MA 02 LINEAR ALGEBRA II
REVIEW OF LECTURES – III

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Section: C7.
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How is it going? This is the second week. If you were expecting a 'full-throttle' action coming in, and felt that last week was rather 'moderate', you can count that before long we reach top speed. No abrupt acceleration, though. That's due to the nature of the subject. Remember, we are laying bricks (page 9 of "Review of Lectures – II") — not exactly a kind of deed that can be finished overnight. Maybe another couple of weeks, then it will completely 'take off'. So, don't let it happen you take it too easy and fall behind.

Last time we reviewed how to find the eigenvalues of a given matrix A. We focused on the case A is 2×2. So, now you have been told two different narratives about eigenvalues. Remember, on Day 1 ("Review of Lectures – I"), I threw

$$(*) \qquad A\boldsymbol{x} \;=\; 3\boldsymbol{x}.$$

I asked you to identify an eigenvalue of A. You guys knew that 3 is an eigenvalue of A. Awesome. But then in the last lecture ("Review of Lectures – II") you didn't see this stuff at all. The last lecture was about some algebraic maneuver to physically calculate the eigenvalues of A, which partially had to do with determinants. Meanwhile $(*)$ does not involve determinants. So you want to see how exactly those two narratives coexist within the same theory (namely, the theory of eigenvalues), and also hear if one has a bearing on the other. Excellent. If this is what you had in mind coming in, awesome, you are right on the target.

Before everything I need to quickly get it over with something which is *technically* a requirement. Namely, the meaning of the right-hand side of $(*)$. So, $3\boldsymbol{x}$ means what? The following answers it:

- **Definition (Scalar multiplication).** Let s be a scalar (a number). Then

$$\boxed{\; s \begin{bmatrix} p \\ q \end{bmatrix} \;=\; \begin{bmatrix} sp \\ sq \end{bmatrix}. \;}$$

This one is "s times a vector". What we have seen earlier is "s times a matrix". The same underlying concept.

Paraphrase:

$$\boxed{\begin{array}{lll} \text{If} \quad \boldsymbol{x} = \begin{bmatrix} p \\ q \end{bmatrix} & \text{and} & s \ : \ \text{a scalar} \\ \qquad\quad \Longrightarrow & & s\boldsymbol{x} = \begin{bmatrix} sp \\ sq \end{bmatrix}. \end{array}}$$

Example 1. (1)

$$2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

(2)

$$3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

(3)

$$\frac{1}{4} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{5}{4} \end{bmatrix}.$$

(4)

$$6 \begin{bmatrix} \frac{-1}{6} \\ \frac{7}{6} \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}.$$

- When it comes to '1' being multiplied (trivial one):

Example 2.

$$1 \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}.$$

- An obvious generalization of Example 2 is

$$1 \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

Paraphrase:

$$\boxed{\text{If} \quad \boldsymbol{x} = \begin{bmatrix} p \\ q \end{bmatrix} \quad \Longrightarrow \quad 1\boldsymbol{x} = \boldsymbol{x}.}$$

- Ones where '0' is involved (trivial ones):

Example 3. $$0\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \qquad 5\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

- An obvious generalization of Example 3 is

$$0\begin{bmatrix}p\\q\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \qquad s\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Notation. We denote $\begin{bmatrix}0\\0\end{bmatrix}$ as $\mathbf{0}$:

$$\mathbf{0} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Then we can paraphrase the above as

$$\boxed{\begin{array}{l}\text{If} \quad \boldsymbol{x} = \begin{bmatrix}p\\q\end{bmatrix} \qquad \text{and} \qquad s \ : \ \text{a scalar} \\ \qquad \Longrightarrow \qquad\qquad 0\boldsymbol{x} = \mathbf{0}, \qquad s\,\mathbf{0} = \mathbf{0}.\end{array}}$$

- **Definition (negation).**

$$\boxed{-\begin{bmatrix}p\\q\end{bmatrix} = \begin{bmatrix}-p\\-q\end{bmatrix}.}$$

Paraphrase:

$$\boxed{\text{If} \quad \boldsymbol{x} = \begin{bmatrix}p\\q\end{bmatrix} \qquad \Longrightarrow \qquad -\boldsymbol{x} = \begin{bmatrix}-p\\-q\end{bmatrix}.}$$

Example 4a. $$\left(-1\right)\begin{bmatrix} 6 \\ 11 \end{bmatrix} = \begin{bmatrix} -6 \\ -11 \end{bmatrix}.$$

Example 4b. $$-\begin{bmatrix} 6 \\ 11 \end{bmatrix} = \begin{bmatrix} -6 \\ -11 \end{bmatrix}.$$

• Examples 4a, 4b indicate that the negative of a vector and the $\left(-1\right)$ times the same vector are equal. This is true in general. Namely:

$$\boxed{\left(-1\right)\begin{bmatrix} p \\ q \end{bmatrix} = -\begin{bmatrix} p \\ q \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\text{If} \quad \boldsymbol{x} = \begin{bmatrix} p \\ q \end{bmatrix} \quad \Longrightarrow \quad \left(-1\right)\boldsymbol{x} = -\boldsymbol{x}.}$$

Exercise 1. Write each of the following in the form $\begin{bmatrix} p \\ q \end{bmatrix}$.

(1) $8\begin{bmatrix} 1 \\ 6 \end{bmatrix}$. (2) $\dfrac{1}{3}\begin{bmatrix} 10 \\ -12 \end{bmatrix}$. (3) $6\begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$.

(4) $\left(-4\right)\begin{bmatrix} -2 \\ -1 \end{bmatrix}$. (5) $1\begin{bmatrix} 10 \\ 3 \end{bmatrix}$. (6) $0\begin{bmatrix} 15 \\ -20 \end{bmatrix}$.

(7) $\dfrac{13}{6}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Exercise 2. Write each of the following in the form $\begin{bmatrix} p \\ q \end{bmatrix}$.

(1) $-\begin{bmatrix} 4 \\ -4 \end{bmatrix}$. (2) $-\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

- So far no matrices involved (only vectors). So, how about

Example 4. $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}.$$

Example 5. $$\begin{bmatrix} 3 & 2 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- An obvious generalizations of Example 4, 5 are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- Remember, from "Review of Lectures – II", that we use the notations I and O:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Keeping these notations (and $\mathbf{0}$ as defined above) intact, we can paraphrase the above as

$$\boxed{\text{If} \quad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \Longrightarrow \qquad I\boldsymbol{x} = \boldsymbol{x}.}$$

$$\boxed{\begin{array}{l} \text{If} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \\ \Longrightarrow \qquad A\mathbf{0} = \mathbf{0}, \quad O\boldsymbol{x} = \mathbf{0}. \end{array}}$$

Exercise 3. Calculate (1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \end{bmatrix}$. (2) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \end{bmatrix}$.

(3) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Exercise 4 (Complex structure). For $\boldsymbol{x} = \begin{bmatrix} p \\ q \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} r \\ s \end{bmatrix}$ define

$$\boxed{\boldsymbol{x} * \boldsymbol{y} = \begin{bmatrix} pr - qs \\ qr + ps \end{bmatrix}.}$$

(1) Prove $\boldsymbol{x} * \boldsymbol{y} = \boldsymbol{y} * \boldsymbol{x}$.

(2) Identify a vector $\mathbf{1}$ satisfying the property

$$\mathbf{1} * \boldsymbol{y} = \boldsymbol{y} \qquad \left(\text{for an arbitrary} \quad \boldsymbol{y} = \begin{bmatrix} r \\ s \end{bmatrix}\right).$$

(3) Prove that the vector which you found in (1) is <u>unique</u>.

(4) Let $\boldsymbol{x} = \begin{bmatrix} p \\ q \end{bmatrix}$. Assume $\boldsymbol{x} \neq \mathbf{0}$. Let $\mathbf{1}$ be the vector which you found in (2). Identify $\boldsymbol{y}$ satisfying the property

$$\boldsymbol{x} * \boldsymbol{y} = \mathbf{1}.$$

(5) $$\begin{bmatrix} \cos\theta \\ \\ \sin\theta \end{bmatrix} * \begin{bmatrix} \cos\phi \\ \\ \sin\phi \end{bmatrix} = \begin{bmatrix} \cos\left(\boxed{}\right) \\ \\ \sin\left(\boxed{}\right) \end{bmatrix} \qquad \left(\text{explain}\right).$$

(6) Let $\mathbf{1}$ be as above. Find one vector $\boldsymbol{i}$ such that $\boldsymbol{i} * \boldsymbol{i} = -\mathbf{1}$.

[<u>**Hints for Exercise 4**</u>]**:** (1) Do the swaps $p \longleftrightarrow r$, and $q \longleftrightarrow s$, in each of $pr - qs$ and $qr + ps$, and confirm that the outcomes remain unchanged.

(2) $\boldsymbol{x} * \boldsymbol{y} = \begin{bmatrix} pr - qs \\ qr + ps \end{bmatrix}$ is otherwise written as $\begin{bmatrix} p & -q \\ q & p \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$.

(3) Suppose $\mathbf{1}$ and $\mathbf{1'}$ satisfy $\mathbf{1} * \boldsymbol{y} = \boldsymbol{y}$ and $\mathbf{1'} * \boldsymbol{y} = \boldsymbol{y}$ for an arbitrary $\boldsymbol{y}$. Your goal is to prove $\mathbf{1} = \mathbf{1'}$. For that matter, first $\mathbf{1} * \mathbf{1'} =$? Also $\mathbf{1'} * \mathbf{1} =$? Then use (2): $\mathbf{1} * \mathbf{1'} = \mathbf{1'} * \mathbf{1}$. (4) Multiply $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}^{-1}$ to the two sides of $\boldsymbol{x} * \boldsymbol{y} = \mathbf{1}$ from the left, taking into account $\boldsymbol{x} * \boldsymbol{y} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$. (5) Addition formulas.

(6) Use (5).

- Now, with all this, let's take a look at the following from page 1 one more time:

$$(*) \qquad A\boldsymbol{x} = 3\boldsymbol{x}.$$

What can you see? Can you find one $\underline{\text{ obvious }}$ $\boldsymbol{x}$ that fulfills $(*)$? Yes, $\boldsymbol{x} = \mathbf{0}$. Indeed, if $\boldsymbol{x} = \mathbf{0}$, then $A\boldsymbol{x}$ becomes $A\mathbf{0}$, which is $\mathbf{0}$ (as we have just seen above). Also, $3\boldsymbol{x}$ becomes $3\mathbf{0}$, which is $\mathbf{0}$ (as we have just seen above). In short, for $\boldsymbol{x} = \mathbf{0}$ the two sides of $(*)$ both become $\mathbf{0}$, and hence $(*)$ is satisfied. We call $\boldsymbol{x} = \mathbf{0}$ $\underline{\text{ the trivial solution }}$ to $(*)$. The connotation of the adjective 'trivial' is 'insubstantial', 'uninteresting'. So, $\underline{\text{ we are interested in ‘non-trivial’ solutions }}$ $\boldsymbol{x}$ of $(*)$. So, let me quiz you.

Pop quiz. What is $\underline{\text{ the trivial solution }}$ to $A\boldsymbol{x} = 3\boldsymbol{x}$?

Answer. $\boldsymbol{x} = \mathbf{0}$. $\square$

The next item is the pinnacle of today's lecture — Non-trivial solutions.

- **Condition for the existence of non-trivial solutions for $A\boldsymbol{x} = 3\boldsymbol{x}$.**

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Consider

$$(*) \qquad \boxed{A\boldsymbol{x} = 3\boldsymbol{x}.}$$

a. " *Non-trivial solutions for $(*)$ may or maynot exist, depending on A.* "

b. " *If $\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix}$ is factored as $(\lambda - 3)(\lambda - k)$ where k is some number (k can be 3 or some other number), then a non-trivial solution for $(*)$ exists.*

c. " *Otherwise, a non-trivial solution for $(*)$ does not exist.* "

- By the way, you remember that $\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix}$ in item b. above is called the characteristic polynomial of A.

Of course, in the above you can replace 3 with any number, and the same set of statements holds. For example:

- **Condition for the existence of non-trivial solutions for $\boldsymbol{Ax = 2x}$.**

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

> Consider
>
> $$(*) \qquad \boxed{A\boldsymbol{x} = 2\boldsymbol{x}.}$$
>
> a. " *Non-trivial solutions for* $(*)$ *may or maynot exist, depending on* A. "
>
> b. " *If* $\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix}$ *is factored as* $(\lambda - 2)(\lambda - k)$ *where* k *is some number* (k *can be* 2 *or some other number*), *then a non-trivial solution for* $(*)$ *exists.*
>
> c. " *Otherwise, a non-trivial solution for* $(*)$ *does not exist.* "

- Now I am going to illustrate these using a concrete example.

Example 6 (= Example 9 in "Review of Lectures – II"). Consider

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}.$$

Like we did last time, let's form

$$\begin{vmatrix} \lambda - 3 & -1 \\ -2 & \lambda - 4 \end{vmatrix} \qquad \left(= \text{the characteristic polynomial of } A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \right).$$

We need to simplify it, and factor it. The result is below $\big($here the steps are omitted, we did it last time$\big)$:

$$\begin{vmatrix} \lambda-3 & -1 \\ -2 & \lambda-4 \end{vmatrix} = \big(\lambda-2\big)\big(\lambda-5\big).$$

So, what does this entail?

Question. With the same A as above, namely, $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$:

(1) Does $A\boldsymbol{x} = 3\boldsymbol{x}$ have a non-trivial solution $\boldsymbol{x}$?

— The answer is ‘no’.

(2) Does $A\boldsymbol{x} = 2\boldsymbol{x}$ have a non-trivial solution $\boldsymbol{x}$?

— The answer is ‘yes’.

(3) Does $A\boldsymbol{x} = 4\boldsymbol{x}$ have a non-trivial solution $\boldsymbol{x}$?

— The answer is ‘no’.

(4) Does $A\boldsymbol{x} = 5\boldsymbol{x}$ have a non-trivial solution $\boldsymbol{x}$?

— The answer is ‘yes’.

(5) Does $A\boldsymbol{x} = 7\boldsymbol{x}$ have a non-trivial solution $\boldsymbol{x}$?

— The answer is ‘no’.

(6) Does $A\boldsymbol{x} = -\boldsymbol{x}$ have a non-trivial solution $\boldsymbol{x}$?

— The answer is ‘no’.

(7) Does $A\boldsymbol{x} = \boldsymbol{x}$ have a non-trivial solution $\boldsymbol{x}$?

— The answer is ‘no’.

(8) Does $A\boldsymbol{x} = \boldsymbol{0}$ have a non-trivial solution $\boldsymbol{x}$?

— The answer is ‘no’.

So, why is it that your answer was 'yes', only for

$$A\boldsymbol{x} = 2\boldsymbol{x} \qquad \text{and} \qquad A\boldsymbol{x} = 5\boldsymbol{x},$$

and no other?

— Yes, that was because the characteristic polynomial of A, which is

$$\begin{vmatrix} \lambda-3 & -1 \\ -2 & \lambda-4 \end{vmatrix},$$

was factored as $\quad \left(\lambda-2\right)\left(\lambda-5\right).\quad$ So,

(∗) $$\boxed{A\boldsymbol{x} = \lambda_0\boldsymbol{x}}$$

has a non-trivial solution, precisely when λ_0 is either 2 or 5, but no other.

- So far so good. But don't you want to find the actual non-trivial solutions for

$$A\boldsymbol{x} = 2\boldsymbol{x} \qquad \text{and} \qquad A\boldsymbol{x} = 5\boldsymbol{x},$$

each? Is that feasible? Yes indeed. That's next.

- **Finding non-trivial solutions for $\boldsymbol{A\boldsymbol{x} = 2\boldsymbol{x}}$ $\left(\textbf{where } \boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}\right)$.**

Since $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, the equation $A\boldsymbol{x} = 2\boldsymbol{x}$ is

(#) $$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}.$$

Our job is to find a <u>non-trivial</u> solution $\begin{bmatrix} x \\ y \end{bmatrix}$ for (#).

For that matter, simplify each of the two sides of (#):

$$\begin{bmatrix} 3x + 1y \\ 2x + 4y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}.$$

So

$$\begin{cases} 3x + \ \ y \ = \ 2x, \\ 2x + 4y \ = \ 2y. \end{cases}$$

Shift the terms:

$$\begin{cases} \ \ x + \ \ y \ = \ 0, \\ 2x + 2y \ = \ 0. \end{cases}$$

Notice that this last pair of equations are essentially identical. What I mean is, you multiply 2 to the two sides of the first equation and get the second equation. So there is some redundancy here. So, we can actually delete the second equation. So

$$\boxed{x + y \ = \ 0.}$$

Can you solve it? Sure. How about

$$x \ = \ 0, \ \ y \ = \ 0.$$

Yes this works. You can write it as $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Oops, this isn't allowed. This is the trivial solution $\mathbf{0}$. My bad. But you can also offer another solution, say

$$x \ = \ 1, \ \ y \ = \ -1.$$

Yes of course. You can write it as $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This one is indeed a non-trivial solution, here we go. But you can also offer

$$x \ = \ 2, \ \ y \ = \ -2.$$

Or the same to say $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. This is another non-trivial solution. Or

$$x \ = \ 3, \ \ y \ = \ -3,$$

namely, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$. This is yet another non-trivial solution. Or more generally

$$x \ = \ k, \ \ y \ = \ -k,$$

namely, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix}$. Here, you have to add $k \neq 0$ just in order to exclude the trivial solution. $\square$

- **Finding non-trivial solutions for $\boldsymbol{A x = 5x}$ $\left(\textbf{where } \boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}\right)$.**

Since $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, the equation $A\boldsymbol{x} = 5\boldsymbol{x}$ is

(##) $$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}.$$

First, simplify (##):

$$\begin{bmatrix} 3x + 1y \\ 2x + 4y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix}.$$

So

$$\begin{cases} 3x + \ y = 5x, \\ 2x + 4y = 5y. \end{cases}$$

Shift the terms:

$$\begin{cases} -2x + \ y = 0, \\ 2x - \ y = 0. \end{cases}$$

Again, this last pair of equations are essentially identical. So delete one of them:

$$\boxed{2x - y = 0.}$$

First, $x = 0$, $y = 0$ works. Namely, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Oops, this isn't allowed. This is the trivial solution $\mathbf{0}$. The same mistake again. My bad. But then how about

$$x = 1, \ y = 2.$$

Yes of course. You can write it as $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. This is a non-trivial solution. But more generally

$$x = k, \ y = 2k$$

works. Namely, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ 2k \end{bmatrix}$. Here, once again, you have to say $k \neq 0$ just in order to exclude the trivial solution. $\square$

- **Summary of Example 6** $\left(\textbf{the matrix in reference is } \boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}\right).$

For $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, the equation

$$(*) \qquad \boxed{A\boldsymbol{x} = \lambda_0 \boldsymbol{x}}$$

has a non-trivial solution, precisely when λ_0 is either 2 or 5, but no other.

(i) For $\lambda_0 = 2$, non-trivial solutions of

$$\boxed{A\boldsymbol{x} = 2\boldsymbol{x}}$$

are of the form $\begin{bmatrix} k \\ -k \end{bmatrix}$, with $k \neq 0$.

(ii) For $\lambda_0 = 5$, non-trivial solutions of

$$\boxed{A\boldsymbol{x} = 5\boldsymbol{x}}$$

are of the form $\begin{bmatrix} k \\ 2k \end{bmatrix}$, with $k \neq 0$.

- Now you remember that $\lambda = 2$ and $\lambda = 5$ are called the <u>eigenvalues</u> of the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$. But here is what's more:

(ii) The vectors $\begin{bmatrix} k \\ -k \end{bmatrix}$, with $k \neq 0$, are called <u>eigenvectors</u> of A associated with the eigenvalue $\lambda = 2$. $\left(\text{Rationale: } A\begin{bmatrix} k \\ -k \end{bmatrix} = 2\begin{bmatrix} k \\ -k \end{bmatrix}.\right)$

(ii) The vectors $\begin{bmatrix} k \\ 2k \end{bmatrix}$, with $k \neq 0$, are called <u>eigenvectors</u> of A associated with the eigenvalue $\lambda = 5$. $\left(\text{Rationale: } A\begin{bmatrix} k \\ 2k \end{bmatrix} = 5\begin{bmatrix} k \\ 2k \end{bmatrix}.\right)$

• More generally:

Definition (Eigenvalues, eigenvectors). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose

$$\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \qquad \left(= \text{ the characteristic polynomial of } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

is factored as $\Big(\lambda - \lambda_1\Big)\Big(\lambda - \lambda_2\Big)$ with some numbers λ_1 and λ_2:

$$\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \Big(\lambda - \lambda_1\Big)\Big(\lambda - \lambda_2\Big).$$

Then $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are called the <u>eigenvalues</u> of A.

In this case, the equations

$$\boxed{A\boldsymbol{x} = \lambda_1 \boldsymbol{x}} \qquad \text{and} \qquad \boxed{A\boldsymbol{x} = \lambda_2 \boldsymbol{x}}$$

both have non-trivial solutions $\boldsymbol{x}$.

(i) Any non-trivial solutions for

$$\boxed{A\boldsymbol{x} = \lambda_1 \boldsymbol{x}}$$

is called an <u>eigenvector</u> of A associated with $\lambda = \lambda_1$.

(ii) Any non-trivial solutions for

$$\boxed{A\boldsymbol{x} = \lambda_2 \boldsymbol{x}}$$

is called an <u>eigenvector</u> of A associated with $\lambda = \lambda_2$.

• Note that it can happen $\lambda_1 = \lambda_2$. We will see such an example later $\big($Example 8$\big)$.

Example 7 $\big($= Example 10 in "Review of Lectures – II"$\big)$.

Let's find the eigenvalues and eigenvectors of

$$B = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}.$$

For that matter, form

$$\begin{vmatrix} \lambda-3 & -3 \\ -1 & \lambda-5 \end{vmatrix} \qquad \left(= \text{ the characteristic polynomial of } B = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \right).$$

Last time we dealt with the same matrix as B, and already got

$$\begin{vmatrix} \lambda-3 & -3 \\ -1 & \lambda-5 \end{vmatrix} = \big(\lambda-2\big)\big(\lambda-6\big).$$

So the matrix B has two eigenvalues: $\lambda = 2$ and $\lambda = 6$. In other words:

(∗) $$\boxed{B\boldsymbol{x} = \lambda_0 \boldsymbol{x}}$$

has a non-trivial solution, precisely when λ_0 is either 2 or 6, but no other.

- **Finding eigenvectors of B associated with $\lambda = 2$** $\left(\textbf{where } B = \begin{bmatrix} \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} \end{bmatrix}\right)$**.**

Since $B = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$, the equation $B\boldsymbol{x} = 2\boldsymbol{x}$ is

(@) $$\begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}.$$

First, simplify (@):

$$\begin{bmatrix} 3x + 3y \\ 1x + 5y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}.$$

So

$$\begin{cases} 3x + 3y = 2x, \\ \ \ x + 5y = 2y. \end{cases}$$

Shift the terms:

$$\begin{cases} x + 3y = 0, \\ x + 3y = 0. \end{cases}$$

So two identical equations came out. Delete one of them:

$$\boxed{x + 3y = 0.}$$

First, $x = 0$, $y = 0$ works. Namely, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. But this is the trivial solution. On the other hand,

$$x = 3, \ y = -1$$

works. Namely, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. This is a non-trivial solution. More generally

$$x = 3k, \ y = -k$$

with $k \neq 0$ works. Namely, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3k \\ -k \end{bmatrix}$ $\left(k \neq 0\right)$. $\square$

- **Finding eigenvectors of $\boldsymbol{B}$ associated with $\boldsymbol{\lambda = 6}$ $\left(\text{where } \boldsymbol{B} = \begin{bmatrix} \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} \end{bmatrix}\right)$.**

Since $B = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$, the equation $B\boldsymbol{x} = 6\boldsymbol{x}$ is

(@)
$$\begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix}.$$

First, simplify (@):

$$\begin{bmatrix} 3x + 3y \\ 1x + 5y \end{bmatrix} = \begin{bmatrix} 6x \\ 6y \end{bmatrix}.$$

So

$$\begin{cases} 3x + 3y = 6x, \\ x + 5y = 6y. \end{cases}$$

Shift the terms:

$$\begin{cases} -3x + 3y = 0, \\ \quad x - \ y = 0. \end{cases}$$

So two essentially identical equations came out. Delete one of them:

$$\boxed{x - y = 0.}$$

First, $x = 0, \ \ y = 0$ works. Namely, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. But this is the trivial solution. On the other hand,

$$x = 1, \ y = 1$$

works. Namely, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This is a non-trivial solution. More generally

$$x = k, \ y = k$$

with $k \neq 0$ works. Namely, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} \ \left(k \neq 0\right)$. $\square$

- **Summary of Example 7** $\left(\textbf{the matrix in reference is } \boldsymbol{B} = \begin{bmatrix} \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} \end{bmatrix}\right)$**.**

The matrix $B = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$ has two eigenvalues: $\lambda = 2$ and $\lambda = 6$. Namely:

$$(*) \qquad \boxed{B\boldsymbol{x} = \lambda_0 \boldsymbol{x}}$$

has a non-trivial solution, precisely when λ_0 is either 2 or 6, but no other.

(i) Eigenvectors of B associated with $\lambda_0 = 2$ are $\begin{bmatrix} 3k \\ -k \end{bmatrix} \ \left(k \neq 0\right)$.

These vectors are the non-trivial solutions for $(*)$ with $\lambda_0 = 2$.

(ii) Eigenvectors of B associated with $\lambda_0 = 6$ are $\begin{bmatrix} k \\ k \end{bmatrix} \ \left(k \neq 0\right)$.

These vectors are the non-trivial solutions for $(*)$ with $\lambda_0 = 6$.

Example 8. Let's find the eigenvalues and eigenvectors of

$$C = \begin{bmatrix} 6 & -1 \\ 1 & 8 \end{bmatrix}.$$

For that matter, form

$$\begin{vmatrix} \lambda - 6 & 1 \\ -1 & \lambda - 8 \end{vmatrix} \qquad \left(= \text{the characteristic polynomial of } C = \begin{bmatrix} 6 & -1 \\ 1 & 8 \end{bmatrix} \right).$$

Let's calculate this, and then factor:

$$\begin{aligned} \begin{vmatrix} \lambda - 6 & 1 \\ -1 & \lambda - 8 \end{vmatrix} &= \left(\lambda - 6\right)\left(\lambda - 8\right) - 1 \cdot \left(-1\right) \\ &= \lambda^2 - 14\lambda + 48 - \left(-1\right) \\ &= \lambda^2 - 14\lambda + 49. \end{aligned}$$

Let's factor it:

$$\lambda^2 - 14\lambda + 49 = \left(\lambda - 7\right)^2.$$

So the matrix C has only one eigenvalue: $\lambda = 7$. In other words,

(∗) $$\boxed{C\boldsymbol{x} = \lambda_0 \boldsymbol{x}}$$

has a non-trivial solution, precisely when λ_0 is 7, but no other.

- **Finding eigenvectors of C associated with $\lambda = 7$** $\left(\textbf{where } C = \begin{bmatrix} \mathbf{6} & \mathbf{-1} \\ \mathbf{1} & \mathbf{8} \end{bmatrix}\right)$.

  Since $C = \begin{bmatrix} 6 & -1 \\ 1 & 8 \end{bmatrix}$, the equation $C\boldsymbol{x} = 7\boldsymbol{x}$ is

$$\begin{bmatrix} 6 & -1 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 7 \begin{bmatrix} x \\ y \end{bmatrix}.$$

This is

$$\begin{cases} 6x - y = 7x, \\ x + 8y = 7y. \end{cases}$$

Shift the terms:

$$\begin{cases} -x - y = 0, \\ x + y = 0. \end{cases}$$

Two essentially identical equations came out. Delete one of them:

$$\boxed{x + y = 0.}$$

So, $\begin{bmatrix} k \\ -k \end{bmatrix}$ $\left(k \neq 0\right)$ works. $\square$

- **Summary of Example 8** $\left(\textbf{the matrix in reference is } \boldsymbol{C} = \begin{bmatrix} \mathbf{6} & \mathbf{-1} \\ \mathbf{1} & \mathbf{8} \end{bmatrix}\right)$.

The matrix $C = \begin{bmatrix} 6 & -1 \\ 1 & 8 \end{bmatrix}$ has only one eigenvalue: $\lambda = 7$. Namely:

(∗) $$\boxed{C\boldsymbol{x} = \lambda_0 \boldsymbol{x}}$$

has a non-trivial solution, precisely when λ_0 is 7, but no other.

(i) Eigenvectors of C associated with $\lambda_0 = 7$ are $\begin{bmatrix} k \\ -k \end{bmatrix}$ $\left(k \neq 0\right)$.

These vectors are the non-trivial solutions for (∗) with $\lambda_0 = 7$.

Exercise 5. Find the eigenvalue(s). Find the eigenvectors associated with each of the eigenvalues.

(1) $\begin{bmatrix} 3 & 2 \\ -9 & -6 \end{bmatrix}$. (2) $\begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$. (3) $\begin{bmatrix} 7 & -1 \\ 1 & 9 \end{bmatrix}$. (4) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.