Linear Algebra II   Recitation 1

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1. Perform each of the following multiplications:

(1) $\begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ -1 & 0 \end{bmatrix}$

(3) AB, where $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(4) AB, where $A = B = \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}$

2. Perform the multiplication in each of (1) and (2). In each of (1) and (2), detect one eigenvalue of A.

(1) $A\boldsymbol{x}$, $A = \begin{bmatrix} 1 & 2 \\ -6 & 8 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(2) $A\boldsymbol{x}$, $A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} a & b \\ b & 2a \end{bmatrix}$. Assume that 2 is an eigenvalue of A and that the vector $\boldsymbol{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector of A, associated with the eigenvalue 2. Determine the two numbers a and b.

4. (a) Perform the following multiplication:

$$AB, \qquad \text{where} \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 6 \\ 10 & 9 \end{bmatrix}.$$

(b) Does the answer for (a) coincide with either A or B? (If 'yes', then indicate which one is.)

(c) Is A or B in (a) the identity matrix? (If 'yes', then indicate which one is.)

5. (a) Perform the following multiplication:

$$PQ, \qquad \text{where} \qquad P = \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix}, \qquad Q = \begin{bmatrix} -5 & -3 \\ -3 & -2 \end{bmatrix}.$$

(b) Does the answer for (a) coincide with the identity matrix?

(c) Knowing the answer for (b), would you say QP is also the identity matrix?

(d) Find Q^{-1}, as well as P^{-1}.

6. Find the inverse P^{-1} of P, if exists:

(1) $P = \begin{bmatrix} 2 & 5 \\ 4 & -2 \end{bmatrix}$ (2) $P = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$

7. Let

$$P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}, \qquad Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

(1a) Perform the matrix multiplication PAQ.
(1b) Is your answer in (1a) a diagonal matrix?
(2a) Perform the matrix multiplication PQ.
(2b) Is your answer in (2a) the identity matrix?
(3) Can you conclude that PAP^{-1} is a diagonal matrix?
(4) Can you conclude that A is diagonalizable?

8. Write each of the following in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(1) $3\begin{bmatrix} -4 & 2 \\ 6 & 5 \end{bmatrix}$ (2) $-\begin{bmatrix} -6 & -8 \\ 3 & 4 \end{bmatrix}$

9. Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. Assume that A has the property that for any 2×2 matrix B, the equation $AB = BA$ holds. Prove that there is a scalar s such that $A = sI$.

10. Calculate:

(1) $\begin{vmatrix} 1 & 6 \\ 1 & 3 \end{vmatrix}$ (2) $\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$

11. (1a) Calculate the characteristic polynomial of $A = \begin{bmatrix} -3 & -2 \\ 6 & 5 \end{bmatrix}$.

(1b) Find the eigenvalues of A in (1a) (if any).

(2a) Calculate the characteristic polynomial of $B = \begin{bmatrix} -2 & -4 \\ 1 & -6 \end{bmatrix}$.

(2b) Find the eigenvalues of B in (2a) (if any).

Linear Algebra II Recitation 1 Answer

1. (1) $\begin{bmatrix} 2 & 3 \\ 18 & 15 \end{bmatrix}$

   (2) $\begin{bmatrix} 5 & 7 \\ -20 & -28 \end{bmatrix}$

   (3) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (4) $\begin{bmatrix} \frac{-1-\sqrt{5}}{4} & \frac{-\sqrt{10-2\sqrt{5}}}{4} \\ \frac{\sqrt{10-2\sqrt{5}}}{4} & \frac{-1-\sqrt{5}}{4} \end{bmatrix}$

2. (1) $\begin{bmatrix} 8 \\ 12 \end{bmatrix} = 4\boldsymbol{x}$. 4 is an eigenvalue of A.

   (2) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\boldsymbol{x}$. 1 is an eigenvalue of A.

3. $a = \frac{6}{7}$, $b = \frac{-4}{7}$.

4. (a) $\begin{bmatrix} 2 & 6 \\ 10 & 9 \end{bmatrix}$.

   (b) Yes. The answer for (a) coincides with B.

   (c) Yes. $A = I$ is the identity matrix.

5. (a) $PQ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (b) Yes, the answer for (a) coincides with I.

   (c) Yes, QP also equals I.

   (d) $Q^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix}$ $(= P)$, $P^{-1} = \begin{bmatrix} -5 & -3 \\ -3 & -2 \end{bmatrix}$ $(= Q)$.

6. (1) $\begin{bmatrix} \frac{1}{12} & \frac{5}{24} \\ \frac{1}{6} & \frac{-1}{12} \end{bmatrix}$ (2) Does not exist.

7. (1a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

   (1b) Yes, the answer for (1a) is a diagonal matrix.

   (2a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (2b) Yes, the answer for (2a) is the identity matrix.

   (3) Yes, PAP^{-1} is a diagonal matrix. (Indeed, by (2a-b), $P^{-1} = Q$.)

   (4) Yes, A is diagonalizable.

8. (1) $\begin{bmatrix} -12 & 6 \\ 18 & 15 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & 8 \\ -3 & -4 \end{bmatrix}$

9. Omitted.

10. (1) -3 (2) 5

11. (1a) $\lambda^2 - 2\lambda - 3$.
(1b) $-1, \quad 3$.
(2a) $\lambda^2 + 8\lambda + 16$.
(2b) -4.