MATH 190A: Homework #2

Due on Jan 22, 2025 at 12:00pm

Professor McKernan

Section A02 8:00AM - 8:50AM
Section Leader: Zhiyuan Jiang

Source Consulted: Textbook, Lecture, Discussion

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Problem 1

Let X be a set. Give X the topology $\mathcal{T}$ where every finite set is closed, plus X. If Y is a subset of X then determine

(i) The interior of Y.

Proof. If $X \setminus Y$ is finite, then $Y \in \mathcal{T}$ and so $\text{int}(Y) = Y$. Otherwise, $X \setminus Y$ is infinite and so Y does not contain a set whose complement is finite. Thus, $\text{int}(Y) = \emptyset$. $\square$

(ii) The closure of Y.

Proof. If Y is finite, then Y is closed so $Y = \overline{Y}$. If Y is infinite, then there is no finite set that contains Y so $\overline{Y} = X$. $\square$

(iii) The boundary of Y.

Proof. By (i) and (ii),

$$\partial Y = \begin{cases} \emptyset & \text{if } Y \text{ is finite and } X \setminus Y \text{ is finite} \\ Y & \text{if } Y \text{ is finite and } X \setminus Y \text{ is infinite} \\ X \setminus Y & \text{if } Y \text{ is infinite and } X \setminus Y \text{ is finite} \\ X & \text{if } Y \text{ is infinite and } X \setminus Y \text{ is infinite} \end{cases}$$

$\square$

Problem 2

Let $a < b \in \mathbb{R}$ and let $Y = [a, b) \subset \mathbb{R}$. What is

(i) The interior of Y?

Proof. Obviously, the largest open set contained in Y is (a, b), so $\text{int}(Y) = (a, b)$. $\square$

(ii) The closure of Y?

Proof. Obviously, the smalled closed set contained in Y is $[a, b]$, so $\overline{Y} = [a, b]$. $\square$

(iii) The boundary of Y?

Proof.

$$\partial Y = \overline{Y} \backslash \text{int}(Y) = \{a, b\}.$$

$\square$

Problem 3

Let (X, d) be a metric space, let $a \in X$ be a point of X and let $r > 0$ be a positive real. Let

$$B = \overline{B_r}(a) = \{x \in X \mid d(a, x) \leq r\}.$$

Show that B is closed in X. $\overline{B_r}(a)$ is called the *closed ball of radius r centred about a.*

Proof. Let $x \in X \backslash B$. Let $\epsilon = d(a, x) - r > 0$. Then $B_\epsilon(x) \subset X \backslash B$, so $X \backslash B$ is covered by a collection of open sets. Thus, $X \backslash B$ is open and the result follows. $\square$

Problem 4

True or false? If true then give a proof and if false then give a counterexample.

(i) Let (X, d) be a metric space, let $a \in X$ be a point of X and let $r > 0$ be a positive real. Let $Y = B_r(a)$ be the *open ball of radius r centred about a.* Then the closure of Y is the closed ball of radius r centred about a.

Proof. True. By problem 3, $\overline{B}_r(a)$ is closed, so it suffices to show $\overline{B}_r(a)$ is the smallest closed set that contains $B_r(a)$. Suppose not. Let C be a closed set such that $B_r(a) \subset C \subset \overline{B}_r(a)$. Then there exists some x such that $d(a, x) = r$ and $x \notin C$. But then there does not exist $s > 0$ such that $B_s(x) \cap B_r(a) \neq \emptyset$. Thus $X \backslash C$ is not open and so C is not closed, contradiction. ☐

(ii) If $(X, \mathcal{T})$ is a topological space and $Y \subset X$ is a subset then

$$\overline{X \setminus Y} = X \setminus \operatorname{int}(Y).$$

Proof. True. Since $C = \overline{X \setminus Y}$ is the smalled closed set that contains $X \setminus Y$, we know $X \backslash C$ is the largest open set that contains Y. Thus, $X \backslash C = \operatorname{int}(Y)$ and the result follows. ☐

(iii) If $(X, \mathcal{T})$ is a topological space and $Y \subset X$ and $Z \subset X$ are two subsets then

$$\operatorname{int}(Y \cup Z) = \operatorname{int}(Y) \cup \operatorname{int}(Z).$$

Proof. False. Let $X = \mathbb{R}$ and $Y = [0, 1]$ and $Z = [1, 2]$. Then $\operatorname{int}(Y \cup Z) = (0, 2)$ but $\operatorname{int}(Y) \cup \operatorname{int}(Z) = (0, 1) \cup (1, 2) = (0, 1) \cup [1, 2] = (0, 2)$. ☐

(iv) The integers $\mathbb{Z}$ are dense in the reals $\mathbb{R}$.

Proof. False. Let $x \in \mathbb{R} \backslash \mathbb{Z}$. There exists some $n \in \mathbb{Z}$ such that $n < x < n + 1$. Let $r = \min(x - n, n - x + 1)$. Then $B_r(x) \cap \mathbb{Z} = \emptyset$, so $\mathbb{Z}$ is not dense in $\mathbb{R}$ by lemma 4.12. ☐

(v) The rationals $\mathbb{Q}$ are dense in the reals $\mathbb{R}$.

Proof. True. Let $x \in \mathbb{R}$ and $r > 0$. Fix q such that $\frac{1}{q} < r$. There exists some p such that $\frac{p-1}{q} \leq x \leq \frac{p}{q} < x + r$. Thus $\frac{p}{q} \in B_r(x) \cap \mathbb{Q}$, so $\mathbb{Q}$ is dense. ☐

Problem 5

Show that every open subset of $\mathbb{R}$ is a disjoint union of open intervals.

Proof. Let U be an open set of $\mathbb{R}$. Then U is a union of a collection of open intervals. Suppose $I_1 = (a, b)$ and $I_2 = (c, d)$ are two open intervals in U such that $I_1 \cap I_2 \neq \emptyset$. Then $I_1 \cup I_2 = (\min(a, c), \max(b, d))$ is an open interval contained in U. Thus, U is a disjoint union of open intervals. $\square$

Problem 6

Let $(X, \mathcal{T})$ be a topological space. Starting with any subset $Y \subset X$ (and any X) what is the maximum number of distinct subsets one can obtain by taking the closure and the complement (as many times as you please, in whatever order you please)?

Proof. 14? $\square$