MATH 188: Homework #6

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Problem 1

The following exercise gives another proof of Cayley's formula, and at the same time provides new information that our proof doesn't give.

Let $n \geq 1$ and let $x_1, \ldots, x_n$ be variables. Given a labeled tree T with vertices $1, \ldots, n$, define the monomial $x(T) = x_1^{d_1} \cdots x_n^{d_n}$ where d_i is the degree of vertex i, i.e., the number of edges containing i. Define $\mathbf{C}_n = \sum_T x(T)$ where the sum is over all labeled trees T with vertices $1, \ldots, n$. Also define

$$\mathbf{D}_n = x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}.$$

(a) Given a polynomial $p(x_1, \ldots, x_n)$, let $p^{(i)}$ be the result of plugging in $x_i = 0$ into the partial derivative $\frac{\partial p}{\partial x_i}$, i.e., the coefficient of x_i if you think of the other variables as constants. If $n \geq 2$, show that

$$\mathbf{C}_n^{(n)} = (x_1 + x_2 + \cdots + x_{n-1})\mathbf{C}_{n-1},$$

$$\mathbf{D}_n^{(n)} = (x_1 + x_2 + \cdots + x_{n-1})\mathbf{D}_{n-1}.$$

Proof. Since a tree is connected, all vertices has positive degree. Hence, we have

$$\mathbf{C}_n^{(n)} = \frac{\partial}{\partial x_n} \sum_T x(T) \Bigg|_{x_n=0} = \sum_T d_n x_1^{d_1} \cdots x_n^{d_n - 1} \Big|_{x_n=0} = \sum_{T; d_n=1} x_1^{d_1} \cdots x_{n-1}^{d_{n-1}}. \tag{1}$$

But then given a T with $d_n = 1$, suppose j is the only neighbor of n. Then, $x(T) = x_j x(T_{n-1})$, where $T_{n-1} = T - \{n\}$ a labeled tree with vertex set $[n-1]$. On the other hand, given a labeled tree T_{n-1} with vertex set $[n-1]$, we may choose a vertex which connects to n and get T with $d_n = 1$, with $x(T) = x_j x(T_{n-1})$. It now follows that

$$\begin{aligned}
\sum_{T; d_n=1} x_1^{d_1} \cdots x_{n-1}^{d_{n-1}} &= \sum_{j=1}^{n-1} \sum_{\substack{T; d_n=1, \\ \{j,n\} \in e(T)}} x_1^{d_1} \cdots x_{n-1}^{d_{n-1}} \\
&= \sum_{j=1}^{n-1} x_j \sum_{T_{n-1}} x(T_{n-1}) = \sum_{j=1}^{n-1} x_j \mathbf{C}_{n-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbf{D}_n^{(n)} &= x_1 x_2 \cdots x_{n-1} (x_1 + x_2 + \cdots + x_n)^{n-2} \\
&+ (n-2) x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-3} |_{x_n=0} \\
&= x_1 x_2 \cdots x_{n-1} (x_1 + x_2 + \cdots + x_{n-1})^{n-2} \\
&= (x_1 + x_2 + \cdots + x_{n-1}) \mathbf{D}_{n-1}.
\end{aligned}$$

□

Problem 1 continued on next page. . .

(b) Assuming that $\mathbf{C}_{n-1} = \mathbf{D}_{n-1}$ show that $\mathbf{C}_n^{(i)} = \mathbf{D}_n^{(i)}$ for all $i = 1, \ldots, n$.

Proof. Define $\mathbf{C}_{[n]-\{i\}} = \sum_{T_{[n]-\{i\}}} x(T_{[n]-\{i\}})$, where the sum is over all labeled trees $T_{[n]-\{i\}}$ with vertices $[n]-\{i\}$. Also define $\mathbf{D}_{[n]-\{i\}} = x_1 x_2 \cdots x_n x_i^{-1}(x_1 + x_2 + \cdots + x_n - x_i)^{n-3}$.
Using the same argument in (a), we may show that

$$\mathbf{C}_n^{(i)} = \sum_{T; d_i = 1} \prod_{j \neq i} x_j^{d_j} = \sum_{j=1, j\neq i}^{n} x_j \mathbf{C}_{[n]-\{i\}},$$

for all i. On the other hand, for all i,

$$\begin{aligned}\mathbf{D}_n^{(i)} = x_i^{-1} x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2} \\ &+ (n-2) x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-3}|_{x_i = 0} \\ &= x_1 x_2 \cdots x_n x_i^{-1}(x_1 + x_2 + \cdots + x_n - x_i)^{n-2} \\ &= (x_1 + x_2 + \cdots + x_n - x_i)\mathbf{D}_{[n]-\{i\}}.\end{aligned}$$

Note that the only differences between $\mathbf{C}_{n-1}, \mathbf{C}_{[n]-\{i\}}$ and between $\mathbf{D}_{n-1}, \mathbf{D}_{[n]-\{i\}}$ are the indexing of the variables. Hence, $\mathbf{C}_{n-1} = \mathbf{D}_{n-1}$ also implies that $\mathbf{C}_{[n]-\{i\}} = \mathbf{D}_{[n]-\{i\}}$. It now follows that

$$\mathbf{C}_n^{(i)} = \sum_{j=1, j\neq i}^{n} x_j \mathbf{C}_{[n]-\{i\}} = \sum_{j=1, j\neq i}^{n} x_j \mathbf{D}_{[n]-\{i\}} = \mathbf{D}_n^{(i)}.$$

, for all i. □

(c) Conclude that $\mathbf{C}_n = \mathbf{D}_n$ for all $n \geq 1$.

Proof. We proceed by induction on n. When $n = 1$, there are only one label tree, which is a singleton. Hence, $\mathbf{C}_1 = 1 = x_1 x_1^{-1} = \mathbf{D}_1$. Suppose $n \geq 2$. By induction and (b), we have $\mathbf{C}_{n-1}^{(i)} = \mathbf{D}_{n-1}^{(i)}$ for all $i \in [n]$. But then

$$\begin{aligned}\mathbf{C}_n &= \sum_T x(T) \\ &= \sum_{i=1}^{n} \sum_{T; d_i = 1} x(T) \\ &= \sum_{i=1}^{n} \mathbf{C}_n^{(i)} \\ &= \sum_{i=1}^{n} \mathbf{D}_n^{(i)} \\ &=\end{aligned}$$

□

Problem 2

How many ways are there to list the letters of the word MATHEMATICS so that no two consecutive letters are the same?

Proof. The repeated characters in the word MATHEMATICS are A, M, and T. Let S_A, S_M, S_T each be the set of ways to list MATHEMATICS with consecutive A, M, T, respectively. By inclusion-exclusion, the number of arrangements of MATHEMATICS with consecutive same characters is

$$|S_A \cup S_M \cup S_T| = |S_A| + |S_M| + |S_T| - |S_A \cap S_M| - |S_A \cap S_T| - |S_T \cap S_T| + |S_A \cap S_M \cap S_T|.$$

Since A, M, and T each appears exactly twice in MATHEMATICS,

$$|S_A \cup S_M \cup S_T| = 3|S_A| - 3|S_A \cap S_M| + |S_A \cap S_M \cap S_T|,$$

by symmetry. Notice that to count elements in S_A, we may view AA as a single character, and

$$|S_A| = \frac{10!}{2!2!}.$$

Similarily, to count elements in $S_A \cap S_M$, we may AA and MM as characters and get

$$|S_A \cap S_M| = \frac{9!}{2!}.$$

Using the same idea, we get

$$|S_A \cap S_M \cap S_T| = 8!.$$

Hence,

$$|S_A \cup S_M \cup S_T| = 3 \cdot \frac{10!}{2!2!} - 3 \cdot \frac{9!}{2!} + 8!.$$

In total there are $\frac{11!}{2!2!2!}$ arrangements of MATHEMATICS, so the number of arrangements of MATHEMATICS with no consecutive repeated characters is

$$\frac{11!}{2!2!2!} - 3 \cdot \frac{10!}{2!2!} + 3 \cdot \frac{9!}{2!} - 8! = 2772000.$$

□

Problem 3

Let $n \geq 2$ be an integer. We have n married couples ($2n$ people in total).

(a) How many ways can we have the $2n$ people stand in a line so that no person is standing next to their spouse?

Proof. Let L be the set of all arrangements of $2n$ people in a line. Put $A = A_1 \cup \cdots \cup A_n$, where A_i is set of ways to line up n couples with the ith couple standing next to each other. Let $S \subseteq [n]$. Define

$$f(S) = |\{x \in A \mid x \in A_i \text{ if and only if } i \in S\}|,$$
$$g(S) = |\{x \in A \mid x \in A_i \text{ if } i \in S\}|.$$

Note that $g(\emptyset) = |A|$ and $f(\emptyset) = 0$. To calculate $g(S) = |\cap_{i \in S} A_i|$, we may view each couple in S as a unit of people and account the ordering of each unit. Then, we would have $2n - |S|$ unit of people, with $|S|$ units each having 2 arrangements, so

$$g(S) = (2n - |S|)!2^{|S|}.$$

By inclusion-exclusion,

$$|A| = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} (-1)^{|S|-1}(2n - |S|)!2^{|S|} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (2n-k)!2^k$$

Since we are calculating the case where no person stands next to their spouse, we have

$$|L \backslash A| = |L| - |A| = (2n)! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (2n-k)!2^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2n-k)!2^k.$$

□

(b) Same as (a), but replace "line" by "circle".

Proof. Let L be the set of all arrangements of $2n$ people in a line. Put $A = A_1 \cup \cdots \cup A_n$, where A_i is set of ways to line up n couples with the ith couple standing next to each other. Let $S \subseteq [n]$. Define

$$f(S) = |\{x \in A \mid x \in A_i \text{ if and only if } i \in S\}|,$$
$$g(S) = |\{x \in A \mid x \in A_i \text{ if } i \in S\}|.$$

Note that $g(\emptyset) = |A|$ and $f(\emptyset) = 0$. To calculate $g(S) = |\cap_{i \in S} A_i|$, we may view each couple in S as a unit of people and account the ordering of each unit. Then, we would have $2n - |S|$ unit of people, with $|S|$ units each having 2 arrangements, so

$$g(S) = \frac{1}{2n - |S|} \cdot (2n - |S|)!2^{|S|} = (2n - |S| - 1)!2^{|S|}.$$

Note that we divide by $2n - |S|$ to disregard shifting the circle. By inclusion-exclusion,

$$|A| = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} (-1)^{|S|-1}(2n - |S| - 1)!2^{|S|} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (2n-k-1)!2^k$$

Hence, the number of ways to have n couples stand in a circle with no person standign next to their spouse is

$$|L \backslash A| = \frac{1}{2n} \cdot (2n)! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (2n-k-1)!2^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2n-k-1)!2^k.$$

□

Problem 4

Let q be a prime power and n a positive integer. Let V be an n-dimensional $\mathbf{F}_q$-vector space and let P be the poset whose elements are linear subspaces of V with the ordering $X \leq Y$ if X is contained in Y. Show that the Möbius function of P is given by

$$\mu(X,Y) = (-1)^d q^{\binom{d}{2}}$$

where $d = \dim Y - \dim X$.

Proof. For $X \leq Y$, it suffices to show that

$$\delta_{X,Y} = \sum_{U \in [X,Y]} (-1)^{d_u} q^{\binom{d_u}{2}},$$

where $d_u = \dim U - \dim X$. Put $x = \dim X$ and $y = \dim Y$. The number of vector spaces of dimension k in the interval $[X,Y]$ is $\begin{bmatrix} d \\ k-x \end{bmatrix}_q$. It now follows that

$$\begin{aligned} \sum_{U \in [X,Y]} (-1)^{d_u} q^{\binom{d_u}{2}} &= \sum_{k=x}^{y} \begin{bmatrix} d \\ k-x \end{bmatrix}_q (-1)^{k-x} q^{\binom{k-x}{2}} \\ &= \sum_{i=0}^{d} \begin{bmatrix} d \\ i \end{bmatrix}_q (-1)^i q^{\binom{i}{2}}. \end{aligned}$$

By Theorem 3.2.4 from Sagen,

$$\sum_{i=0}^{d} \begin{bmatrix} d \\ i \end{bmatrix}_q (-1)^i q^{\binom{i}{2}} = \begin{cases} \prod_{i=0}^{d-1} (1-q^i) & d > 0 \\ 1 & d = 0 \end{cases} = \delta_{X,Y}.$$

□

Problem 5

Let Π_n be the poset of set partitions of $[n]$ and let μ be its Möbius function. Write a formula for the number of connected labeled graphs with vertex set $[n]$ using μ.

Proof. Let $\mathcal{G}$ be the set of all labeled graphs with vertex set $[n]$, and let $P = \{S_1, \ldots, S_m\} \in \Pi_n$. Define

$$f(P) = |\{G \in \mathcal{G} \mid i, j \text{ connected in } G \text{ if and only if } i, j \in S_k \text{ for some } k\}|,$$

$$g(P) = |\{G \in \mathcal{G} \mid i, j \text{ connected in } G \text{ only if } i, j \in S_k \text{ for some } k\}| = \prod_{k=1}^{m} 2^{\binom{|S_k|}{2}} = 2^{\sum_{k=1}^{m} \binom{|S_k|}{2}}.$$

Note that $f(\{[n]\})$ is the number of connected labeled graphs with vertex set $[n]$ and $g(\{[n]\}) = 2^{\binom{n}{2}}$. By definition, $g(P) = \sum_{Q \leq P} f(Q)$. It now follows by the Möbius inversion that

$$f(\{[n]\}) = \sum_{P \in \Pi_n} \mu(P, \{[n]\}) g(P) = \sum_{P \in \Pi_n} \mu(P, \{[n]\}) 2^{\sum_{S \in P} \binom{|S|}{2}}.$$

$\square$

Problem 6

$F(x)=\sum_{n\geq 0} f_n x^n$ is a formal power series that satisfies the following identity:

$$F(x)=\exp\left(\frac{x}{2}(F(x)+1)\right).$$

Find a formula for f_n.

Proof. We first note that $f_0=F(0)=1$. Adding 1 then multiplying $\frac{x}{2}$ on both sides of the given identity yields

$$\frac{x}{2}(F(x)+1)=\frac{x}{2}\left[\exp\left(\frac{x}{2}(F(x)+1)\right)+1\right].$$

Take $G(x)=\frac{1}{2}(e^x+1)$ and $A(x)=\frac{x}{2}(F(x)+1)$. Since $A(0)=0$ and $G(0)\neq 0$, the Lagrange inversion formula gives

$$\begin{aligned}\frac{n+1}{2}[x^n]F(x) &= (n+1)[x^{n+1}]A(x)\\ &= [x^n](G(x)^{n+1})\\ &= [x^n]\frac{1}{2^{n+1}}(e^x+1)^{n+1}\\ &= [x^n]\frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}e^{kx}\\ &= \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{k^n}{n!}\end{aligned}$$

That is, for $n\geq 1$,

$$f_n=\frac{1}{(n+1)!}\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{k}{2}\right)^n.$$

$\square$

Problem 7

Reminder: Lagrange's version of the Taylor remainder theorem says this: if $f(x)$ is an infinitely differentiable function whose Taylor series at 0 converges at $x = r$, then there exists ξ between 0 and r such that

$$f(r) - \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} r^i = \frac{f^{(n+1)}(\xi)}{(n+1)!} r^{n+1}.$$

Use the Taylor remainder theorem to show that

$$\left| \frac{1}{e} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} \right| \leq \frac{1}{(n+1)!}$$

and conclude from this that the number of derangements of n objects is inside the closed interval

$$\left[\frac{n!}{e} - \frac{1}{n+1}, \frac{n!}{e} + \frac{1}{n+1} \right].$$

In particular, show that it is the closest integer to $n!/e$.

Proof. Consider $f(r) = e^{-r}$. When $r = 1$, Taylor remainder theorem yields

$$e^{-1} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \frac{(-1)^{n+1} e^{-\xi}}{(n+1)!},$$

for some ξ between 0 and 1. But then $|e^{-\xi}| \leq 1$, so

$$\left| \frac{1}{e} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} \right| = \left| \frac{(-1)^{n+1} e^{-\xi}}{(n+1)!} \right| \leq \frac{1}{(n+1)!}.$$

Let $D(n)$ be the number of derangements of size n. It now follows from Theorem 6.14 that

$$\frac{n!}{e} - \frac{1}{n+1} \leq D(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \leq \frac{n!}{e} + \frac{1}{n+1}.$$

Obviously $\frac{n!}{e}$ is not an integer. Since $D(n)$ is an integer and $\frac{1}{n+1} \leq \frac{1}{2}$ for all $n \geq 1$, $D(n)$ is the closest integer to $\frac{n!}{e}$. $\square$

Problem 8

Let d_n be the number of derangements of $[n]$, and let

$$D(x) = \sum_{n \geq 0} \frac{d_n}{n!} x^n.$$

(a) Using the structure interpretation for products of EGF, show that

$$D(x)e^x = \frac{1}{1-x}.$$

Proof. Let $D(S)$ denote the set of derangements of S. Define structures $\alpha(S) = D(S)$ and $\beta(S) = \{0\}$. Hte product of two structures is

$$(\alpha \cdot \beta)(S) = \bigsqcup_{T \subseteq S} D(T) \times \{0\} \simeq \bigsqcup_{T \subseteq S} D(T).$$

But then given a derangement of some subset $T \subseteq S$, we get a permutation σ of S with $\sigma(i) = i$ if and only if $i \in S \backslash T$. On the other hand, given a permutation σ of S, we get a derangement of $T = \{i \in S \mid \sigma(i) \neq i\} \subseteq S$. Hence, $\bigsqcup_{T \subseteq S} D(T) \simeq S_n$, the symmetry group of degree n. It now follows that $(\alpha \cdot \beta)(S) = |S|!$, and so

$$D(x)e^x = E_{\alpha \cdot \beta}(x) = \sum_{n \geq 0} x^n = \frac{1}{1-x}.$$

$\square$

(b) Show how this implies the formula we previously obtained:

$$d_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}.$$

Proof. Rearranging the result of (a), we get

$$D(x) = \frac{1}{1-x} e^{-x},$$

and so

$$d_n = n![x^n] \frac{1}{1-x} e^{-x} = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}.$$

$\square$

Problem 9

For a positive integer n, define

$$f(n) = |\{i \in \mathbb{Z} \mid 1 \le i \le n, \gcd(n,i) = 1\}|.$$

(a) Show that

$$n = \sum_{d|n} f(d)$$

where the sum is over all positive integers d that divide n.

Proof.

$$\begin{aligned}
\sum_{d|n} f(d) &= \sum_{d|n} f(n/d) \\
&= \sum_{d|n} |\{i \in \mathbb{Z} \mid 1 \le i \le n/d, \gcd(n/d, i) = 1\}| \\
&= \sum_{d|n} |\{i \in \mathbb{Z} \mid 1 \le i \le n, \gcd(n, i) = d\}| \\
&= n.
\end{aligned}$$

□

(b) Use Möbius inversion to show that

$$f(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

where the product is over the primes p that divide n.

Proof. Define $g(n) = n$, for $n \in \mathbb{N}$. In (a), we already established that $g(n) = \sum_{d|n} f(d)$. By the Möbius inversion formula,

$$\begin{aligned}
f(n) &= \sum_{d|n} g(d)\mu(d,n) \\
&= \sum_{d|n} d\mu(d,n) \\
&= \sum_{d|n} \frac{n}{d}\mu\left(\frac{n}{d}, n\right) \\
&= n\sum_{d|n} \frac{1}{d}\mu\left(\frac{n}{d}, n\right) \\
&= n \sum_{\substack{d = p_1 \cdots p_k \\ p_i | n \text{ and distinct}}} \frac{(-1)^k}{d} \\
&= n \sum_{\substack{d = p_1 \cdots p_k \\ p_i | n \text{ and distinct}}} \left(-\frac{1}{p_1}\right) \cdots \left(-\frac{1}{p_k}\right) \\
&= n \prod_{p|n} \left(1 - \frac{1}{p}\right).
\end{aligned}$$

□

Problem 10

There are n people sitting at a circular table. How many ways can they rearrange seats so that no one sits next to someone they were sitting next to before?

Proof. idk. $\square$

Problem 11

Let q be a prime power and let N_n be the number of monic irreducible polynomials of degree n with coefficients in $\mathbf{F}_q$:

(a) Using that polynomials over a field satisfy unique factorization, show that

$$(1-qx)^{-1} = \prod_{d\geq 1}(1-x^d)^{-N_d}$$

Proof. By the binomial theorem,

$$(1-x^d)^{-N_d} = \sum_{k\geq 0}\binom{-N_d}{k}(x^d)^k.$$

Note that $\binom{-N_d}{k}$ is the number of ways to pick a multiset of size k from N_d elements. Given a monic polynomial, we may view its factorization as a multiset of irreducible polynomials. Hence, $[x^{dk}](1-x^d)^{-N_d}$ is the number of ways to pick a monic polynomial whose factorization is k irreducible polynomials of degree d. But then

$$[x^n]\prod_{d\geq 1}(1-x^d)^{-N_d} = [x^n]\prod_{d\geq 1}\sum_{k\geq 0}\binom{-N_d}{k}(x^d)^k$$

is just the number monic polynomials of degree n. Since there are n undetermined coefficients in a monic polynomial of degree n, there are q^n monic polynomials of degree n. In other words,

$$\prod_{d\geq 1}(1-x^d)^{-N_d} = (1+qx+q^2x^2+\cdots) = (1-qx)^{-1}.$$

□

(b) Take the logarithmic derivative of (a) and compare the coefficient of x^{n-1} to get

$$q^n = \sum_{d|n} dN_d.$$

Proof.

$$\mathcal{L}((1-qx)^{-1}) = q(1-qx)(1-qx)^{-2} = q(1-qx)^{-1}.$$

$$\begin{aligned}\mathcal{L}\left(\prod_{d\geq 1}(1-x^d)^{-N_d}\right) &= \sum_{d\geq 1} N_d\mathcal{L}((1-x^d)^{-1}) \\ &= \sum_{d\geq 1} dN_d x^{d-1}(1-x^d)^{-1} \\ &= \sum_{d\geq 1} dN_d(x^{d-1}+x^{2d-1}+x^{3d-1}+\cdots).\end{aligned}$$

Hence,

$$\begin{aligned}q^n &= [x^{n-1}]q(1-qx)^{-1} \\ &= [x^{n-1}]\sum_{d\geq 1} dN_d x^{d-1}(1-x^d)^{-1} \\ &= [x^n]\sum_{d\geq 1} dN_d(x^d+x^{2d}+x^{3d}+\cdots) \\ &= \sum_{d|n} dN_d.\end{aligned}$$

□

Problem 11 continued on next page...

(c) Use Möbius inversion to get a formula for N_n.

Proof. Since $q^n = \sum_{d|n} dN_d$, by Möbius inversion,

$$\begin{aligned} nN_d &= \sum_{d|n} q^d \mu(d, n) \\ &= \sum_{d|n} q^{n/d} \mu(n/d, n) \\ &= \sum_{\substack{d=p_1\cdots p_k \\ p_i|n \text{ and distinct}}} (-1)^k q^{n/d} \end{aligned}$$

□