C3.8 Analytic Number Theory: Sheet #3

Due on November 24, 2025 at 12:00pm

Professor B. Green

Ray Tsai

Problem 1

Evaluate $\zeta(0)$ and $\zeta(-1)$. (You may want to use the facts that $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$ and that $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.)

Proof. Note that $\Xi(s) = \Xi(1-s)$, where $\Xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Thus

$$\zeta(0) = \lim_{\epsilon\to 0} \frac{\pi^{-(1+\epsilon)/2}\Gamma((1+\epsilon)/2)\zeta(1+\epsilon)}{\pi^{\epsilon/2}\Gamma(-\epsilon/2)} = \pi^{-1/2}\Gamma(1/2)\cdot \lim_{\epsilon\to 0}\frac{\zeta(1+\epsilon)}{\Gamma(-\epsilon/2)} = \pi^{-1/2}\Gamma(1/2)\cdot\lim_{\epsilon\to 0} -\frac{\epsilon\zeta(1+\epsilon)}{2\Gamma(1-\epsilon/2)}.$$

But then ζ has a simple pole at $s=1$, so $\lim_{\epsilon\to 0}\epsilon\zeta(1+\epsilon) = 1$. Thus we have

$$\zeta(0) = -\frac{1}{2}\pi^{-1/2}\Gamma(1/2).$$

Since

$$\Gamma(1/2) = \int_0^\infty e^{-t}t^{-1/2}\,dt = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi},$$

we have $\zeta(0) = -\frac{1}{2}$.

Similarly,

$$\zeta(-1) = \frac{\pi^{-1}\Gamma(1)\zeta(2)}{\pi^{1/2}\Gamma(-1/2)} = \frac{\zeta(2)}{-2\pi^{3/2}\Gamma(1/2)} = \frac{1}{-2\pi^2}\cdot\frac{\pi^2}{6} = -\frac{1}{12}.$$

□

Problem 2

(i) Assume $\Re s > 0$. Calculate the Mellin transform $\tilde{W}(s)$, where $W(x) = 1$ for $0 < x < 1$ and $W(x) = 0$ for $x \geq 1$.

Proof. Note that

$$\tilde{W}(s) = \int_0^1 x^{s-1}\, dx = \frac{1}{s}.$$

$\square$

(ii) Define

$$W_*(x) := \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \tilde{W}(s) x^s\, ds,$$

where the integral is defined to be

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \tilde{W}(s) x^s\, ds$$

(that is, the 'Cauchy principal value' of the indefinite integral). By considering $x = 1$, show that W_* is not identically equal to W.

Proof. Note that $W(1) = 0$. But then

$$W_*(1) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{1}{s}\, ds.$$

Put $s = 2 + it$ for $t \in \mathbb{R}$ and we have

$$\int_{2-i\infty}^{2+i\infty} \frac{1}{s}\, ds = i \int_{-\infty}^{\infty} \frac{1}{2+it}\, dt = i \int_{-\infty}^{\infty} \frac{2}{4+t^2}\, dt + \int_{-\infty}^{\infty} \frac{t}{4+t^2}\, dt.$$

Since $\frac{t}{4+t^2}$ is an odd function, its integral is 0. But then

$$\int_{2-i\infty}^{2+i\infty} \frac{1}{s}\, ds = i \int_{-\infty}^{\infty} \frac{2}{4+t^2}\, dt = 4i \left(\lim_{t\to\infty} \frac{1}{2} \arctan\left(\frac{t}{2}\right) - \frac{1}{2}\arctan(0) \right) = i\pi.$$

But then $W_*(1) \neq 0 = W(1)$. $\square$

Problem 3

Prove directly from the Euler product that $\zeta(s) \neq 0$ for $\Re s > 1$.

Proof. Suppose $s = a + bi$ where $a > 1$. Then

$$|\zeta(s)| = \prod_p (1 - p^{-a})^{-1}.$$

But then for $\zeta(s)$ to be 0, we must have $1/p^a \to \infty$ as $p \to \infty$, which is impossible. $\square$

Problem 4

Define a function $W : \mathbb{R} \to \mathbb{R}$ by

$$W(x) = \begin{cases} \exp\left(\dfrac{1}{x^2 - 1}\right) & |x| < 1, \\ 0 & |x| \geq 1, \end{cases}$$

Show that W is smooth.

Proof. $W(x)$ is clearly smooth on $|x| > 1$. On $(-1, 1)$, $W(x)$ is a composition of smooth functions, so it is smooth. Since $W(x)$ is an even function, it suffices to show that

$$\lim_{x \to 1^-} W^{(n)}(x) = 0,$$

for all $n \in \mathbb{Z}_{\geq 0}$. By induction, we have

$$W^{(n)}(x) = \frac{P_n(x)}{(x^2 - 1)^{2n}} \cdot \exp\left(\frac{1}{x^2 - 1}\right),$$

for all $n \in \mathbb{Z}_{\geq 0}$. Put $t = 1/(1 - x^2)$ and note that $t \to \infty$ as $x \to 1^-$. Thus,

$$\lim_{x \to 1^-} W^{(n)}(x) = \lim_{t \to \infty} P_n(1) \cdot \frac{t^{2n}}{e^t} = 0.$$

This completes the proof. $\square$

Problem 5

Define functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ by setting $F_1(x) = 1$ if $|x| \leq 1$, and 0 otherwise; and $F_2(x) = 1 - |x|$ if $|x| \leq 1$, and 0 otherwise. Show that $\int |\hat{F}_1(\xi)|\, d\xi$ is infinite, but that $\int |\hat{F}_2(\xi)|\, d\xi$ is finite.

Proof. Note that

$$\hat{F}_1(\xi) = \int_{-1}^{1} e^{-i\xi x}\, dx = \frac{i}{\xi}(e^{-i\xi} - e^{i\xi}) = \frac{2\sin\xi}{\xi}.$$

Hence,

$$\int_{\infty}^{-\infty} |\hat{F}_1(\xi)|\, d\xi = \int_{-\infty}^{\infty} \frac{2|\sin\xi|}{|\xi|}\, d\xi = 4\int_0^{\infty} \frac{|\sin\xi|}{|\xi|}\, d\xi$$

For $n \in \mathbb{N}$,

$$\int_{(n-1)\pi}^{n\pi} \frac{|\sin\xi|}{|\xi|}\, d\xi \geq \frac{1}{n\pi}\int_{(n-1)\pi}^{n\pi} |\sin\xi|\, d\xi = \frac{2}{n\pi}.$$

Thus,

$$\int_{\infty}^{-\infty} |\hat{F}_1(\xi)|\, d\xi \geq \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} = \infty.$$

On the other hand,

$$\hat{F}_2(\xi) = \int_{-1}^{1} (1 - |x|)e^{-i\xi x}\, dx = \frac{2(1-\cos\xi)}{\xi^2}.$$

Thus,

$$\int_{\infty}^{-\infty} |\hat{F}_2(\xi)|\, d\xi = 4\int_0^{\infty} \frac{1-\cos\xi}{\xi^2}\, d\xi.$$

For $n \in \mathbb{N}$,

$$\int_{(n-1)\pi}^{n\pi} \frac{1-\cos\xi}{\xi^2}\, d\xi \geq \frac{1}{n^2\pi^2}\int_{(n-1)\pi}^{n\pi} 1 - \cos\xi\, d\xi = \frac{1}{n^2\pi}.$$

But then

$$\int_{\infty}^{-\infty} |\hat{F}_2(\xi)|\, d\xi = 4\sum_{n=1}^{\infty}\frac{1}{n^2\pi} = \frac{2\pi}{3} < \infty.$$

□

Problem 6

Let $\chi : \mathbb{N} \to \{-1, 0, 1\}$ be the function defined by $\chi(n) = 0$ if $2 \mid n$, $\chi(n) = 1$ if $n \equiv 1 \pmod 4$, and $\chi(n) = -1$ if $n \equiv 3 \pmod 4$.

(i) Show that χ is completely multiplicative.

Proof. Let $a, b \in \mathbb{N}$. If $2 \mid ab$, then $2 \mid a$ or $2 \mid b$, so $\chi(ab) = 0 = \chi(a)\chi(b)$. Suppose ab is odd. If $a \equiv b \pmod 4$, then $ab \equiv 1 \pmod 4$, so $\chi(ab) = 1 = \chi(a)\chi(b)$. If $a \equiv -b \equiv 1 \pmod 4$, then $ab \equiv -1 \pmod 4$, so $\chi(ab) = -1 = \chi(a)\chi(b)$. Thus $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{N}$. □

(ii) Define

$$L(s, \chi) := \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}.$$

Evaluate $\lim_{s \to 1^+} L(s, \chi)$.

Proof. Since χ is completely multiplicative,

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=4k+1} n^{-s} - \sum_{n=4k+3} n^{-s} = \sum_{n=4k+1} n^{-s} - (n+2)^{-s} = \sum_{n=4k+1} \frac{(n+2)^s - n^s}{(n^2+2n)^s}.$$

But then

$$\lim_{s \to 1^+} L(s, \chi) = \sum_{n=4k+1} \frac{(n+2) - n}{(n^2+2n)} = \sum_{n=4k+1} \frac{2}{n(n+2)} = \sum_{n=4k+1} \left(\frac{1}{n} - \frac{1}{n+2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

□

(iii) Deduce that $\lim_{s \to 1^+} \sum_p \chi(p)p^{-s}$ converges.

Proof. Note that

$$\log L(s, \chi) = \sum_p -\log(1 - \chi(p)p^{-s}).$$

By the expansion of $\log(1 - x)$, we get

$$-\log(1 - \chi(p)p^{-s}) = \chi(p)p^{-s} + \chi(p^2)p^{-2s} + \chi(p^3)p^{-3s} + \cdots = \chi(p)p^{-s} + \sum_{k=2}^{\infty} \chi(p^k)p^{-ks}.$$

Note that

$$\left|\sum_p \sum_{k=2}^{\infty} \chi(p^k)p^{-ks}\right| \le \sum_p \sum_{k=2}^{\infty} |p^{-ks}| \le \sum_{k \ge 1} \frac{1}{k^2} < \infty$$

converges for any $s > 1$. But then by (b),

$$\lim_{s \to 1^+} \log L(s, \chi) = \lim_{s \to 1^+} \sum_p \chi(p)p^{-s} + \lim_{s \to 1^+} \sum_p \sum_{k=2}^{\infty} \chi(p^k)p^{-ks} = \frac{\pi}{4} < \infty.$$

The result now follows. □

Problem 6 continued on next page. . .

(iv) Conclude that there are infinitely many primes congruent to 1 mod 4, and also infinitely many primes congruent to 3 mod 4.

Proof. Note that $\chi^{-1}(0) = \{2\}$, so there are infinitely many primes congruent to ± 1 mod 4. But then by (iii)

$$\lim_{s \to 1^+} \sum_p \chi(p) p^{-s} = \lim_{s \to 1^+} \sum_{p=4k+1} p^{-s} - \lim_{s \to 1^+} \sum_{p=4k+3} p^{-s} < \infty.$$

If either there are finitely many primes congruent to 1 mod 4 or finitely many primes congruent to 3 mod 4, then the sum above would've diverged, contradiction. □

Problem 7

Show that $\zeta(s)$ does not vanish for real s in the interval $[0, 1]$.

Proof. □