

# MATH 140B: Homework #8

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## Problem 1

If  $0 < x < \frac{\pi}{2}$ , prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

*Proof.* Consider the function  $f(x) = x - \sin x$ . Since  $\cos x < 1 = (x)'$  in  $(0, \pi/2)$ ,  $f'(x) = x - \cos x > 0$  in  $(0, \pi/2)$ , so  $f$  is strictly increasing in  $(0, \pi/2)$ . But then  $f(x) > f(0) = 0$  for all  $x \in (0, \pi/2)$ . It now follows that  $\frac{\sin x}{x} < 1$ .

Now consider  $g(x) = \frac{\sin x}{x}$ .  $g'(x) = \frac{x \cos x - \sin x}{x^2}$ . We now show that  $x < \tan x = \frac{\sin x}{\cos x}$  in  $(0, \pi/2)$ . Put  $h(x) = \tan x - x$ . Since  $|\cos x| < 1$  in  $(0, \pi/2)$ ,  $h'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} - 1 = \frac{1}{\cos^2 x} - 1 > 1$ . But then  $h(0) = 0$  and  $h$  is strictly increasing, so  $\tan x - x > 0$  in  $(0, \pi/2)$ . It now follows that  $g'(x) < \frac{\tan x \cos x - \sin x}{x^2} = 0$  and  $g(\pi/2) = \frac{2}{\pi}$ , and thus  $\frac{2}{\pi} < \frac{\sin x}{x}$  for all  $0 < x < \frac{\pi}{2}$ .  $\square$

## Problem 2

For  $n = 0, 1, 2, \dots$  and  $x$  real, prove that

$$|\sin nx| \leq n|\sin x|.$$

*Proof.* We proceed by induction on  $n$ . The base case  $n = 0$  is trivial. Suppose  $n \geq 1$ .

$$\begin{aligned} |\sin nx| &= \left| \frac{1}{2i}(e^{nix} - e^{-nix}) \right| \\ &= \left| \frac{1}{2i}[(e^{(n-1)ix} - e^{-(n-1)ix})(e^{ix} + e^{-ix}) + (e^{(n-1)ix} + e^{-(n-1)ix})(e^{ix} - e^{-ix})] \right| \\ &= |\sin(n-1)x \cdot \cos x + \cos(n-1)x \cdot \sin x| \\ &\leq |\sin(n-1)x \cdot \cos x| + |\cos(n-1)x \cdot \sin x| \\ &\leq |\sin(n-1)x| + |\sin x| \end{aligned}$$

By induction,

$$|\sin nx| = |\sin(n-1)x| + |\sin x| \leq (n-1)|\sin x| + |\sin x| = n|\sin x|.$$

□

### Problem 3

Put  $s_N = 1 + (\frac{1}{2}) + \cdots + (\frac{1}{N})$ . Prove that

$$\lim_{N \rightarrow \infty} (s_N - \log N)$$

exists. (The limit, often denoted by  $\gamma$ , is called Euler's constant. Its numerical value is 0.5772.... It is not known whether  $\gamma$  is rational or not.)

*Proof.* Let  $f_n = s_n - \log n$ . Since  $\frac{1}{x}$  is a decreasing function,  $\int_n^{n+1} \frac{1}{x} dx \geq \frac{1}{n+1}$ . Thus,

$$f_{n+1} - f_n = \frac{1}{n+1} - (\log(n+1) - \log n) = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x} dx \leq 0,$$

and so  $\{f_n\}$  is a monotonically decreasing sequence. But then  $\int_1^n \frac{1}{x} dx \leq \sum_{k=1}^{n-1} \frac{1}{k}$ . Hence,

$$f_n = \sum_{k=1}^n \frac{1}{k} - \int_1^n \frac{1}{x} dx \geq \frac{1}{n} > 0,$$

so  $f_n$  is bounded below. The result now follows from Theorem 3.14. □

## Problem 4

Prove that  $\sum 1/p$  diverges; the sum extends over all primes.

*Proof.* Given  $N$ , let  $p_1, \dots, p_k$  be those primes that divide at least one integer at most  $N$ . Each  $n \leq N$  is a product of powers of  $p_j$ 's. Since  $\prod_{j=1}^k \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \dots\right)$  is the sum of all inverses of numbers whose factorization consists of only powers of  $p_j$ 's,

$$\begin{aligned} \sum_{n=1}^N \frac{1}{n} &= \sum_{n=1}^N \frac{1}{p_1^{l_1} p_2^{l_2} \dots p_k^{l_k}} \\ &\leq \prod_{j=1}^k \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \dots\right) \\ &= \prod_{j=1}^k \left(1 - \frac{1}{p_j}\right)^{-1}. \end{aligned}$$

We now show that  $e^{2x} \geq (1-x)^{-1}$  for  $x \in (0, 1/2)$ . Put  $f(x) = (1-x)e^{2x}$ . Since  $f'(x) = (1-2x)e^{2x} > 0$  for  $x \in (0, 1/2)$  and  $f(0) = 1$ , we have  $f(x) \geq 1$  in  $(0, 1/2)$ , and thus  $e^{2x} \geq (1-x)^{-1}$ . Hence, we have

$$\prod_{j=1}^k \left(1 - \frac{1}{p_j}\right)^{-1} \leq \exp \sum_{j=1}^k \frac{2}{p_j}.$$

The logarithmic function is monotonically increasing, so we get

$$\frac{1}{2} \log \left( \sum_{n=1}^N \frac{1}{n} \right) \leq \sum_{j=1}^k \frac{1}{p_j}.$$

Since  $k \rightarrow \infty$  as  $N \rightarrow \infty$  and  $\sum_{n=1}^N \frac{1}{n}$  diverges,  $\sum_{j=1}^k \frac{1}{p_j}$  diverges. □

## Problem 5

Suppose  $f \in \mathcal{R}$  on  $[0, A]$  for all  $A < \infty$ , and  $f(x) \rightarrow 1$  as  $x \rightarrow \infty$ . Prove that

$$\lim_{t \rightarrow 0} t \int_0^\infty e^{-tx} f(x) dx = 1 \quad (t > 0).$$

*Proof.* Pick  $\epsilon > 0$ . There exists  $A$  such that  $|f(x) - 1| < \epsilon$  for all  $x \geq A$ . Since  $|e^{-tx}| < 1$  for all  $t > 0$ ,

$$\lim_{t \rightarrow 0^+} t \left| \int_0^A e^{-tx} f(x) dx \right| \leq \lim_{t \rightarrow 0^+} t \int_0^A |f(x)| dx = 0.$$

On the other hand, for  $t > 0$ ,

$$e^{-At}(1 - \epsilon) \leq \left| \int_A^\infty t e^{-tx} (1 - \epsilon) dx \right| \leq t \left| \int_A^\infty e^{-tx} f(x) dx \right| \leq \left| \int_A^\infty t e^{-tx} (1 + \epsilon) dx \right| \leq e^{-At}(1 + \epsilon).$$

Thus,  $t \left| \int_A^\infty e^{-tx} f(x) dx \right| = e^{-At}$ , as  $\epsilon$  is arbitrary. It now follows that

$$\begin{aligned} \lim_{t \rightarrow 0^+} t \left| \int_0^\infty e^{-tx} f(x) dx \right| &= \lim_{t \rightarrow 0^+} t \left| \int_0^A e^{-tx} f(x) dx + \int_A^\infty e^{-tx} f(x) dx \right| \\ &\leq \lim_{t \rightarrow 0^+} t \left| \int_0^A e^{-tx} f(x) dx \right| + \lim_{t \rightarrow 0^+} t \left| \int_A^\infty e^{-tx} f(x) dx \right| \\ &= \lim_{t \rightarrow 0^+} t \left| \int_A^\infty e^{-tx} f(x) dx \right| \\ &= \lim_{t \rightarrow 0^+} e^{-At} \\ &= 1. \end{aligned}$$

□

## Problem 6

If  $\alpha$  is real and  $-1 < x < 1$ , prove Newton's binomial theorem

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

*Proof.* Since

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!} x^{n+1}}{\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n} \right| = \lim_{n \rightarrow \infty} \frac{n-\alpha}{n+1} |x| < 1,$$

the series on the right converges in  $(-1, 1)$  by the ratio test. Let  $f(x)$  denote the function on the right-hand side. By Theorem 8.1,  $f(x)$  is differentiable. Note that

$$f'(x) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^n.$$

Hence, we have

$$\begin{aligned} (1+x)f'(x) &= \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^n + \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^{n+1} \\ &= \alpha + \sum_{n=1}^{\infty} \left( \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} \right) x^n \\ &= \alpha + \sum_{n=1}^{\infty} (n+\alpha-n) \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n \\ &= \alpha + \alpha \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n \\ &= \alpha f(x). \end{aligned}$$

□