MATH 140A: Homework #7

Due on Mar 4, 2024 at 23:59pm

Professor Seward

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Problem 1

Investigate the behavior (convergence or divergence) of $\sum a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$;

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \cdots + \sqrt{n} - \sqrt{n-1} + \sqrt{n+1} - \sqrt{n} \\ &= \sqrt{n+1} - 1.\end{aligned}$$

Hence, $\lim_{n\to\infty}(\sqrt{n+1} - 1) = \infty$. $\square$

(b) $a_n = \left(\sqrt{n+1} - \sqrt{n}\right)/n$;

Proof. Notice

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{n\sqrt{n+1} + \sqrt{n}} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}.$$

By Theorem 3.28, $\sum \frac{1}{n^{\frac{3}{2}}}$ converges, and thus $\sum a_n$ converges by the compatison test. $\square$

(c) $a_n = (\sqrt[n]{n} - 1)^n$;

Proof. Since

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{n} - 1 = 1 - 1 = 0,$$

$\sum a_n$ converges by the root test. $\square$

(d) $a_n = 1/(1 + z^n)$, for complex values of z.

Proof. Suppose $|z| \le 1$. Since

$$|a_n| = \left|\frac{1}{1+z^n}\right| \ge \frac{1}{1+|z|^n} \ge \frac{1}{2},$$

a_n does not converge to 0, and thus $\sum a_n$ diverges.

Suppose $|z| > 1$. Notice

$$|a_n| = \left|\frac{1}{1+z^n}\right| \le \frac{1}{|z^n|} = \left|\frac{1}{z}\right|^n.$$

Since $\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{z}\right|^n} = \frac{1}{|z|} < 1$, $\sum \left|\frac{1}{z}\right|^n$ converges by the root test, and thus a_n converges by the comparison test. $\square$

Problem 2

Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$, if $a_n \geq 0$.

Proof. Note that both $\sum a_n$ and $\sum \frac{1}{n^2}$ converges absolutely. By the Cauchy-Schwarz inequality,

$$\left(\sum \frac{\sqrt{a_n}}{n}\right)^2 \leq \sum a_n \sum \frac{1}{n^2}.$$

Since $\sum a_n \sum \frac{1}{n^2}$ converges, $\sum \frac{\sqrt{a_n}}{n}$ is bounded. But then $\sum \frac{\sqrt{a_n}}{n}$ is a series of nonnegative terms, so it converges. $\square$

Problem 3

If $\sum a_n$ converges and if (b_n) is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Proof. Since (b_n) is monotonic and bounded, (b_n) converges. Put $A = \sum_{n=1}^{\infty} a_n$ and $B = \lim b_n$. Let $c_n = b_n - B$ if b_n monotonically decreases. Otherwise, let $c_n = B - b_n$. In this way, we guarantee c_n is monotonically decreasing and $c_n \to 0$. Then,

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n (b_n - B + B),$$

so $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n \pm AB$, depending on whether b_n monotonically increases or decreases. Since $\sum a_n$ converges, it follows from Theorem 3.42 that $\sum a_n c_n$ converges, and thus $\sum a_n b_n$ converges. $\square$

Problem 4

Find the radius of convergence of each of the following power series:

(a) $\sum n^3 z^n$

Proof. Since

$$\limsup_{n\to\infty} \sqrt[n]{n^3} = \limsup_{n\to\infty} (\sqrt[n]{n})^3 = 1,$$

the radius of convergence is 1. □

(b) $\sum \frac{2^n}{n!} z^n$

Proof. Since

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \left|\frac{2}{n+1}\right| = 0$$

The radius of convergence is ∞. □

(c) $\sum \frac{2^n}{n^2} z^n$

Proof. Since

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} 2\left(\frac{n}{n+1}\right)^2 = 2$$

The radius of convergence is $\frac{1}{2}$. □

(d) $\sum \frac{n^3}{3^n} z^n$.

Proof. Since

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^2 = \frac{1}{3}$$

The radius of convergence is 3. □

Problem 5

Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Proof. Since infinitely many of a_n are distinct from zero, $\limsup_{n\to\infty} |a_n z^n| \geq \limsup_{n\to\infty} |z|^n \geq 1$ when $z > 0$. But then $|a_n z^n|$ does not converge to 0, so $\sum a_n z^n$ diverges when $|z| \geq 1$. $\square$

Problem 6

Prove the following analogue of Theorem 3.10(b): If (E_n) is a sequence of closed, nonempty, and bounded sets in a *complete* metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n\to\infty} \text{diam } E_n = 0,$$

then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.

Proof. Since E_n is nonempty, let (p_n) be a sequence such that $a_n \in E_n$ for all n. Let K_N contain the points $p_n, p_{N+1}, \ldots$. Since $K_N \subset E_N$ and E_N is bounded,

$$\lim_{n\to\infty} \text{diam } K_n \leq \lim_{n\to\infty} \text{diam } E_n = 0,$$

and so (p_n) is a Cauchy sequence. Since X is complete, p_n converges to some point $p \in X$. Note that p is a limit point of every E_n. But then E_n is closed, so $p \in E_n$ for all n, that is, $p \in \bigcap_{n=1}^{\infty} E_n$. Suppose for the sake of contradiction that $\bigcap_{k=1}^{\infty} E_k$ contains two distinct points p, q. But since $\bigcap_{k=1}^{\infty} E_k \subset E_n$ for all n,

$$0 < \text{diam} \bigcap_{k=1}^{\infty} E_k \leq \text{diam } E_n,$$

and so diam E_n does not converge to 0, contradiction. $\square$

Problem 7

Suppose X is a nonempty complete metric space, and (G_n) is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_{n=1}^{\infty} G_n$ is not empty (In fact, it is dense in X). *Hint*: Find a shrinking sequence of neighborhoods E_n such that $\overline{E}_n \subset G_n$, and apply Exercise 3.21.

Proof. We inductively construct sequence of open sets (E_n). Since G_1 is open and nonempty, let $x_1 \in G_1$. There exists small enough $\epsilon_1 > 0$ such that diam $N_{\epsilon_1}(x_1) < 1$ and $\overline{N_{\epsilon_1}(x_1)} \subset G_1$. Put $E_1 = N_{\epsilon_1}(x_1)$.

Suppose that E_n is constructed. Since G_{n+1} is dense and open, $E_n \cap G_{n+1}$ is nonempty and open. Let $x_{n+1} \in E_n \cap G_{n+1}$. There exists small enough ϵ_{n+1} such that diam $N_{\epsilon_{n+1}}(x_{n+1}) < \frac{1}{n+1}$ and $\overline{N_{\epsilon_{n+1}}(x_{n+1})} \subset E_n \cap G_{n+1}$. Put $E_{n+1} = N_{\epsilon_{n+1}}(x_{n+1})$.

Thus, we have constructed a sequence of closed, nonempty, and bounded sets $\overline{E_n}$. Since $G_n \supset \overline{E_n} \supset \overline{E_{n+1}}$ and

$$\lim_{n\to\infty} \text{diam } \overline{E_n} = \lim_{n\to\infty} \text{diam } E_n \leq \lim_{n\to\infty} \frac{1}{n} = 0,$$

it follows from Exercise 3.21 that $\bigcap_{n=1}^{\infty} G_n \supset \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$. $\square$