Linear Algebra II Recitation 6

Yasuyuki Kachi & Shunji Moriya

1. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, prove $(AB)^T = B^T A^T$.

2. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. In the following, we always identify a vector $\begin{bmatrix} u \\ v \end{bmatrix}$ with the point (u, v) on the coordinate plane.

   (1) Let $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Prove that $A\boldsymbol{x}$ is the point obtained by rotating $\boldsymbol{x}$ by the angle θ around the origin.

   (Hint: calculate the product $A\boldsymbol{x}$ using the polar coordinate expression $x = r\cos\phi,\ y = r\sin\phi$.)

   (2) Using (1), find the point obtained by rotating $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by the angle $\dfrac{\pi}{3}$ around the origin.

3. For the following matrices, make $Q^{-1}AQ$ a diagonal matrix with some <u>orthogonal</u> matrix Q. Write out both $Q^{-1}AQ$ and Q.

   (1) $A = \begin{bmatrix} 3 & 6 \\ 6 & -2 \end{bmatrix}$ (2) $A = \begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

4. Find the characteristic polynomial and eigenvalues of the following matrices. Find the eigenvectors associated with each of the eigenvalues.

   (1) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$ (2) $A = \begin{bmatrix} 5 & -4 & -2 \\ 6 & -5 & -2 \\ 3 & -3 & 2 \end{bmatrix}$

   Here, as in the case of 2×2 matrices, the characteristic polynomial of a 3×3 matrix A is defined by

   $$\chi_A(\lambda) = \det(\lambda I - A), \qquad \text{where } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   An eigenvalue of A is a solution of $\chi_A(\lambda) = 0$. An eigenvector of A associated with an eigenvalue λ is a non-zero vector $\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which satisfies $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Recitation 6, Answer

1. Omitted.

2. (1) Omitted.

   (2) $\left(\frac{1-2\sqrt{3}}{2}, \frac{2+\sqrt{3}}{2}\right)$

3. (1) $Q^{-1}AQ = \begin{bmatrix} 7 & 0 \\ 0 & -6 \end{bmatrix}$ where $Q = \begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{-3}{\sqrt{13}} \end{bmatrix}$.

   (2) $Q^{-1}AQ = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ where $Q = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}$.

4. (1) $\chi_A(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 4)$.
   Eigenvalues: $\lambda = 1, 2$ and 4.

   Eigenvectors associated with $\lambda = 1$: $k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $(k \neq 0)$.

   Eigenvectors associated with $\lambda = 2$: $k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $(k \neq 0)$.

   Eigenvectors associated with $\lambda = 4$: $k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $(k \neq 0)$.

   (2) $\chi_A(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2$.
   Eigenvalues: $\lambda = -1, 1$ and 2.

   Eigenvectors associated with $\lambda = -1$: $k \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ $(k \neq 0)$.

   Eigenvectors associated with $\lambda = 1$: $k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $(k \neq 0)$.

   Eigenvectors associated with $\lambda = 2$: $k \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ $(k \neq 0)$.