Math 109 HW 3

Ray Tsai

10/17/2022

1.

Proposition 1. *The additive inverse of integer n is unique.*

Proof. Let b, c be some integers such that $n + b = 0, n + c = 0$. We will show that the additive inverse of integer n is unique.

$$n + b = 0 \quad n + c = 0 \tag{1}$$
$$n + b + (-n) = -n \quad n + c + (-n) = -n \tag{2}$$
$$b = -n = c \tag{3}$$

Thus, the additive inverse of integer n is unique. □

2. (a)

Proposition 2. *For all real number x, there is a 2×2 matrix over $\mathbb{R}$ such that its determinant is x.*

Proof. Let $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, where a, b, c, d are real numbers. We will show that, for all real number x, there is a 2×2 matrix over $\mathbb{R}$ such that its determinant is x.

Let $ad = k, bc = l$ where k, l are real numbers.

$$ad - bc = k - l \tag{4}$$

Let $k - l$ be a real number x.

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \tag{5}$$
$$= k - l \tag{6}$$
$$= x \tag{7}$$

Thus, the determinant of a 2 x 2 matrix over $\mathbb{R}$ can be any real number x. □

(b)

Proposition 3. *There exist a different 2 x 2 matrix over $\mathbb{R}$ such that its determinant is the same as the matrix in part(a).*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$, where a, b, c, d are real numbers. We will show that B and A have the same determinant.

$$det(A) = ad - bc = da - bc = det(B) \tag{8}$$

Thus, A is not a unique matrix that has its determinant. $\square$

3.

Proposition 4. *For all $x \in \mathbb{R}$, we have $x^2 \geq 0$.*

Proof. Let $x = |x|$ when $x \geq 0$ and $x = -|x|$ when $x < 0$, according to the definition provided above. We will show that $x^2 \geq 0$.

We can separate x^2 into two cases where $x \geq 0$ or $x < 0$.

$$x^2 = \begin{cases} |x|^2, & \text{if } x \geq 0 \\ (-|x|)^2, & \text{if } x < 0 \end{cases} \tag{9}$$

Since $|x| \geq 0$,

$$|x| \cdot |x| \geq 0 \cdot |x| \tag{10}$$

$$|x|^2 \geq 0 \tag{11}$$

Hence, $x^2 = |x|^2 \geq 0$ when $x \geq 0$.
When $x < 0$,

$$x^2 = (-|x|)^2 \tag{12}$$

$$= (-1)^2|x|^2 \tag{13}$$

$$= |x|^2 \geq 0 \tag{14}$$

Thus, for all $x \in \mathbb{R}$, we have $x^2 \geq 0$. $\square$

4.

Proposition 5. *For all $x \in \mathbb{R}$, if $x^2 = x$, then $x < 2$.*

Proof. Let $x \in \mathbb{R}$. We will show that if $x^2 = x$, then $x < 2$.

$$x^2 = x \tag{15}$$

$$x^2 + (-x) = x + (-x) \tag{16}$$

$$x(x-1) = 0 \tag{17}$$

From the contrapositive of HW 3 Fact 4, we know that if $x(x-1) = 0$ then $x - 1 = 0$ or $x = 0$. If $x - 1 = 0$ then $x - 1 + 1 = x = 1$. Thus, x can be 0 or 1, both of which are smaller than 2. Thus, if $x^2 = x$, then $x < 2$. $\square$

5.

Proposition 6. *If n is an integer, then $n^2 + 3n + 1$ is odd.*

Proof. Let n be an integer. We will show that $n^2 + 3n + 1$ is odd.

$$n^2 + 3n + 1 = n(n + 3) + 1 \tag{18}$$

From HW3 Fact 3, we know that all integers are even or odd. Thus, we can split $n(n + 3) + 1$ into 2 cases, n is even and n is odd.
If n is even, let n be $2k$ for some integer k by HW3 Fact 1.

$$\begin{aligned} n(n + 3) + 1 &= 2k(2k + 3) + 1 && (19) \\ &= 2(2k^2 + 3k) + 1 && (20) \end{aligned}$$

Let $2k^2 + 3k$ be some integer l.

$$2(2k^2 + 3k) + 1 = 2l + 1 \tag{21}$$

Therefore, if n is even, $n^2 + 3n + 1$ is odd by HW3 Fact 2. If n is odd, let n be $2k + 1$ for some integer k by HW3 Fact 2.

$$\begin{aligned} n(n + 3) + 1 &= (2k + 1)(2k + 4) + 1 && (22) \\ &= 2(2k + 1)(k + 1) + 1 && (23) \end{aligned}$$

Let $(2k + 1)(k + 1)$ be some integer l.

$$2(2k + 1)(k + 1) + 1 = 2l + 1 \tag{24}$$

Therefore, if n is odd, $n^2 + 3n + 1$ is odd by HW3 fact 1.
Thus, for all integer n, $n^2 + 3n + 1$ is odd. □

6.

Proposition 7. *For all integer a, b. If $a + b$ is even, then $a - b$ is even.*

Proof. Let a, b be some integers. We will show that if $a + b$ is even, then $a - b$ is even.
By HW3 Fact 1, let $a + b$ be an even integer $2k$ for some integer k.

$$\begin{aligned} a - b &= a + b - 2b && (25) \\ &= 2k - 2b && (26) \\ &= 2(k - b) && (27) \end{aligned}$$

Let $k - b$ be some integer l.

$$2(k - b) = 2l \tag{28}$$

Thus, if $a + b$ is even, then $a - b$ is even by HW3 fact 1. □

7.

Proposition 8. *Let a, b be integers. If ab is even, then a or b is even.*

Proof. We will prove by contradiction. Suppose for sake of contradiction that there exist some even integer ab where both a and b are odd. By HW3 Fact 2, let a and b be some odd integers $2k+1$ and $2l+1$ for some integers k, l.

$$\begin{aligned} ab &= (2k+1)(2l+1) && (29) \\ &= 4kl + 2l + 2k + 1 && (30) \\ &= 2(2kl + l + k) + 1 && (31) \end{aligned}$$

Let $2kl + l + k$ be some integer m.

$$2(2kl + l + k) + 1 = 2m + 1 \tag{32}$$

By HW3 Fact 2, this shows that if both a and b are odd integers then ab is odd, which contradicts our initial assumption. Thus, if ab is even, then a or b is even. $\square$