MATH 220A: Homework #7

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Professor Ebenfelt

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Problem 1

Show that the closure of a totally bounded set is totally bounded.

Proof. Suppose not. Let X be a totally bounded set. Let $\epsilon > 0$. There exist finite number of points $x_1, \ldots, x_n \in X$ such that $X \subset \bigcup_{i=1}^n B_{\epsilon/2}(x_i)$. But then

$$\overline{X} \subset \overline{\bigcup_{i=1}^n B_{\epsilon/2}(x_i)} \subset \bigcup_{i=1}^n \overline{B_{\epsilon/2}(x_i)} \subset \bigcup_{i=1}^n B_{\epsilon}(x_i).$$

□

Problem 2

We say that $f : X \to \mathbb{C}$ is bounded if there is a constant $M > 0$ with $|f(x)| \le M$ for all $x \in X$. Show that if f and g are bounded uniformly continuous (Lipschitz) functions from X into $\mathbb{C}$, then so is fg.

Proof. Since there exist M, N such that $|f(x)| \le M$ and $|g(x)| \le N$ for all $x \in X$,

$$|fg(x)| = |f(x)g(x)| \le |f(x)||g(x)| \le MN$$

for all $x \in X$, and thus fg is bounded. Now, let $\epsilon > 0$. Since f and g are uniformly continuous, there exists ν such that $|f(x) - f(y)| < \epsilon/(M + N)$ and $|g(x) - g(y)| < \epsilon/(M + N)$ whenever $d(x, y) < \delta$. Then,

$$\begin{aligned} |fg(x) - fg(y)| &= |f(x)g(x) + f(x)g(y) - f(x)g(y) - f(y)g(y)| \\ &= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\ &\le |f(x)||(g(x) - g(y))| + |g(y)||(f(x) - f(y))| \\ &< \epsilon M/(M + N) + \epsilon N/(M + N) < \epsilon, \end{aligned}$$

whenever $d(x, y) < \delta$. Thus, fg is uniformly continuous.

Suppose that f and g are Lipschitz functions. Then, there exists K such that $|f(x) - f(y)|, |g(x) - g(y)| \le Kd(x, y)$ for all $x, y \in X$. Through the same calculation as above, we have

$$|fg(x) - fg(y)| \le |f(x)||(g(x) - g(y))| + |g(y)||(f(x) - f(y))| \le K(M + N)d(x, y),$$

and thus fg is Lipschitz. $\square$

Problem 3

Suppose $f : X \to \Omega$ is uniformly continuous; show that if $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Ω. Is this still true if we only assume that f is continuous? (Prove or give a counterexample.)

Proof. Let d and ρ each denote the metric on X and Ω, respectively. Pick $\epsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \epsilon$. Since $\{x_n\}$ is Cauchy, there exists N such that $d(x_n, x_m) < \delta$ whenever $n, m \geq N$. But then $\rho(f(x_n), f(x_m)) < \epsilon$ whenever $n, m \geq N$, and thus $\{f(x_n)\}$ is Cauchy.

If f is only continuous, then the statement is not necessarily true. Consider the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ and function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is Cauchy as it converges to 0. We also know that f is continuous. But then $\{f(n)\}_{n\in\mathbb{N}} \to \infty$ so it is not Cauchy. $\square$

Problem 4

Recall the definition of a dense set (1.14). Suppose that Ω is a complete metric space and that $f : (D, d) \to (\Omega, \rho)$ is uniformly continuous, where D is dense in (X, d). Use the last problem to show that there is a uniformly continuous function $g : X \to \Omega$ with $g(x) = f(x)$ for every x in D.

Proof. Let $x \in X$. Since D is dense in X, there exists a sequence $\{x_n\} \subseteq D$ such that $x_n \to x$, and so $\{x_n\}$ is Cauchy. Since f is uniformly continuous, $\{f(x_n)\}$ is also Cauchy, by the result of the previous problem. Since Ω is complete, $\{f(x_n)\}$ converges to some $y \in \Omega$. Define $g : X \to \Omega$ by $g(x) = y$. Note that $g(x) = f(x)$ for all $x \in D$, as $g(x) = \lim_{n\to\infty} f(x_n) = f(x)$.

We claim that g is uniformly continuous. Pick $\epsilon > 0$. Since f is uniformly continuous, there exists δ such that $\rho(f(x), f(y)) < \frac{\epsilon}{3}$ whenever $d(x, y) < \delta$. Suppose $x, y \in X$ with $d(x, y) < \frac{\delta}{3}$. There exist sequences $\{x_n\}, \{y_n\} \subseteq D$ with $x_n \to x$ and $y_n \to y$, and thus there exists N_1 such that $d(x_n, x), d(y_n, y) < \frac{\delta}{3}$ whenever $n \geq N_1$. Since $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) < \delta$, we have $\rho(f(x_n), f(y_n)) < \frac{\epsilon}{3}$ for all $n \geq N_1$. Since $f(x_n) \to g(x)$ and $f(y_n) \to g(y)$, there exists N_2 such that $\rho(f(x_n), g(x)), \rho(f(y_n), g(y)) < \frac{\epsilon}{3}$ whenever $n \geq N_2$. It now follows that for all $d(x, y) < \frac{\delta}{3}$, we may find $n \geq \max(N_1, N_2)$ such that

$$\rho(g(x), g(y)) \leq \rho(g(x), f(x_n)) + \rho(f(x_n), f(y_n)) + \rho(f(y_n), g(y)) < \epsilon.$$

$\square$

Problem 5

Let G be an open subset of $\mathbb{C}$ and let P be a polygon in G from a to b. Use Theorems 5.15 and 5.17 to show that there is a polygon $Q \subseteq G$ from a to b which is composed of line segments that are parallel to either the real or imaginary axes.

Proof. Since P is a polygon, $P = [z_1, z_2] \cup [z_n, z_{n+1}]$ is a union of finitely many line intervals, where $z_1 = a, z_2, \ldots, z_n, z_{n+1} = b \in G$. But then each $[z_k, z_{k+1}]$ is compact, so P is compact. By theorem 5.17, we have $d(\mathbb{C} \backslash G, P) > 0$. For each interval $[z_k, z_{k+1}]$ in P, define function $f_k : [z_k, z_{k+1}] \to \mathbb{R}$ as the Manhattan distanct from $z \in [z_k, z_{k+1}]$ to z_k on the complex plane, i.e. $f_k(z) = |Re(z) - Re(z_k)| + |Im(z) - Im(z_k)|$. We claim that f_k is continuous. Pick $\epsilon > 0$ and let $z \in [z_k, z_{k+1}]$. Let $\nu \in (0, \epsilon/\pi)$. Since every point in $[z_k, z_{k+1}]$ are on the same line, $Re(w) - Re(z_k)$ have the same sign for all $w \in [z_k, z_{k+1}]$, and thus $||Re(z) - Re(z_k)| - |Re(w) - Re(z_k)|| = |Re(z) - Re(w)|$. Hence, for all $w \in B_\nu(z) \cap [z_k, z_{k+1}]$,

$$\begin{aligned}
|f_k(z) - f_k(w)| &= |(|Re(z) - Re(z_k)| + |Im(z) - Im(z_k)|) - (|Re(w) - Re(z_k)| + |Im(w) - Im(z_k)|)| \\
&\leq |(|Re(z) - Re(z_k)| - |Re(w) - Re(z_k)|)| + |(|Im(z) - Im(z_k)| - |Im(w) - Im(z_k)|)| \\
&= |Re(z) - Re(w)| + |Im(z) - Im(w)| < \pi d(z, w) < \epsilon,
\end{aligned}$$

where the last inequality follows from the fact that the perimeter of a triangle inscribed in a circle is less than the circumference of the circle. Thus, f_k is continuous for all k. By theorem 5.15, f_k is uniformly continuous, so there exists δ such that for all $z, w \in [z_k, z_{k+1}]$ with $d(z, w) < \delta$, we have $|f_k(z) - f_k(w)| < d(\mathbb{C} \backslash G, P)$. We may now partition $[z_k, z_{k+1}]$ into finitely many intervals of length less than δ, with endpoints $z_k = w_0, w_1, \ldots, w_m = z_{k+1}$. Since $|f_k(w_i) - f_k(w_{i+1})| < d(\mathbb{C} \backslash G, P)$ for all i,

$$[Re(w_i), Re(w_{i+1})] \cup [Im(w_i), Im(w_{i+1})] \subset G.$$

The result now follows. □

Problem 6

Let $\{f_n\}$ be a sequence of uniformly continuous functions from (X, d) into (Ω, ρ) and suppose that $f = u - \lim f_n$ exists. Prove that f is uniformly continuous. If each f_n is a Lipschitz function with constant M_n and $\sup M_n < \infty$, show that f is a Lipschitz function. If $\sup M_n = \infty$, show that f may fail to be Lipschitz.

Proof. Pick $\epsilon > 0$. There exists n such that $\rho(f_n(x), f(x)) < \epsilon/3$ for all $x \in X$. Since f_n is uniformly continuous, there exists δ such that $\rho(f_n(x), f_n(y)) < \epsilon/3$ whenever $d(x, y) < \delta$. Then, whenever $d(x, y) < \delta$,

$$\rho(f(x), f(y)) \leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f_n(y), f(y)) < \epsilon,$$

and thus f is uniformly continuous.

Suppose that each f_n is Lipschitz with constant M_n and $\sup M_n < \infty$. Given $x, y \in X$, there exists n such that $\rho(f_n(z), f(z)) < d(x, y)$ for all $z \in X$. It now follows that

$$\rho(f(x), f(y)) \leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f_n(y), f(y)) < (M_n + 2)d(x, y).$$

However, this does not work in the general case. Consider $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. Given any $K \in \mathbb{R}$, pick $\epsilon \in (0, \frac{1}{K^2})$ and we have

$$|f(\epsilon) - f(0)| = \left|\sqrt{\epsilon} - 0\right| = \sqrt{\epsilon} > K\epsilon.$$

Hence, f is not Lipschitz. But then by the Weierstrass approximation theorem, there exists a sequence of polynomials $\{p_n\}$ on $[0, 1]$ such that $p_n \to f$ uniformly. Since p_n' is continuous on a compact set, $\sup_{x \in [a,b]} |p_n'(x)| < \infty$. Put $M_n = 2 \sup_{x \in [a,b]} |p_n'(x)|$ and we have

$$|p_n(x) - p_n(y)| < M_n |x - y|$$

for all $x, y \in [0, 1]$, which makes p_n Lipschitz.

$\square$