Math 158 HW1

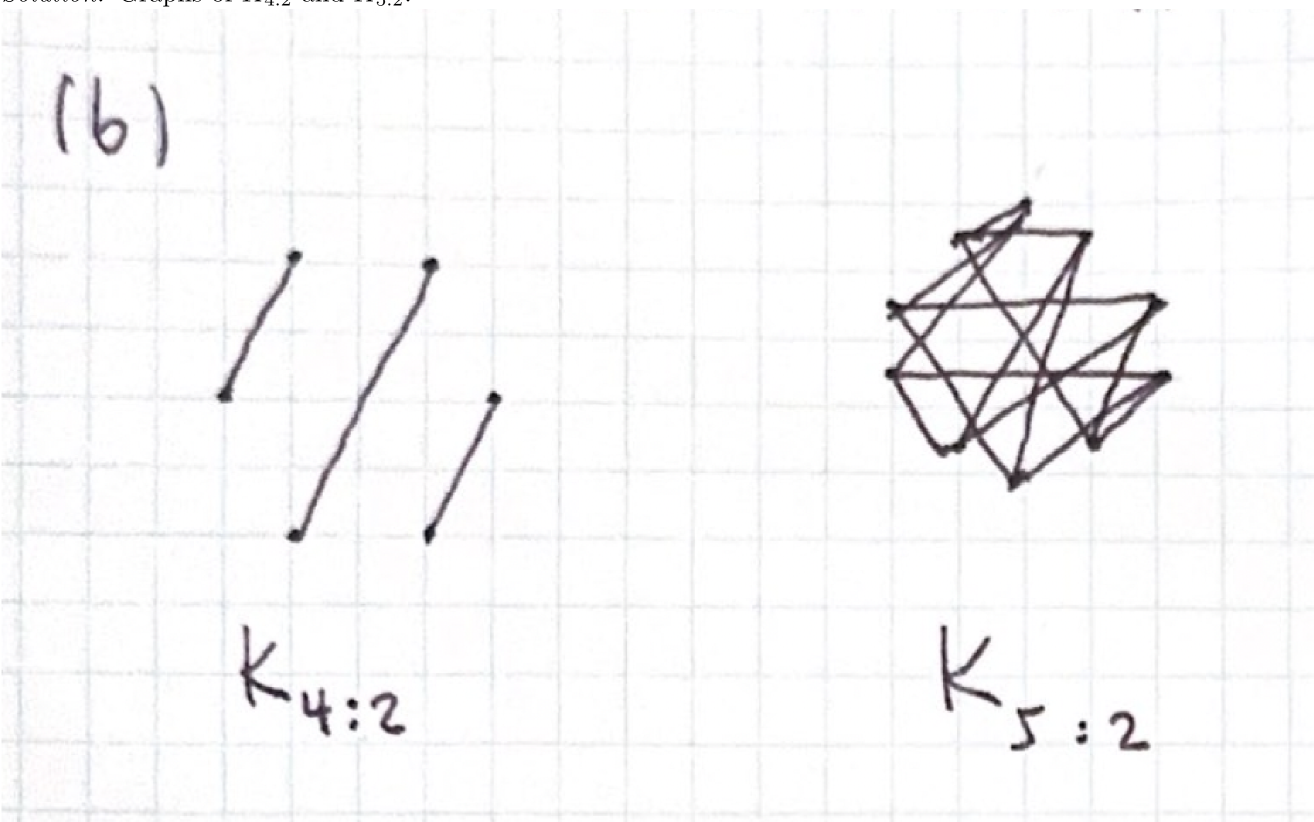
Question 1. Let $K_{n:r}$ denote the Kneser graph, whose vertex set is the set of r-element subsets of an n-element set, and where two vertices form an edge if the corresponding sets are disjoint.

(a) Describe $K_{n:1}$ for $n \geq 1$.

Solution. Since $\forall v, u \in V(K_{n:1})$, $v \cap u = \emptyset$. Thus, $\forall v, u \in V(K_{n:1})$, $\{v, u\} \in E(K_{n:1})$, which makes $K_{n:1}$ a K_n complete graph. $\square$

(b) Draw $K_{4:2}$ and $K_{5:2}$.

Solution. Graphs of $K_{4:2}$ and $K_{5:2}$:



$\square$

(c) Determine $|E(K_{n:r})|$ for $n \geq 2r \geq 1$.

Solution. For each $v \in V(K_{n:r}$, v forms edges with other vertices whose vertex set is r of the other $n-r$ elements that are not in the vertex set of v, which implies that $d_{K_{n:r}}(v) = \binom{n-r}{r}$. Since there are $\binom{n}{r}$ vertices in $K_{n:r}$, by the Handshake Theorem, we have $|E(K_{n:r})| = \binom{n}{r}\binom{n-r}{r}/2$. $\square$

Question 2. Let G be a digraph such that every vertex has a positive in-degree. Prove that G contains a directed cycle.

Proof. We will prove this by contradiction. Let $v \in V(G)$. Suppose for the sake of contradiction that G does not contain any directed cycle. Starting from v, we can find a path P by tracing back to a vertex with an edge directed to the current vertex we're on. We then add the vertex to P and go to that vertex, and we repeat the previous actions. Since every vertex in G has a positive in-degree, we can always find another vertex that has a directed edge to the current vertex we're on and not in P. However, this makes G have infinitely many vertices, which is a contradiction. Therefore, G contains a directed cycle. $\square$

Question 3. Let g be an n-vertex graph with $n \geq 2$ and $\delta(G) \geq (n-1)/2$. Prove that G is connected and the diameter of G is at most two.

Proof. We will first prove that G is connected by contradiction. Suppose for the sake of contradiction that G is disconnected. Let $n = |V(G)|$, $v \in V(G)$, H be the component of G that contains v. Since $d_G(v) \geq \delta(G) \geq (n-1)/2$, we have $|V(H)| \geq (n-1)/2 + 1 = (n+1)/2$, which implies that other components in G contain at most $n - (n+1)/2 = (n-1)/2$ vertices. However, this shows that $\Delta(G - V(H)) \leq (n-1)/2 - 1 < (n-1)/2$, which contradicts $\delta(G) \geq (n-1)/2$ because H is disconnected to $G - V(H)$. Therefore, G is connected.
We will now prove that the diameter of G is at most two. Let $u, w \in V(G)$. If $u \in N(w)$, then $d_G(u,w) = 1$. If $u \notin N(w)$, then $N(u), N(w) \in V(G) \backslash \{u, w\}$. Since $|N(u)|, |N(w)| \geq \delta(G) \geq (n-1)/2$, we have $|N(u)| + |N(w)| > n - 2 = |V(G) \backslash \{u, w\}|$. Hence, $|N(u)| \cap |N(w)| \neq \emptyset$, which means that $d_G(u,w) = 2$. Therefore, the diameter of G is at most two. $\square$

Question 4. Let P and Q be the longest paths in a connected graph G. Prove that

$$V(P) \cap V(Q) \neq \emptyset.$$

Proof. We will prove this by contradiction. Let P, Q be the longest paths in a connected graph G, with $\{p_1, p_2, \dots, p_{n+1}\}$ and $\{q_1, q_2, \dots, q_{n+1}\}$ as their vertex sets respectively, and $n = |E(P)| = |E(Q)|$. Suppose for the sake of contradiction that $V(P) \cap V(Q) = \emptyset$. Since G is connected, there must be a path R that starts from p_i and ends at q_j, for some $1 \leq i, j \leq n+1$. Let $m = d_G(p_i, q_j)$. Since $p_i \neq q_j$, we have $m \geq 1$. Let P' be the longer path between $p_1p_2 \dots p_i$ and $p_ip_{i+1} \dots p_{n+1}$, Q' be the longer path between $q_1q_2 \dots q_j$ and $q_jq_{j+1} \dots q_{n+1}$. By connecting P', Q', and R, we get a new path S. Since $|E(P')|, |E(Q')| \geq n/2$, $|E(R)| = m \geq 1$, we have $|E(S)| \geq n+1$, which contradicts that P, Q are the longest paths on G. Therefore, if P, Q are the longest paths in a connected graph, then $V(P) \cap V(Q) \neq \emptyset$. $\square$

Question 5. Prove that a graph G of minimum degree at least $k \geq 2$ containing no triangles contains a cycle of length at least $2k$.

Proof. Let P be the longest path in G, say $v_1 v_2 \dots v_t$. Then $N(v_1) \subseteq V(P)$ or else we get a longer path. Since G does not contain any triangles, if $v_p, v_q \in N(v_1)$ for some $p > q$, then $p - q \geq 2$. Since $|N(v_1)| \geq \delta(G) \geq k$ and $d_P(v_p, v_q) \geq 2$ for all $v_p, v_q \in N(v_1)$, $t \geq 2k$ and v_1 has a neighbor v_i for some $i \geq 2k$. Then, the cycle $v_1 v_2 \dots v_i v_1$ has length at least $2k$. $\square$