Linear Algebra II Recitation 5

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1. (1) Let $f(x) = x^2 - 5x + 2$ and $A = \begin{bmatrix} 0 & 10 \\ -1 & 7 \end{bmatrix}$. Calculate $f(A)$.

   (2) Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$. Calculate $\chi_A(A)$ (without resorting any theorem).

   (3) Let
$$A = \begin{bmatrix} 42384031609 & -33649837 \\ 7017128346 & 1843570134 \end{bmatrix}.$$
   Find $\chi_A(A)$. No calculation. Cite an appropriate theorem.

2. Let a, b, c and d be real numbers. Let $X = \begin{bmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{bmatrix}$.

   (1) Find the characteristic polynomial $\chi_X(\lambda)$ of X.

   (2) Use (1) and Cayley-Hamilton's Theorem to write X^4 as a scalar times X^2.

   (3) Let $A = (a^2 + b^2 + c^2 + d^2)I + 2aX + 2X^2$. Prove
$$A^T = (a^2 + b^2 + c^2 + d^2)I - 2aX + 2X^2$$

   (4) Prove that AA^T is a scalar times the identity matrix.

   (5) Find $\det A$.

   (The outline of solution was explained by Prof.Kachi at the lecture of July, 9 so you are also requested to give the details such as the calculation process omitted at the lecture.)

3. (Another way to find the n-th power) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

   (1) Find two constant numbers p, q such that for any integer $n \geq 0$ the following equation holds.
$$A^{n+2} = p\,A^{n+1} + q\,A^n$$

   (2) Write $A^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$. Obtain a recurrence relation for each of a_n, b_n, c_n, and d_n from the result of (1).

   (3) By solving the recurrence relation of (2), find A^n.

4. Solve the following simultaneous linear equations (or systems of linear equations).

(1) $\begin{bmatrix} 2 & -1 & 5 \\ 0 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(2) $\begin{bmatrix} 2 & -1 & 9 \\ -1 & 1 & -3 \\ 1 & -3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Recitation 5, Answer

1. (1) $\begin{bmatrix} -8 & 20 \\ -2 & 6 \end{bmatrix}$

   (2) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

   (3) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, by Cayley-Hamilton's theorem.

2. (1) $\chi_X(\lambda) = \lambda^3 + (b^2 + c^2 + d^2)\lambda$

   (2) $X^4 = -(b^2 + c^2 + d^2)X^2$

   (3) Omitted.

   (4) $AA^T = (a^2 + b^2 + c^2 + d^2)^2 I$, the proof is omitted

   (5) $\det A = (a^2 + b^2 + c^2 + d^2)^3$

3. (1) $p = q = 1$

   (2) $a_{n+2} = a_{n+1} + a_n$, and the same recurrence relation holds for b_n, c_n, and d_n. (The sequence $\{a_n\}$ is called *Fibonacci sequence*.)

   (3) $A^n = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha^{n-1} - \beta^{n-1} & \alpha^n - \beta^n \\ \alpha^n - \beta^n & \alpha^{n+1} - \beta^{n+1} \end{bmatrix}$, where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.

4. (1) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3k \\ -k \\ k \end{bmatrix}$ where k is any (real or complex) number.

   (2) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6k \\ -3k \\ k \end{bmatrix}$ where k is any (real or complex) number.

Solution of a 3-term recurrence relation

Problem. Find a closed formula for the general (n-th) term of a sequence $\{a_n\}_n$ which satisfies a 3-term recurrence relation

$$a_{n+2} = pa_{n+1} + qa_n \qquad \cdots\cdots (*)$$

where p, q are constant numbers.

Solution. We consider the *characteristic equation*

$$x^2 = px + q$$

of $(*)$. Let α, β be the solutions of the equation. We have

$$\begin{cases} a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n) & \cdots (1) \\ a_{n+2} - \beta a_{n+1} = \alpha(a_{n+1} - \beta a_n) & \cdots (2) \end{cases}$$

By the equation (1), we see the sequence $\{a_{n+1} - \alpha a_n\}$ is a geometric sequence, and its n-th term is given by

$$a_{n+1} - \alpha a_n = \beta^{n-1}(a_2 - \alpha a_1).$$

Similary we can find the n-th term of $\{a_{n+1} - \beta a_n\}$ from (2). By eliminating a_{n+1} from these two explicit formulae, we can find the n-th term of $\{a_n\}$.