MA 02 LINEAR ALGEBRA II

REVIEW OF LECTURES – X

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Section: C7.
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• Symmetric matrices.

We continue to study 2×2 matrices. Nothing stops us from taking a stab at their larger-size counterparts ($n \times n$ matrices). Going from 2×2 to $n \times n$ is not completely *vis-à-vis*: While the basic properties (associativiaty, distributivity, the identity matrix, *etc.*) are entirely *vis-à-vis*, we are going to run into some delicate classification problem which we haven't previously been saddled by: the eigenvalue/diagonalization problem in the $n \times n$ case with $n \geq 3$ is much subtler than in its 2×2 counterparts. To cope with it, we rely on the notion of "vector spaces". So far we haven't made a refernce to it, but that was only because up until now we were mainly looking at the 2×2 case.

In "Linear Algebra I", we did some abridged version of the theory of 'vector spaces'. It might have struck you as 'pretentious' (for lack of a better term) — full of abstruse formalism. Such an "uninviting" impression notwithstanding, the notion of "vector spaces" is indispensable in math, as in the entire math depends on it. It is probably unrealistic to anticipate your professor (me in this case) to make a strong case that that is true, in one semester (/quarter). But I try to at least give you some supporting evidence in due course.

One benefit of integrating the notion of abstract vector spaces into our discourse is that matrices are naturally recalibrated as 'linear transforms'. All that is seemingly just a minor tweak of wording, but such amenability on our part will be rewarded: For an $n \times n$ matrix A, it is not always true that two eigenvectors associated with one eigenvalue of a matrix A are scalar multiples of each other. Therefore, in order to form Q to diagonalize A (provided it is feasible) you need to choose a 'basis' (one key concept underpinning "vector spaces") of the so-called 'eigenspace' of A. At this point, without adequate understanding of the notion of vector spaces, you would not be able to move forward.

"*Spectral theory*" is one pinnacle of this course. Today we cover the 2×2 case (archetype). We prove that a 2×2 symmetric matrix A with real number entires is diagonalizable by a real orthogonal matrix Q. Proof the same statement for an $n \times n$ matrix *bona fide* requires the notion of abstract vector spaces.

Definition (Transpose).

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define its <u>transpose</u> as $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

So, the transpose of A means you just swap the top-right and the bottom-left entries of A.

Repeat: $\boxed{\text{If} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Longrightarrow \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.}$

Formula. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$,

(1)
$$\left(A^T\right)^T = A.$$

(2)
$$\left(A + B\right)^T = A^T + B^T.$$

(3) If t is a scalar, then
$$\left(tA\right)^T = tA^T.$$

(4)
$$\left(AB\right)^T = B^T A^T.$$

Exercise 1. Prove part (4) of the above formula.

Definition (Symmetric matrix).

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies
$$A^T = A,$$
then we say A is a <u>symmetic matrix</u>.

• Clearly, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a symmetric matrix precisely when the top-right entry and the bottom-left entry of A are equal: $b = c$.

• In other words, a general form of a 2×2 symmetric matrix is

$$\boxed{A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.}$$

Example 1. $\begin{bmatrix} 3 & 5 \\ 5 & -1 \end{bmatrix}$ is a symmetric matrix.

Example 2. $\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$ is a symmetric matrix.

Example 3. $\begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix}$ is a symmetric matrix.

Example 4. $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ is a symmetric matrix.

Example 5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a symmetric matrix.

Example 6. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a symmetric matrix.

Formula 2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix, not necessarily symmetric. Then

(1) $A + A^T$ is symmetric.

(2) AA^T is symmetric.

(3) $A^T A$ is symmetric.

Example 7. $A = \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix}$ is $\underline{\text{ not }}$ a symmetric matrix. Its transpose is

$$A^T = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}.$$

This A^T is $\underline{\text{ not }}$ a symmetric matrix either. However,

(1) $$A + A^T = \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 7 & 0 \end{bmatrix}$$

is a symmetric matrix.

(2) $$AA^T = \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 6 \\ 6 & 9 \end{bmatrix}$$

is a symmetric matrix.

(3) $$A^T A = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 8 & 16 \end{bmatrix}$$

is a symmetric matrix.

Note. As this example shows, in general $\boxed{AA^T \neq A^T A.}$

The next subject is related to the last lecture ("Review of Lecutres – IX"):

Orthogonal matrices revisited.

We have already previously defined orthogonal matrices, in "Review of Lecutres – IX". Let's recall:

Definition 1 (Orthogonal matrices). Matrices of the forms

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},$$

$\big(\theta:$ a real number$\big)$ are called orthogonal matrices.

- There is actually another, equivalent, definition of orthogonal matrices.

Definition 2 (Alternative definition of orthogonal matrices).

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies

$$\boxed{AA^T = I,}$$

then we say A is an orthogonal matrix .

The above "alternative definition" (Definition 2) seemingly differs from our original definition (Definition 1 in the previous page). However, it turns out that those two definitions are mathematically equivalent :

$$\boxed{\text{Definition 1} \underset{\text{equivalent}}{\iff} \text{Definition 2.}}$$

Proof of equivalence between the two definitions (of orthogonal matrices).

We need to prove

(i) [Definition 1 $\implies$ Definition 2] and

(ii) [Definition 2 $\implies$ Definition 1].

∘ First, let's prove (i). This part is easier of the two. For

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

we have

$$
\begin{aligned}
AA^T &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \Big(\cos\theta\Big)\Big(\cos\theta\Big) + \Big(-\sin\theta\Big)\Big(-\sin\theta\Big) & \Big(\cos\theta\Big)\Big(\sin\theta\Big) + \Big(-\sin\theta\Big)\Big(\cos\theta\Big) \\ \Big(\sin\theta\Big)\Big(\cos\theta\Big) + \Big(\cos\theta\Big)\Big(-\sin\theta\Big) & \Big(\sin\theta\Big)\Big(\sin\theta\Big) + \Big(\cos\theta\Big)\Big(\cos\theta\Big) \end{bmatrix} \\
&= \begin{bmatrix} \Big(\cos\theta\Big)^2 + \Big(\sin\theta\Big)^2 & 0 \\ 0 & \Big(\sin\theta\Big)^2 + \Big(\cos\theta\Big)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Next, for

$$
B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},
$$

we have

$$
\begin{aligned}
BB^T &= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \Big(\cos\theta\Big)\Big(\cos\theta\Big) + \Big(\sin\theta\Big)\Big(\sin\theta\Big) & \Big(\cos\theta\Big)\Big(\sin\theta\Big) + \Big(\sin\theta\Big)\Big(-\cos\theta\Big) \\ \Big(\sin\theta\Big)\Big(\cos\theta\Big) + \Big(-\cos\theta\Big)\Big(\sin\theta\Big) & \Big(\sin\theta\Big)\Big(\sin\theta\Big) + \Big(-\cos\theta\Big)\Big(-\cos\theta\Big) \end{bmatrix} \\
&= \begin{bmatrix} \Big(\cos\theta\Big)^2 + \Big(\sin\theta\Big)^2 & 0 \\ 0 & \Big(\sin\theta\Big)^2 + \Big(\cos\theta\Big)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Here, in the above, we relied on

Formula. $\boxed{\Big(\cos\theta\Big)^2 + \Big(\sin\theta\Big)^2 = 1.}$

- Second, let's prove (ii). Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies $AA^T = I$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

That is

$$\begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So

$$(*) \qquad \begin{cases} a^2 + b^2 = 1, \\ c^2 + d^2 = 1, \\ ac + bd = 0. \end{cases}$$

Now we rely on the following, from "Review of Lecutres – IX":

Fact. Let a and b be real numbers. Suppose they satisfy

$$a^2 + b^2 = 1.$$

Then there exists a real number θ such that

$$a = \cos\theta, \qquad b = \sin\theta.$$

If you use this, you conclude

$$a = \cos\theta, \qquad b = \sin\theta$$

from $a^2 + b^2 = 1$ $\big(\text{in } (*)\big)$, and also conclude

$$c = \cos\phi, \qquad d = \sin\phi.$$

from $c^2 + d^2 = 1$ $\big(\text{in } (*)\big)$.

Now substitute these into the equation

$$ac + bd = 0$$

(in $(*)$):

$$\Big(\cos\theta\Big)\Big(\cos\phi\Big) + \Big(\sin\theta\Big)\Big(\sin\phi\Big) = 0.$$

In other words

$$\cos\Big(\theta - \phi\Big) = 0.$$

(Here we used one of the 'trig identities'. See below.) So

$$\theta - \phi = \frac{\pi}{2} + k\pi,$$

that is,

$$\phi = \theta - \frac{\pi}{2} - k\pi,$$

with some integer k. Then it follows that

$$\begin{aligned} c &= \cos\phi \\ &= \cos\Big(\theta - \frac{\pi}{2} - k\pi\Big) \\ &= \pm\sin\theta \end{aligned}$$

(double sign is '+' when k is even, '−' when k is odd), and

$$\begin{aligned} d &= \sin\phi \\ &= \sin\Big(\theta - \frac{\pi}{2} - k\pi\Big) \\ &= \mp\cos\theta \end{aligned}$$

(double sign is '−' when k is even, '+' when k is odd). So we conclude

$$A = \begin{bmatrix} a & b \\ \pm b & \mp a \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \pm\sin\theta & \mp\cos\theta \end{bmatrix} \quad \Big(\text{double sign in the same order}\Big). \qquad \square$$

- In the above proof, we have used one of the following:

Formula.

$$\cos\Big(\theta+\phi\Big) = \Big(\cos\theta\Big)\Big(\cos\phi\Big) - \Big(\sin\theta\Big)\Big(\sin\phi\Big),$$
$$\sin\Big(\theta+\phi\Big) = \Big(\sin\theta\Big)\Big(\cos\phi\Big) + \Big(\cos\theta\Big)\Big(\sin\phi\Big),$$

$$\cos\Big(\theta-\phi\Big) = \Big(\cos\theta\Big)\Big(\cos\phi\Big) + \Big(\sin\theta\Big)\Big(\sin\phi\Big),$$
$$\sin\Big(\theta-\phi\Big) = \Big(\sin\theta\Big)\Big(\cos\phi\Big) - \Big(\cos\theta\Big)\Big(\sin\phi\Big).$$

- **Spectral theory – I (Archetype).**

  Now we are ready to talk about 'spectral theory'. Here is the main statement:

Spectral Theorem (2 × 2 case). Let A be a 2×2 symmetric matrix:

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

Suppose a, b, c and d are all real numbers. Then there exists an orthogonal matrix

$Q = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, where p, q, r and s are all real numbers, such that

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

where λ_1 and λ_2 are the eigenvalues of A $\big($they may or maynot coincide$\big)$.

Proof of Spectral theorem.

First, with a suitable scalar t, we can make

$$A - tI = \begin{bmatrix} a-t & b \\ b & d-t \end{bmatrix}$$

"traceless" :

$$\operatorname{tr}\Big(A - tI\Big) = 0.$$

Or, in a plain language,

$$\Big(a-t\Big) + \Big(d-t\Big) = 0.$$

Let's call this new matrix B:

$$B = A - tI = \begin{bmatrix} a' & b \\ b & d' \end{bmatrix}.$$

So

$$\operatorname{tr} B = 0,$$

or, in a plain language,

$$a' + d' = 0.$$

Next, with a suitable scalar s, we can make

$$sB = \begin{bmatrix} sa' & sb \\ sb & sd' \end{bmatrix}$$

satisfy

$$sa' = \cos\theta, \qquad sb = \sin\theta \quad \Big(\text{for some real number } \theta\Big).$$

$\Big($Indeed, $\quad s = \Big(\sqrt{a'^2 + b^2}\Big)^{-1}.\Big)$ Then from $\quad a' + d' = 0 \quad$ we also have

$$sd' = -\cos\theta$$

for the same θ. Let's call this new matrix C:

$$C = sB = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

We already know that C has eigenvalues ± 1, and moreover

$$\boldsymbol{x}_\pm = \begin{bmatrix} \sin\theta \\ -\Big(\cos\theta\Big) \pm 1 \end{bmatrix}$$

is an eigenvector of C associated with $\lambda = \pm 1$ $\big($the double sign in the same order$\big)$.

Here, recall that for a vector $\boldsymbol{u} = \begin{bmatrix} u \\ v \end{bmatrix}$, we define its <u>norm</u> as

$$\|\boldsymbol{u}\| = \sqrt{u^2 + v^2}\,.$$

Also recall that, if $\boldsymbol{u} \neq \mathbf{0}$, then

$$\left\| \frac{1}{\|\boldsymbol{u}\|}\boldsymbol{u} \right\| = 1.$$

Now, $\|\boldsymbol{x}_\pm\| = \sqrt{2 \mp 2\cos\theta}\,.$ Accordingly, set

$$\boldsymbol{y}_\pm = \frac{1}{\|\boldsymbol{x}_\pm\|}\boldsymbol{x}_\pm = \frac{1}{\sqrt{2 \mp 2\cos\theta}} \begin{bmatrix} \sin\theta \\ -\Big(\cos\theta\Big) \pm 1 \end{bmatrix}$$

$\big($double sign in the same order still being intact$\big)$. Then $\boldsymbol{y}_\pm$ is another eigenvector of C associated with $\lambda = \pm 1$. $\big($Indeed, $\boldsymbol{y}_\pm$ is a non-zero scalar multiple of $\boldsymbol{x}_\pm$, and $\boldsymbol{x}_\pm$ is an eigenvector of C associated with $\lambda = \pm 1.\big)$ Now, set

$$(*) \qquad Q = \begin{bmatrix} \boldsymbol{y}_+ & \boldsymbol{y}_- \end{bmatrix} = \begin{bmatrix} \dfrac{\sin\theta}{\sqrt{2 - 2\cos\theta}} & \dfrac{\sin\theta}{\sqrt{2 + 2\cos\theta}} \\[2ex] \dfrac{-\Big(\cos\theta\Big) + 1}{\sqrt{2 - 2\cos\theta}} & \dfrac{-\Big(\cos\theta\Big) - 1}{\sqrt{2 + 2\cos\theta}} \end{bmatrix}.$$

Then Q is an orthogonal matrix. Indeed, by the trig "half-angle formula", the above Q is alternatively written as follows:

$$(**) \qquad Q = \begin{bmatrix} \cos\dfrac{\theta}{2} & \sin\dfrac{\theta}{2} \\ \sin\dfrac{\theta}{2} & -\cos\dfrac{\theta}{2} \end{bmatrix} \qquad \bigl(\text{provided } 0 < \theta < \pi\bigr).$$

Note that, although the second expression $(**)$ is clearly more elegant than the first expression $(*)$, it has pros and cons: When you've already gotten a hold of the values of $\sin\theta$ and $\cos\theta$ $\bigl($namely, you've gotten a hold of the matrix $C\bigr)$, you still don't immediately know the values of $\sin\dfrac{\theta}{2}$ and $\cos\dfrac{\theta}{2}$. So you would have to rely on the expression $(*)$ in order to figure them out. So, that way you will end up relying on $(*)$ anyway. $\square$

Exercise 2. (1) For $\boldsymbol{x}_{\pm} = \begin{bmatrix} \sin\theta \\ -\bigl(\cos\theta\bigr) \pm 1 \end{bmatrix}$, prove

$$\|\boldsymbol{x}_{\pm}\| = \sqrt{2 \mp 2\cos\theta} \qquad \bigl(\text{double sign in the same order}\bigr).$$

(2) Prove that $(*)$ and $(**)$ are two mutually mathematically equivalent expressions of the same matrix. Namely, prove (2a), (2b), (2c) and (2d) below:

$$\text{(2a)} \qquad \frac{\sin\theta}{\sqrt{2 - 2\cos\theta}} = \cos\frac{\theta}{2} \qquad \bigl(0 < \theta < \pi\bigr),$$

$$\text{(2b)} \qquad \frac{\sin\theta}{\sqrt{2 + 2\cos\theta}} = \sin\frac{\theta}{2} \qquad \bigl(0 < \theta < \pi\bigr),$$

$$\text{(2c)} \qquad \frac{-\bigl(\cos\theta\bigr) + 1}{\sqrt{2 - 2\cos\theta}} = \sin\frac{\theta}{2} \qquad \bigl(0 < \theta < \pi\bigr), \quad \text{and}$$

$$\text{(2d)} \qquad \frac{-\bigl(\cos\theta\bigr) - 1}{\sqrt{2 + 2\cos\theta}} = -\cos\frac{\theta}{2} \qquad \bigl(0 < \theta < \pi\bigr).$$

Example 8. Let

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.$$

Let's make $Q^{-1}AQ$ a diagonal matrix, with some $\underline{\text{ orthogonal }}$ matrix Q.

Step 1. Find a suitable scalar t such that $A - tI$ is traceless: $\operatorname{tr}\Big(A - tI\Big) = 0$. In this case, the trace of the original matrix A is

$$5 + 2 = 7.$$

So $\dfrac{1}{2}$ times it, that is, $\dfrac{7}{2}$, will do:

$$t = \frac{7}{2}.$$

Accordingly the matrix $B = A - tI$ becomes

$$\begin{aligned} B &= A - \frac{7}{2}I \\ &= \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \frac{7}{2} & 0 \\ 0 & \frac{7}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2} & 2 \\ 2 & -\frac{3}{2} \end{bmatrix}. \end{aligned}$$

Step 2. Find a suitable scalar s such that

$$s \cdot \frac{3}{2} = \cos\theta, \qquad s \cdot 2 = \sin\theta \qquad \Big(\text{for some real number } \theta\Big).$$

This s is simply found as

$$
\begin{aligned}
s &= \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 + 2^2}} \\
&= \frac{1}{\sqrt{\frac{9}{4} + 4}} \\
&= \frac{1}{\sqrt{\frac{25}{4}}} \\
&= \sqrt{\frac{4}{25}} = \frac{2}{5}.
\end{aligned}
$$

Accordingly the matrix $C = sB$ $\big($where B is found in Step 1$\big)$ becomes

$$
\begin{aligned}
C &= \frac{2}{5} B \\
&= \frac{2}{5} \begin{bmatrix} \frac{3}{2} & 2 \\ 2 & -\frac{3}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}.
\end{aligned}
$$

So, let's agree that C is indeed of the form

$$
C = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},
$$

where

$$
\cos\theta = \frac{3}{5}, \qquad \text{and} \qquad \sin\theta = \frac{4}{5}.
$$

In particular, C is indeed an orthogonal matrix.

Step 3. Recall that the matrix Q in Exercise 2 above $\big($page 12$\big)$ itself is an orthogonal matrix, and with which we can accomplish

$$Q^{-1}CQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\big(\lambda = \pm 1$ are the two eigenvalues of $C\big)$. Multiply $s^{-1} = \dfrac{5}{2}$ to the two sides:

$$Q^{-1}BQ = \begin{bmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{5}{2} \end{bmatrix}.$$

Add $tI = \dfrac{7}{2}I$ to the two sides:

$$\begin{aligned} Q^{-1}AQ &= \begin{bmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} + \begin{bmatrix} \frac{7}{2} & 0 \\ 0 & \frac{7}{2} \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

This is the final result, the diagonalized matrix we sought. We need to provide the concrete shape of Q as well.

Step 4. Keeping in mind

$$\cos\theta = \frac{3}{5}, \qquad \text{and} \qquad \sin\theta = \frac{4}{5}$$

$\big($from Step 2 above$\big)$, we may find Q using $(*)$ $\big($in page 11$\big)$:

$$Q = \begin{bmatrix} \dfrac{\sin\theta}{\sqrt{2 - 2\cos\theta}} & \dfrac{\sin\theta}{\sqrt{2 + 2\cos\theta}} \\[2ex] \dfrac{-\Big(\cos\theta\Big) + 1}{\sqrt{2 - 2\cos\theta}} & \dfrac{-\Big(\cos\theta\Big) - 1}{\sqrt{2 + 2\cos\theta}} \end{bmatrix}$$

$$
= \begin{bmatrix} \dfrac{\dfrac{4}{5}}{\sqrt{2 - 2 \cdot \dfrac{3}{5}}} & \dfrac{\dfrac{4}{5}}{\sqrt{2 + 2 \cdot \dfrac{3}{5}}} \\ \dfrac{-\dfrac{3}{5} + 1}{\sqrt{2 - 2 \cdot \dfrac{3}{5}}} & \dfrac{-\dfrac{3}{5} - 1}{\sqrt{2 + 2 \cdot \dfrac{3}{5}}} \end{bmatrix}
$$

$$
= \begin{bmatrix} \dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \\ \dfrac{1}{\sqrt{5}} & -\dfrac{2}{\sqrt{5}} \end{bmatrix}.
$$

- To summarize:

Diagonalization result. $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$ is diagonalized by an orthogonal matrix Q as follows:

$$
Q^{-1}AQ = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{where} \quad Q = \begin{bmatrix} \dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \\ \dfrac{1}{\sqrt{5}} & -\dfrac{2}{\sqrt{5}} \end{bmatrix}.
$$

Exercise 3. Make $Q^{-1}AQ$ a diagnonal matrix, with some orthogonal matrix Q. Write out both $Q^{-1}AQ$ and Q.

(1) $A = \begin{bmatrix} 3 & 6 \\ 6 & -2 \end{bmatrix}$. (2) $A = \begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$.

(3) $A = \begin{bmatrix} 10 & 7 \\ 7 & 10 \end{bmatrix}$.