MA 02 LINEAR ALGEBRA II

REVIEW OF LECTURES – VI

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Section: C7.
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Recipé to diagonalize a given matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose A has $\underline{\underline{\text{two distinct}}}$ eigenvalues

$$\chi_A(\lambda) = \Big(\lambda - \lambda_1\Big)\Big(\lambda - \lambda_2\Big) \qquad \Big(\underline{\underline{\lambda_1 \neq \lambda_2}}\Big).$$

Suppose

- ○ $\boldsymbol{x} = \begin{bmatrix} p \\ r \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = \lambda_1$.
- ○ $\boldsymbol{y} = \begin{bmatrix} q \\ s \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = \lambda_2$.

Then set

$$Q = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Then

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

In other words, set $P = Q^{-1}$, and

$$PAP^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Question. What's glaring in this statement?

— Yes, the underlined parts:

$$\text{“}\underline{\underline{\text{two distinct eigenvalues}}}\text{”.} \qquad \text{“}\underline{\underline{\lambda_1 \neq \lambda_2.}}\text{”}$$

Yes, that's right. $\underline{\underline{\text{We already know how to diagonalize a given matrix}}}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ following the recipé in the previous page, $\underline{\underline{\text{provided it has two distinct eigenvalues}}}$. Matrices that fall into this category $\big($“Category I”$\big)$ have been taken care of.

- Meanwhile, have we seen $\underline{\underline{\text{a matrix}}}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\underline{\underline{\text{which has only one eigenvalue}}}$, or the same to say, $\underline{\underline{\text{a matrix}}}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\underline{\underline{\text{such that } \chi_A(\lambda) \text{ is of the form}}}$ $\underline{\underline{\big(\lambda - \lambda_1\big)^2}}$ $\big($“Category II”$\big)$? Yes we have:

(i) $\begin{bmatrix} 6 & -1 \\ 1 & 8 \end{bmatrix}$ $\big($Example 8 from “Review of Lectures – III”$\big)$,

(ii) $\begin{bmatrix} 8 & -9 \\ 1 & 2 \end{bmatrix}$ $\big($Problem [I], part (2) from “Quiz – II”$\big)$,

(iii) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\big($Exercise 5, part (4) from “Review of Lectures – III”$\big)$.

The characteristic polynomials for these matrices:

$$\text{(i)} \ \big(\lambda - 7\big)^2, \qquad \text{(ii)} \ \big(\lambda - 5\big)^2, \qquad \text{(iii)} \ \big(\lambda + 1\big)^2.$$

Once again, these three matrices fall into “Category II”. So, are “Category II” matrices diagonalizable?

— The answer is, "it depends". For example, (iii) is a diagonal matrix.

A Quick Fact. A diagonal matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is diagonalizable.

Indeed, nothing stops us from artificially rewriting A as IAI^{-1}. In other words, $A = PAP^{-1}$ with $P = I$. Now suppose A is diagonal. Then PAP^{-1} is diagonal with $P = I$. So A is diagonalizable. $\square$

Exercise 1. True or false:

(1) A diagonal matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is diagonalizable. (?)

(2) A diagonalizable matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is diagonal. (?)

• In view of "Quick Fact" above, (iii) in the previous page is diagonalizable. But what about (i) and (ii)? In short, our question boils down to the following:

Question. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

" *Suppose A has only one eigenvalue λ_1. Thus*

$$\chi_A(\lambda) = \left(\lambda - \lambda_1\right)^2.$$

Suppose A is not diagonal. Is A diagonalizable? "

The answer is, tada..

Answer. Never.

Can you prove it? — Here is how it goes.

Proof. Suppose $\chi_A(\lambda) = \left(\lambda - \lambda_1\right)^2$. Suppose A is diagonalizable. Agree

$$(*) \qquad Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix},$$

for a suitable $Q = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. The right-hand side of $(*)$ is $\lambda_1 I$. So

$$(**) \qquad Q^{-1}AQ = \lambda_1 I.$$

In $(**)$, multiply Q from the left, and multiply Q^{-1} from the right:

$$QQ^{-1}AQQ^{-1} = Q\left(\lambda_1 I\right)Q^{-1}.$$

The left-hand side of $(**)$ is simplified as A, whereas the right-hand side of $(**)$ is simplified as $\lambda_1 I$. So

$$A = \lambda_1 I.$$

So A is indeed a diagonal matrix. $\square$

- **Thumbnail sketch of mathematical logic — Contrapositive.**

  You might challenge me: "The implication

$$\boxed{\begin{array}{l} \chi_A(\lambda) = \left(\lambda - \lambda_1\right)^2, \quad A: \text{ diagonalizable} \\ \qquad\qquad\qquad \Longrightarrow \qquad A: \text{ diagonal.} \end{array}}$$

is what was proved above. What we wanted to actually prove is below."

$$\boxed{\begin{array}{l} \chi_A(\lambda) = \left(\lambda - \lambda_1\right)^2, \quad A: \underline{\underline{\text{not}}} \text{ diagonal} \\ \qquad\qquad \Longrightarrow \qquad A: \underline{\underline{\text{not}}} \text{ diagonalizable.} \end{array}}$$

— Yes indeed. Good point. You are absolutely right. But the truth is, the above two boxes are actually one and the same. They are mutually contrapositive statements.

Vocabulary lesson. Please familiarize yourself with the word

"contrapositive".

The contrapositive of the implication $\left[A \implies B\right]$ is

$$\left[\text{not } B \implies \text{not } A\right].$$

If $\left[A \implies B\right]$ (the original implication) is correct, then $\left[\text{not } B \implies \text{not } A\right]$ (the contrapositive) is also correct, and vice versa.

In mathematics, we often opt to prove the contrapositive of the implication which we want to prove. And it suffices for the purpose. The above was one of those instances.

• In any case, let's duplicate what we have just proved (the second box in the previous page):

Fact. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose A falls into "Category II", namely, A has only one eigenvalue:

$$\chi_A(\lambda) = \left(\lambda - \lambda_1\right)^2.$$

Suppose furthermore that A is not diagonal. Then A is not diagonalizable.

Namely, no matter what P you choose, PAP^{-1} is not diagonal.

• So, all that said, what's <u>the second best alternative</u> to diagonalization for those "Category II" matrices?

• **Triangularization. Jordan canonical form.**

The second best alternative is

" *triangularization into a* <u>*Jordan canonical form*</u> ".

Let's use the same example as before $\big($(i), (ii) listed in page 2$\big)$. Out of the blue:

(i) $A = \begin{bmatrix} 6 & -1 \\ 1 & 8 \end{bmatrix}$ is "triangularized" as

$$Q^{-1}AQ = \begin{bmatrix} 7 & 0 \\ 1 & 7 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} k & -1 \\ -k+1 & 1 \end{bmatrix}.$$

(ii) $\begin{bmatrix} 8 & -9 \\ 1 & 2 \end{bmatrix}$ is "triangularized" as

$$Q^{-1}AQ = \begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} k+1 & 3 \\ \frac{k}{3} & 1 \end{bmatrix}.$$

Here, in the above, k $\big($inside $Q\big)$ stands for an arbitrary scalar.

• Now, in each of (i) and (ii), $Q^{-1}AQ$ is of the form $\begin{bmatrix} r & 0 \\ 1 & r \end{bmatrix}$. We say that they are the Jordan canonical form of A.

Definition (Jordan cell/block).

A matrix of the form

$$\begin{bmatrix} r & 0 \\ 1 & r \end{bmatrix}$$

$\big($where r is a scalar$\big)$ is called a <u>Jordan cell</u>, or a <u>Jordan block</u>.

Definition (Jordan canonical form).

For a given matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, suppose you manage to find an appropriate Q such that

$$Q^{-1}AQ = \begin{bmatrix} r & 0 \\ 1 & r \end{bmatrix}.$$

Then this resulting matrix $\begin{bmatrix} r & 0 \\ 1 & r \end{bmatrix}$ is called <u>the Jordan canonical form</u> of A.

In that case we also say we <u>have triangularized A into its Jordan canonical form</u>.

See Remark on page 16.

• Now, an obvious question here is, for a given A which is not diagonal and belongs to "Category II", how to find Q to make $Q^{-1}AQ$ into a Jordan canonical form. Another question that needs to be addressed is whether that is always feasible. For that matter, let's digest the following theorem, which is rather theoretical:

Theorem. Assume that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ belongs to "Category II", namely, A has only one eigenvalue $\lambda = \lambda_1$. Then A is written as either

(a) $$A = \begin{bmatrix} \lambda_1 + xy & -x^2 \\ y^2 & \lambda_1 - xy \end{bmatrix} \quad \text{or}$$

(b) $$A = \begin{bmatrix} \lambda_1 - xy & x^2 \\ -y^2 & \lambda_1 + xy \end{bmatrix},$$

using suitable x, y.

• The gist of proof of this theorem is actually the so-called "projective geometry" (not too advanced, something on par with undergraduate, math major, abstract algebra). So we skip the proof here. Rather, let's just trust the validity of this theorem. Below I demonstrate the feasibility of triangularization of A in each of (a) and (b) above into a Jordan canonical form.

Proof $\Bigg($of feasibililty of converting $\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$ into a Jordan canonical form,

when $\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$ is not diagonal and falls into "Category II"$\Bigg)$.

Assume that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ falls into "Category II" $\big($namely, it has only one eigenvalue$\big)$ and moreover it is not diagonal. By Theorem in the previous page, either

(a) $$A = \begin{bmatrix} \lambda_1 + xy & -x^2 \\ y^2 & \lambda_1 - xy \end{bmatrix} \quad \text{or}$$

(b) $$A = \begin{bmatrix} \lambda_1 - xy & x^2 \\ -y^2 & \lambda_1 + xy \end{bmatrix},$$

using suitable x, y. In each of (a) and (b), if suppose $x = 0$ and $y = 0$, then A is clearly diagonal, which contradicts our assumption made above. Thus at least one of x and y is non-zero. First, in both cases (a) and (b) we have

$$\chi_A(\lambda) = \big(\lambda - \lambda_1\big)^2.$$

So the only eigenvalue of A is $\lambda = \lambda_1$. An eigenvector of A associated with this eigenvalue is $\begin{bmatrix} x \\ y \end{bmatrix}$ $\big($in both cases$\big)$. So let's just hypothetically write

$$Q = \begin{bmatrix} p & x \\ q & y \end{bmatrix}.$$

Notice that the right column of Q is the eigenvector of A. Then by calculation

(a) $$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 \\ py - qx & \lambda_1 \end{bmatrix}, \quad \text{and}$$

(b) $$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 \\ -py + qx & \lambda_1 \end{bmatrix},$$

respectively.

We can make this a Jordan canonical form, simply by choosing p and q suitably so that the left-bottom entry equals 1:

(a) $$py - qx \;=\; 1, \qquad \text{and}$$

(b) $$-py + qx \;=\; 1,$$

respectively. $\square$

- Let's take another look at the last part of this proof: With

$$Q \;=\; \begin{bmatrix} p & x \\ q & y \end{bmatrix},$$

whose right-column is the eigenvector of A, and whose left-column is 'undecided' $\big($so temporarily p and q are inserted$\big)$, you can make $Q^{-1}AQ$ into a Jordan canonical form, provided that the condition

$$\det Q \;=\; \begin{vmatrix} p & x \\ q & y \end{vmatrix} \;=\; 1 \qquad \big(\text{case (a)}\big),$$

$$\det Q \;=\; \begin{vmatrix} p & x \\ q & y \end{vmatrix} \;=\; -1 \qquad \big(\text{case (b)}\big)$$

is met. You can choose p and q suitably so you can force the matrix Q to have determinant 1 $\big($case (a)$\big)$, and -1 $\big($case (b)$\big)$.

Now, there is one final question we need to answer: That is, can we decide just by staring at the original matrix A whether the case (a) holds or the case (b) holds? Yes:

- ○ Case (a) means that the top-right entry of A is ≤ 0, bottom-left ≥ 0.
- ○ Case (b) means that the top-right entry of A is ≥ 0, bottom-left ≤ 0.

So, all this is going to be crystallized into the following recipé:

Recipé to triangularize a given (non-diagonal) matrix in "Category II".

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose A has $\underline{\underline{\text{only one}}}$ eigenvalue $\lambda = \lambda_1$:

$$\chi_A(\lambda) = \left(\lambda - \lambda_1\right)^2.$$

Suppose A is $\underline{\underline{\text{not diagonal.}}}$ Then we have the dichotomy:

- Case (a) : b $\big(=$ top-right of $A\big)$ ≤ 0, c $\big(=$ bottom-left of $A\big)$ ≥ 0.
- Case (b) : b $\big(=$ top-right of $A\big)$ ≥ 0, c $\big(=$ bottom-left of $A\big)$ ≤ 0.

Set

$$\boldsymbol{y} = \begin{bmatrix} \sqrt{|b|} \\ \pm\sqrt{|c|} \end{bmatrix},$$

where $\pm$ is chosen suitably so $\boldsymbol{y}$ is an eigenvector of A associated with the eigenvalue $\lambda = \lambda_1$, namely, $A\boldsymbol{y} = \lambda_1 \boldsymbol{y}$.

Then set

$$Q = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} p & \sqrt{|b|} \\ q & \pm\sqrt{|c|} \end{bmatrix},$$

with the appropriate choice of '$\pm$'. Choose p and q arbitrarily such that

$$\det Q = 1 \quad \big(\text{case (a)}\big), \qquad \det Q = -1 \quad \big(\text{case (b)}\big)$$

Then

$$Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{bmatrix}.$$

In other words, set $P = Q^{-1}$, and

$$PAP^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{bmatrix}.$$

Pop Quiz. Let's decide whether

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ -2 & 13 \end{bmatrix}$$

falls into "Category II", namely, A has only one eigenvalue. Moreover, if that's true, then let's triangularize A into a Jordan canonical form.

Solution for Pop Quiz. Let's do it step by step.

Step 1. Let's find the eigenvalue(s) of A.

$$\begin{aligned} \chi_A\left(\lambda\right) &= \det\left(\lambda I - A\right) \\ &= \begin{vmatrix} \lambda - 9 & -2 \\ 2 & \lambda - 13 \end{vmatrix} \\ &= \left(\lambda - 9\right)\left(\lambda - 13\right) - \left(-2\right) \cdot 2 \\ &= \lambda^2 - 22\lambda + 117 + 4 \\ &= \lambda^2 - 22\lambda + 121 \\ &= \left(\lambda - 11\right)^2. \end{aligned}$$

So, A has only one eigenvalue, which is $\lambda = 11$. Awesome.

Step 2. Let's find $\boldsymbol{y}$ as in Recipé in page 10. Since $b = 2$, $c = -2$,

$\boldsymbol{y} = \begin{bmatrix} \sqrt{|b|} \\ \pm\sqrt{|c|} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \pm\sqrt{2} \end{bmatrix}$. Of these, an eigenvector of A is

$$\boldsymbol{y} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}.$$

Step 3. Let's find out whether Case (a) or Case (b) holds in Recipé in page 10. Clearly the top-right of the matrix $A = \begin{bmatrix} 9 & 2 \\ -2 & 13 \end{bmatrix}$ is ≥ 0, so case (b) holds.

Step 4. Form Q. The right column of Q is $\boldsymbol{y}$ which we found in Step 2, so

$$Q = \begin{bmatrix} p & \sqrt{2} \\ q & \sqrt{2} \end{bmatrix}.$$

Step 5. Decide p and q. Since we are in Case (b), so pick just any number for each of p and q with which this matrix Q has determinant -1:

$$\det Q = \begin{vmatrix} p & \sqrt{2} \\ q & \sqrt{2} \end{vmatrix} = -1.$$

So, I would just choose $p = 0$, $q = \dfrac{1}{\sqrt{2}}$. So

$$Q = \begin{bmatrix} 0 & \sqrt{2} \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix}.$$

Answer. $A = \begin{bmatrix} 9 & 2 \\ -2 & 13 \end{bmatrix}$ is triangularized into a Jordan canonical form as

$$Q^{-1}AQ = \begin{bmatrix} 11 & 0 \\ 1 & 11 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} 0 & \sqrt{2} \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix}.$$

Alternative answer. $A = \begin{bmatrix} 9 & 2 \\ -2 & 13 \end{bmatrix}$ is triangularized into a Jordan canonical form as

$$PAP^{-1} = \begin{bmatrix} 11 & 0 \\ 1 & 11 \end{bmatrix}, \qquad \text{where} \qquad P = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

• The above method of solving the Pop Quiz (page 11) works, as long as you remember the content of Recipé in page 10. That's one way to go around it. But what if the same question was on the test, and you didn't remember it? Good news: No panic. Even if you haven't memorized the same recipé — which I myself actually don't even bother to (more below) — you can still solve it by just knowing the general '*tenor*' of it. You find an alternative solution in the next page. That method doesn't rely on Recipé in page 10.

Here (to the extent that I brought it up) just some nuts-and-bolts: I wrote up that recipé by myself, needless to say. I didn't copy it off some textbook or anything. What I mean is, I was able to spell it out because I've worked out the proof of the feasibility of triangularization myself (as manifested in page 8–top of page 9). Recipé in page 10 naturally fell into place as a by-product of that proof. In my judgment it was worthy to highlight, so I decided to include it. Indeed, it gives us one exact prescription to triangularize a given matrix into its Jordan canonical form. So, if you happen to be a '*minimalist*' (more below), then just focus on "Second (alternative) solution" in the next page. Of course, you still retain the option to use Recipé in page 10. Either way.

Let me digress: I don't consult anything when writing up my notes for this class. No 'cheat sheet'. Everything is in my head. Maybe one or two exceptions along the way throughout the semester, just to check up on something, that's about it. But that doesn't mean that everything I write up for this class is *literally* in my head. Rather, I have a complete understanding of the ideas that underlie the subject (linear algebra in this case). With it I can reconstruct everything, and what you see in my notes are exactly that. The above proof of feasibility of triangularization is something that we working mathematicians can reproduce without peeking at anything. But it doesn't mean that we memorize it by heart. Actually our second nature is to *minimize* memorization. We know how to reproduce things whenever needed. So, we mathematicians are the *minimalists*. Now, that is when we are teaching an elementary level class (this class is rather elementary for us — sorry to break it to you — but that's only because we are pros). At more advanced, cutting edge corners of math, every research depends on other, past, research results. So every research article comes with 'references/bibliography'. So whenever we need to rely on something in our research we just know which article(s) authored by which individual(s) to refer to, or at least know how to search what's (potentially) relevant.

Back to Recipé in page 10: That is not exactly something I myself feel compelled to memorize. So, naturally, I wouldn't dare insist you to remember it. Instead, I actually recommend you to follow what's next:

Second (alternative) solution for Pop Quiz in page 11.

Once again, $A = \begin{bmatrix} 9 & 2 \\ -2 & 13 \end{bmatrix}$. We can actually employ the following method to solve the same pop quiz.

Step 1. Let's find the eigenvalue(s) of A. This part is the same. So, let me just show the result:

$$\chi_A(\lambda) = \det\Big(\lambda I - A\Big)$$

$$= \begin{vmatrix} \lambda - 9 & -2 \\ 2 & \lambda - 13 \end{vmatrix} = \Big(\lambda - 11\Big)^2.$$

So, A has only one eigenvalue, which is $\lambda = 11$.

Step 2. Let's find one eigenvector $\boldsymbol{y} = \begin{bmatrix} x \\ y \end{bmatrix}$ of A associated with $\lambda = 11$.

Since $A = \begin{bmatrix} 9 & 2 \\ -2 & 13 \end{bmatrix}$, the equation $A\boldsymbol{x} = 11\boldsymbol{x}$ is

(@) $$\begin{bmatrix} 9 & 2 \\ -2 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 11 \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$\begin{cases} 9x + 2y = 11x, \\ -2x + 13y = 11y. \end{cases}$$

Shift the terms:

$$\begin{cases} -2x + 2y = 0, \\ -2x + 2y = 0. \end{cases}$$

So two identical equations came out. These equations are the same as

$$\boxed{x - y = 0.}$$

Clearly $x = 1, \quad y = 1$ works. Thus:

- $\boldsymbol{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda = 11$.

Step 3. Form $Q = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix}$, where $\boldsymbol{y}$ is the vector which you just found, and $\boldsymbol{x} = \begin{bmatrix} p \\ q \end{bmatrix}$ is another vector:

$$Q = \begin{bmatrix} p & 1 \\ q & 1 \end{bmatrix}.$$

We are going to decide p and q.

Step 4. Calculate Q^{-1}:

$$Q^{-1} = \frac{1}{p-q} \begin{bmatrix} 1 & -1 \\ -q & p \end{bmatrix}.$$

Step 5. Calculate $Q^{-1}AQ$:

$$\begin{aligned}
Q^{-1}AQ &= \frac{1}{p-q} \begin{bmatrix} 1 & -1 \\ -q & p \end{bmatrix} \begin{bmatrix} 9 & 2 \\ -2 & 13 \end{bmatrix} \begin{bmatrix} p & 1 \\ q & 1 \end{bmatrix} \\
&= \frac{1}{p-q} \begin{bmatrix} 11 & -11 \\ -9q - 2p & -2q + 13p \end{bmatrix} \begin{bmatrix} p & 1 \\ q & 1 \end{bmatrix} \\
&= \frac{1}{p-q} \begin{bmatrix} 11p - 11q & 0 \\ -9pq - 2p^2 - 2q^2 + 13pq & -9q - 2p - 2q + 13p \end{bmatrix} \\
&= \frac{1}{p-q} \begin{bmatrix} 11p - 11q & 0 \\ -2p^2 + 4pq - 2q^2 & 11p - 11q \end{bmatrix} \\
&= \frac{1}{p-q} \begin{bmatrix} 11(p-q) & 0 \\ -2(p-q)^2 & 11(p-q) \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ -2(p-q) & 11 \end{bmatrix}.
\end{aligned}$$

Step 6 (Finishing touch).

Choose p and q so the bottom-left corner of the matrix that came out in Step 5 equals 1:

$$-2\left(p-q\right) = 1.$$

So I would just choose $p = 0$, $q = \frac{1}{2}$. So

$$Q = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

Answer. $A = \begin{bmatrix} 9 & 2 \\ -2 & 13 \end{bmatrix}$ is triangularized into a Jordan canonical form as follows:

$$Q^{-1}AQ = \begin{bmatrix} 11 & 0 \\ 1 & 11 \end{bmatrix}, \qquad \text{where} \qquad Q = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

Alternative answer. $A = \begin{bmatrix} 9 & 2 \\ -2 & 13 \end{bmatrix}$ is triangularized into a Jordan canonical form as

$$PAP^{-1} = \begin{bmatrix} 11 & 0 \\ 1 & 11 \end{bmatrix}, \qquad \text{where} \qquad P = \begin{bmatrix} -2 & 2 \\ 1 & 0 \end{bmatrix}.$$

Remark. In some (many) literatures, a Jordan cell is

$$\begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix} \qquad \left(\text{as opposed to} \quad \begin{bmatrix} r & 0 \\ 1 & r \end{bmatrix}\right).$$

Theoretically speaking, there is no intrinsic difference between them. You have to use a coin toss to decide. Most importantly, once you decided it, you have to stick with it throughout. In this case, I tossed a coin and decided to call

$$\begin{bmatrix} r & 0 \\ 1 & r \end{bmatrix}$$

a Jordan cell. So I'm going to stick with it throughout. So I ask you to follow it. It's not like one way is standard and the other way is idiosyncratic.

Exercise 2. Triangularize each of the following matrix into a Jordan canonical form. If not feasible, then say 'not feasible'.

(1) $A = \begin{bmatrix} 14 & -27 \\ 3 & -4 \end{bmatrix}$. (2) $A = \begin{bmatrix} 12 & 9 \\ -16 & -12 \end{bmatrix}$.

(3) $A = \begin{bmatrix} -4 & 5 \\ -20 & 16 \end{bmatrix}$. (4) $A = \begin{bmatrix} 3\sqrt{2} & -2 \\ 4 & -\sqrt{2} \end{bmatrix}$.

(5) $A = \begin{bmatrix} \sqrt{2}+\sqrt{3} & 2 \\ -1 & -\sqrt{2}+\sqrt{3} \end{bmatrix}$.