MA 02 LINEAR ALGEBRA II

SOLUTION FOR PRACTICE EXAM – FINAL A (07/23)

July 23 (Tue), 2024

Section: C7.

Instructors: Yasuyuki Kachi (lecture) & Shunji Moriya (recitation)

[I] (15pts) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Suppose $\det A \neq 0$.

(1) True or false :

$$\left(ABA^{-1}\right)^3 = AB^3A^{-1}.$$

- The answer is 'True'.

(2) Let $f(x) = x^3 + x^2 - 2x - 1$.

True or false :

$$f\left(ABA^{-1}\right) = Af(B)A^{-1}.$$

- The answer is 'True'.

[II] (15pts) Find the characteristic polynomial, the eigenvalues, and then eigenvectors associated with each of the eigenvalues of A. Then diagonalize A:

$$A = \begin{bmatrix} -3 & -2 \\ 6 & 5 \end{bmatrix}.$$

Give the matrix Q with which $Q^{-1}AQ$ equals the diagonal matrix.

[Solution]:

$$\chi_A(\lambda) = \begin{vmatrix} \lambda - (-3) & -(-2) \\ -6 & \lambda - 5 \end{vmatrix}$$

$$= (\lambda + 3)(\lambda - 5) - 2 \cdot (-6)$$

$$= \lambda^2 - 2\lambda - 15 + 12$$

$$= \lambda^2 - 2\lambda - 3.$$

We may factor this as

$$\chi_A(\lambda) = \Big(\lambda + 1\Big)\Big(\lambda - 3\Big).$$

Thus the eigenvalues of A are $\qquad \lambda = -1, \quad 3.$

- Eigenvector of A associated with $\lambda = -1$: Solve the equation

$$A \begin{bmatrix} x \\ y \end{bmatrix} = (-1) \begin{bmatrix} x \\ y \end{bmatrix}.$$

Namely,

$$\begin{bmatrix} -3 & -2 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (-1) \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{cases} -3x - 2y = -x, \\ 6x + 5y = -y. \end{cases}$$

That is

$$\begin{cases} -2x - 2y = 0, \\ 6x + 6y = 0. \end{cases}$$

So

$$x + y = 0.$$

Thus one eigenvector of A associated with the eigenvalue $\lambda = -1$ is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

- Eigenvector of A associated with $\lambda = 3$: Solve the equation

$$A \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}.$$

Namely,

$$\begin{bmatrix} -3 & -2 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{cases} -3x - 2y = 3x, \\ 6x + 5y = 3y. \end{cases}$$

That is

$$\begin{cases} -6x - 2y = 0, \\ 6x + 2y = 0. \end{cases}$$

So

$$3x + y = 0.$$

Thus one eigenvector of A associated with the eigenvalue $\lambda = 3$ is

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

- Juxtapose those two eigenvectors: $Q = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$. With this Q, we can diagonalize A as follows:

$$Q^{-1} A Q = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}.$$

[III] (15pts) Find the characteristic polynomial, the eigenvalues, and then eigenvectors associated with each of the eigenvalues of A. Then diagonalize A:

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}.$$

Give the matrix Q with which $Q^{-1}AQ$ equals the diagonal matrix.

$\left[\,\underline{\text{Solution}}\,\right]$: $$\chi_A(\lambda) = \begin{vmatrix} \lambda - 4 & -(-1) \\ -1 & \lambda - 1 \end{vmatrix}$$
$$= (\lambda - 4)(\lambda - 1) - 1 \cdot (-1)$$
$$= \lambda^2 - 5\lambda + 4 + 1$$
$$= \lambda^2 - 5\lambda + 5.$$

Via quadratic formula, we may factor this as

$$\chi_A(\lambda) = \left(\lambda - \frac{5 + \sqrt{5}}{2}\right)\left(\lambda - \frac{5 - \sqrt{5}}{2}\right).$$

Thus the eigenvalues of A are $\lambda = \dfrac{5 \pm \sqrt{5}}{2}$.

- Eigenvector of A associated with $\lambda = \dfrac{5 \pm \sqrt{5}}{2}$ (in what follows double sign in the same order):

Solve the equation

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \frac{5 \pm \sqrt{5}}{2} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Namely,

$$\begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{5 \pm \sqrt{5}}{2} \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{cases} 4x - y = \dfrac{5 \pm \sqrt{5}}{2} x, \\ \\ x + y = \dfrac{5 \pm \sqrt{5}}{2} y. \end{cases}$$

That is

$$\begin{cases} \dfrac{3 \mp \sqrt{5}}{2} x \quad - \qquad\qquad 2y \quad = 0, \\ \\ \qquad\qquad x \quad + \dfrac{-3 \mp \sqrt{5}}{2} y = 0. \end{cases}$$

These two equations are one and the same. So

$$x \quad + \frac{-3 \mp \sqrt{5}}{2} y = 0.$$

Thus one eigenvector of A associated with the eigenvalue $\lambda = \dfrac{5 \pm \sqrt{5}}{2}$ is

$$\begin{bmatrix} \dfrac{3 \pm \sqrt{5}}{2} \\ \\ 1 \end{bmatrix}.$$

- Juxtapose those two eigenvectors: $Q = \begin{bmatrix} \dfrac{3 + \sqrt{5}}{2} & \dfrac{3 - \sqrt{5}}{2} \\ \\ 1 & 1 \end{bmatrix}$. With this Q, we can diagonalize A as follows:

$$Q^{-1}AQ = \begin{bmatrix} \dfrac{5+\sqrt{5}}{2} & 0 \\ 0 & \dfrac{5-\sqrt{5}}{2} \end{bmatrix}.$$

[IV] (20pts) State Cayley-Hamilton's theorem for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then prove it.

$\left[\underline{\text{Answer}}\right]$: Let $\chi_A(\lambda)$ be the characteristic polynomial of A. Then $\chi_A(A) = O$.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The characteristic polynomial of A is

$$\begin{aligned} \chi_A(\lambda) &= \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \\ &= \lambda^2 - \left(a + d\right)\lambda + \left(ad - bc\right). \end{aligned}$$

$\chi_A(A)$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 - \left(a+d\right)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(ad - bc\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - bd \\ ac + cd - ac - cd & bc + d^2 - ad - d^2 + ad - bc \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad \text{q.e.d.}$$

[V] (40pts) Let b, c, d be real numbers. Let

$$X = \begin{bmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{bmatrix}.$$

(1) Calculate X^2. Then calculate

$$A = \left(b^2 + c^2 + d^2\right) I + 2X^2.$$

Show work.

[Answer]:

$$X^2 = \begin{bmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{bmatrix} \begin{bmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -c^2 - d^2 & bc & bd \\ bc & -b^2 - d^2 & cd \\ bd & cd & -b^2 - c^2 \end{bmatrix}.$$

$$A = \left(b^2 + c^2 + d^2\right) I + 2X^2$$

$$= \begin{bmatrix} b^2+c^2+d^2 & 0 & 0 \\ 0 & b^2+c^2+d^2 & 0 \\ 0 & 0 & b^2+c^2+d^2 \end{bmatrix} + 2 \begin{bmatrix} -c^2 - d^2 & bc & bd \\ bc & -b^2 - d^2 & cd \\ bd & cd & -b^2 - c^2 \end{bmatrix}$$

$$= \begin{bmatrix} b^2-c^2-d^2 & 2bc & 2bd \\ 2bc & -b^2+c^2-d^2 & 2cd \\ 2bd & 2cd & -b^2-c^2+d^2 \end{bmatrix}.$$

(2) $$X^3 = -\left(\boxed{\quad b^2 + c^2 + d^2 \quad} \right) X.$$

(3) Assume $b^2 + c^2 + d^2 = 1$. Calculate A^2, where A is in (1) above.

Show work.

$\boxed{\text{Solution}}$: Under the assumption,

$$A = I + 2X^2 \quad \text{and} \quad X^3 = -X.$$

Thus

$$\begin{aligned} A^2 &= \left(I + 2X^2\right)^2 \\ &= I + 4X^2 + 4X^4 \\ &= I + 4X^2 + 4XX^3 \\ &= I + 4X^2 + 4X\left(-X\right) \\ &= I + 4X^2 - 4X^2 \\ &= I. \end{aligned}$$

(4) Find the eigenvalues of A. Identify the one with multiplicity 2, if any.

$\boxed{\text{Answer}}$: $\lambda = 1, \ -1.$

The multiplicity of the eigenvalue $\lambda = -1$ is 2.

Indeed, from $A^2 = I$, it follows that an eigenvalue λ of A satisfies $\lambda^2 = 1$. So $\lambda = 1$ and $\lambda = -1$ are the only candidates for the eigenvalues of A. Here we claim that both $\lambda = 1$ and $\lambda = -1$ are the eigenvalues of A. Indeed, for $b = 1$ $\big($which forces $c = d = 0$ under our assumption $b^2 + c^2 + d^2 = 1\big)$,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Clearly the multiplicity of $\lambda = -1$ is 2. Then the same remains true even when b, c and d are general such that $b^2 + c^2 + d^2 = 1$, because the coefficients of the characteristic polynomial of A are in themselves polynomials in b, c and d. q.e.d.