# Probability and Random Process

# Binomial Distributions

This lesson presents a basic definition of a binomial distribution along with notation and methods for finding probability values.

Binomial probability distributions allow us to deal with circumstances in which the outcomes belong to two relevant categories such as acceptable/defective or survived/died or true/false.

# Binomial Probability Distribution

A binomial probability distribution results from a procedure that meets all the following requirements:

1. The procedure has a fixed number of trials.
2. The trials must be independent. (The outcome of any individual trial doesn't affect the probabilities in the other trials.)
3. Each trial must have all outcomes classified into two categories (commonly referred to as success and failure).
4. The probability of a success remains the same in all trials.

# Notation for Binomial Probability Distributions

S and F (success and failure) denote the two possible categories of all outcomes; $p$ and $q$ will denote the probabilities of S and F, respectively, so

$$P(\mathrm{S}) = p$$ ($p$ = probability of success)

$$P(\mathrm{F}) = 1 - p = q$$ ($q$ = probability of failure)

# Notation (continued)

| | |
|---|---|
| $n$ | denotes the fixed number of trials. |
| $x$ | denotes a specific number of successes in $n$ trials, so $x$ can be any whole number between 0 and $n$, inclusive. |
| $p$ | denotes the probability of success in one of the $n$ trials. |
| $q$ | denotes the probability of failure in one of the $n$ trials. |
| $P(x)$ | denotes the probability of getting exactly $x$ successes among the $n$ trials. |

# Caution

- Be sure that $x$ and $p$ both refer to the same category being called a success.

- When sampling without replacement, consider events to be independent if $n < 0.05N$.

# Example

- When an adult is randomly selected, there is a 0.85 probability that this person knows what Twitter is.
- Suppose we want to find the probability that exactly three of five randomly selected adults know of Twitter.
- Does this procedure result in a binomial distribution?

Yes. There are five trials which are independent. Each trial has two outcomes and there is a constant probability of 0.85 that an adult knows of Twitter.

# Method 1: Using the Binomial Probability Formula

$$P(x) = \frac{n!}{(n-x)!x!} \cdot p^x \cdot q^{n-x}$$

$$\text{for } x = 0, 1, 2, \ldots, n$$

where

$n$ = number of trials

$x$ = number of successes among $n$ trials

$p$ = probability of success in any one trial

$q$ = probability of failure in any one trial ($q = 1 - p$)

# Method 2: Using Technology

STATDISK, Minitab, Excel and the TI-83 Plus calculator can all be used to find binomial probabilities.

**EXCEL**

| | A | B |
|---|---|---|
| 1 | 0 | 0.000977 |
| 2 | 1 | 0.014648 |
| 3 | 2 | 0.087891 |
| 4 | 3 | 0.263672 |
| 5 | 4 | 0.395508 |
| 6 | 5 | 0.237305 |

**TI-83 PLUS Calculator**

| L1 | L2 | L3 2 |
|---|---|---|
| 0 | 9.8E-4 | ------ |
| 1 | .01465 | |
| 2 | .08789 | |
| 3 | .26367 | |
| 4 | .39551 | |
| 5 | .2373 | |
| ------ | | |

L2(7) =

# Method 3: Using Tables

Part of a Table is shown below. With $n$ = 12 and $p$ = 0.80 in the binomial distribution, the probabilities of 4, 5, 6, and 7 successes are 0.001, 0.003, 0.016, and 0.053 respectively.

| $n$ | $x$ | $p$<br>0.80 |
|---|---|---|
| | 4 | 0.001 |
| | 5 | 0.003 |
| | 6 | 0.016 |
| | 7 | 0.053 |

→

| $x$ | $p$ |
|---|---|
| 4 | 0.001 |
| 5 | 0.003 |
| 6 | 0.016 |
| 7 | 0.053 |

# Strategy for Finding Binomial Probabilities

- Use computer software or a TI-83/84 Plus calculator, if available.
- If neither software nor the TI-83/84 Plus calculator is available, use a Table, if possible.
- If neither software nor the TI-83/84 Plus calculator is available and the probabilities can't be found using a Table, use the binomial probability formula.

# Example

We have:

$$n = 5, \quad x = 3, \quad p = 0.85, \quad q = 0.15$$

$$\boldsymbol{P}(3) = \frac{5!}{(5-3)!\,3!} * 0.85^{3} * 0.15^{(5-3)}$$

$$= \frac{5!}{2!3!} * 0.614125 * 0.0225$$

$$= (10)(0.614125)(0.0225)$$

$$= 0.138$$

# Rationale for the Binomial Probability Formula

$$P(x) = \underbrace{\frac{n!}{(n-x)!x!}}_{\text{The number of outcomes with exactly } x \text{ successes among } n \text{ trials}} \cdot p^x \cdot q^{n-x}$$

# Rationale for the Binomial Probability Formula

$$P(x) = \underbrace{\frac{n!}{(n-x)!x!}}_{\substack{\text{Number of outcomes with} \\ \text{exactly } x \text{ successes} \\ \text{among } n \text{ trials}}} \cdot \underbrace{p^x \cdot q^{n-x}}_{\substack{\text{The probability of } x \\ \text{successes among } n \\ \text{trials for any one} \\ \text{particular order}}}$$

# Binomial Distribution: Formulas

**Mean** $\mu = n \cdot p$

**Variance** $\sigma^2 = n \cdot p \cdot q$

**Std. Dev.** $\sigma = \sqrt{n \cdot p \cdot q}$

Where

$n$ = number of fixed trials

$p$ = probability of success in one of the $n$ trials

$q$ = probability of failure in one of the $n$ trials

# Example

McDonald's has a 95% recognition rate. A special focus group consists of 12 randomly selected adults.

For such a group, find the mean and standard deviation.

$$\mu = np = 12(0.95) = 11.4$$

$$\sigma = \sqrt{npq} = \sqrt{12(0.95)(0.05)} = 0.754983 = 0.8 \ \text{(rounded)}$$

# Example - continued

Use the range rule of thumb to find the minimum and maximum usual number of people who would recognize McDonald's.

$$\mu + 2\sigma = 11.4 + 2(0.8) = 13 \text{ people}$$

$$\mu - 2\sigma = 11.4 - 2(0.8) = 9.8 \text{ people}$$

If a particular group of 12 people had all 12 recognize the brand name of McDonald's, that would ***not*** be unusual.

# Poisson Distribution

The Poisson distribution is a discrete probability distribution that applies to occurrences of some event over a specified interval. The random variable $x$ is the number of occurrences of the event in an interval. The interval can be time, distance, area, volume, or some similar unit.

## Formula

$$P(x) = \frac{\mu^x \cdot e^{-\mu}}{x!}$$

where $e \approx 2.71828$

$\mu =$ mean number of occurrences of the event over the interval

# Requirements of the Poisson Distribution

- The random variable $x$ is the number of occurrences of an event over some interval.
- The occurrences must be random.
- The occurrences must be independent of each other.
- The occurrences must be uniformly distributed over the interval being used.

## Parameters

- The mean is $\mu$ .
- The standard deviation is $\sigma = \sqrt{\mu}$.

# Differences from a Binomial Distribution

The Poisson distribution differs from the binomial distribution in these fundamental ways:

- The binomial distribution is affected by the sample size $n$ and the probability $p$, whereas the Poisson distribution is affected only by the mean $\mu$.
- In a binomial distribution the possible values of the random variable $x$ are $0, 1, \ldots, n$, but a Poisson distribution has possible $x$ values of $0, 1, 2, \ldots$ , with no upper limit.

# Example

For a recent period of 100 years, there were 530 Atlantic hurricanes. Assume the Poisson distribution is a suitable model.

a. Find $\mu$, the mean number of hurricanes per year.

b. If $P(x)$ is the probability of $x$ hurricanes in a randomly selected year, find $P(2)$.

# Example

a. Find $\mu$, the mean number of hurricanes per year.

$$\mu = \frac{\text{number of hurricanes}}{\text{number of years}} = \frac{530}{100} = 5.3$$

b. If $P(x)$ is the probability of $x$ hurricanes in a randomly selected year, find $P(2)$.

$$P(2) = \frac{\mu^x \cdot e^{-\mu}}{x!} = \frac{5.3^2 (2.71828)^{-5.3}}{2!} = 0.0701$$

# Poisson as an Approximation to the Binomial Distribution

The Poisson distribution is sometimes used to approximate the binomial distribution when *n* is large and *p* is small.

Rule of Thumb to Use the Poisson to Approximate the Binomial

- $n \geq 100$
- $np \leq 10$

# Poisson as an Approximation to the Binomial Distribution -

If both of the following requirements are met,

- $n \geq 100$
- $np \leq 10$

then use the following formula to calculate $\mu$,

**Value for $\mu$**

$$\mu = n \cdot p$$

# Example

In the Maine Pick 4 game, you pay \$0.50 to select a sequence of four digits, such as 2449.

If you play the game once every day, find the probability of winning at least once in a year with 365 days.

The chance of winning is $p = \dfrac{1}{10,000}$

Then, we need $\mu$: $\mu = np = 365 \cdot \dfrac{1}{10,000} = 0.0365$

# Example - continued

Because we want the probability of winning "at least" once, we will first find $P(0)$.

$$P(0) = \frac{0.0365^{0}(2.71828)^{-0.0365}}{0!} = 0.9642$$

There is a 0.9642 probability of no wins, so the probability of at least one win is:

$$1 - 0.9642 = 0.0358$$