FU08 - Automata and Languages
Exercise 2

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December 8, 2024

Question 1: Answer the following question

For the set $S = \{1, 4, 7, 8\}$ and the relations:

$$\begin{aligned}
R1 &= \{(1,7),(1,8),(7,4),(4,7)\} \\
R2 &= \{(1,7),(7,1),(7,4),(4,7),(4,4)\} \\
R3 &= \{(1,1),(1,7),(4,4),(7,1),(7,4),(4,7),(7,7),(8,8)\} \\
R4 &= \{(1,1),(4,4),(7,7),(8,8)\}
\end{aligned}$$

a. Complete the following table (by writing yes or no):

	Reflexive	Symmetric	Transitive	Equivalence
R1				
R2				
R3				
R4				

b. Find the following:
i. the reflexive closure for R1 and R2
ii. the transitive closure for R2 and R4
iii. the symmetric closure for R2 and R3
iv. the {reflexive, symmetric, transitive }-closure for R1, and R3.

Solution:
a.

	Reflexive	Symmetric	Transitive	Equivalence
R1	no	no	no	no
R2	no	yes	no	no
R3	yes	yes	no	no
R4	yes	yes	yes	yes

b.
i. The reflexive closure for R1 is $R1^* = \{(1,7);(1,8);(7,4);(4,7);(1,1);(4,4);(7,7);(8,8)\}$
The reflexive closure for R2 is $R2^* = \{(1,7);(7,1);(7,4);(4,7);(4,4);(1,1);(7,7);(8,8)\}$
ii. The transitive closure for R2 is $R2^+ = \{(1,7);(7,1);(7,4);(4,7);(4,4);(1,1);(1,4);(7,7);(4,1)\}$
The transitive closure for R4 is $R4^+ = \{(1,1);(4,4);(7,7);(8,8)\}$
iii. The symmetric closure for R2 is $R2_{\text{symmetric}} = \{(1,7);(7,1);(7,4);(4,7);(4,4)\}$
The symmetric closure for R3 is $R3_{\text{symmetric}} = \{(1,1);(1,7);(4,4);(7,1);(7,4);(4,7);(7,7);(8,8)\}$
iv. The {reflexive, symmetric, transitive}-closure is the equivalence closure. Hence,
the equivalence closure for R1 is $R1_{\text{equiv}} = \{(1,7);(1,8);(7,4);(4,7);(1,1);(4,4);(7,7);(8,8);(1,4);(7,1);(8,1)\}$
the equivalence closure for R3 is $R3_{\text{equiv}} = \{(1,1);(1,7);(4,4);(7,1);(7,4);(4,7);(7,7);(8,8);(1,4);(4,1)\}$

Question 2: Answer the following question

Consider the set $S = \{a, b, c, d, e, f\}$ and the relations:
$f1 = \{(a, a), (a, b), (c, d), (e, f)\}$
$f2 = \{(a, b), (b, c), (c, d), (e, d)\}$
$f3 = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f)\}$
$f4 = \{(a, f), (b, b), (c, d), (e, f)\}$
Complete the following table (by writing yes or no):

	Function	1-to-1	Onto	1-to-1 correspondence
f1				
f2				
f3				
f4				

Solution:

	Function	1-to-1	Onto	1-to-1 correspondence
f1	no	no	no	no
f2	yes	no	no	no
f3	yes	yes	yes	yes
f4	yes	no	no	no

Question 3: Prove by induction on n that:

$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$

Solution:
• Base case: $n = 0 \implies P(0) \equiv 0^3 = 0^2$ (true).
• Since the base case is true, we assume the inductive hypothesis $P(k)$ is true for some k. Hence, $P(k) \equiv$ for some k, the following equality holds: $\sum_{i=0}^{k} i^3 = \left(\sum_{i=0}^{k} i\right)^2$. Now, we prove that $P(k+1)$ is also true.

Observe that, $P(k+1) \equiv \sum_{i=0}^{k+1} i^3 = \left(\sum_{i=0}^{k+1} i\right)^2$ (*)

$\Leftrightarrow$ LHS $= \sum_{i=0}^{k} i^3 + (k+1)^3 = \left(\sum_{i=0}^{k} i\right)^2 + (k+1)^3$
Hence, plug LHS back into (*), we obtain:

$$\begin{aligned}
\left(\sum_{i=0}^{k} i\right)^2 + (k+1)^3 &= \left(\sum_{i=0}^{k+1} i\right)^2 \\
\Leftrightarrow (k+1)^3 &= \left(\sum_{i=0}^{k+1} i\right)^2 - \left(\sum_{i=0}^{k} i\right)^2 \\
\Leftrightarrow (k+1)^3 &= \left[\left(\sum_{i=0}^{k+1} i\right) - \left(\sum_{i=0}^{k} i\right)\right] . \left[\left(\sum_{i=0}^{k+1} i\right) + \left(\sum_{i=0}^{k} i\right)\right] \\
\Leftrightarrow (k+1)^3 &= \left[\left(\sum_{i=0}^{k} i\right) + (k+1) - \left(\sum_{i=0}^{k} i\right)\right] . \left[\left(\sum_{i=0}^{k} i\right) + (k+1) + \left(\sum_{i=0}^{k} i\right)\right] \\
\Leftrightarrow (k+1)^3 &= (k+1) . \left(\frac{k(k+1)}{2} + (k+1) + \frac{k(k+1)}{2}\right) \\
\Leftrightarrow (k+1)^3 &= (k+1)\left((k+1) + k(k+1)\right) = (k+1)(k+1)(k+1) \blacksquare
\end{aligned}$$

Hence, $P(k+1)$ is true, proving $P(n)$ true $\forall n$ by mathematical induction.

Question 4: Answer the following question

Prove that for a finite set A, $\left|2^{\mathrm{A}}\right| = 2^{|\mathrm{A}|}$

Solution: Say, the finite set A has the number of elements is n. Hence, the statement to be proven becomes: Given a set A, $|A| = n$; prove that $|2^A| = 2^n$.
We know that 2^A indicates the power set of the set A. So, $|2^A|$ indicates the size of the power set of the set A, or $|\mathscr{P}(A)|$.

• Base case: $n = 1$, say $A = \{a\}$. $|\mathscr{P}(A)| = 2^1 = 2 = \{a, \emptyset\}$ (true).
• Since the base case is true, we assume the inductive hypothesis $P(k)$ is true for some k. Hence, we show that $P(k+1)$ is true. Or, given a set A, $|A| = k+1$; prove that $|\mathscr{P}(A)| = 2^{k+1}$.

Since $P(k)$ is assumed to be true, $|A| = k$ and $|\mathscr{P}(A)| = 2^k$. Let the addition set of 1 element making $|A| = k+1$ is $T = \{z\}$. Note that, T and A are assumed to be disjoint.
Now, the power set of $(A \cup T)$ contains 2 components. The set of subsets of T and the union of the set of subsets of T and the element (or subsets) of the original set A.
$\implies |\mathscr{P}(A \cup T)| = 2^k.2^k = 2^{k+1}$ ■.

Hence, $P(k+1)$ is also true. Proving $P(n)$ true $\forall n$ by mathematical induction.

Question 5: Answer the following question

Show that if S_1 and S_2 are finite sets with $|S_1| = n$ and $|S_2| = m$, then $|S_1 \cup S_2| \leqslant n + m$.

Solution:
According to inclusion-exclusion principle for two sets S_1 and S_2:

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$$
$$|S_1 \cup S_2| = n + m - |S_1 \cap S_2|$$

We know that, $|S_1 \cap S_2| \geqslant 0$ ($= 0$ when S_1 and S_2 are disjoint sets.) Then, $-|S_1 \cap S_2| \leqslant 0$. Hence,

$$|S_1 \cup S_2| = n + m - |S_1 \cap S_2| \leqslant n + m \blacksquare$$

Hence, $|S_1 \cup S_2| \leqslant n + m \ \forall n, m$.