MA 02 LINEAR ALGEBRA II

REVIEW OF LECTURES – II

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Section: C7.
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So, we just saw some glimpse of eigenvalues. Today I explain how to actually calculate eigenvalues. Excited? We are going to stick with the 2×2 case, though. (We worry about the larger size case later.) For that matter, we need to do some more reviews, we need to cover some preliminary materials. Let's jump-start:

- Right off the bat, check this out:

$$\begin{bmatrix} \frac{2}{5} & \frac{-4}{5} \\ \frac{-1}{5} & \frac{1}{5} \end{bmatrix}.$$

This is just a matrix. What's the big deal? All the entries are in fractions. They all have the denominator 5. So, we would much rather write it like

$$\frac{1}{5}\begin{bmatrix} 2 & -4 \\ -1 & 1 \end{bmatrix}.$$

This way we don't have to write the denominator 5 multiple times. Technically, though, this is "a scalar multiplied to a matrix". So we should set a rule about "how you multiply a scalar to a matrix". I'm sure this was covered in Linear Algebra I. But a quick refresher wouldn't hurt. Here we go: So, we would want a rule that allows us to do something like

$$\begin{bmatrix} \frac{2}{5} & \frac{-4}{5} \\ \frac{-1}{5} & \frac{1}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -4 \\ -1 & 1 \end{bmatrix}.$$

Good news: Nothing stops us from setting the following rule which also validates this:

- **Definition (Scalar multiplication).** Let s be a scalar (a number). Then

$$\boxed{s\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\begin{array}{llll} \text{If} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \text{and} & s : \text{ a scalar} \\ \Longrightarrow & & sA = \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}. \end{array}}$$

- In the above, s doesn't have to be a fraction. s can be any real number.

  Why don't we throw the following too, something to be paired with the above:

- **Definition (negation).**

$$\boxed{- \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\text{If} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Longrightarrow \quad -A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}.}$$

Example 1. (1) $$3 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}.$$

(2) $$4 \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix}.$$

(3) $$\frac{1}{7} \begin{bmatrix} 5 & 7 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} & 1 \\ \frac{-1}{7} & 0 \end{bmatrix}.$$

(4) $$\frac{9}{2} \begin{bmatrix} \frac{2}{9} & 2 \\ \frac{4}{9} & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 2 & \frac{1}{2} \end{bmatrix}.$$

• More examples. 'Trivial' ones:

Example 2. $$1\begin{bmatrix}0 & -2\\ 6 & 3\end{bmatrix} = \begin{bmatrix}0 & -2\\ 6 & 3\end{bmatrix}.$$

• An obvious generalization of Example 2 is

$$1\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

Paraphrase:

$$\boxed{\text{If} \quad A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \Longrightarrow \quad 1A = A.}$$

Example 3. $$0\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \qquad 8\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

• An obvious generalization of Example 3 is

$$0\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \qquad s\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

• We denote $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ as O. Then we can paraphrase it as:

$$\boxed{\begin{array}{l} \text{If} \quad A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad \text{and} \quad s : \text{a scalar} \\ \Longrightarrow \qquad 0A = O, \quad sO = O. \end{array}}$$

Example 4a. $$\left(-1\right)\begin{bmatrix}3 & 4\\ 5 & 9\end{bmatrix} = \begin{bmatrix}-3 & -4\\ -5 & -9\end{bmatrix}.$$

Example 4b. $$-\begin{bmatrix}3 & 4\\ 5 & 9\end{bmatrix} = \begin{bmatrix}-3 & -4\\ -5 & -9\end{bmatrix}.$$

• Examples 4a, 4b indicate that the negative of a matrix and the $\left(-1\right)$ times the same matrix are equal. This is true in general. Namely:

$$\boxed{\left(-1\right)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\text{If} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \left(-1\right)A = -A.}$$

Exercise 1. Write each of the following in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(1) $3\begin{bmatrix} -4 & 2 \\ 6 & 5 \end{bmatrix}$. (2) $\dfrac{1}{2}\begin{bmatrix} 10 & 12 \\ 8 & 4 \end{bmatrix}$. (3) $\dfrac{1}{8}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(4) $\left(-2\right)\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$. (5) $1\begin{bmatrix} 7 & -5 \\ \frac{1}{2} & 1 \end{bmatrix}$. (6) $0\begin{bmatrix} 124 & 242 \\ 163 & 89 \end{bmatrix}$.

(7) $1000\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Exercise 2. Write each of the following in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(1) $-\begin{bmatrix} -6 & -8 \\ 3 & 4 \end{bmatrix}$. (2) $-\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Exercise 3. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\big($the transpose of $A\big)$. Assume $A^T = -A$. Prove that there is a scalar s such that $A = s\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

And that was just an intro. The actual first order of business today is the <u>determinants</u>. Like I said, let's stick with the 2×2 case. The morale is, if you don't understand the 2×2 case, you will not understand the larger size case.

Definition (Determinant).

The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined as follows:

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.}$$

- We also frequently use the following notation:

The notation 'det' .

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A is often written as $\det A$. So

$$\boxed{\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.}$$

- **Remark.** Clearly, the determinant of a matrix is *not* a matrix. Rather, the determinant of a matrix is a scalar $\big($as the next few examples will illustrate it$\big)$:

Example 5. (1) For $A = \begin{bmatrix} 7 & 5 \\ 2 & 1 \end{bmatrix}$, its determinant is

$$\begin{aligned} \det A = \begin{vmatrix} 7 & 5 \\ 2 & 1 \end{vmatrix} &= 7 \cdot 1 - 5 \cdot 2 \\ &= -3. \end{aligned}$$

(2) For $A = \begin{bmatrix} -6 & 2 \\ 8 & -4 \end{bmatrix}$, its determinant is

$$\begin{aligned} \det A = \begin{vmatrix} -6 & 2 \\ 8 & -4 \end{vmatrix} &= (-6) \cdot (-4) - 2 \cdot 8 \\ &= 8. \end{aligned}$$

(3) For $A = \begin{bmatrix} -2 & 4 \\ -3 & 6 \end{bmatrix}$, its determinant is

$$\det A = \begin{vmatrix} -2 & 4 \\ -3 & 6 \end{vmatrix} = \big(-2\big)\cdot 6 - 4\cdot\big(-3\big)$$

$$= 0.$$

(4) For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, its determinant is

$$\det A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\cdot 1 - 0\cdot 0$$

$$= 1.$$

- Now, do you agree with the following?

Formula 1.

$$\boxed{\begin{vmatrix} sa & sb \\ sc & sd \end{vmatrix} = s^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}.}$$

Formula 1 paraphrased. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\boxed{\det\big(sA\big) = s^2 \det A.}$$

Exercise 4. Calculate:

(1) $\begin{vmatrix} 1 & 6 \\ 1 & 3 \end{vmatrix}$. (2) $\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$. (3) $\begin{vmatrix} 2 & 5 \\ \frac{3}{10} & 4 \end{vmatrix}$.

(4) $\det A$, where $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$.

(5a) $\quad \det A, \quad$ where $\quad A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$

(5b) $\quad \det B, \quad$ where $\quad B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \left(= \dfrac{1}{\sqrt{2}} A \quad \text{where } A \text{ is in (5a).}\right)$

(6a) $\quad \det A, \quad$ where $\quad A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$

(6b) $\quad \det B, \quad$ where $\quad B = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \left(= \dfrac{1}{2} A \quad \text{where } A \text{ is in (6a).}\right)$

(7a) $\quad \det A, \quad$ where $\quad A = \begin{bmatrix} -1+\sqrt{5} & -\sqrt{10+2\sqrt{5}} \\ \sqrt{10+2\sqrt{5}} & -1+\sqrt{5} \end{bmatrix}.$

(7b) $\quad \det B, \quad$ where $\quad B = \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix} \left(= \dfrac{1}{4} A \quad \text{where } A \text{ is in (7a).}\right)$

Exercise 5. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. True or false:

(1a) $\quad \left(-2\right)A = -\left(2A\right).$ $\qquad$ (1b) $\quad \left(-2\right)A = 2\left(-A\right).$

(2a) $\quad -\left(-1\right)A = A.$ $\qquad$ (2b) $\quad -\left(-A\right) = A.$

(3) $\quad 3\left(7A\right) = 21A.$

(4a) $\quad 0A = A.$ $\qquad$ (4b) $\quad 3O = O.$

(5a) $\quad \det\left(5A\right) = 5\det A.$ $\qquad$ (5b) $\quad \det\left(-A\right) = -\det A.$

- Sorry to be repetitious, but once again:

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.}$$

This $ad - bc$ looks familiar. We've seen it last time. Do you remember where? Yes, the "inversion formula" from the previous lecture. Let me recite:

Inversion formula. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the inverse of A exists provided $ad - bc \neq 0$, and

$$\boxed{A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.}$$

Last time I didn't say the following because I didn't want to overstuff the lecture. But this formula looks a little crammed, because of the fractions. Thanks to the rule that we have just adopted, we can actually write

$$\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \left(\text{instead of} \quad \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}\right).$$

So here we go:

Inversion formula Paraphrased – I. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the inverse of A exists provided $ad - bc \neq 0$, and

$$\boxed{A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.}$$

Better still:

Inversion formula Paraphrased – II. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the inverse of A exists provided $\det A \neq 0$, and

$$\boxed{A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} .}$$

Clearly these two versions have the same content, but the second one uses the notation 'det'.

• So far this is pretty darn basic. But please don't be misled about the level of difficulty. Students often grumble like "my professor started the class with a 3rd grade level math, and then at one point (s)he was covering something that was above my head, and the next thing I know I just had no clue what was going on, I got so lost." The undertone is the professor is merciless. Actually, if that sutdent had carefully followed what has been covered from Day 1, not taking anything for granted, no matter how easy it appeared, that would not have happened. So, like I said, no, please don't take the level of difficulty for granted. But the student is not blameworthy in one respect, namely, all that has to do with the nature of math . They say "*hate the game, not the players*," The nature of math, the way I like to describe it, has some parallels to it: math is a unique scientific discipline, in the sense it does not depend on any other scientific discipline. On the other hand, physics relies on math. So, naturally a part of physics courses is math. On the contrary, technically a math course does not go outside of math. As a part of that picture, math is like a dictionary, where the latter defines something as plain as 'apple' using words, because a dictionary is closed within lexicons. Similarly, one aspect of math is you have to begin the lecture at the level you tell people a descriptive definition of an apple, pretending that they don't know what an apple is even though in reality they indeed know what an apple is. That part is redundant, if you received a specialized training in math, as in if you are a math major. Now, in our class too, that is 'necessary'. Well, to me, that is not really a bad news. Now, here is another important aspect of math, to which the above dictionary metaphor *does not* apply: In math, you make sure everything is presented in a logically consistent fashion. Avoid defining the concept A using the concept B and then define the concept B using the concept A. In math, that is called a 'tautology'.* As a branch of science, math possesses these two distinctive features, and any math class is bound by them. We build everything from complete scratch (a *tabula rasa* approach).

*It means a 'catch 22', a circular argument.

• Next let's familiarize ourselves with the determinant of a matrix that involves a letter. I don't want you to be afraid to see letters inside the determinant formations.

Example 6. Let's calculate $\begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix}$.

It goes as follows:

$$\begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix} = x \cdot 1 - 1 \cdot 1$$

$$= x - 1.$$

Example 7. Let's calculate $\begin{vmatrix} x-2 & 4 \\ 3x & x^2 \end{vmatrix}$.

It goes as follows:

$$\begin{vmatrix} x-2 & 4 \\ 3x & x^2 \end{vmatrix} = \Big(x-2\Big)x^2 - 4 \cdot 3x$$

$$= x^3 - 2x^2 - 12x.$$

Example 8a. Let's calculate $\begin{vmatrix} x & -y \\ y & x \end{vmatrix}$.

It goes as follows:

$$\begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x \cdot x - \Big(-y\Big) \cdot y$$

$$= x^2 + y^2.$$

Example 8b. Let's calculate $\begin{vmatrix} x & y \\ y & -x \end{vmatrix}$.

It goes as follows:

$$\begin{vmatrix} x & y \\ y & -x \end{vmatrix} = x \cdot \Big(-x\Big) - y \cdot y$$

$$= -x^2 - y^2.$$

• Remember, we are shooting for the <u>eigenvalues</u> today. The following is the penultimate step toward that goal.

Definition (characteristic polynomial). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We define the <u>characteristic polynomial</u> of A as the following determinant:

$$\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix}.$$

Here, the Greek letter λ is used. There is no logical reason why we should use λ $\big($it could have been $x\big)$, but this is what everybody else does. So let's stick with it.

• **Warning.** (i) the determinant of A

is one thing,

(ii) the characteristic polyomial of A

is another. <u>Please make sure to understand that those two are different.</u>

Let's take a look at some examples.

Example 9. Let's calculate the characteristic polynomial of

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}.$$

It goes as follows:

$$\begin{aligned} \begin{vmatrix} \lambda - 3 & -1 \\ -2 & \lambda - 4 \end{vmatrix} &= \big(\lambda - 3\big)\big(\lambda - 4\big) - \big(-1\big) \cdot \big(-2\big) \\ &= \lambda^2 - 7\lambda + 12 - 2 \\ &= \lambda^2 - 7\lambda + 10. \end{aligned}$$

Now, if you are told to 'factor' this, can you oblige? Yes:

$$\lambda^2 - 7\lambda + 10 = \big(\lambda - 2\big)\big(\lambda - 5\big).$$

What do you see?

— Yes, after factorizaiton, 2 and 5 came out as the roots of

$$\lambda^2 - 7\lambda + 10 = 0.$$

Breaking news — the two numbers 2 and 5 are the eigenvalues of A.

More gnenerally:

- **Method how to find eigenvalues.**

> Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Set the characteristic polynomial of A to equal 0:
>
> $$\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = 0.$$
>
> This is a quadratic equation. Solve it. The roots are the eigenvalues of A.

In Example 9, we found the eigenvalues of $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ following the above method. They are 2 and 5.

Example 10. Let's calculate the characteristic polynomial of $B = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$. Let's also find eigenvalues of B (if any).

It goes as follows. First, the characteristic polynomial of B:

$$\begin{aligned} \begin{vmatrix} \lambda - 3 & -3 \\ -1 & \lambda - 5 \end{vmatrix} &= (\lambda - 3)(\lambda - 5) - (-3) \cdot (-1) \\ &= \lambda^2 - 8\lambda + 15 - 3 \\ &= \lambda^2 - 8\lambda + 12. \end{aligned}$$

As for the eigenvalues of B, let's factor it:

$$\lambda^2 - 8\lambda + 12 = (\lambda - 2)(\lambda - 6).$$

So we just found the eigenvalues of B, which are 2 and 6.

Exercise 6. (1a) Calculate the characteristic polynomial of $A = \begin{bmatrix} -3 & -2 \\ 6 & 5 \end{bmatrix}$.

(1b) Find the eigenvalues of A in (1a) $\big($if any$\big)$.

(2a) Calculate the characteristic polynomial of $B = \begin{bmatrix} -2 & -4 \\ 1 & -6 \end{bmatrix}$.

(2b) Find the eigenvalues of B in (2a) $\big($if any$\big)$.