MA 02 LINEAR ALGEBRA II

REVIEW OF LECTURES – VIII

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Section: C7.
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- **A matrix being substituted into polynomials.**

We have just familiarized ourselves with polynomials. Today, $f(x)$, $g(x)$, $h(x)$, *etc.* stand for polynomials. A general form of a polynomial (in descending order):

$$f(x) = a \qquad \Big(\text{degree } 0 \text{ polynomial}\Big),$$

$$f(x) = ax + b \qquad \Big(\text{degree } 1 \text{ polynomial}\Big),$$

$$f(x) = ax^2 + bx + c \qquad \Big(\text{degree } 2 \text{ polynomial}\Big),$$

$$f(x) = ax^3 + bx^2 + cx + d \qquad \Big(\text{degree } 3 \text{ polynomial}\Big),$$

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e \qquad \Big(\text{degree } 4 \text{ polynomial}\Big),$$

$$\vdots$$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \Big(\text{degree } n \text{ polynomial}\Big).$$

- Another name for degree 0, 1, 2, 3, 4, 5, 6, polynomials:
  - ◦ Degree 0 polynomial = <u>constant</u> polynomial,
  - ◦ Degree 1 polynomial = <u>linear</u> polynomial,
  - ◦ Degree 2 polynomial = <u>quadratic</u> polynomial,
  - ◦ Degree 3 polynomial = <u>cubic</u> polynomial,
  - ◦ Degree 4 polynomial = <u>quartic</u> polynomial,
  - ◦ Degree 5 polynomial = <u>quintic</u> polynomial,
  - ◦ Degree 6 polynomial = <u>sextic</u> polynomial.

- Today, we want to define

$$\boxed{f(A)}$$

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. How do you make sense of it? Yes.

Rule. $\underline{\underline{\text{Simply replace } x \text{ with } A.}}$

Example 1. Let

$$f(x) = 2x, \qquad A = \begin{bmatrix} 2 & 7 \\ 8 & 6 \end{bmatrix}.$$

Then

$$\begin{aligned} f(A) &= 2A \\ &= 2\begin{bmatrix} 2 & 7 \\ 8 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 14 \\ 16 & 12 \end{bmatrix}. \end{aligned}$$

Example 2. Let

$$f(x) = x^2, \qquad A = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}.$$

Then

$$\begin{aligned} f(A) &= A^2 \\ &= \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -3 & -2 \\ 6 & 1 \end{bmatrix}. \end{aligned}$$

Example 3. Let

$$f(x) = 2x^2 - 3x, \qquad A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} f(A) &= 2A^2 - 3A \\ &= 2\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^2 - 3\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \\ &= 2\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \\ &= 2\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}. \end{aligned}$$

- The next one is a little bit tricky.

Example 4. Let

$$f(x) = 4x + 5, \qquad A = \begin{bmatrix} 6 & 5 \\ 2 & 7 \end{bmatrix}.$$

Then

$$\begin{aligned} f(A) &= 4A + 5 \\ &= 4\begin{bmatrix} 6 & 5 \\ 2 & 7 \end{bmatrix} + 5 \quad ?? \end{aligned}$$

Oh, wait. This one doesn't make sense. We have defined a scalar 'times' a matrix, but not a matrix 'plus' a scalar. What should we do? $\underline{\underline{\text{Yes, you should furnish } 5I.}}$ So

$$
\begin{aligned}
f(A) &= 4A + 5I \\
&= 4\begin{bmatrix} 6 & 5 \\ 2 & 7 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 24 & 20 \\ 8 & 28 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 29 & 20 \\ 8 & 33 \end{bmatrix}.
\end{aligned}
$$

Example 5. Let

$$
f(x) = 1 + \sqrt{2}\,x + x^2, \qquad A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.
$$

Then

$$
\begin{aligned}
f(A) &= I + \sqrt{2}\,A + A^2 \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^2 \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}.
\end{aligned}
$$

- More generally:

Definition. Consider a <u>polynomial</u>:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_0, a_1, a_2, \cdots, a_n$ are scalars. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We define <u>the substitution of f at A</u> as

$$\boxed{f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n.}$$

Exercise 1. Calculate $f(A)$:

(1) $A = \begin{bmatrix} -2 & 2 \\ -1 & 6 \end{bmatrix}$, $f(x) = -2x$. (2) $A = \begin{bmatrix} 3 & -2 \\ 5 & -7 \end{bmatrix}$, $f(x) = x^2$.

(3) $A = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$, $f(x) = x^2 + 3x + 5$.

- **Substitution $f(A)$ and polynomial arithmetic.**

  For two polynomials $f(x)$ and $g(x)$, we already know we can

  (i) add up $f(x)$ and $g(x)$, (ii) subtract $g(x)$ from $f(x)$,

  (iii) multiply a constant to $f(x)$, (iv) multiply out $f(x)$ and $g(x)$.

So, the following make sense:

$$\begin{aligned} (f \pm g)(x) &= f(x) \pm g(x) && \text{(dounle sign in the same order)}, \\ (cf)(x) &= cf(x) && (c \text{ is a constant}), \\ (fg)(x) &= f(x)g(x). \end{aligned}$$

Example 6. For $f(x) = 1 + x^2$, and $g(x) = x + x^3$, we have

$$\begin{aligned}
\Big(f+g\Big)(x) &= \Big(1+x^2\Big) + \Big(x+x^3\Big) \\
&= 1 + x + x^2 + x^3, \\
\Big(f-g\Big)(x) &= \Big(1+x^2\Big) - \Big(x+x^3\Big) \\
&= 1 - x + x^2 - x^3, \\
\Big(2f\Big)(x) &= 2\Big(1+x^2\Big) \\
&= 2 + 2x^2, \\
\Big(5g\Big)(x) &= 5\Big(x+x^3\Big) \\
&= 5x + 5x^3, \\
\Big(fg\Big)(x) &= \Big(1+x^2\Big)\Big(x+x^3\Big) \\
&= x + 2x^3 + x^5.
\end{aligned}$$

Formula. Let $f(x)$ and $g(x)$ be polynomials. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{aligned}
\Big(f \pm g\Big)(A) &= f(A) \pm g(A) && \Big(\text{dounle sign in the same order}\Big), \\
\Big(cf\Big)(A) &= cf(A) && \Big(c \text{ is a constant}\Big), \\
\Big(fg\Big)(A) &= f(A)g(A).
\end{aligned}$$

Corollary. Let $f(x)$, $g(x)$ and A be as above. Then $f(A)$ and $g(A)$ commute:

$$\boxed{f(A)\,g(A) \;=\; g(A)\,f(A).}$$

Example 7. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. Let

$$f(x) \;=\; 2 + x + x^2, \qquad g(x) \;=\; 1 - x^2 + x^4.$$

Then

$$\begin{aligned}
f(A) &= 2I + A + A^2 \\
&= 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^2 \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
g(A) &= I - A^2 + A^4 \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^2 + \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^4 \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix}.
\end{aligned}$$

Let's confirm by direct calculation that $f(A)$ and $g(A)$ commute:

$$\begin{aligned}
f(A)\,g(A) &= \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -2 & 6 \end{bmatrix}, \\
g(A)\,f(A) &= \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -2 & 6 \end{bmatrix}.
\end{aligned}$$

- **Cayley-Hamilton's theorem.**

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The characteristic polynomial of A is

$$\begin{aligned} \chi_A(\lambda) &= \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \\ &= \lambda^2 - \Big(a + d\Big)\lambda + \Big(ad - bc\Big). \end{aligned}$$

After seeing all this, aren't you curious as to what happens when you substitute A right into the characteristic polynomial of A? Don't you expect that something special happens? Why don't we just do that: By direct calculation:

$$\begin{aligned} &\chi_A(A) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 - \Big(a + d\Big)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \Big(ad - bc\Big)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= \begin{bmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - bd \\ ac + cd - ac - cd & bc + d^2 - ad - d^2 + ad - bc \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Wow, the zero matrix O came out. This result is worth highlighting:

Formula (Cayley–Hamilton's Theorem).

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let $\chi_A(\lambda)$ be the characteristic polynomial of A. Then

$$\boxed{\chi_A(A) = O.}$$

Exercise 2. Calculate $f(A)$.

(1) $f(x) = x^2 - 5x + 2, \quad A = \begin{bmatrix} 0 & 10 \\ -1 & 7 \end{bmatrix}.$

(2) $f(x) = x^3 - 10x^2 + 31x, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$

Exercise 3. Let $A = \begin{bmatrix} -4 & -3 \\ 9 & 6 \end{bmatrix},$

$$f(x) = 1 - x + x^2, \qquad g(x) = 1 + x + x^2.$$

(1) Find $\big(fg\big)(x)$. Write your answer in an expanded form.

(2) Calculate $f(A)$, and $g(A)$ each.

(3) Calculate $\big(fg\big)(x)$ using the result of (1).

(4) Independently of (3), calculate $f(A)g(A)$ using the result of (2).

(5) Independently of (3), (4), calculate $g(A)f(A)$ using the result of (2).

Exercise 4. Let

$$A = \begin{bmatrix} 42384031609 & -33649837 \\ 7017128346 & 1843570134 \end{bmatrix}.$$

Find $\chi_A(A)$. No calculation. Cite an appropriate theorem.

Exercise 5. Let

$$A = \begin{bmatrix} \sqrt{3} - \sqrt{7} - \sqrt{11} & 2\sqrt{5} + \sqrt{17} \\ 2\sqrt{5} - \sqrt{17} & \sqrt{3} - \sqrt{7} + \sqrt{11} \end{bmatrix}.$$

Find $\chi_A(A)$. No calculation. Cite an appropriate theorem.

- **Cayley–Hamilton's Theorem for larger size matrices.**

Cayley–Hamilton's Theorem holds true for an $n \times n$ size matrix, not just for 2×2. Today we are not going to prove it. Let me just highlight the result:

Formula (Cayley–Hamilton's Theorem).

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be an $n \times n$ matrix. Let $\chi_A(\lambda)$ be the characteristic polynomial of A:

$$\chi_A(\lambda) = \det\Big(\lambda I - A\Big) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}.$$

Then

$$\boxed{\chi_A(A) = O.}$$

Exercise 6. Let a, b, c and d be real numbers. Let $X = \begin{bmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{bmatrix}$.

(1) Find the characteristic polynomial $\chi_X(\lambda)$ of X.

(2) Use (1) and Cayley–Hamilton's Theorem to write X^4 as a scalar times X^2.

(3) Let $A = \Big(a^2 + b^2 + c^2 + d^2\Big) I + 2aX + 2X^2$. Prove

$$A^T = \Big(a^2 + b^2 + c^2 + d^2\Big) I - 2aX + 2X^2.$$

(4) Prove that AA^T is a scalar times the identity matrix.

(5) Find $\det A$.