MA 02 LINEAR ALGEBRA II

REVIEW OF LECTURES – VIII SUPPLEMENT

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Section: C7.
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This is a supplement to "Review of Lectures – VIII", a crash course on polynomials. I'm sure you are familiar with polynomials. As elementary as it seems, dealing with polynomials requires care. What's included in this supplement is a bare minimum, so I expect you to be proficient with these.

Polynomials are made of monomials. So you have to know what monomials are. Spellings:

"polynomial," "monomial."

First, we call each of

$$1, \quad x, \quad x^2, \quad x^3, \quad x^4, \quad x^5, \quad \cdots,$$

a monomial in x. Here, we include 1 in the list because $1 = x^0$. Next, we also call each of

$$3, \quad -2x, \quad 4x^2, \quad \sqrt{2}\,x^3, \quad \frac{1}{4}x^4, \quad 7x^5, \quad \cdots,$$

a monomial in x.

More generally, suppose a is a constant real number, and n is an integer with $n \geq 0$. Then we call

$$a\,x^n$$

a monomial in x.

Note that neither of

$$\boxed{x^{-1}} \qquad \text{nor} \qquad \boxed{x^{\frac{1}{2}}}$$

is a monomial, because the exponent is either negative or a non-integer.

Next, a <u>polynomial</u> is a finite sum of monomials. Thus

$$1 + x, \quad 2x + x^2, \quad -\frac{4}{5} + \sqrt{3}\,x^3, \quad x + x^4 + x^7, \quad -6 + 2x - 8x^3 + x^5,$$

are examples of polynomials in x. On the other hand, none of

$$\sqrt{x}, \quad 1 + x^{\frac{3}{2}}, \quad \frac{1}{x}, \quad \sqrt{2 + x^2}, \quad \frac{3}{4 - x}, \quad 2x^{-2} + x^2$$

is a polynomial in x.

• **Note.** There is no general rule that we must obey when it comes to the order of terms in polynomial expressions. So, you may write either

$$2 + x, \qquad \text{or} \qquad x + 2.$$

These two are one and the same. Similarly, you may write either

$$4 - 3x + x^2, \qquad -3x + x^2 + 4, \qquad x^2 + 4 - 3x,$$

$$4 + x^2 - 3x, \qquad x^2 - 3x + 4, \qquad \text{or} \qquad -3x + 4 + x^2.$$

These six are all one and the same. Usually, though, we prefer to write polynomials either in the <u>ascending order</u> or in the <u>descending order</u> of exponents. So,

$$4 - 3x + x^2 \qquad \Big(\text{ascending order}\Big),$$

and

$$x^2 - 3x + 4 \qquad \Big(\text{descending order}\Big),$$

are equally preferable.

Exercise 1. Permute the order of terms, if necessary, to make each of the given polynomials in the ascending order.

(1) $2x + x^4 - \dfrac{1}{2}x^2.$ (2) $-\dfrac{4}{3}x^3 - 5x^2 - 4x^5.$

(3) $x^8 + 5x^6 - 10x^9 + 3.$ (4) $x + \sqrt{5}\,x^5 + \sqrt{3}\,x^3 - \sqrt{2}\,x^2.$

(5) $x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$

[**Answers**]:

(1) $2x - \dfrac{1}{2}x^2 + x^4.$ (2) $-5x^2 - \dfrac{4}{3}x^3 - 4x^5.$

(3) $3 + 5x^6 + x^8 - 10x^9.$ (4) $x - \sqrt{2}\,x^2 + \sqrt{3}\,x^3 + \sqrt{5}\,x^5.$

(5) $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9.$

Exercise 2. Permute the order of terms, if necessary, to make each of the given polynomials in the descending order.

(1) $6x^2 + x^3.$ (2) $\dfrac{1}{2}x^3 + \dfrac{1}{3}x^4 - x.$

(3) $x^7 + 5x^5 - 10x^8 + 4x^6.$ (4) $1 - x^2 + x^4 - x^6 + x^8 - x^{10}.$

[**Answers**]:

(1) $x^3 + 6x^2.$ (2) $\dfrac{1}{3}x^4 + \dfrac{1}{2}x^3 - x.$

(3) $-10x^8 + x^7 + 4x^6 + 5x^5.$ (4) $-x^{10} + x^8 - x^6 + x^4 - x^2 + 1.$

- **Polynomial addition.**

We may add two polynomials and the outcome is a polynomial. Let me use some examples to illustrate it.

Example 1. Let's calculate

$$\Big(x + 2\Big) + \Big(x^3 + 4x\Big).$$

This is just uncover the parentesis and reorder the terms $\big($in either the ascending or descending order$\big)$. So,

$$\begin{aligned}\Big(x^2 + 2\Big) + \Big(x^3 + 4x\Big) &= x^2 + 2 + x^3 + 4x \\ &= x^3 + x^2 + 4x + 2.\end{aligned}$$

$\big($This answer is clearly written in the descending order. You can write it in the ascending order instead. You don't have to give both.$\big)$

Example 2. Let's calculate

$$\left(x^4 + \frac{1}{5}x^2\right) + \Big(2x^4 - 6x^3 + x\Big).$$

This one looks pretty similar to Example 1 above, but this one involves more than what was required in Example 1. Indeed, let's just first uncover the parenthesis, and reorder terms:

$$\begin{aligned}&\left(x^4 + \frac{1}{5}x^2\right) + \Big(2x^4 - 6x^3 + x\Big) \\ &= x^4 + \frac{1}{5}x^2 + 2x^4 - 6x^3 + x \\ &= x^4 + 2x^4 - 6x^3 + \frac{1}{5}x^2 + x.\end{aligned}$$

Realize that there are two monomials that involve x^4. You have to combine them. x^4 and $2x^4$ make $3x^4$. So the above is simplified as

$$3x^4 - 6x^3 + \frac{1}{5}x^2 + x.$$

This is the final answer.

$\star$ If you like, you can do the above as follows:

$$\begin{array}{rcccccccc}
 & x^4 & & & + & \frac{1}{5}x^2 & & \\
 & 2x^4 & - & 6x^3 & & & + & x \\
+) & \hline
 & 3x^4 & - & 6x^3 & + & \frac{1}{5}x^2 & + & x.
\end{array}$$

$\star$ Let's do a similar example, but in a different format.

Example 3. Let's find $f(x) + g(x)$, where

$$f(x) = x^4 + \frac{7}{2}x^3 - \frac{5}{2}x^2 - x, \qquad \text{and}$$

$$g(x) = 6x^4 + \frac{1}{2}x^3 - x^2 - 3x + 2.$$

This is essentially the same type of a problem as the previous ones, but the difference is that two polynomials are given names as $f(x)$ and $g(x)$. Still, the same method works. Here is how it goes:

$f(x) + g(x)$

$$= \left(x^4 + \frac{7}{2}x^3 - \frac{5}{2}x^2 - x\right) + \left(6x^4 + \frac{1}{2}x^3 - x^2 - 3x + 2\right)$$

$$= x^4 + \frac{7}{2}x^3 - \frac{5}{2}x^2 - x + 6x^4 + \frac{1}{2}x^3 - x^2 - 3x + 2$$

$$\left(\text{uncovered parentheses}\right)$$

$$= x^4 + 6x^4 + \frac{7}{2}x^3 + \frac{1}{2}x^3 - \frac{5}{2}x^2 - x^2 - x - 3x + 2$$

$$\left(\text{re-ordered terms}\right)$$

$$= \left(x^4 + 6x^4\right) + \left(\frac{7}{2}x^3 + \frac{1}{2}x^3\right) + \left(-\frac{5}{2}x^2 - x^2\right) + \left(-x - 3x\right) + 2$$

$$= 7x^4 + 4x^3 + \left(-\frac{7}{2}x^2\right) + \left(-4x\right) + 2$$

$$= 7x^4 + 4x^3 - \frac{7}{2}x^2 - 4x + 2.$$

⋆ If you like, you can do the above as follows:

$$\begin{array}{rcccccccccc}
 & x^4 & + & \frac{7}{2}x^3 & - & \frac{5}{2}x^2 & - & x & & \\
 & 6x^4 & + & \frac{1}{2}x^3 & - & x^2 & - & 3x & + & 2 \\
+) & \hline
 & 7x^4 & + & 4\,x^3 & - & \frac{7}{2}x^2 & - & 4x & + & 2
\end{array}$$

Exercise 3. Do

(1) $\Big(x^7 + 3\,x^5 + 2\,x^3\Big) + \Big(-\,x^6 - x^4 - 2\,x^2\Big).$

(2) $\Big(x^4 + 9\,x^3 + 1\Big) + \Big(-\,x^4 - x^3 - 5\,x^2 + 2x + 3\Big).$

(3) $\left(\frac{1}{2}x^3 + \frac{1}{3}x\right) + \left(\frac{1}{3}x^3 - \frac{1}{4}x\right).$

(4) $f\,(x) + g\,(x),$ where

$$\begin{aligned} f\,(x) &= x^6 + 8x^5 + 12x^4 + 36x^3 + 9x^2, \\ g\,(x) &= x^8 - 3x^6 - 8x^4 - 24x^3 + 45x - 120. \end{aligned}$$

(5) $f\,(x) + g\,(x),$ where

$$\begin{aligned} f\,(x) &= x^7 + x^5 + x^3 + x, \\ g\,(x) &= x^8 + x^6 + x^4 + x^2 + 1. \end{aligned}$$

$\Big[\,$**Answers**$\,\Big]$**:**

(1) $x^7 - x^6 + 3x^5 - x^4 + 2x^3 - 2\,x^2.$ (2) $8\,x^3 - 5x^2 + 2x + 4.$

(3) $\frac{5}{6}\,x^3 + \frac{1}{12}\,x.$

(4) $x^8 - 2x^6 + 8x^5 + 4x^4 + 12x^3 + 9x^2 + 45x - 120.$

(5) $x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$

- **Polynomial subtraction.**

We may subtract one polynomial from another polynomial. The outcome is a polynomial. This is very similar to polynomial additions. Let's do some example.

Example 4. Let's do

$$\Big(x^3 + 2x^2\Big) - \Big(3x + 4\Big).$$

When you uncover the parenteses, you have to be careful. Namely:

$$\Big(x^3 + 2x^2\Big) - \Big(3x + 4\Big) = x^3 + 2x^2 - 3x - 4.$$

In the above, notice that all the terms inside the second parenthesis got negated after uncovering that parenthesis. That's the correct way to do it.

Example 5. Let's do

$$\Big(x^6 - 8x^4 + 3\,x^2\Big) - \Big(2x^6 - 3x^4\Big).$$

Once again, when you uncover the parenteses,

$$\begin{aligned}
&\Big(x^6 - 8x^4 + 3\,x^2\Big) - \Big(2x^6 - 3x^4\Big) \\
&\qquad = x^6 - 8x^4 + 3x^2 - 2x^6 + 3x^4 \\
&\qquad \Big(\text{all the terms in the second parenthesis got negated}\Big) \\
&\qquad = x^6 - 2x^6 - 8x^4 + 3x^4 + 3x^2 \\
&\qquad \Big(\text{re-ordered terms}\Big) \\
&\qquad = \Big(x^6 - 2x^6\Big) + \Big(-8x^4 + 3x^4\Big) + 3x^2 \\
&\qquad = -x^6 - 5x^4 + 3x^2.
\end{aligned}$$

$\star$ If you like, you can do it like

$$\begin{array}{rrrrrr} & x^6 & - & 8x^4 & + & 3x^2 \\ & 2x^6 & - & 3x^4 & & \\ -) & \multicolumn{5}{l}{\rule{6cm}{0.4pt}} \\ & -x^6 & - & 5x^4 & + & 3x^2 \end{array}$$

Exercise 4. Do

(1) $\left(x^3 + 11\,x^2 + 21\,x\right) - \left(-\,x^2 - x + 4\right)$.

(2) $\left(-\,x^7 + 5x^6 + x^3 - 6\right) - \left(-\,2x^6 - 3x^4 + 7\,x^3 + 2x + 5\right)$.

(3) $\left(\dfrac{3}{2}\,x^4 + \dfrac{7}{4}\,x^2\right) - \left(\dfrac{1}{6}\,x^4 - \dfrac{1}{4}\,x^2 + 1\right)$.

(4) $f\left(x\right) - g\left(x\right)$, where

$$\begin{aligned} f\left(x\right) &= x^5 + 4x^4 + 16x^3 + 22x^2 + 18x, \\ g\left(x\right) &= x^6 - 31x^4 - 62x^2 + 72x + 56. \end{aligned}$$

(5) $f\left(x\right) - g\left(x\right)$, where

$$f\left(x\right) = x^{13} + x^9 + x^5 + x, \qquad g\left(x\right) = x^{11} + x^7 + x^3.$$

[**Answers**]:

(1) $x^3 + 12x^2 + 22x - 4$. (2) $-x^7 + 7x^6 + 3x^4 - 6x^3 - 2x - 11$.

(3) $\dfrac{4}{3}x^4 + 2x^2 - 1$. (4) $-x^6 + x^5 + 35x^4 + 16x^3 + 84x^2 - 54x - 56$.

(5) $x^{13} - x^{11} + x^9 - x^7 + x^5 - x^3 + x$.

- **Constsant multiplication.**

  We may multiply a constant with a polynomial.

Example 6. Let's do

$$10\left(2\,x^3 + 3x^2 + 4\,x + 5\right).$$

This is simply multiply 10 to each of the terms. So

$$10\left(2\,x^3 + 3x^2 + 4\,x + 5\right) = 20\,x^3 + 30x^2 + 40\,x + 50.$$

Example 7. Let's do

$$-2\left(12\,x^5 - 21x^3 + 48\,x\right).$$

This is simply multiply -2 to each of the terms. So

$$-2\left(12\,x^5 - 21x^3 + 48\,x\right) = -24\,x^5 + 42x^3 - 96\,x.$$

Example 8. Sometimes you see something like

$$\frac{2x^4 - 11x^2 + 14x - 9}{2}.$$

This is the same as

$$\frac{1}{2}\left(2x^4 - 11x^2 + 14x - 9\right).$$

The answer is, of course,

$$x^4 - \frac{11}{2}x^2 + 7x - \frac{9}{2}.$$

Exercise 5. Simplify:

(1) $6\left(x^7 + 7\,x^6 + 21\,x^5\right)$. (2) $-4\left(-\,x^2 + 5x + 3\right)$.

(3) $\dfrac{8x^{10} - 20x^8 + 24x^6 - 12x^4}{4}$. (4) $\dfrac{1}{3}\left(\dfrac{3}{5}x^4 + \dfrac{3}{7}x^3 + \dfrac{3}{25}x^2 + \dfrac{3}{65}x\right)$.

(5) $3\,f\left(x\right)$ where $f\left(x\right) = x^5 + 4x^4 + 16x^3 + 22x^2 + 18x$.

$\left[\,\underline{\textbf{Answers}}\,\right]$: (1) $6x^7 + 42x^6 + 126x^5$. (2) $4x^2 - 20x - 12$.

(3) $2x^{10} - 5x^8 + 6x^6 - 3x^4$. (4) $\dfrac{1}{5}x^4 + \dfrac{1}{7}x^3 + \dfrac{1}{25}x^2 + \dfrac{1}{65}x$.

(5) $3x^5 + 12x^4 + 48x^3 + 66x^2 + 54x$.

- **A monomial multipled to a polynomial.**

More generally, we may multiply a monomial with a polynomial. The outcome is a polynomial. This process is called an <u>expansion</u>.

Example 9. Let's expand

$$x\left(x^4 + 3x^3 + 5x^2\right).$$

This is simply raise the exponent in each x-to-the-power inside the parenthesis by 1. So

$$x\left(x^4 + 3x^3 + 5x^2\right) = x^5 + 3x^4 + 5x^3.$$

Example 10. Let's expand

$$x^2\left(8x^3 + 24x^2 + 6x + 3\right).$$

This time you raise the exponent in each x-to-the-power inside the parenthesis by 2. So

$$x^2\left(8x^3 + 24x^2 + 6x + 3\right) = 8x^5 + 24x^4 + 6x^3 + 3x^2.$$

Example 11. Let's expand

$$2x\left(x^2 + 8x + 5\right).$$

This requires you to do two things at once, namely, (i) raise the exponent in each x-to-the-power inside the parenthesis by 1, and then (ii) multiply 2 to each of the terms. So

$$2x\left(x^2 + 8x + 5\right) = 2x^3 + 16x^2 + 10x.$$

Example 12. Let's expand

$$5x^4\left(6x^5 + 12x^3 + 8x^2 + 7\right).$$

This is similar, do two things at once: namely, (i) raise the exponent in each x-to-the-power inside the parenthesis by 4, and then (ii) multiply 5 to each of the terms. So

$$5x^4\left(6x^5 + 12x^3 + 8x^2 + 7\right) = 30x^9 + 60x^7 + 40x^6 + 35x^4.$$

Exercise 6. Expand

(1) $x\left(x^2 - x + 4\right)$. (2) $x^3\left(2x^4 + 8x^3 + 5x\right)$.

(3) $6x\left(-\dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 - \dfrac{1}{6}x\right)$.

(4) $\dfrac{1}{4}x^2\left(x^4 + 2x^3 + 3x^2 + 2x + 1\right)$.

[**Answers**]:

(1) $x^3 - x^2 + 4x$. (2) $2x^7 + 8x^6 + 5x^4$.

(3) $-2x^4 + 3x^3 - x^2$. (4) $\dfrac{1}{4}x^6 + \dfrac{1}{2}x^5 + \dfrac{3}{4}x^4 + \dfrac{1}{2}x^3 + \dfrac{1}{4}x^2$.

- **Multiplying out two polynomials.**

Even more generally, we may multiply out two polymials. The outcome is a polynomial. This process is once again called an expansion .

Example 13. Let's expand

$$\Big(x + 1\Big)\Big(x^2 + 1\Big).$$

It goes as follows.

$$\Big(\boxed{x} + \boxed{1}\Big) \quad \Big(\boxed{x^2 + 1}\Big)$$

multiply

multiply

$$= \; x\,\Big(x^2 + 1\Big) \;+\; 1\,\Big(x^2 + 1\Big)$$

$$= \Big(x^3 + x\Big) \;+\; \Big(x^2 + 1\Big)$$

$$= x^3 + x + x^2 + 1$$

$$= x^3 + x^2 + x + 1.$$

In short,

$$\Big(x + 1\Big)\Big(x^2 + 1\Big) \;=\; x^3 + x^2 + x + 1.$$

$\star$ As you can see, the above consists of two multiplications, both "monomial times polynomial" type, followed by one polynomial addition.

Example 14. Let's expand

$$\Big(2x^2 + 3\Big)\Big(x^2 - 3x + 4\Big).$$

It goes as follows.

$$\Big(\boxed{2x} + \boxed{3}\Big)\ \Big(\boxed{x^2 - 3x + 4}\Big)$$

multiply

multiply

$$= 2x\ \Big(x^2 - 3x + 4\Big) \quad + \quad 3\ \Big(x^2 - 3x + 4\Big)$$

$$= \Big(2x^3 - 6x^2 + 8x\Big) \quad + \quad \Big(3x^2 - 9x + 12\Big)$$

$$= 2x^3 - 6x^2 + 8x + 3x^2 - 9x + 12$$

$$\Big(\text{uncovered parentheses}\Big)$$

$$= 2x^3 - 6x^2 + 3x^2 + 8x - 9x + 12$$

$$\Big(\text{re-ordered terms}\Big)$$

$$= 2x^3 - 3x^2 - x + 12.$$

In short,

$$\Big(2x + 3\Big)\Big(x^2 - 3x + 4\Big) \;=\; 2x^3 - 3x^2 - x + 12.$$

⋆ Once again, the procedure consists of two multiplications, both "monomial times polynomial" type, followed by one polynomial addition. The following recaptures the same algorithm:

$$
\begin{array}{rrrrrrrr}
 & & & x^2 & - & 3\,x & + & 4 \\
 & & & & & 2\,x & + & 3 \\
\times) & & & & & & & \\
\hline
 & & & 3\,x^2 & - & 9\,x & + & 12 \\
2\,x^3 & - & & 6\,x^2 & + & 8\,x & & \\
\hline
2\,x^3 & - & & 3\,x^2 & - & x & + & 12
\end{array}
$$

Let's dissect. Take a look at the following which is a mere duplication of the above, but with some highlighted boxes:

$$
\begin{array}{rrrrrrrr}
 & & & \boxed{x^2 \quad - \quad 3\,x \quad + \quad 4} \\
 & & & 2\,x \quad \boxed{+ \quad 3} \\
\times) & & & \\
\hline
 & & & \boxed{3\,x^2 \quad - \quad 9\,x \quad + \quad 12} \\
2\,x^3 & - & & 6\,x^2 \quad + \quad 8\,x \\
\hline
2\,x^3 & - & & 3\,x^2 \quad - \quad x \quad + \quad 12
\end{array}
$$

Ignore the parts that are not enclosed by the boxes. Only look at those highlighted by the boxes. This is just constant 3 $\big($which is a monomial$\big)$ multiplied to the polynomial $x^2 - 3x + 4$. The outcome is highlighted in the lowest box. Next, look at the following, I have simply changed the locations of the boxes:

$$
\begin{array}{rr}
 & \boxed{x^2 - 3x + 4} \\
 & \boxed{2x} + 3 \\
\times) & \\
\hline
 & 3x^2 - 9x + 12 \\
 & \boxed{2x^3 - 6x^2 + 8x} \\
\hline
 & 2x^3 - 3x^2 - x + 12
\end{array}
$$

Again, ignore everything else but only look at the highlighted parts. This is just monomial $2x$ multiplied to the polynomial $x^2 - 3x + 4$. The outcome is highlighted in the lowest box. Finally, just one more time I'm going to change the locations of the boxes:

$$
\begin{array}{rr}
 & x^2 - 3x + 4 \\
 & 2x + 3 \\
\times) & \\
\hline
 & \boxed{3x^2 - 9x + 12} \\
 & \boxed{2x^3 - 6x^2 + 8x} \\
\hline
 & \boxed{2x^3 - 3x^2 - x + 12}
\end{array}
$$

Only look at the last three lines as highlighted. This is just polynomial additions. The outcome is highlighted in the lowest line in the box. This is exactly the final outcome.

Let's do another example.

Example 15. Let's expand

$$\Big(-\,2x^2\,+\,5x\,-\,3\Big)\Big(x^2\,+\,2x\,-\,7\Big).$$

It goes as follows.

$$\Big(\ \boxed{-\,2x^2}\ \ \boxed{+\,5x}\ \ \boxed{-\,3}\ \Big)\ \Big(\ \boxed{x^2\,+\,2x\,-\,7}\ \Big)$$

multiply

multiply

multiply

$$= \;-2x^2\,\Big(x^2+2x-7\Big) \;+\; 5x\,\Big(x^2+2x-7\Big) \;-\; 3\,\Big(x^2+2x-7\Big)$$

$$= \;\Big(-\,2\,x^4-4x^3+14\,x^2\Big) \;+\; \Big(5\,x^3+10x^2-35\,x\Big) \;+\; \Big(-\,3x^2-6x+21\Big)$$

$$= \;-\,2\,x^4\,-\,4x^3\,+\,14\,x^2\,+\,5\,x^3\,+\,10x^2\,-\,35\,x\,-\,3x^2\,-\,6x\,+\,21$$

(uncovered parentheses)

$$= \;-\,2x^4\,-\,4x^3\,+\,5x^3\,+\,14x^2\,+\,10x^2\,-\,3x^2\,-\,35x\,-\,6x\,+\,21$$

(re-ordered terms)

$$= \;-\,2x^4\,+\,x^3\,+\,21x^2-\,41x\,+\,21.$$

In short,

$$\Big(-\,2x^2\,+\,5x\,-\,3\Big)\Big(x^2\,+\,2x\,-\,7\Big) \;=\; -\,2x^4\,+\,x^3\,+\,21x^2-\,41x\,+\,21.$$

⋆ Just like the previous example, we can do it the following way:

$$
\begin{array}{rrrrrrrrrr}
 & & & & & x^2 & + & 2\,x & - & 7 \\
 & & & & - & 2\,x^2 & + & 5\,x & - & 3 \\
\hline
 & & & & - & 3\,x^2 & - & 6\,x & + & 21 \\
 & & 5\,x^3 & + & 10 & x^2 & - & 35\,x & & \\
-2\,x^4 & - & 4\,x^3 & + & 14 & x^2 & & & & \\
\hline
-2\,x^4 & + & x^3 & + & 21 & x^2 & - & 41\,x & + & 21
\end{array}
$$

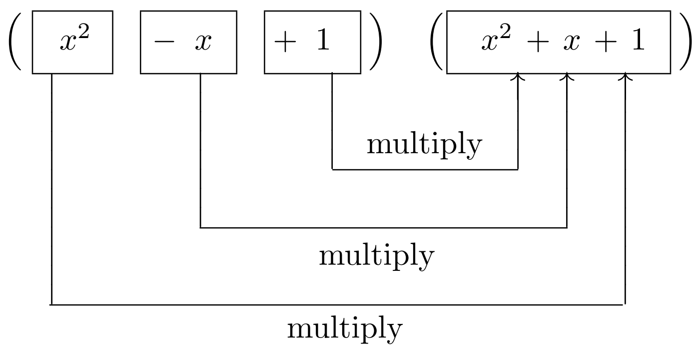
$\times)$

You can do it either way, whichever way suits you better. Let's do more examples.

Example 16. Let's expand

$$\Big(x^2 - x + 1\Big)\Big(x^2 + x + 1\Big).$$

It goes as follows.



$$
\begin{aligned}
&= x^2 \Big(x^2 + x + 1\Big) \quad - \quad x \Big(x^2 + x + 1\Big) \quad + \quad 1 \Big(x^2 + x + 1\Big) \\
&= \Big(x^4 + x^3 + x^2\Big) \quad + \quad \Big(- x^3 - x^2 - x\Big) \quad + \quad \Big(x^2 + x + 1\Big) \\
&= x^4 + x^3 + x^2 - x^3 - x^2 - x + x^2 + x + 1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big(\text{uncovered parentheses}\Big) \\
&= x^4 + x^3 - x^3 + x^2 - x^2 + x^2 - x + x + 1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big(\text{re-ordered terms}\Big) \\
&= x^4 + x^2 + 1.
\end{aligned}
$$

In short,

$$
\Big(x^2 - x + 1\Big)\Big(x^2 + x + 1\Big) \;=\; x^4 + x^2 + 1.
$$

$\star$ The following is an alternative way:

$$
\begin{array}{rccccccccc}
 & & & & & x^2 & + & x & + & 1 \\
 & & & & & x^2 & - & x & + & 1 \\
\times) & \hline
 & & & & & x^2 & + & x & + & 1 \\
 & & - & x^3 & - & x^2 & - & x & & \\
 & x^4 & + & x^3 & + & x^2 & & & & \\
\hline
 & x^4 & & & + & x^2 & & & + & 1
\end{array}
$$

- Let's do

$$(0)\qquad \Big(x - 1\Big) \cdot 1,$$

$$(1)\qquad \Big(x - 1\Big)\Big(x + 1\Big),$$

$$(2)\qquad \Big(x - 1\Big)\Big(x^2 + x + 1\Big),$$

$$(3)\qquad \Big(x - 1\Big)\Big(x^3 + x^2 + x + 1\Big),$$

$$(4)\qquad \Big(x - 1\Big)\Big(x^4 + x^3 + x^2 + x + 1\Big),$$

$$(5)\qquad \Big(x - 1\Big)\Big(x^5 + x^4 + x^3 + x^2 + x + 1\Big),$$

$$(6)\qquad \Big(x - 1\Big)\Big(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\Big),$$

$$\vdots$$

Let's try (5). Let's do it the second way:

$$\begin{array}{rcccccccccccc}
 & & x^5 & + & x^4 & + & x^3 & + & x^2 & + & x & + & 1 \\
 & & & & & & & & & & x & - & 1 \\
\times) & & & & & & & & & & & & \\
\hline
 & & -x^5 & - & x^4 & - & x^3 & - & x^2 & - & x & - & 1 \\
x^6 & + & x^5 & + & x^4 & + & x^3 & + & x^2 & + & x & & \\
\hline
x^6 & & & & & & & & & & & - & 1
\end{array}$$

In short,

$$(5)\qquad \Big(x - 1\Big)\Big(x^5 + x^4 + x^3 + x^2 + x + 1\Big) = x^6 - 1.$$

By extrapolation:

$$
\begin{array}{lll}
(0) & \Big(x - 1\Big) \cdot 1 & = x - 1, \\
(1) & \Big(x - 1\Big)\Big(x + 1\Big) & = x^2 - 1, \\
(2) & \Big(x - 1\Big)\Big(x^2 + x + 1\Big) & = x^3 - 1, \\
(3) & \Big(x - 1\Big)\Big(x^3 + x^2 + x + 1\Big) & = x^4 - 1, \\
(4) & \Big(x - 1\Big)\Big(x^4 + x^3 + x^2 + x + 1\Big) & = x^5 - 1, \\
(5) & \Big(x - 1\Big)\Big(x^5 + x^4 + x^3 + x^2 + x + 1\Big) & = x^6 - 1, \\
(6) & \Big(x - 1\Big)\Big(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\Big) & = x^7 - 1, \\
 & \vdots &
\end{array}
$$

⋆ A minor tweak:

$$
\begin{array}{lll}
(0) & \Big(1 - x\Big) \cdot 1 & = 1 - x, \\
(1) & \Big(1 - x\Big)\Big(1 + x\Big) & = 1 - x^2, \\
(2) & \Big(1 - x\Big)\Big(1 + x + x^2\Big) & = 1 - x^3, \\
(3) & \Big(1 - x\Big)\Big(1 + x + x^2 + x^3\Big) & = 1 - x^4, \\
(4) & \Big(1 - x\Big)\Big(1 + x + x^2 + x^3 + x^4\Big) & = 1 - x^5, \\
(5) & \Big(1 - x\Big)\Big(1 + x + x^2 + x^3 + x^4 + x^5\Big) & = 1 - x^6, \\
(6) & \Big(1 - x\Big)\Big(1 + x + x^2 + x^3 + x^4 + x^5 + x^6\Big) & = 1 - x^7, \\
 & \vdots &
\end{array}
$$

⋆ Further tweak:

$$
\begin{array}{lll}
(0) & \Big(1 + x\Big) \cdot 1 & = \ 1 + x, \\
(1) & \Big(1 + x\Big)\Big(1 - x\Big) & = \ 1 - x^2, \\
(2) & \Big(1 + x\Big)\Big(1 - x + x^2\Big) & = \ 1 + x^3, \\
(3) & \Big(1 + x\Big)\Big(1 - x + x^2 - x^3\Big) & = \ 1 - x^4, \\
(4) & \Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4\Big) & = \ 1 + x^5, \\
(5) & \Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4 - x^5\Big) & = \ 1 - x^6, \\
(6) & \Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4 - x^5 + x^6\Big) & = \ 1 + x^7,
\end{array}
$$

$$\vdots$$

Exercise 7. Expand

(1) $\Big(x + 3\Big)\Big(x + 4\Big)$. (2) $\Big(x + 5\Big)\Big(x - 2\Big)$.

(3) $\Big(x^2 - 7\Big)\Big(x^2 - 3x + 1\Big)$. (4) $\Big(x^3 - 4x^2 + 2\Big)\Big(x^2 + 3x - 8\Big)$.

[**Answers**]**:**

(1) $x^2 + 7x + 12$. (2) $x^2 + 3x - 10$.

(3) $x^4 - 3x^3 - 6x^2 + 21x - 7$. (4) $x^5 - x^4 - 20x^3 + 34\,x^2 + 6x - 16$.

Exercise 8. Expand

(1) $$\Big(x - 1\Big)\Big(x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\Big).$$

(2) $$\Big(x - 1\Big)\Big(x^{40} + x^{39} + x^{38} + x^{37} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} + x^{30} + x^{29} + x^{28} + x^{27} + x^{26} + x^{25} + x^{24} + x^{23} + x^{22} + x^{21} + x^{20} + x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\Big).$$

(3) $$\Big(1 - x\Big)\Big(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + x^{20} + x^{21} + x^{22} + x^{23}\Big).$$

(4) $$\Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9 + x^{10} - x^{11} + x^{12} - x^{13} + x^{14}\Big).$$

(5) $$\Big(1 + x\Big)\Big(1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9 + x^{10} - x^{11} + x^{12} - x^{13} + x^{14} - x^{15} + x^{16} - x^{17} + x^{18} - x^{19} + x^{20} - x^{21} + x^{22} - x^{23} + x^{24} - x^{25} + x^{26} - x^{27} + x^{28} - x^{29} + x^{30} - x^{31} + x^{32} - x^{33} + x^{34} - x^{35} + x^{36} - x^{37} + x^{38} - x^{39} + x^{40} - x^{41} + x^{42} - x^{43} + x^{44} - x^{45} + x^{46} - x^{47}\Big).$$

[**Answers**]:

(1) $x^{21} - 1.$ (2) $x^{41} - 1.$

(3) $1 - x^{24}.$ (4) $1 + x^{15}.$

(5) $1 - x^{48}.$

- **Squaring Polynomials.**

Next, squaring polynomials. For a polynomial $f(x)$, $f(x)^2$ means $f(x) \cdot f(x)$.

Example 17. Let's expand

$$\left(x^2 + 3\right)^2.$$

This is the same as $\left(x^2 + 3\right)\left(x^2 + 3\right)$. So

$$\begin{aligned}\left(x^2 + 3\right)\left(x^2 + 3\right) &= x^2\left(x^2 + 3\right) + 3\left(x^2 + 3\right) \\ &= x^4 + 3x^2 + 3x^2 + 9 \\ &= x^4 + 6x^2 + 9.\end{aligned}$$

Now, some of you might say "why not use the binomial formula?". That's excellent point. Yes, binomial formula is

$$\boxed{\left(a + b\right)^2 = a^2 + 2ab + b^2.}$$

Substitute a with x^2, and substitute b with 3:

$$\begin{aligned}\left(x^2 + 3\right)^2 &= \left(x^2\right)^2 + 2 \cdot x^2 \cdot 3 + 3^2 \\ &= x^4 + 6x^2 + 9.\end{aligned}$$

Example 18. Let's expand

$$\left(x^2 + x + 2\right)^2.$$

This time you have to do it like $\left(x^2 + x + 2\right)\left(x^2 + x + 2\right)$. It goes as follows:

$$\begin{aligned}
&\left(x^2+x+2\right)\left(x^2+x+2\right)\\
&\qquad = x^2\left(x^2+x+2\right)+x\left(x^2+x+2\right)+2\left(x^2+x+2\right)\\
&\qquad = \left(x^4+x^3+2x^2\right)+\left(x^3+x^2+2x\right)+\left(2x^2+2x+4\right)\\
&\qquad = x^4+x^3+2x^2+x^3+x^2+2x+2x^2+2x+4\\
&\qquad = x^4+x^3+x^3+2x^2+x^2+2x^2+2x+2x+4\\
&\qquad = x^4+2x^3+5x^2+4x+4.
\end{aligned}$$

You can certainly do it this way:

$$\begin{array}{rcccccccccc}
 & & & & & x^2 & + & x & + & 2 \\
 & & & & & x^2 & + & x & + & 2 \\
\times) & & & & & & & & & \\
\hline
 & & & & & 2\,x^2 & + & 2\,x & + & 4 \\
 & & & x^3 & + & x^2 & + & 2\,x & & \\
 & x^4 & + & x^3 & + & 2\,x^2 & & & & \\
\hline
 & x^4 & + & 2\,x^3 & + & 5\,x^2 & + & 4\,x & + & 4
\end{array}$$

To conclude,

$$\left(x^2+x+2\right)^2 = x^4+2x^3+5x^2+4x+4.$$

Example 19. Let's expand

$$\left(x^3+4x^2-3x+2\right)^2.$$

Let's just do it in the second way.

$$
\begin{array}{rrrrrrrrr}
 & & & x^3 & +\ 4\,x^2 & -\ 3\,x & +\ 2 \\
 & & & x^3 & +\ 4\,x^2 & -\ 3\,x & +\ 2 \\
\hline
 & & & 2\,x^3 & +\ 8\,x^2 & -\ 6\,x & +\ 4 \\
 & & -\ 3\,x^4 & -\ 12\,x^3 & +\ 9\,x^2 & -\ 6\,x & \\
 & 4\,x^5 & +\ 16\,x^4 & -\ 12\,x^3 & +\ 8\,x^2 & & \\
x^6 & +\ 4\,x^5 & -\ 3\,x^4 & +\ 2\,x^3 & & & \\
\hline
x^6 & +\ 8\,x^5 & +\ 10\,x^4 & -\ 20\,x^3 & +\ 25\,x^2 & -\ 12\,x & +\ 4
\end{array}
$$

To conclude,

$$\Big(x^3 + 4x^2 - 3x + 2\Big)^2 \;=\; x^6 + 8x^5 + 10x^4 - 20x^3 + 25x^2 - 12x + 4.$$

Example 20. Let's expand $\Big(x^4 - 5x^3 - 6\,x + 4\Big)^2.$

As before, we can handle it like

$$
\begin{array}{rrrrrrrrr}
 & & & & x^4 & -\ 5\,x^3 & & -\ 6\,x & +\ 4 \\
 & & & & x^4 & -\ 5\,x^3 & & -\ 6\,x & +\ 4 \\
\hline
 & & & & 4\,x^4 & -\ 20\,x^3 & & -\ 24\,x & +\ 16 \\
 & & & -\ 6\,x^5 & +\ 30\,x^4 & & +\ 36\,x^2 & -\ 24\,x & \\
 & -\ 5\,x^7 & +\ 25\,x^6 & & +\ 30\,x^4 & -\ 20\,x^3 & & & \\
x^8 & -\ 5\,x^7 & & -\ 6\,x^5 & +\ 4\,x^4 & & & & \\
\hline
x^8 & -\ 10\,x^7 & +\ 25\,x^6 & -\ 12\,x^5 & +\ 68\,x^4 & -\ 40\,x^3 & +\ 36\,x^2 & -\ 48\,x & +\ 16
\end{array}
$$

To conclude,

$$\left(x^4 - 5x^3 - 6x + 4\right)^2$$

$$= x^8 - 10x^7 + 25x^6 - 12x^5 + 68x^4 - 40x^3 + 36x^2 - 48x + 16.$$

Example 21. Let's expand $\left(x^4 + x^3 + x^2 + x + 1\right)^2$.

The same deal:

$$\begin{array}{rccccccccccccccccc}
 & & & & & & & & x^4 & + & x^3 & + & x^2 & + & x & + & 1 \\
 & & & & & & & & x^4 & + & x^3 & + & x^2 & + & x & + & 1 \\
\hline
 & & & & & & & & x^4 & + & x^3 & + & x^2 & + & x & + & 1 \\
 & & & & & & x^5 & + & x^4 & + & x^3 & + & x^2 & + & x & & \\
 & & & & x^6 & + & x^5 & + & x^4 & + & x^3 & + & x^2 & & & & \\
 & & x^7 & + & x^6 & + & x^5 & + & x^4 & + & x^3 & & & & & & \\
x^8 & + & x^7 & + & x^6 & + & x^5 & + & x^4 & & & & & & & & \\
\hline
\end{array}$$

$$x^8 + 2x^7 + 3x^6 + 4x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$$

To conclude,

$$\left(x^4 + x^3 + x^2 + x + 1\right)^2$$

$$= x^8 + 2x^7 + 3x^6 + 4x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1.$$

Exercise 9. Expand

(1) $\left(x^3 - 8x\right)^2.$ (2) $\left(2x^2 - 3x + 5\right)^2.$

(3) $\left(x^3 + x^2 - 4\right)^2.$ (4) $\left(x^2 + \dfrac{1}{2}x + \dfrac{1}{3}\right)^2.$

(5) $\left(1 + x + x^2 + x^3 + x^4 + x^5\right)^2.$

[**Answers**]:

(1) $x^6 - 16x^4 + 64x^2.$ (2) $4x^4 - 12x^3 + 29x^2 - 30x + 25.$

(3) $x^6 + 2x^5 + x^4 - 8x^3 - 8x^2 + 16.$

(4) $x^4 + x^3 + \dfrac{11}{12}x^2 + \dfrac{11}{3}x + \dfrac{11}{9}.$

(5) $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 5x^6 + 4x^7 + 3x^8 + 2x^9 + x^{10}.$

- **Product of three or more polynomials.**

Example 22. How about

$$\left(x + 1\right)\left(x^2 + x + 1\right)\left(x^4 + x^2 + 1\right)?$$

There are three polynomials involved. This one you have to do it step by step, namely, you first do the boxed part:

$$\boxed{\left(x + 1\right)\left(x^2 + x + 1\right)}\ \left(x^4 + x^2 + 1\right)$$

and then you multiply the outcome with the third factor $x^4 + x^2 + 1.$ Let's do it:

Step 1. Do $\Big(x + 1\Big)\Big(x^2 + x + 1\Big)$:

$$
\begin{array}{rrrrrrr}
 & & x^2 & + & x & + & 1 \\
\times) & & & & x & + & 1 \\
\hline
 & & x^2 & + & x & + & 1 \\
x^3 & + & x^2 & + & x & & \\
\hline
x^3 & + & 2x^2 & + & 2x & + & 1
\end{array}
$$

In short,

$$\Big(x + 1\Big)\Big(x^2 + x + 1\Big) = x^3 + 2x^2 + 2x + 1.$$

Step 2. Do $\Big(x^3 + 2x^2 + 2x + 1\Big)\Big(x^4 + x^2 + 1\Big)$:

$$
\begin{array}{rrrrrrrrrrrrrrr}
 & & & & & & x^4 & & & + & x^2 & & & + & 1 \\
 & & & & & & & & x^3 & + & 2x^2 & + & 2x & + & 1 \\
\hline
 & & & & & & x^4 & & & + & x^2 & & & + & 1 \\
 & & & & 2x^5 & & & + & 2x^3 & & & + & 2x & & \\
 & & 2x^6 & & & + & 2x^4 & & & + & 2x^2 & & & & \\
x^7 & & & + & x^5 & & & + & x^3 & & & & & & \\
\hline
x^7 & + & 2x^6 & + & 3x^5 & + & 3x^4 & + & 3x^3 & + & 3x^2 & + & 2x & + & 1
\end{array}
$$

In short,

$$\Big(x^3 + 2x^2 + 2x + 1\Big)\Big(x^4 + x^2 + 1\Big)$$
$$= x^7 + 2x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 2x + 1.$$

To conclude,

$$\Big(x + 1\Big)\Big(x^2 + x + 1\Big)\Big(x^4 + x^2 + 1\Big)$$
$$= x^7 + 2x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 2x + 1.$$

Exercise 10. Expand:

(1) $\Big(x - 1\Big)\Big(x + 1\Big)^2.$

(2) $\Big(x - 1\Big)\Big(x - 3\Big)\Big(x^2 - 3\Big).$

(3) $\Big(x - 1\Big)\Big(x + 1\Big)\Big(x^2 + 1\Big).$

(4) $\Big(x^2 - \sqrt{2}\,x + 1\Big)\Big(x^2 + \sqrt{2}\,x + 1\Big)\Big(x^2 - 1\Big).$

(5) $\Big(x - \sqrt{2} - 1\Big)\Big(x + \sqrt{2} - 1\Big)\Big(x - \sqrt{2} + 1\Big)\Big(x + \sqrt{2} + 1\Big).$

[**Answers**]:

(1) $x^3 + x^2 - x - 1.$

(2) $x^4 - 4x^3 + 12x - 9.$

(3) $x^4 - 1.$

(4) $x^6 - x^4 + x^2 - 1.$

(5) $x^4 - 6\,x^2 + 1.$

- **Raising products.**

  The following has some bearings on certain types of polynomial multiplications:

$\boxed{1}$ (row 0)

1/ \1

$\boxed{1}$ $\boxed{1}$ (row 1)

1/ 2\ /1 \2

$\boxed{1}$ $\boxed{3}$ $\boxed{2}$ (row 2)

1/ 3\ /1 3\ /1 \3

$\boxed{1}$ $\boxed{6}$ $\boxed{11}$ $\boxed{6}$ (row 3)

1/ 4\ /1 4\ /1 4\ /1 \4

$\boxed{1}$ $\boxed{10}$ $\boxed{35}$ $\boxed{50}$ $\boxed{24}$ (row 4)

1/ 5\ /1 5\ /1 5\ /1 5\ /1 \5

$\boxed{1}$ $\boxed{15}$ $\boxed{85}$ $\boxed{225}$ $\boxed{274}$ $\boxed{120}$ (row 5)

1/ 6\ /1 6\ /1 6\ /1 6\ /1 6\ /1 \6

$\boxed{1}$ $\boxed{21}$ $\boxed{175}$ $\boxed{735}$ $\boxed{1624}$ $\boxed{1764}$ $\boxed{720}$ (row 6)

1/ 7\ /1 7\ /1 7\ /1 7\ /1 7\ /1 7\ /1 \7

I'm sure you are familiar with Pascal's triangle. The above is a variation of it. This can be used to get the expansions

$$
\begin{aligned}
&\Big(x+1\Big),\\
&\Big(x+1\Big)\Big(x+2\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big)\Big(x+5\Big),\\
&\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big)\Big(x+5\Big)\Big(x+6\Big),\\
&\qquad\vdots \qquad\qquad\qquad\qquad\qquad \ddots
\end{aligned}
$$

Namely:

$$\Big(x+1\Big)\Big(x+2\Big) = x^2 + 3x + 2,$$

$$\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big) = x^3 + 6x^2 + 11x + 6,$$

$$\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big)$$
$$= x^4 + 10x^3 + 35x^2 + 50x + 24,$$

$$\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big)\Big(x+5\Big)$$
$$= x^5 + 15x^4 + 85x^3 + 225x^2 + 274x + 120,$$

$$\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big)\Big(x+5\Big)\Big(x+6\Big)$$
$$= x^6 + 21x^5 + 175x^4 + 735x^3 + 1624x^2 + 1764x + 720.$$

Exercise 11. Expand:

$$\Big(x+1\Big)\Big(x+2\Big)\Big(x+3\Big)\Big(x+4\Big)\Big(x+5\Big)\Big(x+6\Big)\Big(x+7\Big).$$

⋆ As for this, the triangle in the previous page is apparently shown only up to the sixth row. You have to extend it to the seventh row.

[**Answer**]:

$$x^7 + 28x^6 + 322x^5 + 1960x^4 + 6769x^3 + 13132x^2 + 13068x + 5040.$$