C8.3 Combinatorics: Sheet #2

Due on November12, 2025 at 12:00pm

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Problem 1

What are the 50th, 51st and 52nd elements of $\mathbb{N}^{(3)}$ in the colex order? What about in lex?

Proof. Notice that are $\binom{k-1}{2}$ elements in $\mathbb{N}^{(3)}$ with $k+2$ as the largest number. Thus, there are

$$\sum_{k=1}^{n} \binom{k-1}{2}$$

elements $x \in \mathbb{N}^{(3)}$ such that $x \leq_{colex} (k, k+1, k+2)$. Note that

$$\sum_{k=1}^{6} \binom{k-1}{2} = 35, \quad \sum_{k=1}^{7} \binom{k-1}{2} = 56.$$

Thus the 50th, 51st and 52nd elements have 8 as their largest number. Counting down from $(6, 7, 8)$ now gives us the 50th, 51st and 52nd elements in colex order as $(5, 6, 8)$, $(1, 7, 8)$ and $(2, 7, 8)$. $\square$

Problem 2

Let $\mathcal{F} \subset [10]^{(3)}$, and suppose $|\mathcal{F}| = 29$.

(a) What is the minimum possible size of $\partial\mathcal{F}$?

Proof. By Kruskal-Katona Theorem, the family $\mathcal{F} \subset [10]^{(3)}$ with the minimum possible shadow is the family consisting of the first 29 elements in colex order, which is the family

$$[7]^{(3)} \backslash \{765, 764, 763, 762, 761, 754\} = [6]^{(3)} \cup \{ab7 : ab \in [4]^{(2)}\} \cup \{753, 752, 751\}.$$

The shadow of this family is

$$[6]^{(2)} \cup \{a7 : a \in [5]\},$$

which has size 20. □

(b) Find a family that achieves this minimum.

Proof. See part (a). □

Problem 3

Suppose that $\mathcal{F} \subset [n]^{(r)}$, and let $\mathcal{A}$ denote the first $|\mathcal{F}|$ elements of $[n]^{(r)}$ in colex order. If $|\partial\mathcal{F}| = |\partial\mathcal{A}|$ must we have $\mathcal{F} = \mathcal{A}$ (possibly after relabelling elements)?

Proof. No. Consider the case $r = 2$, where $\mathcal{F} = \{13, 23, 14, 24\}$. Then $\mathcal{A} = \{13, 23, 14, 24\}$ and $|\partial\mathcal{F}| = |\partial\mathcal{A}| = 4$. But $\mathcal{F} \neq \mathcal{A}$ for any relabelling of elements. $\square$

Problem 4

The *upper shadow* $\partial^+(\mathcal{F})$ of a set $\mathcal{F} \subset [n]^{(r)}$ is the set

$$\partial^+(\mathcal{F}) := \{A \in [n]^{(r+1)} : A \supset B \text{ for some } B \in \mathcal{F}\}.$$

Give a version of the Kruskal-Katona Theorem for the upper shadow.

Proof. Let $\mathcal{F} \subseteq [n]^{(r)}$ and let $\mathcal{A}$ be the family consisting of the last $|\mathcal{F}|$ elements of $[n]^{(r)}$ in colex order. We will show that $|\partial^+\mathcal{F}| \geq |\partial^+\mathcal{A}|$.

For any family of subsets $\mathcal{S}$ of $[n]$, let $\mathcal{S}^C := \{[n] \backslash S : S \in \mathcal{S}\}$, and note that $|\mathcal{S}| = |\mathcal{S}^C|$. Since $\partial\mathcal{F}^C \subseteq [n]^{(n-r-1)}$ consists of all the $(n-r-1)$-element subsets of $[n]$ that are disjoint from $\mathcal{F}$, its complement are all the $(r+1)$-element subsets that contains $\mathcal{F}$. In other words, $\partial^+\mathcal{F} = (\partial\mathcal{F}^C)^C$. Let $\mathcal{A}$ be the family consisting of the last $|\mathcal{F}|$ elements of $[n]^{(r)}$ in colex order. Then $\mathcal{A}^C$ is the first $|\mathcal{F}|$ elements of $[n]^{(n-r)}$. It now follows from Kruskal-Katona Theorem that

$$|\partial^+\mathcal{F}| = |(\partial\mathcal{F}^C)^C| = |\partial\mathcal{F}^C| \geq |\partial\mathcal{A}^C| = |(\partial^+\mathcal{A}^C)^C| = |\partial^+\mathcal{A}|.$$

$\square$

Problem 5

Give a proof of Hall's Theorem using Dilworth's Theorem.

Proof. Let G be a bipartite graph with parts A and B such that $|A| \leq |B|$. If there is a complete matching in G from A to B, then $\Gamma(S) \geq |S|$ for all $S \subseteq A$.

Now suppose that G satisfies the Hall's Condition. Define poset $(P, \leq)$, where $P = V(G)$ and $x \leq y$ if $x \in A$, $y \in B$, and $\{x, y\} \in E(G)$. Note that each chain has lengh at most 2. Suppose that there is an antichain $\mathcal{X}$ of size $|B| + 1$. Then $A \cap \mathcal{X}$ is nonempty. By the Hall's Condition,

$$|B \cap \mathcal{X}| + \Gamma(A \cap \mathcal{X}) \geq |B \cap \mathcal{X}| + |A \cap \mathcal{X}| = |\mathcal{X}| = |B| + 1.$$

But then $\Gamma(A \cap \mathcal{X}) \subseteq B$, so

$$(B \cap \mathcal{X}) \cap \Gamma(A \cap \mathcal{X}) \neq \emptyset.$$

This implies that there is an edge in $\mathcal{X}$, contradiction. Hence, B is the maximum antichain in P. Dilworth's Theorem now furnishes a set $\mathcal{C}$ of $|B|$ chains that cover P, and note that each $b \in B$ is in exactly one chain in $\mathcal{C}$. Since $\mathcal{C}$ covers A, there is a chain $C_a = \{a, b\} \in \mathcal{C}$ for some $b \in B$. Then $\mathcal{M} = \bigcup_{a \in A} C_a$ is a set of disjoint edges that saturates A. $\square$

Problem 6

Prove that in any sequence of $n^2 + 1$ real numbers there is an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $n + 1$.

Proof. Let $x_1, x_2, \ldots, x_{n^2+1}$ be the sequence of distinct real numbers. Define a poset $(P, \leq)$ by letting P be the set of numbers in the sequence and $x_i < x_j$ in the poset if $x_i < x_j$ and $i < j$. Suppose that there is no decreasing subsequence of length $n+1$. Then the maximum antichain in P has size at most n. By Dilworth's Theorem, n chains are needed to cover P. But then there must be a chain of length $n+1$, otherwise n chains would not be enough to cover $n^2 + 1$ elements. The result now follows. $\square$

Problem 7

We say that $\mathcal{A} \subset \mathcal{P}[n]$ is a *downset* if, for every $A \in \mathcal{A}$, every subset of A belongs to $\mathcal{A}$. Prove that if $\mathcal{A}$ is a downset then the average size of sets in $\mathcal{A}$ is at most $n/2$.

Proof. By proposition 10, $\mathcal{P}[n]$ may be partitioned into symmetric chains, which also gives a partition of $\mathcal{A}$ into chains. Let $\mathcal{C}$ be a symmetric chain that contains some element $A \in \mathcal{A}$. Since $\mathcal{A}$ is a downset, every element below A in $\mathcal{C}$ is also in $\mathcal{A}$. Thus the average weight of $\mathcal{C}$ is does not increase when restricted to $\mathcal{A}$. But then the average weight of a symmetric chain is $n/2$. $\square$

Problem 8

Prove that every intersecting family $\mathcal{F} \subset \mathcal{P}[n]$ is contained in an intersecting family of size 2^{n-1}.

Proof. We proceed by induction on n. The base case is trivial. Suppose $n \geq 2$. Let $\mathcal{F}_{<n} = \mathcal{F} \cap P[n-1]$ and $\mathcal{F}_n = \mathcal{F} \setminus P[n-1]$. By induction, $\mathcal{F}_{<n}$ is contained in an intersecting family $\mathcal{S} \subset P[n-1]$ of size 2^{n-2}. Note that any element of $P[n-1] \setminus \mathcal{S}$ is disjoint from some element of $\mathcal{S}$. Consider the family $\mathcal{S}' = \{A \cup \{n\} : A \in \mathcal{S}\}$. By defnition $\mathcal{S}'$ intersects with any element of $\mathcal{S}$. Notice that we also have $\mathcal{F}_n \subseteq \mathcal{S}'$, otherwise there exists $A \cup \{n\} \in \mathcal{F}_n$ such that $A \notin \mathcal{S}$, which implies that $A \cup \{n\}$ is disjoint from some element of $\mathcal{S}$, contradiction. It now follows that $\mathcal{S} \cup \mathcal{S}'$ is a intersecting family of size 2^{n-1} that contains $\mathcal{F}$. $\square$