

**MA 02 LINEAR ALGEBRA II**  
**REVIEW OF LECTURES – VII**

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**Section:** C7.

**Instructors:** Yasuyuki Kachi (lecture) & Shunji Moriya (recitation)

• **Double sign.**

Today we are going to see the double sign ‘ $\pm$ ’ everywhere. I’m sure you are completely familiar with it. Still, there are some subtle aspects of it, which people sometimes get confused about. In this class, we cannot afford to be sloppy about those aspects. So today let me start with that, which itself is not linear algebra but something more basic. Then once I’m convinced you and I are on the same page, at that point let’s resume to linear algebra. Sounds good? I want to use some concrete examples, and you are going to see what I’m talking about. First, check this out:

**Example 1.** Needless to say,

$$2 \pm 3$$

means

$$2 + 3 \quad \text{and} \quad 2 - 3,$$

stuffed together in one place. So the expression  $2 \pm 3$  represents two numbers: 5, and  $-1$ . This example is no brainer.

**Exercise 1.** (1)  $\frac{3 \pm 11}{2}$  represents what number(s)?

(2)  $2 \pm \sqrt{3}$  represents what number(s)?

(3) Use ‘ $\pm$ ’ to confine the following two numbers within one expression: 6 and 1.

• As easy as the above may sound, we need to be very cautious when it comes to

“ double sign in the same order ”.

First and foremost, this is an ‘enforcement’. I enforce this from time to time by necessity during my lectures. And that also means you enforce this from time to time by necessity in your papers. Examples below best capture the crux of the matter:

**Example 2.** Suppose “double sign in the same order” is enforced. Then

$$1 - \left( \pm \sqrt{2} \right)$$

is simplified as

$$1 \mp \sqrt{2}.$$

Notice that I used

$$' \mp '.$$

Let's dissect:

- '±' means '+' first, '-' second.
- '∓' means '-' first, '+' second.

Repeat: Under “double sign in the same order”,

$$1 - \left( \pm \sqrt{2} \right) = 1 \mp \sqrt{2}.$$

**Example 3.** Suppose “double sign in the same order” is enforced. Then

$$\pm 7 \pm 2$$

means

$$7 + 2 \quad \text{and} \quad -7 - 2,$$

which are +9 and -9. So these are put together as

$$\pm 9.$$

Repeat: Under “double sign in the same order”,

$$\pm 7 \pm 2 = \pm 9.$$

**Example 4.** Suppose “double sign in the same order” is enforced. Then

$$\pm 13 \mp 8$$

means

$$+13 - 8 \quad \text{and} \quad -13 + 8,$$

which are  $+5$  and  $-5$ . So these are put together as

$$\pm 5.$$

Repeat: Under “double sign in the same order”,

$$\pm 13 \mp 8 = \pm 5.$$

**Example 5.** Suppose “double sign in the same order” is enforced. Then

$$\pm 8 - \left( \mp 2 \right)$$

is simplified as what? Yes, this is simplified as

$$\pm 10.$$

Indeed, negating  $\mp$  will give rise to  $\pm$ , so this is the same as  $\pm 8 \pm 2$ .

**Example 6.** Under “double sign in the same order” being enforced,

$$(a) \quad \pm 13 \pm 3\sqrt{3} = \pm \left( 13 + 3\sqrt{3} \right).$$

$$(b) \quad \pm 13 \mp 3\sqrt{3} = \pm \left( 13 - 3\sqrt{3} \right).$$

Indeed, as for (a), the two sides both represent  $+13 + 3\sqrt{3}$ , and  $-13 - 3\sqrt{3}$ , in this order. As for (b), the two sides both represent  $+13 - 3\sqrt{3}$ , and  $-13 + 3\sqrt{3}$ , in this order.

**Exercise 2.** Double sign in the same order. True or false:

$$(1) \quad 8 - \left(1 \pm \sqrt{5}\right) = 7 \pm \sqrt{5} \quad (?) \qquad (2) \quad \pm\sqrt{2} \mp 2\sqrt{2} = \pm 3\sqrt{2} \quad (?)$$

$$(3) \quad \mp 4 \pm \sqrt{6} = \mp \left(4 - \sqrt{6}\right) \quad (?)$$

**Exercise 3.** Double sign in the same order. Simplify:

$$(1) \quad 2\sqrt{2} - \left(\mp 3 + 2\sqrt{2}\right). \qquad (2) \quad \pm 3\sqrt{6} - \left(1 \pm 4\sqrt{6}\right).$$

$$(3) \quad \frac{-5 \pm \sqrt{10}}{2} + \frac{-3 \pm 3\sqrt{10}}{2}.$$

**Example 7.** Under “double sign in the same order” being enforced,

$$\begin{aligned} \frac{-4 \pm \sqrt{21}}{2} \cdot \frac{4 \pm \sqrt{21}}{2} &= \frac{\left(-4 + (\pm \sqrt{21})\right)\left(4 + (\pm \sqrt{21})\right)}{2 \cdot 2} \\ &= \frac{-4 \cdot 4 + (\pm \sqrt{21})(\pm \sqrt{21})}{4} \\ &= \frac{-4^2 + \left(\sqrt{21}\right)^2}{4} \\ &= \frac{-16 + 21}{4} = \frac{5}{4}. \end{aligned}$$

**Example 8.** Under “double sign in the same order” being enforced,

$$\begin{aligned} \frac{7 \pm 5\sqrt{2}}{2} \cdot \frac{7 \mp 5\sqrt{2}}{2} &= \frac{\left(7 + (\pm 5\sqrt{2})\right)\left(7 - (\pm 5\sqrt{2})\right)}{2 \cdot 2} \\ &= \frac{7 \cdot 7 - (\pm 5\sqrt{2})(\pm 5\sqrt{2})}{4} \\ &= \frac{7^2 - \left(5\sqrt{2}\right)^2}{4} \\ &= \frac{49 - 50}{4} = -\frac{1}{4}. \end{aligned}$$

**Exercise 4.** Double sign in the same order. Simplify:

$$(1) \quad \pm a \mp a. \quad (2) \quad \pm a \pm 2a. \quad (3) \quad \pm 4a \mp 9a.$$

$$(4) \quad \left( \pm a \right)^2. \quad (5) \quad \left( \pm a \right) \left( \mp a \right).$$

**Exercise 5.** Double sign in the same order. Simplify:

$$(1) \quad \left( 1 \pm \sqrt{2} \right) \left( 1 \mp \sqrt{2} \right). \quad (2) \quad \frac{\sqrt{6} \pm 2\sqrt{5}}{2} \cdot \frac{\sqrt{6} \mp 2\sqrt{5}}{2}.$$

$$(3) \quad \left( \pm 6 \pm \sqrt{11} \right) \left( \mp 6 \pm \sqrt{11} \right).$$

**Exercise 6.** Double sign in the same order. Expand:

$$(1) \quad a \left( c \pm d \right). \quad (2) \quad a \left( c \mp d \right).$$

$$(3) \quad \pm b \left( c \pm d \right). \quad (4) \quad \mp b \left( c \pm d \right).$$

$$(5a) \quad \left( a \pm b \right) \left( c \pm d \right). \quad (5b) \quad \left( a \pm b \right)^2.$$

$$(6a) \quad \left( a \pm b \right) \left( c \mp d \right). \quad (6b) \quad \left( a \pm b \right) \left( a \mp b \right).$$

$$(7a) \quad \left( a + b \right) \left( \pm c \pm d \right). \quad (7b) \quad \left( a + b \right) \left( \pm a \pm b \right).$$

$$(8a) \quad \left( a - b \right) \left( \pm c \pm d \right). \quad (8b) \quad \left( a - b \right) \left( \pm a \pm b \right).$$

$$(9) \quad \left( 1 \pm b \mp c \right) \left( 1 \mp b \pm c \right).$$

$$(10) \quad \left( a \pm b \pm c \pm d \right) \left( a \pm b \mp c \mp d \right) \left( a \mp b \pm c \mp d \right) \left( a \mp b \mp c \pm d \right).$$

- **More eigenvalues.**

Let's take a look:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

What about it? Let's calculate the eigenvalues of  $A$ . I almost see this coming: "Again?" True, we have already practiced a ton of this. Why not move on to the next topic? Not so fast. Let's form the characteristic polynomial of  $A$ :

$$\begin{aligned} \chi_A(\lambda) &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 3)(\lambda - 2) - (-1) \cdot (-1) \\ &= \lambda^2 - 5\lambda + 6 - 1 \\ &= \lambda^2 - 5\lambda + 5. \end{aligned}$$

So, based on this, can you find the eigenvalues of  $A$ ? Sure. This one doesn't seem to factor, though. But wait, we can always resort to the 'Quadratic Formula', right? Now, don't get freaked out, this is a good place to review that formula:

**Quadratic formula.**

The general quadratic equation is of the form

$$ax^2 + bx + c = 0 \quad (a \neq 0).$$

Here,  $x$  is the unknown, and  $a$ ,  $b$  and  $c$  are knowns. Its roots are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- Now, can you pull this formula directly from the equation? I'm sure you can. I leave it as an exercise.

**Exercise 7.** Derive “Quadratic Formula” above, directly from the equation.

- Now, back to the matrix  $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ . We were shooting for the eigenvalues of  $A$ . For that matter, as usual, we need to solve

$$(*) \quad \lambda^2 - 5\lambda + 5 = 0.$$

So  $a = 1$ ,  $b = -5$ ,  $c = 5$ . Note that the unknown is  $\lambda$ , instead of  $x$ . By Quadratic Formula,

$$\begin{aligned} \lambda &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} \\ &= \frac{5 \pm \sqrt{5}}{2}. \end{aligned}$$

So, the two eigenvalues are

$$\lambda = \frac{5 + \sqrt{5}}{2} \quad \text{and} \quad \lambda = \frac{5 - \sqrt{5}}{2}.$$

All right. So, although we thought  $\chi_A(\lambda)$  doesn't factor, it indeed factors, as

$$\begin{aligned} \chi_A(\lambda) &= \lambda^2 - 5\lambda + 5 \\ &= \left( \lambda - \frac{5 + \sqrt{5}}{2} \right) \left( \lambda - \frac{5 - \sqrt{5}}{2} \right). \end{aligned}$$

$$\left( \text{Note: You don't have to write it as } \left( \lambda - \frac{5 \pm \sqrt{5}}{2} \right) \left( \lambda - \frac{5 \mp \sqrt{5}}{2} \right). \right)$$

Next, can we find eigenvectors of  $A$ ? Sure. Here we go, let's find

$$\circ \quad \text{an eigenvector } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ of } A \text{ associated with } \lambda = \frac{5 + \sqrt{5}}{2},$$

and also

$$\circ \quad \text{an eigenvector } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ of } A \text{ associated with } \lambda = \frac{5 - \sqrt{5}}{2}.$$

Good news: We don't have to do it separately. The key is to throughout use the  
double sign '±', and keep the following intact:

“double sign in the same order”.

Since  $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ , the equation  $A\mathbf{x} = \frac{5 \pm \sqrt{5}}{2}\mathbf{x}$  is

$$(\#) \quad \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{5 \pm \sqrt{5}}{2} \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{cases} 3x + y = \frac{5 \pm \sqrt{5}}{2}x, \\ x + 2y = \frac{5 \pm \sqrt{5}}{2}y. \end{cases}$$

Shift the terms:

$$\begin{cases} \frac{1 \mp \sqrt{5}}{2}x + y = 0, \\ x + \frac{-1 \mp \sqrt{5}}{2}y = 0. \end{cases}$$



These equations are the essentially identical. You may or may not see it immediately.

But actually the second equation is obtained by just multiplying  $\frac{-1 \mp \sqrt{5}}{2}$  to the two sides of the first equation. Indeed:

$$\begin{aligned} \frac{1 \mp \sqrt{5}}{2} \cdot \frac{-1 \mp \sqrt{5}}{2} &= \frac{\left(1 + (\mp \sqrt{5})\right)\left(-1 + (\mp \sqrt{5})\right)}{2 \cdot 2} \\ &= \frac{-1^2 + \left(\mp \sqrt{5}\right)^2}{4} \\ &= \frac{-1 + 5}{4} = 1. \end{aligned}$$

So, ignore the second equation:

$$\boxed{\frac{1 \mp \sqrt{5}}{2} x + y = 0.}$$

Clearly  $x = 1$ ,  $y = \frac{-1 \pm \sqrt{5}}{2}$  works. Thus:

$$\begin{aligned} \circ \quad \mathbf{x}_{\pm} &= \begin{bmatrix} 1 \\ \frac{-1 \pm \sqrt{5}}{2} \end{bmatrix} \text{ is an eigenvector of } A \text{ associated with the eigenvalue} \\ \lambda &= \frac{5 \pm \sqrt{5}}{2}. \end{aligned}$$

**Diagonalization result.**  $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$  is diagonalized as follows:

$$Q^{-1}AQ = \begin{bmatrix} \frac{5 + \sqrt{5}}{2} & 0 \\ 0 & \frac{5 - \sqrt{5}}{2} \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} 1 & 1 \\ \frac{-1 + \sqrt{5}}{2} & \frac{-1 - \sqrt{5}}{2} \end{bmatrix}.$$

- **Non-real eigenvalues.** Let's take a look:

$$B = \begin{bmatrix} 2 & 1 \\ -8 & 7 \end{bmatrix}.$$

What about it? Let's calculate the eigenvalues of  $B$ . “*Aren't we done with that?*” I know. I hate to be the bearer of bad news, but I still need to do a certain particular kind, and this one falls into it. But actually this isn't bad at all. You will see it. Trust me. So, as usual, form the characteristic polynomial of  $B$  as a starter:

$$\begin{aligned} \chi_B(\lambda) &= \det(\lambda I - B) \\ &= \begin{vmatrix} \lambda - 2 & -1 \\ 8 & \lambda - 7 \end{vmatrix} \\ &= (\lambda - 2)(\lambda - 7) - (-1) \cdot 8 \\ &= \lambda^2 - 9\lambda + 14 + 8 \\ &= \lambda^2 - 9\lambda + 22. \end{aligned}$$

So, based on this, can you find the eigenvalues of  $B$ ? Well, just like the last one, this one doesn't seem to factor, but we can always resort to ‘Quadratic Formula’, right? Sure. So we end up having to solve

$$(\#) \quad \lambda^2 - 9\lambda + 22 = 0.$$

Throw  $a = 1$ ,  $b = -9$ ,  $c = 22$  into the Quadratic Formula:

$$\begin{aligned} \lambda &= \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \cdot 1 \cdot 22}}{2 \cdot 1} \\ &= \frac{9 \pm \sqrt{-7}}{2}. \end{aligned}$$

Uh-huh. So, the two eigenvalues of  $B$  are

$$\lambda = \frac{9 + \sqrt{-7}}{2} \quad \text{and} \quad \lambda = \frac{9 - \sqrt{-7}}{2}.$$

All right. So, we got

$$\begin{aligned} \chi_B(\lambda) &= \lambda^2 - 9\lambda + 22 \\ &= \left( \lambda - \frac{9 + \sqrt{-7}}{2} \right) \left( \lambda - \frac{9 - \sqrt{-7}}{2} \right). \end{aligned}$$

Oh, but wait... On a second look,  $-7$  is sitting inside the square-root symbol. Is that allowed?

• **Complex numbers — Thumbnail sketch.**

Sure, why not?  $\sqrt{-7}$  is an example of a complex number. You remember complex numbers from high school. Recall that a number of the form

$$(\diamond) \quad \boxed{a + \sqrt{-1} b} \quad \left( a, b : \underline{\text{real}} \text{ numbers} \right)$$

is called a complex number. In case you are second-guessing how come something like  $\sqrt{-7}$  and  $(\diamond)$  are cut from the same cloth, check this out:

$$\sqrt{-7} = 0 + \sqrt{-1} \cdot \sqrt{7}.$$

In a similar vein,

$$\frac{9 + \sqrt{-7}}{2}, \quad \text{and} \quad \frac{9 - \sqrt{-7}}{2},$$

are complex numbers, because they are rewritten as

$$\frac{9}{2} + \sqrt{-1} \cdot \frac{\sqrt{7}}{2}, \quad \text{and} \quad \frac{9}{2} - \sqrt{-1} \cdot \frac{\sqrt{7}}{2},$$

respectively.

- Now, here is my point:

“ The two complex numbers

$$\boxed{\lambda = \frac{9 + \sqrt{-7}}{2}} \quad \text{and} \quad \boxed{\lambda = \frac{9 - \sqrt{-7}}{2}}$$

are the legit roots of the equation

(#)

$$\boxed{\lambda^2 - 9\lambda + 22 = 0.}$$

Nothing stops us from viewing

$$\boxed{\lambda = \frac{9 + \sqrt{-7}}{2}} \quad \text{and} \quad \boxed{\lambda = \frac{9 - \sqrt{-7}}{2}}$$

as the legit eigenvalues of the matrix

$$B = \begin{bmatrix} 2 & 1 \\ -8 & 7 \end{bmatrix}. \quad ”$$

— Uh-huh. All right.

So, can I ask you a favor? Can you please just chew and swallow the above, as in take my word for it? I know, I should better give you a crash course on complex numbers. As far as that department goes, the above is just the bare minimum. But it actually suffices for the time being (for the rest of today’s lecture). The truth is, there are aspects of complex numbers you guys might not have learned yet and will ultimately have bearings on the eigenvalue business. I plan to address those as necessary.

So for now I want to go ahead and proceed with finding the eigenvectors of  $B$ . Here we go:

- an eigenvector  $\mathbf{x}_{\pm} = \begin{bmatrix} x \\ y \end{bmatrix}$  of  $A$  associated with  $\lambda = \frac{9 \pm \sqrt{-7}}{2}$ .

Once again: We don't have to do it separately. The key is to throughout use the double sign '±', and keep the following intact:

“double sign in the same order”.

Since  $B = \begin{bmatrix} 2 & 1 \\ -8 & 7 \end{bmatrix}$ , the equation  $B\mathbf{x} = \frac{9 \pm \sqrt{-7}}{2}\mathbf{x}$  is

$$(\textcircled{a}) \quad \begin{bmatrix} 2 & 1 \\ -8 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{9 \pm \sqrt{-7}}{2} \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{cases} 2x + y = \frac{9 \pm \sqrt{-7}}{2}x, \\ -8x + 7y = \frac{9 \pm \sqrt{-7}}{2}y. \end{cases}$$

Shift the terms:

$$\begin{cases} \frac{-5 \mp \sqrt{-7}}{2}x + y = 0, \\ -8x + \frac{5 \mp \sqrt{-7}}{2}y = 0. \end{cases}$$

These equations are the essentially identical. Indeed, the second equation is obtained by just multiplying  $\frac{5 \mp \sqrt{-7}}{2}$  to the two sides of the first equation:

$$\begin{aligned} \frac{-5 \mp \sqrt{-7}}{2} \cdot \frac{5 \mp \sqrt{-7}}{2} &= \frac{\left(-5 + (\mp \sqrt{-7})\right)\left(5 + (\mp \sqrt{-7})\right)}{2 \cdot 2} \\ &= \frac{-5^2 + \left(\mp \sqrt{-7}\right)^2}{4} \\ &= \frac{-25 - 7}{4} = -8. \end{aligned}$$

So, ignore the second equation:

$$\frac{-5 \mp \sqrt{-7}}{2} x + y = 0.$$

Clearly  $x = 1$ ,  $y = \frac{5 \pm \sqrt{-7}}{2}$  works. Thus:

$$\circ \quad \mathbf{x}_{\pm} = \begin{bmatrix} 1 \\ \frac{5 \pm \sqrt{-7}}{2} \end{bmatrix} \text{ is an eigenvector of } B \text{ associated with the eigenvalue}$$

$$\lambda = \frac{9 \pm \sqrt{-7}}{2}.$$

**Diagonalization result.**  $B = \begin{bmatrix} 2 & 1 \\ -8 & 7 \end{bmatrix}$  is diagonalized as follows:

$$Q^{-1}BQ = \begin{bmatrix} \frac{9 + \sqrt{-7}}{2} & 0 \\ 0 & \frac{9 - \sqrt{-7}}{2} \end{bmatrix}, \text{ where } Q = \begin{bmatrix} 1 & 1 \\ \frac{5 + \sqrt{-7}}{2} & \frac{5 - \sqrt{-7}}{2} \end{bmatrix}.$$

**Exercise 8.** Find the characteristic polynomial, the eigenvalues, and then eigenvectors associated with each of the eigenvalues. Then diagonalize the matrix.

$$(1) \quad A = \begin{bmatrix} 3 & 3 \\ 4 & 1 \end{bmatrix}. \quad (2) \quad A = \begin{bmatrix} 1 & -1 \\ 13 & 6 \end{bmatrix}.$$

$$(3) \quad A = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & 2\sqrt{3} \end{bmatrix}. \quad (4) \quad A = \begin{bmatrix} \frac{1+\sqrt{-3}}{2} & \frac{3-\sqrt{-3}}{2} \\ \frac{3-\sqrt{-3}}{2} & \frac{1+\sqrt{-3}}{2} \end{bmatrix}.$$

$$(5) \quad A = \begin{bmatrix} -\alpha^5 - 1 & -\alpha^2 + \frac{1 - \sqrt{-7}}{2\alpha^2} \\ -\alpha^2 + \frac{1 - \sqrt{-7}}{2\alpha^2} & \frac{3 - \sqrt{-7}}{2} \end{bmatrix}, \text{ where}$$

$$\alpha = \cos \frac{2\pi}{7} + \sqrt{-1} \sin \frac{2\pi}{7}. \text{ Note: } \alpha^7 = 1, \quad \alpha^4 + \alpha^2 + \alpha = \frac{-1 + \sqrt{-7}}{2}.$$

