MA 02 LINEAR ALGEBRA II

REVIEW OF LECTURES – IV

June 25 (Tue), 2024

Section: C7.
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On Day 1, I briefly talked about 'diagonalization'. (I'm sure you remember that.) I said (in essence)

"*diagonalization* and *eigenvalues* have to do with each other"

(see "Review of Lectures – I", page 16). But then on the next two lectures I have backpedalled a bit, directed our attention exclusively on one side, namely, eigenvalues. I haven't revealed how exactly those two notions — diagonalization and eigenvalues — are related. Detailed discussion about their relation is our next agenda. We'll end up spending the next couple of class meetings for that. Sounds good? Now, to that end we need some more refreshers (as usual): Today we are going to review three general properties on matrices and determinants that are all very important. The first one is called 'product formula'. Here we go:

- **Product formula.**

Example 1. Let's consider the following two matrices

$$A = \begin{bmatrix} 2 & 1 \\ -4 & 7 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 6 & -3 \\ 5 & -1 \end{bmatrix}.$$

These two are just randomly picked. Let's calculate the determinant for each of A and B:

$$\begin{aligned} \det A &= \begin{vmatrix} 2 & 1 \\ -4 & 7 \end{vmatrix} \\ &= 2 \cdot 7 - 1 \cdot (-4) = 18, \end{aligned}$$

$$\begin{aligned} \det B &= \begin{vmatrix} 6 & -3 \\ 5 & -1 \end{vmatrix} \\ &= 6 \cdot (-1) - (-3) \cdot 5 = 9. \end{aligned}$$

Mmm. So far so good. But then, independently of those, why don't we calculate AB (not the determinant yet, just A times B):

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 1 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2\cdot 6 + 1\cdot 5 & 2\cdot\big(-3\big) + 1\cdot\big(-1\big) \\ \big(-4\big)\cdot 6 + 7\cdot 5 & \big(-4\big)\cdot\big(-3\big) + 7\cdot\big(-1\big) \end{bmatrix} \\ &= \begin{bmatrix} 17 & -7 \\ 11 & 5 \end{bmatrix}. \end{aligned}$$

Do you know where I am going? Yes. I want you to now calculate the determinant of this last one $\begin{bmatrix} 17 & -7 \\ 11 & 5 \end{bmatrix}$, and see what happens. Voilà:

$$\begin{aligned} \det\Big(AB\Big) &= 17\cdot 5 - \big(-7\big)\cdot 11 \\ &= 162. \end{aligned}$$

To summarize,

$$\det A = 18, \qquad \det B = 9, \qquad \det\Big(A\,B\Big) = 162.$$

Realize $18\cdot 9 = 162$. Question: Is that a coincidence? My answer: No, that is not a coincidence. The same is actually true for any A and B. Below is the first highlight of the day:

Formula 1 (Product Formula for 2×2).

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, we have

$$\boxed{\det\Big(AB\Big) = \Big(\det A\Big)\Big(\det B\Big).}$$

- Here is another example $\big($of the same nature$\big)$ that epitomizes this:

Example 2. $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$.

Once again these are random choices. Agree

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \\ &= 1 \cdot \big(-1\big) - 3 \cdot 2 = -7, \end{aligned}$$

$$\begin{aligned} \det B &= \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} \\ &= 2 \cdot 5 - 1 \cdot 4 = 6. \end{aligned}$$

Independently of these,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 + 3 \cdot 4 & 1 \cdot 1 + 3 \cdot 5 \\ 2 \cdot 2 + \big(-1\big) \cdot 4 & 2 \cdot 1 + \big(-1\big) \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 16 \\ 0 & -3 \end{bmatrix}, \end{aligned}$$

so

$$\begin{aligned} \det\Big(AB\Big) &= 14 \cdot \big(-3\big) - 16 \cdot 0 \\ &= -42. \end{aligned}$$

In sum:

$$\det A = -7, \qquad \det B = 6, \qquad \det\Big(AB\Big) = -42.$$

Notice $\Big(-7\Big) \cdot 6 = -42.$ So this is in sync with what the formula says:

$$\det\Big(AB\Big) = \Big(\det A\Big)\Big(\det B\Big).$$

- Let me recite the formula one more time:

Formula 1 (= Product Formula) duplicated.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, we have

$$\boxed{\det\Big(AB\Big) = \Big(\det A\Big)\Big(\det B\Big).}$$

- How come is this formula true, though? Let's dissect. First agree

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \det B = \begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - qr.$$

Meanwhile, agree

$$AB = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}.$$

So

$$\begin{aligned} \det\Big(AB\Big) &= \begin{vmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{vmatrix} \\ &= \Big(ap + br\Big)\Big(cq + ds\Big) - \Big(aq + bs\Big)\Big(cp + dr\Big). \end{aligned}$$

Bases on these, let's agree that the content of Formula 1 is a mere paraphrase of the following:

Formula 1′ (Spelled out version of Product Formula).

$$(*) \qquad \boxed{\begin{aligned} &\Big(ap + br\Big)\Big(cq + ds\Big) - \Big(aq + bs\Big)\Big(cp + dr\Big) \\ &\qquad = \Big(ad - bc\Big)\Big(ps - qr\Big). \end{aligned}}$$

So, like I said, $(*)$ above is "the spelt-out version" of Formula 1, meaning:

" *In order to prove* Formula 1, *it suffices to prove* $(*)$ (Formula $1'$). "

Good news: Verifying $(*)$ is straightforward. This is mundane enough that I can leave it as an exercise:

Exercise 1. Prove Formula 1, by way of proving $(*)$ (Formula $1'$).

[**Hint for Exercise 1**] **:** First, just say "it suffices to prove $(*)$ in Formula $1'$, so you will only prove $(*)$ in Formula $1'$ ". For the proof of $(*)$, break up the formation on each side of $(*)$, and then cancel all the terms that are subject to cancel. Check that the two sides become equal.

• So, you would think that Formula 1 is one of those "*ad nauseum*", which you are compelled to memorize, in fact, the level of sophistication is subpar, at best. The truth is, although this formula itself appears to be rather rudimentary, it has larger size counterparts, and those are indeed non-trivial. If you just hear it, then you are probably not sure what I am talking about. So, even though this is a sneak preview, and we will definitely come back to this topic later on, for now let's just take a peek at what each of the 3×3 and the 4×4 counterparts says (below). How trivial or non-trivial are they? Decide for yourself.

• **Spelt-out version of 'Product Formula' for 3×3.**

$$\begin{aligned}
&\Big(a_1p_1 + a_2p_2 + a_3p_3 \Big)\Big(b_1q_1 + b_2q_2 + b_3q_3 \Big)\Big(c_1r_1 + c_2r_2 + c_3r_3 \Big) \\
+ &\Big(a_1q_1 + a_2q_2 + a_3q_3 \Big)\Big(b_1r_1 + b_2r_2 + b_3r_3 \Big)\Big(c_1p_1 + c_2p_2 + c_3p_3 \Big) \\
+ &\Big(a_1r_1 + a_2r_2 + a_3r_3 \Big)\Big(b_1p_1 + b_2p_2 + b_3p_3 \Big)\Big(c_1q_1 + c_2q_2 + c_3q_3 \Big) \\
- &\Big(a_1r_1 + a_2r_2 + a_3r_3 \Big)\Big(b_1q_1 + b_2q_2 + b_3q_3 \Big)\Big(c_1p_1 + c_2p_2 + c_3p_3 \Big) \\
- &\Big(a_1p_1 + a_2p_2 + a_3p_3 \Big)\Big(b_1r_1 + b_2r_2 + b_3r_3 \Big)\Big(c_1q_1 + c_2q_2 + c_3q_3 \Big) \\
- &\Big(a_1q_1 + a_2q_2 + a_3q_3 \Big)\Big(b_1p_1 + b_2p_2 + b_3p_3 \Big)\Big(c_1r_1 + c_2r_2 + c_3r_3 \Big) \\
&= \Big(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3 \Big) \\
&\quad \cdot \Big(p_1q_2r_3 + p_2q_3r_1 + p_3q_1r_2 - p_3q_2r_1 - p_1q_3r_2 - p_2q_1r_3 \Big) .
\end{aligned}$$

- **Spelt-out version of ‘Product Formula’ for 4×4.**

$$
\begin{array}{l}
\quad(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
+(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
+(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
+(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
+(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
+(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
+(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
+(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
+(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
+(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
+(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
+(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
-(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
-(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
-(a_1p_1+a_2q_1+a_3r_1+a_4s_1)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
-(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
-(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
-(a_1p_2+a_2q_2+a_3r_2+a_4s_2)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
-(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_4+c_2q_4+c_3r_4+c_4s_4)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)\\
-(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_4+b_2q_4+b_3r_4+b_4s_4)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
-(a_1p_3+a_2q_3+a_3r_3+a_4s_3)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_4+d_2q_4+d_3r_4+d_4s_4)\\
-(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_1+b_2q_1+b_3r_1+b_4s_1)(c_1p_2+c_2q_2+c_3r_2+c_4s_2)(d_1p_3+d_2q_3+d_3r_3+d_4s_3)\\
-(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_2+b_2q_2+b_3r_2+b_4s_2)(c_1p_3+c_2q_3+c_3r_3+c_4s_3)(d_1p_1+d_2q_1+d_3r_1+d_4s_1)\\
-(a_1p_4+a_2q_4+a_3r_4+a_4s_4)(b_1p_3+b_2q_3+b_3r_3+b_4s_3)(c_1p_1+c_2q_1+c_3r_1+c_4s_1)(d_1p_2+d_2q_2+d_3r_2+d_4s_2)
\end{array}
$$

$$
\begin{array}{rl}
= & \Big(\ a_1b_2c_3d_4+a_1b_3c_4d_2+a_1b_4c_2d_3+a_2b_1c_4d_3+a_2b_4c_3d_1+a_2b_3c_1d_4\\
& +a_3b_1c_2d_4+a_3b_2c_4d_1+a_3b_4c_1d_2+a_4b_1c_3d_2+a_4b_3c_2d_1+a_4b_2c_1d_3\\
& -a_1b_2c_4d_3-a_1b_4c_3d_2-a_1b_3c_2d_4-a_2b_1c_3d_4-a_2b_3c_4d_1-a_2b_4c_1d_3\\
& -a_3b_1c_4d_2-a_3b_4c_2d_1-a_3b_2c_1d_4-a_4b_1c_2d_3-a_4b_2c_3d_1-a_4b_3c_1d_2\ \Big)\\
\cdot & \Big(\ p_1q_2r_3s_4+p_1q_3r_4s_2+p_1q_4r_2s_3+p_2q_1r_4s_3+p_2q_4r_3s_1+p_2q_3r_1s_4\\
& +p_3q_1r_2s_4+p_3q_2r_4s_1+p_3q_4r_1s_2+p_4q_1r_3s_2+p_4q_3r_2s_1+p_4q_2r_1s_3\\
& -p_1q_2r_4s_3-p_1q_4r_3s_2-p_1q_3r_2s_4-p_2q_1r_3s_4-p_2q_3r_4s_1-p_2q_4r_1s_3\\
& -p_3q_1r_4s_2-p_3q_4r_2s_1y-p_3q_2r_1s_4-p_4q_1r_2s_3-p_4q_2r_3s_1-p_4q_3r_1s_2\ \Big).
\end{array}
$$

Exercise 2. For $A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$, and $B = \begin{bmatrix} 6 & 5 \\ 8 & 3 \end{bmatrix}$, calculate

(1) $\det A$, (2) $\det B$, (3) $\Big(\det A\Big)\Big(\det B\Big)$ based on (1–2),

(4) AB, and (5) $\det\Big(AB\Big)$ based on (4).

Confirm that the answer for (3) and the answer for (5) coincide.

- **Matrix addition, subtraction.**

Next item on today's menu is matrix addition and subtraction. In one line:

" *Matrix addition and subtraction are done entry-wise.* "

Definition (Matrix addition/subtraction).

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, define

$$A + B = \begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix},$$

$$A - B = \begin{bmatrix} a-p & b-q \\ c-r & d-s \end{bmatrix}.$$

- In case you wonder: I somehow did matrix multiplications first, before additions and subtractions. Maybe that's a little unorthodox. The truth is, it doesn't matter, either way is completely viable. In this case, I threw a certain topic on Day 1 for the purpose of setting the tone, and that necessitated me to do matrix multiplications right off the bat. At that time I could dispense with $\big($matrix$\big)$ additions and subtractions, but that trend didn't last long. Some nuts and bolts — how to linearly align various items is $\big($primarily$\big)$ dictated by the logical interdependence of the items, yet there are always some leeways, because $\big($apparently$\big)$ not all items are logically tied to one another. This is one of those instances.

• Anyhow, let's take a look at some examples:

Example 3. For $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$, we have

$$A + B = \begin{bmatrix} 1 + (-3) & 2 + (-2) \\ 2 + 4 & 1 + 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 6 & 3 \end{bmatrix},$$

$$A - B = \begin{bmatrix} 1 - (-3) & 2 - (-2) \\ 2 - 4 & 1 - 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -2 & -1 \end{bmatrix}.$$

Quiz 1. For A and B as in Example 3 above,

(1) do each of

$$\det A, \qquad \det B, \qquad \det \Big(A + B\Big) \qquad \text{and} \qquad \det \Big(A - B\Big).$$

(2) <u>True or false</u> :

$$\det A + \det B \qquad \text{equals} \qquad \det \Big(A + B\Big) \qquad (?)$$

(3) <u>True or false</u> :

$$\det A - \det B \qquad \text{equals} \qquad \det \Big(A - B\Big) \qquad (?)$$

Solution. Let's do it together. First, as for (1):

$$\det A = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 1 \cdot 1 - 2 \cdot 2 = -3,$$

$$\det B = \begin{vmatrix} -3 & -2 \\ 4 & 2 \end{vmatrix}$$

$$= (-3) \cdot 2 - (-2) \cdot 4 = 2,$$

$$\det\Big(A+B\Big) = \begin{vmatrix} -2 & 0 \\ 6 & 3 \end{vmatrix}$$

$$= \big(-2\big)\cdot 3 \;-\; 0\cdot 6 \;=\; -6,$$

$$\det\Big(A-B\Big) = \begin{vmatrix} 4 & 4 \\ -2 & -1 \end{vmatrix}$$

$$= 4\cdot\big(-1\big) \;-\; 4\cdot\big(-2\big) \;=\; 4.$$

So what do you see?

$$\det A \;+\; \det B \;=\; -1 \qquad \text{whereas} \qquad \det\Big(A+B\Big) \;=\; -6,$$

so they are not equal. So, the answer for (2) is 'false'. Likewise

$$\det A \;-\; \det B \;=\; -5 \qquad \text{whereas} \qquad \det\Big(A-B\Big) \;=\; 4,$$

so they are not equal. So, the answer for (3) is 'false'. □

- It is worthwhile to stress this aspect of the determinants:

Warning.

" *While the determinant operation is compatible with matrix multiplications,*

it is not compatible with matrix additions and subtracitons. "

Repeat: In general,

$\det A \;+\; \det B$	and	$\det\Big(A+B\Big)$	are not equal.
$\det A \;-\; \det B$	and	$\det\Big(A-B\Big)$	are not equal.

Exercise 3. For $A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}$, and $B = \begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix}$, calculate

(1) $A + B$, (2) $\det\big(A+B\big)$ based on (1), (3) $\det A + \det B$.

Do the answer for (2) and the answer for (3) coincide? Also calculate

(4) $A - B$, (5) $\det\big(A-B\big)$ based on (4), (6) $\det A - \det B$.

Do the answer for (5) and the answer for (6) coincide?

- Having agreed with the above scope, let's move on to the next pop quiz.

Quiz 2. Recall $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (the identity matrix), so

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

(Agree?) Meanwhile, let

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}.$$

Do the subtraction $\lambda I - A$.

Solution. This is a piece of cake. Right? So, once again

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad \text{and} \qquad A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}.$$

So the answer for $\lambda I - A$ is

$$\lambda I - A = \begin{bmatrix} \lambda - 3 & -1 \\ -4 & \lambda - 6 \end{bmatrix}.$$

Good job. $\square$

• So we have just solved the quiz. For some reason, I want you to take another look at the answer:

$$\lambda I - A = \begin{bmatrix} \lambda-3 & -1 \\ -4 & \lambda-6 \end{bmatrix}.$$

A little bit out of the blue, why don't we just take the determinant of this? Sure:

$$\det\left(\lambda I - A\right) = \begin{vmatrix} \lambda-3 & -1 \\ -4 & \lambda-6 \end{vmatrix}.$$

Let me quiz you.

What's the name of this?

— Yes, it is <u>the characteristic polynomial</u> of $A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$. Uh-huh. So, if I say that, more generally, the characteristic polynomial of any matrix A is always written as $\det\left(\lambda I - A\right)$, would I be correct? Yes, of course. Namely:

Fact. <u>Let</u> $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. <u>Then the characteristic polynomial</u> $\begin{vmatrix} \lambda-a & -b \\ -c & \lambda-d \end{vmatrix}$ <u>of A is otherwise written as</u> $\boxed{\det\left(\lambda I - A\right).}$

Warning 2. Don't falsely simplify $\det\left(\lambda I - A\right)$ as $\det\left(\lambda I\right) - \det A$. That's not allowed. In fact, a formation like $\det\left(\lambda I\right) - \det A$ is of little significance. Meanwhile <u>the role played by the formation</u>

$$\det\left(\lambda I - A\right)$$

<u>in linear algebra is paramount.</u>

- Since $\det\left(\lambda I - A\right)$ so frequently appears, it makes sense to designate a symbol:

Notation $\left(\textbf{Characteristic polynomial } \boldsymbol{\chi_A(\lambda)}\right)$**.**

From now on, we denote the characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$\boxed{\chi_A\left(\lambda\right).}$$

So, can you describe $\chi_A\left(\lambda\right)$ as a determinant? — Yes:

$$\boxed{\chi_A\left(\lambda\right) \;=\; \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix}\;.}$$

Or, the same to say

$$\boxed{\chi_A\left(\lambda\right) \;=\; \det\left(\lambda I - A\right).}$$

Example 4. What is $\chi_A\left(\lambda\right)$ for $A = \begin{bmatrix} 8 & 3 \\ 6 & 5 \end{bmatrix}$? Calculate it.

— Well, this is a pice of cake. We are familiar with this, except this new notation $\chi_A\left(\lambda\right)$. Here we go:

$$\begin{aligned} \chi_A\left(\lambda\right) &= \begin{vmatrix} \lambda - 8 & -3 \\ -6 & \lambda - 5 \end{vmatrix} \\ &= \left(\lambda - 8\right)\left(\lambda - 5\right) - \left(-3\right)\cdot\left(-6\right) \\ &= \lambda^2 - 13\lambda + 22. \end{aligned}$$

Exercise 4. Express $\chi_A\left(\lambda\right)$ as a determinant. Then calculate it.

(1) $A = \begin{bmatrix} 6 & 4 \\ 6 & 1 \end{bmatrix}$. (2) $A = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix}$.

- **Associativity Law.**

In matrix arithmetic, we often deal with the situation where three matrices are involved. Suppose we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad C = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

and we would like to multiply them together, as in ABC. We have actually seen it on Day 1 — what we have seen is more like PAP^{-1}, but this falls into the template of "three matrices being multiplied together". So let's talk about just ABC, which is more general.

As innocuous as it seems, if you stop and think twice, we have to actually worry about the following. There are two obvious ways to calculate it:

(i) Calculate ABC as $\Big(AB\Big)C$. (ii) Calculate ABC as $A\Big(BC\Big)$.

A natural question here is, whether these two match. This is something we need to analyze. The answer is, that is indeed the case. In fact,

$$\begin{aligned}
\Big(AB\Big)C &= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) \begin{bmatrix} x & y \\ z & w \end{bmatrix} \\
&= \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \\
&= \begin{bmatrix} apx + brx + aqz + bsz & apy + bry + aqw + bsw \\ cpx + drx + cqz + dsz & cpy + dry + cqw + dsw \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
A\Big(BC\Big) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} px + qz & py + qw \\ rx + sz & ry + sw \end{bmatrix} \\
&= \begin{bmatrix} apx + aqz + brx + bsz & apy + aqw + bry + bsw \\ cpx + cqz + drx + dsz & cpy + cqw + dry + dsw \end{bmatrix}.
\end{aligned}$$

If you look at these, you see that $\Big(AB\Big)C$ and $A\Big(BC\Big)$ indeed coincide.

- To highlight the result:

Formula (Associativity Law). For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \qquad C = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

we have

$$\boxed{\Big(AB\Big)C \;=\; A\Big(BC\Big).}$$

- So what does this entail? Yes, when it comes to ABC, we don't ever have to worry about placing parenthesis either over the AB part, or over the BC part. This is just like

$$2 \cdot 3 \cdot 5.$$

You immediately say 30 is the answer. I bet you probably did it this way: First $2 \cdot 3 = 6$, and then $6 \cdot 5 = 30$. But like I said "probably". Indeed, some of you might have done it the following way: First $3 \cdot 5 = 15$, and then $2 \cdot 15 = 30$. But we all know that, either way you do it you'll get the same answer. Below is the mathematically precise way to compile what I just said:

$$\Big(2 \cdot 3\Big) \cdot 5 \;=\; 2 \cdot \Big(3 \cdot 5\Big).$$

More generally, if a, b and c are numbers $\big($real numbers, to be precise$\big)$, then

$$\Big(a \cdot b\Big) \cdot c \;=\; a \cdot \Big(b \cdot c\Big).$$

So, you would say Formula above *merely* says that the same is true for matrices. Well, that's true. So then you might say Formula above is not worthy to isolate because that's no surprise. Well, you know, I have at least two ways to retort. One is nothing is certain when you generalize something from a number to something else, you are going to lose some of the properties that the numbers possessed. For example, I stressed that, in general

$$AB \;\neq\; BA$$

for matrices A and B. Needless to say, $ab = ba$ for numbers a and b.

Another retort is there is actually a number system wherein

$$\Big(a \cdot b\Big) \cdot c \;\neq\; a \cdot \Big(b \cdot c\Big).$$

It's just that we haven't seen it yet in this class. Numbers in such number system are called Cayley numbers. But you don't really have to know what it is. If you are interested, we can chat about it, but not here, not right now.

Definition. Keeping Formula above in mind, we define <u>the triple product</u>

$$ABC$$

as either $A\Big(BC\Big)$, or equivalently, $\Big(AB\Big)C$.

- Next when it comes to "quadruple product": For

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad D = \begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix},$$

there are apparently five different ways to calculate $ABCD$:

(i) Calculate $ABCD$ as $\Big(\,(\,AB\,)\,C\,\Big)\,D$.

(ii) Calculate $ABCD$ as $\Big(\,A\,(\,BC\,)\,\Big)\,D$.

(iii) Calculate $ABCD$ as $A\Big(\,(\,BC\,)\,D\,\Big)$.

(iv) Calculate $ABCD$ as $A\Big(\,B\,(\,CD\,)\,\Big)$.

(v) Calculate $ABCD$ as $\Big(\,AB\,\Big)\Big(\,CD\,\Big)$.

Do you see that all these five (i–v) coincide?

Exercise 5. Explain why (i–v) all coincide.

$\Big[\,$<u>**Hint for Exercise 5**</u>$\,\Big]$ **:** First explain why (i) and (ii) are the same. For that matter, it suffices to say (i) and (ii) are both $\Big(ABC\Big)D$. Next, explain why (iii) and (iv) are the same $\big($same logic$\big)$. Next, explain why (i) and (v) are the same $\big($set $AB = E\big)$. Finally, explain why (iv) and (v) are the same $\big($set $CD = F\big)$.

It is worth highlighting the content of Exercise 5 $\big($below$\big)$:

Corollary. Let A, B, C, D be as above. Then

$$\begin{aligned}\Big(\,(\,AB\,)\,C\,\Big)\,D \;&=\; \Big(\,A\,(\,BC\,)\,\Big)\,D \;=\; A\Big(\,(\,BC\,)\,D\,\Big)\\ &=\; A\Big(\,B\,(\,CD\,)\,\Big) \;=\; \Big(\,AB\,\Big)\Big(\,CD\,\Big).\end{aligned}$$

Definition. Keeping Corollary above in mind, we define <u>the quadruple product</u> $ABCD$ as the five mutually equal matrices as above

$$\begin{aligned}ABCD \;=\; \Big(\,(\,AB\,)\,C\,\Big)\,D \;&=\; \Big(\,A\,(\,BC\,)\,\Big)\,D \;=\; A\Big(\,(\,BC\,)\,D\,\Big)\\ &=\; A\Big(\,B\,(\,CD\,)\,\Big) \;=\; \Big(\,AB\,\Big)\Big(\,CD\,\Big).\end{aligned}$$

• **Consecutive product.** We may extend the above idea, and may define a consecutive product for an arbitrary number of matrices.

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}, \quad \cdots, \quad A_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}.$$

Define the product $A_1\,A_2\,A_3\,\cdots A_{k-1}A_k$ as

$$\begin{aligned}A_1\,A_2\,A_3\,\cdots A_{k-1}A_k \;&=\; \Bigg(\Big(\Big(\,\cdot\cdot\,(\,(A_1\,A_2)\,A_3\,)\,\cdots\,)A_{k-2}\Big)\,A_{k-1}\Bigg)\,A_k\\ &=\; A_1\Bigg(\,A_2\,\Big(\,A_3\,(\,\cdots\,(\,A_{k-2}\,(A_{k-1}\,A_k)\,)\,\cdot\cdot\,)\Big)\Bigg).\end{aligned}$$

Exercise 6. Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Calculate

(1) AB. (2) BC. (3) CD.
(4) ABC. (5) BCD. (6) $ABCD$.

- **Powers.** In the product

$$A_1 A_2 A_3 \cdots A_k,$$

suppose $A_1, A_2, A_3, \cdots, A_k$ are mutually identical, call it A. Then we might as well just write it as A^k. In other words:

Definition. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, consider

$$\begin{aligned} A^1 &= A, \\ A^2 &= AA, \\ A^3 &= AAA, \\ A^4 &= AAAA, \\ A^5 &= AAAAA, \\ &\vdots \end{aligned}$$

- So

$$A^k = \underbrace{A\ A\ A\ \cdots\ A}_{k}.$$

Example 5. Remember that $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ satisfies

$$IA = A$$

no matter what $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is. So, nothing stops us from setting $A = I$, and that way we get

$$II = I.$$

In other words,

$$I^2 = I.$$

From this we also get

$$\begin{aligned} I\,I^2 &= I\,I \\ &= I. \end{aligned}$$

In other words, $I^3 = I.$ From this we also get

$$\begin{aligned} I\,I^3 &= I\,I \\ &= I. \end{aligned}$$

In other words, $I^4 = I.$ And this goes on and on.

So, in short, for the identity matrix I, we have

$$I^k = I, \quad \text{for} \quad k = 1, 2, 3, \cdots .$$

Stated in other words,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{for} \quad k = 1, 2, 3, \cdots .$$

• One last topic to wrap up today's class. In the matrix multiplication formation, suppose some constant is being multiplied to each of the matrices, like

$$\Big(tA\Big)\Big(uB\Big).$$

Then we can pull those constants to the left. Below is the precise statement:

Formula. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Let t and u be scalars. Then

$$\Big(tA\Big)\Big(uB\Big) = (tu)\Big(AB\Big).$$

Corollary. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and t a scalar. Then for $k = 1, 2, 3, \cdots,$

$$\Big(t\,A\Big)^k = t^k\,A^k.$$

Example 6. For $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, we have

$$A^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}.$$

Exercise 7. For $k = 1, 2, 3, \cdots,$ find

(1) $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^k.$ (2) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^k.$ (3) $\begin{bmatrix} 2 & a \\ 0 & 2 \end{bmatrix}^k.$

Exercise 8. (1) True or False. $\Big(AB\Big)^2 = A^2B^2.$

(2) Suppose $AB = BA$. True or False. $\Big(AB\Big)^2 = A^2B^2.$