Problem A. *(Adopted from MIT EECS 6.S955)*
Given a graph $G = (V, E)$, $V = \{1, \ldots, n\}$, $E \subseteq V \times V$. A common problem is *graph layout*, where we choose positions of the vertices in V on the plane $\mathbb{R}^2$ respecting the connectivity of G. Assume that $(i, i) \notin E \ \ \forall i \in V$ and $(i, j) = (j, i) \ \ \forall (i, j) \in E$.
Take $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^2$ to be the positions of the vertices in V; these are the unknowns in graph layout. Suppose a finite positions of vertices are specified in $V_0 \subseteq V$ (labeled as $\vec{v}_k^0 \ \forall k \in V_0$). Derive two linear systems satisfied by (x_i, y_i) of $\vec{v}_i$'s to minimize the Dirichlet energy:

$$\min E(\vec{v}_1, \ldots, \vec{v}_n) = \sum_{(i,j)\in E} \|\vec{v}_i - \vec{v}_j\|_2^2$$
$$\text{s.t } \vec{v}_k = \vec{v}_k^0 \quad \forall k \in V_0.$$

Implement and compare both gradient descent and conjugate gradients for solving this system. Use a preconditioner of choice for conjugate gradients. Show improve of convergence.

Solution .
Since $\vec{v_i} \in \mathbb{R}^2$, let $\vec{v_i} = (x_i, y_i)$. This implies $||\vec{v_i} - \vec{v_j}||_2^2 = (x_i - x_j)^2 + (y_i - y_j)^2$. Then

$$\begin{aligned}
E(\vec{v}_1, \ldots, \vec{v}_n) &= \sum_{(i,j)\in E} \|\vec{v}_i - \vec{v}_j\|_2^2 \\
&= \sum_{(i,j)\in E} (x_i - x_j)^2 + (y_i - y_j)^2 \\
&= E_x + E_y \quad \text{where} \begin{cases} E_x \equiv \sum_{(i,j)\in E} (x_i - x_j)^2 \\ E_y \equiv \sum_{(i,j)\in E} (y_i - y_j)^2 \end{cases}
\end{aligned}$$

Rewriting our objective

$$\min E_x + E_y$$
$$\text{s.t } \ (x_k, y_k) = \vec{v}_k = \vec{v}_k^0 \quad \forall k \in V_0.$$

We solve for E_x. Define $U = V \backslash V_0$. Then

$$\min \sum_{(i,j)\in E} (x_i - x_j)^2 \equiv \min \sum_{(i,u)\in E} (x_u - x_i)^2 \ \ \forall u \in U$$

$$\begin{aligned}
\frac{\partial E_x}{\partial x_u} &= 2 \sum_{i\in N(u)} (x_u - x_i) = 0 \\
\iff& \sum_{i\in N(u)} (x_u - x_i) = 0 \\
\iff& \delta(u) x_u - \sum_{i\in N(u)} x_i = 0 \\
\iff& \delta(u) x_u - \sum_{i\in N(u)\cap U} x_i = \sum_{j\in N(u)\cap V_0} x_j
\end{aligned}$$

Denote the cardinality of U as $m = |U|$. Let

1. $L \in \mathbb{R}^{m\times m}$; $L_{ii} = \delta(i)$; $L_{ij} = \begin{cases} -1 & \forall i \in N(j) \text{ for } j \in U \\ 0 & \text{otherwise} \end{cases}$ (the Laplacian matrix)

2. $\vec{x} \in \mathbb{R}^{m\times 1} = (x_{u_1}, \dots, x_{u_m})^\top \quad \forall u \in U$

3. $\vec{b_x} \in \mathbb{R}^{m\times 1} = (x_{j_1}, \dots, x_{j_m})^\top \quad \forall j \in N(i) \cap V_0$ with $i \in U$

This defines a linear system $E_x \equiv L\vec{x} - \vec{b_x}$. The work for E_y is analogous.

With our assumption $(i, j) = (j, i) \quad \forall (i, j) \in E$, L is intuitively symmetric. Also, the Laplacian graph matrix is always (strictly or not) diagonally dominant with non-negative diagonal entries. These combined guarantees positive semi-definiteness.

Complexity
The method of both steepest descent and conjugate gradients both depends on the matrix-vectors products operations. More often, the graph Laplacian matrix is sparse. Let t be the number of non-zero entries of L.[1] Then the complexity of:

- Gradient descent: $O(t\kappa)$
- Conjugate gradients: $O(t\sqrt{\kappa})$

with $\kappa \equiv \mathrm{cond}(L)$.

Preconditioning
Let us decompose $L = D + A + A^\top$ where D is the diagonal matrix consists of the diagonal values of L, A is the strictly lower triangular matrix consists of corresponding part of L. Then, the SOR iterative scheme is

$$(\omega^{-1}D + A)\vec{x_{k+1}} = ((\omega^{-1} - 1)D - A^\top)\vec{x_k} + \vec{b_x}$$

for some constant $\omega \in (0, 2)$.
Show of convergence.
By simply letting $M \equiv (\omega D + A)$ and $N \equiv ((\omega^{-1} - 1)D - A^\top)$ yields the classic Jacobi's iterative scheme

$$M\vec{x_{k+1}} = N\vec{x_k} + \vec{b_x}$$

which converges to $L^{-1}\vec{b_x}$ when $\rho(M^{-1}N) < 1$.
Improve of convergence.
show later

■

References

[1] J. R. Shewchuk, "An introduction to the conjugate gradient method without the agonizing pain," tech. rep., USA, 1994.