c,d]$. There exists $x = \frac{(b-a)y-(cb-ad)}{d-c} \in [a,b]$ such that $f(x) = y$, and so $f$ is surjective.

Thus, $f$ is an one-to-one correspondence. ☐

3. The points of the closed interval $[a,b]$ and the open interval $(c,d)$.

*Proof.* Define $f : [a,b] \to (c,d)$ to be

$$f(x) = \begin{cases} c + \frac{d-c}{n+2}, & x = a + \frac{b-a}{n}, n \in \mathbb{N} \\ \frac{c+d}{2}, & x = a \\ \frac{(d-c)x+(cb-ad)}{b-a}, & \text{otherwise.} \end{cases}$$

Note that the product of $f$ of different cases would not be equal.

Obviously, $f(x) \in (c,d)$ for all $x \in [a,b]$. Let $k, m \in [a,b]$. If $k = m = a$, then $f(k) = f(m) = \frac{c+d}{2}$. If $k = m = a + \frac{b-a}{n}$, for some $n \in \mathbb{N}$, then $f(k) = f(m) = c + \frac{d-c}{n+2}$. Otherwise, $\frac{(d-c)k+(cb-ad)}{b-a} = \frac{(d-c)m+(cb-ad)}{b-a}$, which implies that $f(k) = f(m)$. Therefore, $f$ is well defined.

Suppose that $f(k) = f(m)$. If $f(k) = f(m) = \frac{c+d}{2}$, then $k = m = a$. If $f(k) = f(m) = c + \frac{d-c}{n+2}$ for some $n \in \mathbb{N}$, then $k = m = a + \frac{b-a}{n}$. If $f(k) = \frac{(d-c)k+(cb-ad)}{b-a} = \frac{(d-c)m+(cb-ad)}{b-a} = f(m)$, Then $k = m$, by the results we obtained from previous parts. Thus, $f$ is injective.

Let $y \in (c,d)$. There exists

$$x = \begin{cases} a + \frac{b-a}{n}, & y = c + \frac{d-c}{n+2}, n \in \mathbb{N} \\ a, & y = \frac{c+d}{2} \\ \frac{(b-a)x+(ad-cb)}{d-c}, & \text{otherwise,} \end{cases}$$

such that $f(x) = y$. Thus, $f$ is surjective.

Therefore, $f$ is bijective. ☐

4. The points of the closed interval $[a,b]$ and $\mathbb{R}$

*Proof.* Consider $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ and $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$. Since tan and $\tan^{-1}$ are inverses of each other, they are bijective. We can then use the function $f$ we defined in part 3 to get a bijective mapping from $[a,b]$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, we get a bijection $(\tan \circ f) : [a,b] \to \mathbb{R}$,

$$(\tan \circ f)(x) = \begin{cases} \tan(-\frac{\pi}{2} + \frac{\pi}{n+2}), & x = a + \frac{b-a}{n}, n \in \mathbb{N} \\ 0, & x = a \\ \tan(\frac{2\pi x - \pi(b+a)}{2(b-a)}), & \text{otherwise.} \end{cases}$$

☐

# Problem 3

Fix $b > 1, y > 0$, and prove that there is a unique real $x$ such that $b^x = y$.

*Proof.* We first show that for any positive integer $n$, $b^n - 1 \geq n(b-1)$. We show that $b^n > 1$ by induction on $n$. We already know $b > 1$. For $n > 1$, $b^n = b \cdot b^{n-1} > 1$, since $b^{n-1} > 1$ by induction. Thus,

$$b^n - 1 = (b-1)(b^{n-1} + \cdots + b + 1) \geq (b-1)n. \tag{1}$$

By Theorem 1.21, we know that there exists a unique $a \in \mathbb{R}^+$ such that $a^n = b$. Suppose that $a \leq 1$. We show that $a^n \leq 1$ by induction on $n$. For $n > 1$, we know that $a^n = a \cdot a^{n-1} \leq 1$, since $a^{n-1} \leq 1$ by induction. Thus, $a$ must be greater than 1. Then, by (1), we know that $b - 1 = a^n - 1 \geq (a-1)n = (b^{\frac{1}{n}} - 1)n$.

Let $t > 1$. Suppose that $n > \frac{b-1}{t-1}$, then $nt - n > b - 1$. Since $n \geq 1$, we know $t > b$. Note that since $a^n > 1$ for all $n \in \mathbb{N}$, $b = b^{\frac{1}{n}} \cdot a^{n-1} \geq b^{\frac{1}{n}}$. Thus, we get

$$t > b \geq b^{\frac{1}{n}}. \tag{2}$$

Let $w \in \mathbb{R}$. Suppose that $b^w < y$. Let $t = y \cdot b^{-w} > b^w \cdot b^{-w} = 1$. By (2), there exists $n > \frac{b-1}{t-1}$, such that $t = y \cdot b^{-w} > b^{\frac{1}{n}}$, and so $y > b^{w+\frac{1}{n}}$. Suppose that $b^w < y$. Let $t = b^w y^{-1}$. Similarly, there exists $n > \frac{b-1}{t-1}$, such that $t = b^w y^{-1} > b^{\frac{1}{n}}$, and so $b^{w-\frac{1}{n}} > y$.

Let $A$ be the set of all $w$ such that $b^w < y$. We will show that $x = \sup A$ satisfies $b^x = y$. Suppose for the sake of contradiction that $b^x < y$. Then, by the result we obtained above, we know there exists a large enough $n \in \mathbb{N}$, such that $b^x < b^{x+\frac{1}{n}} < n$. This implies that there exists $x + \frac{1}{n} \in A$, which contradicts that $x = \sup A$. Suppose for the sake of contradiction that $b^x > y$. Then, by the result we obtained above, there exists a large enough $n \in \mathbb{N}$, such that $b^x > b^{x-\frac{1}{n}} > y$, contradicting the fact that $x = \sup A$. Thus, $b^x = y$.

Suppose that $b^z = b^x = y$. $x \not> z$, otherwise $b^z < b^x$, contradiction. Similarly, we also know $x \not< z$. Therefore, $x$ is unique. $\square$

# Problem 4

If $x, y$ are complex, prove that
$$||x| - |y|| \leq |x - y|.$$

*Proof.* We sqaure both sides. On the right-hand-side, we have

$$\begin{aligned} |x - y|^2 &= (x - y)\overline{(x - y)} \\ &= |x|^2 + |y|^2 - y\overline{x} - x\overline{y} \end{aligned}$$

Note that $\overline{x\overline{y}} = y\overline{x}$, so $y\overline{x} + x\overline{y} = 2\,\mathrm{Re}(x\overline{y})$. On the left-hand-side, we have

$$\begin{aligned} (|x| - |y|)^2 &= |x|^2 + |y|^2 - 2|x||y| \\ &= |x|^2 + |y|^2 - 2|x||\overline{y}| \\ &= |x|^2 + |y|^2 - 2|x\overline{y}| \end{aligned}$$

Since $\mathrm{Re}(x\overline{y}) \leq |x\overline{y}|$,

$$\begin{aligned} |x|^2 + |y|^2 - 2|x\overline{y}| &\leq |x|^2 + |y|^2 - 2\,\mathrm{Re}(x\overline{y}) \\ &= |x|^2 + |y|^2 - y\overline{x} - x\overline{y}, \end{aligned}$$

and thus $||x| - |y|| \leq |x - y|$. $\square$

# Problem 5

If z is a complex number such that $|z| = 1$, that is, such that $z\overline{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2$$

*Proof.*

$$\begin{aligned} |1 + z|^2 + |1 - z|^2 &= (1 + z)\overline{(1 + z)} + (1 - z)\overline{(1 - z)} \\ &= 1 + z + \overline{z} + z\overline{z} + 1 - z - \overline{z} + z\overline{z} \\ &= 4. \end{aligned}$$

□

# Problem 6

Prove that
$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$$
if $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

*Proof.*

$$\begin{aligned}|x+y|^2 + |x-y|^2 &= |x|^2 + |y|^2 + 2x \cdot y + |x|^2 + |y|^2 - 2x \cdot y \\ &= 2|x|^2 + 2|y|^2.\end{aligned}$$

Interpreting geometrically, if $x, y$ were the neighboring sides of a para;;elogram, then $x+y$ and $x-y$ are its diagonals. Thus, the equation suggests that the sum of the squares of the sides is equal to the sum of the square of the diagonals. □