MATH 140A: Homework #8

Due on Mar 8, 2024 at 23:59pm

Professor Seward

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Problem 1

Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{(1+a_n)}$ diverges.

Proof. Note that if $a_n > 1$, then $\frac{a_n}{a_n+1} = 1 - \frac{1}{a_n+1} > \frac{1}{2}$. On the other hand, if $a_n \leq 1$, we have $\frac{a_n}{a_n+1} \geq \frac{a_n}{2}$. If there are infinitely many n such that $a_n > 1$, then the series obviously diverges, as it would be greater than the sum of infinitely many $\frac{1}{2}$. Hence, we may assume there exists $N \geq 0$ such that $a_n \leq 1$ for all $n \geq N$. But then

$$\sum \frac{a_n}{(1+a_n)} \geq \sum_{n=1}^{N-1} \frac{a_n}{(1+a_n)} + \frac{1}{2} \sum_{n=N}^{\infty} a_n.$$

Since $\sum a_n$ diverges, $\frac{1}{2} \sum_{n=N}^{\infty} a_n$ diverges, by comparison test. The result now follows. □

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

Proof. We first note that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.$$

Fix $\epsilon \in (0, 1)$. Since S_n is increasing and unbounded, $\frac{s_N}{s_{N+k}} \to 0$. Hence, we may find large enough k such that $\frac{s_N}{s_{N+k}} < 1 - \epsilon$. But then $\sum_{n=N+1}^{N+k} \frac{a_n}{s_n} \geq \epsilon$, which fails to meet the Cauchy criterion. □

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

Proof. Since

$$\frac{a_n}{s_n^2} \leq \frac{a_n}{s_{n-1} s_n} = \frac{1}{s_{n-1}} - \frac{1}{s_n},$$

the consecutive terms cancel out, and we get $\sum_{n=1}^{N} \frac{a_n}{s_n^2} \leq \sum_{n=2}^{N} \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{a_1} - \frac{1}{s_N}$. But then s_n is increasing and unbounded, and thus

$$\frac{1}{a_1} \leq \lim_{N \to \infty} \sum \frac{a_n}{s_n^2} \leq \lim_{N \to \infty} \frac{1}{a_1} - \frac{1}{s_N} = \frac{1}{a_1}.$$

Hence, the series converges to $\frac{1}{a_1}$. □

Problem 2

Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

Proof. Let $A = \sum_{n=1}^{\infty} a_n$. We know $r_n = A - s_n$, where s_n is the sum of the first $n-1$ terms of a_n. Note that $r_n < r_m$, as $s_n > s_m$. Hence,

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m + \cdots + a_{n-1}}{r_m} = \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m}.$$

Let $\epsilon \in (0,1)$. Since $r_n \to 0$, for any integer N, we may find large enough $n \geq N$, such that

$$\sum_{m=N}^{n} > 1 - \frac{r_n}{r_N} > \epsilon.$$

The result now follows from the Cauchy criterion.

$\lim_{n\to\infty} \sum_{k=m}^{n} \frac{a_n}{r_n}$ □

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Proof. Since $a_n > 0$,

$$0 < \frac{a_n}{\sqrt{r_n}} = \frac{2(r_n - r_{n+1})}{2\sqrt{r_n}} < \frac{2(r_n - r_{n+1})}{\sqrt{r_n} + \sqrt{r_{n+1}}} = 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right).$$

Note that $\sum_{n=1}^{N} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2(\sqrt{r_1} - \sqrt{r_{N+1}})$. But then $r_n \to 0$, so

$$0 \leq \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}} \leq \sum_{n=1}^{\infty} 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right) = 2\sqrt{r_1}.$$

Therefore, the series converges to $2\sqrt{r_1}$, by the comparison test. □

Problem 3

Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Proof. Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series. Let $A_N = \sum_{n=1}^N |a_n|$ and $B_N = \sum_{n=1}^N |b_n|$, and $C_N = \sum_{n=1}^N |c_n| = \sum_{n=1}^N |\sum_{k=1}^n a_k b_{n-k}|$. Since $|c_n|$ is nonnegative, it suffices to show that that C_n is bounded. Hence,

$$\begin{aligned}
C_N &= \sum_{n=1}^N \left| \sum_{k=1}^n a_k b_{n-k} \right| \\
&\leq \sum_{n=1}^N \sum_{k=1}^n |a_k||b_{n-k}| \\
&= \sum_{k=1}^N |a_k| \sum_{j=1}^{N-k} |b_j| \\
&= \sum_{k=1}^N |a_k| B_{N-k} \\
&\leq \sum_{k=1}^N |a_k| B_N \\
&= A_N B_N,
\end{aligned}$$

and the result follows. $\square$

Problem 4

Associate to each sequence $a = (\alpha_n)$ in which α_n is 0 or 2, the real number

$$\chi(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $\chi(a)$ is precisely the Cantor set described in Theorem 2.44.

Proof. We continue to use the notations $E_1, E_2, \dots$ and P defined in Theorem 2.44. Given some a, we first show that $\sum_{k=1}^{n} \frac{\alpha_k}{3^k} = \inf I_n$ for some interval I_n of E_n by induction on n. Since a_1 is either 0 or 2, $\frac{\alpha_1}{3}$ is obviously the lower end point of some interval in E_1. Suppose $n > 1$. By induction, we know $\sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} = \sup I_{n-1}$ for some interval $I_{n-1} \subset E_n$. Since $I_{n-1} \cap E_n$ is a union of 2 intervals, put I_{n_1} to be the lower interval of $I_{n-1} \cap E_n$ and let I_{n_2} be the upper one. Note that $\inf I_{n_1} = \inf I_{n-1}$ and $\sup I_{n_2} = \sup I_{n-1}$. If $a_n = 0$, then $\sum_{k=1}^{n} \frac{\alpha_k}{3^k} = \inf I_{n-1} = \inf I_{n_1}$ and we are done. Suppose $a_n = 2$. Note that the width of I_{n-1} is 3^{n-1}, and the width of I_{n_2} is 3^{-n}. Since $\sup I_{n-1} = \sup I_{n_2}$, we get $\inf I_{n_2} = \sup I_{n-1} - 3^{-n} = \inf I_{n-1} + \frac{2}{3} \cdot 3^{-n}$. But then $\sum_{k=1}^{n} \frac{\alpha_k}{3^k} = \inf I_{n-1} + \frac{2}{3} \cdot 3^{-n} = \inf I_{n_2}$, and this completes the induction. Since all E_n are closed and $E_1 \supset E_2 \supset \dots$, we have $\sum_{k=1}^{n} \frac{\alpha_k}{3^k} \in E_m$, for all positive integer $m \le n$. Hence, we have $\chi(a) \in E_n$, for all n, and thus $\chi(a) \in P$.

We now show the converse. Let $x \in P$. We construct a sequence $a = (\alpha_n)$ by putting $a_n = 0$ if x is in the lower interval of $I_{n-1} \cap E_n$, where $I_{n-1} \subset E_{n-1}$ is the interval which contains x. Otherwise, if x is in the upper interval of $I_{n-1} \cap E_n$, put $a_n = 2$. From the first part, we already know $\chi(a) \in P$. We show that $\sum_{k=1}^{n} \frac{\alpha_k}{3^k}$ is in the same interval $I_n \subset E_n$ that contains x by induction on n. The base case is trivial. Suppose $n > 1$. By induction, $\sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} \in I_{n-1}$. Note that $\sum_{k=1}^{n} \frac{\alpha_k}{3^k}$ will be in either the upper or lower interval of $I_{n-1} \cap E_n$, by the first part of the proof. But then by the construction of α_n, $\sum_{k=1}^{n} \frac{\alpha_k}{3^k}$ will be in the upper interval if x is in the upper one and vice versa, and this completes the induction. It follows that $\chi(a)$ shares the same interval I_n with x, for all n. Fix $\epsilon > 0$. Since P contains no segments, $I_n \subset B_\epsilon(x)$ for large enough n, where $I_n \subset E_n$ is the interval that contains x. But then $\chi(a) \in I_n$, and thus $|\chi(a) - x| < \epsilon$. The result now follows. $\square$

Problem 5

Suppose (p_n) and (q_n) are Cauchy sequences in a metric space X. Show that the sequence $(d(p_n, q_n))$ converges. *Hint*: For any m, n,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

Proof. Fix $\epsilon > 0$. Since (p_n) and (q_n) are Cauchy sequences, there exists integer N such that $d(p_n, p_m) < \frac{\epsilon}{2}$ and $d(q_n, q_m) < \frac{\epsilon}{2}$, for $m, n \geq N$. But then

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) < d(p_m, q_m) + \epsilon.$$

Since the inequality still holds if we swap m, n, we get

$$|d(p_n, q_n) - d(p_m, q_m)| < \epsilon.$$

Hence, $(d(p_n, q_n))$ is also a Cauchy sequence. Since $(d(p_n, q_n))$ is in $\mathbb{R}$, the result now follows from Theorem 3.11. $\square$