MATH 140B: Homework #1

Due on Apr 12, 2024 at 23:59pm

Professor Seward

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Problem 1

Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing on (a, b), and let g be its inverse function. Prove that g is differentiable and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Proof. Suppose for contradiction that there exists $x, y \in (a, b)$ such that $y > x$ but $f(y) < f(x)$. Since f is differentiable in (x, y), there exists $w \in (x, y)$ such that $(y - x)f'(w) = f(y) - f(x)$, by the Mean Value Theorem. But then $f'(w) < 0$, contradiction.

Pick any arbitrary closed set $S \subset (a, b)$. By Theorem 2.41, S is compact. By Theorem 4.14, $f(S)$ is compact and thus closed in the domain of g. By Theorem 4.8, g is continuous. Let $y = f(x)$ and $s = f(t)$, such that $s \neq y$. Since g is continuous, $t \to x$ as $s \to y$. Note that $\frac{1}{f'(x)}$ exists. Hence,

$$g'(f(x)) = \lim_{s \to y} \frac{g(s) - g(y)}{s - y} = \lim_{t \to x} \frac{1}{\frac{f(t) - f(x)}{t - x}} = \frac{1}{\lim\limits_{t \to x} \frac{f(t) - f(x)}{t - x}} = \frac{1}{f'(x)},$$

and g is differentiable, as f is strictly increasing and thus injective. $\square$

Problem 2

If

$$C_0 + \frac{C_1}{2} + \ldots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \ldots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Define $f(x) = \sum_{k=0}^{n} \frac{C_k x^{k+1}}{k+1}$. f is differentiable as it is a real polynomial, and $f'(x) = C_0 + C_1 x + \ldots + C_{n-1}x^{n-1} + C_n x^n$. But then $f(0) = f(1) = 0$, and the result now follows from the mean value theorem. $\square$

Problem 3

Suppose

(a) f is continuous for $x \geq 0$,

(b) $f'(x)$ exists for $x > 0$,

(c) $f(0) = 0$,

(d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

Proof. Notice that $g(x) = \frac{f(x)-f(0)}{x-0}$. By the mean value theorem, there exists $w \in (0, x)$ such that $f'(w) = g(x)$. Since f' is monotonically increasing and $x > w$, $f'(x) \geq f'(w) = \frac{f(x)}{x}$, and so $xf'(x) - f(x) \geq 0$. But then g is differentiable and $g'(x) = \frac{xf'(x)-f(x)}{x^2} \geq 0$, by Theorem 5.3. Pick any a, b such that $b > a > 0$. By the mean value theorem, $g(b) - g(a) = (b-a)g'(p) \geq 0$, for some $p > 0$, and the result follows. □

Problem 4

Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is uniformly differentiable on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

Proof. Since f' is continuous on a compact set, f' is uniformly continuous, by Theorem 4.19. That is, there exists $\delta > 0$ such that $|f'(y) - f'(x)| < \epsilon$, for all $|y - x| < \delta$, $y, x \in [a, b]$. By the mean value theorem, for any $x, t \in [a, b]$ such that $0 < |x - t| < \delta$,

$$\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| = |f'(w) - f'(x)| < \epsilon,$$

for some w between x and t, as $|w - x| < |t - x| < \delta$.

Since this holds for all components, it also holds for vector-valued functions, by Theorem 3.4. $\square$

Problem 5

Let f be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Proof. Since f is continuous and differentiable on $\mathbb{R} \backslash 0$, for all $x \neq 0$ we have

$$\frac{f(x) - f(0)}{x} = f'(w),$$

for some w between 0 and x. But then

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} f'(w) = 3,$$

since $w \to 0$ as $x \to 0$. The result now follows. $\square$

Problem 6

Suppose f is a real, three times differentiable function on $[-1, 1]$, such that

$$f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$. Note that equality holds for $\frac{x^3+x^2}{2}$. *Hint:* Use Theorem 5.15, with $\alpha = 0$ and $\beta = \pm 1$, to show that there exists $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6. \tag{1}$$

Proof. Define function P over $[-1, 1]$ as

$$P(x) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!} \cdot x^k = \frac{f''(0)}{2} \cdot x^2.$$

By Taylor's Theorem, there exists $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f(1) = P(1) + \frac{f^{(3)}(s)}{6}, \quad \text{and} \quad f(-1) = P(-1) - \frac{f^{(3)}(t)}{6}.$$

Note that $P(1) = P(-1)$. Combining both equations, we get

$$f(1) - f(-1) = 1 = \frac{f^{(3)}(s)}{6} + \frac{f^{(3)}(t)}{6},$$

and (1) follows. Since the average of $f^{(3)}(s)$ and $f^{(3)}(t)$ is 3, one of them must be at least 3. □

Problem 7

Suppose f is differentiable on $[a,b]$, $f(a)=0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$. Prove that $f(x)=0$ for all $x \in [a,b]$.

Proof. Fix $x_0 \in [a,b]$, and let

$$M_0 = \sup |f(x)|, \quad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. To achieve this, we may pick $x_0 = a + \frac{1}{2A}$. Hence, it remains to show that $f = 0$ on $[x_0, b]$. Again, we may pick $x_1 = x_0 + \frac{1}{2A} = a + 2 \cdot \frac{1}{2A}$. Then, $A(x_1 - x_0) = \frac{1}{2} < 1$, and thus it remains to show that $f = 0$ on $[x_1, b]$, and so on. Let natural number $n > 2A(b-a)$. Since $x_n = a + n \cdot \frac{1}{2A} > b$, the above process would reach b and terminate after n steps, which concludes that $f(x) = 0$ for all $x \in [a,b]$. $\square$