# MATH 220A: Homework #7

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# Problem 1

Let $I(r) = \int_\gamma \frac{e^{iz}}{z}\,dz$ where $\gamma : [0,\pi] \to \mathbb{C}$ is defined by $\gamma(t) = re^{it}$. Show that $\lim_{r\to\infty} I(r) = 0$.

*Proof.* Note that $\gamma'(t) = ire^{it}$ and so

$$|I(r)| = \left|\int_0^\pi \frac{e^{ire^{it}}}{re^{it}} \cdot ire^{it}\,dt\right| = \left|i\int_0^\pi e^{ire^{it}}\,dt\right| \le \int_0^\pi \left|e^{ire^{it}}\right|\,dt = \int_0^\pi \left|e^{r(i\cos(t)-\sin(t))}\right|\,dt = \int_0^\pi e^{-r\sin(t)}\,dt.$$

Pick $\epsilon > 0$. There exists integer $N > -\log(\epsilon)$ such that for all $r > N$ and $t \in [0,\pi]$,

$$\left|e^{-r\sin(t)}\right| \le e^{-r} < e^{-N} < \epsilon.$$

Hence, $e^{-r\sin(t)}$ uniformly converges to 0 on $[0,\pi]$, and thus

$$\lim_{r\to\infty} \int_0^\pi e^{-r\sin(t)}\,dt = 0.$$

The result now follows. $\square$

# Problem 2

Let $\gamma(t) = 2e^{it}$ for $-\pi \leq t \leq \pi$ and find $\int_\gamma (z^2-1)^{-1}\, dz$.

*Proof.* Note that $\gamma'(t) = 2ie^{it}$. We then have

$$\int_\gamma (z^2-1)^{-1}\, dz = \int_{-\pi}^{\pi} \frac{2ie^{it}}{(2e^{it})^2 - 1}\, dt = \int_{-\pi}^{\pi} \frac{2ie^{it}}{4e^{2it}-1}\, dt.$$

Let $u = 2e^{it} + 1$, then $du = 2ie^{it}\, dt$ and so

$$\int_{-\pi}^{\pi} \frac{2ie^{it}}{4e^{2it}-1}\, dt = \int_1^1 \frac{1}{u(u-2)}\, dt = 0.$$

$\square$

# Problem 3

Show that if $F_1$ and $F_2$ are primitives for $f : G \to \mathbb{C}$ and $G$ is connected, then there is a constant $c$ such that $F_1(z) = c + F_2(z)$ for each $z$ in $G$.

*Proof.* Suppose $F_1' = F_2' = f$. Then

$$\frac{d}{dz}(F_1(z) - F_2(z)) = F_1'(z) - F_2'(z) = 0,$$

so the function $F_1(z) - F_2(z)$ is constant, and the result now follows. □

# Problem 4

Show that the function defined by (2.2) is continuous.

*Proof.* Pick $\epsilon > 0$. Since $\varphi$ is continuous in a compact set, $\varphi$ is uniformly continuous. Thus, there exists $\delta > 0$ such that for all $s \in [a, b]$, $|\varphi(s,t) - \varphi(s,x)| < \frac{\epsilon}{b-a}$ for all $x, t \in [c, d]$ and $|x - t| < \delta$. It now follows that for all $s \in [a, b]$ and $|t - x| < \delta$,

$$|g(t) - g(x)| = \left| \int_a^b \varphi(s,t) - \varphi(s,x)\, ds \right| \leq \int_a^b |\varphi(s,t) - \varphi(s,x)|\, ds < \frac{\epsilon}{b-a} \cdot (b-a) < \epsilon.$$

$\square$

# Problem 5

Prove the following analogue of Leibniz's rule (this exercise will be frequently used in the later sections.) Let $G$ be an open set and let $\gamma$ be a rectifiable curve in $G$. Suppose that $\varphi : \{\gamma\} \times G \to \mathbb{C}$ is a continuous function and define $g : G \to \mathbb{C}$ by

$$g(z) = \int_\gamma \varphi(w, z)\, dw$$

then $g$ is continuous. If $\frac{\partial \varphi}{\partial z}$ exists for each $(w, z)$ in $\{\gamma\} \times G$ and is continuous, then $g$ is analytic and

$$g'(z) = \int_\gamma \frac{\partial \varphi}{\partial z}(w, z)\, dw. \tag{1}$$

*Proof.* Fix $z_0 \in G$. Pick $\epsilon > 0$. Note that $\gamma : [a, b] \to G$, for some interval $[a, b]$. We first show that $g$ is continuous. Put $L = \int_\gamma |dw|$. Since $\gamma$ is continuous on a compact set, its image $\{\gamma\}$ is compact. For $r > 0$ such that the closed ball $\overline{B_r(z_0)} \subset G$, $\varphi$ is uniformly continuous on $\{\gamma\} \times \overline{B_r(z_0)}$. Thus, there exists $\delta_r > 0$ such that $|\varphi(s, z) - \varphi(s, w)| < \frac{\epsilon}{L}$ for all $s \in \{\gamma\}$ and $z, w \in \overline{B_r(z_0)}$ with $d(z, w) < \delta_r$. It now follows that for all $s \in \{\gamma\}$ and $z \in \overline{B_r(z_0)}$ with $d(z, z_0) < \delta_r$,

$$|g(z) - g(z_0)| = \left| \int_\gamma \varphi(s, z) - \varphi(s, z_0)\, ds \right| \le \int_\gamma |\varphi(s, z) - \varphi(s, z_0)|\, |ds| < \frac{\epsilon}{L} \cdot L = \epsilon.$$

Now suppose that $\varphi' = \frac{\partial \varphi}{\partial z}$ exists for each $(w, z)$ in $\{\gamma\}$ and is continuous. It suffices to verify (1), as the continuity of $g'$ follows from (1) and the first part of the proof. Since $\varphi'$ is uniformly continuous on $\{\gamma\} \times \overline{B_r(z_0)}$, there exists $\delta'_r > 0$ such that $|\varphi'(s, w) - \varphi'(s, z)| < \epsilon/L$ for all $s \in \{\gamma\}$ and $w, z \in \overline{B_r(z_0)}$ with $d(w, z) < \delta'_r$. This implies that for all for $s \in \{\gamma\}$ and $d(z, z_0) < \delta'_r$,

$$\left| \int_{z_0}^{z} [\varphi'(s, w) - \varphi'(s, z_0)]\, dw \right| \le \frac{\epsilon(z - z_0)}{L}. \tag{2}$$

Given a fixed $s \in \{\gamma\}$, $\Phi(z) = \varphi(s, z) - z\varphi'(s, z_0)$ is a primitive of $\varphi'(s, z) - \varphi'(s, z_0)$. It now follows from (2) and the funamental theorem of calculus that

$$|\varphi(s, z) - \varphi(s, z_0) - (z - z_0)\varphi'(s, z_0)| \le \frac{\epsilon(z - z_0)}{L}.$$

By the definition of $g$, we have

$$\left| \frac{g(z) - g(z_0)}{z - z_0} - \int_\gamma \varphi'(s, z_0)\, ds \right| \le \int_\gamma \left| \frac{\varphi(s, z) - \varphi(s, z_0)}{z - z_0} - \varphi'(s, z_0) \right| |ds| < \frac{\epsilon}{L} \cdot L = \epsilon,$$

for $d(z, z_0) < \delta'_r$. $\square$

# Problem 6

Suppose that $\gamma$ is a rectifiable curve in $\mathbb{C}$ and $\varphi$ is defined and continuous on $\{\gamma\}$. Use Exercise 2 to show that

$$g(z) = \int_\gamma \frac{\varphi(w)}{w-z}\,dw$$

is analytic on $\mathbb{C} - \{\gamma\}$ and

$$g^{(n)}(z) = n! \int_\gamma \frac{\varphi(w)}{(w-z)^{n+1}}\,dw. \tag{3}$$

*Proof.* Define $\phi(w,z) = \frac{\varphi(w)}{w-z}$ for $w \in \{\gamma\}$ and $z \in \mathbb{C} - \gamma$. Note that $\phi$ is continuous on $\{\gamma\} \times (\mathbb{C} - \gamma)$, as $\varphi$ and $\frac{1}{w-z}$ are continuous. Since $\frac{\partial \phi}{\partial z} = \frac{\varphi(w)}{(w-z)^2}$ exists and is continuous, $g$ is analytic on $\mathbb{C} - \gamma$ and $g'(z) = \int_\gamma \frac{\varphi(w)}{(w-z)^2}\,dw$, by the previous exercise. We now proceed by induction on $n$ to show (3). The base case is done. Suppose $n > 1$. By induction,

$$g^{(n)}(z) = \frac{\partial}{\partial z}\left[(n-1)! \int_\gamma \frac{\varphi(w)}{(w-z)^n}\,dw\right].$$

Since $\frac{\partial}{\partial z}\frac{\varphi(w)}{(w-z)^n} = \frac{n\varphi(w)}{(w-z)^{n+1}}$ exists and is continuous,

$$g^{(n)}(z) = (n-1)! \int_\gamma \frac{\partial}{\partial z}\frac{\varphi(w)}{(w-z)^n}\,dw = n! \int_\gamma \frac{\varphi(w)}{(w-z)^{n+1}}\,dw.$$

$\square$