MATH 140A: Homework #9

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Professor Seward

Ray Tsai

A16848188

Problem 1

Suppose f is a real function defined on $\mathbb{R}^1$ which satisfies

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

Proof. No. Consider $f(x) = \begin{cases} 1 & , x = 0 \\ 0 & , x \neq 0 \end{cases}$. Since $f(x)$ is constant at all points other than $x = 0$, the condition stated above holds for f. But then $\lim_{x \to 0} f(x) = 0 \neq 1 = f(0)$, so f is not continuous. $\square$

Problem 2

If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subseteq \overline{f(E)}$$

for every set $E \subseteq X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Proof. Note that $f(E) \subseteq \overline{f(E)}$, so it suffices to show $f(E') \subseteq \overline{f(E)}$. Let $p \in E'$. Since f is continuous, for all $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that $d_Y(f(p), f(q)) < 1/n$ for all $d_X(p, q) < \delta_n$. But then p is a limit point of E, so we may pick $q_n \in E \cap N_{\delta_n}(p) \backslash \{p\}$ for each $n \in \mathbb{N}$. Now, for arbitrary $\epsilon > 0$, we may find $f(q_n) \in f(E)$, such that $d_Y(f(q_n), f(p)) < 1/n < \epsilon$, for large enough n. Hence, $f(p)$ is a limit point of $f(E)$, and the result follows.

Consider $X = \mathbb{Q}$ and $Y = \mathbb{R}$. Take $f : \mathbb{Q} \hookrightarrow \mathbb{R}$ to be the natural inclusion. Since $\mathbb{Q}$ is closed in itself but $\overline{\mathbb{Q}} = \mathbb{R}$ in $\mathbb{R}$, $f(\overline{\mathbb{Q}}) = f(\mathbb{Q}) = \mathbb{Q} \subset \mathbb{R}$. $\square$

Problem 3

Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Proof. Since $\{0\}$ is closed in $\mathbb{R}$, $Z(f) = f^{-1}(0)$ is closed in X, by Theorem 4.8. $\square$

Problem 4

Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$ (In other words, a continuous function is determined by its values on a dense subset of its domain).

Proof. Let $x \in X$. Fix $\epsilon > 0$. There exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$, for any $y \in X$ with $d_X(x, y) < \delta$. Since E is dense in X, there exists $e \in E$ such that $d_X(x, e) < \delta$. But then $d_Y(f(x), f(e)) < \epsilon$. Since ϵ was arbitrary, $f(E)$ is dense in $f(X)$.

We now prove the second question. Let $p \in X$. Since E is dense in X, pick $p_n \in E \cap (N_{\frac{1}{n}} \backslash \{p\})$, for each $n \in \mathbb{N}$. But then

$$f(p) = \lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} g(p_n) = g(p).$$

$\square$

Problem 5

If f is a real continuous function defined on a closed set $E \subseteq \mathbb{R}^1$, prove that there exist continuous real functions on $\mathbb{R}^1$ such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called continuous extensions of f from E to $\mathbb{R}^1$.)

Proof. By exercise 2.29, E^c is a union of countably many disjoint segments, say $E^c = \bigcup_{n=1}(a_n, b_n)$, where $a_i < b_i \leq a_{i+1}$ for all i. Define $g_n : (a_n, b_n) \to \mathbb{R}$ as

$$g_n(x) \mapsto \begin{cases} f(b_n), & a_n = -\infty \\ f(a_n), & b_n = \infty \\ f(a_n) + \frac{f(b_n)-f(a_n)}{b_n-a_n} \cdot (x - a_n), & \text{otherwise} \end{cases}.$$

If we plotted out the two dimension graph of g, the image of each segment (a_i, b_i) is a straight line which connects $(a_i, f(a_i))$ and $(b_i, f(b_i))$. Note that we g_n is continuous, as it is either constant or a linear polynomial. Now define $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) \mapsto \begin{cases} g_n(x), & x \in (a_n, b_n) \\ f(x), & \text{otherwise} \end{cases}.$$

Obvisouly, $g(x) = f(x)$ for all $x \in E$. Fix $\epsilon > 0$, and let $p \in \mathbb{R}$. Suppose $p \notin E$. Since E^c is open, $N_r(p) \subseteq E^c$ for some $r > 0$. Since $g_n(p)$ is continuous, there exists $\delta \in (0, r)$, such that $|g_n(p) - g_n(q)| < \epsilon$ for all $q \in N_\delta(p)$. But then $q \in E^c$, so $g(q) = g_n(q)$, and thus g is continuous at p.

Now suppose $p \in E$. Since f is continuous, there eixsts $\delta > 0$ such that $|g(p) - g(q)| = |f(p) - f(q)| < \epsilon$, for all $q \in (p - \delta, p + \delta) \cap E$. We may assume $(p - \delta, p + \delta)$ intersects with some (a_i, b_i), otherwise we are done.

Suppose some $(a_i, b_i) \subset (p - \delta, p + \delta)$. Since $a_i, b_i \in E$ and, by construction, $\min(f(a_i), f(b_i)) \leq f(q) \leq \max(f(a_i), f(b_i))$ for $q \in (a_i, b_i)$, we have $|g(p) - g(q)| < \epsilon$ for $q \in (a_i, b_i)$.

Hence, we only have to care about the case where the ends of $(p - \delta, p + \delta)$ overlap with other segments. Suppose $(p - \delta, p + \delta)$ partially intersects with the bottom part of some (a_i, b_i), then we shrunk the segment to $(p - \delta, p + \delta_a)$, with $a_i = p + \delta_a$. Similarly, suppose $(p - \delta, p + \delta)$ partially intersects with the top part of some (a_i, b_i), then we shrunk the segment to $(p - \delta_b, p + \delta)$, with $b_i = p + \delta_b$.

Thus, if $\delta_a, \delta_b > 0$, we would end up with some neighborhood $N_p = (p - \delta_b, p + \delta_a)$, where $\delta_a, \delta_b \leq \delta$, such that $|g(p) - g(q)| < \epsilon$ for all $q \in N_p$.

Suppose $\delta_a = 0$. Then $p = a_i$, for some i. But then $\lim_{x \to a_i} g_i(x) = g_i(a_i)$. Similarly, suppose $\delta_b = 0$. Then $p = b_i$, for some i. But then $\lim_{x \to b_i} g_i(x) = g_i(b_i)$. Hence, in either case, we may still find some $\delta'_a, \delta'_b > 0$, such that $|g(p) - g(q)| < \epsilon$ for $q \in (p - \delta'_b, p + \delta'_a)$, and thus $g(p)$ is continuous at p. ☐

Problem 6

If f is defined on E, the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers and f is real-valued, the graph of f is a subset of the plane. Suppose E is compact and prove that f is continuous on E if and only if its graph is compact.

Proof. Suppose $f : E \to Y$ is continuous. Define $\phi : E \to E \times Y$ that sends x to $(x, f(x))$. Fix $\epsilon > 0$. There exists $\nu > 0$ such that $d_Y(f(x), f(y)) < \epsilon/2$ for $y \in N_\nu(x)$. Take $\delta = \min(\nu, \epsilon/2)$. We then have

$$d(\phi(x), \phi(y)) = d_E(x, y) + d_Y(f(x), f(y)) < \min(\nu, \epsilon/2) + \epsilon/2 \leq \epsilon,$$

for all $y \in E$ with $d_E(x, y) < \delta$. Since ϕ is a continuous mapping of a compact metrix space, $\phi(E)$ is compact, by By Theorem 4.14.

We now show the converse. Since $d(\phi(x), \phi(y)) \geq d_Y(f(x), f(y))$, it is obvious that f is continuous if ϕ is continuous, as the $\delta > 0$ that can be used to show the continui of ϕ also applies to show that of f. Hence, suppose f is not continuous at some point p, then ϕ is not continuous at p. Then, there exists a sequence p_n that converges to p while $\phi(p_n)$ does not converge to $\phi(p)$. We may assume that some subsequence $\phi(p_{n_i})$ converges, otherwise $\phi(E)$ is not compact, by Theorem 3.6. But then $\phi(p_{n_i})$ converges to some point k, where $f(p) \neq k$. Since f is well-defined, $(p, k) \neq (p, f(p))$ is a limit point not contained in $\phi(E)$. Hence, $\phi(E)$ is not closed, and thus it is not compact, by Theorem 2.34. $\square$

Problem 7

Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}^1$. Prove that f is bounded on E. Show that the conclusion is false if the boundedness of E is omitted from the hypothesis.

Proof. Suppose for the sake of contradiction that f is unbounded on E. Then, we may pick p_n such that $|f(p_n)| > n$, for each $n \in \mathbb{N}$. Note that $f(p_n) \to \infty$. But p_n is a sequence in a bounded set E, so there exists a convergent subsequence p_{n_i}, by Theorem 3.6. Fix $\epsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $|x - y| < \delta$. Since p_{n_i} is a Cauchy sequence by Theorem 3.11, there exists integer N such that $|p_{n_i} - p_{n_j}| < \delta$ for $i, j \geq N$ and thus $|f(p_{n_i}) - f(p_{n_j})| < \epsilon$. Hence, $|f(p_{n_i})| \leq |f(p_{n_i}) - f(p_{n_j})| + |f(p_{n_j})| < \epsilon + |f(p_{n_j})|$. Fixing j, we have $\lim_{i\to\infty} |f(p_{n_i})| \leq \epsilon + |f(p_{n_j})|$, contradicting our choice of p_n.

Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = x$. For $\epsilon > 0$, there exists $\delta = \epsilon$ such that $|f(x) - f(y)| = |x - y| < \epsilon$ for all $|x - y| < \delta$. It follows that f is uniformly continuous but unbounded, so the conclusion is false if E is not bounded. $\square$