MA 02 LINEAR ALGEBRA II

REVIEW OF LECTURES – V

June 26 (Wed), 2024

Section: C7.
Instructors: Yasuyuki Kachi (lecture) & Shunji Moriya (recitation)

- **Distributive Law.**

The following is another aspect of matrix arithmetic. We often deal with 'expansions', like

$$A\Big(B+C\Big), \qquad \Big(B+C\Big)D, \qquad A\Big(B+C\Big)D,$$
$$A\Big(B-C\Big), \qquad \Big(B-C\Big)D, \qquad A\Big(B-C\Big)D,$$

etc. Can we expand these like numbers? As in

$$A\Big(B+C\Big) \;=\; AB + AC, \qquad \Big(B-C\Big)D \;=\; BD - CD,$$

and like? The answer is, "yes".

Formula 1 (Distributive Laws). For

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad D = \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix},$$

the following hold:

(1) $$A\Big(B+C\Big) \;=\; AB + AC,$$

(2) $$\Big(B+C\Big)D \;=\; BD + CD,$$

(3) $$A\Big(B+C\Big)D \;=\; ABD + ACD,$$

(4) $$A\Big(B-C\Big) \;=\; AB - AC,$$

(5) $$\Big(B-C\Big)D \;=\; BD - CD,$$

(6) $$A\Big(B-C\Big)D \;=\; ABD - ACD.$$

- **More on Distributive Laws.**

There is always the next level. Let's take a look at the following:

$$\Big(A+B\Big)\Big(C+D\Big).$$

Quiz. Is the following the right way to do it?

$$\Big(A+B\Big)\Big(C+D\Big) \;=\; AC \,+\, AD \,+\, BC \,+\, BD.$$

If so, then prove it. Use Formula in the previous page.

Solution. Yes. Indeed, let's temporarily call $C+D$ as E. So

$$\Big(A+B\Big)\Big(C+D\Big)$$

is $\Big(A+B\Big)E$. According to part (2) of the previous formula, this is expanded as

$$\Big(A+B\Big)E \;=\; AE \,+\, BE.$$

But since E was the temporary name for $C+D$, so $AE + BE$ is

$$A\Big(C+D\Big) \;+\; B\Big(C+D\Big).$$

Now according to part (1) of the previous formula, this is expanded as

$$AC \,+\, AD \,+\, BC \,+\, BD. \qquad \Box$$

— In short, this quiz was the combination of part (1) and part (2) of Formula in the previous page. Let's highlight:

Corollary 1. Let A, B, C and D be as above. Then

$$\boxed{\Big(A+B\Big)\Big(C+D\Big) \;=\; AC \,+\, AD \,+\, BC \,+\, BD.}$$

Exercise 1.

(a) Prove part (1) and part (2) of Formula 1 in page 1. As for part (1), physically calculate each of

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} + \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \right), \qquad \text{and}$$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$$

separately, and verify that they match. Part (2) is completely similar.

(b) Prove that part (3) of Formula 1 in page 1 <u>is equivalent to</u> parts (1–2) of the same formula. So, prove "(1) implies (2–3)" <u>and</u> "(2–3) imply (1)" both.

Example 1. Can we expand

$$\Big(A+B\Big)\Big(A+B\Big) \ ?$$

Sure. This is a special case of Corollary. Namely, the case $C = A$ and $D = B$:

$$\Big(A+B\Big)\Big(A+B\Big) \ = \ AA + AB + BA + BB.$$

But can't we rewrite this as

$$\Big(A+B\Big)^2 \ = \ A^2 + AB + BA + B^2?$$

Sure. But then we should be careful. There is no room for further simplification. AB and BA are usually not the same. So you might be tempted to combine them and throw $2AB$ for it, but you can't. Let's highlight:

Corollary 2. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$,

$$\boxed{\Big(A+B\Big)^2 \ = \ A^2 + AB + BA + B^2.}$$

Example 2. Next, can we expand

$$\Big(A+B\Big)\Big(A-B\Big) \ ?$$

Sure. But as for this, we will be benefited by highlighting the following first:

Formula 2 (Distributive & Associative Laws Involving Scalars – I). For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

the following hold:

(1) $$-\Big(A+B\Big) \ = \ -A \ - \ B.$$

(2) $$\Big(-A\Big)B \ = \ -\Big(AB\Big).$$

(3) $$A\Big(-B\Big) \ = \ -\Big(AB\Big).$$

More generally, for a scalar t,

(4) $$t\Big(A+B\Big) \ = \ tA \ + \ tB.$$

(5) $$\Big(tA\Big)B \ = \ t\Big(AB\Big).$$

(6) $$A\Big(tB\Big) \ = \ t\Big(AB\Big).$$

(Clearly part (1–3) are the case $t = -1$ of part (4–6).)

Example 2 resumed. Back to

$$\Big(A+B\Big)\Big(A-B\Big),$$

how do we go about? Yes. Let's temporarily call $A+B$ as E. So this is $E\Big(A-B\Big)$. According to part (1) of Formula 1 in page 1, this is expanded as

$$E\Big(A-B\Big) \ = \ EA \ - \ EB.$$

But since E was the temporary name for $A+B$, so this is

$$\Big(A+B\Big)A \;-\; \Big(A+B\Big)B.$$

Now this is expanded as

$$AA \;+\; BA \;-\; \Big(AB \;+\; BB\Big),$$

that is,

$$AA \;+\; BA \;+\; \bigg(-\Big(AB \;+\; BB\Big)\bigg),$$

that is,

$$AA \;+\; BA \;+\; \Big(-\;AB \;-\; BB\Big)$$

$\big($where we have used part (1) of Formula 2 in page 4$\big)$, that is,

$$AA \;+\; BA \;-\; AB \;-\; BB.$$

Rewrite AA as A^2, also BB as B^2, so we arrive at

$$\Big(A+B\Big)\Big(A-B\Big) \;=\; A^2 \;+\; BA \;-\; AB \;-\; B^2. \qquad \square$$

Here, again, we should be careful. There is no room for further simplification. AB and BA are usually not the same. You might be tempted to cancel them, but you can't.

Corollary 3. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$,

$$\boxed{\Big(A+B\Big)\Big(A-B\Big) \;=\; A^2 \;+\; BA \;-\; AB \;-\; B^2.}$$

Example 3.

$$A\Big(I - B\Big)C \;=\; AIC \;-\; ABC$$

$$=\; AC \;-\; ABC.$$

Example 3a. Let's tweak it:

$$A\Big(2I - B\Big)C \;=\; A\Big(2I\Big)C \;-\; ABC$$

$$=\; 2AC \;-\; ABC.$$

Example 3b. In the above, suppose $C \;=\; A^{-1}$:

$$A\Big(2I - B\Big)A^{-1} \;=\; 2AA^{-1} \;-\; ABA^{-1}$$

$$=\; 2I \;-\; ABA^{-1}.$$

Example 4.

$$\Big(ABA^{-1}\Big)^2 \;=\; ABA^{-1}ABA^{-1}$$

$$=\; ABIBA^{-1}$$

$$=\; ABBA^{-1}$$

$$=\; AB^2A^{-1}.$$

Example 4a. Let's tweak it:

$$4I \;+\; \Big(ABA^{-1}\Big)^2 \;=\; 4I \;+\; AB^2A^{-1}.$$

Example 4b. Meanwhile

$$A\Big(4I \;+\; B^2\Big)A^{-1} \;=\; A\Big(4I\Big)A^{-1} \;+\; AB^2A^{-1}$$

$$=\; 4AIA^{-1} \;+\; AB^2A^{-1}$$

$$=\; 4I \;+\; AB^2A^{-1}.$$

Example 4c. If you combine the results of Examples 4a and 4b, you see

$$(\#) \qquad 4I + \left(ABA^{-1}\right)^2 = A\left(4I + B^2\right)A^{-1}.$$

A little shift of a paradigm: Think of $4I + B^2$ as $f\left(B\right)$, where $f\left(x\right) = 4 + x^2$. Likewise, think of $4I + \left(ABA^{-1}\right)^2$ as $f\left(ABA^{-1}\right)$. Then $(\#)$ is rewritten as

$$(\#)' \qquad f\left(ABA^{-1}\right) = Af\left(B\right)A^{-1}.$$

More on this later.

Formula 3 (Distributive & Associative Laws Involving Scalars – II). For

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad D = \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix},$$

the following hold:

$$(1) \qquad \left(-A\right)\left(B+C\right) = -AB - AC.$$

$$(2) \qquad \left(B+C\right)\left(-D\right) = -BD - CD.$$

More generally, for a scalar t,

$$(3) \qquad \left(tA\right)\left(B+C\right) = tAB + tAC.$$

$$(4) \qquad \left(B+C\right)\left(tD\right) = tBD + tCD.$$

(Clearly part (1–2) are the case $t = -1$ of part (3–4).) Also, for scalars t and u,

$$(5) \qquad A\left(tB+uC\right) = tAB + uAC.$$

$$(6) \qquad \left(tB+uC\right)D = tBD + uCD.$$

(Clearly part (1–2), and part (4–5), of Formula 1 in page 1 are the case $t = 1$, $u = 1$; and the case $t = 1$, $u = -1$, of part (5–6) above, respectively.)

- **The inverse and the determinant.**

Later we will utilize the following formula (proof is an exercise: Exercise 5 below.)

Formula 4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose $\det A \neq 0$ so A^{-1} exists. Then

$$\boxed{\det\left(A^{-1}\right) = \frac{1}{\det A}\ .}$$

- **Diagonalization revisited.**

As I announced last time, today we return to the topic of diagonalization. Let me pull the same example from Day 1:

Example 5 (= the same example as "Review of Lectures – I", pages 14–15)**.**

Let's consider

$$A = \begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}.$$

The following may be out of the blue, but bear with me. This matrix A is clearly *not* a diagonal matrix. However, form PAQ as follows:

$$(*) \qquad \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}}_{\substack{\| \\ P}} \underbrace{\begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}}_{\substack{\| \\ A}} \underbrace{\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}}_{\substack{\| \\ Q}}.$$

Here the choices of P and Q are deliberate. Don't ask me what prompted me to make these choices for P and Q. All I can say right now is, with these choices of P and Q, once we calculate this PAQ, it will become

$$PAQ = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

Is there anything that stands out? Yes, this last matrix is a <u>diagonal matrix</u>.

But that's not the end of it. P and Q are actually $\underline{\text{ inverses of each other }}$. Indeed, let's form PQ:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}}_{\substack{\| \\ P}} \underbrace{\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}}_{\substack{\| \\ Q}}.$$

Once you calculate this, you will end up getting

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

the identity matrix. So in other words, $Q = P^{-1}$. So, PAQ is PAP^{-1}. Now, this way we arrive at the following:

$$\boxed{\begin{array}{c} A = \begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \\[2ex] \Longrightarrow \qquad PAP^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}. \end{array}}$$

So again, A is not diagonal but PAP^{-1} is.

On Day 1, I didn't explain how I came up with P and Q, where one is the inverse of the other. Today is the day. First, breaking news:

Fact. $$A = \begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}$$

$\big($the same A as above$\big)$ has 4 and 5 as its eigenvalue. This is due to the fact that 4 and 5 are the two diagonal entries of the 'diagonalized' matrix PAP^{-1}.

An obvious question here is "how come?" To let the cat out of the bag, the theory that we have been developing is tailor-made for this.We have been persistently laying bricks, and now we can capitalize on it, we are going to readily apply them. Clue:

Clue. Agree

$$P\Big(\lambda I - A\Big)P^{-1} \;=\; \lambda I \;-\; PAP^{-1}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \lambda - 4 & 0 \\ 0 & \lambda - 5 \end{bmatrix}.$$

In short, $P\Big(\lambda I - A\Big)P^{-1} = \begin{bmatrix} \lambda - 4 & 0 \\ 0 & \lambda - 5 \end{bmatrix}.$ Let's take the determinant:

$$(@) \qquad \det\bigg(P\Big(\lambda I - A\Big)P^{-1}\bigg) = \begin{vmatrix} \lambda - 4 & 0 \\ 0 & \lambda - 5 \end{vmatrix}.$$

1. First, the right-hand side of (@) is clearly $\Big(\lambda - 4\Big)\Big(\lambda - 5\Big)$.

2. Second, by Product Formula ("Review of Lectures – IV"), the left-hand side of (@) is broken up as

$$\Big(\det P\Big)\bigg(\det\Big(\lambda I - A\Big)\bigg)\Big(\det P^{-1}\Big) = \Big(\det P\Big)\bigg(\det\Big(\lambda I - A\Big)\bigg)\frac{1}{\det P}$$

$$= \Big(\det P\Big)\frac{1}{\det P}\bigg(\det\Big(\lambda I - A\Big)\bigg)$$

$\Big[$here $\det\Big(\lambda I - A\Big)$ and $\dfrac{1}{\det P}$ are interchangeable because both are scalars$\Big]$

$$= \det\Big(\lambda I - A\Big)$$

$$= \chi_A(\lambda) \quad \Big(\text{the characteristic polynomial of } A\Big).$$

From 1. and 2. above, we conclude $\chi_A(\lambda) = \Big(\lambda - 4\Big)\Big(\lambda - 5\Big)$. $\square$

• Let me recite 'Fact' below:

Fact.
$$A = \begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}$$

$\big($the same A as above$\big)$ has 4 and 5 as its eigenvalue. This is due to the fact 4 and 5 are the two diagonal entries of the 'diagonalized' matrix $P\,A\,P^{-1}$:

$$P\,A\,P^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

Next, here is another fact:

Fact 2. In

$$Q = P^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix},$$

the two columns are the eigenvectors:

- $\begin{bmatrix} 7 \\ -3 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = 4$.
- $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = 5$.

• Let's verify these:

$$\begin{aligned}
\begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} 7 \\ -3 \end{bmatrix} &= \begin{bmatrix} \big(-2\big) \cdot 7 + \big(-14\big) \cdot \big(-3\big) \\ 3 \cdot 7 + 11 \cdot \big(-3\big) \end{bmatrix} \\
&= \begin{bmatrix} 28 \\ -12 \end{bmatrix} = 4 \begin{bmatrix} 7 \\ -3 \end{bmatrix}, \\
\begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} &= \begin{bmatrix} \big(-2\big) \cdot \big(-2\big) + \big(-14\big) \cdot 1 \\ 3 \cdot \big(-2\big) + 11 \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} -10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \qquad \square
\end{aligned}$$

• But the real question is why the matrix Q, whose columns are eigenvectors of A, has the ability to make $Q^{-1}AQ \ \big(= \ PAP^{-1}\big)$ diagonal.

The answer is very simple. It is as follows. For simplicity, let's denote

$$\boldsymbol{x} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

So

- ○ $\boldsymbol{x}$ is an eigenvector of A associated with the eigenvalue $\lambda = 4$,
- ○ $\boldsymbol{y}$ is an eigenvector of A associated with the eigenvalue $\lambda = 5$,

and moreover $Q = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix}$. Then

$$\begin{aligned} AQ &= A \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} \\ &= \begin{bmatrix} A\boldsymbol{x} & A\boldsymbol{y} \end{bmatrix} \\ &= \begin{bmatrix} 4\,\boldsymbol{x} & 5\,\boldsymbol{y} \end{bmatrix}. \end{aligned}$$

Here, the matrix $\begin{bmatrix} 4\,\boldsymbol{x} & 5\,\boldsymbol{y} \end{bmatrix}$ is actually rewritten as

$$\begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

$\Bigg($Indeed, physically calculate $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 4p & 5q \\ 4r & 5s \end{bmatrix}$ comes out.$\Bigg)$

Here, remember $\begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} = Q$. So, in short,

$$AQ = Q \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

Multiply Q^{-1} from the left to the both sides of this last identity, and we obtain

$$Q^{-1}AQ = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}. \quad \square$$

• We can now extrapolate the above picture, and establish a method for diagonalizing a given matrix A.

Recipé to diagonalize a given matrix.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Suppose A has <u>two distinct</u> eigenvalues

$$\chi_A(\lambda) = \Big(\lambda - \lambda_1\Big)\Big(\lambda - \lambda_2\Big) \qquad \Big(\lambda_1 \neq \lambda_2\Big).$$

Suppose

- ○ $\boldsymbol{x} = \begin{bmatrix} p \\ r \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = \lambda_1$.
- ○ $\boldsymbol{y} = \begin{bmatrix} q \\ s \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = \lambda_2$.

Then set

$$Q = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Then

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

In other words, set $P = Q^{-1}$, and

$$PAP^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Example 6. Let's diagonalize the matrix

$$A = \begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}.$$

Step 1. Find the eigenvalues. This is routine:

$$\begin{aligned} \chi_A(\lambda) &= \det\left(\lambda I - A\right) \\ &= \begin{vmatrix} \lambda+4 & -6 \\ -7 & \lambda+5 \end{vmatrix} \\ &= \left(\lambda+4\right)\left(\lambda+5\right) - \left(-6\right)\cdot\left(-7\right) \\ &= \lambda^2 + 9\lambda + 20 - 42 \\ &= \lambda^2 + 9\lambda - 22 \\ &= \left(\lambda-2\right)\left(\lambda+11\right). \end{aligned}$$

So, the eigenvalues of A are

$$\lambda = 2 \qquad \text{and} \qquad \lambda = -11.$$

Step 2. Find eigenvectors of A associated with each of the two eigenvalues of A (**Step 2a** and **Step 2b** below).

Step 2a. Find an eigenvector of A associated with $\lambda = 2$.

Since $A = \begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}$, the equation $A\boldsymbol{x} = 2\boldsymbol{x}$ is

(@)
$$\begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{bmatrix} -4x + 6y \\ 7x - 5y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}.$$

That is,

$$\begin{cases} -4x + 6y = 2x, \\ 7x - 5y = 2y. \end{cases}$$

Shift the terms:

$$\begin{cases} -6x + 6y = 0, \\ 7x - 7y = 0. \end{cases}$$

So two essentially identical equations came out. These equations are the same as

$$\boxed{x - y = 0.}$$

Clearly

$$x = 1, \ y = 1$$

works. Thus:

- $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = 2$.

Step 2b. Find an eigenvector of A associated with $\lambda = -11$.

Since $A = \begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}$, the equation $A\boldsymbol{x} = -11\boldsymbol{x}$ is

(@) $$\begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -11 \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{bmatrix} -4x + 6y \\ 7x - 5y \end{bmatrix} = \begin{bmatrix} -11x \\ -11y \end{bmatrix}.$$

That is,

$$\begin{cases} -4x + 6y = -11x, \\ 7x - 5y = -11y. \end{cases}$$

Shift the terms:

$$\begin{cases} 7x + 6y = 0, \\ 7x + 6y = 0. \end{cases}$$

So two identical equations came out. Delete one of them:

$$\boxed{7x + 6y = 0.}$$

Clearly

$$x = 6, \; y = -7$$

works. Thus:

- $\boldsymbol{y} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = -11$.

Step 3. Form

$$Q = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix},$$

using $\boldsymbol{x}$ from Step 2a, and $\boldsymbol{y}$ from Step 2b:

$$Q = \begin{bmatrix} 1 & 6 \\ 1 & -7 \end{bmatrix}.$$

Answer. $A = \begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}$ is diagonalized as follows:

$$Q^{-1}AQ = \begin{bmatrix} 2 & 0 \\ 0 & -11 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} 1 & 6 \\ 1 & -7 \end{bmatrix}.$$

- Just in case, let's find $Q^{-1} = P$. First, note

$$\det Q = \begin{vmatrix} 1 & 6 \\ 1 & -7 \end{vmatrix}$$
$$= 1 \cdot \big(-7\big) - 6 \cdot 1 = -13.$$

So

$$P = \begin{bmatrix} 1 & 6 \\ 1 & -7 \end{bmatrix}^{-1} = \frac{1}{\det Q} \begin{bmatrix} -7 & -6 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{1}{-13} \begin{bmatrix} -7 & -6 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{1}{13} \begin{bmatrix} 7 & 6 \\ 1 & -1 \end{bmatrix}.$$

Accordingly, we may paraphrase our answer as follows:

Alternative Answer. $A = \begin{bmatrix} -4 & 6 \\ 7 & -5 \end{bmatrix}$ is diagonalized as follows:

$$PAP^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & -11 \end{bmatrix}, \qquad \text{where} \qquad P = \frac{1}{13} \begin{bmatrix} 7 & 6 \\ 1 & -1 \end{bmatrix}.$$

Note. You may instead throw $P = \begin{bmatrix} 7 & 6 \\ 1 & -1 \end{bmatrix}$ in "Alternative Answer". This is acceptable, as long as you are aware of the fact that this P and Q above are no longer inverses of each other. Indeed, when you replace P with a non-zero scalar multiple of P, it does not affect PAP^{-1}.

Exercise 2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. True or false :

(1a) $\left(A+I\right)^2 = A^2 + 2A + I.$ (1b) $\left(A+B\right)^2 = A^2 + 2AB + B^2.$

(1c) Suppose $AB = BA$. Then $\left(A+B\right)^2 = A^2 + 2AB + B^2.$

(2a) $\left(A+I\right)\left(A-I\right) = A^2 - I.$ (2b) $\left(A+B\right)\left(A-B\right) = A^2 - B^2.$

(2c) Suppose $AB = BA$. Then $\left(A+B\right)\left(A-B\right) = A^2 - B^2.$

Exercise 3. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Expand

(1) $\left(A+2I\right)^2.$ (2) $\left(A+4I\right)\left(A-7I\right).$ (3) $\left(A-I\right)\left(A^2+A+I\right).$

Exercise 4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Suppose $\det A \neq 0$.

True or false :

(1) $\left(ABA^{-1}\right)^3 = AB^3A^{-1}.$

(2) $2I + 3\left(ABA^{-1}\right) + \left(ABA^{-1}\right)^3 = A\left(2I + 3B + B^3\right)A^{-1}.$

Exercise 5. Prove Formula 4: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $\det A \neq 0$,

$$\det\left(A^{-1}\right) = \frac{1}{\det A}.$$

Exercise 6. Diagonalize:

(1) $A = \begin{bmatrix} 8 & 1 \\ 8 & 6 \end{bmatrix}.$ (2) $B = \begin{bmatrix} 3 & 3 \\ 9 & -23 \end{bmatrix}.$