MATH 140B: Homework #6

Due on May 17, 2024 at 23:59pm

Professor Seward

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Problem 1

Suppose g and f_n ($n = 1, 2, 3, \dots$) are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \le g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^\infty g(x)\, dx < \infty.$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx = \int_0^\infty f(x)\, dx.$$

Proof. We first show that $\int_0^T f_n(x)\, dx$ converges for all n. Define $h_{n,k} = \int_{1/k}^T f_n(x)\, dx$. Pick $\epsilon > 0$. Since $\int_0^T g(x)\, dx$ exists, there exists N such that $\int_0^{1/n} g(x)\, dx < \epsilon$ for all $n \ge N$. Let $\beta > \alpha \ge N$. Then,

$$|h_{n,\beta} - h_{n,\alpha}| = \left| \int_{1/\beta}^{1/\alpha} f_n(x)\, dx \right| \le \int_{1/\beta}^{1/\alpha} |f_n(x)|\, dx \le \int_{1/\beta}^{1/\alpha} g(x)\, dx < \epsilon,$$

and so $h_{n,k}$ converges. Since $|f|$ is also bounded by g, we may apply the same argument to show that $\int_0^T f(x)\, dx$ converges.

We now show that $\int_0^\infty f_n(x)\, dx$ converges for all n. Define $u_{n,k} = \int_0^k f_n(x)\, dx$. Pick $\epsilon > 0$. Since $\int_0^\infty g(x)\, dx$ exists, there exists N such that $\int_n^\infty g(x)\, dx < \epsilon$ for all $n \ge N$. Let $\beta > \alpha \ge N$. Then,

$$|u_{n,\beta} - u_{n,\alpha}| = \left| \int_\alpha^\beta f_n(x)\, dx \right| \le \int_\alpha^\beta |f_n(x)|\, dx \le \int_\alpha^\beta g(x)\, dx < \epsilon,$$

and so $u_{n,k}$ converges. Since $|f|$ is also bounded by g, we may apply the same argument to show that $\int_0^\infty f(x)\, dx$ converges.

Let $I_n(t) = \int_t^\infty f_n(x)\, dx$ and let $I(t) = \int_t^\infty f(x)\, dx$. We show that $I_n \to I$ uniformly on $(0, \infty)$. Again, pick $\epsilon > 0$. There exists $t_1, t_2 \in (0, \infty)$, $t_2 > t_1$, such that $\int_0^{t_1} g(x)\, dx < \epsilon/6$ and $\int_{t_2}^\infty g(x)\, dx < \epsilon/6$. Since f_n converges to f uniformly, there exists N such that $|f_n(x) - f(x)| < \epsilon/3(t_2 - t_1)$ for all $n \ge N$ and $x \in [t_1, t_2]$. Hence, for all $n \ge N$ and $t \in (0, \infty)$, let $t' \in (0, \min(t, t_1))$ and we have

$$\begin{aligned}
|I(t) - I_n(t)| &\le \int_{t'}^\infty |f(x) - f_n(x)|\, dx \\
&= \int_{t'}^{t_1} |f(x) - f_n(x)|\, dx + \int_{t_1}^{t_2} |f(x) - f_n(x)|\, dx + \int_{t_2}^\infty |f(x) - f_n(x)|\, dx \\
&\le 2\int_0^{t_1} g(x)\, dx + \int_{t_1}^{t_2} |f(x) - f_n(x)|\, dx + 2\int_{t_2}^\infty g(x)\, dx \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

Thus, $I_n \to I$ uniformly on $(0, \infty)$. By Theorem 7.11,

$$\lim_{n\to\infty} \lim_{t\to 0} I_n(t) = \lim_{t \to 0} I(t).$$

$\square$

Problem 2

Assume that (f_n) is a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \le f_n(x) \le 1$ for all x and all n.

Prove that there is a function f and a sequence (n_k) such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

for every $x \in \mathbb{R}^1$. (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem.*)

Proof. Since (f_n) is pointwise bounded, Theorem 7.23 yields a subsequence (f_{n_k}) such that $(f_{n_k}(q))$ converges for every $q \in \mathbb{Q}$. Define $f : \mathbb{Q} \to \mathbb{R}$ such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x),$$

and we extend f to $\mathbb{R} \to \mathbb{R}$ by defining

$$f(x) = \sup_{q\le x} f(q), \quad (q \in \mathbb{Q}).$$

We first show that f is monotonically increasing. Since f_{n_k} is monotonically increasing, f is monotonically increasing on $\mathbb{Q}$. Let $x, y \in \mathbb{R}$. There eixsts rational $s \in (x, y)$. Then, for any $r, t \in \mathbb{Q}$ such that $r < x < s < y < t$, $f(r) < f(s) < f(t)$, and thus

$$f(x) = \sup_{q\le x} f(q) \le f(s) \le \sup_{q\le y} f(q) = f(y).$$

By Theorem 4.30, f has at most countably many discontinuous points. We show that f pointwise converges on every continuous point. Let $x \in \mathbb{R}$ be a continuous point of f. Pick $\epsilon > 0$. There exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/10$ for all $|x - y| < \delta$. Let $s, t \in (x - \delta, x + \delta)$ be rationals such that $s < x < t$. Then $|f(t)-f(s)| < \epsilon/5$. There exists N_1, N_2 such that $|f(t)-f_{n_k}(t)| < \epsilon/5$ for all $k \ge N_1$ and $|f(s)-f_{n_k}(s)| < \epsilon/5$ for all $k \ge N_2$. Put $N = \max(N_1, N_2)$. It now follows that

$$\begin{aligned}
|f(x) - f_{n_k}(x)| &\le |f(x) - f(t)| + |f(t) - f_{n_k}(t)| + |f_{n_k}(t) - f_{n_k}(x)| \\
&\le |f(x) - f(t)| + |f(t) - f_{n_k}(t)| + |f_{n_k}(t) - f_{n_k}(s)| \\
&\le |f(x) - f(t)| + |f(t) - f_{n_k}(t)| + |f_{n_k}(t) - f(t)| + |f(t) - f(s)| + |f(s) - f_{n_k}(s)| \\
&< \epsilon/10 + \epsilon/5 + \epsilon/5 + \epsilon/5 + \epsilon/5 < \epsilon,
\end{aligned}$$

for all $k \ge N$, and thus $f(x) = \lim_{k\to\infty} f_{n_k}(x)$.

It remains to guarantee the pointwise convergence of the discontinuous points of f. Again by Theorem 7.23, there exists a subsequence $(f_{n_{k_i}})$ which converges on every discontinuous points of f, as there are at most countably many of them. Now define $f' : \mathbb{R} \to \mathbb{R}$,

$$f'(x) = \begin{cases} \lim_{i\to\infty} f_{n_{k_i}}(x), & \text{if } f \text{ is discontinuous at } x \\ f(x), & \text{otherwise} \end{cases}.$$

Then,

$$f'(x) = \lim_{i\to\infty} f_{n_{k_i}}(x),$$

for all $x \in \mathbb{R}$. ☐

Problem 3

Suppose f is a real continuous function on $\mathbb{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$, and (f_n) is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

Proof. Pick $\epsilon > 0$. Since (f_n) is equicontinuous on $[0, 1]$, there exists $\delta > 0$ such that

$$|f(nx) - f(ny)| < \epsilon,$$

whenever $|x - y| < \delta$, for all n and $x, y \in [0, 1]$. Let $s, t \in [0, \infty]$. Put integer $n > \max(|s - t|/\delta, s, t)$. Since $|\frac{s}{n} - \frac{t}{n}| = |s - t|/n < \delta$ and $\frac{s}{n}, \frac{t}{n} \in [0, 1]$, we have

$$|f(s) - f(t)| < \epsilon.$$

But then ϵ is arbitrary, so f is constant on $[0, \infty]$. $\square$

Problem 4

Let (f_n) be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)\,dt \quad (a \leq x \leq b).$$

Prove that there exists a subsequence (F_{n_k}) which converges uniformly on $[a, b]$.

Proof. We show that (F_n) is pointwise bounded and equicontinuous on $[a, b]$.

Since (f_n) is uniformly bounded, there exists K such that $|f_n(x)| < K$ for all x and n. Hence,

$$|F_n(x)| \leq \int_a^x |f_n(t)|\,dt < K(x-a),$$

and so F_n is point-wise bounded.

Pick $\epsilon > 0$. Let $\delta = \epsilon/K(b-a)$. Then,

$$|F_n(x) - F_n(y)| \leq \int_x^y |f_n(t)|\,dt \leq K(y-x) < \epsilon,$$

for all $n \in \mathbb{N}$ and $x, y \in [a, b]$ such that $\delta > y - x > 0$. Hence, (F_n) is equicontinuous.

The result now follows from Theorem 7.25. $\square$

Problem 5

Let K be a compact metric space, let S be a subset of $\mathscr{C}(K)$. Prove that S is compact (with respect to the metric defined in Definition 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on K.)

Proof. Suppose S is compact. S is closed so S is uniformly closed by definition.

Pick $\epsilon > 0$ and pick $\nu \in (0, \epsilon/2)$. Consider the open cover $\{B_\nu(f)\}_{f \in S}$. There exists $\{f_i\}_{i=1}^n$ such that $\bigcup_{i=1}^n B_\nu(f_i) \supset S$. Since K is compact, f is uniformly continuous for all $f \in \mathscr{C}(K)$. Hence, for each f_i, there exists δ_i such that $|f_i(x) - f_i(y)| < \epsilon - 2\nu$ for all $x, y \in K$ such that $d(x,y) < \delta_i$. Put $\delta = \min_{1 \le i \le n} \delta_i$. Let $g \in S$. Then $\sup_{x \in K} |g(x) - f_i(x)| < \nu$ for some f_i. But then

$$\begin{aligned}
|g(x) - g(y)| &\le |g(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - g(y)| \\
&< \nu + \epsilon - 2\nu + \nu = \epsilon.
\end{aligned}$$

Define $\Phi(x) = \nu + \max_{1 \le i \le n} |f_i(x)|$. Then, $f(x) < \Phi(x)$ for all $f \in S$, so S is pointwise bounded.

Now suppose that S is uniformly closed, pointwise bounded, and equicontinuous. Let $T \subset S$ be an infinite subset of S. By Theorem 7.25, T contains a uniformly convergent sequence, which converges to some $f \in S$ as S is uniformly closed. Hence, every infinite subset of S has a limit point in S, and thus S is compact. $\square$

Problem 6

If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n \, dx = 0 \quad (n = 0, 1, 2, \ldots),$$

prove that $f(x) = 0$ on $[0,1]$.

Proof. By the Weierstrass Theorem, there exists a sequence of polynomials P_n which converges to f uniformly on $[0,1]$. By the given identity,

$$\int_0^1 f(x)P_n(x)\, dx = 0.$$

But then

$$\int_0^1 f^2(x)\, dx = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)\, dx = 0,$$

and the result now follows. □