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Math 140A A02 **Homework 1** **Professor Mohammadi**

Question A. Let
$$E = \left\{ \frac{5n+8}{11n} : n \in \mathbb{N} \right\}.$$
Compute $\sup E$ and $\inf E$. Justify your answer.

Solution. We will show that $\sup E = \frac{13}{11}$ and $\inf E = \frac{5}{11}$. Since $n \in \mathbb{N}$,

$$\begin{aligned} n &\geq 1 \\ 1 \geq \frac{1}{n} &\geq 0 \\ \frac{8}{11} \geq \frac{8}{11n} &\geq 0 \\ \frac{13}{11} \geq \frac{5n+8}{11n} &\geq \frac{5}{11}, \end{aligned}$$

and thus $\frac{13}{11}$ and $\frac{5}{11}$ are a upper bound and a lower bound of E respectively. Let $s < \frac{13}{11}$. Since $\frac{13}{11} \in E$, s is not a upper bound of E. Therefore, $\sup E = \frac{13}{11}$.

We will now show $\inf E = \frac{5}{11}$ by contradiction. Suppose for the sake of contradiction that there exists a lower bound l of E such that $l > \frac{5}{11}$. Then, for any $n \in \mathbb{N}$,

$$\begin{aligned} \frac{5n+8}{11n} &\geq l \\ \frac{8}{11l-5} &\geq n, \end{aligned}$$

contradiction as $\mathbb{N}$ is unbounded above. Therefore, $\inf E = \frac{5}{11}$. $\square$

Question B. Let S and T be two bounded subsets of the real numbers. Prove that

$$\sup(T \cup S) = \max\{\sup T, \sup S\}.$$

Proof. Assume without loss of generality that $\max\{\sup T, \sup S\} = \sup T$. For all $s \in S$ and $t \in T$, since $\sup T \geq t$ and $\sup T \geq \sup S \geq s$, we know $\sup T \geq x$, for all $x \in T \cup S$, which shows that $\sup T$ is an upper bound of $T \cup S$. Let $k < \sup T$. Then there exists some $p \in T \subseteq T \cup S$ such that $p > k$, and thus k is not an upper bound of $T \cup S$. Therefore, the statement of the question holds. $\square$

Question C. Let S and T be two bounded, nonempty, subsets of the set of positive real numbers. Define $ST := \{st : s \in S, t \in T\}$ and $S + T := \{s + t : s \in S, t \in T\}$. Prove that

$$\sup(ST) = (\sup S)\dot{(}\sup T) \text{ and } \sup(S + T) = \sup S + \sup T.$$

Proof. We first show that $\sup(ST) = (\sup S)\dot{(}\sup T)$. Let $t \in T$, $s \in S$. Since $s < \sup S$ and $t < \sup T$, we have $st < (\sup S)t < (\sup S)(\sup T)$, and thus $(\sup S)(\sup T)$ is an upper bound of ST. Let $k \in \mathbb{R}^+$, such that $k < (\sup S)(\sup T)$. Since $\frac{k}{\sup S} < \sup T$, there exists $t \in T$ such that $\frac{k}{\sup S} < t < \sup T$. Then, we also know that since $\frac{k}{t} < \sup S$, there exists $s \in S$, such that $\frac{k}{t} < s < \sup S$. Rearranged, we get $k < st \in ST$, which shows that k is not an upper bound of ST, and thus $\sup(ST) = (\sup S)(\sup T)$.

We now show that $\sup(S+T) = \sup S + \sup T$. Let $t \in T$, $s \in S$. Since $s < \sup S$ and $t < \sup T$, we have $s + t < \sup S + \sup T$, and thus $\sup S + \sup T$ is an upper bound of $S + T$. Let $k \in \mathbb{R}^+$, such that $k < \sup S + \sup T$. Since $k - \sup T < \sup S$, there exists $s \in S$ such that $k - \sup T < s$. Since $k - s < \sup T$, there exists $t \in T$ such that $k - s < t$, and thus we know there exist $s + t \in S + T$ such that $k < s + t < \sup S + \sup T$. Therefore, $\sup(S + T) = \sup S + \sup T$. $\square$

Question D. Let F be the set of all rational functions

$$\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} \tag{1}$$

where the coefficients are real numbers and $b_m \neq 0$.

(i) Define addition and multiplication of two elements in F to be the usual addition and multiplication of functions. Show that with this addition and multiplication, F is a field.

Proof. Let $A = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \,|\, a_n, \ldots, a_0 \in \mathbb{R}\}$, $B = \{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0 \,|\, b_m, \ldots, b_0 \in \mathbb{R} - \{0\}\}$. Let $a = \frac{f_1}{g_1}, b = \frac{f_2}{g_2}, c = \frac{f_3}{g_3} \in F$.

Associativity: Since

$$(a+b)+c = \frac{f_1 g_2 g_3 + f_2 g_1 g_3 + f_3 g_1 g_2}{g_1 g_2 g_3} = a + (b+c)$$

and

$$(ab)c = \frac{f_1 f_2 f_3}{g_1 g_2 g_3} = a(bc),$$

F is associative under $+$ and $\times$.

Commutativity: Since

$$a + b = \frac{f_1 g_2 + f_2 g_1}{g_1 g_2} = b + a$$

and

$$ab = \frac{f_1 f_2}{g_1 g_2} = ba,$$

F is commutative under $+$ and $\times$.

Additive and multiplicative identity: Since

$$a + 0 = 0 + a = a$$

and

$$a \cdot 1 = 1 \cdot a = a,$$

F has additive and multiplicative identity.

Additive inverses: For every a, we have $a^{-1} = -a \in F$, so that $a + (-a) = 0$.

Multiplicative inverses: For every $a \neq 0$, we have $a^{-1} = \frac{g_1}{f_1} \in F$. Note that $f_1 \in B$. Then, we have $aa^{-1} = \frac{f_1}{g_1} \cdot \frac{g_1}{f_1} = 1$.

Distributivity: Since

$$a(b+c) = \frac{f_1}{g_1} \cdot \frac{f_2 g_3 + f_3 g_2}{g_2 g_3} = (\frac{f_1}{g_1} \cdot \frac{f_2}{g_2}) + (\frac{f_1}{g_1} \cdot \frac{f_3}{g_3}) = (ab) + (ac),$$

F is distributive.

The above qualities show that F is a field under addition and multiplication. $\square$

(ii) We can define an order on F as follows. A rational function like (1) is positive if and only if a_n and b_m have the same sign, i.e. $a_n b_m > 0$. Now given two rational functions $\frac{p}{q}$ and $\frac{f}{g}$ we define:

$$\frac{p}{q} > \frac{f}{g} \text{ if and only if } \frac{p}{q} - \frac{f}{g} > 0.$$

Show with this ordering and the operations in part (i), F is an ordered field.

Proof. We continue using the defined sets A, B and elements $a, b, c \in F$ from part (i).

We first show that F is an ordered set. Let $n_1, m_1 \in \mathbb{R}$, $m_1 \neq 0$, each be the leading coefficient of f_1, g_1. Since $\mathbb{R}$ is an ordered set, we know $n_1 m_1$ must be either positive, negative, or equal to 0. This indicates that for all $f \in F$, f must be either positive, negative, or equal to 0. Since $a - b \in F$, it must be either positive, negative, or equal to 0. Therefore, since $a, b \in F$, one and only one of the following statements

$$a > b, \quad b > a, \quad a = b$$

is true.

Suppose $a > b$ and $b > c$, then $\frac{f_1 g_2 - f_2 g_1}{g_1 g_2} > 0$ and $\frac{f_2 g_3 - f_3 g_2}{g_2 g_3} > 0$. Combining two equations, we get $\frac{f_1 g_2 g_3 - f_2 g_1 g_3 + f_2 g_1 g_3 - f_3 g_1 g_2}{g_1 g_2 g_3} > 0$. It follows that

$$\frac{f_1 g_3 - f_3 g_1}{g_1 g_3} = a - c > 0.$$

Thus, F is an ordered set since it meets the two required conditions.

Suppose $c > b$. We know $a + c = \frac{f_1 g_3 + f_3 g_1}{g_1 g_3}$ and $a + b = \frac{f_1 g_2 + f_2 g_1}{g_1 g_2}$. Since $c > b$, we rearrange and get $f_3 g_2 > f_2 g_3$. Thus

$$\begin{aligned}
f_3 g_2 &> f_2 g_3 \\
f_3 g_2 g_1 &> f_2 g_3 g_1 \\
f_1 g_2 g_3 + f_3 g_2 g_1 &> f_1 g_2 g_3 + f_2 g_3 g_1 \\
\frac{f_1 g_3 + f_3 g_1}{g_1 g_3} &> \frac{f_1 g_2 + f_2 g_1}{g_1 g_2} && \text{dividing } g_1 g_2 g_3 \text{ on both sides} \\
a + c &> a + b.
\end{aligned}$$

Suppose a, b are positive. Let $n_1, n_2, m_1, m_2 \in \mathbb{R} - \{0\}$ each be the leading coefficient of f_1, f_2, g_1, g_2, we get $n_1 m_1, n_2 m_2 > 0$. Since the leading coefficient of the product of two polynomials is the product of the leading coefficients of the two polynomials, we know that the leading coefficient of $f_1 f_2$ and $g_1 g_2$ are $n_1 n_2$ and $m_1 m_2$, respectively. Since $n_1 m_1, n_2 m_2 > 0$, $n_1 n_2 m_1 m_2 > 0$, and thus $ab = \frac{f_1 f_2}{g_1 g_2}$ is also positive.

Since all the conditions are met, F is an ordered field. $\square$

(iii) Write the following polynomials in order of increasing size using the order defined in (ii): $x^2, -x^5, 2, x + 6, 3 - 2x$.

Solution. Since

$$\begin{gathered} x^2 - (x+6) = x^2 - x - 6 > 0, \\ x + 6 - 2 = x + 4 > 0, \\ 2 - (-2x+3) = 2x - 1 > 0, \\ -2x + 3 - (-x^5) = x^5 - 2x + 3 > 0, \end{gathered}$$

we have

$$x^2 > x + 6 > 2 > -2x + 3 > -x^5,$$

by the transitivity of ordered sets. □

(iv) Show that $x > a$ for all $a \in R$.

Proof. Let $a \in \mathbb{R}$. Since $x - a$ has a leading coefficient of 1, the statement holds true. □

Question E1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Proof. Let $r = \frac{m}{n}$, for $m, n \in \mathbb{Z}$, $\gcd(m, n) = 1$. We first show $r + x$ to be irrational. Suppose for the sake of contradiction that $r + x = \frac{p}{q}$, for $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$. Then $x = \frac{p}{q} - \frac{m}{n} = \frac{mq+np}{nq} \in \mathbb{Q}$, contradiction.

We now show rx to be irrational. Suppose for the sake of contradiction that $rx = \frac{k}{l}$, for $k, l \in \mathbb{Z}$, $\gcd(k, l) = 1$. Then $x = \frac{\frac{k}{l}}{\frac{m}{n}} = \frac{kn}{lm} \in \mathbb{Q}$, contradiction.

Therefore, both $r + x$ and rx are irrational. $\square$

Question E2. Prove that there is no rational number whose square is 12.

Proof. Let $p = \frac{m}{n}$, for $m, n \in \mathbb{Z}$, $\gcd(m, n) = 1$. Suppose for the sake of contradiction that $p^2 = 12$. We know $m^2 = 12n^2$, and so $m = 2k$, for $k \in \mathbb{Z}$. We then have $k^2 = 3n^2$, which implies that $3|k$. This shows that $m = 6l$, for $6 \in \mathbb{Z}$. Substituting it back into the equation, we get $3l^2 = n^2$, which shows that $3|m, n$, contradiction. Therefore, the statement of the question holds true. $\square$

Question E5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Proof. Let $k = \inf A$, $b \in -A$. Since $-b \in A$, we know $k < -b$. Therefore, $-k > b$, and thus $-k$ is an upper bound of $-A$. Let $m \in \mathbb{R}$, such that $m < -k$. Since $-m > k$, we know there exists $a \in A$, such that $-m > a$. Since $-a \in -A$ and $-a > m$, m is not an upper bound of $-A$. Therefore, $k = -\sup(-A)$. $\square$

Question E8. Prove that no order can be defined in the complex field that turns it into an ordered field.

Proof. Let $a, b \in \mathbb{C}$. Suppose for the sake of contradiction that there exists some ordering such that $a > b$. We then have

$$\begin{aligned} a &> b \\ ia &> ib \\ -a &> -b \\ a &< b, \end{aligned}$$

contradiction. Thus, the statement holds true. $\square$