Superimposed Extremal Graphs

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1 Introduction

Given graph G with n vertices, let $G_1, \dots, G_m$ be subgraphs of G. Let F be a graph with at least one edge. Our goal is to determine the maximum sum of the number of edges over all G_i's, i.e. $\sum_{i=1}^m e(G_i)$, with the constraint of $E(G_i) \cap E(G_j)$ not including some graph F for all distinct i, j.

2 Content

- Examine the case where $G_1, \dots, G_m$ are induced
  - The case $F = K_3$.
  - Color-critical F.
  - Generalize to any non-bipartite F.
- Examine the non-induced case
  - The case $F = K_3$.

3 Induced Case

In this section, we assume that $G_1, \dots, G_m$ are induced subgraphs of G. Given graph H, let $\mathcal{T}(H)$ be the graph with an additional vertex connecting to all vertices in H.

3.1 Triangle Case

Theorem 3.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j. Then*

$$\sum_{i=1}^{n} e(G_i) \leq n \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = \cdots = G_n = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

Lemma 3.2. *Suppose $E(G_1) \cap E(G_2)$ does not include K_3. Then*

$$e(G_1) + e(G_2) \leq 2 \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$, unless n is odd and $G_1 = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and $G_2 = \mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$.

Proof. Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both G_1 and G_2. Let $A = V(G_1)\backslash C$, and let $B = V(G_2)\backslash C$. For simplicity, put $a = |A|$, $b = |B|$, and $c = |C|$. We may assume that $a + b + c = n$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to a, b, c. Since G_1, G_2 are induced graphs, we have $\{u, v\} \in E(G_1)$ if and only if $\{u, v\} \in E(G_2)$, for $u, v \in C$. This implies the subgraph of G_1 induced by C is identical to the subgraph of G_2 induced by C. In other words, $E(G_1[C]) = E(G_2[C]) = E(G_i) \cap E(G_j)$, which is triangle-free. By Mantel's Theorem, $e(G_1[C]) \le \left\lfloor \frac{c^2}{4} \right\rfloor$, with equality if and only if $G_1[C] = K_{\lceil \frac{c}{2} \rceil, \lfloor \frac{c}{2} \rfloor}$. Hence, we may write

$$\begin{aligned} e(G_1) + e(G_2) &\le \binom{|V(G_1)|}{2} + \binom{|V(G_2)|}{2} - 2\left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor\right] \\ &= \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \left\lfloor \frac{c^2}{4} \right\rfloor\right]. \qquad (1) \end{aligned}$$

Define $f(a, b, c)$ as the function on the right-hand-side of (1). We show that $f(a, b, c)$ attains its maximum at $a = b = 0$ and $c = n$. Note that

$$\begin{aligned} f(a, b-2, c+2) - f(a, b, c) &= \binom{a+c+2}{2} - \binom{a+c}{2} \\ &\quad - 2\left[\binom{c+2}{2} - \binom{c}{2} - \left\lfloor \frac{(c+2)^2}{4} \right\rfloor + \left\lfloor \frac{c^2}{4} \right\rfloor\right] \\ &= 2(a+c) + 1 - 2[2c + 1 - (c+1)] \\ &= 2a + 1 > 0. \end{aligned}$$

By symmetry, $f(a-2, b, c+2) > f(a, b, c)$, and thus f attains its maximum when c is $n-1$ or n, that is, $a + b \le 1$. Equation (1) now yields,

$$e(G_1) + e(G_2) \le f(a, b, c) \le 2\left\lfloor \frac{n^2}{4} \right\rfloor.$$

Assume that $a = 0$. When $c = n$, the equality holds only if $G_1 = G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. If $c = n-1$, then the equality holds only if n is odd and $G_1 = G[C] = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and G_2 is G_1 with all vertices connected with the only remaining vertex, that is, $G_2 = \mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$. $\square$

We now give the proof for Theorem 3.1:

Proof of Theorem 3.1. We may assume that $n > 1$. Put $G_{n+i} = G_i$. By Lemma 3.2,

$$\sum_{i=1}^{n} e(G_i) = \frac{1}{2}\sum_{i=1}^{n}(e(G_i) + e(G_{i+1})) \le \frac{1}{2}\sum_{i=1}^{n} 2\left\lfloor \frac{n^2}{4} \right\rfloor = n\left\lfloor \frac{n^2}{4} \right\rfloor.$$

Suppose the equality holds. By Lemma 3.2, we are done if n is even. Suppose n is odd and $G_i = \mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$ for some i. By Lemma 3.2, one of G_i and G_{i+1} is $K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ and the other is $\mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor})$, for all i. Hence, $G_{i+1} = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}, G_{i+2} = \mathcal{T}(K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}), \ldots$ and the alternation proceeds. But then $G_{n+i} = G_i = K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ as n is odd, and this contradiction completes the proof. $\square$

3.2 Color-Critical Case

We may generalize the triangle case to any color-critical F in the same manner. Let Turán graph $T_r(n)$ denote the the complete balanced r-partite graph. We know for a fact that, for large enough n, if F is a $(r+1)$-color-critical graph with $r \geq 2$, then $\mathrm{ex}(n, F) = \mathrm{ex}(n, K_{r+1})$, and the extremal graph is $T_r(n)$.

Theorem 3.3. *Let F be a $(r+1)$-color-critical graph with $r \geq 2$. Suppose that $E(G_i) \cap E(G_j)$ is F-free for distinct i, j. For large enough n,*

$$\sum_{i=1}^{n} e(G_i) \leq n \cdot \mathrm{ex}(n, F),$$

with equality if and only if $G_1 = G_2 = \cdots = G_n = T_r(n)$.

Lemma 3.4. *Let F be a $(r+1)$-color-critical graph with $r \geq 2$. Suppose $E(G_1) \cap E(G_2)$ does not include F. For large enough n,*

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n, F),$$

with equality if and only if $G_1 = G_2 = T_r(n)$, unless $r = 2$, n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2 = \mathcal{T}(G_1)$.

Proof. Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both G_1 and G_2. Let $A = V(G_1) \backslash C$, and let $B = V(G_2) \backslash C$. For simplicity, put $a = |A|$, $b = |B|$, and $c = |C|$. We may assume that $a + b + c = n$. By the same argument in Lemma 3.2, $E(G_1[C]) = E(G_2[C]) = E(G_i) \cap E(G_j)$, which is F-free. Thus,

$$E(G_1[C]) \leq \mathrm{ex}(n, F) = \mathrm{ex}(n, K_{r+1}),$$

with equality if and only if $G_1[C] = T_r(c)$. Hence,

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \mathrm{ex}(c, K_{r+1})\right]. \tag{2}$$

Define $f(a, b, c)$ as the function on the right-hand-side of (2). We show that $f(a, b, c)$ attains its maximum at $a = b = 0$ and $c = n$. Note that

$$\begin{aligned} f(a, b-2, c+2) - f(a, b, c) &= \binom{a+c+2}{2} - \binom{a+c}{2} \\ &\quad - 2\left[\binom{c+2}{2} - \binom{c}{2} - \mathrm{ex}(c+2, K_{r+1}) + \mathrm{ex}(c, K_{r+1})\right] \\ &= 2a - 2c - 1 + 2[\mathrm{ex}(c+2, K_{r+1}) - \mathrm{ex}(c, K_{r+1})]. \end{aligned}$$

Since $r \geq 2$,

$$\begin{aligned} \mathrm{ex}(c+2, K_{r+1}) - \mathrm{ex}(c, K_{r+1}) &= \mathrm{ex}(c+2, K_{r+1}) - \mathrm{ex}(c+1, K_{r+1}) \\ &\qquad + \mathrm{ex}(c+1, K_{r+1}) - \mathrm{ex}(c, K_{r+1}) \\ &= \left(c + 2 - \left\lceil \frac{c+2}{r} \right\rceil\right) + \left(c + 1 - \left\lceil \frac{c+1}{r} \right\rceil\right) \\ &\geq 2c + 3 - \left(\left\lceil \frac{c+2}{2} \right\rceil + \left\lceil \frac{c+1}{2} \right\rceil\right) = c + 1, \end{aligned}$$

so $f(a, b-2, c+2) - f(a, b, c) \geq 2a + 1 > 0$. By symmetry, $f(a-2, b, c+2) > f(a, b, c)$, and thus f attains its maximum when c is $n-1$ or n, that is, $a+b \leq 1$. Equation (2) now yields,

$$e(G_1) + e(G_2) \leq \max\left[2 \cdot \mathrm{ex}(n, K_{r+1}),\ 2 \cdot \mathrm{ex}(n-1, K_{r+1}) + n - 1\right].$$

Assume that $a = 0$. Since

$$\begin{aligned} 2 \cdot \mathrm{ex}(n, K_{r+1}) - [2 \cdot \mathrm{ex}(n-1, K_{r+1}) + n - 1] &= 2\left(n - \left\lceil \frac{n}{r} \right\rceil\right) - n + 1 \qquad (3)\\ &\geq n + 1 - 2\left\lceil \frac{n}{2} \right\rceil \geq 0, \end{aligned}$$

we have

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n, K_{r+1}). \qquad (4)$$

If $c = n$, the equality for (4) holds only if $G_1 = G_2 = T_r(n)$. Suppose $c = n-1$ and the equality holds. Observe that the equation (3) is equal to zero only when $r = 2$ and n is odd. Hence, if $c = n-1$, the equality for (4) could only be achieved when $r = 2$, n is odd, $G_1 = T_r(n-1)$, and $G_2 = \mathcal{T}(G_1)$. □

Theorem 3.3 now follows from Lemma 3.5 and the same arument as in Theorem 3.1.

3.3 Non-bipartite Case

Theorem 3.5. *Let F be $(r+1)$-colorable, with $r \geq 2$. Suppose that $E(G_i) \cap E(G_j)$ is F-free for distinct i, j. For large enough n,*

$$\sum_{i=1}^{n} e(G_i) \leq n \cdot \mathrm{ex}(n, F),$$

with equality if and only if $G_1 = G_2 = \cdots = G_n$ are n-vertex extremal graphs for F.

By the same argument as in Theorem 3.1, it suffices to prove the following lemma:

Lemma 3.6. *Let F be $(r+1)$-colorable, with $r \geq 2$. Suppose $E(G_1) \cap E(G_2)$ does not include F. For large enough n,*

$$e(G_1) + e(G_2) \leq 2 \cdot \mathrm{ex}(n, F),$$

with equality if and only if $G_1 = G_2$ are n-vertex extremal graphs for F, unless n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2 = \mathcal{T}(G_1)$.

Proof. Let $C = V(G_1) \cap V(G_2)$, the set of vertices in both G_1 and G_2. Let $A = V(G_1) \backslash C$, and let $B = V(G_2) \backslash C$. For simplicity, put $a = |A|$, $b = |B|$, $c = |C|$, and $r = \chi(F)$.

We now find an upper bound of $e(G_1) + e(G_2)$ with respect to a, b, c. Since G_1, G_2 are induced graphs, we have $E(G_1[C]) = E(G_2[C]) = E(G[C]) = E(G_i) \cap E(G_j)$, which is F-free. Hence, we may write

$$e(G_1) + e(G_2) \leq \binom{a+c}{2} + \binom{b+c}{2} - 2\left[\binom{c}{2} - \mathrm{ex}(c, F)\right]. \qquad (5)$$

Define $f(a,b,c)$ as the function on the right-hand-side. We show that $f(a,b,c)$ attains its maximum at $a = b = 0$ and $c = n$. By a theorem of Simonovits, for large enough c, $\text{ex}(c,F) = \text{ex}(c,K_{r+1}) + \text{ex}(c,\tilde{F})$, where $\tilde{F}$ is the family of residue subgraphs of F after F is embedded into $T_r(c)$. Hence, we may write

$$\begin{aligned} f(a,b-2,c+2) - f(a,b,c) &= \binom{a+c+2}{2} - \binom{a+c}{2} \\ &\quad - 2\left[\binom{c+2}{2} - \binom{c}{2} - \text{ex}(c+2,F) + \text{ex}(c,F)\right] \\ &\geq 2a - 2c - 1 + 2[\text{ex}(c+2,K_{r+1}) - \text{ex}(c,K_{r+1})] > 0, \end{aligned}$$

as shown in the proof of Lemma 3.4. By symmetry, we also have $f(a-2,b,c+2) > f(a,b,c)$. Thus, f attains its maximum when c is $n-1$ or n. Equation (5) now yields,

$$e(G_1) + e(G_2) \leq \max\left[2\cdot\text{ex}(n,F), 2\cdot\text{ex}(n-1,F) + n - 1\right].$$

Assume that $a = 0$. Since

$$\begin{aligned} 2\cdot\text{ex}(n,F) - [2\cdot\text{ex}(n-1,F) + n - 1] &\geq 2[\text{ex}(n,K_{r+1}) - \text{ex}(n-1,K_{r+1})] \qquad (6)\\ &\quad - n + 1 \qquad (7)\\ &= 2\left(n - \left\lceil\frac{n}{r}\right\rceil\right) - n + 1 \qquad (8)\\ &\geq n + 1 - 2\left\lceil\frac{n}{2}\right\rceil \geq 0, \end{aligned}$$

we have

$$e(G_1) + e(G_2) \leq 2\cdot\text{ex}(n,F). \tag{9}$$

If $c = n$, the equality for (9) holds only if $G_1 = G_2$ are n-vertex extramal graphs for F. Suppose $c = n-1$ and the equality holds. Observe that equation (6) is equal to zero only when $r = 2$ and n is odd. Hence, if $c = n-1$, the equality for (9) could only be achieved when $r = 2$, n is odd, G_1 is an $(n-1)$-vertex extremal graph for F, and $G_2 = \mathcal{T}(G_1)$. □

4 Non-induced Case

We now remove the assumption that $G_1, \ldots, G_m$ are induced subgraphs. Again, we first consider the triangle-free case.

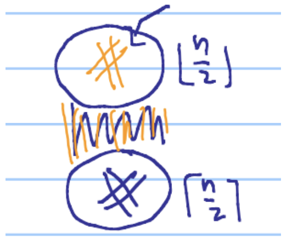
4.1 Triangle-Free Case

Theorem 4.1. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j. Then,*

$$\sum_{i=1}^{m} e(G_i) \leq \binom{n}{2} + (m-1)\left\lfloor\frac{n^2}{4}\right\rfloor.$$

The natural extremal construction is to simply put $G_1 = K_n$ and the rest as $K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$. However, even for $m = 2$ there are multiple extremal constructions.

For example, put G_1 as $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and connect all possible pairs of vertices on the left part. On the other hand, put G_2 as $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and connect all possible pairs of vertices on the right part.



Then, $E(G_1) \cap E(G_2)$ is triangle-free and

$$\begin{aligned} e(G_1) + e(G_2) &= 2e(G_1 \cap G_2) + e(G_1 \Delta G_2) \\ &= 2\left\lfloor \frac{n^2}{4} \right\rfloor + \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor = \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

Here we introduce the notation of *compression* of $G_1, \ldots, G_m$, which is the graph obtained by moving all edges in only one G_i to G_1. Performing compression for the case $m = 2$, we get

$$e(G_1) + e(G_2) = e(G_1) + e(G_1 \cap G_2) \leq \binom{n}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = K_n$ and $G_2 = G_1 \cap G_2 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. That is, the extremal graphs for $m = 2$ are isomorphic, up to compression.

We use the notion of compression to solve for $m = 3, 4$:

Theorem 4.2. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j. Then,*

$$e(G_1) + e(G_2) + e(G_3) \leq \binom{n}{2} + 2\left\lfloor \frac{n^2}{4} \right\rfloor,$$

with equality if and only if $G_1 = K_n$ and $G_2, G_3 = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ after compression.

Proof. See Chase Wilson's proof. □

TODO: solve $m = 4$.

Theorem 4.3. *Suppose that $E(G_i) \cap E(G_j)$ does not include K_3 for distinct i, j. Then,*

$$\sum_{i=1}^{m} e(G_i) \leq (1 + o_m(1))m \left\lfloor \frac{n^2}{4} \right\rfloor,$$

as $m \to \infty$.

Proof. forgot... □