# Superimposed Extremal Graphs

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## 1 Introduction

Given graph $G$ with $n$ vertices, let $G_1, \ldots, G_m$ be subgraphs of $G$. Let $F$ be a graph with at least one edge. Our goal is to determine the maximum sum of the number of edges over all $G_i$'s, i.e. $\sum_{i=1}^m e(G_i)$, with the constraint of $E(G_i) \cap E(G_j)$ not including some graph $F$ for all distinct $i, j$.

## 2 Content

- Examine the case where $G_1, \ldots, G_m$ are induced
  - The case $F = K_3$.
  - Color-critical $F$.
  - Generalize to any non-bipartite $F$.
- Examine the non-induced case
  - The case $F = K_3$.

## 3 Induced Case

In this section, we assume that $G_1, \ldots, G_m$ are induced subgraphs of $G$. Given graph $H$, let $\mathcal{T}(H)$ be the graph with an additional vertex connecting to all vertices in $H$.

### 3.1 Triangle Case

**Theorem 3.1.** *Suppose that $E(G_i) \cap E(G_j)$ does not include $K_3$ for distinct $i, j$. Then*

$$\sum_{i=1}^n e(G_i) \leq n \left\lfloor \frac{n^2}{4} \right\rfloor,$$

*with equality if and only if $G_1 = G_2 = \cdots = G_n = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

**Lemma 3.2.** *Suppose $E(G_1) \cap E(G_2)$ does not include $K_3$. Then*

$$e(G_1) + e(G_2) \leq 2 \left\lfloor \frac{n^2}{4} \right\rfloor,$$

*with equality if and only if $G_1 = G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$, unless $n$ is odd and $G_1 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and $G_2 = \mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$.*

*Proof.* Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both $G_1$ and $G_2$. Let $A = V(G_1)\backslash C$, and let $B = V(G_2)\backslash C$. For simplicity, put $a = |A|$, $b = |B|$, and $c = |C|$. We may assume that $a + b + c = n$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to $a, b, c$. Since $G_1, G_2$ are induced graphs, we have $\{u, v\} \in E(G_1)$ if and only if $\{u, v\} \in E(G_2)$, for $u, v \in C$. This implies the subgraph of $G_1$ induced by $C$ is identical to the subgraph of $G_2$ induced by $C$. In other words, $E(G_1[C]) = E(G_2[C]) = E(G_i) \cap E(G_j)$, which is triangle-free. By Mantel's Theorem, $e(G_1[C]) \leq \left\lfloor \frac{c^2}{4} \right\rfloor$, with equality if and only if $G_1[C] = K_{\lceil \frac{c}{2} \rceil, \lfloor \frac{c}{2} \rfloor}$. Hence, we may write

$$\begin{aligned} e(G_1) + e(G_2) &\leq \binom{|V(G_1)|}{2} + \binom{|V(G_2)|}{2} - 2\left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor\right] \\ &= \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor\right]. \end{aligned} \tag{1}$$

Define $f(a, b, c)$ as the function on the right-hand-side of (1). We show that $f(a, b, c)$ attains its maximum at $a = b = 0$ and $c = n$. Note that

$$\begin{aligned} f(a, b-2, c+2) - f(a, b, c) &= \binom{a+c+2}{2} - \binom{a+c}{2} \\ &\quad - 2\left[\binom{c+2}{2} - \binom{c}{2} - \left\lfloor \frac{(c+2)^2}{4} \right\rfloor + \left\lfloor \frac{c^2}{4} \right\rfloor\right] \\ &= 2(a+c) + 1 - 2[2c + 1 - (c+1)] \\ &= 2a + 1 > 0. \end{aligned}$$

By symmetry, $f(a-2, b, c+2) > f(a, b, c)$, and thus $f$ attains its maximum when $c$ is $n-1$ or $n$, that is, $a + b \leq 1$. Equation (1) now yields,

$$e(G_1) + e(G_2) \leq f(a, b, c) \leq 2\left\lfloor \frac{n^2}{4} \right\rfloor.$$

Assume that $a = 0$. When $c = n$, the equality holds only if $G_1 = G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. If $c = n-1$, then the equality holds only if $n$ is odd and $G_1 = G[C] = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and $G_2$ is $G_1$ with all vertices connected with the only remaining vertex, that is, $G_2 = \mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$. □

We now give the proof for Theorem 3.1:

*Proof of Theorem 3.1.* We may assume that $n > 1$. Put $G_{n+i} = G_i$. By Lemma 3.2,

$$\sum_{i=1}^{n} e(G_i) = \frac{1}{2}\sum_{i=1}^{n}(e(G_i) + e(G_{i+1})) \leq \frac{1}{2}\sum_{i=1}^{n} 2\left\lfloor \frac{n^2}{4} \right\rfloor = n\left\lfloor \frac{n^2}{4} \right\rfloor.$$

Suppose the equality holds. By Lemma 3.2, we are done if $n$ is even. Suppose $n$ is odd and $G_i = \mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$ for some $i$. By Lemma 3.2, one of $G_i$ and $G_{i+1}$ is $K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and the other is $\mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$, for all $i$. Hence, $G_{i+1} = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}, G_{i+2} = \mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}), \ldots$ and the alternation proceeds. But then $G_{n+i} = G_i = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ as $n$ is odd, and this contradiction completes the proof. □

### 3.2 Color-Critical Case

We may generalize the triangle case to any color-critical $F$ in the same manner. Let Turán graph $T_r(n)$ denote the the complete balanced $r$-partite graph. We know for a fact that, for large enough $n$, if $F$ is a $(r+1)$-color-critical graph with $r \geq 2$, then $\mathrm{ex}(n, F) = \mathrm{ex}(n, K_{r+1})$, and the extremal graph is $T_r(n)$.

**Theorem 3.3.** *Let $F$ be a $(r+1)$-color-critical graph with $r \geq 2$. Suppose that $E(G_i) \cap E(G_j)$ is $F$-free for distinct $i, j$. For large enough $n$,*

$$\sum_{i=1}^{n} e(G_i) \leq n \cdot \mathrm{ex}(n, F),$$

*with equality if and only if $G_1 = G_2 = \cdots = G_n = T_r(n)$.*

**Lemma 3.4.** *Let $F$ be a $(r+1)$-color-critical graph with $r \geq 2$. Suppose $E(G_1) \cap E(G_2)$ does not include $F$. For large enough $n$,*

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n, F),$$

*with equality if and only if $G_1 = G_2 = T_r(n)$, unless $r = 2$, $n$ is odd, $G_1$ is an $(n-1)$-vertex extremal graph for $F$, and $G_2 = \mathcal{T}(G_1)$.*

*Proof.* Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both $G_1$ and $G_2$. Let $A = V(G_1) \backslash C$, and let $B = V(G_2) \backslash C$. For simplicity, put $a = |A|$, $b = |B|$, and $c = |C|$. We may assume that $a + b + c = n$. By the same argument in Lemma 3.2, $E(G_1[C]) = E(G_2[C]) = E(G_i) \cap E(G_j)$, which is $F$-free. Thus,

$$E(G_1[C]) \leq \mathrm{ex}(n, F) = \mathrm{ex}(n, K_{r+1}),$$

with equality if and only if $G_1[C] = T_r(c)$. Hence,

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \mathrm{ex}(c, K_{r+1})\right]. \tag{2}$$

Define $f(a, b, c)$ as the function on the right-hand-side of (2). We show that $f(a, b, c)$ attains its maximum at $a = b = 0$ and $c = n$. Note that

$$\begin{aligned} f(a, b-2, c+2) - f(a, b, c) &= \binom{a+c+2}{2} - \binom{a+c}{2} \\ &\quad - 2\left[\binom{c+2}{2} - \binom{c}{2} - \mathrm{ex}(c+2, K_{r+1}) + \mathrm{ex}(c, K_{r+1})\right] \\ &= 2a - 2c - 1 + 2[\mathrm{ex}(c+2, K_{r+1}) - \mathrm{ex}(c, K_{r+1})]. \end{aligned}$$

Since $r \geq 2$,

$$\begin{aligned} \mathrm{ex}(c+2, K_{r+1}) - \mathrm{ex}(c, K_{r+1}) &= \mathrm{ex}(c+2, K_{r+1}) - \mathrm{ex}(c+1, K_{r+1}) \\ &\qquad + \mathrm{ex}(c+1, K_{r+1}) - \mathrm{ex}(c, K_{r+1}) \\ &= \left(c + 2 - \left\lceil \frac{c+2}{r} \right\rceil\right) + \left(c + 1 - \left\lceil \frac{c+1}{r} \right\rceil\right) \\ &\geq 2c + 3 - \left(\left\lceil \frac{c+2}{2} \right\rceil + \left\lceil \frac{c+1}{2} \right\rceil\right) = c + 1, \end{aligned}$$

so $f(a, b-2, c+2) - f(a, b, c) \geq 2a + 1 > 0$. By symmetry, $f(a-2, b, c+2) > f(a, b, c)$, and thus $f$ attains its maximum when $c$ is $n-1$ or $n$, that is, $a+b \leq 1$. Equation (2) now yields,

$$e(G_1) + e(G_2) \leq \max\left[2 \cdot \mathrm{ex}(n, K_{r+1}),\ 2 \cdot \mathrm{ex}(n-1, K_{r+1}) + n - 1\right].$$

Assume that $a = 0$. Since

$$\begin{aligned} 2 \cdot \mathrm{ex}(n, K_{r+1}) - [2 \cdot \mathrm{ex}(n-1, K_{r+1}) + n - 1] &= 2\left(n - \left\lceil \frac{n}{r} \right\rceil\right) - n + 1 \\ &\geq n + 1 - 2\left\lceil \frac{n}{2} \right\rceil \geq 0, \end{aligned} \tag{3}$$

we have

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n, K_{r+1}). \tag{4}$$

If $c = n$, the equality for (4) holds only if $G_1 = G_2 = T_r(n)$. Suppose $c = n-1$ and the equality holds. Observe that the equation (3) is equal to zero only when $r = 2$ and $n$ is odd. Hence, if $c = n-1$, the equality for (4) could only be achieved when $r = 2$, $n$ is odd, $G_1 = T_r(n-1)$, and $G_2 = \mathcal{T}(G_1)$. $\square$

Theorem 3.3 now follows from Lemma 3.5 and the same arument as in Theorem 3.1.

### 3.3 Non-bipartite Case

**Theorem 3.5.** *Let $F$ be $(r+1)$-colorable, with $r \geq 2$. Suppose that $E(G_i) \cap E(G_j)$ is $F$-free for distinct $i, j$. For large enough $n$,*

$$\sum_{i=1}^{n} e(G_i) \leq n \cdot \mathrm{ex}(n, F),$$

*with equality if and only if $G_1 = G_2 = \cdots = G_n$ are $n$-vertex extremal graphs for $F$.*

By the same argument as in Theorem 3.1, it suffices to prove the following lemma:

**Lemma 3.6.** *Let $F$ be $(r+1)$-colorable, with $r \geq 2$. Suppose $E(G_1) \cap E(G_2)$ does not include $F$. For large enough $n$,*

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n, F),$$

*with equality if and only if $G_1 = G_2$ are $n$-vertex extremal graphs for $F$, unless $n$ is odd, $G_1$ is an $(n-1)$-vertex extremal graph for $F$, and $G_2 = \mathcal{T}(G_1)$.*

*Proof.* Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both $G_1$ and $G_2$. Let $A = V(G_1) \backslash C$, and let $B = V(G_2) \backslash C$. For simplicity, put $a = |A|$, $b = |B|$, $c = |C|$, and $r = \chi(F)$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to $a, b, c$. Since $G_1, G_2$ are induced graphs, we have $E(G_1[C]) = E(G_2[C]) = E(G[C]) = E(G_i) \cap E(G_j)$, which is $F$-free. Hence, we may write

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \mathrm{ex}(c, F)\right]. \tag{5}$$

Define $f(a,b,c)$ as the function on the right-hand-side. We show that $f(a,b,c)$ attains its maximum at $a = b = 0$ and $c = n$. By a theorem of Simonovits, for large enough $c$, $\mathrm{ex}(c,F) = \mathrm{ex}(c,K_{r+1}) + \mathrm{ex}(c,\tilde{F})$, where $\tilde{F}$ is the family of residue subgraphs of $F$ after $F$ is embedded into $T_r(c)$. Hence, we may write

$$\begin{aligned} f(a,b-2,c+2) - f(a,b,c) = &\binom{a+c+2}{2} - \binom{a+c}{2} \\ &- 2\left[\binom{c+2}{2} - \binom{c}{2} - \mathrm{ex}(c+2,F) + \mathrm{ex}(c,F)\right] \\ &\geq 2a - 2c - 1 + 2[\mathrm{ex}(c+2,K_{r+1}) - \mathrm{ex}(c,K_{r+1})] > 0, \end{aligned}$$

as shown in the proof of Lemma 3.4. By symmetry, we also have $f(a-2,b,c+2) > f(a,b,c)$. Thus, $f$ attains its maximum when $c$ is $n-1$ or $n$. Equation (5) now yields,

$$e(G_1) + e(G_2) \leq \max\left[2 \cdot \mathrm{ex}(n,F), 2 \cdot \mathrm{ex}(n-1,F) + n - 1\right].$$

Assume that $a = 0$. Since

$$\begin{aligned} 2 \cdot \mathrm{ex}(n,F) - [2 \cdot \mathrm{ex}(n-1,F) + n - 1] &\geq 2[\mathrm{ex}(n,K_{r+1}) - \mathrm{ex}(n-1,K_{r+1})] && (6) \\ &\quad - n + 1 && (7) \\ &= 2\left(n - \left\lceil \frac{n}{r} \right\rceil\right) - n + 1 && (8) \\ &\geq n + 1 - 2\left\lceil \frac{n}{2} \right\rceil \geq 0, \end{aligned}$$

we have

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n,F). \tag{9}$$

If $c = n$, the equality for (9) holds only if $G_1 = G_2$ are $n$-vertex extramal graphs for $F$. Suppose $c = n-1$ and the equality holds. Observe that equation (6) is equal to zero only when $r = 2$ and $n$ is odd. Hence, if $c = n-1$, the equality for (9) could only be achieved when $r = 2$, $n$ is odd, $G_1$ is an $(n-1)$-vertex extremal graph for $F$, and $G_2 = \mathcal{T}(G_1)$. □

# 4 Non-induced Case

We now remove the assumption that $G_1, \ldots, G_m$ are induced subgraphs. Again, we first consider the triangle-free case.

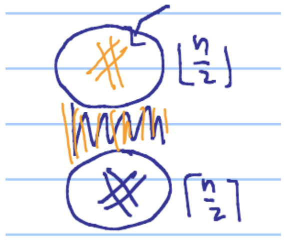
## 4.1 Triangle-Free Case

**Theorem 4.1.** *Suppose that $E(G_i) \cap E(G_j)$ does not include $K_3$ for distinct $i,j$. Then,*

$$\sum_{i=1}^{m} e(G_i) \leq \binom{n}{2} + (m-1)\left\lfloor \frac{n^2}{4} \right\rfloor.$$

The natural extremal construction is to simply put $G_1 = K_n$ and the rest as $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. However, even for $m = 2$ there are multiple extremal constructions.

For example, put $G_1$ as $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and connect all possible pairs of vertices on the left part. On the other hand, put $G_2$ as $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and connect all possible pairs of vertices on the right part.




Then, $E(G_1) \cap E(G_2)$ is triangle-free and

$$\begin{aligned} e(G_1) + e(G_2) &= 2e(G_1 \cap G_2) + e(G_1 \Delta G_2) \\ &= 2\left\lfloor \frac{n^2}{4} \right\rfloor + \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor = \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

Here we introduce the notation of *compression* of $G_1, \ldots, G_m$, which is the graph obtained by moving all edges in only one $G_i$ to $G_1$. Performing compression for the case $m = 2$, we get

$$e(G_1) + e(G_2) = e(G_1) + e(G_1 \cap G_2) \leq \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = K_n$ and $G_2 = G_1 \cap G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. That is, the extremal graphs for $m = 2$ are isomorphic, up to compression.

We use the notion of compression to solve for $m = 3, 4$:

**Theorem 4.2.** *Suppose that $E(G_i) \cap E(G_j)$ does not include $K_3$ for distinct $i, j$. Then,*

$$e(G_1) + e(G_2) + e(G_3) \leq \binom{n}{2} + 2\left\lfloor \frac{n^2}{4} \right\rfloor,$$

*with equality if and only if $G_1 = K_n$ and $G_2, G_3 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ after compression.*

*Proof.* See Chase Wilson's proof. □

TODO: solve $m = 4$.

**Theorem 4.3.** *Suppose that $E(G_i) \cap E(G_j)$ does not include $K_3$ for distinct $i, j$. Then,*

$$\sum_{i=1}^{m} e(G_i) \leq (1 + o_m(1))m \left\lfloor \frac{n^2}{4} \right\rfloor,$$

*as $m \to \infty$.*

*Proof.* forgot... □