Game Theory: Week 1 Assignment

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Problem 1: The Scope of Zermelo's Theorem (4 Marks)

The proof of Theorem 2.4 in the material uses several fundamental properties of the game of chess.

(a) (2 marks) Identify the general properties a game must satisfy for the logic behind Theorem 2.4 to be applicable.

(b) (2 marks) Provide three distinct examples of games that satisfy these properties, and three distinct non-examples. For each non-example, clearly identify which property is violated.

Problem 2: Equivalence of Statements (3 Marks)

Consider the following two statements:

- **Statement A (Theorem 2.4):** In chess, one and only one of the following must be true: (i) White has a winning strategy, (ii) Black has a winning strategy, or (iii) Both players have a strategy to guarantee at least a draw.
- **Statement B:** Every individual game of chess, when played out, must result in a win for White, a win for Black, or a draw.

Are these two statements logically equivalent? Provide a rigorous explanation for your answer.
Hint: Focus on the difference between strategy and outcome

Problem 3: Analysis of "Long Chess" (6 Marks)

The proof of Theorem 2.4 assumes that chess is a finite game. We now relax this assumption. Consider a modified game called **'Long Chess'**, where rules that guarantee finiteness (e.g., the "three-fold repetition" and "50-move" rules) are removed. In 'Long Chess', any game that continues for an infinite number of moves is declared a draw by definition.

(a) (4 marks) Prove that Theorem 2.4 still holds for 'Long Chess' i.e. prove that even in this potentially infinite game, it must be that either White has a winning strategy, Black has a winning strategy, or both players can force a draw.

(b) (2 marks) Further, prove that if a player has a winning strategy in 'Long Chess', they must also has a winning strategy in standard, finite chess.

Hint: The state of a chess board (piece positions, castling rights, en passant possibilities) can be described by a finite amount of information.

Problem 4: Classification of Combinatorial Games (6 Marks)

A broad class of two-player sequential games, often called **Combinatorial Games**, can be characterized by the following axioms:

(i) There are two players, conventionally named **Left** and **Right**.

(ii) From any given game position, Left and Right may have different sets of available moves.

(iii) The players take turns moving. A player may only make a move that is available to them.

(iv) The game is guaranteed to terminate; no infinite plays are possible.

(v) The game ends when a player whose turn it is cannot make a move. This player loses.

An example of such kind of game is Domineering

Using a proof by induction, prove that every position in any such game falls into exactly one of the following four outcome classes:

Class $\mathcal{L}$: Left has a winning strategy, regardless of who moves first from this position.

Class $\mathcal{R}$: Right has a winning strategy, regardless of who moves first.

Class $\mathcal{N}$: The **N**ext player to move from this position has a winning strategy.

Class $\mathcal{P}$: The **P**revious player (i.e., the second player to move) has a winning strategy.

Hint: For your inductive step, consider a position G. Its class will depend on the classes of the positions reachable by Left and the positions reachable by Right. For example, what can you conclude about the class of G if Left can move to a position in Class $\mathcal{P}$?

Problem 5: The Island of the Blue-Eyed People (6 Marks)

This problem explores the practical consequences of common knowledge. Consider the following scenario:

Setup: An isolated island is inhabited by a tribe of 100 people, each of whom is a perfect logician. Each person can see the eye color of every other person, but is unaware of their own.

Rules: It is forbidden to communicate about eye color. If a person deduces that they have blue eyes, they must leave the island on a boat that departs at midnight that same day.

One day, an all-knowing visitor arrives and makes a single, public statement heard by everyone:

'There is at least one person with blue eyes on this island.'

The visitor then leaves. Assume that, unknown to the islanders, there are in fact exactly **three** blue-eyed people.

Will anyone leave the island? If so, on which day will they leave, and how do they deduce their own eye color? Provide a rigorous, step-by-step explanation for your conclusion. Your argument should be built inductively, by analyzing the cases for one, two, and then three blue-eyed people.