

# C3.8 Analytic Number Theory: Sheet #2

Due on November 4, 2025 at 12:00pm

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**Problem 1**

Evaluate the sum  $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$ .

*Proof.* By Proposition 3.1(ii),

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\sum_{n=1}^{\infty} n^{-2}} = \frac{6}{\pi^2}.$$

□

**Problem 2**

Give a simple description of the function  $\phi \star 1$ .

*Proof.* Note that

$$\phi \star 1 = \sum_{d|n} \phi(n/d).$$

But then  $\phi(n/d)$  is the number of  $m \leq n$  such that  $\gcd(m, n) = d$ . Thus  $\phi \star 1(n) = n$ .  $\square$

## Problem 3

Establish the following Dirichlet series:

$$(i) \sum_n \tau(n)n^{-s} = \zeta(s)^2 \text{ for } \operatorname{Re} s > 1;$$

*Proof.* Note that

$$1 \star 1 = \sum_{d|n} 1 = \tau(n).$$

The result now follows from Proposition 3.1(i).  $\square$

$$(ii) \sum_n \phi(n)n^{-s} = \frac{\zeta(s-1)}{\zeta(s)} \text{ for } \operatorname{Re} s > 2;$$

*Proof.* Since

$$\phi \star 1 = \sum_{d|n} \phi(n/d) = n,$$

by Proposition 3.1(i) we have

$$\zeta(s) \sum_n \phi(n)n^{-s} = \sum_n n \cdot n^{-s} = \zeta(s-1).$$

The result now follows.  $\square$

$$(iii) \sum_n \sigma(n)n^{-s} = \zeta(s)\zeta(s-1);$$

*Proof.* By (ii),  $\zeta(s-1)$  is the Dirichlet series for  $\phi \star 1 = n$ . But then

$$n \star 1 = \sum_{d|n} d = \sigma(n).$$

The result now follows from Proposition 3.1(i).  $\square$

- (iv) If  $\lambda(n)$  is the Liouville function, that is to say the unique completely multiplicative function equal to  $-1$  on the primes, then

$$\sum_n \lambda(n)n^{-s} = \frac{\zeta(2s)}{\zeta(s)} \text{ for } \operatorname{Re} s > 1.$$

*Proof.* Let  $d | n$ . Notice

$$\lambda(d) + \frac{\lambda(n)}{\lambda(d)} = \frac{\lambda(d)^2 + \lambda(n)}{\lambda(d)} = \frac{1 + \lambda(n)}{\lambda(d)} = \begin{cases} 2/\lambda(d) & \text{if } n \text{ is a square number} \\ 0 & \text{otherwise} \end{cases}$$

Thus  $\lambda \star 1(n) = 0$  if  $n$  is not a square number. Suppose now  $n$  is a square number. Then  $n$  has an odd number of divisors  $d$ . That is if  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ , then  $|\alpha| = \alpha_1 + \cdots + \alpha_k$  is odd. But then  $d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ ,  $\lambda(d) = (-1)^{\sum_{i=1}^k \beta_i}$ . Thus

$$\lambda \star 1(n) = \sum_{d|n} \lambda(d) = \sum_{0 \leq \beta_i \leq \alpha_i} (-1)^{\sum_{i=1}^k \beta_i} = \sum_{i=1}^{|\alpha|} (-1)^i = 1.$$

It now follows that

$$\zeta(s) \sum_n \lambda(n)n^{-s} = \sum_{\substack{n=1, \\ n=k^2}}^{\infty} n^{-s} = \sum_{n=1}^{\infty} n^{-2s} = \zeta(2s).$$

$\square$

## Problem 4

Obtain an asymptotic for  $\sum_{n < X} \tau(n)$

*Proof.* Note that

$$\sum_{n < X} \tau(n) = \sum_{n < X} \sum_{ab=n} 1 = \sum_{ab < X} 1 = 2 \sum_{\substack{n \leq \sqrt{X} \\ n \leq \lfloor X/n \rfloor}} \lfloor X/n \rfloor - \mathbb{1}_{X=k^2} \cdot (\sqrt{X})^2.$$

Since

$$\sum_{n \leq \sqrt{X}} \lfloor X/n \rfloor = (1 + o(1))X \sum_{n \leq \sqrt{X}} 1/n = (1 + o(1)) \left( \frac{1}{2} \log X + \gamma \right) X,$$

we have

$$\sum_{n < X} \tau(n) = (1 + o(1))X \log X.$$

□

**Problem 5**

True or false? There is a constant  $C$  such that  $\tau(n) \leq \log^C n$  for all sufficiently large  $n$ . Justify your answer.

*Proof.* False. Suppose it is true and that  $n$  is large. Consider the product of the first  $k$  primes  $n = p_1 p_2 \cdots p_k$ .

$$\tau(n) = (1+1)(1+1)\cdots(1+1) = 2^k.$$

But then by the Prime Number Theorem,  $p_k \sim k \log k$ . Thus

$$\tau(n) = 2^k \leq \log^C n \leq \log^C p_k^k = (1 + o(1))(k^2 \log k)^C,$$

contradiction as  $\lim_{k \rightarrow \infty} \frac{2^k}{(k^2 \log k)^C} = \infty$  for fixed  $C$ . □

## Problem 6

Show that

$$\sum_n \Lambda(n) \left\lfloor \frac{Y}{n} \right\rfloor = \sum_{n \leq Y} \log n.$$

By considering  $Y = X$  and  $Y = X/2$ , use this to prove that

$$\sum_{X/2 < n \leq X} \Lambda(n) \ll X.$$

*Proof.* By Legendre's formula, the exponent of the largest power of prime  $p$  that divides  $n!$  is

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

Thus

$$\sum_n \Lambda(n) \left\lfloor \frac{Y}{n} \right\rfloor = \sum_{p \leq Y} \nu_p(Y!) \log p = \log \prod_{p \leq Y} p^{\nu_p(Y!)} = \log Y! = \sum_{n \leq Y} \log n.$$

Notice

$$\begin{aligned} \sum_n \Lambda(n) \left\lfloor \frac{X}{n} \right\rfloor - 2 \sum_n \Lambda(n) \left\lfloor \frac{X/2}{n} \right\rfloor &= \left( \sum_{n \leq X/2} \Lambda(n) \left\lfloor \frac{X}{n} \right\rfloor + \sum_{X/2 < n \leq X} \Lambda(n) \left\lfloor \frac{X}{n} \right\rfloor \right) - 2 \sum_{n \leq X/2} \Lambda(n) \left\lfloor \frac{X/2}{n} \right\rfloor \\ &= \sum_{n \leq X/2} \Lambda(n) \left( \left\lfloor \frac{X}{n} \right\rfloor - 2 \left\lfloor \frac{X/2}{n} \right\rfloor \right) + \sum_{X/2 < n \leq X} \Lambda(n) \left\lfloor \frac{X}{n} \right\rfloor \\ &\geq \sum_{X/2 < n \leq X} \Lambda(n), \end{aligned}$$

as  $\left\lfloor \frac{X}{n} \right\rfloor - 2 \left\lfloor \frac{X/2}{n} \right\rfloor \geq 0$  for  $n > X/2$  and  $\left\lfloor \frac{X}{n} \right\rfloor = 1$  for  $X/2 < n \leq X$ . But then on the LHS, we have

$$\sum_n \Lambda(n) \left\lfloor \frac{X}{n} \right\rfloor - 2 \sum_n \Lambda(n) \left\lfloor \frac{X/2}{n} \right\rfloor = \log X! - 2 \log(X/2)! = \log \frac{X!}{(X/2)!(X/2)!}.$$

By the Stirling's Formula,

$$\frac{X!}{(X/2)!(X/2)!} = (1 + o(1)) 2^X \sqrt{\frac{2}{\pi X}}.$$

Thus

$$\log \frac{X!}{(X/2)!(X/2)!} = O(X).$$

The result now follows from combining all of the above.  $\square$

## Problem 7

Write  $L(X) := \sum_{n \leq X} \lambda(n)$  and  $M(X) := \sum_{n \leq X} \mu(n)$ . Establish the relations

$$L(X) = \sum_{d \leq \sqrt{X}} M\left(\frac{X}{d^2}\right) \quad \text{and} \quad M(X) = \sum_{d \leq \sqrt{X}} \mu(d)L\left(\frac{X}{d^2}\right),$$

and hence conclude that the statements  $L(X) = o(X)$  and  $M(X) = o(X)$  are equivalent.

*Proof.* By Problem 3(iv),

$$\lambda \star 1(n) = \begin{cases} 1 & \text{if } n \text{ is a square number} \\ 0 & \text{otherwise} \end{cases}.$$

Thus let  $\text{sq}(n)$  denote the indicator function of the set of square numbers. The Möbius inversion formula then yields

$$\lambda(n) = \mu \star \text{sq}(n) = \sum_{d|n} \mu(n/d) \text{sq}(d).$$

Hence,

$$L(X) = \sum_{n \leq X} \sum_{d|n} \mu(n/d) \text{sq}(d) = \sum_{d \leq X} \text{sq}(d) \sum_{n \leq X/d} \mu(n) = \sum_{d \leq \sqrt{X}} \sum_{n \leq X/d^2} \mu(n) = \sum_{d \leq \sqrt{X}} M\left(\frac{X}{d^2}\right).$$

But then

$$\sum_{d \leq \sqrt{X}} \mu(d)L\left(\frac{X}{d^2}\right) = \sum_{d^2 k \leq X} \mu(d)\lambda(k) = \sum_{n \leq X} \sum_{d^2|n} \mu(d)\lambda\left(\frac{n}{d^2}\right).$$

Note that

$$\begin{aligned} \sum_{d^2|n} \mu(d)\lambda\left(\frac{n}{d^2}\right) &= \sum_{d^2|n} \mu(d) \sum_{j|n/d^2} \mu(n/jd^2) \text{sq}(j) \\ &= \sum_{d^2|n} \mu(d) \sum_{(jd)^2|n} \mu(n/(jd)^2) \\ &= \sum_{m^2|n} \mu(n/m^2) \sum_{d|m} \mu(d) \\ &= \sum_{m^2|n} \mu(n/m^2) \cdot \delta(m) = \mu(n). \end{aligned}$$

Thus,

$$M(X) = \sum_{d \leq \sqrt{X}} \mu(d)L\left(\frac{X}{d^2}\right).$$

Suppose  $L(X) = o(X)$  and that  $M(X) = \Omega(X)$ . Then

$$L(X) = \sum_{d \leq \sqrt{X}} M\left(\frac{X}{d^2}\right) = \Omega\left(X \sum_{d \leq \sqrt{X}} \frac{1}{d^2}\right) = \Omega(X),$$

contradiction. On the other hand, suppose  $M(X) = o(X)$ . Fix  $\epsilon > 0$ . Since  $|M(Y)/Y| \leq 1$ , we may pick  $N$  large enough such that

$$\left| \frac{1}{X} \sum_{N \leq d \leq \sqrt{X}} M\left(\frac{X}{d^2}\right) \right| = \left| \sum_{N \leq d \leq \sqrt{X}} \frac{1}{d^2} \cdot \frac{M\left(\frac{X}{d^2}\right)}{X/d^2} \right| \leq \left| \sum_{d \geq N} \frac{1}{d^2} \right| < \epsilon/2.$$

We also have

$$\lim_{X \rightarrow \infty} \frac{1}{X} \sum_{d \leq N} M\left(\frac{X}{d^2}\right) = \lim_{X \rightarrow \infty} \sum_{d \leq N} \frac{1}{d^2} \cdot \frac{M\left(\frac{X}{d^2}\right)}{X/d^2} = 0$$

But then

$$\frac{L(X)}{X} = \frac{1}{X} \sum_{d \leq N} M\left(\frac{X}{d^2}\right) + \frac{1}{X} \sum_{N \leq d \leq \sqrt{X}} M\left(\frac{X}{d^2}\right) < \epsilon,$$

as  $X \rightarrow \infty$ . Since  $\epsilon$  was arbitrary, we have  $L(X) = o(X)$ . □

## Problem 8

Give an asymptotic for  $\sum_{n \leq X} \phi(n)$ . (*Hint. Using the answer to Question 2, or otherwise, first establish that the expression to be estimated is  $\sum_{d \leq X} \mu(d) \sum_{m \leq X/d} m$ .*)

*Proof.* Since  $\phi * 1 = n$ , the Möbius inversion formula yields

$$\phi(n) = n * \mu(n) = \sum_{d|n} \mu(d) \cdot \frac{n}{d}.$$

Thus we have

$$\sum_{n \leq X} \phi(n) = \sum_{n \leq X} \sum_{d|n} \mu(d) \cdot \frac{n}{d}.$$

Notice that for a fixed  $d \leq X$ , we will sum up  $n/d$  over all  $n \leq X$  such that  $d | n$ , and times it by  $\mu(d)$ . In other words, we will sum up all integers  $m \leq X/d$  and multiply it by  $\mu(d)$ . Thus the double sum can be written as

$$\begin{aligned} \sum_{n \leq X} \phi(n) &= \sum_{d \leq X} \mu(d) \sum_{m \leq X/d} m \\ &= \sum_{d \leq X} \mu(d) \cdot \frac{(1 + \lfloor X/d \rfloor) \lfloor X/d \rfloor}{2} \\ &= \frac{1}{2} \sum_{d \leq X} (1 + o(1)) \mu(d) \cdot (X/d) + \frac{1}{2} \sum_{d \leq X} (1 + o(1)) \mu(d) \cdot (X/d)^2 \\ &= \frac{X}{2} \sum_{d \leq X} (1 + o(1)) \frac{\mu(d)}{d} + \frac{X^2}{2} \sum_{d \leq X} (1 + o(1)) \frac{\mu(d)}{d^2}. \end{aligned}$$

By Problem 1,  $\sum_{d \leq X} (1 + o(1)) \frac{\mu(d)}{d^2} \rightarrow \frac{6}{\pi^2}$  as  $X \rightarrow \infty$ . On the other hand,

$$\sum_{d \leq X} (1 + o(1)) \frac{\mu(d)}{d} \leq \sum_{d \leq X} (1 + o(1)) \frac{1}{d} = O(\log X).$$

Thus we may conclude that

$$\sum_{n \leq X} \phi(n) = \frac{3}{\pi^2} X^2 + O(X \log X).$$

□