FU08 - Automata and Languages
Exercise 1

Nguyen Tuan Dung
s1312004

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Question 1: Answer the following question

Give an example of a relation that is:
(a) reflexive, transitive, but not symmetric
(b) reflexive, symmetric, but not transitive
(c) transitive, symmetric, but not reflexive

Solution:
(a) An example of a relation that is reflexive, transitive, but not symmetric is the relation R = {(0,0); (0,1); (0,2); (1,1); (1,2); (2,2)} on the set A = {0,1,2}. R is the mathematical relation of $\leqslant$ on the set A.
Reflexive: $\forall a \in A : (a, a) \in R$.
Transitive: $\forall ((a, b) \in R) \land ((b, c) \in R)$ also $(a, c) \in R$.
Not symmetric: $(1, 2) \in R$ but $(2, 1) \notin R$.

(b) An example of a relation that is reflexive, symmetric, but not transitive is the relation R = {(0,1); (1,0); (1,2); (2,1); (0,0); (1,1); (2,2)} on the set A = {0,1,2}.
Reflexive: The relation R contains (0,0); (1,1) and (2,2).
Symmetric: The relation R contains (0,1); (1,0) also (1,2); (2,1).
Not transitive: The relation R contains (0,1) and (1,2) but does not contain (0,2).

(c) An example of a relation that is transitive, symmetric, but not reflexive is the relation R = {(0,1); (1,0); (0,0); (1,1)} on the set A = {0,1,2}.
Transitive: The relation R satisfies (0,1), (1,0), (0,0), etc...
Symmetric: The relation R contains (0,1), (1,0).
Not reflexive: The relation R lacks (2,2) to satisfy the condition of reflexivity.

Question 2: Answer the following question

Consider the relation between two sets defined by:
$S_1 \equiv S_2$ if and only if $|S_1| = |S_2|$.
Show that this is an equivalence relation.

Solution:
To show that a relation is equivalence, we show that it is both reflexive, symmetric and transitive.
Reflexive: A set is equivalence to itself (i.e $S_1 \equiv S_1 \Leftrightarrow |S_1| = |S_1|$).
Symmetric: Since equality is an symmetric relation, if $|S_1| = |S_2| \Leftrightarrow |S_2| = |S_1|$.
Transitive: Say, $(|S_1| = |S_2|) \land (|S_2| = |S_3|)$, since equality is a transitive relation, hence $|S_1| = |S_3|$.
From the above proof, we see that the relation mentioned is Reflexive, Symmetric, Transitive, hence satisfies the condition of being an equivalence relation.

Question 3: Answer the following question

Let R be an equivalence relation on a set A.
For each $a \in A$ the equivalence class of a is denoted by $[a]$ =$\{b : aRb\}$. Show that for all $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Solution:
Say, $[a] \cap [b] \neq \emptyset$ then $\exists k \in A$ with $k \in ([a] \cap [b])$. Hence, $((a, k) \in R) \land ((b, k) \in R)$. Since R is an equivalence relation as mentioned $(k, b) \in R$, this implies that $(a, b) \in R$ (i.e transitivity).
• We know that $(a, b) \in R$. Let an arbitrary element $x \in [b] \implies (b, x) \in R$. Since R is an equivalence relation, $(a, x) \in R$ (i.e transitivity). According to definition $x \in [a]$. And since $x \in [b]$, then $[b] \subseteq [a]$. (1)
• Again, $(b, a) \in R$ (symmetric). Let an arbitrary element $y \in [a] \implies (a, y) \in R$. Since R is an equivalence relation, $(b, y) \in R$ (i.e transitivity). According to definition $y \in [b]$. And since $y \in [a]$, then $[a] \subseteq [b]$. (2)
From (1) (2) $\implies [a] = [b]$ $\forall (a, b) \in R$ (3).
Hence, we know that $\forall (a, b) \in R, [a] = [b]$ when $\exists x \in [a] \cap [b]$ (which means $[a], [b]$ are joint sets). (4)

$$(3), (4) \implies \forall (a, b) \in R \left[\begin{array}{l} [a] = [b] \\ [a] \cap [b] = \emptyset \end{array} \right.$$