Linear Algebra II Recitation 4

Yasuyuki Kachi & Shunji Moriya

1. Double sign in the same order. Simplify:

   (1) $\pm a \mp a$ (2) $\pm a \pm 2a$

   (3) $(1 \pm \sqrt{2})(1 \mp \sqrt{2})$ (4) $(\pm 6 \pm \sqrt{11})(\mp 6 \pm \sqrt{11})$

   (5) $(a \pm b)^2$

2. Find the characteristic polynomial, the eigenvalues, and then eigenvectors associated with each of the eigenvalues. Then diagonalize the matrix.

   (1) $A = \begin{bmatrix} 3 & 3 \\ 4 & 1 \end{bmatrix}$ (2) $A = \begin{bmatrix} 1 & -1 \\ 13 & 6 \end{bmatrix}$

3. Let A be a 2×2 matrix. Suppose A has two distinct eigenvalues λ_1, λ_2. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be eigenvectors of A associated with λ_1 and λ_2, respectively.

   (1) Let c_1, c_2 be two numbers. Calculate $A(c_1\boldsymbol{x} + c_2\boldsymbol{y})$ and express the result using $\lambda_1, \lambda_2, c_1, c_2, \boldsymbol{x}, \boldsymbol{y}$ (and without using A).

   (2) Prove that $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent. In other words, prove the implication : $c_1\boldsymbol{x} + c_2\boldsymbol{y} = \mathbf{0} \ \Rightarrow \ c_1 = c_2 = 0$.

   (Hint : by multiplying the equation by A, we have $A(c_1\boldsymbol{x}+c_2\boldsymbol{y}) = \mathbf{0}$. Then, determine c_1 and c_2 by solving the simultaneous equation $\begin{cases} c_1\boldsymbol{x} + c_2\boldsymbol{y} = \mathbf{0} \\ A(c_1\boldsymbol{x} + c_2\boldsymbol{y}) = \mathbf{0} \end{cases}$.)

   (3) Prove that the inverse of the 2×2 matrix $Q = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix}$ exists.

   (Hint : you can use (2) and the conditions on the invertibility of matrices which you learned in Linear Algebra I)

   The claim of (3) implies that any 2×2 matrix which has two distinct eigenvalues is diagonalizable.

Recitation 4, Answer

1. (1) 0 (2) $\pm 3a$
   (3) -1 (4) -25
   (5) $a^2 \pm 2ab + b^2$

2. (1) $\chi_A(\lambda) = \lambda^2 - 4\lambda - 9$. $\lambda = 2 \pm \sqrt{13}$. $\boldsymbol{x} = \begin{bmatrix} 3 \\ -1 \pm \sqrt{13} \end{bmatrix}$.

   $$Q^{-1}AQ = \begin{bmatrix} 2+\sqrt{13} & 0 \\ 0 & 2-\sqrt{13} \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} 3 & 3 \\ -1+\sqrt{13} & -1-\sqrt{13} \end{bmatrix}.$$

   (2) $\chi_A(\lambda) = \lambda^2 - 7\lambda + 19$. $\lambda = \frac{7 \pm 3\sqrt{-3}}{2}$. $\boldsymbol{x} = \begin{bmatrix} 1 \\ \frac{-5 \mp 3\sqrt{-3}}{2} \end{bmatrix}$.

   $$Q^{-1}AQ = \begin{bmatrix} \frac{7+3\sqrt{-3}}{2} & 0 \\ 0 & \frac{7-3\sqrt{-3}}{2} \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} 1 & 1 \\ \frac{-5-3\sqrt{-3}}{2} & \frac{-5+3\sqrt{-3}}{2} \end{bmatrix}.$$

3. (1) $c_1\lambda_1\boldsymbol{x} + c_2\lambda_2\boldsymbol{y}$.
   (2) Omitted.
   (3) Omitted

Comments on your answer to Recitation 4

- For (2) of problem no.2, some of you wrote "Since the eigenvalues are complex numbers, A is not diagonalizable".
  —This is wrong. A 2×2 matrix which has two distinct eigenvalues is always diagonalizable (even if they are complex numbers).

- For the same problem, please write complex numbers as $a + bi$ (a, b real numbers) in your result. For example, $\frac{1}{1+2i}$ is calculated as follows.

$$\frac{1}{1+2i} = \frac{1}{1+2i} \times \frac{1-2i}{1-2i} = \frac{1-2i}{1^2-(2i)^2} = \frac{1-2i}{5}$$

- For (2) of problem no.3, some of you wrote "As $\boldsymbol{x}$ and $\boldsymbol{y}$ are associated with two distinct eigenvectors, $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent".
  —What the problem requires is nothing but the proof of this fact, so the proof falls into circular reasoning (becomes meaningless) if you cite the fact.