FU08 - Automata and Languages
Exercise 11

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January 29, 2025

Question 1: Answer the following question

Let $\sum = \{a, b\}$, define $n_a, n_b : \sum^* \longrightarrow N$ (where N is the set of natural numbers) such that $n_a(w)$ is the number of a's in w and $n_b(w)$ is the number of b's in w. Show that the following languages are not regular.

a. $L_1 = \{w : n_a(w) = n_b(w)\}$
b. $L_2 = \{w : n_a(w) \neq n_b(w)\}$

Solution:
a. Let us assume that L_1 is a regular language.
We choose the pumping number as m. We use contradiction to prove that $L_1 = a^m b^m$ is not regular.
Say, the input string is $w = a^m b^m \Leftrightarrow |w| = 2m \geqslant m$.

Now, we write $w = xyz \Leftrightarrow a^m b^m = xyz = \overbrace{aaa...aaa...aaa...a}^{m}\overbrace{bbb...bb...bbb...b}^{m}$.
With $x = aaa...aaa$; $y = aaa...aa...$; $z = aaa...a\ bbb...bb...b$; $|xy| \leqslant m$; $|y| \geqslant 1$

Let $y = a^\alpha$ for $\alpha \in [1; m] \implies w = a^m b^m = \overbrace{a^{m-\alpha-\beta}}^{x}\ \overbrace{a^\alpha}^{y}\ \overbrace{a^\beta b^m}^{z}$; this ensures that $w \in L_1$.
Using pumping lemma, we doubly pump y.
Hence,

$$\begin{aligned} w &= a^m b^m \\ &= \overbrace{a^{m-\alpha-\beta}}^{x}\ \overbrace{a^\alpha a^\alpha}^{y}\ \overbrace{a^\beta b^m}^{z} \\ &= a^{m+\alpha} b^m \in L_1 \end{aligned}$$

Since $m + \alpha \neq m \Leftrightarrow n_a(w) \neq n_b(w)$ because of $\alpha \geqslant 1$.
This contradicts our assumption. Proving that our assumption is wrong.
Hence, we have proven that L_1 is non-regular.

b. Again, let us assume that L_2 is a regular language.
We choose the pumping number as m. We use contradiction to prove that $L_2 = a^m b^{m+k}$ is not regular.
Say, the input string is $w = a^m b^{m+k} \Leftrightarrow |w| = 2m + k \geqslant m$ and $k \geqslant 1$, this can goes vice versa since the input string is symmetric.

Now, we write $w = xyz \Leftrightarrow a^m b^{m+k} = xyz = \overbrace{aaa...aaa...aaa...a}^{m}\overbrace{bbb...bb...bbb...b}^{m+k}$.
With $x = aaa...aaa$; $y = aaa...aa...$; $z = aaa...a\ bbb...bb...b$; $|xy| \leqslant m$; $|y| \geqslant 1$

Let $y = a^\alpha$ for $\alpha \in [1; m] \implies w = a^m b^{m+k} = \overbrace{a^{m-\alpha-\beta}}^{x}\ \overbrace{a^\alpha}^{y}\ \overbrace{a^\beta b^{m+k}}^{z}$; this ensures that $w \in L_2$.
Using pumping lemma, we pump y k times.
Hence,

$$\begin{aligned} w &= a^m b^{m+k} \\ &= \overbrace{a^{m-\alpha-\beta}}^{x}\ \overbrace{a^\alpha a^k}^{y}\ \overbrace{a^\beta b^{m+k}}^{z} \\ &= a^{m+k} b^{m+k} \in L_2 \end{aligned}$$

Since $m + k = m + k \Leftrightarrow n_a(w) = n_b(w)$.
This contradicts our assumption. Proving that our assumption is wrong.
Hence, we have proven that L_2 is non-regular.

Question 2: Answer the following question

Show that the language $\{0^n10^n : n \geqslant 1\}$ is not a regular language.

Solution:

Let $L = \{0^n10^n : n \geqslant 1\}$, we assume that L is a regular language.
We choose the pumping number as m. We use contradiction to prove that $L = \{0^n10^n : n \geqslant 1\}$ is not regular.
Say, the input string is $w = 0^m10^m \Leftrightarrow |w| = 2m + 1 > m$.

Now, we write $w = xyz \Leftrightarrow 0^m10^m = xyz = \overbrace{\underbrace{000...00}_{x}\ \underbrace{0...0}_{y}}^{m}\ \underbrace{1\,\overbrace{000...000}^{m}...0}_{z}$ with $|xy| = m$; $|z| = m + 1$

Let $y = 0^\alpha$ for $\alpha \in [1; m] \implies w = 0^{m-\alpha}0^\alpha10^m \in L$. Using pumping lemma, we doubly pump y.

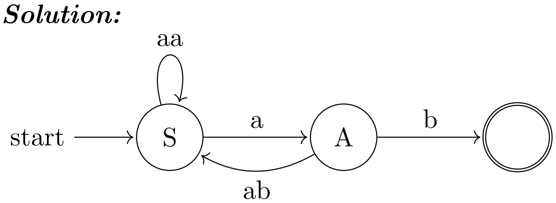
Hence, $w = \overbrace{0^{m-\alpha}}^{x}\ \overbrace{0^\alpha 0^\alpha}^{y}\ \overbrace{10^m}^{z} = 0^{m+\alpha}10^m$ must $\in L$.
This is mathematically wrong since $m + \alpha \neq m$ because of $\alpha \geqslant 1$.
This contracdicts our assumption. Proving that our assumption is wrong.
Hence, we have proven that L is non-regular.

Question 3: Answer the following question

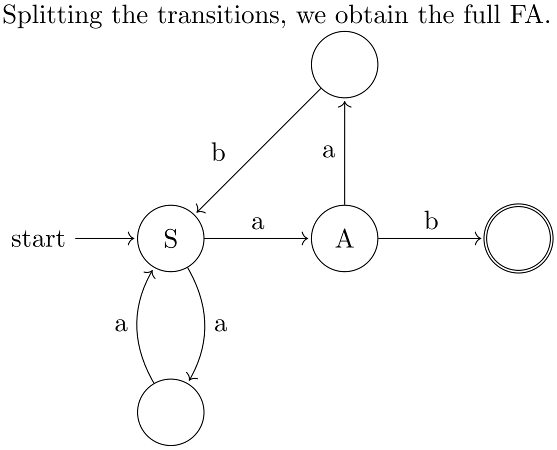
Construct a finite automaton that accepts the language generated by the grammar:

$$S \longrightarrow aaS \mid aA$$
$$A \longrightarrow abS \mid b$$

Solution:



Splitting the transitions, we obtain the full FA.



This is the final finite automaton that accepts the language above.

Question 4: Answer the following question

Construct a right-linear grammar that accept the language:

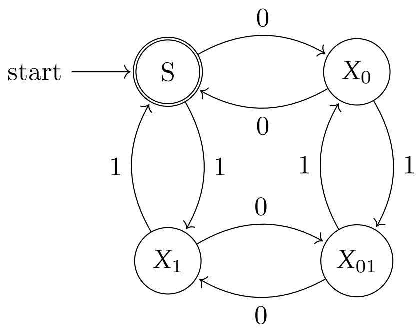
L = { w: w has both an even numberr of 0's and an even number of 1's }

Solution:

Define states:

- S: w has even number of both 0 and 1
- X_0: w has odd number of 0
- X_1: w has odd number of 1
- X_{01}: w has odd number of both 0 and 1

We construct a finite automaton to represent this language.



With this finite automaton, we can write a right-linear grammar:

$$
\begin{aligned}
S &\longrightarrow 0X_0 \mid 1X_1 \mid \lambda \\
X_0 &\longrightarrow 0S \mid 1X_{01} \\
X_1 &\longrightarrow 0X_{01} \mid 1S \\
X_{01} &\longrightarrow 0X_1 \mid 1X_0
\end{aligned}
$$