Torben
20 september, 2011 [S] 11.5 opg. 28, 41 Calculus E2011

Følgende er en opsamling fra TØ 21-09-2011.

Opg. 28

Vi får givet $\ln(x+yz) = 1 + xy^2z^3$ og skal finde $\frac{\partial z}{\partial x}$ og $\frac{\partial z}{\partial y}$. Vi benytter samme fremgangsmetode som i eks. 9 s. 786 i [S]. Derfor lad $F(x,y,z) = \ln(x+yz) - xy^2z^3 - 1$. Vi finder så:

$$F_z = \frac{\partial F}{\partial z} = \frac{y}{x+yz} - 3xy^2z^2$$
$$F_x = \frac{\partial F}{\partial x} = \frac{1}{x+yz} - y^2z^3$$
$$F_y = \frac{\partial F}{\partial y} = \frac{z}{x+yz} - 2xyz^3$$

Ligning 7 s. 785 i [S] giver nu

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \cdots = \frac{y^2z^3(x+yz)-1}{y-3xy^2z^2(x+yz)}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \cdots = \frac{2xyz^3(x+yz)-z}{y-3xy^2z^2(x+yz)}$$

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Antag først at alle funktioner har anden ordens partielle afledte. Vi skal vise $z(x+at, x-at) = f(x+at) + g(x-at)$ altid løser bølgeligningen: $\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$. Dette vil vi gøre ved først at "udregne" VS og derefter HS.

Først lader vi $u(x,t) = x+at$ og $v(x,t) = x-at$. Dermed fås $z(u,v) = f(u(x,t)) + g(v(x,t))$. Da vi kan partiel differetiere hvert led for sig fås

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 g}{\partial t^2}$$

Nu er det bare at regne løs ved at anvende den generelle version af kædereglen(f er en funktion af en variabel der hver er en funktion af 2 variable):

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} \\ &= a\frac{\partial f}{\partial u} \\ \frac{\partial^2 f}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) \\ &= a\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial u}\right) \\ &= a\left(\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)\right)\frac{\partial u}{\partial t} \\ &= a^2\frac{\partial^2 f}{\partial u^2}\end{aligned}$$

Tilsvarende fås

$$\begin{aligned}
\frac{\partial g}{\partial t} &= \frac{\partial g}{\partial v}\frac{\partial v}{\partial t} \\
&= -a\frac{\partial g}{\partial v} \\
\frac{\partial^2 v}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial t}\right) \\
&= -a\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial v}\right) \\
&= -a\left(\frac{\partial}{\partial v}\left(\frac{\partial g}{\partial v}\right)\right)\frac{\partial v}{\partial t} \\
&= a^2\frac{\partial^2 g}{\partial v^2}
\end{aligned}$$

Vi kan nu få venstresiden til:

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 g}{\partial t^2} = a^2\frac{\partial^2 f}{\partial u^2} + a^2\frac{\partial^2 g}{\partial v^2} = a^2\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}\right) \tag{1}$$

Vi tager nu fat på HS. Igen kan vi partiel differetiere hvert led. Dvs.:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2}$$

Så er det igen bare at regne løs:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) \\
&= \left(\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)\right)\frac{\partial u}{\partial x} \\
&= \frac{\partial^2 f}{\partial u}
\end{aligned}$$

Og

$$\begin{aligned}
\frac{\partial^2 g}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial v}\right) \\
&= \left(\frac{\partial}{\partial v}\left(\frac{\partial g}{\partial v}\right)\right)\frac{\partial v}{\partial x} \\
&= \frac{\partial^2 g}{\partial v^2}
\end{aligned}$$

Så HS bliver

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} = \frac{\partial^2 f}{\partial u} + \frac{\partial^2 g}{\partial v^2} \tag{2}$$

Ved at sammenholde (2) med (1) fås

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= a^2 \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} \right) \\ &= a^2 \frac{\partial^2 z}{\partial x^2} \end{aligned}$$

Som skulle vises!