Linear Algebra II   Recitation 2

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1. Find the eigenvalues. Find the eigenvectors associated with each of the eigenvalues.

   (1) $\begin{bmatrix} 3 & 2 \\ -9 & -6 \end{bmatrix}$ (2) $\begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$ (3) $\begin{bmatrix} a & 1 \\ 3 & a \end{bmatrix}$

2. For $A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$, and $B = \begin{bmatrix} 6 & 5 \\ 8 & 3 \end{bmatrix}$, calculate

   (1) $\det A$,   (2) $\det B$,   (3) $(\det A)(\det B)$ based on (1-2),

   (4) AB, and   (5) $\det(AB)$ based on (4).

3. For $A = \begin{bmatrix} 101 & 100 \\ 100 & 99 \end{bmatrix}$, and $B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$, calculate the followings.

   (1) $\det A$ (2) $\det B$ (3) $\det(A^2)$

   (4) $\det(A^2B)$ (5) $\det(A^3B^2A^4)$

4. For $k = 1, 2, 3, \ldots$, find each of the following powers.
   (Guess a general form and prove that your expectation is true by mathematical induction.)

   (1) $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^k$ (2) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^k$ (3) $\begin{bmatrix} 2 & a \\ 0 & 2 \end{bmatrix}^k$

5. *Let n be an integer with $n \geq 2$ and $A = [a_{ij}]$ an $n \times n$ matrix (so the (i,j)-entry of A is a_{ij} for any pair $i, j = 1, \ldots, n$). Assume that A has the property that for any $n \times n$ matrix B, the equation $AB = BA$ holds. Prove that there is a scalar s such that $A = sI$. Here, I is the $n \times n$ identity matrix.

   (Hint: The idea of a proof is similar to the case of 2×2 matrices, while it is more difficult to write down the proof clearly.)

*Problem no.5 is excluded from the problems for submission.

Recitation 2, Answer

1. (1) Eigenvalues: $\lambda = 0, -3$.

   Eigenvectors associated with $\lambda = 0$: $\begin{bmatrix} 2k \\ -3k \end{bmatrix} \quad (k \neq 0)$.

   Eigenvectors associated with $\lambda = -3$: $\begin{bmatrix} k \\ -3k \end{bmatrix} \quad (k \neq 0)$.

   (2) Eigenvalues: $\lambda = 1, 6$.

   Eigenvectors associated with $\lambda = 1$: $\begin{bmatrix} 4k \\ -k \end{bmatrix} \quad (k \neq 0)$.

   Eigenvectors associated with $\lambda = 6$: $\begin{bmatrix} k \\ k \end{bmatrix} \quad (k \neq 0)$.

   (3) Eigenvalues: $\lambda = a \pm \sqrt{3}$.

   Eigenvectors associated with $\lambda = a + \sqrt{3}$: $\begin{bmatrix} k \\ \sqrt{3}k \end{bmatrix} \quad (k \neq 0)$.

   Eigenvectors associated with $\lambda = a - \sqrt{3}$: $\begin{bmatrix} k \\ -\sqrt{3}k \end{bmatrix} \quad (k \neq 0)$.

2. (1) -6 (2) -22 (3) 132

   (4) $\begin{bmatrix} 8 & 14 \\ -6 & 6 \end{bmatrix}$ (5) 132

3. (1) -1 (2) 2 (3) 1

   (4) 2 (5) -4

4. (1) $\begin{bmatrix} 2^k & 0 \\ 0 & 5^k \end{bmatrix}$ (2) $\begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix}$ (3) $\begin{bmatrix} 2^k & k2^{k-1}a \\ 0 & 2^k \end{bmatrix}$

5. Omitted.