

C8.3 Combinatorics: Sheet #1

Due on October 28, 2025 at 12:00pm

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Problem 1

Write down all antichains contained in $\mathcal{P}(1)$ and $\mathcal{P}(2)$. How many different antichains are there in $\mathcal{P}(3)$?

Proof. The antichains in $\mathcal{P}(1)$ are $\{\emptyset\}$ and $\{1\}$. The antichains in $\mathcal{P}(2)$ are $\{\emptyset\}$, $\{1\}$, $\{2\}$, $\{1, 2\}$, and $\{12\}$. There are 20 antichains in $\mathcal{P}(3)$. \square

Problem 2

- (a) Look up Stirling's Formula. Use it to find an asymptotic estimate for $\binom{n}{n/2}$ of the form $(1 + o(1))f(n)$ when n is even.

Proof. By Stirling's Formula,

$$\binom{n}{n/2} = \frac{n!}{(n/2)!(n/2)!} = \frac{(1 + o(1))\sqrt{2\pi n}(n/e)^n}{(1 + o(1))\pi n(n/2e)^n} = (1 + o(1))2^n \sqrt{\frac{2}{\pi n}}.$$

□

- (b) Now do the same for $\binom{n}{pn}$ where $p \in (0, 1)$ is a constant and pn is an integer. Write your answer in terms of the binary entropy function

$$H(p) = -p \log p - (1 - p) \log(1 - p)$$

Proof. By Stirling's Formula,

$$\begin{aligned} \binom{n}{pn} &= \frac{n!}{(pn)!((1-p)n)!} \\ &= \frac{(1 + o(1))\sqrt{2\pi n}(n/e)^n}{(1 + o(1))\sqrt{2\pi pn}(pn/e)^{pn} \sqrt{2\pi(1-p)n}((1-p)n/e)^{(1-p)n}} \\ &= (1 + o(1)) \frac{1}{\sqrt{2\pi p(1-p)n}} \cdot \frac{(n/e)^n}{(pn/e)^{pn} (n(1-p)/e)^{(1-p)n}} \\ &= (1 + o(1)) \frac{1}{\sqrt{2\pi p(1-p)n}} \cdot \frac{n^n}{(pn)^{pn} (n(1-p))^{(1-p)n}} \\ &= (1 + o(1)) \frac{1}{\sqrt{2\pi p(1-p)n}} \cdot \frac{2^{n \log n}}{2^{pn \log(pn)} 2^{(1-p)n \log(n(1-p))}} \\ &= (1 + o(1)) \frac{2^{n \log n - pn(\log n + \log p) - (1-p)n(\log n + \log(1-p))}}{\sqrt{2\pi p(1-p)n}} \\ &= (1 + o(1)) \frac{2^{nH(p)}}{\sqrt{2\pi p(1-p)n}}. \end{aligned}$$

□

Problem 3

Let $k \leq n/2$, and suppose that \mathcal{F} is an antichain in $\mathcal{P}[n]$ such that every $A \in \mathcal{F}$ has $|A| \leq k$. Prove that $|\mathcal{F}| \leq \binom{n}{k}$.

Proof. Let $\mathcal{P}_k[n]$ be the set of all subsets of $[n]$ of size $k \leq n$. For $1 \leq k \leq n/2$, consider the bipartite subgraph G_k of the discrete cube Q_n induced by $[n]^{(k-1)} \sqcup [n]^{(k)}$. Note that there is edge between $A \in [n]^{(k-1)}$ and $B \in [n]^{(k)}$ if and only if $A \subseteq B$.

We now verify the conditions of Hall's Theorem to show that there is a matching saturating $[n]^{(k-1)}$. Let $S \subseteq [n]^{(k-1)}$ and let $T = \Gamma(S)$. Notice that each $A \in S$ has $n - k + 1$ neighbors in T , whereas each $B \in T$ has $k - 1$ neighbors in $[n]^{(k-1)}$. But then

$$|S| \cdot (n - k + 1) = e(S, T) \leq |T| \cdot k.$$

Since $k \leq n/2$, we have $|S| \leq |T| \cdot k / (n - k + 1) \leq |T|$. Hall's Theorem now furnishes a matching in G_k saturating $[n]^{(k-1)}$, for any $1 \leq k \leq n/2$. By connecting the matchings between G_k for $1 \leq k \leq n/2$, we get $\binom{n}{k}$ chains that partition $\mathcal{P}_k[n]$. It now follows that \mathcal{F} intersects with any of these chains in at most one element, and so $|\mathcal{F}| \leq \binom{n}{k}$. \square

Problem 4

Let (P, \leq) be a poset. Suppose that every chain in P has at most k elements. Prove that P can be written as the union of k antichains.

Proof. For $x \in P$, define $h(x)$ as the length of the longest chain containing x as the maximal element. Notice that if $x > y$ then $h(x) > h(y)$, as we may append x to the end of any chain containing y . This implies x and y are incomparable if $h(x) = h(y)$. But then for any $x \in P$ we have $h(x) \leq k$. Thus for $1 \leq n \leq k$, $A_n = \{x \in P \mid h(x) = n\}$ is an antichain. The result now follows. \square

Problem 5

Suppose $\mathcal{F} \subset \mathcal{P}[n]$ is a set system containing no chain with $k + 1$ sets.

- (a) Prove that $\sum_{i=0}^n \frac{|\mathcal{F}_i|}{\binom{n}{i}} \leq k$, where $\mathcal{F}_i = \mathcal{F} \cap [n]^{(i)}$ for each i .

Proof. Since every chain in \mathcal{F} has at most k elements, the proof of Problem 4 furnishes a partition of \mathcal{F} into k antichains A_1, \dots, A_k . By the LYM inequality, for $1 \leq j \leq k$ we have

$$\sum_{i=0}^n \frac{|A_j \cap [n]^{(i)}|}{\binom{n}{i}} \leq 1. \quad (1)$$

But then

$$\sum_{i=0}^n \frac{|\mathcal{F}_i|}{\binom{n}{i}} = \sum_{i=0}^n \sum_{j=1}^k \frac{|A_j \cap [n]^{(i)}|}{\binom{n}{i}} = \sum_{j=1}^k \sum_{i=0}^n \frac{|A_j \cap [n]^{(i)}|}{\binom{n}{i}} \leq k. \quad (2)$$

□

- (b) What is the maximum possible size of such a system?

Proof. By the LYM inequality, equality holds in (1) if and only if $A_j = [n]^{(i)}$ for some i . Thus, equality can be achieved when $\mathcal{F} = \bigsqcup_{i \in I} [n]^{(i)}$ for some $I \subseteq [n]$ of size k . □

Problem 6

Let \mathcal{A} be an antichain in $\mathcal{P}[n]$ that is not of the form $[n]^{(r)}$. Must there exist a maximal chain disjoint from \mathcal{A} ?

Proof. For $A \in \mathcal{A}$, the fraction of maximal chains in $\mathcal{P}[n]$ that contain A is

$$\frac{|A|!(n-|A|)!}{n!} = \frac{1}{\binom{n}{|A|}}.$$

Since each maximal chain intersects with at most one element of \mathcal{A} , the fraction of maximal chains that intersect with \mathcal{A} is

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} = \sum_{i=0}^n \sum_{A \in \mathcal{A} \cap [n]^{(i)}} \frac{1}{\binom{n}{i}} = \sum_{i=0}^n \frac{|\mathcal{A} \cap [n]^{(i)}|}{\binom{n}{i}}.$$

But then \mathcal{A} is not of the form $[n]^{(r)}$, so by the LYM inequality, the above sum is strictly less than 1. This completes the proof. \square

Problem 7

Let (P, \leq) be an infinite poset. Must P contain an infinite chain or antichain?

Proof. Suppose P contains no infinite antichain and no infinite chain. Define $h(x)$ as the length of the longest chain containing x as the maximal element. Notice that if $x > y$ then $h(x) > h(y)$, as we may append x to the end of any chain containing y . This implies x and y are incomparable if $h(x) = h(y)$. Thus $A_n = \{x \in P \mid h(x) = n\}$ is an antichain for $n \in \mathbb{N}$ and $P = \bigsqcup_{n \in \mathbb{N}} A_n$. Since each A_n is finite, there must be infinitely many n such that A_n is non-empty. But then $h(x)$ is unbounded on P , so there must exist an infinite chain in P , contradiction. \square