MATH 140B: Homework #3

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Professor Seward

Ray Tsai

A16848188

Problem 1

Suppose f is a bounded real function on $[a, b]$ and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

Proof. f is not necessarily integrable. Consider

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}.$$

$f^2(x) = 1$ is obviously continuous, as it is constant. Since both rationals and irrationals are dense in $\mathbb{R}$,

$$U(P, f) = \sum_{i=0}^{n} \Delta x_i = b - a, \quad L(P, f) = \sum_{i=0}^{n} -\Delta x_i = a - b,$$

for any partition P. But then $U(P, f) - L(P, f) = 2(b - a)$, and thus $f \notin \mathscr{R}$.

Suppose $f^3 \in \mathscr{R}$. Since f is bounded, we may assume $|f| < M$. Define $\phi(x) = \sqrt[3]{x}$. Note that $\phi(f^3(x)) = f(x)$. Since x^3 is a continuous 1-1 mapping on $[-M^{1/3}, M^{1/3}]$, its inverse $\phi = \sqrt[3]{x}$ is continuous on $[-M, M]$, by Theorem 4.17. But then by Theorem 6.11, $f(x) = \phi(f^3(x)) \in \mathscr{R}$ on $[a, b]$. $\square$

Problem 2

Let P be the Cantor set constructed in Theorem 2.44. Let f be a bounded real function on $[0,1]$ which is continuous at every point outside P. Prove that $f \in \mathscr{R}$ on $[0,1]$.

Proof. We first show that P can be covered by finitely many segments whose total length can be made as small as desired. Pick $\epsilon > 0$. Note that $P = \bigcap_{n=1}^{\infty} E_n$, where E_n is the union of 2^n intervals, each of length 3^{-n}. Pick n large enough such that $\frac{2^n}{3^n} < \epsilon$. We know P can be covered by E_n. Put $\nu \in (0, \epsilon - \frac{2^n}{3^n})$. Let C be the union of segments, where each segment corresponds to an interval in E_n with both endpoints extended by $\frac{\nu}{2^{n+1}}$. Then, C is a open cover of P and the total length of all segments in C is $\frac{2^n}{3^n} + \nu < \epsilon$.

We may assume that $C = \bigcup_{i=1}^{2^n}(u_i, v_i)$ and the intervals $[u_i, v_i]$ are pairwise disjoint. Let $M = \sup|f(x)|$. Put $K = [0,1] \backslash C$. Since K is compact, f is uniformly continuous on K. Hence, there exists $\delta > 0$ such that $|f(s) - f(t)| < \epsilon$ whenever $s, t \in K$ and $|s - t| < \delta$.

Now consider a partition $\rho = \{x_0, x_1, \ldots, x_k\}$ of $[0,1]$ such that each u_i, v_i occurs in ρ and no point of any segment (u_i, v_i) occurs in ρ. Additionally, if x_{i-1} is not one of the u_j, then $\Delta x_i < \delta$.

Note that $M_i - m_i \le 2M$ for every i, and that $M_i - m_i \le \epsilon$ unless x_{i-1} is one of the u_j. Hence,

$$\begin{aligned} U(\rho, f) - L(\rho, f) &= \sum_{i=1}^{k}(M_i - m_i)\Delta x_i \\ &= \sum_{x_{i-1}=u_j}^{k} (M_i - m_i)\Delta x_i + \sum_{x_{i-1}\neq u_j}^{k} (M_i - m_i)\Delta x_i \\ &< 2M\epsilon + \epsilon = (2M+1)\epsilon, \end{aligned}$$

and the result follows from Theorem 6.6. □

Problem 3

Suppose f is a real function on $(0,1]$ and $f \in \mathscr{R}$ on $[c,1]$ for every $c > 0$. Define

$$\int_0^1 f(x)\,dx = \lim_{c \to 0} \int_c^1 f(x)\,dx$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0,1]$, show that this definition of the integral agrees with the old one.

Proof.

$$\lim_{c \to 0} \int_c^1 f(x)\,dx = \int_0^1 f(x)\,dx - \lim_{c \to 0} \int_0^c f(x)\,dx,$$

so it remains to show that $\lim_{c \to 0} \int_0^c f(x)\,dx = 0$. Since $f \in \mathscr{R}$, we may assume $|f(x)| \leq M$ for $x \in [0,1]$. Pick $\epsilon > 0$. Then, given any partition $P = \{x_0, \ldots, x_n\}$, we have $\delta = \frac{\epsilon}{nM}$ such that,

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \leq nMc < \epsilon, \quad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i > -nMc > -\epsilon,$$

for all $c \in (0, \delta)$. But then

$$\left| \int_0^c f(x) \right| < \epsilon,$$

and the result follows. □

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Proof. Define $f(x) = (-1)^n (n+1)$ if $x \in (\frac{1}{n+1}, \frac{1}{n}]$, for $n \in \mathbb{N}$. Suppose $c = \frac{1}{n+1}$. Then,

$$\int_c^1 f(x)\,dx = \sum_{k=1}^{n} \frac{(-1)^k}{k}, \quad \int_c^1 |f(x)|\,dx = \sum_{k=1}^{n} \frac{1}{k}.$$

As $c \to 0$, $n \to \infty$, and thus $\int_c^1 f(x)\,dx$ converges but not $\int_c^1 |f(x)|\,dx$. □

Problem 4

Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)\,dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series.)

Proof. Consider the partition $P = \{1, \ldots, n\}$. Note that as $f \geq 0$, both the $\int_1^n f(x)\,dx$ and $\sum_{k=1}^n f(k)$ are monotonically increasing with respect to n. Hence, it suffices to show that the integral and summation are bounded together. Since f is monotonically decreasing,

$$U(P, f) = \sum_{k=1}^{n-1} f(k), \quad L(P, f) = \sum_{k=2}^{n} f(k).$$

Note that since f is at least 0 and monotonically decreasing, $\lim_{n \to \infty} f(n) \in \mathbb{R}$. We then get

$$f(n) + \sum_{k=2}^n f(k) \leq f(n) + \int_1^n f(x)\,dx \leq \sum_{k=1}^n f(k) \leq f(1) + \int_2^n f(x)\,dx.$$

But then $\int_1^\infty f(x)\,dx$ and $\sum_{n=1}^\infty f(n)$ are bounded together, and the result follows. □

Problem 5

Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

Proof. Fix u. Define $f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$. Then, $f'(v) = v^{q-1} - u, f''(v) = (q-1)v^{q-2} \geq 0$, for $v \geq 0$. Hence, $f(v)$ reaches minimum at $v = u^{\frac{1}{q-1}}$. Note that $p = \frac{q}{q-1}$. But then

$$\begin{aligned} f(v) &= \frac{u^p}{p} + \frac{v^q}{q} - uv \\ &\geq \frac{u^p}{p} + \frac{u^{\frac{q}{q-1}}}{q} - u^{\frac{q}{q-1}} \\ &= \left(\frac{1}{p} + \frac{1}{q} - 1\right) u^p = 0, \end{aligned}$$

and the result follows. □

(b) If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

Proof. By (a),

$$\int_a^b \frac{f^p}{p} d\alpha + \int_a^b \frac{g^q}{q} d\alpha = \int_a^b \frac{f^p}{p} + \frac{g^q}{q} d\alpha \geq \int_a^b fg d\alpha.$$

But then

$$1 = \frac{1}{p} + \frac{1}{q} = \int_a^b fg d\alpha.$$

□

(c) If f and g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left|\int_a^b fg d\alpha\right| \leq \left(\int_a^b |f|^p d\alpha\right)^{1/p} \left(\int_a^b |g|^q d\alpha\right)^{1/q}.$$

This is Hölder's inequality. When $p = q = 2$, it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

Problem 5 continued on next page. . .

Proof. Put $F = \int_a^b |f|^p d\alpha$, $G = \int_a^b |g|^p d\alpha$. Since $f, g \in \mathscr{R}(\alpha)$, $|f|, |g| < M \in \mathbb{R}$. Note that $F = 0$ implies $\int_a^b |f| d\alpha = 0$. Thus

$$0 = M \int_a^b |f| \, d\alpha \geq \int_a^b |f||g| \, d\alpha \geq \left| \int_a^b fg d\alpha \right|,$$

and the inequality holds.

Hence, we may assume $F, G > 0$. Substituting f as $\frac{|f|}{F^{1/p}}$ and g as $\frac{|g|}{G^{1/q}}$, we get

$$\int_a^b \frac{|f||g|}{F^{1/p} G^{1/q}} \, d\alpha \leq= 1.$$

But then

$$\left| \int_a^b |fg| \, d\alpha \right| \leq \int_a^b |f||g| \, d\alpha \leq F^{1/p} G^{1/q}.$$

$\square$

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 6.7 and 6.8.

Proof. Since the equality holds for any finite interval, and thus the inequality also holds if the improper integrals converge.

Suppose the improper integral of f or g diverge, the right-hand side of the inequality tends to infinity, and thus the inequality still holds. $\square$