MATH 173A: Homework #4

Due on Nov 10, 2024 at 23:59pm

Professor Cloninger

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Problem 1

(a) Find an expression for the orthogonal projection of a point $x \in \mathbb{R}^n$ onto the convex set

$$B = \{z \in \mathbb{R}^n : 0 \leq z_i \leq 1 \text{ for each } i = 1, ..., n\}.$$

You need to show your work, and justify your answer. The expression can be written piecewise, and per dimension if it's easier / more compact. **Hint:** It might be helpful to sketch B, when $n = 2$ (i.e., in 2 dimensions), and use the sketch to help you figure out what the projection should be.

Proof. For $x \in \mathbb{R}^n$, we need to find $\Pi_B(x) = \arg\min_{z \in B} \|z - x\| = \arg\min_{z \in B} \sum_i (z_i - x_i)^2$. Notice that we may decouple this minimization problem across n dimension by minimizing each z_i independently. That is, for all i

$$z_i = \arg\min_{a \in [0,1]} (a - x_i)^2 = \min(\max(0, x_i), 1).$$

□

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by

$$f(x) = \|Ax\|_2^2 + a^T x$$

where $A \in \mathbb{R}^{n \times n}$ is a positive definite matrix, and $a \in \mathbb{R}^n$. Write a projected gradient descent algorithm to solve

$$\min_{x \in \Omega} f(x)$$

for $\Omega = B$, with B from part (a). You do not need to specify the step size for this problem.

Proof. Note that

$$\nabla f(x) = 2A^T Ax + a,$$

and thus the projected gradient descent algorithm is

$$x^{(k+1)} = \Pi_\Omega \left(x^{(k)} - \mu \nabla f(x^{(k)}) \right) = \Pi_B \left(x^{(k)} - \mu(2A^T Ax^{(k)} + a) \right).$$

More explicitly, for all i,

$$x_i^{(k+1)} = \min \left(\max \left(0, x_i^{(k)} - 2\mu(A^T Ax^{(k)} + a)_i \right), 1 \right).$$

□

(c) Repeat part (b) but for $\Omega = B_2^n = \{z \in \mathbb{R}^n : \|z\|_2 \leq 1\}$.

Proof. Notice

$$\Pi_\Omega(x) = \begin{cases} \frac{x}{\|x\|_2} & \text{if } \|x\| > 1, \\ x & \text{if } \|x\| \leq 1. \end{cases}$$

Hence, the projected gradient descent algorithm is

$$x^{(k+1)} = \Pi_\Omega \left(x^{(k)} - \mu \nabla f(x^{(k)}) \right) = \Pi_B \left((I - 2\mu A^T A)x^{(k)} - \mu a \right),$$

which is $\frac{(I-2\mu A^T A)x^{(k)} - \mu a}{\|(I-2\mu A^T A)x^{(k)} - \mu a\|_2}$ if $\|x\| > 1$ and $(I - 2\mu A^T A)x^{(k)} - \mu a$ otherwise. □

Problem 2

Consider the *hollow* sphere S in $\mathbb{R}^n$, i.e., the set $S := \{x \in \mathbb{R}^n : \|x\|_2^2 = 1\}$. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = x^T Qx$$

where Q is an $n \times n$ symmetric matrix. For this problem you may use the fact that $\nabla f(x) = 2Qx$.

(a) For an arbitrary point $y \in \mathbb{R}^n$, $\Pi(y)$ be the projection of y onto S. Find an expression for $\Pi(y)$ and give a short argument (i.e., proof) for why this is the correct expression. Make sure to handle the case $y = 0$ (i.e., the zero vector).

Proof. I claim that $\Pi(y) = \frac{y}{\|y\|_2}$ if $y \neq 0$ and $\Pi(0)$ can be any point in S. Note that the reverse triangle-inequality yields a lower bound

$$\|x - y\|_2 \geq |\|x\|_2 - \|y\|_2| = |1 - \|y\|_2|,$$

for $x \in \Omega$. Obvisouly, any $x \in \Omega$ achieves the lower bound when $y = 0$. Suppose $y \neq 0$. Obviously $\frac{y}{\|y\|_2} \in \Omega$. Since

$$\left\| \frac{y}{\|y\|_2} - y \right\| = \left\| \left(\frac{1}{\|y\|_2} - 1 \right) y \right\| = \|y\|_2 \left| \frac{1}{\|y\|_2} - 1 \right| = |1 - \|y\|_2|$$

achieves the lower bound, $\Pi(y) = \frac{y}{\|y\|_2}$. □

(b) Is S a convex set?

Proof. S is not a convex set. Consider $x = (1, 0)$ and $y = (-1, 0)$. Then $0 = \frac{1}{2}(1, 0) + \frac{1}{2}(-1, 0) \notin S$. □

(c) Write a projected gradient descent algorithm, with constant step size μ, for

$$\min_{x \in \mathbb{R}^n} x^T Qx \qquad \text{subject to} \qquad \|x\|_2^2 = 1.$$

Proof. Note that $\nabla f(x) = 2Qx$, and thus the projected gradient descent algorithm is

$$x^{(k+1)} = \Pi_S \left((I - 2\mu Q)x^{(k)} \right),$$

which is equal to $\frac{(I-2\mu Q)x^{(k)}}{\|(I-2\mu Q)x^{(k)}\|}$ if $x^{(k)} \neq 0$ and any point in S if $x^{(k)} = 0$. □

(d) Is the projected gradient descent algorithm guaranteed to converge to the solution for small enough μ? If not, can you give an example of Q and an initialization $x^{(0)}$ where the algorithm won't converge?

Proof. Consider $Q = \text{diag}(1, 0)$ and $x^{(0)} = (1, 0)$. Then $x^{(k+1)} = \Pi_S \left(\begin{bmatrix} 1 - 2\mu & 0 \\ 0 & 1 \end{bmatrix} x^{(k)} \right)$. Since $x^{(0)}$ only have the first entry non-zero,

$$x^{(k+1)} = \frac{1 - 2\mu}{|1 - 2\mu|} x^{(k)} = \left(\frac{1 - 2\mu}{|1 - 2\mu|} \right)^{k+1} x^{(0)},$$

which is equal to $x^{(0)}$ for small enough μ. But then $f(x^{(0)}) = 1$ and $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = 0$, so the algorithm fails to converge. □

In [48]:
```
# import statements
import numpy as np
from matplotlib import pyplot as plt
from sklearn.datasets import fetch_openml
```

Question 3

Consider the $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 2)^2 - x_1 x_2$ and the following constrained optimization problem:

$$\min_{x_1, x_2} f(x_1, x_2) \quad \text{subject to} \quad 0 \leq x_i \leq 1,\ i = 1, 2.$$

Write a projected gradient descent algorithm, with constant step size $\mu = 0.001$ starting at $(0.5, 0.5)$ for 175 iterations, for the above optimization problem. Plot the function value $f(x^{(t)})$ against the iteration t.

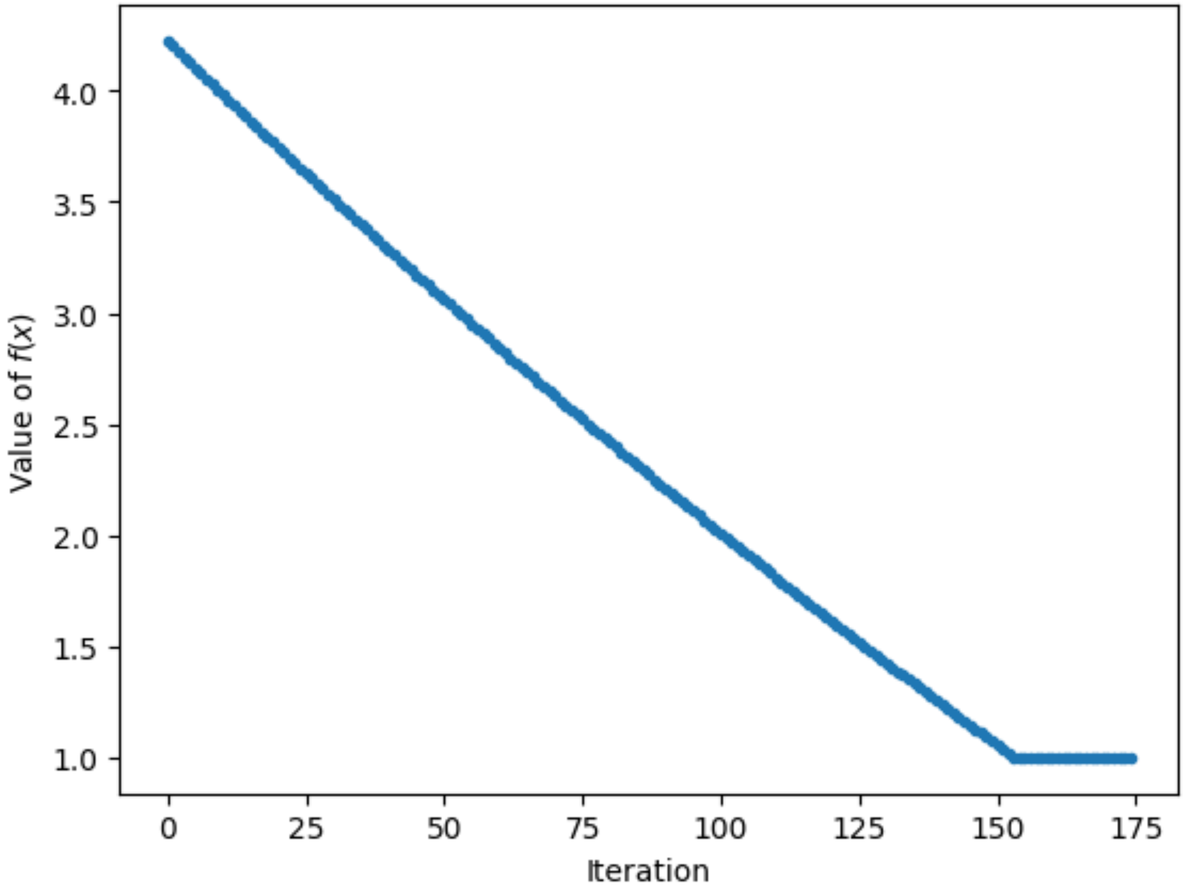
In [49]:
```
mu = 0.001
x = np.array([0.5, 0.5])
x_values = []
iterations = range(175)

def f(x_1, x_2):
    return (x_1 - 2)**2 + (x_2 - 2)**2 - x_1 * x_2

def df(x_1, x_2):
    return np.array([2 * (x_1 - 2) - x_2,  2 * (x_2 - 2) - x_1])

for i in iterations:
    y = x - mu * df(x[0], x[1])
    x = [min(max(0, y[0]), 1), min(max(0, y[1]), 1)]
    x_values.append(f(x[0], x[1]))

plt.plot(iterations, x_values, marker='.')
plt.xlabel("Iteration")
plt.ylabel(r"Value of $f(x)$")
plt.show()
```



Question 4

In this coding question, you'll implement a classifier with logistic regression

$$F(w) = \frac{1}{N}\sum_{i=1}^{N}\log(1+e^{-\langle w,x_i\rangle y_i}).$$

We will redo the MNIST coding question from HW3.5 but using different versions of gradient descent

$$w^{(t+1)} = w^{(t)} - \mu p^{(t)}.$$

This time, we will differentiate 4's and 9's. That is, instead of classifying images of 0 and 1, you classify the images of 4 and 9.

Loading MNIST Data

In this section, you will learn to load MNIST data. If you do not have tensorflow available on your jupyter notebook, uncomment the next cell, run it, restart the kernel, and comment the next cell once more.

In [50]:
```
# !pip3 install scikit-learn
```

In [51]:
```
# this cell will take a minute to run depending on your internet connection
X, y = fetch_openml('mnist_784', version=1, return_X_y=True) # getting data
print('X shape:', X.shape, 'y shape:', y.shape)
```

```
X shape: (70000, 784) y shape: (70000,)
```

In [52]:
```
# this cell processes some of the data

# if this returns an error of the form "KeyError: 0", then try running the f
# X = X.values # this converts X from a pandas dataframe to a numpy array

X = X.values
digits = {j:[] for j in range(10)}
for j in range(len(y)): # takes data assigns it into a dictionary
    digits[int(y[j])].append(X[j].reshape(28,28))
digits = {j:np.stack(digits[j]) for j in range(10)} # stack everything to be
for j in range(10):
    print('Shape of data with label', j, ':', digits[j].shape )
```

```
Shape of data with label 0 : (6903, 28, 28)
Shape of data with label 1 : (7877, 28, 28)
Shape of data with label 2 : (6990, 28, 28)
Shape of data with label 3 : (7141, 28, 28)
Shape of data with label 4 : (6824, 28, 28)
Shape of data with label 5 : (6313, 28, 28)
Shape of data with label 6 : (6876, 28, 28)
Shape of data with label 7 : (7293, 28, 28)
Shape of data with label 8 : (6825, 28, 28)
Shape of data with label 9 : (6958, 28, 28)
```

In [53]:
```
# this cell would stack 100 examples from each class together
# this cell also ensures that each pixel is a flot between 0 and 1 instead o
data = []

for i in range(10):
    flattened_images = digits[i][:100].reshape(100,-1)
    data.append(flattened_images)

data = np.vstack(data)
data = data.astype('float32') / 255.0
```

Data PreProcess

In [54]:
```
x_4 = digits[4][:500].reshape(500,-1)
x_9 = digits[9][:500].reshape(500,-1)

x_4_test = digits[4][500:1000].reshape(500,-1)
x_9_test = digits[9][500:1000].reshape(500,-1)

x_train = np.vstack((x_4, x_9))
x_train = x_train.astype('float32') / 255.0

x_test = np.vstack((x_4_test, x_9_test))
x_test = x_test.astype('float32') / 255.0

y_train = np.hstack((-1 * np.ones(500), np.ones(500)))
y_test = np.hstack((-1 * np.ones(500), np.ones(500)))
```

Defining $F(w)$ and $\nabla F(w)$

In [55]:
```
def F(w):
    sum = 0
    N = len(x_train)
    for i in range(N):
        sum += np.log(1 + np.exp(-y_train[i] * np.dot(w, x_train[i])))
    return sum / N

def dF(w):
    sum = 0
    N = len(x_train)
    for i in range(N):
        sum += -y_train[i] * np.exp(-y_train[i] * np.dot(w, x_train[i])) * x
    return sum / N
```

L^2 Gradient Descent

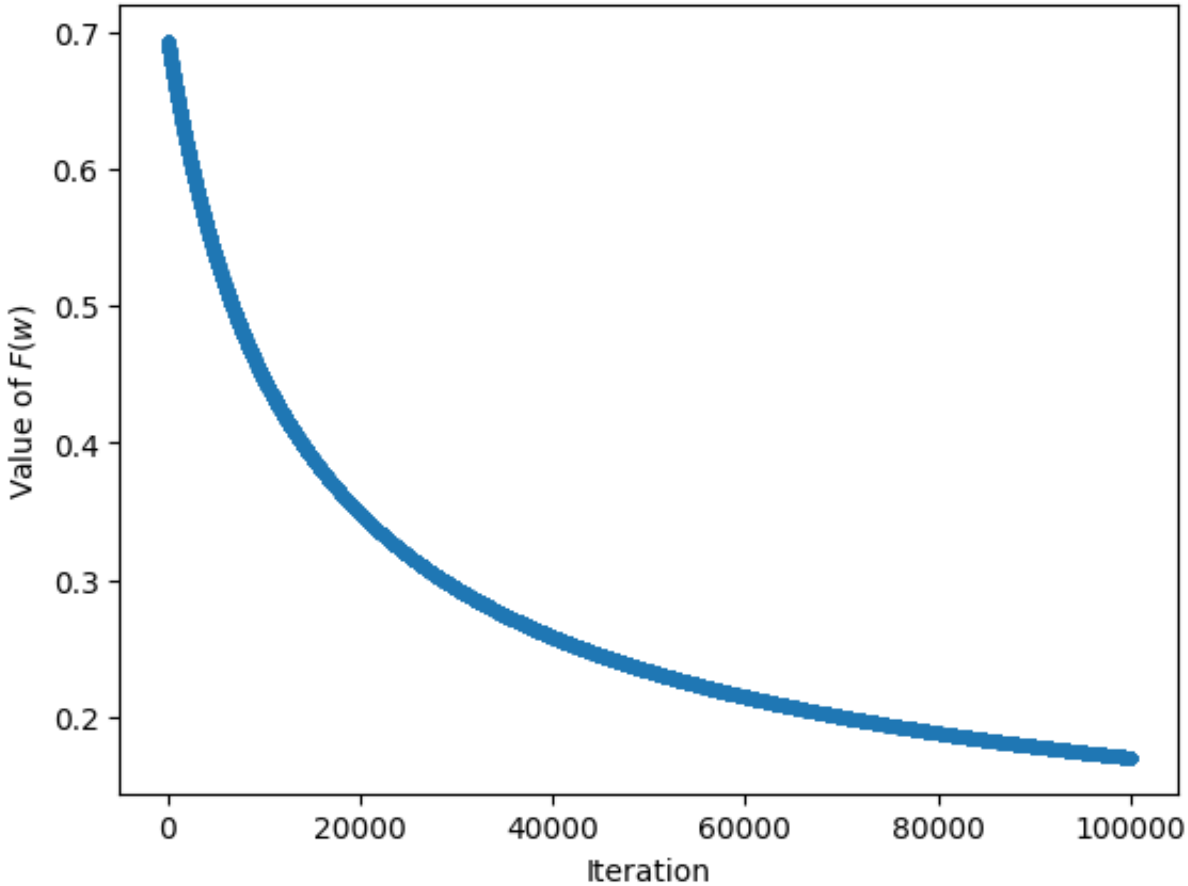
In [56]:
```
mu = 1e-4
T = 100000

w_values = []
iterations = range(T)

w = np.zeros(x_train.shape[1])
for i in iterations:
    w_values.append(F(w))
    w = w - mu * dF(w)

plt.plot(iterations, w_values, marker='.')
plt.xlabel("Iteration")
```

```
plt.ylabel(r"Value of $F(w)$")
plt.show()
```



Error Rate

In [ ]:
```
error = 0

for i in range(1000):
    if np.dot(w, x_test[i]) > 0:
        y_test[i] = 1
    else:
        y_test[i] = -1
    error += (y_test[i] != y_train[i])

print("Error rate:", error / 1000 * 100, "%")
```

```
Error rate: 4.2 %
```

Part 4A

Implement and run L^∞ gradient descent with step size $\mu = 10^{-6}$. Run your algorithm for at least 10000 iterations and initialize with $w^{(0)} = 0$ (i.e. the zero vector). **Recall:** L^∞

gradient descent uses steps

$$p^{(t)} = \textnormal{sign}(\nabla F(w^{(t)}))\|\nabla F(w^{(t)})\|_1.$$

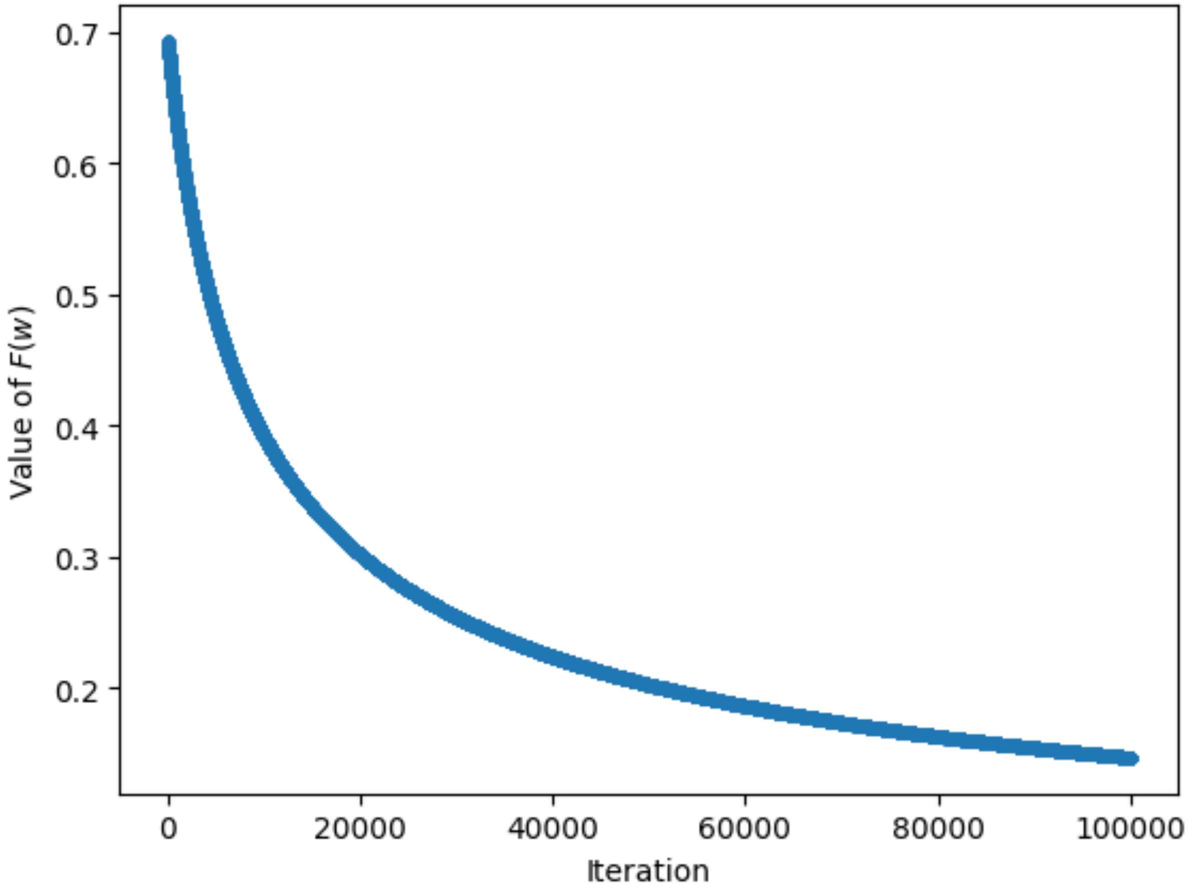
In [58]:
```
mu = 1e-6
T = 100000

w_values = []
iterations = range(T)

def p(w):
    f = dF(w)
    return np.sign(f) * np.linalg.norm(f, 1)

w = np.zeros(x_train.shape[1])
for i in iterations:
    w_values.append(F(w))
    w = w - mu * p(w)

plt.plot(iterations, w_values, marker='.')
plt.xlabel("Iteration")
plt.ylabel(r"Value of $F(w)$")
plt.show()
```



Error Rate

In [59]:
```
error = 0

for i in range(1000):
    if np.dot(w, x_test[i]) > 0:
        y_test[i] = 1
    else:
        y_test[i] = -1
    error += (y_test[i] != y_train[i])

print("Error rate:", error / 1000 * 100, "%")
```

```
Error rate: 5.7 %
```

Part 4b

Implement and run L^1 gradient descent (aka. coordinate descent) with step size $\mu = 10^{-4}$. Run your algorithm for at least 10000 iterations and initialize with $w^{(0)} = 0$ (i.e. the zero vector). **Recall:** L^1 gradient descent uses steps

$$p^{(t)} = \textnormal{sign}(\nabla_{j^*} F(w^{(t)}))\|\nabla F(w^{(t)})\|_\infty e_{j^*},$$

where j^* is the location of the largest entry of the gradient, and e_j is the zero vector with a 1 in the j^{th} entry.

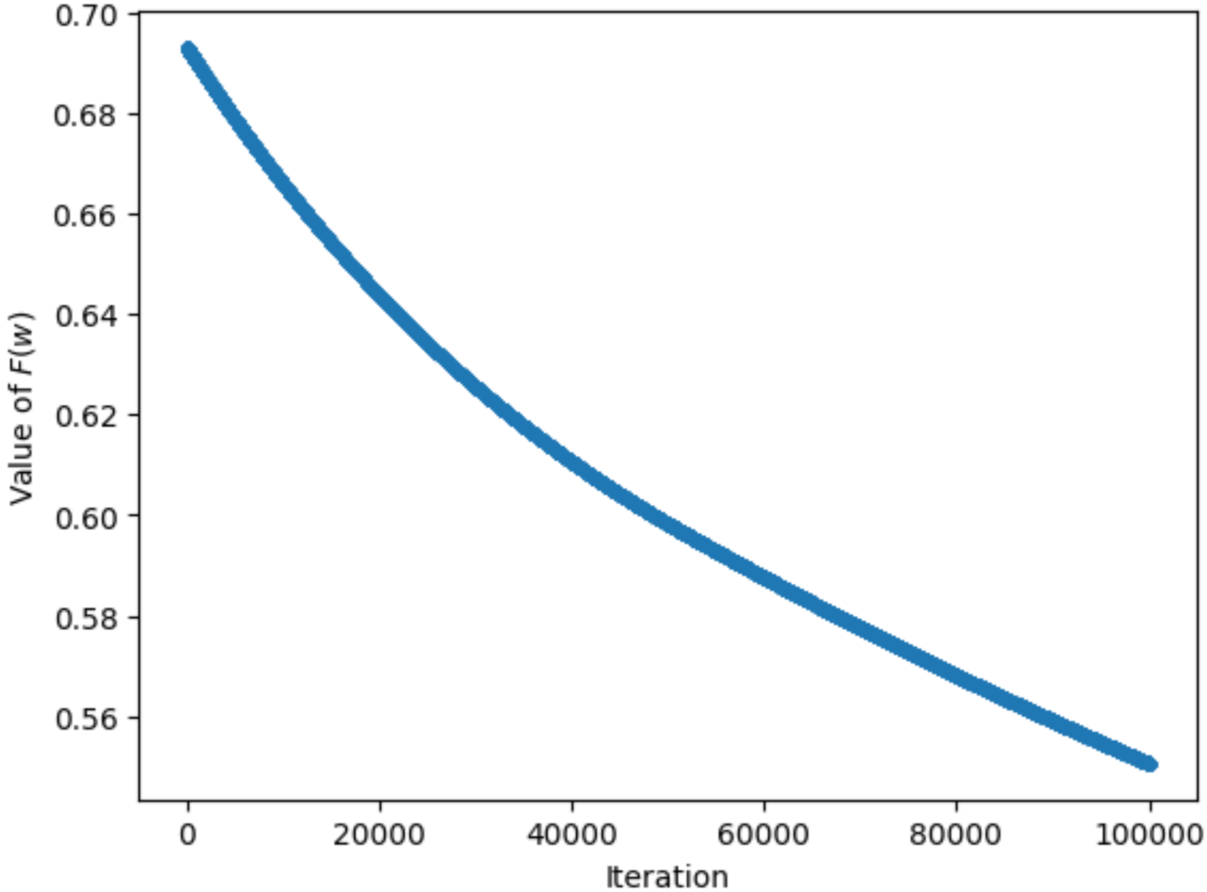
In [60]:
```
mu = 1e-4
T = 100000

w_values = []
iterations = range(T)
running_sum = np.zeros(x_train.shape[1])

def p(w):
    f = dF(w)
    j = 0
    for i in range(len(f)):
        if abs(f[i]) > abs(f[j]):
            j = i
    running_sum[j] += 1
    return f[j] * np.eye(len(f))[j]

w = np.zeros(x_train.shape[1])
for i in iterations:
    w_values.append(F(w))
    w = w - mu * p(w)

plt.plot(iterations, w_values, marker='.')
plt.xlabel("Iteration")
plt.ylabel(r"Value of $F(w)$")
plt.show()
```



Error Rate

In [61]:
```
error = 0

for i in range(1000):
    if np.dot(w, x_test[i]) > 0:
        y_test[i] = 1
    else:
        y_test[i] = -1
    error += (y_test[i] != y_train[i])

print("Error rate:", error / 1000 * 100, "%")
```

```
Error rate: 9.700000000000001 %
```

Part 4c

Compare these two descent plots of $F(w)$, along with the analogous plot for gradient descent from HW3.5. Which performs best, and do you have an argument for why? Do you think the performance would change with different step sizes?

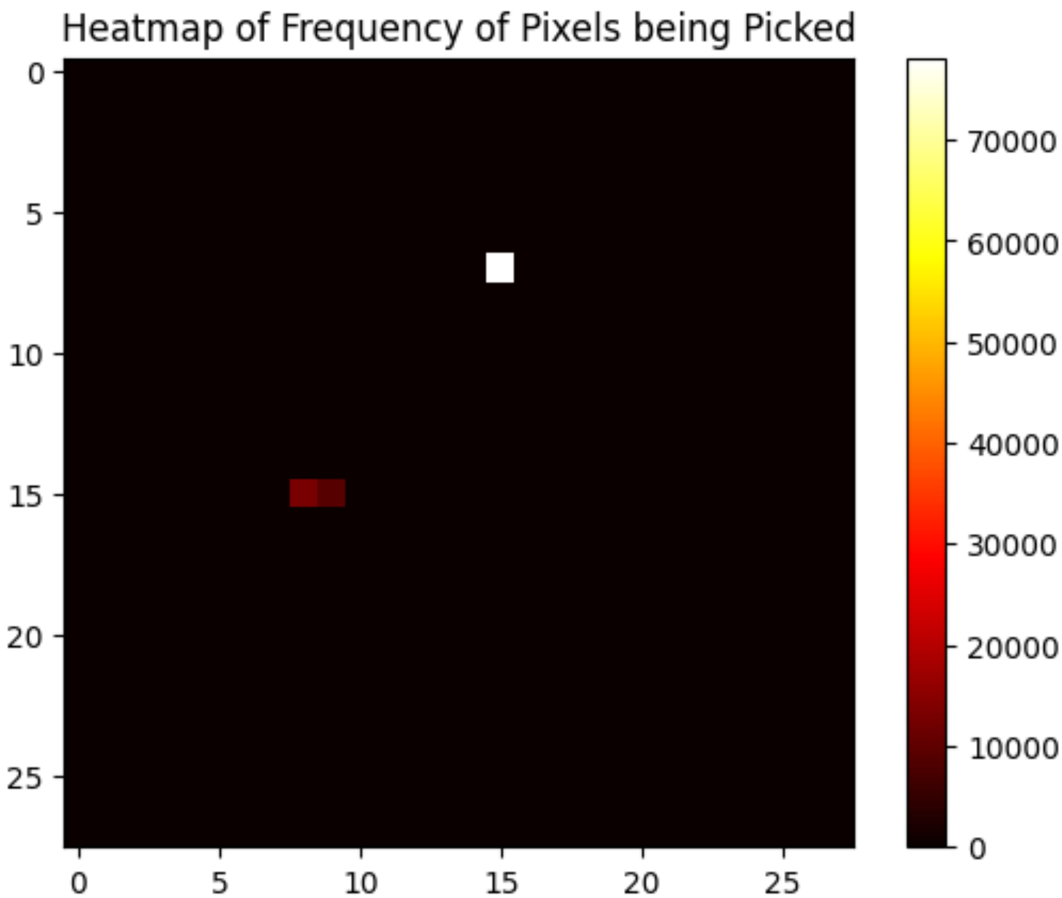
Answer

Judging from the plots, the L^2 and L^∞ gradient descent converges the quickest, and the L^1 version converges the slowest, so I would rank their performances in the order of $L^2 = L^\infty > L^1$. I believe L^1 gradient descent performs the worst because it only targets one coordinate at a time. On the other hand, L^∞ and L^2 gradient descent covers all possible descent direction, so they perform better than L^1. I think step sizes would affect the performances to an extent, but the rank of their performances would not change given the optimal step sizes.

Part 4d

For the coordinate descent problem, rerun gradient descent but store a running sum of which entry of $p^{(t)}$ is nonzero at each iteration (not the actual value of the direction vector, just e_{j^*}). This will result in a size 784 vector of mostly zeros, and should have integers at various entries whose sum equals the number of iterations. Reshape this vector to be a 28x28 image and display the result. Why do you think these are the pixels that were chosen in the gradient? How can you use this to interpret the algorithm and its results?

In [68]:
```
plt.imshow(running_sum.reshape(28, 28), cmap='hot')
plt.colorbar()
plt.title("Heatmap of Frequency of Pixels being Picked")
plt.show()
```



I think these pixels are chosen because they are the most significant features distinguishing 4's and 9's. The most chosen pixels (white and red) are positioned around the area where a 4 would have a shape angle while a 9 would be round. Hence, the L^1 gradient descent, though not being the most performant, can be used to analyze what/where the most influential features are.