MATH 100A: Homework #6

Due on November 16, 2023 at 12:00pm

Professor McKernan

Section A02 5:00PM - 5:50PM
Section Leader: Castellano

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

If G is a cyclic group and N is a subgroup of G, show that G/N is a cyclic group.

Proof. Since G is cyclic, there exists some $a \in G$ that generates G. In other words, for all $g \in G$, $g = a^i$, for some $i \in \mathbb{Z}$. Since G is abelian, any subgroup N of G is normal. Consider $G/N = \{gN \mid g \in G\} = \{a^i N \mid i \in \mathbb{Z}\}$. We first show that $a^i N = (aN)^i$ for all positive i by induction on i. The base case is trivial. Suppose that $i > 1$. Then $a^i N = (a^{i-1}N)aN = (aN)^{i-1}aN = (aN)^i$, by induction. Now consider $(aN)^{-1}$. Since $(aN)^{-1}$ is the inverse of aN, $aN(aN)^{-1} = (aN)^0 = N = (aN)(a^{-1}N)$, and so we know $a^{-1}N = (aN)^{-1}$ and $N = a^0 N = (aN)^0$. Then, with the same induction argument, we can show that $a^{-i}N = (aN)^{-i}$ for all positive i, and thus all $gN \in G/N$ can be generated by aN. Therefore, G/N is also a cyclic group, with generator aN. $\square$

Problem 2

If G is an abelian group and N is a subgroup of G, show that G/N is an abelian group.

Proof. Since G is abelian, any subgroup N of G is normal. Let $gN, hN \in G/N$, for some $g, h \in G$. Since $(gN)(hN) = ghN = hgN = (hN)(gN)$, G/N is abelian. $\square$

Problem 3

If G is a group and $Z(G)$ the center of G, show that if $G/Z(G)$ is cyclic, then G is abelian.

Proof. Suppose that $G/Z(G)$ is generated by some element $gZ(G)$, where $g \in G$. Then, $G/Z(G) = \{(gZ(G))^i \,|\, i \in \mathbb{Z}\} = \{g^i Z(G) \,|\, i \in \mathbb{Z}\}$. Let $a, b \in G$. Then, $a = g^i z$ and $b = g^j z'$, for some $i, j \in \mathbb{Z}$, $z, z' \in Z(G)$. This immediately follows that

$$ab = (g^i z)(g^j z') = (g^{i+j})zz' = (g^j)(g^i)z'z = (g^j z')(g^i z) = ba,$$

and thus G is abelian. $\square$

Problem 4

If G is a group and $N \triangleleft G$ is such that G/N is abelian, prove that $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.

Proof. Let $aN, bN \in G/N$. Since G/N is abelian, $abN = aNbN = bNaN = baN$, and thus $aba^{-1}b^{-1} \in N$. $\square$

Problem 5

If G is a group and $N \lhd G$ is such that

$$aba^{-1}b^{-1} \in N$$

for all $a, b \in G$, prove that G/N is abelian.

Proof. Let $aN, bN \in G/N$. Since $bab^{-1}a^{-1} \in N$,

$$NaNb = Nab = N(bab^{-1}a^{-1})ab = Nba = NbNa,$$

and thus G/N is abelian. □

Problem 6

Let G be an abelian group (possibly infinite) and let the set $T = \{a \in G \,|\, a^m = e, m > 1 \text{ depending on } a\}$. Prove that:

(a) T is a subgroup of G.

Proof. We first note that T is nonempty, as $e^l = e$ for all l, so $e \in T$. It suffices to show T is closed under multiplication and taking inverses. Let $a, b \in T$. We know $a^m = b^n = e$ for some positive m, n. Let $k = mn$. Since G is abelian, $(ab)^k = a^k b^k = (a^m)^n (b^n)^m = e$, and so $ab \in T$. Since $(a^{-1})^m = (a^m)^{-1} = e$, $a^{-1} \in T$, and we are done. □

(b) G/T has no element – other than its identity element – of finite order.

Proof. We may assume there exists $g \in G \backslash T$, otherwise we are done. Then, g is of infinite order. Note that for $i \in N$, $g^i \notin T$, otherwise there exists m such that $(g^i)^m = g^{im} = e$, contradiction. Since G is abelian, T is normal, and thus $(gT)^i = g^i T$ for all $i \in \mathbb{N}$. This immediately follows that $g^i \notin T$ so $g^i T \neq T$, which implies that gT is of infinite order. □

Problem 7

Let G be the group of all real-valued functions on the unit interval $[0, 1]$, where we define, for $f, g \in G$, addition by $(f+g)(x) = f(x)+g(x)$ for every $x \in [0, 1]$. If $N = \{f \in G \mid f\left(\frac{1}{4}\right) = 0\}$, prove that $G/N \simeq$ real numbers under $+$.

Proof. We first show that N is a normal subgroup. Note that since $e(x) = 0 \in N$, N is nonempty. Let $f, g \in N$. Since $(f+g)\left(\frac{1}{4}\right) = f\left(\frac{1}{4}\right) + g\left(\frac{1}{4}\right) = 0$, $f+g \in N$. We also know that $f^{-1}\left(\frac{1}{4}\right) = -f\left(\frac{1}{4}\right) = 0$, so $f^{-1} \in N$. Since addition is commutative, G is abelian, and thus N is indeed a normal subgroup. Define $\phi : G \to \mathbb{R}$ as $\phi(f) = f\left(\frac{1}{4}\right)$. Let $f, g \in G$, such that $f = g$. Then, we know $\phi(f) = f\left(\frac{1}{4}\right) = g\left(\frac{1}{4}\right) = \phi(g)$, so ϕ is well defined. Since $\phi(f+g) = (f+g)\left(\frac{1}{4}\right) = f\left(\frac{1}{4}\right) + g\left(\frac{1}{4}\right) = \phi(f) + \phi(g)$, ϕ is homomorphic. For all $x \in \mathbb{R}$, there exists $h \in G$, such that $\phi(h) = h\left(\frac{1}{4}\right) = x$, and thus ϕ is onto. Let $k \in N$. Then $\phi(k) = 0$, and so $k \in \text{Ker } \phi$. Suppose that $k \in \text{Ker } \phi$, then $\phi(k) = k\left(\frac{1}{4}\right) = 0$, which implies that $k \in N$. Therefore, Ker $\phi = N$, and $G/N \simeq \mathbb{R}$, by the First Isomorphism Theorem. $\square$

Problem 8

If G_1, G_2 are two groups and $G = G_1 \times G_2 = \{(a, b) \mid a \in G_1, b \in G_2\}$, where we define $(a, b)(c, d) = (ab, cd)$, show that:

(a) $N = \{(a, e_2) \mid a \in G_1\}$, where e_2 is the unit element of G_2, is a normal subgroup of G.

Proof. N is obviously not empty. Let $a, a' \in G_1$. Since $(a, e_2)(a', e_2) = (aa', e_2) \in G$ and (a^{-1}, e_2), the inverse of (a, e_2), is also in G, N is a subgroup. Let $g = (g_1, g_2) \in G$. Since $g(a, e_2)g^{-1} = (g_1 a g_1^{-1}, e_2) \in N$, N is normal. $\square$

(b) $N \simeq G_1$

Proof. Define $\phi : N \to G_1$ as $\phi((a, e_2)) = a$. ϕ is obviously well defined. Let $x = (a, e_2), y = (a', e_2)$ Since $\phi(xy) = aa' = \phi(x)\phi(y)$, ϕ is a homomorphism. For $g_1 \in G_1$, we have $\phi((g_1, e_2)) = g_1$, and so ϕ is surjective. Since G_1 has an unique identity element e_1, Ker $\phi = \{(e_1, e_2)\}$, and so ϕ is injective. We have shown that ϕ is a bijective homomorphism, so $N \simeq G_1$. $\square$

(c) $G/N \simeq G_2$

Proof. Define $\psi : G \to G_2$ as $\psi((a, b)) = b$. Let $x = (a, b), y = (a', b')$. Since $\psi(xy) = bb' = \psi(x)\psi(y)$, ψ is a homomorphism. For $g_2 \in G_2$, we have $\psi((g_1, g_2)) = g_2$ for some $g_1 \in G_1$, so ψ is surjective. Since Ker $\psi = \{(a, e_2) \mid a \in G_1\} = N$, $G/N \simeq G_2$, by the First Isomorphism Theorem. $\square$

Problem 9

If G is a group and $N \lhd G$, show that if $a \in G$ has finite order $o(a)$, then Na in G/N has finite order m, where $m \,|\, o(a)$.

Proof. Define $\phi : G \to G/N$ as $\phi(a) = aN$. We already know ϕ is a surjective homomorphism. Suppose that $a \in G$ is of finite order $o(a)$. Since ϕ is a homomorphism, $(Na)^{o(a)} = (\phi(a))^{o(a)} = \phi(a^{o(a)}) = e$, and thus Na is of some order $m \leq o(a)$. This means that $(Na)^m = Na^m = N$, and so $a^m \in N$. Suppose for the sake of contradiction that $m \nmid o(a)$. Then, $o(a) = pm + q$, for some $p, q \in \mathbb{Z}$, $0 < q < m$. Since $e = a^{o(a)} = (a^m)^p a^q \in N$, we know $a^q \in N$. However, we then get $(Na)^q = Na^q = N$, which implies Na is of order $q < m$, contradiction. Therefore, $m \,|\, o(a)$. ☐

Problem 10

If φ is a homomorphism of G onto G' and $N \lhd G$, show that $\varphi(N) \lhd G'$.

Proof. For all g, we know $\varphi(g)\varphi(N)(\varphi(g))^{-1} = \varphi(gNg^{-1})$. However, $gNg^{-1} \subset N$, and so $\phi(gNg^{-1}) \subset \phi(N)$. Therefore, $\phi(N) \lhd G'$. $\square$