MA 02 LINEAR ALGEBRA II

REVIEW OF LECTURES – I

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Section: C7.

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Welcome to MA02, "Linear Algebra II". I know you have already taken MA01, "Linear Algebra I". And you are eager to take another dose of linear algebra. That's commendable, because you can bet this 'sequel' will be centered around a deeper side of linear algebra. So, what is linear algebra? I always throw this question on Day 1 of just about any course I teach "what is [*the course title*]?" Everyone knows that linear algebra is basically about matrices. Actually sometimes we hear the term 'matrix algebra' and mean it to be the same as 'linear algebra'. So are they synonymous? First of all, matrix algebra sounds rudimentary. You have to know how to multiply matrices out, like

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 1 & 5 \end{bmatrix}.$$

But then not much else, or what? Determinants? Sure. But then towards the end of that class (I'm talking about "Linear Algebra I") you have also learned the notion of 'vector spaces' in that class (well, most of you, and if you haven't, don't worry I will cover it very thoroughly). Actually I bet you have had an impression 'why suddenly this?' As a matter of fact, your class suddenly switched gears and addressed completely something else, except you keep hearing the word 'vectors', and a part of which looked something as primitive as

$$\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x},$$

and also something that is as "meaningless" (quote-unquote) as

$$1 \cdot \boldsymbol{x} = \boldsymbol{x}.$$

So you really don't know what this is all about or why this is a part of the curricula. All this sounds just playing with words. So you are like "all I need to know is how to manipulate matrices, how to calculate determinants and that should suffice as a course content for my graduation requirement". Judging my undertone, you probably know what I am going to say next.

"*No, not so fast!*"

A pop quiz: Have you ever heard the word 'eigenvalues'? What is an eigenvalue? If suppose

$$A\boldsymbol{x} = 3\boldsymbol{x}$$

where A is a matrix, and $\boldsymbol{x}$ is a vector, then 3 is called what? Yep, an eigenvalue for A. You probably have heard the word 'eigenvalue' somewhere along the way, probably along with another, related, word 'eigenvector'. By the way, in the above, $\boldsymbol{x}$ is called an eigenvector of A, associated with the eigenvalue 3.

Another pop quiz: Suppose

$$B\boldsymbol{y} = \boldsymbol{y},$$

where B is a matrix and $\boldsymbol{y}$ a vector. Can you detect an eigenvalue of B? Yes, 1 is an eigenvalue of B. Also, $\boldsymbol{y}$ is an eigenvector of B, associated with the eigenvalue 1. But in this equation you don't see 1 anywhere. Where does 1 come from? Yes, $\boldsymbol{y}$ on the right-hand side is actually $1\boldsymbol{y}$. But hey, that's something you were dismissive about earlier. In reality, you are actually dependent on this: $1 \cdot \boldsymbol{y} = \boldsymbol{y}$. Without it you cannot answer my question. So should we still dismiss it? You might still argue like this is trivial, or like I'm just pulling your leg: $1 \cdot \boldsymbol{y}$ is just $\boldsymbol{y}$ and that's been that way for thousands of years. It is going to be that way until the end of the human species. Okay. So, then, can you *prove* it? More on this later.

Anyway, it is true that you keep hearing the word 'eigenvalues', and that certainly makes you think 'eigenvalue' is an important concept in math. But have you ever second-guessed why that is important, or it is really *that* important? My take? I say "you have the right to know if and why the concept of eigenvalues is important." "First off, it is an absolute truth that the concept of eigenvalues is indeed super-duper important in math. The fact that this is emphasized in Linear Algebra class is not by some freakish fluke, it is for a reason." Uh-huh. But why is it so important then? Did you get to hear that part as well in your Linear Algebra I course? Maybe not. Wanna hear that? If you say 'no thanks', then maybe you are not as motivated as you are expected to be. What did I say a few minutes ago? I said for you to sign up for this class is really commendable. But you must have a strong reason why you decided to take this class in the first place. Yes, the reason might be that it is a part of the requirement, but you don't just take classes only because of the requirements, right? Because your genuine intellectual curiosity should also drive you to sign up for classes. I certainly don't mind it, if your primary reason you are here in this room is to earn credits, fulfill requirements, and graduate. But I also want to hear, and hopefully this is not just my wishful thinking, that you wanted to know the answer to all the above and more. Then congratulations, you are in the right place. So like I said, I welcome you.

So, why don't we jump-start some crash course on matrix multiplications. Today I want to stay basic, so let's just review some 'easy' stuffs (bear with me):

- **Rule – I.**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad \underline{\text{ is calculated as }} \quad \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.$$

- Repeat:

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.}$$

- Paraphrase:

$$\boxed{\begin{aligned} & A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \\ & \Longrightarrow \quad AB = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}. \end{aligned}}$$

Now, this rule may not be too self-evident — just with casual scanning you don't really get the idea how you should go about it (that's if you see it for the first time here, which is not the case with you, though). So a breakdown:

- **Break-down:** First off $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called a <u>matrix</u> (<u>matrices</u> in plural).

Sure, everybody knows that. But this one is more specifically called a <u>2×2 matrix</u>.

In other words, $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$ becomes a 2×2 matrix once you have filled numbers in the boxes. Second:

- A and B are both 2×2 matrices $\Longrightarrow$ AB is a 2×2 matrix.

  In other words:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

(i) Let us find $\diamondsuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{\diamondsuit} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

Since $\diamondsuit$ is in the top-left, accordingly highlight the portion of A and B, like

$$\begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} & \begin{matrix} q \\ s \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{\diamondsuit} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

Pretend that the two highlighted parts are the vectors $\begin{bmatrix} a, & b \end{bmatrix}$ and $\begin{bmatrix} p, & r \end{bmatrix}$. Do their "dot product":

$$a\,p \quad + \quad b\,r.$$

That is going to be sitting in the $\diamondsuit$ spot:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

(ii) Next, let us find $\heartsuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} & \boxed{\heartsuit} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

Since $\heartsuit$ is in the top-right, accordingly highlight the portion of A and B, like

$$\begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix} \begin{bmatrix} \begin{matrix} p \\ r \end{matrix} & \boxed{\begin{matrix} q \\ s \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{ap + br} & \boxed{\heartsuit} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

Pretend that the two highlighted parts are the vectors $\begin{bmatrix} a, & b \end{bmatrix}$ and $\begin{bmatrix} q, & s \end{bmatrix}$.

Do their "dot product":

$$a\,q \quad + \quad b\,s.$$

That is going to be sitting in the $\heartsuit$ spot:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{} & \boxed{} \end{bmatrix}.$$

(iii) Similarly, we can find $\clubsuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{\clubsuit} & \boxed{} \end{bmatrix}$$

by highlighting

$$\begin{bmatrix} a & b \\ \boxed{c \quad d} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} & \begin{matrix} q \\ s \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{\clubsuit} & \boxed{} \end{bmatrix}.$$

The highlighted parts are the vectors $\begin{bmatrix} c, & d \end{bmatrix}$ and $\begin{bmatrix} p, & r \end{bmatrix}$. Do their "dot product":

$$c\,p \quad + \quad d\,r.$$

That is going to be sitting in the $\clubsuit$ spot:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{} \end{bmatrix}.$$

(iv) Finally, we can find $\spadesuit$ in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{\spadesuit} \end{bmatrix}$$

by highlighting

$$\begin{bmatrix} a & b \\ \boxed{c \quad d} \end{bmatrix} \begin{bmatrix} p & \boxed{\begin{matrix} q \\ s \end{matrix}} \\ r & \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{\spadesuit} \end{bmatrix}.$$

The highlighted parts are the vectors $\begin{bmatrix} c, & d \end{bmatrix}$ and $\begin{bmatrix} q, & s \end{bmatrix}$. Do their "dot product":

$$c\,q \quad + \quad d\,s.$$

That is going to be sitting in the $\spadesuit$ spot:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{cq+ds} \end{bmatrix}.$$

- <u>In sum, calculating</u> $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ <u>takes four steps</u> :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \boxed{\diamondsuit} & \boxed{\heartsuit} \\ \boxed{\clubsuit} & \boxed{\spadesuit} \end{bmatrix}.$$

<u>Those four steps</u> : $\diamondsuit$, $\heartsuit$, $\clubsuit$ <u>and</u> $\spadesuit$, <u>are performed independently.</u>

- **Alternative perspective.** Below is another way to look at it.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

is like

$$\underbrace{\begin{bmatrix} \boxed{\begin{matrix} a & b \\ c & d \end{matrix}} \end{bmatrix}}_{\begin{matrix} \| \\ A \end{matrix}} \begin{bmatrix} \underbrace{\boxed{\begin{matrix} p \\ r \end{matrix}}}_{\begin{matrix} \| \\ \boldsymbol{x} \end{matrix}} & \underbrace{\boxed{\begin{matrix} q \\ s \end{matrix}}}_{\begin{matrix} \| \\ \boldsymbol{y} \end{matrix}} \end{bmatrix},$$

which is basically

$$A \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix}.$$

And this is going to be converted to

$$\begin{bmatrix} A\boldsymbol{x} & A\boldsymbol{y} \end{bmatrix},$$

and that's precisely the above highlighted multiplication rule. Repeat:

- **Rule – I paraphrased.**

$$A \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} & A\boldsymbol{y} \end{bmatrix}.$$

- Here, it requires to set the rule how to do $A\boldsymbol{x}$ and $A\boldsymbol{y}$ each. Obviously, how to do $A\boldsymbol{y}$ is parallel to how to do $A\boldsymbol{x}$. So let's just set the rule how to do $A\boldsymbol{x}$.

- **Rule – II.**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix} \quad \underline{\text{is calculated as}} \quad \begin{bmatrix} ap + br \\ cp + dr \end{bmatrix}.$$

- Repeat:

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} ap + br \\ cp + dr \end{bmatrix}.}$$

- Paraphrase:

$$\boxed{\begin{array}{ll} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} p \\ r \end{bmatrix} & \\ \Longrightarrow & A\boldsymbol{x} = \begin{bmatrix} ap + br \\ cp + dr \end{bmatrix}. \end{array}}$$

Break down (just in case). This time around, it is just two steps, instead of four. So, we are going to do

$$\begin{bmatrix} \boxed{a \quad b} \\ \boxed{c \quad d} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{\quad \diamondsuit \quad} \\ \boxed{\quad \clubsuit \quad} \end{bmatrix}.$$

(i) To find $\diamondsuit$, do the "dot product" of the vectors $\begin{bmatrix} a, & b \end{bmatrix}$ and $\begin{bmatrix} p, & r \end{bmatrix}$:

$$a\,p \quad + \quad b\,r.$$

In short,

$$\begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{\quad ap + br \quad} \\ \boxed{\quad \clubsuit \quad} \end{bmatrix}.$$

(ii) Next, to find $\clubsuit$, do the "dot product" of the vectors $\begin{bmatrix} c, & d \end{bmatrix}$ and $\begin{bmatrix} p, & r \end{bmatrix}$:

$$c\,p \quad + \quad d\,r.$$

In short,

$$\begin{bmatrix} a \quad b \\ \boxed{c \quad d} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{\quad ap + br \quad} \\ \boxed{\quad cp + dr \quad} \end{bmatrix}.$$

And that's it.

Example 1. For $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$, we have

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 + 2 \cdot \big(-1\big) & 1 \cdot \big(-1\big) + 2 \cdot 8 \\ 4 \cdot 2 + 2 \cdot \big(-1\big) & 4 \cdot \big(-1\big) + 2 \cdot 8 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 15 \\ 6 & 12 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 1 + \big(-1\big) \cdot 4 & 2 \cdot 2 + \big(-1\big) \cdot 2 \\ \big(-1\big) \cdot 1 + 8 \cdot 4 & \big(-1\big) \cdot 2 + 8 \cdot 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 2 \\ 31 & 14 \end{bmatrix}. \end{aligned}$$

- **Case in point:** $\underline{\ AB \text{ and } BA \text{ need not be equal}\ }$.

Example 2. For $A = \begin{bmatrix} 5 & -2 \\ 6 & -4 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, we have

$$\begin{aligned} A\boldsymbol{x} &= \begin{bmatrix} 5 & -2 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 5 \cdot 3 + \big(-2\big) \cdot 4 \\ 6 \cdot 3 + \big(-4\big) \cdot 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}. \end{aligned}$$

- **Warning.** in this case "$\boldsymbol{x}A$" does not make sense.

Exercise 1. Perform each of the following multiplications:

(1) $\begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 5 \end{bmatrix}$. (2) $\begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ -1 & 0 \end{bmatrix}$.

(3) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 1 \\ 1 & \frac{-3}{2} \end{bmatrix}$. (4) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(5) AB, where $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$,

(6) AB, where $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

(7) AB, where $A = B = \begin{bmatrix} \frac{-1+\sqrt{5}}{4} & \frac{-\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{-1+\sqrt{5}}{4} \end{bmatrix}$.

Exercise 2. Perform each of the following multiplications:

(1) $\begin{bmatrix} 3 & \frac{1}{2} \\ \frac{5}{2} & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. (2) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} p \\ q \end{bmatrix}$.

Exercise 3. Perform the multiplication in each of (1) and (2). In each of (1) and (2), detect one eigenvalue of A.

(1) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 1 & 2 \\ -6 & 8 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(2) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

- Next I want to recall one specific matrix which is important.

Definition. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called <u>the identity matrix</u>. We usually use the letter " I " for this:

$$\boxed{I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.}$$

Example 3. Let's do IA and AI for

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad A = \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix}.$$

Here we go:

$$\begin{aligned} IA &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 + 0 \cdot 7 & 1 \cdot 4 + 0 \cdot (-3) \\ 0 \cdot 2 + 1 \cdot 7 & 0 \cdot 4 + 1 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} AI &= \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 1 + 4 \cdot 0 & 2 \cdot 0 + 4 \cdot 1 \\ 7 \cdot 1 + (-3) \cdot 0 & 7 \cdot 0 + (-3) \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & -3 \end{bmatrix}. \end{aligned}$$

What do you see? The outcomes of IA and AI are both just A:

$$IA = A, \qquad AI = A.$$

As you know (if you remember it from Linear Algebra I), this is not by coincidence. It is true that I multiplied to A from the left, and from the right, will both produce A. How ever innocuous, this is worth highlighting:

Fact. For $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\boxed{IA = A, \qquad AI = A.}$$

A good analogy: In the context of matrix multiplications, the identity matrix 'I' serves the same role as '1' (the number) does in the usual number multiplications. Namely, we always have $1\,a = a$, and $a\,1 = a$. (right?) Now, below is another important definition.

Definition. Suppose two matrices P and Q satisfy $PQ = I$. Then $QP = I$ is true. In this situation, we say P and Q are <u>inverses of each other</u>. Write

$$P = Q^{-1}, \qquad \text{and} \qquad Q = P^{-1}.$$

Example 4. Let $P = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$. Then

$$\begin{aligned} PQ &= \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot 2 + 1 \cdot (-7) & 4 \cdot (-1) + 1 \cdot 4 \\ 7 \cdot 2 + 2 \cdot (-7) & 7 \cdot (-1) + 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I. \end{aligned}$$

In short, $PQ = I$. Hence, QP also equals I, and moreover, by definition, P and Q are inverses of each other: $Q = P^{-1}$, $P = Q^{-1}$. About the part "QP also equals I", if you are skeptical:

$$\begin{aligned} QP &= \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 4 + (-1) \cdot 7 & 2 \cdot 1 + (-1) \cdot 2 \\ (-7) \cdot 4 + 4 \cdot 7 & (-7) \cdot 1 + 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I. \end{aligned}$$

• By the way, P^{-1} is pronounced "P inverse". It makes sense to provide how to invert a matrix, along with the precise condition when that is feasible, which is below:

Inversion formula. The inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists provided $ad - bc \neq 0$, and

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.}$$

Exercise 4. (1a) Perform the following multiplication:

$$AB, \qquad \text{where} \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 6 \\ 10 & 9 \end{bmatrix}.$$

(1b) Does the answer for (1a) coincide with either A or B? (If 'yes', then indicate which one is.)

(1c) Is A or B in (1a) the identity matrix? (If 'yes', then indicate which one is.)

(2a) Perform the following multiplication:

$$AB, \qquad \text{where} \qquad A = \begin{bmatrix} -1 & -3 \\ 14 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(2b) Does the answer for (2a) coincide with either A or B? (If 'yes', then indicate which one is.)

(2c) Is A or B in (2a) the identity matrix? (If 'yes', then indicate which one is.)

Exercise 5. (a) Perform the following multiplication:

$$PQ, \qquad \text{where} \qquad P = \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix}, \qquad Q = \begin{bmatrix} -5 & -3 \\ -3 & -2 \end{bmatrix}.$$

(b) Does the answer for (a) coincide with the identity matrix?

(c) Knowing the answer for (b), would you say QP is also the identity matrix?

(d) Find Q^{-1}, as well as P^{-1}.

Exercise 6. Find the inverse P^{-1} of P, if exists:

(1) $P = \begin{bmatrix} 2 & 5 \\ 4 & -2 \end{bmatrix}$. (2) $P = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$. (3) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(4) $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. (5) $P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

• In the above we have only dealt with 2×2 matrices. Nothing stops us from considering 3×3 matrices, 4×4 matrices, 5×5 matrices, and so on. Today let's not do their multiplications, inverse, *etc.* Rather, let's just recall what they are.

Below are the general forms of 3×3, 4×4 and 5×5 matrices, respectively:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} : \text{ a } \underline{\ 3 \times 3 \text{ matrix}\ }.$$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} : \text{ a } \underline{\ 4 \times 4 \text{ matrix}\ }.$$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{bmatrix} : \text{ a } \underline{\ 5 \times 5 \text{ matrix}\ }.$$

You can similarly define 6×6, 7×7, or more generally $n \times n$, matrices.

• And so much of a crash course for one day. Let's get back to our original discussion (continuation from the bottom of page 2). Do you remember 'diagonalization'? First

$$\begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

are all examples of <u>diagonal matrices</u> .

- A general form of diagonal matrices are

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \underline{\text{ a diagonal } 2 \times 2 \text{ matrix }},$$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : \underline{\text{ a diagonal } 3 \times 3 \text{ matrix }}.$$

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} : \underline{\text{ a diagonal } 4 \times 4 \text{ matrix }}.$$

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix} : \underline{\text{ a diagonal } 5 \times 5 \text{ matrix }}.$$

Meanwhile,

$$A = \begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}$$

is clearly *not* a diagonal matrix. I have the good news: Let's squeeze A by a pair of matrices, like

$$(*) \qquad \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}}_{\substack{\parallel \\ P}} \underbrace{\begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}}_{\substack{\parallel \\ A}} \underbrace{\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}}_{\substack{\parallel \\ Q}}.$$

If you calculate this, then it becomes

$$\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix},$$

which is indeed a diagonal matrix. Here, in $(*)$, P and Q are related. Namely, delete A from $(*)$ so consider

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}}_{\substack{\| \\ P}} \underbrace{\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}}_{\substack{\| \\ Q}}.$$

As you can see, this actually equals

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

which is the identity matrix. So this tells us that P and Q in $(*)$ are inverses of each other: $Q = P^{-1}$.

So, $(*)$ is indeed $P\,A\,P^{-1}$. To summarize what we have found so far:

$$\boxed{\begin{array}{c} A = \begin{bmatrix} -2 & -14 \\ 3 & 11 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \\[1em] \Longrightarrow \qquad P\,A\,P^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}. \end{array}}$$

so A is not diagonal but $P\,A\,P^{-1}$ is. Then we say A is <u>diagonalizable</u>.

- **Warning.** <u>Not all matrices</u> $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ <u>are diagonalizable</u>.

Here is an example of a non-diagonalizable matrix:

Example 5. The matrix

$$B = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

is non-diagonalizable. Namely, no matter what matrix P you choose, you can never make $P\,B\,P^{-1}$ a diagonal matrix.

Today I won't explain why the matrix B in Example 5 above is non-diagonalizable. Just take my word for it. But we are going to return to this later.

Anyhow, here is a type of a problem which you commonly see in linear algebra exams:

- **A classic linear algebra exam problem.**

  A matrix A is concretely given.

  > "*Decide whether A is diagonalizable. If so, then diagonalize it, namely, find P such that PAP^{-1} is diagonal.*"

  You should count that I am going to include problems with this format in future exams. But why is this so common as an exam problem? There is a reason for that. Actually I don't seem to have a one-line explanation why. So let's hold onto that thought, why this type of things are so prevalent.

- So, today — besides some rudiments on matrix multiplications and inverses — we have seen two things: One is eigenvalues, and another diagonalizations. Actually these two are somehow related to each other. I will explain how exactly so, as the semester progresses.

  Now, when it comes to 'diagonalizations', there is an innocuous-looking fact which is actually 'deep' in disguise. The framework within which that fact is best laid out is called 'spectral theory'. In order to cover the 'spectral theory' we need to thoroughly build some foundations, 'brick-by-brick'. One of my goals here is actually to throw light on this 'spectral theory'. This is the part of linear algebra which is 'enamouring'. More on that later.

Exercise 7. Let

$$P = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}, \qquad Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

(1a) Perform the matrix multiplication PAQ.

(1b) Is your answer in (1a) a diagonal matrix?

(2a) Perform the matrix multiplication PQ.

(2b) Is your answer in (2a) the identity matrix?

(3) Can you conclude that PAP^{-1} is a diagonal matrix?

(4) Can you conclude that A is diagonalizable?