Probability and Random Process

$\mu = E(x)$ $\quad \sigma^2 = E[(x-\mu)^2]$

$= \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\, dx$

Continuous Type Random Variable

1.) Normal (Gaussian) Distribution :-

X is a normal or Gaussian random with parameters μ and σ^2 if its density function is given by

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{--- (1)}$$

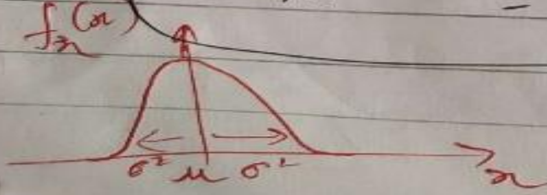
This is a bell-shaped curve, symmetric around the parameter μ. and its distribution function is given by

$$F_x(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

$$\triangleq G\left(\frac{x-\mu}{\sigma}\right)$$

where the function

$$\boxed{G(x) \triangleq \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}$$



$N(\mu, \sigma^2)$

$\frac{1}{\ldots} \rightarrow$ is the height of the curve's peak.

$\mu \to$ is the position of the center peak.
$\sigma \to$ control the width of the "bell"

Since $f_x(x)$ depends on two parameters μ and σ^2, the notation $x \sim N(\mu, \sigma^2)$ will be used to represent the Gaussian or normal p.d.f.

The constant $\sqrt{2\pi\sigma^2}$ in (1) is the normalization constant that maintains the area under $f_x(x)$ to unity. (Now proof it)

The special case $x \sim N(0, 1)$ is often referred to as the standard normal random variable.

application
(Maxwell $\Rightarrow$ velocity of molecules $N(\mu, \sigma^2)$)

& Hagen $\Rightarrow$ theory of error $\Rightarrow$ error is $N(\mu, \sigma^2)$

* In signal processing they serve to define gaussian filters.

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Uniform distribution

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Uniform Distribution :⇒

X is said to be uniformly distributed in the interval (a, b), $-\infty < a < b < \infty$, if

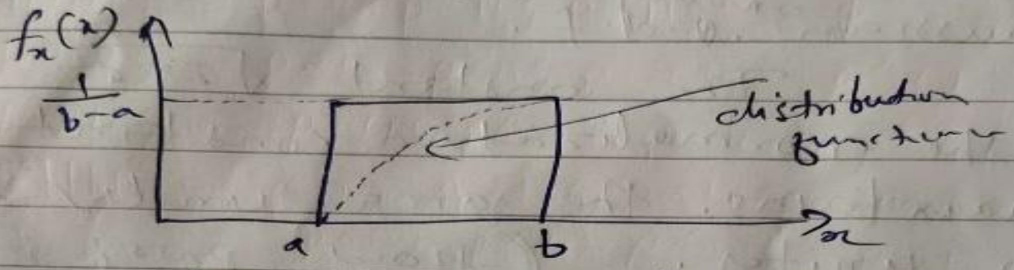


fig:- Uniform density function

$$\text{if } f_x(x) = \begin{cases} \frac{1}{b-a} & ; \ a \le x \le b \\ 0 & ; \ \text{otherwise} \end{cases}$$

The distribution function of X is

$$F_x(x) = \begin{cases} 0 & ; \ x < a \\ \frac{x-a}{b-a} & ; \ a < x < b \\ 1 & ; \ x \ge b \end{cases}$$

The Binomial Probability law

A **Bernoulli trial** involves performing an experiment once and noting whether a particular event A occurs. The outcome of the Bernoulli trial is said to be a "success" if A occurs and a "failure" otherwise.

We will view the outcome of a single Bernoulli trial as the outcome of a toss of a coin for which the probability of head (success) is $p = P(A)$. The probability of K successes in n Bernoulli trial is then equal to the probability of k heads in n tosses of the coin.

Exa' ⇒ Suppose that a coin is tossed three times. If we assume that the tosses are independent and the probability of heads is p, then the probability for the sequences of heads and trials is -

0 0 0
0 1 0
0 1 1
1 0 0
1 0 1
1 1 0
1 1 1

$$P\{HHH\} = P\{H\}P\{H\}P\{H\} = p^3$$
$$P\{HHT\} = P\{H\}P\{H\}P\{T\} = p^2(1-p)$$
$$P\{HTH\} = P\{H\}P\{T\}P\{H\} = p^2(1-p)$$
$$P\{THH\} = P\{T\}P\{H\}P\{H\} = p^2(1-p)$$
$$P\{HTT\} = P(H)P\{T\}P\{T\} = p(1-p)^2$$
$$P(THT) = P\{T\}P(H)P(T) = p(1-p)^2$$
$$P(TTH) = P(T)P(T)P(H) = p(1-p)^2$$
$$P(TTT) = T\ T\ T = (1-p)^3$$

Let k be the number of heads in three trials, then

$$P[k=0] = P\{TTT\} = (1-p)^3$$

$$P[k=1] = P\{TTH, THT, HTT\} = 3p(1-p)^2$$
$$P[k=2] = P\{HHT, HTH, THH\} = 3p^2(1-p)$$
$$P[k=3] = P\{HHH\} = p^3$$

Theorem: Let k be the number of successes in n independent Bernoulli trials, then the probabilities of k are given by the binomial probability law:

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, \ldots, n$

where $P_n(k)$ is the probability of k successes in n trials, and

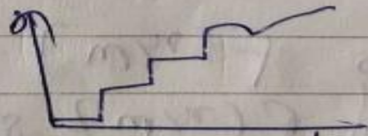
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$n!$ factorial $n! = n(n-1)\cdots(2)(1)$

$0! = 1$

Poisson · Binomial distribution

$P(Y=k) =$ Same



Staircase distribution

See by yourself

1.-> Poisson distribution

2.-> Gamma distribution

3.-> Chi-square — " —

4.-> Rayleigh — " —

5.-> Beta — " —

1. Let A in ① be identified as the event $\{X \le x\}$ for the random variable X. The resulting probability $P\{X \le x \mid M\}$ is defined as the conditional distribution function of X, which we denote $F_x\left(\frac{x}{M}\right)$

Conditional Distribution :⇒

We know that the probability of an event A assuming M is given by

$$P(A/M) = \frac{P(AM)}{P(M)} \quad \text{—①} \quad \text{where } P(M) \neq 0$$

↳ 1.

The conditional distribution $F(x/M)$ of a random variable X, assuming M is defined as the conditional probability of the event $\{X \le x\}$.

$$F(x/M) = P\{X \le x \mid M\} = \frac{P\{X \le x, M\}}{P(M)}$$

Here $\{X \le x, M\}$ is the intersection of the events $\{X \le x\}$ and M, that is, the event consisting of all outcomes ξ such that $X(\xi) \le x$ and $\xi \in M$.

So $F(x/M)$ is same as $F_x(x)$
so $F(x/M)$ should follow all properties of $F_x(x)$

1.) $F(\infty/m) = 1$ & $F(-\infty/m) = 0$

2.) $P\{x_1 < X \le x_2\} = F(x_2/m) - F(x_1/m)$

The conditional density $f(x/m)$ is the derivative of $F(x/m)$

$$f(x/m) = \frac{d}{dx}(F(x/m))$$

The function is non-negative & its area equals 1.

Exer 1 ⟹ Determine the conditional $F(x/m)$ of the random variable $X(f_i) = 10i$ of the fair-die experiment, where $M = \{f_2, f_4, f_6\}$ is the event "even".

Sol:

$$S = \{f_1, f_2, ----, f_6\}$$

$$X = 10i$$

$$S_X = \{10, 20, 30, ----, 60\}$$

if $x \geq 60$, then $\{X \leq x, m\} = m$

$$F(x/m) = \frac{P(X \leq 60, m)}{P(m)} = \frac{P(f_2, f_4, f_6)}{P(m)}$$

$$= \frac{P(m)}{P(m)} = 1$$

if $40 \leq x < 60$, then $\{X \leq x, m\} = \{f_2, f_4\}$

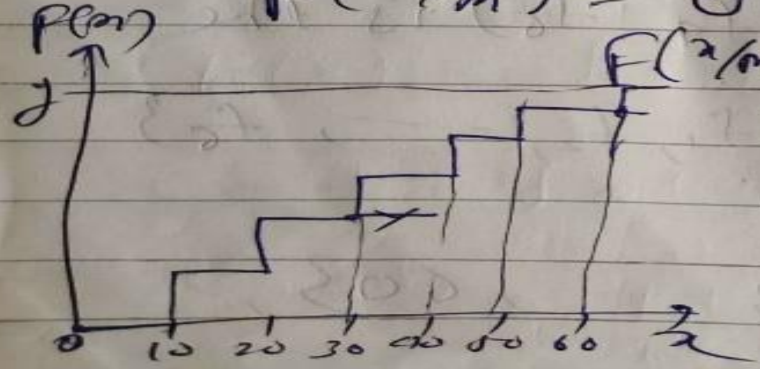
$$F(x/m) = \frac{P\{f_2, f_4\}}{P(m)} = \frac{2/6}{3/6} = \frac{2}{3}$$

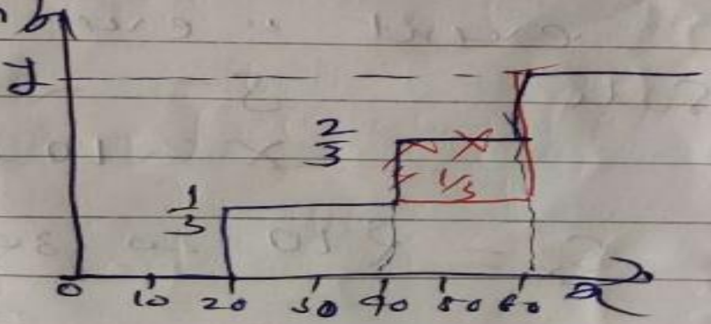
if $20 \leq x < 40$ then $\{X \leq x, m\} = \{f_2\}$

$$F(x/m) = \frac{P(f_2)}{P(m)} = \frac{1/6}{3/6} = \frac{1}{3}$$

if $x < 20$, then $\{X \leq x, m\} = \{\phi\}$

$$F(x/m) = 0$$





total probability

$$P(B) = P(B/A_1)\,P(A_1) + \cdots + P(B/A_n)\,P(A_n)$$

substitute $B = \{X \le x\}$

$$F(x) = F(x/A_1)\,P(A_1) + \cdots + F(x/A_n)\,P(A_n)$$

Bays theorem.

$$f(x) = f(x/A_1)\,P(A_1) + \cdots + f(x/A_n)\,P(A_n)$$

$A_1, A_2, \cdots$ form a partition of S.

Bay's theorem

$$P(B/A) = \frac{P(A/B)\,P(B)}{P(B)}$$

$$P(A_i/B) = \frac{P(B/A_i)\,P(A_i)}{P(B)}$$

if $B = \{X \le x\}$

$$P(A/X = x) = \frac{P\left(\frac{X \le x}{A_i}\right) \cdot P(A_i)}{P(X \le x)}$$

$$\Rightarrow P(A/X = x) = \frac{f(x/A_i)\,P(A_i)}{f(x)}$$

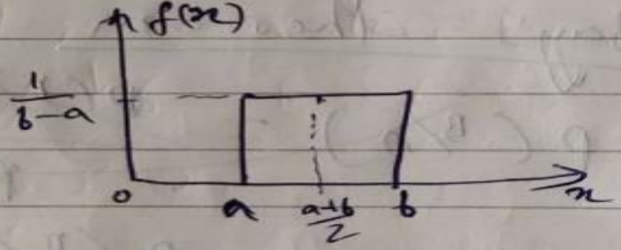
Mean (or expected value) and variance :⇒

Mean (or expected value) :⇒ The mean (or μ_x) of a random variable X is by definition the integral

$$E\{X\} = \int_{-\infty}^{\infty} x\, f(x)\, dx \quad \text{--- (1)}$$

It is a number 4 denoted by μ_x or η_x or η (or central tendency)

Ex. If X is uniform in the interval (a, b) find the mean.



Soln. ⇒

$$f(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \le x \le b \\ 0 & x > b \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{a} x f(x)\, dx + \int_{a}^{b} x f(x)\, dx + \int_{b}^{\infty} x f(x)\, dx$$

$$= 0 + \frac{1}{b-a}\int_{a}^{b} x\, dx + 0$$

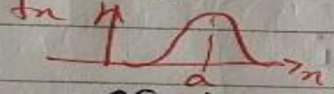
$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{2(b-a)}\left[b^2 - a^2\right]$$

$$= \frac{b+a}{2} \qquad \left(\because\ a^2 - b^2 = (a-b)(a+b)\right)$$

$$= \frac{a+b}{2}$$

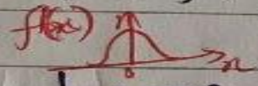
~~We ltm note~~ ⇒ **Note:** ⇒

1.) If the vertical line $x = a$ is an axis of symmetry of $f(x)$ then $E\{X\} = a$.



$E(x) = a$

2.) If $f(-x) = f(x)$, then $E\{X\} = 0$.



Discrete type: → For discrete type random variables the integral can be written as a sum.

~~Suppo~~

$$E\{X\} = \sum_i p_i x_i \quad ; \quad p_i = P\{X = x_i\}$$

proof from eqn ① Suppose that X takes the values x_i with probability p_i. In this case

$$f(x) = \sum_i p_i \, \delta(x - x_i)$$

so $$E\{X\} = \int_{-\infty}^{\infty} x \sum_i p_i \, \delta(x - x_i) \, dx$$

$$= \sum_i P_i \int_{-\infty}^{\infty} \cancel{x}\, \delta(x - x_i)\, dx$$

we know that

$$\int_{-\infty}^{\infty} x\, \delta(x - x_i)\, dx = x_i$$

$$x = 5$$

$$\left(\text{e.g. } \int_{-\infty}^{\infty} 5 \cdot \underline{\delta(0)}\, dx = 5\right.$$

$$\int_{-\infty}^{\infty} \delta(0)\, dx = 1$$

$$\boxed{E\{X\} = \sum_i P_i x_i}$$

Ex: If X takes the values 1, 2, ..., 6 with probability 1/6 find E{X}.

Sol: $E(X) = \sum_i P_i x_i$

$$= \frac{1}{6}[1+2+\cdots+6]$$

$$= 3.5$$

Conditional mean ⟹ The conditional mean of a random variable X assuming an event M is given ~~by the inte~~

then $E\{X/M\} = \int_{-\infty}^{\infty} x f(x/M)\, dx$

for discrete ~~case~~ type random variable.

$$E\{x/m\} = \sum_i x_i \, P\left\{\frac{x = x_i}{m}\right\}$$

Properties of mean or expection (E)

(i) Linearity

In particular,

if $E(y) = E(ax + b)$

then $E\{y\} = aE\{x\} + b$ $\left(\because E\{ax\} = aE\{x\}\right)$

$(y = g(x))$

or $c_1 g_1(x) + c_2 g_2(x) + \cdots$

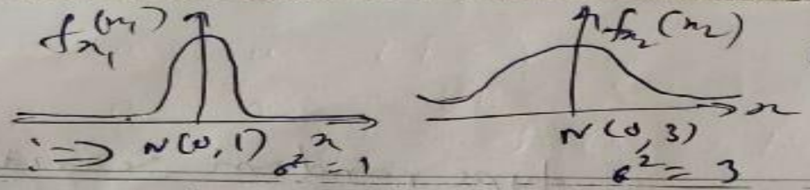
$$E[c_1 g_1(x) + c_2 g_2(x) + \cdots] = c_1 E[g_1(x)] + c_2 E[g_2(x)] + \cdots$$

2. Complex random variable

if $Z = x + jy$, then

$$E\{Z\} = E\{x + jy\} = E\{x\} + jE\{y\}$$

$N(\mu, \sigma^2)$



Variance :⇒

see book page 144

For a random variable X with mean μ, $x-\mu$ represents the deviation of the random variable from its mean. Since this deviation can be either positive or negative, consider the quantity $(X-\mu)^2$, and its average value $E[(X-\mu)^2]$ represents the average square deviation of X around its mean.

Define $\sigma_x^2 \triangleq E[(X-\mu)^2] > 0$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f_x(x)\, dx > 0$$

σ_x^2 is ~~sta~~ variance of random variable X.
& σ_x is standard deviation of X.

Skewness :⇒ skewness is usually described as a measure of a dataset's symmetry – or lack of symmetry.
[A perfectly symmetrical data set will have a skewness of 0. The normal distribution has a skewness of 0.]
The skewness is defined as

$$S \text{ or } \mu_3 = \sum \frac{(X_i - \bar{X})^3}{N\sigma^3}$$

where N is the sample size. X_i is the ith X value. $\bar{X}$ is mean of X. σ is the sample standard deviation.