MATH 188: Homework #5

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Problem 1

Let B_n be the set of binary strings of length n, i.e., of words of length n in the alphabet $\{0, 1\}$. Define a simple graph H_n whose vertex set is B_n and where two binary strings are connected by an edge if and only if they differ in exactly one position.

Let V be the real vector space with basis $\{v_x \mid x \in B_n\}$. It's not convenient to pick an ordering of the vertices to write down the adjacency matrix. Instead, we will think of the adjacency matrix A of H_n as a linear operator on V:

$$A \sum_{x \in B_n} c_x v_x = \sum_{x \in B_n} c_x \sum_{\substack{y \text{ such that} \\ \{y,x\} \text{ is an edge}}} v_y.$$

(a) For each $x \in B_n$, define $E_x \in V$ by

$$E_x = \sum_{y \in B_n} (-1)^{x \cdot y} v_y,$$

where $x \cdot y = x_1 y_1 + \ldots + x_n y_n$. Show that this is an eigenvector for A with eigenvalue $n - 2|x|$ where $|x|$ is the number of x_i equal to 1.

Proof. Let $x \in B_n$. Let $N(x)$ denote the set of neighbors of x in H_n, and let x_i denote the ith digit of x. We first note that

$$E_x = \sum_{\substack{y \in B_n \\ x \cdot y \text{ is even}}} v_y - \sum_{\substack{y \in B_n \\ x \cdot y \text{ is odd}}} v_y$$

Hence, we may write

$$\begin{aligned}
AE_x &= A \sum_{\substack{y \in B_n \\ x \cdot y \text{ is even}}} v_y - A \sum_{\substack{y \in B_n \\ x \cdot y \text{ is odd}}} v_y && (1)\\
&= \sum_{\substack{y \in B_n \\ x \cdot y \text{ is even}}} \sum_{y' \in N(y)} v_{y'} - \sum_{\substack{y \in B_n \\ x \cdot y \text{ is odd}}} \sum_{y' \in N(y)} v_{y'} && (2)\\
&= \sum_{\substack{y \in B_n \\ x \cdot y \text{ is even}}} \left(\sum_{\substack{y' \in N(y) \\ x \cdot y' \text{ is even}}} v_{y'} + \sum_{\substack{y' \in N(y) \\ x \cdot y' \text{ is odd}}} v_{y'} \right) - \sum_{\substack{y \in B_n \\ x \cdot y \text{ is odd}}} \left(\sum_{\substack{y' \in N(y) \\ x \cdot y' \text{ is even}}} v_{y'} + \sum_{\substack{y' \in N(z) \\ x \cdot y' \text{ is odd}}} v_{y'} \right). && (3)
\end{aligned}$$

Given a $y' \in B_n$. Let $y \in N(y')$ such that $y_i \neq y'_i$. Suppose $x \cdot y'$ is even. Notice that $x \cdot y$ remains even if and only if $x_i = 0$. Similarly, when $x \cdot y'$ is odd, $x \cdot y$ remains odd if and only if $x_i = 0$.

Hence, y' has $n - |x|$ number of neighbors y such that $x \cdot y' \equiv x \cdot y \pmod 2$ and $|x|$ number of neighbors y such that $x \cdot y' \not\equiv x \cdot y \pmod 2$

But then for all $y' \in B_n$, (3) adds $v_{y'}$ $(n - 2|x|)$ times if $x \cdot y'$ is even and adds $-v_{y'}$ $(n - 2|x|)$ times if if $x \cdot y'$ is odd. It now follows that

$$AE_x = (n - 2|x|) \left(\sum_{\substack{y \in B_n \\ x \cdot y \text{ is even}}} v_y - \sum_{\substack{y \in B_n \\ x \cdot y \text{ is odd}}} v_y \right) = (n - 2|x|) E_x.$$

□

Problem 1 continued on next page...

(b) For $x \in B_n$ and a non-negative integer d, give a formula for the number of closed walks of length d in H_n beginning at x. [You may assume without proof that $\{E_x \mid x \in B_n\}$ is linearly independent.]

Proof. Let $x, y \in B_n$. We first show that

$$\#\text{closed walks of } x \text{ of length } d = \#\text{closed walks of } y \text{ of length } d. \tag{4}$$

Suppose x and y differ at positions $i_1, i_2, \ldots, i_N$. Let W be a length-d closed walk of x, say $w_1 w_2 \ldots w_d$. For each w_i in W, we flip w's i_kth digit, for $i \leq k \leq N$. Let W' be the resulting sequence of strings. Obviously, $w'_1 = w'_d = y$. Since we flip the same positions of each string in W, the operation above perserves the difference of positions between each string. That is, for each consecutive strings w'_i and w'_{i+1} in W', w'_i and w'_{i+1} differ at only 1 position. Hence, $\{w'_i, w'_{i+1}\}$ is an edge in H_n. But then W' is a closed walk of y of length d. Note that applying the operation again on W' yields W, and thus the equality of (4).

But then (4) also implies that the number of closed walks of length d beginning at any vertex is the same. By Proposition 4.6, the number of closed walks in H_n of length

$$tr(A^d) = \sum_{x \in B_n} (n - 2|x|)^d = \sum_{i=0}^{n} \binom{n}{i} (n - 2i)^d$$

Hence,

$$\#\text{closed walks of } x \text{ of length } d = 2^{-n} \sum_{i=0}^{n} \binom{n}{i} (n - 2i)^d$$

$\square$

Problem 2

(a) Fix a positive integer k. Construct a directed graph for which walks (between certain vertices) can be interpreted as binary strings with exactly k zeroes. Explain clearly how this interpretation works, including how the length of the walk relates to the length of the binary string.

Proof. $\square$

(b) Construct a directed graph for which walks (between certain vertices) can be interpreted as binary strings in which no symbol ever appears 3 times in a row. Explain clearly how this interpretation works, including how the length of the walk relates to the length of the binary string.

Proof. Consider the following graph G:

$$\begin{tikzcd}
v_0 \arrow[r, bend left] \arrow[d] & v_1 \arrow[l, bend left] \arrow[d] \\
v_0' \arrow[ur] & v_1' \arrow[ul]
\end{tikzcd}$$

We now show a bijection between the walks in G and the binary strings in which no symbol ever appears 3 times in a row.

Given a walk in G, $\square$

Problem 3

(a) We have n distinguishable tables. We want to paint each one either red, blue, or green such that an odd number of them are red and an even number of them are blue. How many ways can this be done?

Proof. Suppose S is a set of tables. Let $\alpha(S)$ be the set of ways to paint the tables red according to our rules, so $|\alpha(S)| = 1$ if $|S|$ is odd and $|\alpha(S)| = 0$ otherwise. Similarly, let $|\beta(S)|$ be the set of ways to paint the table blue according to our rules, so $|\beta(S)| = 1$ if $|S|$ is even and $|\beta(S)| = 0$ otherwise. Finally, let $\gamma(S)$ be the set of ways to paint the table green according to our rules, so $|\gamma(S)| = 1$. Hence,

$$E_\gamma(x) = \sum_{n \geq 0} \frac{x^n}{n!} = e^x.$$

$$E_\beta(x) = \sum_{n \geq 0} \frac{x^{2n}}{(2n)!} = (e^x + e^{-x})/2.$$

$$E_\alpha(x) = \sum_{n \geq 0} \frac{x^{2n+1}}{(2n+1)!} = e^x - \sum_{n \geq 0} \frac{x^{2n}}{(2n)!} = (e^x - e^{-x})/2.$$

It now follows that

$$E_{\alpha\cdot\beta\cdot\gamma}(x) = E_\alpha E_\beta E_\gamma(x) = \frac{1}{4}e^x(e^x + e^{-x})(e^x - e^{-x}) = \frac{1}{4}(e^{3x} - e^{-x}) = \frac{1}{4}\sum_{n \geq 0} \frac{(3^n - (-1)^n)x^n}{n!}.$$

Hence, there are

$$n![x^n]E_{\alpha\cdot\beta\cdot\gamma}(x) = \frac{(3^n + (-1)^{n+1})}{4}$$

ways to color the the n tables. □

(b) Continuing with (a), we add the colors white and yellow, but the total number of tables which are white or yellow must be even (in symbols: #white tables + #yellow tables is even). How many ways are there to choose colors?

Proof. Let $\delta(S)$ be the set of ways to paint the tables either white or yellow according to our rules, so $|\delta(S)| = 2^{|S|}$ if $|S|$ is even and $|\delta(S)| = 0$ otherwise. Hence,

$$E_\delta(x) = \sum_{n \geq 0} \frac{2^n x^n}{n!} = e^{2x}.$$

It now follows that

$$E_{\alpha\cdot\beta\cdot\gamma\cdot\delta}(x) = E_{\alpha\cdot\beta\cdot\gamma}(x)E_\delta(x) = \frac{1}{4}(e^{5x} - e^x) = \frac{1}{4}\sum_{n \geq 0} \frac{(5^n - 1)x^n}{n!}.$$

Hence, there are

$$n![x^n]E_{\alpha\cdot\beta\cdot\gamma\cdot\delta}(x) = \frac{5^n - 1}{4}$$

ways to color n tables with our rules. □

Problem 4

Let $b_{n,k}$ be the number of set partitions of $[n]$ with k blocks such that every block has an even (and positive) number of elements and let b_n be the same, but with no restriction on the number of blocks.

(a) Find a formula for the EGF $B_k(x) = \sum_{n\geq 0} b_{n,k} \frac{x^n}{n!}$.

Proof. Consider the selection structure

$$\beta(S) = \begin{cases} \{1\} & \text{if } |S| > 0 \text{ and } |S| \text{ is even} \\ \emptyset & \text{otherwise} \end{cases},$$

which picks out nonempty subsets of even size. But then, $\beta^k(S)$ is the set of ordered set partitions of S with k blocks of even sizes. Hence,

$$\sum_{n\geq 0} k! b_{n,k} \frac{x^n}{n!} = E_{\beta^k}(x) = E_\beta(x)^k = \left(\frac{e^x + e^{-x}}{2} - 1\right)^k,$$

and thus

$$B_k(x) = \sum_{n\geq 0} b_{n,k} \frac{x^n}{n!} = \frac{(e^x + e^{-x} - 2)^k}{2^k k!}$$

□

(b) Find a formula for the EGF $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$.

Proof. We continue using the selection structure β from (a). Then, $e^\beta(S)$ can be interpreted as the set of partitions of $[n]$ such that all blocks are of even sizes. It now follows that

$$B(x) = E_{e^\beta}(x) = \exp(E_\beta(x)) = \exp\left(\frac{e^x + e^{-x}}{2} - 1\right).$$

□

Problem 5

Let a_n be the number of functions $f : [n] \to [n]$ such that $f \circ f = f$. Find a simple formula for the EGF $A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$.

Proof. For each $f : [n] \to [n]$, we associate a directed graph G_f with vertex set $[n]$ and edge set $\{(i, j) : f(i) = j\}$. Suppose $f \circ f = f$. Say we have $i, j \in [n]$ such that $f(i) = j$. Then, $f(j) = f(f(i)) = f(i) = j$. But then each connected component in G are of either of the following form:

$$\begin{tikzcd} i \arrow[loop above] \end{tikzcd} \qquad\qquad \begin{tikzcd} i \arrow[r] & j \arrow[loop above] \end{tikzcd}$$

Hence, G_f is partitioned into components of sizes 1 or 2. Note that given a component of G_f of size 2, there are 2 possible orientations. Now define structure

$$\alpha(S) = \begin{cases} S & \text{if } |S| = 1 \text{ or } 2 \\ \emptyset & \text{otherwise} \end{cases},$$

which picks out the components which could be in G_f, with each element in $\alpha(S)$ being the vertex in the component which self-loops. Then, $e^{\alpha}([n])$ can be interpreted as the set of possible graphs G_f, where $f : [n] \to [n]$ such that $f \circ f = f$. Thus, $e^{\alpha}([n])$ is also interpreted as the set of $f : [n] \to [n]$ such that $f \circ f = f$. It now follows that

$$A(x) = E_{e^{\alpha}}(x) = \exp(E_{\alpha}(x)) = \exp\left(x^2 + x\right) = e^{x^2} e^x.$$

□

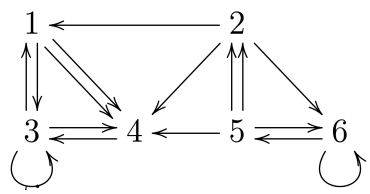
Problem 6

Explain why it is impossible to have a directed graph for which walks of length $2n$ can be interpreted as balanced sets of n pairs of parentheses.

Proof. $\square$

Problem 7

Let A be the adjacency matrix of the following directed graph:



Express $F_{A;1,4}(x)$ as a rational function.

Proof. We first note that

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

By Theorem 4.9, we have

$$F_{A;1,4}(x) = -\frac{\det((I_6 - xA);4,1)}{\det(I_6 - xA)}.$$

By MATLAB,

$$\det(I_6 - xA) = 4x^6 + 6x^5 + 6x^4 - x^3 - 2x^2 - 2x + 1,$$

$$\det((I_6 - xA);4,1) = x(-2x^4 + 3x^3 + x^2 + 3x - 2).$$

Hence,

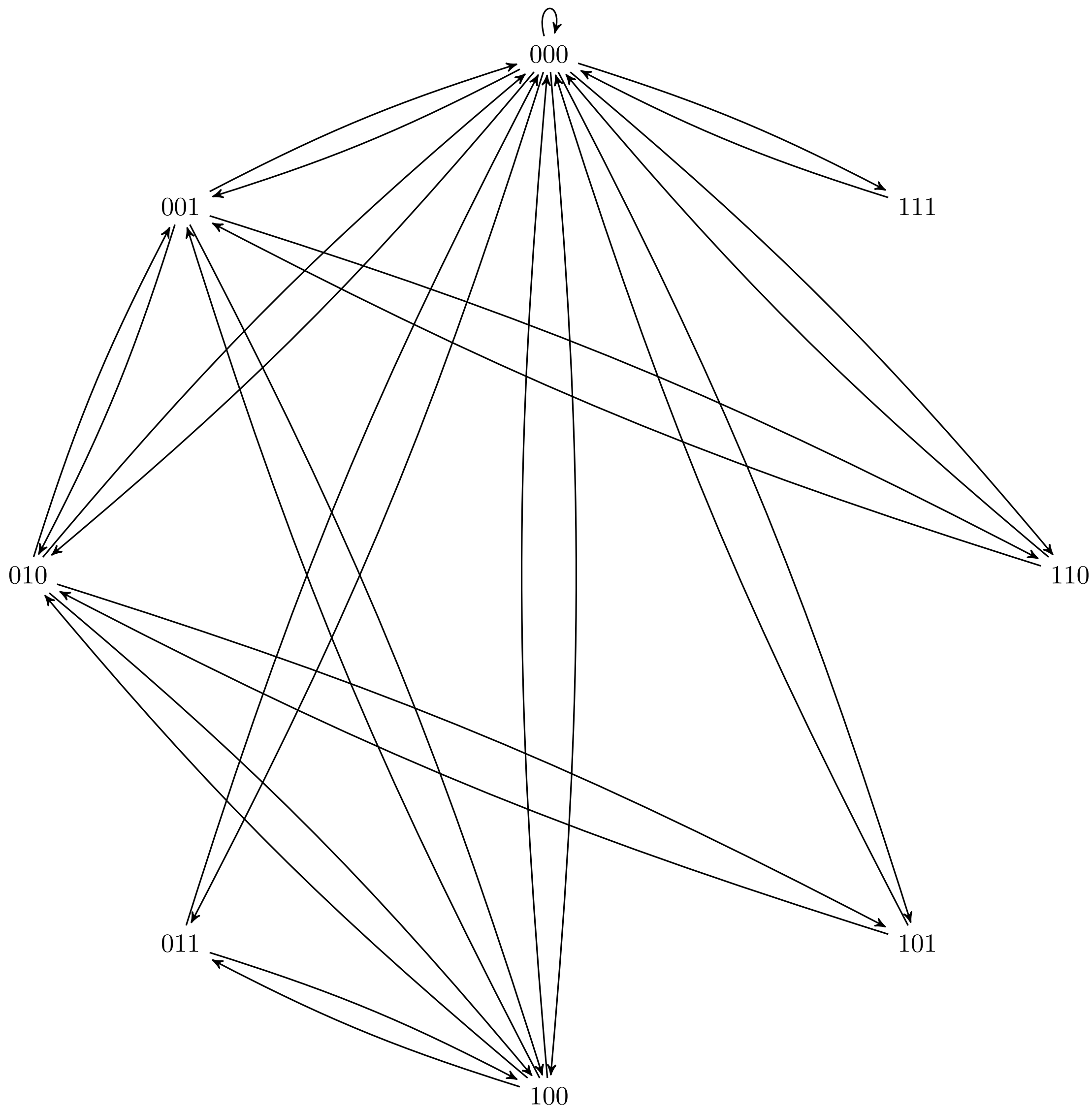
$$F_{A;1,4}(x) = -\frac{x(-2x^4 + 3x^3 + x^2 + 3x - 2)}{4x^6 + 6x^5 + 6x^4 - x^3 - 2x^2 - 2x + 1} = \frac{x(x-2)}{2x^3 + 2x^2 + x - 1}.$$

$\square$

Problem 8

Compute G_3 in Example 4.11.

Proof. Here is G_3:



□

Problem 9

Given a sequence $(a_n)_{n\geq 0}$ we define its q-EGF to be

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{[n]_q!}.$$

What is the q-analogue of Proposition 5.2?

Proof. Suppose we have $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{[n]_q!}$ and $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{[n]_q!}$. Then,

$$A(x) + B(x) = \sum_{n\geq 0} (a_n + b_n) \frac{x^n}{[n]_q!} \tag{5}$$

$$A(x)B(x) = \sum_{n\geq 0} \sum_{i=0}^{n} \frac{a_i}{[i]_q!} \frac{b_{n-i}}{[n-i]_q!} x^n = \sum_{n\geq 0} \left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q a_i b_{n-1} \right) \frac{x^n}{[n]_q!}. \tag{6}$$

Define q-structure α_q, which takes as input a finite vector space V of $\mathbf{F}_q$ and outputs a finite set $\alpha_q(V)$, with the property that $|\alpha_q(V)| = |\alpha_q(U)|$ whenever $|V| = |U|$. Let α_q, β_q be q-structures. Define the addition and multiplication of q-structures to be

$$(\alpha_q + \beta_q)(V) = \alpha_q(V) \sqcup \beta_q(V)$$
$$(\alpha_q \cdot \beta_q)(V) = \bigsqcup_{S \text{ subspace of } V} \alpha_q(S) \times \beta_q(S^{\perp}).$$

Define the q-EGF associated to structure α_q to be

$$E_{q_\alpha}(x) = \sum_{n\geq 0} |\alpha(\mathbf{F}_q^n)| \frac{x^n}{[n]_q!}.$$

We show that

$$E_{\alpha_q+\beta_q}(x) = E_{\alpha_q}(x) + E_{\beta_q}(x), \quad E_{\alpha_q\cdot\beta_q}(x) = E_{\alpha_q}(x)E_{\beta_q}(x).$$

For the sum, we have $|(\alpha+\beta)(\mathbf{F}_q^n)| = |(\alpha)(\mathbf{F}_q^n)| + |(\beta)(\mathbf{F}_q^n)|$, which is the coefficient of $E_{\alpha_q}(x) + E_{\beta_q}(x)$ by (5). On the other hand , we have

$$|(\alpha \cdot \beta)(\mathbf{F}_q^n)| = \sum_{S \text{ subspace of } \mathbf{F}_q^n} |\alpha(S)| \cdot |\beta(S^{\perp})| = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q |\alpha(\mathbf{F}_q^i)| \cdot |\beta(\mathbf{F}_q^{n-i})|.$$

for the product, which is the coefficient of $E_{\alpha_q}(x)E_{\beta_q}(x)$ by (6). □

Problem 10

Let n be a positive integer. Given a group of n people, we want to divide them into nonempty committees and choose a leader for each committee, and also choose one of the committees to be in charge of all the others. Let h_n be the number of ways to do this and set $h_0 = 1$. Give a simple expression for the exponential generating function $H(x) = \sum_{n \geq 0} h_n \frac{x^n}{n!}$.

Proof. Define

$$\alpha(S) = S$$

which outputs the possible leader of a committee S. Then, e^α can be interpreted as the set of possible committee divisions of n people, with each committee having a leader. Hence,

$$H(x) = E_{e^\alpha} = \exp(E_\alpha(x)) = \exp\left(\sum_{n\geq 0} \frac{nx^n}{n!}\right) = \exp\left(\sum_{n \geq 1} \frac{x^n}{(n-1)!}\right) = \exp(xe^x).$$

□

Problem 11

Let h_n be the number of bijections $f : [n] \to [n]$ with the property that $f \circ f \circ f$ is the identity function. Give a simple expression for the exponential generating function $H(x) = \sum_{n \geq 0} h_n \frac{x^n}{n!}$.

Proof. Suppose bijection $f : [n] \to [n]$ has the property that $f \circ f \circ f$ is the identity function. Then, f is a permutation on $[n]$ whose cycles all have length 1 or 3. Let $\alpha(S)$ be the set of cyclic orderings of S if $|S| = 1$ or 3 and $\alpha(S) = \emptyset$ otherwise. Then, $E_\alpha = x + \frac{x^3}{3}$. But then e^{alpha} can be interpreted as the set of bijections f with $f \circ f \circ f$ being the identity. It now follows that

$$E_{e^\alpha}(x) = \exp\left(x + \frac{x^3}{3}\right).$$

$\square$

Problem 12

Given $G(x)$ with $G(0) \neq 0$, define its *logarithmic derivative* to be $\mathcal{L}(G) = \frac{DG(x)}{G(x)}$.

(a) Show that for any $F(x)$, we have $\mathcal{L}(\exp(F(x))) = DF(x)$.

Proof. We first note that

$$D\exp(F(x)) = DF(x)\sum_{n\geq 1}\frac{(F(x))^{n-1}}{(n-1)!} = (DF(x))\exp(F(x)).$$

Hence,

$$\mathcal{L}(\exp(F(x))) = \frac{D\exp(F(x))}{\exp(F(x))} = \frac{(DF(x))\exp(F(x))}{\exp(F(x))} = DF(x).$$

□

(b) Show that if $G_1(0) = G_2(0) = 0$, then $\mathcal{L}(G_1(x)G_2(x)) = \mathcal{L}(G_1(x)) + \mathcal{L}(G_2(x))$.

Proof.

$$\begin{aligned}\mathcal{L}(G_1(x)G_2(x)) &= \frac{D(G_1(x)G_2(x))}{G_1(x)G_2(x)} \\ &= \frac{G_2(x)DG_1(x) + G_1(x)DG_2(x)}{G_1(x)G_2(x)} \\ &= \frac{DG_1(x)}{G_1(x)} + \frac{DG_2(x)}{G_2(x)} = \mathcal{L}(G_1(x)) + \mathcal{L}(G_2(x)).\end{aligned}$$

□

(c) Let a_n be the number of involutions of size n and let $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$. From Example 5.11, we have $A(x) = \exp(x + \frac{x^2}{2})$. Apply $\mathcal{L}$ to prove for all $n \geq 0$ that $a_{n+2} = a_{n+1} + (n+1)a_n$.

Proof. By (a) and (b),

$$\frac{DA(x)}{A(x)} = \mathcal{L}(A(x)) = \mathcal{L}(e^x e^{x^2/2}) = \mathcal{L}(e^x) + \mathcal{L}(e^{x^2/2}) = Dx + D\left(\frac{x^2}{2}\right) = 1 + x.$$

Hence,

$$D(A(x)) = (1+x)A(x).$$

Since

$$D(A(x)) = \sum_{n\geq 1} a_n \frac{x^{n-1}}{(n-1)!} = \sum_{n\geq 0} a_{n+1}\frac{x^n}{n!},$$

$$\begin{aligned}(1+x)A(x) &= \sum_{n\geq 0} a_n \frac{x^n}{n!} + \sum_{n\geq 0} a_n \frac{x^{n+1}}{n!} \\ &= a_0 + \sum_{n\geq 0} a_n \frac{x^n}{n!} + \sum_{n\geq 1} a_{n-1}\frac{x^n}{(n-1)!},\end{aligned}$$

for all $n \geq 1$,

$$a_{n+1} = a_n + na_{n-1},$$

and the result now follows. □