Linear Algebra II Recitation 3

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1. Let A, B be 2×2 matrices. <u>True or false</u> :

   (1a) $(A+I)^2 = A^2 + 2A + I$. (1b) $(A+B)^2 = A^2 + 2AB + B^2$.

   (1c) Suppose $AB = BA$. Then $(A+B)^2 = A^2 + 2AB + B^2$.

   (2) Suppose $\det A \neq 0$. Then $(ABA^{-1})^3 = AB^3A^{-1}$.

2. Diagonalize $A = \begin{bmatrix} 8 & 1 \\ 8 & 6 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$.

   (1) Diagonalize A. In other words, find a 2×2 matrix Q such that $Q^{-1}AQ$ is diagonal (and find $Q^{-1}AQ$ for your matrix Q).

   (2) Put $B = Q^{-1}AQ$ for the matrix Q you obtained in (1).
   Prove $A = QBQ^{-1}$.

   (Hint: multiply the defining equation of B by Q from the left and by Q^{-1} from the right.)

   (3) For any integer $n \geq 0$, find the n-th power A^n using (2).

   (Hint: like 1-(2), we see $A^n = QB^nQ^{-1}$, and B^n is easy to compute since B is diagonal.)

4. Triangularize each of the following matrix into a Jordan canonical form. If not feasible, then say 'not feasible'.

   (1) $A = \begin{bmatrix} 14 & -27 \\ 3 & -4 \end{bmatrix}$ (2) $A = \begin{bmatrix} 12 & 9 \\ -16 & -12 \end{bmatrix}$

5. Let $A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$. Diagonalize A, and using the result, find A^n.

Recitation 3, Answer

1. (1a) True. (1b) False (1c) True (2) True.

2. $Q^{-1}AQ = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ where $Q = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix}$.

   Alternatively, $PAP^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ where $P = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 4 & 1 \end{bmatrix}$

3. (1) $Q^{-1}AQ = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$, where $Q = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

   (2) Omitted.

   (3) $A^n = \begin{bmatrix} 2 \cdot 3^n - (-2)^n & 2 \cdot (-2)^n - 2 \cdot 3^n \\ 3^n - (-2)^n & 2 \cdot (-2)^n - 3^n \end{bmatrix}$

4. (1) $Q^{-1}AQ = \begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix}$, where $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & 3\sqrt{3} \\ 0 & \sqrt{3} \end{bmatrix}$.

   (2) $Q^{-1}AQ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, where $Q = \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}$.

5. $Q^{-1}AQ = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, where $Q = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$. $A^n = \frac{1}{2}\begin{bmatrix} 3 - 3^n & 3^n - 1 \\ 3 - 3^{n+1} & 3^{n+1} - 1 \end{bmatrix}$