**MA 02 LINEAR ALGEBRA II**

**REVIEW OF LECTURES – IX**

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**Section:** C7.

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• **Rotations, reflections.**

Today I want to start with one specific type of a matrix. What do the following matrices have in common?

$$\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix},$$

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{4} & -\frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}, \qquad \begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & -\frac{\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{\sqrt{2+\sqrt{2}}}{2} \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

• As for the first five, check this out:

$$\cos\frac{\pi}{3} = \frac{1}{2}, \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \qquad \sin\frac{\pi}{6} = \frac{1}{2},$$

$$\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}, \qquad \sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4},$$

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}, \qquad \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

So

$$\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\dfrac{\pi}{3} & -\sin\dfrac{\pi}{3} \\ \sin\dfrac{\pi}{3} & \cos\dfrac{\pi}{3} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos\dfrac{\pi}{4} & -\sin\dfrac{\pi}{4} \\ \sin\dfrac{\pi}{4} & \cos\dfrac{\pi}{4} \end{bmatrix},$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \cos\dfrac{\pi}{6} & -\sin\dfrac{\pi}{6} \\ \sin\dfrac{\pi}{6} & \cos\dfrac{\pi}{6} \end{bmatrix},$$

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix} = \begin{bmatrix} \cos\dfrac{2\pi}{5} & -\sin\dfrac{2\pi}{5} \\ \sin\dfrac{2\pi}{5} & \cos\dfrac{2\pi}{5} \end{bmatrix},$$

$$\begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & \frac{-\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{\sqrt{2+\sqrt{2}}}{2} \end{bmatrix} = \begin{bmatrix} \cos\dfrac{\pi}{8} & -\sin\dfrac{\pi}{8} \\ \sin\dfrac{\pi}{8} & \cos\dfrac{\pi}{8} \end{bmatrix}.$$

- As for the last three,

$$\begin{aligned} \cos\ 0 &= 1, & \sin\ 0 &= 0, \\ \cos\ \frac{\pi}{2} &= 0, & \sin\ \frac{\pi}{2} &= 1, \\ \cos\ \pi &= -1, & \sin\ \pi &= 0. \end{aligned}$$

So

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\dfrac{\pi}{2} & -\sin\dfrac{\pi}{2} \\ \sin\dfrac{\pi}{2} & \cos\dfrac{\pi}{2} \end{bmatrix},$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix}.$$

In short, the matrices listed on page 1 all fall into

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Let's calculate the characteristic polynomial, and the eigenvalues.

$$\begin{aligned} \chi_A(\lambda) &= \det\Big(\lambda I - A\Big) \\ &= \begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix} \\ &= \Big(\lambda - \cos\theta\Big)\Big(\lambda - \cos\theta\Big) - \Big(\sin\theta\Big)\Big(-\sin\theta\Big) \\ &= \Big(\lambda - \cos\theta\Big)^2 + \Big(\sin\theta\Big)^2 \\ &= \lambda^2 - 2\Big(\cos\theta\Big)\lambda + \Big(\cos\theta\Big)^2 + \Big(\sin\theta\Big)^2. \end{aligned}$$

Here, recall

**Formula.** $$\boxed{\Big(\cos\theta\Big)^2 + \Big(\sin\theta\Big)^2 = 1.}$$

We utilize this and conclude

$$\chi_A(\lambda) = \lambda^2 - 2\Big(\cos\theta\Big)\lambda + 1.$$

**Summary.** $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has the following characteristic polynomial:

$$\boxed{\chi_A(\lambda) = \lambda^2 - 2\Big(\cos\theta\Big)\lambda + 1.}$$

- Can you find the eigenvalue(s) of $A$? Sure. What should we do? Yes, rely on

**Quadratic formula.** The equation

$$\boxed{a\,x^2 \;+\; b\,x \;+\; c \;=\; 0 \qquad \Big(a \neq 0\Big)}$$

$\Big(x:$ _unknown_, $a$, $b$, $c$: _knowns_$\Big)$ are solved as

$$\boxed{x \;=\; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}$$

The equation to solve is

$$\lambda^2 \;-\; 2\Big(\cos\theta\Big)\lambda \;+\; 1 \;=\; 0.$$

So, $a \;=\; 1,\;\; b \;=\; -2\Big(\cos\theta\Big),\;\; c \;=\; 1.$ Accordingly:

$$\begin{aligned}
\lambda \;&=\; \frac{-\Big(-\,2\Big(\cos\theta\Big)\Big) \;\pm\; \sqrt{\Big(-\,2\Big(\cos\theta\Big)\Big)^2 \;-\; 4\cdot 1\cdot 1}}{2\cdot 1} \\
&=\; \frac{2\Big(\cos\theta\Big) \;\pm\; \sqrt{4\Big(\cos\theta\Big)^2 \;-\; 4}}{2} \\
&=\; \frac{2\Big(\cos\theta\Big) \;\pm\; 2\sqrt{\Big(\cos\theta\Big)^2 \;-\; 1}}{2} \\
&=\; \frac{2\Big(\cos\theta\Big) \;\pm\; 2\sqrt{-\Big(\sin\theta\Big)^2}}{2} \qquad\qquad \Big(\text{by Formula}\Big) \\
&=\; \Big(\cos\theta\Big) \;\pm\; \sqrt{-\Big(\sin\theta\Big)^2}\;.
\end{aligned}$$

So, how should we handle $\pm\sqrt{-\left(\sin\theta\right)^2}$ ? Can we simplify it? Yes, there is a dichotomy:

(i) If $\sin\theta$ is 0, then $\pm\sqrt{-\left(\sin\theta\right)^2} = 0$.

(ii) If $\sin\theta$ is non-zero, then $\left(\sin\theta\right)^2$ is positive, thus $-\left(\sin\theta\right)^2$ is negative. Then $\pm\sqrt{-\left(\sin\theta\right)^2}$ is 'non-real'. Write it as $\pm\sqrt{-1}\sin\theta$.

Here, suppose $\sin\theta = 0$ (namely, suppose we are in case (i)). Then $\pm\sqrt{-\left(\sin\theta\right)^2}$ still equals $\pm\sqrt{-1}\sin\theta$, indeed, both equal 0. So in both cases (i) and (ii) we end up getting

$$\pm\sqrt{-\left(\sin\theta\right)^2} = \pm\sqrt{-1}\sin\theta.$$

**Summary 2.** $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has the following eigenvalues:

$$\lambda = \left(\cos\theta\right) \pm \sqrt{-1}\left(\sin\theta\right).$$

• Next, let's find the eigenvectors of $A$, associated with the above eigenvalues. In what follows, the double sign in the same order.

Since $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, the equation $A\boldsymbol{x} = \left(\left(\cos\theta\right) \pm \sqrt{-1}\left(\sin\theta\right)\right)\boldsymbol{x}$ is

$$(\#) \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \left(\left(\cos\theta\right) \pm \sqrt{-1}\left(\sin\theta\right)\right)\begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{cases} \Big(\cos\theta\Big)x - \Big(\sin\theta\Big)y &= \bigg(\Big(\cos\theta\Big) \pm \sqrt{-1}\Big(\sin\theta\Big)\bigg)x, \\ \Big(\sin\theta\Big)x + \Big(\cos\theta\Big)y &= \bigg(\Big(\cos\theta\Big) \pm \sqrt{-1}\Big(\sin\theta\Big)\bigg)y. \end{cases}$$

Shift the terms:

$$\begin{cases} \mp\sqrt{-1}\Big(\sin\theta\Big)x \qquad - \Big(\sin\theta\Big)y &= 0, \\ \Big(\sin\theta\Big)x \mp \sqrt{-1}\Big(\cos\theta\Big)y &= 0. \end{cases}$$

These equations are the essentially identical. You may or may not see it immediately. But actually the second equation is obtained by just multiplying $\pm\sqrt{-1}$ to the two sides of the first equation. Indeed:

$$\begin{aligned} &\Big(\pm\sqrt{-1}\Big)\Big(\mp\sqrt{-1}\Big(\sin\theta\Big)x - \Big(\sin\theta\Big)y\Big) \\ &\qquad = \Big(\pm\sqrt{-1}\Big)\Big(\mp\sqrt{-1}\Big)\Big(\sin\theta\Big)x - \Big(\pm\sqrt{-1}\Big)\Big(\sin\theta\Big)y \\ &\qquad = -\Big(\sqrt{-1}\Big)^2\Big(\sin\theta\Big)x \mp \sqrt{-1}\Big(\sin\theta\Big)y \\ &\qquad = -\Big(-1\Big)\Big(\sin\theta\Big)x \mp \sqrt{-1}\Big(\sin\theta\Big)y \\ &\qquad = \Big(\sin\theta\Big)x \mp \sqrt{-1}\Big(\sin\theta\Big)y. \end{aligned}$$

So, ignore the second equation:

$$\mp\sqrt{-1}\Big(\sin\theta\Big)x - \Big(\sin\theta\Big)y = 0.$$

Assume $\sin\theta \neq 0$, and divide the two sides by $\sin\theta$:

$$\boxed{\mp\sqrt{-1}\,x - y = 0.}$$

Clearly $x = \sqrt{-1}\,,\ \ y = \pm 1$ works. Thus:

- $\boldsymbol{x}_{\pm} = \begin{bmatrix} \sqrt{-1} \\ \pm 1 \end{bmatrix}$ is an eigenvector of $A$ associated with the eigenvalue

$$\lambda = \Big(\cos\theta\Big) \pm \sqrt{-1}\Big(\sin\theta\Big).$$

**Diagonalization result.** $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is diagonalized as follows:

$$Q^{-1}AQ = \begin{bmatrix} \Big(\cos\theta\Big) + \sqrt{-1}\Big(\sin\theta\Big) & 0 \\ 0 & \Big(\cos\theta\Big) - \sqrt{-1}\Big(\sin\theta\Big) \end{bmatrix},$$

$$\text{where} \quad Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1} \\ 1 & -1 \end{bmatrix}.$$

**Example 1.** We can diagonalize

$$A = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix},$$

as follows: As we have already observed:

$$\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}.$$

So the eigenvalues of $A$ are

$$\Big(\cos\frac{\pi}{3}\Big) \pm \sqrt{-1}\Big(\sin\frac{\pi}{3}\Big) = \frac{1}{2} \pm \sqrt{-1}\,\frac{\sqrt{3}}{2}$$
$$= \frac{1 \pm \sqrt{-3}}{2}.$$

Accordingly:

$$Q^{-1}AQ \;=\; \begin{bmatrix} \dfrac{1+\sqrt{-3}}{2} & 0 \\ 0 & \dfrac{1-\sqrt{-3}}{2} \end{bmatrix}, \qquad \text{where} \qquad Q \;=\; \begin{bmatrix} \sqrt{-1} & \sqrt{-1} \\ 1 & -1 \end{bmatrix}.$$

**Example 2.** We can diagonalize

$$A \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

as follows: As we have already observed:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\dfrac{\pi}{2} & -\sin\dfrac{\pi}{2} \\ \sin\dfrac{\pi}{2} & \cos\dfrac{\pi}{2} \end{bmatrix}.$$

So the eigenvalues of $A$ are

$$\begin{aligned} \left(\cos\frac{\pi}{2}\right) \pm \sqrt{-1}\left(\sin\frac{\pi}{2}\right) \;&=\; 0 \pm \sqrt{-1}\cdot 1 \\ &=\; \pm\sqrt{-1}\,. \end{aligned}$$

Accordingly

$$Q^{-1}AQ \;=\; \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}, \qquad \text{where} \qquad Q \;=\; \begin{bmatrix} \sqrt{-1} & \sqrt{-1} \\ 1 & -1 \end{bmatrix}.$$

**Exercise 1.** Diagonalize each of

(1) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, (2) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$,

(3) $\begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}$, (4) $\begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & \frac{-\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{\sqrt{2+\sqrt{2}}}{2} \end{bmatrix}$.

**Example 3.** Let's take a look at

$$A = \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

Actually this $A$ still falls into

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The underlying reason behind it is,

$$3^2 + 4^2 = 5^2.$$

So

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1.$$

In calculus, we have learned the following fundamental fact:

**Fact.** Let $a$ and $b$ be real numbers. Suppose they satisfy

$$a^2 + b^2 = 1.$$

Then there exists a real number $\theta$ such that

$$a = \cos\theta, \qquad b = \sin\theta.$$

The above falls precisely into the case $a = \dfrac{3}{5}$ and $b = \dfrac{4}{5}$. The problem is, for these $a$, $b$, you cannot express $\theta$ concretely as a concrete number times $\pi$. Indeed, $\theta$ has at least three expressions:

$$\theta = \arccos\frac{3}{5} = \arcsin\frac{4}{5} = \arctan\frac{4}{3}.$$

However, none of these three can be simplifed any further. The good news is, nevertheless, we can find the eigenvalues of $A$, and also diagonalize $A$, following the above method.

Let's perform: The eigenvalues of $A = \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ are

$$\left(\cos\theta\right) \pm \sqrt{-1}\left(\sin\theta\right) = \frac{3}{5} \pm \sqrt{-1}\cdot\frac{4}{5} = \frac{3 \pm \sqrt{-1}\cdot 4}{5}.$$

We may diagonalize $A$ as

$$Q^{-1}AQ = \begin{bmatrix} \dfrac{3+\sqrt{-1}\cdot 4}{5} & 0 \\ 0 & \dfrac{3-\sqrt{-1}\cdot 4}{5} \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} \sqrt{-1} & \sqrt{-1} \\ 1 & -1 \end{bmatrix}.$$

**Remark.** In the above, we've seen

$$3^2 + 4^2 = 5^2.$$

There is a way to produce infinitely many triplets $\left(a,\ b,\ c\right)$ where $a$, $b$ and $c$ are all integers, satisfying

$$a^2 + b^2 = c^2.$$

Indeed, just substitute integers $k$ into

$$\begin{cases} a = k^2 - 1, \\ b = 2k, \\ c = k^2 + 1. \end{cases}$$

Substitute $k = 2$ into the above and you'll end up getting $a = 3$, $b = 4$, $c = 5$.

**Exercise 2.** Substitute $k = 3, 4, 5, 6$ each, into the above to produce triplets $\left(a,\ b,\ c\right)$ satisfying $a^2 + b^2 = c^2$. Then write out the matrix $\begin{bmatrix} \frac{a}{c} & \frac{-b}{c} \\ \frac{b}{c} & \frac{a}{c} \end{bmatrix}$ in each of the four cases. Find the eigenvalues. Diagonalize.

- **Kissing cousins.**

  Here is today's second theme:

$$B \;=\; \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

What about it? We have just dealt with it, or what? On a second look, this is actually different from what we have just seen:

$$A \;=\; \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Yes, $B$ is different from $A$. Even if you change $\theta$ suitably, you cannot convert $A$ into $B$. So, in other words, $A$ and $B$ are different animals, though they alwfully resemble each other. Since we have alanyzed $A$, let's also analyze $B$. So, eigenvalues and diagonalizations of $B$. Here we go:

$$\begin{aligned}
\chi_B\left(\lambda\right) \;&=\; \det\left(\lambda I - B\right) \\
&=\; \begin{vmatrix} \lambda - \cos\theta & -\sin\theta \\ -\sin\theta & \lambda + \cos\theta \end{vmatrix} \\
&=\; \left(\lambda - \cos\theta\right)\left(\lambda + \cos\theta\right) - \left(-\sin\theta\right)\left(-\sin\theta\right) \\
&=\; \lambda^2 - \left(\left(\cos\theta\right)^2 + \left(\sin\theta\right)^2\right) \\
&=\; \lambda^2 - 1 \\
&=\; \left(\lambda - 1\right)\left(\lambda + 1\right).
\end{aligned}$$

In the above, we have used

**Formula.** $$\boxed{\left(\cos\theta\right)^2 + \left(\sin\theta\right)^2 \;=\; 1.}$$

**Summary.** $B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ has the following characteristic polynomial:

$$\boxed{\chi_B\left(\lambda\right) = \Big(\lambda - 1\Big)\Big(\lambda + 1\Big).}$$

$B$ has the following eigenvalues:

$$\lambda = \pm 1.$$

• Next, let's find the eigenvectors of $B$, associated with the above eigenvalues. In what follows, the double sign in the same order.

Since $B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, the equation $B\boldsymbol{x} = \pm\boldsymbol{x}$ is

(#)
$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \pm \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{cases} \Big(\cos\theta\Big)x + \Big(\sin\theta\Big)y = \pm x, \\ \Big(\sin\theta\Big)x - \Big(\cos\theta\Big)y = \pm y. \end{cases}$$

Shift the terms:

$$\begin{cases} \Big(\Big(\cos\theta\Big) \mp 1\Big)x + \Big(\sin\theta\Big)y = 0, \\ \Big(\sin\theta\Big)x + \Big(\Big(-\cos\theta\Big) \mp 1\Big)y = 0. \end{cases}$$

These equations are the essentially identical. You may or may not see it immediately.

But actually the second equation is obtained by just multiplying $\dfrac{\left(-\cos\theta\right)\mp 1}{\sin\theta}$ to the two sides of the first equation $\big($where $\sin\theta \neq 0$ is assumed$\big)$. Indeed:

$$\begin{aligned}
&\frac{\left(-\cos\theta\right)\mp 1}{\sin\theta}\left[\left(\left(\cos\theta\right)\mp 1\right)x + \left(\sin\theta\right)y\right] \\
&= \frac{1}{\sin\theta}\cdot\left[\left(\left(-\cos\theta\right)\mp 1\right)\left(\left(\cos\theta\right)\mp 1\right)x + \left(\left(-\cos\theta\right)\mp 1\right)\left(\sin\theta\right)y\right] \\
&= \frac{1}{\sin\theta}\cdot\left[\left(-\left(\cos\theta\right)^2 + \left(\mp 1\right)^2\right)x + \left(\left(-\cos\theta\right)\mp 1\right)\left(\sin\theta\right)y\right] \\
&= \frac{1}{\sin\theta}\cdot\left[\left(1-\left(\cos\theta\right)^2\right)x + \left(\left(-\cos\theta\right)\mp 1\right)\left(\sin\theta\right)y\right] \\
&= \frac{1}{\sin\theta}\cdot\left[\left(\sin\theta\right)^2 x + \left(\left(-\cos\theta\right)\mp 1\right)\left(\sin\theta\right)y\right] \\
&= \left(\sin\theta\right)x + \left(\left(-\cos\theta\right)\mp 1\right)y.
\end{aligned}$$

So, ignore the second equation:

$$\boxed{\left(\left(\cos\theta\right)\mp 1\right)x + \left(\sin\theta\right)y \;=\; 0.}$$

Clearly $x = \sin\theta$, $y = -\left(\cos\theta\right)\pm 1$ works. Thus:

- $\boldsymbol{x}_\pm = \begin{bmatrix} \sin\theta \\ -\left(\cos\theta\right)\pm 1 \end{bmatrix}$ is an eigenvector of $B$ associated with

$$\lambda \;=\; \pm 1.$$

**Diagonalization result.** $B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ is diagonalized as follows:

$$Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} \sin\theta & \sin\theta \\ -\Big(\cos\theta\Big)+1 & -\Big(\cos\theta\Big)-1 \end{bmatrix}.$$

• Now agree that all of the following fall into the $B$-type:

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}, \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}, \qquad \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix},$$

$$\begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{1-\sqrt{5}}{4} \end{bmatrix}, \qquad \begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & \frac{\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{-\sqrt{2+\sqrt{2}}}{2} \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Exercise 3.** Diagonalize each of the above eight matrices.

**Exercise 4.** Do the same as Exercise 2, but with $\begin{bmatrix} \frac{a}{c} & \frac{b}{c} \\ \frac{b}{c} & \frac{-a}{c} \end{bmatrix}$ instead.

**Definition (Orthogonal matrices).** Matrices of the forms

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},$$

$\big(\theta:$ a real number$\big)$ are called orthogonal matrices.

• We are going to explore the properties of orthogonal matrices in the next lecture. Also, I plan to introduce symmetric matrices, and explain how they and orthogonal matrices have bearings of each other in the context of diagonalizability.