MA 02 LINEAR ALGEBRA II

SOLUTION FOR PRACTICE EXAM – FINAL B (07/24)

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Section: C7.
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[I] (20pts) (1) Let $f(x)$ and $g(x)$ be polynomials. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

True or false :
$$f(A)\,g(A) = g(A)\,f(A).$$

[Answer]: True .

(2) Let a, b and c be real numbers. Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a symmetric matrix.

True or false :

There exists an orthogonal matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.

[Answer]: True .

[II] (20pts) Find the characteristic polynomial, the eigenvalues, and then eigenvectors associated with each of the eigenvalues of A. Is A diagonalizable? If not, then find the Jordan canonical form of A.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

[Solution]:
$$\chi_A(\lambda) = \begin{vmatrix} \lambda - 1 & -(-1) \\ -1 & \lambda - 3 \end{vmatrix}$$
$$= (\lambda - 1)(\lambda - 3) - 1 \cdot (-1)$$

$$= \lambda^2 - 4\lambda + 3 + 1$$

$$= \lambda^2 - 4\lambda + 4$$

$$= \left(\lambda - 2\right)^2.$$

Thus the characteristic polynomial of A is a perfect square. $\big($Stated in other words, A has only one eigenvalue.$\big)$ Note that A itself is not a diagonal matrix. Therefore, we conclude that A is not diagonalizable. Accordingly, we follow the recipé in page 10 of "Review of Lectures – VI".

Let a, b, c and d be the four entries of the original matrix A:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then clearly $b < 0$ and $c > 0$. Thus we are in "case (a)" in page 10 of "Review of Lectures – VI". While taking heed of that fact, let's find the vector $\boldsymbol{y}$ in page 10 of "Review of Lectures – VI" as follows:

$$\boldsymbol{y} = \begin{bmatrix} \sqrt{|b|} \\ \pm\sqrt{|c|} \end{bmatrix} = \begin{bmatrix} \sqrt{|1|} \\ \pm\sqrt{|1|} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}.$$

Here, $\pm$ is chosen suitably so $\boldsymbol{y}$ is an eigenvector of A associated with the eigenvalue $\lambda = 2$, namely, $A\boldsymbol{y} = 2\boldsymbol{y}$.

In this case,

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

indeed satisfies $A\boldsymbol{y} = 2\boldsymbol{y}$. $\left(\boldsymbol{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right.$ is disqualified because it does not satisfy $\left. A\boldsymbol{y} = 2\boldsymbol{y}. \right)$

Following the recipé in page 10 of "Review of Lectures – VI", set

$$Q = \begin{bmatrix} p & 1 \\ q & -1 \end{bmatrix},$$

where p and q are real numbers. If p and q satisfy $\det Q = 1$ $\big($note: since $b < 0$, $c > 0$, thus we are in "case (a)" in page 10 of "Review of Lectures – VI"$\big)$, then $Q^{-1}AQ$ is in a Jordan canonical form:

$$Q^{-1}AQ = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}.$$

Let's choose $p = -1$, $q = -0$. To conclude:

$\left[\underline{\text{ Answer }}\right]$: $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ is triangularized into a Jordan canonical form as

$$Q^{-1}AQ = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

[III] (20pts) Find the characteristic polynomial, the eigenvalues, and then eigenvectors associated with each of the eigenvalues of A. Then diagonalize the matrix

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix},$$

with the condition the matrix Q with which $Q^{-1}AQ$ equals the diagonal matrix is <u>orthogonal</u>.

$\left[\underline{\text{ Solution }}\right]$:

Step 1. Find a suitable scalar t such that $A - tI$ is traceless: $\operatorname{tr}\Big(A - tI\Big) = 0$. In this case, the trace of the original matrix A is

$$6 + 3 = 9.$$

So $\dfrac{1}{2}$ times it, that is, $\dfrac{9}{2}$, will do:

$$t = \frac{9}{2}.$$

Accordingly the matrix $B = A - tI$ becomes

$$\begin{aligned} B &= A - \frac{9}{2} I \\ &= \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \frac{9}{2} & 0 \\ 0 & \frac{9}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2} & 2 \\ 2 & -\frac{3}{2} \end{bmatrix}. \end{aligned}$$

Step 2. Find a suitable scalar s such that

$$s \cdot \frac{3}{2} = \cos\theta, \qquad s \cdot 2 = \sin\theta \qquad \Big(\text{for some real number } \theta\Big).$$

This s is simply found as

$$\begin{aligned} s &= \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 + 2^2}} \\ &= \frac{1}{\sqrt{\frac{9}{4} + 4}} \\ &= \frac{1}{\sqrt{\frac{25}{4}}} \\ &= \sqrt{\frac{4}{25}} = \frac{2}{5}. \end{aligned}$$

Accordingly the matrix $C = sB$ $\big($where B is found in Step 1$\big)$ becomes

$$\begin{aligned} C &= \frac{2}{5} B \\ &= \frac{2}{5} \begin{bmatrix} \frac{3}{2} & 2 \\ 2 & -\frac{3}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}. \end{aligned}$$

So, let's agree that C is indeed of the form

$$C = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},$$

where

$$\cos\theta = \frac{3}{5}, \qquad \text{and} \qquad \sin\theta = \frac{4}{5}.$$

In particular, C is indeed an orthogonal matrix.

Step 3. Recall that the matrix Q in Exercise 2 above $\big($page 12$\big)$ itself is an orthogonal matrix, and with which we can accomplish

$$Q^{-1} C Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\big(\lambda = \pm 1$ are the two eigenvalues of $C\big)$. Multiply $s^{-1} = \dfrac{5}{2}$ to the two sides:

$$Q^{-1} B Q = \begin{bmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{5}{2} \end{bmatrix}.$$

Add $tI = \dfrac{9}{2} I$ to the two sides:

$$Q^{-1}AQ = \begin{bmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} + \begin{bmatrix} \frac{9}{2} & 0 \\ 0 & \frac{9}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}.$$

This is the final result, the diagonalized matrix we sought. We need to provide the concrete shape of Q as well.

Step 4. Keeping in mind

$$\cos\theta = \frac{3}{5}, \qquad \text{and} \qquad \sin\theta = \frac{4}{5}$$

$\big($from Step 2 above$\big)$, we may find Q using

$$(*) \qquad Q = \begin{bmatrix} \boldsymbol{y}_+ & \boldsymbol{y}_- \end{bmatrix} = \begin{bmatrix} \dfrac{\sin\theta}{\sqrt{2 - 2\cos\theta}} & \dfrac{\sin\theta}{\sqrt{2 + 2\cos\theta}} \\[2ex] \dfrac{-\Big(\cos\theta\Big) + 1}{\sqrt{2 - 2\cos\theta}} & \dfrac{-\Big(\cos\theta\Big) - 1}{\sqrt{2 + 2\cos\theta}} \end{bmatrix}.$$

$\big($= $(*)$ in page 11 of "Review of Lectures – X"$\big)$:

$$Q = \begin{bmatrix} \dfrac{\sin\theta}{\sqrt{2 - 2\cos\theta}} & \dfrac{\sin\theta}{\sqrt{2 + 2\cos\theta}} \\[2ex] \dfrac{-\Big(\cos\theta\Big) + 1}{\sqrt{2 - 2\cos\theta}} & \dfrac{-\Big(\cos\theta\Big) - 1}{\sqrt{2 + 2\cos\theta}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\frac{4}{5}}{\sqrt{2 - 2\cdot\frac{3}{5}}} & \dfrac{\frac{4}{5}}{\sqrt{2 + 2\cdot\frac{3}{5}}} \\[2ex] \dfrac{-\frac{3}{5} + 1}{\sqrt{2 - 2\cdot\frac{3}{5}}} & \dfrac{-\frac{3}{5} - 1}{\sqrt{2 + 2\cdot\frac{3}{5}}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \\ \dfrac{1}{\sqrt{5}} & -\dfrac{2}{\sqrt{5}} \end{bmatrix}.$$

- To summarize:

Diagonalization result. $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$ is diagonalized by an orthogonal matrix Q as follows:

$$Q^{-1}AQ = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} \dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \\ \dfrac{1}{\sqrt{5}} & -\dfrac{2}{\sqrt{5}} \end{bmatrix}.$$

[IV] (40pts) Let a, b, c and d be real numbers. Let

$$A = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \quad \text{and}$$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(1) $$AQ = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a+b+c+d & a+b-c-d & a-b+c-d & a-b-c+d \\ a+b+c+d & a+b-c-d & -a+b-c+d & -a+b+c-d \\ a+b+c+d & -a-b+c+d & a-b+c-d & -a+b+c-d \\ a+b+c+d & -a-b+c+d & -a+b-c+d & a-b-c+d \end{bmatrix}.$$

(2)
$$Q^2 = QQ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4I.$$

From this it follows

$$Q^{-1} = \frac{1}{4} Q.$$

(3)
$$Q^{-1}AQ = \frac{1}{4} QAQ$$
$$= \frac{1}{4} \begin{bmatrix} 4\big(a+b+c+d\big) & 0 & 0 & 0 \\ 0 & 4\big(a+b-c-d\big) & 0 & 0 \\ 0 & 0 & 4\big(a-b+c-d\big) & 0 \\ 0 & 0 & 0 & 4\big(a-b-c+d\big) \end{bmatrix}$$
$$= \begin{bmatrix} a+b+c+d & 0 & 0 & 0 \\ 0 & a+b-c-d & 0 & 0 \\ 0 & 0 & a-b+c-d & 0 \\ 0 & 0 & 0 & a-b-c+d \end{bmatrix}.$$