MATH 264A: Homework 2

Due on Nov 26, 2024 at 23:59pm

Professor Warnke

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Problem 1

This problem illustrates how variants of the usual switching arguments can sometimes lead to fairly simple proofs (if one is content with bounds instead of asymptotics). Let $\mathcal{N}_i$ denote the set of permutations of $[n]$ with exactly i fixed points. Picking a random permutation $\pi_n \in S_n$ uniformly at random, below we give a simple and robust proof of the non-trivial bound

$$\mathbb{P}(\pi_n \text{ has no fixed points}) \geq \frac{1}{3} - o(1) \quad \text{as } n \to \infty$$

(a) Using a basic switching argument, show that $|\mathcal{N}_1|/|\mathcal{N}_0| \leq 1 + O(1/n)$.

Proof. Define switching operation $\phi : \mathcal{N}_1 \to 2^{\mathcal{N}_0}$ by sending $\pi \in \mathcal{N}_1$ to the set of all possible permutations π' obtained by switching the fixed point k with any other element in $[n]\backslash\{k\}$.

Forward Switching:

Let $\pi \in \mathcal{N}_1$ with $\pi(k) = k$. Notice if permutation π' is defined by

$$\pi'(i) = \begin{cases} \pi(j) & \text{if } i = k \\ \pi(k) = k & \text{if } i = j \\ \pi(i) & \text{otherwise} \end{cases}$$

for some $j \neq k$, then $\pi' \in \mathcal{N}_0$. Since there are $n-1$ choices for j, we have $|\phi(\pi)| = n - 1$.

Reverse Switching:

Number of ways $\pi' \in \mathcal{N}_0$ can be obtained by switching some $\pi \in \mathcal{N}_1$ is

$$|\phi^{-1}(\pi')| = (\#\text{choices for fixed point } k)(\#\text{valid points in } \pi' \text{ to swap with } k)$$

There are n choices for k. Notice that we need to swap k with $\pi'(k)$ to possibly obtain a permutation with k as its only fixed point. If $\pi'(k) = j$ and $\pi'(j) = k$, then swapping k with j will yield a permutation with both k and j as fixed points. Hence, the number of valid points to swap with k is ≤ 1, and thus $|\phi^{-1}(\pi')| \leq n$.

Double Counting:

$$\sum_{\pi \in \mathcal{N}_1} |\phi(\pi)| = \sum_{\pi' \in \mathcal{N}_0} |\phi^{-1}(\pi')| \quad \implies \quad |\mathcal{N}_1|/|\mathcal{N}_0| \leq \frac{n-1}{n} = 1 + O(1/n).$$

□

(b) Using a simple counting argument, show that in a random permutation, the expected number of fixed points is equal to one.

Proof. Let X be the number of fixed points in a random permutation $\pi_n \in S_n$, and let X_i be the indicator for the event that $\pi_n(i) = i$. Then,

$$\mathbb{E}_{\pi_n}[X] = \sum_{i=1}^{n} \mathbb{E}_{\pi_n}[X_i] = \sum_{i=1}^{n} \mathbb{P}(i \text{ is a fixed point}) = \sum_{i=1}^{n} \frac{\#\pi \in S_n \text{ with } \pi(i) = i}{|S_n|} = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

□

(c) By combining these estimates, conclude that

$$\mathbb{P}(\pi_n \text{ has no fixed points}) = \frac{|\mathcal{N}_0|}{n!} \geq \frac{1}{3} - O(1/n).$$

Proof. Since $\sum_{i=0}^{n} \frac{|\mathcal{N}_i|}{n!} = 1$, combining (b) we have

$$2\left(1 - \frac{|\mathcal{N}_0|}{n!} - \frac{|\mathcal{N}_1|}{n!}\right) = 2\sum_{i=2}^{n} \frac{|\mathcal{N}_i|}{n!} \leq \sum_{i=2}^{n} i \cdot \frac{|\mathcal{N}_i|}{n!} = \mathbb{E}_{\pi_n}[X] - \frac{|\mathcal{N}_1|}{n!} = 1 - \frac{|\mathcal{N}_1|}{n!}.$$

Rearranging yields

$$2 \cdot \frac{|\mathcal{N}_0|}{n!} \geq 1 - \frac{|\mathcal{N}_1|}{n!}$$

It now follows $|\mathcal{N}_1| \leq (1 + O(1/n))|\mathcal{N}_0|$ from (a) that

$$2 \cdot \frac{|\mathcal{N}_0|}{n!} \geq 1 - (1 + O(1/n))\frac{|\mathcal{N}_0|}{n!} \quad \Longrightarrow \quad \frac{|\mathcal{N}_0|}{n!} \geq \frac{1}{3} - O(1/n).$$

□