

# MATH 180B: Homework #2

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## Problem 1

Suppose  $U$  and  $V$  are independent and follow the geometric distribution

$$p(k) = p(1-p)^k \quad \text{for } k = 0, 1, \dots$$

Define the random variable  $Z = U + V$ .

- (a) Determine the joint probability mass function  $p_{U,Z}(u, z) = \mathbb{P}\{U = u, Z = z\}$ .

*Proof.* Since  $p_{U,Z}(u, z) = \mathbb{P}\{U = u, Z = z\} = \mathbb{P}\{U = u, V = z - u\}$ , we have

$$p_{U,Z}(u, z) = p(1-p)^u p(1-p)^{z-u} = p^2(1-p)^z.$$

□

- (b) Determine the conditional probability mass function for  $U$  given that  $Z = n$ .

*Proof.* Note that  $p_Z(z) = \sum_{u=0}^z p_U(u)p_U(z-u) = \sum_{u=0}^z p_U(u)p_v(v) = \sum_{u=0}^z p^2(1-p)^z = (z+1)p^2(1-p)^z$ . Thus,

$$\begin{aligned} p_{U|Z}(u|n) &= \frac{p_{U,Z}(u, n)}{p_Z(n)} \\ &= \frac{p^2(1-p)^n}{(n+1)p^2(1-p)^n} \\ &= \frac{1}{n+1}. \end{aligned}$$

□

## Problem 2

Let  $N$  have a Poisson distribution with parameter  $\lambda = 1$ . Conditioned on  $N = n$ , let  $X$  have a uniform distribution over the integers  $0, 1, \dots, n+1$ . What is the marginal distribution for  $X$ ?

*Proof.*

$$\begin{aligned}
 p_X(x) &= \sum_{n=0}^{\infty} p_{X|N}(x|n)p_N(n) \\
 &= \sum_{n=x-1}^{\infty} \frac{1}{n+2} \cdot \frac{1}{n!} e^{-1} \\
 &= e^{-1} \sum_{n=x-1}^{\infty} \frac{1}{(n+2)n!} \\
 &= e^{-1} \sum_{n=x-1}^{\infty} \frac{1}{(n+1)!} - \frac{1}{(n+2)!} \\
 &= e^{-1} \left( \sum_{n=x-2}^{\infty} \frac{1}{(n+2)!} - \sum_{n=x-1}^{\infty} \frac{1}{(n+2)!} \right) \\
 &= e^{-1} \left( \frac{1}{x!} + \sum_{n=x-1}^{\infty} \frac{1}{(n+2)!} - \sum_{n=x-1}^{\infty} \frac{1}{(n+2)!} \right) \\
 &= \frac{1}{e(x!)}.
 \end{aligned}$$

□

### Problem 3

Suppose that upon striking a plate a single electron is transformed into a number  $N$  of electrons, where  $N$  is a random variable with mean  $\mu$  and standard deviation  $\sigma$ . Suppose that each of these electrons strikes a second plate and releases further electrons, independently of each other and each with the same probability distribution as  $N$ . Let  $Z$  be the total number of electrons emitted from the second plate. Determine the mean and variance of  $Z$ .

*Proof.* Note that  $Z = \xi_1 + \cdots + \xi_N$ , where  $\xi_k$  is the number of electrons struck by the  $k$ th electron from the first plate.  $\xi_k$  shares the same distribution as  $N$ , for all  $k$ . Thus,

$$\mathbb{E}[Z] = \mathbb{E}[N]\mathbb{E}[N] = \mu^2,$$

$$\text{Var}(Z) = \mathbb{E}[N]\text{Var}(N) + \mathbb{E}[N]^2\text{Var}(N) = \mu\sigma^2 + \mu^2\sigma^2 = \mu(\mu + 1)\sigma^2.$$

□

## Problem 4

A six-sided die is rolled, and the number  $N$  on the uppermost face is recorded. From a jar containing 10 tags numbered  $1, 2, \dots, 10$  we then select  $N$  tags at random without replacement. Let  $X$  be the smallest number on the drawn tags. Determine  $\mathbb{P}(X = 2)$  and  $\mathbb{E}[X]$ .

*Proof.* Note that  $\mathbb{P}(X = x|N = n) = \frac{\binom{10-x}{n-1}}{\binom{10}{n}}$ . Thus,

$$\begin{aligned} \mathbb{P}(X = 2) &= \sum_{n=1}^6 \mathbb{P}(X = 2|N = n)\mathbb{P}(N = n) \\ &= \frac{1}{6} \sum_{n=1}^6 \mathbb{P}(X = 2|N = n) \\ &= \frac{1}{6} \sum_{n=1}^6 \frac{\binom{8}{n-1}}{\binom{10}{n}} \\ &= \frac{1}{6} \sum_{n=1}^6 \frac{n}{9} - \frac{n^2}{90} \\ &= \frac{119}{540}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X|N]] \\ &= \mathbb{E} \left[ \sum_{x=1}^{11-N} x \mathbb{P}(X = x|N) \right] \\ &= \mathbb{E} \left[ \sum_{x=1}^{11-N} \frac{x \binom{10-x}{N-1}}{\binom{10}{N}} \right] \\ &= \mathbb{E} \left[ \frac{1}{\binom{10}{N}} \sum_{x=1}^{11-N} \binom{x}{1} \binom{10-x}{N-1} \right] \\ &= \mathbb{E} \left[ \frac{11}{N+1} \right] \approx 2.92024 \end{aligned}$$

□

## Problem 5

Suppose that  $\xi_1, \xi_2, \dots$  are independent and identically distributed with  $\mathbb{P}\{\xi_k = \pm 1\} = \frac{1}{2}$ . Let  $N$  be independent of  $\xi_1, \xi_2, \dots$  and follow the geometric probability mass function

$$p_N(k) = \alpha(1 - \alpha)^k \quad \text{for } k = 0, 1, \dots,$$

where  $0 < \alpha < 1$ . Form the random sum  $Z = \xi_1 + \dots + \xi_N$ .

(a) Determine the mean and variance of  $Z$ .

*Proof.* Since the  $\mathbb{E}[N] = \frac{1-\alpha}{\alpha}$  and  $\text{Var}(N) = \frac{1-\alpha}{\alpha^2}$ ,

$$\mathbb{E}[Z] = \mathbb{E}[N]\mathbb{E}[\xi_1] = 0, \quad \text{Var}(Z) = \mathbb{E}[N]\text{Var}(\xi_1) + \mathbb{E}[\xi_1]^2\text{Var}(N) = \frac{1}{\alpha}.$$

□

(b) Evaluate the higher moments  $m_3 = \mathbb{E}[Z^3]$  and  $m_4 = \mathbb{E}[Z^4]$ .

*Proof.* Note that  $\xi_i^2 = 1$  and  $\mathbb{E}[\xi_i] = \mathbb{E}[\xi_i \xi_j] = 0$ . Hence,

$$\mathbb{E}[\xi_i \xi_j \xi_k] = \mathbb{E}[\xi_i \xi_j \xi_k \xi_m] = \mathbb{E}[\xi_i^2 \xi_j \xi_k] = \mathbb{E}[\xi_i^3 \xi_j] = 0,$$

so we only need to care about  $\mathbb{E}[\xi_i^2 \xi_j^2]$  and  $\mathbb{E}[\xi_i^4]$ . We thus get

$$\begin{aligned} \mathbb{E}[Z^3] &= \mathbb{E} \left[ \mathbb{E} \left[ \sum_i \sum_j \sum_k \xi_i \xi_j \xi_k | N \right] \right] \\ &= \mathbb{E} [N(N-1)(N-2)\mathbb{E}[\xi_i \xi_j \xi_k] + N(3N-2)\mathbb{E}[\xi_i]] = 0, \\ \mathbb{E}[Z^4] &= \mathbb{E} \left[ \sum_i \sum_j \sum_k \sum_m \mathbb{E}[\xi_i \xi_j \xi_k \xi_m] | N \right] \\ &= \mathbb{E} [3N(N-1)\mathbb{E}[\xi_i^2 \xi_j^2] + N\mathbb{E}[\xi_i^4]] \\ &= \mathbb{E} [(3N^2 - 2N)\mathbb{E}[1]] \\ &= \mathbb{E} [3N^2 - 2N] = 3(\text{Var}(N) + \mathbb{E}[N]^2) - 2\mathbb{E}[N] = \frac{6 - 5\alpha}{\alpha^2}. \end{aligned}$$

□

## Problem 6

To form a slightly different random sum, let  $\xi_0, \xi_1, \dots$  be independent identically distributed random variables and let  $N$  be a nonnegative integer-valued random variable, independent of  $\xi_0, \xi_1, \dots$ . The first two moments are

$$\mathbb{E}[\xi_k] = \mu, \quad \text{Var}[\xi_k] = \sigma^2,$$

$$\mathbb{E}[N] = \nu, \quad \text{Var}[N] = \tau^2.$$

Determine the mean and variance of the random sum  $Z = \xi_0 + \dots + \xi_N$ .

*Proof.*

$$\mathbb{E}[Z] = \mathbb{E}[\xi_k](\mathbb{E}[N] + 1) = \mu(\nu + 1),$$

$$\text{Var}(Z) = (\mathbb{E}[N] + 1)\text{Var}(\xi_k) + \mathbb{E}[\xi_k]^2\text{Var}(N) = (\nu + 1)\sigma^2 + \mu^2\tau^2.$$

□