MATH 220A: Homework #9

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Professor Ebenfelt

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Problem 1

Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic and $a \in G$ such that

$$|f(a)| \leq |f(z)|$$

for all $z \in G$. Show that either $f(a) = 0$ or f is constant.

Proof. Suppose $f(a) \neq 0$. Consider $g(z) = \frac{1}{f(z)}$. Since g is analytic on G and $g(a) \geq g(z)$ for all $z \in G$, by the maximum modulus principle, g is constant, which also makes f a constant. $\square$

Problem 2

Let G be a region and let f and g be analytic functions on G such that

$$f(z)g(z) = 0$$

for all $z \in G$. Show that either $f \equiv 0$ or $g \equiv 0$.

Proof. If f is constant, then etiher $f \equiv 0$ or $g \equiv 0$ and we are done. Suppose f, g are not constants. There exists a $a \in G$ such that $f(a) = 0$. By Corollary 3.10, there is an $R > 0$ such that $B(a, R) \subset G$ and $f(z) \neq 0$ for all $z \in B(a, R) \backslash \{a\}$. That is, $g(z) = 0$ for all $z \in B(a, R) \backslash \{a\}$. But then the set $\{z \in G : g(z) = 0\}$ has a limit point at a, which implies that $g \equiv 0$ by theorem 3.7, contradiction. $\square$

Problem 3

Show that if γ and σ are closed rectifiable curves having the same initial points, then

(a) $n(\gamma; a) = -n(-\gamma; a)$ for every $a \notin \{\gamma\}$.

Proof. By proposition 1.17,

$$n(-\gamma; a) = \frac{1}{2\pi i}\int_{-\gamma}\frac{dz}{z-a} = -\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} = -n(\gamma; a).$$

□

(b) $n(\gamma + \sigma; a) = n(\gamma; a) + n(\sigma; a)$ for every $a \notin \{\gamma\} \cup \{\sigma\}$.

Proof.

$$\begin{aligned} n(\gamma + \sigma; a) &= \frac{1}{2\pi i}\int_{\gamma+\sigma}\frac{dz}{z-a} \\ &= \frac{1}{2\pi i}\int_0^1 \frac{((\gamma+\sigma)(t))'}{(\gamma+\sigma)(t)-a}\,dt \\ &= \frac{1}{2\pi i}\left(\int_0^{\frac{1}{2}}\frac{(\gamma(2t))'}{\gamma(2t)-a}\,dt + \int_{\frac{1}{2}}^1 \frac{(\sigma(2t-1))'}{\sigma(2t-1)-a}\,dt\right) \end{aligned}$$

Since $\gamma(2t)$ on $[0, 1/2]$ is equivalent to $\gamma(t)$ on $[0, 1]$, and $\sigma(2t-1)$ on $[1/2, 1]$ is equivalent to $\sigma(t)$ on $[0, 1]$,

$$\int_0^{\frac{1}{2}}\frac{(\gamma(2t))'}{\gamma(2t)-a}\,dt = \int_\gamma \frac{dz}{z-a}, \quad \int_{\frac{1}{2}}^1 \frac{(\sigma(2t-1))'}{\sigma(2t-1)-a}\,dt = \int_\sigma \frac{dz}{z-a}.$$

□

The result now follows.

Problem 4

Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p never vanishes in $\{z : |z| \geq R\}$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, show that

$$\int_\gamma \frac{p'(z)}{p(z)}\,dz = 2\pi i n.$$

Proof. Define $q(t) = p(\gamma(t))$ for $t \in [0,1]$. Then,

$$\int_\gamma \frac{p'(z)}{p(z)}\,dz = \int_0^1 \frac{p'(\gamma(t))\gamma'(t)}{p(\gamma(t))}\,dt = \int_0^1 \frac{q'(t)}{q(t)}\,dt.$$

Define

$$g(s) = \int_0^s \frac{q'(t)}{q(t)}\,dt.$$

Note that $g(0) = 0$, $g(1) = \int_\gamma \frac{p'(z)}{p(z)}\,dz$, and $g'(s) = \frac{q'(s)}{q(s)}$ for $s \in [0,1]$. But this gives

$$\frac{d}{dt}e^{-g(t)}q(t) = e^{-g(t)}q'(t) - e^{-g(t)}g'(t)q(t) = e^{-g(t)}\left(q'(t) - \frac{q'(t)}{q(t)} \cdot q(t)\right) = 0,$$

so $e^{-g(t)}q(t) = e^{-g(1)}q(1) = q(0) = p(R)$ for all t. Since $\gamma(0) = \gamma(1)$, we have $q(0) = q(1)$ and thus $e^{-g(1)} = 1$. It now follows that $g(1) = 2\pi i k$ for some $k \in \mathbb{Z}$. $\square$

Problem 5

Suppose $f : G \to \mathbb{C}$ is analytic and define $\varphi : G \times G \to \mathbb{C}$ by

$$\varphi(z,w) = \frac{f(z) - f(w)}{z - w} \quad \text{if } z \neq w \quad \text{and} \quad \varphi(z,z) = f'(z).$$

Prove that φ is continuous and for each fixed w, $z \mapsto \varphi(z,w)$ is analytic.

Proof. Fix $z_0, w_0 \in G$. Suppose $z_0 \neq w_0$. Since f is continuous, and $z - w$ is nonzero and continuous, *varphi* is continuous. Suppose $z_0 = w_0$. Pick $\epsilon > 0$. Since f is analytic, there exists a $\delta_1 > 0$ such that $|w - z_0| < \delta_1$ implies

$$|f'(w) - f'(z_0)| < \epsilon/2.$$

Since f' exists, there exists $\delta_2 > 0$ such that for all $z, w \in G$,

$$\left| \frac{f(z) - f(w)}{z - w} - f'(w) \right| < \epsilon/2 \tag{1}$$

whenever $|z - w| < \delta_2$. Put $\delta \in (0, \min(\delta_1, \delta_2/2))$. Note that for all $z, w \in B_\delta(z_0)$, $|z - w| < 2\delta \leq \delta_2$. Hence,

$$\left| \frac{f(z) - f(w)}{z - w} - f'(z) \right| \leq \left| \frac{f(z) - f(w)}{z - w} - f'(w) \right| + |f'(w) - f'(z_0)| < \epsilon$$

for all $z, w \in B_\delta(z_0)$, $z \neq w$. Thus, φ is continuous at (z_0, z_0).

We now show that for each fixed w, $z \mapsto \varphi(z,w)$ is analytic. If $z \neq w$, $\varphi'(z,w) = \frac{f'(z)(z-w) - f(z) + f(w)}{(z-w)^2}$ is continuous. Suppose $z = w$. We need to show that

$$\lim_{x \to z} \frac{\varphi(x,w) - \varphi(z,w)}{x - z} = \lim_{x \to z} \frac{\frac{f(x)-f(z)}{x-z} - f'(z)}{x - z}$$

exists and is continuous. Consider the power series expansion of f at z.

$$f(x) = f(z) + f'(z)(x - z) + \frac{f''(z)}{2!}(x - z)^2 + \cdots$$

But then

$$\begin{aligned} \lim_{x \to z} \frac{\frac{f(x)-f(z)}{x-z} - f'(z)}{x - z} &= \lim_{x \to z} \frac{[f'(z) + \frac{f''(z)}{2!}(x - z) + \cdots] - f'(z)}{x - z} && (2) \\ &= \lim_{x \to z} \frac{f''(z)}{2!} + \frac{f^{(3)}(z)}{3!}(x - z) + \cdots = \frac{f''(z)}{2!}. && (3) \end{aligned}$$

The result now follows from the analyticity of f. $\square$

Problem 6

Give the details of the proof of Theorem 5.6: Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If $\gamma_1, \ldots, \gamma_m$ are closed rectifiable curves in G such that

$$n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0 \quad \text{for all } w \in \mathbb{C} - G,$$

then for $a \in G - \{\gamma\}$

$$f(a) \sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z-a}\, dz.$$

Proof. We continue from the textbook. By assumption $H \cup G = \mathbb{C}$. Since $n(\gamma_1; w) + \cdots + n(\gamma_m; w)$ is integer valued and continuous, H is open. Define

$$g(z) = \sum_{k=1}^{m} \int_{\gamma_i} \varphi(z, w)\, dw,$$

if $z \in G$ and

$$g(z) = \sum_{k=1}^{m} \int_{\gamma_i} \frac{f(w)}{w-z}\, dw,$$

if $z \in H$. By the proof of Theorem 5.4, $\int_{\gamma_i} \frac{dw}{w-z}$ is well defined for all i and $z \in G \cap H$, and so g is well defined for $z \in G \cap H$. Following the same argument as in the proof of Theorem 5.4, g is entire as it is a finite sum of entire functions. By Theorem 4.4, H contains a neighborhood of ∞ in $\mathbb{C}_\infty$. Since f is bounded on $\{\gamma\}$ and $\lim_{z\to\infty}(w-z)^{-1} = 0$ uniformly for $w \in \{\gamma\}$,

$$\lim_{z\to\infty} g(z) = \sum_{k=1}^{m} \int_{\gamma_i} \lim_{z\to\infty} \frac{f(w)}{w-z}\, dw = 0.$$

Hence there exists $R > 0$ such that $|g(z) \leq 1|$ for all $|z| \geq R$. Since g is bounded on $\overline{B}_R(0)$ it follows that g is a bounded entire function and hence constant by Liouville's Theorem. But then $\lim_{z\to\infty} g(z) = 0$ so $g \equiv 0$. Hence, for $a \in G\backslash\{\gamma\}$,

$$0 = \sum_{k=1}^{m} \int_{\gamma_i} \frac{f(w) - f(a)}{w-a}\, dw = \sum_{k=1}^{m} \int_{\gamma_i} \frac{f(w)}{w-a}\, dw - f(a) \sum_{k=1}^{m} \int_{\gamma_i} \frac{dw}{w-a},$$

and the result now follows from the definition of $n(\gamma_i; a)$. $\square$