MATH 188: Homework #3

Due on May 3, 2024 at 23:59pm

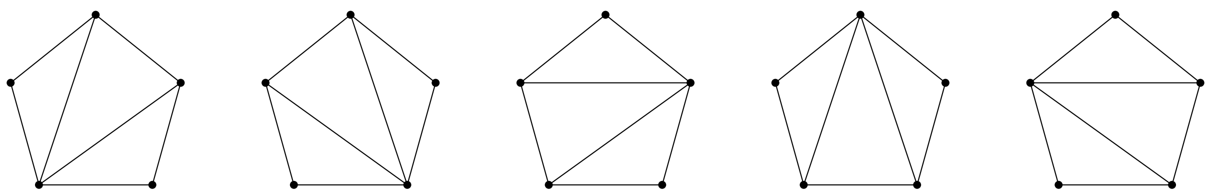
Professor Kunnawalkam Elayavalli

Ray Tsai

A16848188

Problem 1

Let n be a positive integer. Show that the number of ways of triangulating (i.e., drawing diagonals between vertices that do not intersect except at vertices so that the regions are all triangles) a convex polygon with $(n+2)$ vertices is the nth Catalan number C_n. By convention, the "2-gon" and triangle both have exactly one triangulation and here are the 5 triangulations of a pentagon:



Proof. We proceed by induction on n. There is only $C_1 = 1$ way to triangulate a triangle, so the base case is done. Suppose $n > 1$. Index the vertices in counter-clockwise order from 0 to $n+1$, say $v_0, v_1, \ldots, v_{n+1}$. We focus on v_0. The the two clockwise most edges incident to v_0 are $\{v_0, v_1\}$ and $\{v_0, v_k\}$, for some $2 \leq k \leq n+1$. Since there are no edges between $\{v_0, v_1\}$ and $\{v_0, v_k\}$, $v_0 v_1 v_k$ form a triangle. Removing triangle $v_0 v_1 v_k$, we get an k-gon $v_1 v_2 \ldots v_k$ and an $(n-k+3)$-gon $v_k v_{k+1} \ldots v_{n+1} v_0$. By induction, there are $C_{k-2} C_{n-k+1}$ ways to triangulate these two polygons, and thus there are $C_{k-2} C_{n-k+1}$ triangulations of the $(n+2)$-gon which contains the triangle $v_0 v_1 v_k$. Therefore, the total number of triangulations of an $(n+2)$-gon is

$$\sum_{k=2}^{n+1} C_{k-2} C_{n-k+1} = \sum_{i=0}^{n-1} C_i C_{n-i-1} = C_n.$$

$\square$

Problem 2

Consider the following variation of counting balanced parentheses. We have a new symbol $*$. Let a_n be the number of length n strings consisting of left/right parentheses and $*$ such that the result of deleting all of the $*$'s is a balanced set of parentheses ($a_0 = 1$). Let $A(x) = \sum_{n \geq 0} a_n x^n$. Find polynomials $a(x)$, $b(x)$, $c(x)$ in x, not all identically 0, such that

$$a(x)A(x)^2 + b(x)A(x) + c(x) = 0.$$

Proof. Let $P(n)$ be the set of length n strings consisting of parentheses and $*$ such that the result of deleting all of the $*$'s is a balanced set of parentheses. For $n \geq 2$, notice that the end of any string $w \in P(n)$ must either be $*$ or), so $P(n) = P(n-1) \sqcup P_{)}(n)$, where $P_{)}(n)$ is the set of set of $w \in P(n)$ which ends with).

I claim that $|P_{)}(n)| = \sum_{k=0}^{n-2} a_k a_{n-k-2}$. Let $w \in P_{)}(n)$. w ends with). Consider the (that pairs with it. To the left of them is a string in $P(k)$ and in between the two of them is another string in $P(n-k-2)$, where $0 \leq k \leq n-2$. These strings can be chosen independently, so there are $a_k a_{n-k-2}$ ways for this to happen. Since the cases with different k don't overlap, we sum over all possibilities to get

$$|P_{)}(n)| = \sum_{k=0}^{n-2} |P(k)| \cdot |P(n-k-2)| = \sum_{k=0}^{n-2} a_k a_{n-k-2}.,$$

and thus for $n \geq 2$,

$$a_n = a_{n-1} + \sum_{k=0}^{n-2} a_k a_{n-k-2}.$$

Note that $a_0 = a_1 = 1$. It now follows that

$$\begin{aligned}
A(x) &= \sum_{n \geq 0} a_n x^n \\
&= a_0 + a_1 x + \sum_{n \geq 2} a_{n-1} x^n + \sum_{n \geq 2} \left(\sum_{k=0}^{n-2} a_k a_{n-k-2} \right) x^n \\
&= 1 + x + x \sum_{n \geq 1} a_n x^n + x^2 \sum_{n \geq 0} \left(\sum_{k=0}^{n} a_k a_{n-k-2} \right) x^n \\
&= 1 + x + x(A(x) - 1) + x^2 A^2(x).
\end{aligned}$$

Rearranged, we get

$$x^2 A^2(x) + (x-1)A(x) + 1 = 0,$$

and the result now follows. □

Problem 3

Let n be a positive integer. Consider the equation

$$x_1 + x_2 + \ldots + x_8 = 2n.$$

For each of the following conditions, how many solutions are there? Give as simple of a formula as possible.

(a) The x_i are non-negative even integers.

Proof. Let

$$C_{even} = \{(x_1, \ldots, x_8) \mid x_1 + \cdots + x_8 = 2n,\ x_i = 2k_i \text{ for some } k_i \in \mathbb{Z}_{\geq 0}\},$$
$$C_n = \{(y_1, \ldots, y_8) \mid y_1 + \cdots + y_8 = n,\ x_i \in \mathbb{Z}_{\geq 0}\}.$$

We show that $C_n \simeq C_{even}$. Define $f : C_{even} \to C_n$ which sends $(x_1, \ldots, x_8)$ to $(k_1, \ldots, k_8)$ and $g : C_n \to C_{even}$ which sends $(y_1, \ldots, y_8)$ to $(2y_1, \ldots, 2y_8)$. Both f and g are obvisouly well-defined. Since

$$g(f(x_1, \ldots, x_8)) = g(k_1, \ldots, k_8) = (2k_1, \ldots, 2k_8) = (x_1, \ldots, x_8),$$

$$f(g(y_1, \ldots, y_8)) = f(2y_1, \ldots, 2y_8) = (y_1, \ldots, y_8),$$

f is a bijection, and thus $C_n \simeq C_{even}$. But then we know there are $\binom{n+7}{7}$ weak compositions of n with 8 parts, and the result now follows. □

(b) The x_i are positive odd integers.

Proof. Note that

$$\begin{aligned}
\frac{x^8}{(1-x^2)^8} &= \left(x \sum_{a_1 \geq 0} x^{2a_1} \right) \cdots \left(x \sum_{a_8 \geq 0} x^{2a_8} \right) \\
&= \left(\sum_{a_1 \geq 0} x^{2a_1+1} \right) \cdots \left(\sum_{a_9 \geq 0} x^{2a_9+1} \right) \\
&= \sum_{\substack{(k_1, \ldots, k_8) \in \mathbb{Z}^8_{\geq 1}, \\ k_i \text{ odd}}} x^{k_1 + \cdots + k_8},
\end{aligned}$$

so the number of solutions where all x_i's are positive odd integers are

$$[x^{2n}] \frac{x^8}{(1-x^2)^8} = [x^{2n-8}] \frac{1}{(1-x^2)^8} = [x^{n-4}] \frac{1}{(1-x)^8} = \binom{n+3}{7}.$$

□

(c) The x_i are non-negative integers and $x_8 \leq 9$.

Proof. Suppose $x_8 = k$, for some $0 \leq k \leq 9$. Then, there are $\binom{2n-k+6}{6}$ solutions, as there are $\binom{2n-k+6}{6}$ solutions to $x_1 + \cdots + x_7 = 2n - k$. Hence, in total, there are $\sum_{k=0}^{9} \binom{2n-k+6}{6}$ solutions. □

Problem 4

Let k, n be positive integers such that $k \geq n$.

(a) Show that

$$\sum_{(a_1,\ldots,a_n)} a_1a_2\cdots a_n = \binom{n+k-1}{k-n},$$

where the sum is over all compositions of k into n parts.

Proof. Note that

$$\begin{aligned}
\frac{x^n}{(1-x)^{2n}} &= xD\left(\sum_{a_1\geq 0} x^{a_1}\right)\cdots xD\left(\sum_{a_1\geq 0} x^{a_1}\right) \\
&= \left(x\sum_{a_1\geq 1} a_1x^{a_1-1}\right)\cdots\left(x\sum_{a_n\geq 1} a_nx^{a_n-1}\right) \\
&= \left(\sum_{a_1\geq 1} a_1x^{a_1}\right)\cdots\left(\sum_{a_n\geq 1} a_nx^{a_n}\right) \\
&= \sum_{(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 1}} a_1a_2\cdots a_nx^{a_1+\cdots+a_n}.
\end{aligned}$$

Hence,

$$\sum_{\substack{(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 1}\\ a_1+\cdots+a_n=k}} a_1a_2\cdots a_n = [x^k]\frac{x^n}{(1-x)^{2n}} = [x^{k-n}]\frac{1}{(1-x)^{2n}} = \binom{n+k-1}{k-n}.$$

□

(b) Show that

$$\sum_{(a_1,\ldots,a_n)} 2^{a_2-1}3^{a_3-1}\cdots n^{a_n-1} = S(k,n),$$

where the sum is over all compositions of k into n parts.

Proof. Note that

$$\begin{aligned}
F_n(x) &= \left(\frac{x}{1-x}\right)\left(\frac{x}{1-2x}\right)\cdots\left(\frac{x}{1-nx}\right) \\
&= \left(x\sum_{a_1\geq 0} x^{a_1}\right)\left(x\sum_{a_2\geq 0}(2x)^{a_2}\right)\cdots\left(x\sum_{a_n\geq 0}(nx)^{a_n}\right) \\
&= \left(x\sum_{a_1\geq 1} x^{a_1-1}\right)\left(x\sum_{a_2\geq 1}(2x)^{a_2-1}\right)\cdots\left(x\sum_{a_n\geq 1}(nx)^{a_n-1}\right) \\
&= \sum_{(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 1}} 2^{a_2-1}\cdots n^{a_n-1}x^{a_1+\cdots+a_n}.
\end{aligned}$$

Hence,

$$\sum_{\substack{(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 1}\\ a_1+\cdots+a_n=k}} 2^{a_2-1}3^{a_3-1}\cdots n^{a_n-1} = [x^k]F_n(x) = S(k,n).$$

□

Problem 5

(a) Give a closed formula for the number of pairs of subsets S, T of $[n]$ such that $S \subset T$ (i.e., $S \subseteq T$ and $S \neq T$).

Proof. There are $\binom{n}{k}$ ways to pick a subset of size k, and each subset of size k has $2^k - 1$ strict subsets. Hence, the total number of S, T pairs is

$$\sum_{k=0}^{n} \binom{n}{k}(2^k - 1) = \sum_{k=0}^{n} \binom{n}{k} 2^k - \sum_{k=0}^{n} \binom{n}{k} = (1+2)^n - (1+1)^n = 3^n - 2^n,$$

by the binomial theorem. □

(b) Give a closed formula for the number of k-tuples of subsets $(S_1, \ldots, S_k)$ of $[n]$ such that $\bigcup_{i=1}^k S_i = [n]$.

Proof. Let a_n be the number of k-tuples of subsets $(S_1, \ldots, S_k)$ of $[n]$ such that $\bigcup_{i=1}^k S_i = [n]$. Put $a_0 = 1$. We show that $a_n = (2^k - 1)^n$ by induction on n. Given $(S_1, \ldots, S_k)$ such that $\bigcup_{i=1}^k S_i = [n-1]$, we have to add n to at least one of the S_i's to ensure $\bigcup_{i=1}^k S_i = [n]$. Since for each such k-tuple there are $2^k - 1$ ways to do so, we get

$$a_n = (2^k - 1)a_{n-1} = (2^k - 1)^n,$$

by induction. □

Problem 6

Give a closed formula for the number of k-tuples of subsets $(S_1, \ldots, S_k)$ of $[n]$ such that $S_i \subseteq S_{i+1}$ for $i = 1, \ldots, k-1$.

Proof. Notice that the first appearance of any $j \in [n]$ in the tuple determines j's existence in any S_i, as all subsequent sets in the tuple would also contain j. Since each $j \in [n]$ can either first appear in one of the k sets or never appear, there are $k+1$ possible distributions of j in a k-tuple, for each j in $[n]$. Since there are n elements in total, there are $(k+1)^n$ k-tuples of subsets $(S_1, \ldots, S_k)$ of $[n]$ such that $S_i \subseteq S_{i+1}$. $\square$

Problem 7

What is the total number of parts of all compositions of k?

Proof. The possible number of parts of a composition of k is anywhere between $n = 1$ to $n = k$, so the total number of parts of all compositions is

$$\begin{aligned}\sum_{n=1}^{k} \binom{k-1}{n-1} n &= \sum_{n=0}^{k-1} \binom{k-1}{n} (n+1) \\ &= \sum_{n=1}^{k-1} \binom{k-1}{n} n + \sum_{n=0}^{k-1} \binom{k-1}{n}.\end{aligned}$$

Note that

$$(k-1)(x+1)^{k-2} = D(x+1)^{k-1} = \sum_{n=1}^{k-1} \binom{k-1}{n} n x^{n-1}.$$

Hence,

$$\sum_{n=1}^{k} \binom{k-1}{n-1} n = (k-1)(1+1)^{k-2} + (1+1)^{k-1} = (k+1)2^{k-2}.$$

□

Problem 8

Fix an integer $k \geq 2$. Call a composition $(a_1, \ldots, a_n)$ of k doubly even if the number of a_i which are even is also even (i.e., there could be no even a_i, or 2 of them, or 4, etc.). Show that the number of doubly even compositions of k is 2^{k-2}.

Proof. Let E be the set of doubly even compositions of k, and C be the set of compositions of $k-1$. We show that $E \simeq C$. Define $f : E \to C$ as

$$f(a_1, \ldots, a_n) = \begin{cases} (a_1, \ldots, a_n - 1), & \text{if } a_n > 1 \\ (a_1, \ldots, a_{n-1}), & \text{if } a_n = 1 \end{cases}.$$

On the other hand, define $g : C \to E$ as

$$g(a_1, \ldots, a_n) = \begin{cases} (a_1, \ldots, a_n, 1), & \text{if } (a_1, \ldots, a_n) \text{ is doubly even} \\ (a_1, \ldots, a_n + 1), & \text{otherwise} \end{cases}.$$

Note that f is obviously well defined. Let $(a_1, \ldots, a_n) \in C$. If $(a_1, \ldots, a_n)$ is doubly even, then $(a_1, \ldots, a_n, 1)$ is also doubly even. If $(a_1, \ldots, a_n)$ is not doubly even, then $(a_1, \ldots, a_n + 1)$ is doubly even, as incrementing a_n by 1 either increase or decrease the amount of even numbers in the tuple by 1. Hence, g is also well-defined. Since

$$g(f(a_1, \ldots, a_n)) = \begin{cases} (a_1, \ldots, a_{n-1}, 1), & \text{if } a_n = 1 \\ (a_1, \ldots, (a_n - 1) + 1), & \text{if } a_n > 1 \end{cases} = (a_1, \ldots, a_n),$$

$$f(g(a_1, \ldots, a_n)) = \begin{cases} (a_1, \ldots, a_n), & \text{if } (a_1, \ldots, a_n) \text{ is doubly even} \\ (a_1, \ldots, (a_n + 1) - 1), & \text{otherwise} \end{cases} = (a_1, \ldots, a_n),$$

f and g are inverses of each other, and thus $E \simeq C$. Hence, the number of doubly even compositions of k is equal to the number of compositions of $k-1$, which is 2^{k-2}. □

Problem 9

Let $F(n)$ be the number of set partitions of $[n]$ such that every block has size ≥ 2. Prove that

$$B(n) = F(n) + F(n+1),$$

where $B(n)$ is the nth Bell number.

Proof. Let P be the set of all partitions of $[n]$, A_k be the set partitions of $[k]$ such that every block has size ≥ 2, and let S be the set of partition of $[n]$ which contains at least a singleton. It is obvious that $P = A_n \sqcup S$ and $|A_n| = F(n)$. It remains to show that $|S| = F(n+1)$.

Define $f : S \to A_{n+1}$ which puts all singletons of a partition into the same block as $n+1$. On the other hand, define $g : A_{n+1} \to S$ which breaks the block containing $n+1$ into singletons and removes $n+1$.

Let $p, p' \in S$, say $p = p' = \{b_1, \dots, b_l, \{s_1\}, \dots, \{s_k\}\}$, where $|b_i| \geq 2$. Then,

$$f(p) = f(p') = \{b_1, \dots, b_l, \{s_1, \dots, s_k, n+1\}\} \in A_{n+1},$$

so f is well-defined.

Now suppose $q, q' \in A_{n+1}$, say $q = q' = \{b_1, \dots, b_l, \{s_1, \dots, s_k, n+1\}\}$. Note that each block in q, q' has size at least 2. Then,

$$g(q) = g(q') = \{b_1, \dots, b_l, \{s_1\}, \dots, \{s_k\}\},$$

which contains at least one singleton, and thus g is well-defined.

Since

$$g(f(p)) = g(\{b_1, \dots, b_l, \{s_1, \dots, s_k, n+1\}\}) = \{b_1, \dots, b_l, \{s_1\}, \dots, \{s_k\}\} = p,$$

$$f(g(q)) = f(\{b_1, \dots, b_l, \{s_1\}, \dots, \{s_k\}\}) = \{b_1, \dots, b_l, \{s_1, \dots, s_k, n+1\}\} = q,$$

f and g are inverses of each other, and so $S \simeq A_{n+1}$.

But then $|S| = |A_{n+1}| = F(n+1)$, and the result follows. $\square$