MATH 180B: Homework #6

Due on Mar 08, 2024 at 23:59pm

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Problem 1

Which states are transient and which are recurrent in the Markov chain whose transition probability matrix is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{cccccc} \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array} ?$$

Proof. The communicating classes of the matrix are $\{0\}, \{1\}, \{3\}, \{5\}, \{2, 4\}$. Note that $\{5\}$ and $\{2, 4\}$ are closed classes, so states $2, 4, 5$ are recurrent. In addition since $0, 1, 3$ are connected all to closed classes, they will eventually get stuck in those closed classes and never return to the original state, and thus they are transient. $\square$

Problem 2

Determine the communicating classes and period for each state of the Markov chain whose transition probability matrix is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{cccccc} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{array} \right\| \end{array} ?$$

Proof. Since states each of $0, 1$ is not accessible to any other classes, they each form their own classes. Since there exists a cycle $2 \to 3 \to 4 \to 5 \to 2$ which passes through all the rest of the states, they all belong to a class. Hence, the communicating classes are $\{0\}, \{1\}, \{2, 3, 4, 5\}$. Since 0 and 5 are both connected to itself, $\{0\}$ and $\{2, 3, 4, 5\}$ have period 1. Since 1 is not accessible from any states including itself, $d(1) = 0$. ☐

Problem 3

Recall the first return distribution

$$f_{ii}^{(n)} = \Pr\{X_1 \neq i, X_2 \neq j, \ldots, X_{n-1} \neq i, X_n = i \mid X_0 = i\} \text{ for } n = 1, 2, \ldots,$$

with $f_{ii}^{(0)} = 0$ by convention. Using equation (4.16), determine $f_{00}^{(n)}, n = 1, 2, 3, 4$, for the Markov chain whose transition probability matrix is

$$\begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}.$$

Proof. Call that matrix P. Note that

$$P^2 = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 1 \\ 2 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 3 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{array}, \quad P^3 = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{3}{4} \\ 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 2 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ 3 & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{array}, \quad P^4 = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & \frac{3}{8} & \frac{1}{16} & \frac{1}{8} & \frac{7}{16} \\ 1 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ 2 & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \\ 3 & \frac{3}{16} & \frac{1}{8} & \frac{1}{8} & \frac{9}{16} \end{array}.$$

Hence, we get

$$\begin{aligned} P_{00}^1 = 0 &= f_{00}^{(0)} P_{00}^1 + f_{00}^{(1)} P_{00}^0 = f_{00}^{(1)} \\ P_{00}^2 = \frac{1}{4} &= f_{00}^{(0)} P_{00}^2 + f_{00}^{(1)} P_{00}^1 + f_{00}^{(2)} P_{00}^0 = f_{00}^{(2)} \\ P_{00}^3 = \frac{1}{8} &= f_{00}^{(0)} P_{00}^3 + f_{00}^{(1)} P_{00}^2 + f_{00}^{(2)} P_{00}^1 + f_{00}^{(3)} P_{00}^0 = f_{00}^{(3)} \\ P_{00}^4 = \frac{3}{8} &= f_{00}^{(0)} P_{00}^4 + f_{00}^{(1)} P_{00}^3 + f_{00}^{(2)} P_{00}^2 + f_{00}^{(3)} P_{00}^1 + f_{00}^{(4)} P_{00}^0 = \frac{1}{4} \cdot \frac{1}{4} + f_{00}^{(4)}, \end{aligned}$$

and thus $f_{00}^{(1)} = 0, f_{00}^{(2)} = \frac{1}{4}, f_{00}^{(3)} = \frac{1}{8}, f_{00}^{(4)} = \frac{5}{16}$. $\square$

Problem 4

Let $\{\alpha_i : i = 1, 2, \dots\}$ be a probability distribution, and consider the Markov chain whose transition probability matrix is

$$\begin{array}{c|ccccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\
0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
3 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
4 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

What condition on the probability distribution $\{\alpha_i : i = 1, 2, \dots\}$ is necessary and sufficient in order that a limiting distribution exist, and what is this limiting distribution? Assume $\alpha_1 > 0$ and $\alpha_2 > 0$ so that the chain is aperiodic.

Proof. Call that matrix P. We show that $\sum_{k=1}^{\infty} k\alpha_k < \infty$ is the necessary and sufficient condition to the existence of the limiting distribution.

P is obviously aperiodic and irreducible. Hence, if state 0 is recurrent, every state in P is recurrent. Let $R = \inf\{n \geq 1; X_n = 0\}$. Note that $P(R \leq k \mid X_0 = 0) = \sum_{i=1}^{k} f_{00}^{(i)} = \sum_{i=1}^{k} \alpha_i$. Clearly, $\lim_{k\to\infty} \sum_{i=1}^{k} \alpha_i = 1$. It follows that $f_{00} = \lim_{k\to\infty} \sum_{i=1}^{k} f_{00}^{(i)} = 1$, so P is indeed recurrent.

Suppose that $\sum_{k=1}^{\infty} k\alpha_k < \infty$. Since $m_0 = E[R \mid X_0 = 0] = \sum_{k=1}^{\infty} k\alpha_k < \infty$, the state 0 is positively recurrent, with

$$\begin{aligned}
\pi_0 &= \lim_{n\to\infty} P_{00}^{(n)} = \frac{1}{m_0} = \frac{1}{\sum_{k=1}^{\infty} k\alpha_k} \\
\pi_1 &= (1 - \alpha_1)\pi_0 \\
\pi_2 &= (1 - \alpha_1 - \alpha_2)\pi_0 \\
&\vdots \\
\pi_n &= \left(1 - \sum_{i=1}^{n} \alpha_i\right)\pi_0 \\
&\vdots
\end{aligned}$$

Hence, the limiting distribution is $\pi_k = \frac{1 - \sum_{i=1}^{k} \alpha_i}{m_0}$.

Suppose that $\sum_{k=1}^{\infty} k\alpha_k = \infty$. Then, $m_0 = \infty$. It follows that $\pi_0 = \frac{1}{m_0} = 0$, so the limiting distribution does not exist. □

Problem 5

Determine the period of state 0 in the Markov chain whose transition probability matrix is

$$P = \begin{array}{c} \\ 3 \\ 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ -3 \\ -4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 \end{array} \\ \left\| \begin{array}{cccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right\| \end{array}$$

Proof. There are two communicating classes in P, namely $\{1, 2, 3\}$ and $\{0, -1, -2, -3, -4\}$. Note that 0 is accessible to both classes. But since $\{1, 2, 3\}$ is a closed class, there are no path to return to 0 after entering that class. Hence, we may focus on the class $\{0, -1, -2, -3, -4\}$. Since the only path to return to 0 is via the cycle $0 \to -1 \to -2 \to -3 \to -4 \to 0$, the state 0 is of period 5. $\square$

Problem 6

A Markov chain on states $0, 1, \ldots$ has transition probabilities

$$P_{ij} = \frac{1}{i+2} \quad \text{for } j = 0, 1, \ldots, i, i+1.$$

Find the stationary distribution.

Proof. We solve for $\pi = \pi P$ and get $\pi_0 = \sum_{i=0}^{\infty} \frac{x_i}{i+2}$ and $\pi_k = \sum_{i=k}^{\infty} \frac{x_{i-1}}{i+1}$, for $k > 0$. Hence, we get $\pi_k = \frac{\pi_0}{k!}$. Since $\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{k=0}^{\infty} \frac{1}{k!} = e\pi_0 = 1$, we get $\pi_0 = \frac{1}{e}$, and so $\pi_k = \frac{1}{k!e}$. $\square$