A Layman's Introduction to Knots and Jones Polynomials

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Definitions

Knot = A piece-wise linear (or smooth) embedding of S^1 into $\mathbb{R}^3$ or S^3

Link = A p.l. (or smooth) embedding of disjoint circles into $\mathbb{R}^3$ or S^3

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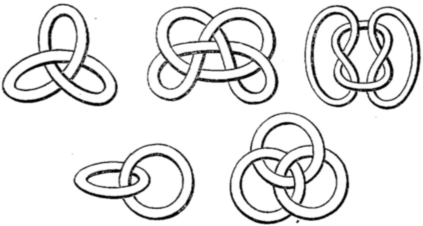

Figure: Illustrations of knots and links, including a trefoil knot, top left, in an 1869 paper by Lord Kelvin on his knotted vortex theory of atoms.

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Knot Diagrams

A picture of a projection of a knot onto a plane

Knot Equivalence

Two knots are equivalent if one knot can be pushed about smoothly, without intersecting itself, to coincide with another knot.

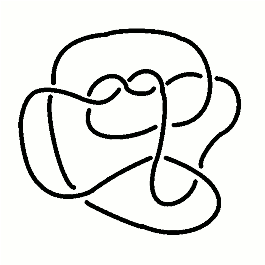


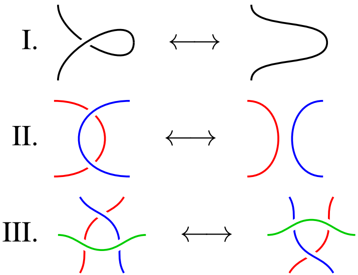
Figure: Deformation to an unknot

unknot = the boundary of a simplicial disk

Reidemeister Moves

Theorem (Reidemeister 1927)

Two knots are equivalent if and only if all their diagrams are connected by a finite sequence of Reidemeister moves of Type I, II or III.

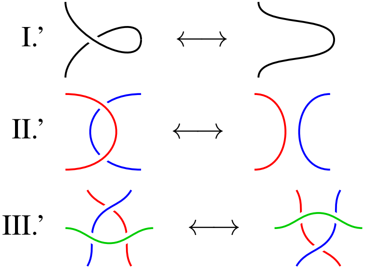


In this case, we also say their diagrams are equivalent.

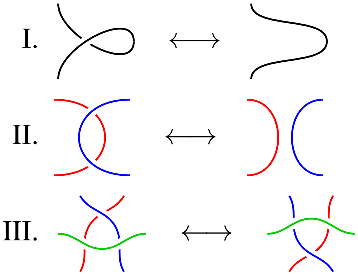
Reidemeister Moves

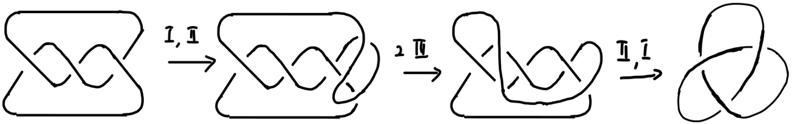
Remark

The following moves can be seen (exercise) to be consequences of the three types of Reidemeister moves.



Reidemeister Moves





Recognition Problem

Given two knots/knot diagrams, determining the (non-)equivalence of two knots.

Unkotting Problem

Given a knot (diagram), determining whether it is the unknot.

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Unkotting Problem

Given a knot (diagram), determining whether it is the unknot.

- Both problems are NP.
- $n =$ the sum of crossing numbers of two diagrams; an upper bound on the number of Reidemeister moves is $2^{2^{2^{\cdots^{n}}}}$, where where the height of the tower of 2s is $10^{1,000,000n}$ (Coward & Lackenby 2014).
- $c =$ the sum of crossing numbers of an unknot diagram; an upper bound on the number of Reidemeister moves required to arrive at the standard unknot is $(236c)^{11}$ (Lackenby 2015).

Unknotting Problem

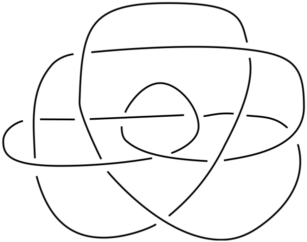

Figure: One of Ochiai's unknots

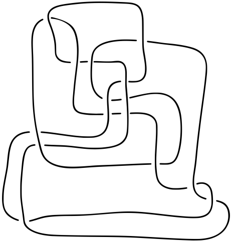

Figure: Thistlethwaite unknot

Not easy, actually very hard, to see both of the knots above are the unknot.

Knot Complements

Knot:	$K \subset S^3$
Regular neighbourhood:	$n(K)$
Knot complement:	$\overline{S^3 - n(K)}$

$$K_1 = K_2 \quad \Longrightarrow \quad \overline{S^3 - n(K_1)} = \overline{S^3 - n(K_2)}$$

$$\text{Invariants of } \overline{S^3 - n(K)} \quad \Longrightarrow \quad \text{Invariants of } K$$

Topological and Geometric Invariants

- $\pi_1(K) = \pi_1(\overline{S^3 - n(K)})$, the knot group of K
- Hyperbolic volume of $\overline{S^3 - n(K)} = \mathrm{vol}(K)$, the volume of K

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Fact: The only knot with infinite cyclic knot group is the unknot.

Diagrammatic Invariant

Idea

Knots are equivalent $\Longleftrightarrow$ Knot diagrams are equivalent

Diagrammatic invariants: invariants that respect Reidemeister moves

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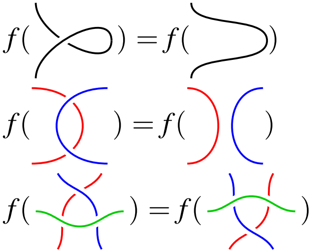
Diagrammatic invariants: invariants that respect Reidemeister moves

A diagrammatic polynomial invariant:

Knot:	$K \subset S^3$
Knot polynomial:	$f(K) \in \mathbb{Z}[t]$ or $\mathbb{Z}[t, t^{-1}]$



Kauffman Bracket

Definition

The Kauffman bracket is a function from unoriented link diagrams in the oriented plane to Laurent polynomials $\mathbb{Z}[A, A^{-1}]$. It maps a diagram D to $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ and is characterized by

1. $\left\langle \bigcirc \right\rangle = 1;$
2. $\left\langle L \cup \bigcirc \right\rangle = (-A^2 - A^{-2})\langle L \rangle;$
3. $\left\langle \times \right\rangle = A \left\langle \asymp \right\rangle + A^{-1} \left\langle)(\right\rangle.$

Or if you tilt your head $\frac{\pi}{2}$,

3'. $\left\langle \times \right\rangle = A \left\langle)(\right\rangle + A^{-1} \left\langle \asymp \right\rangle.$

KAUFFMAN BRACKET

Observation

Given a knot diagram, if you change all $\times$ to either $\asymp$ or $)($, then you arrive at an unlink.

1. $\left\langle \bigcirc \right\rangle = 1$;
2. $\left\langle L \cup \bigcirc \right\rangle = (-A^2 - A^{-2})\langle L \rangle$;
3. $\left\langle \times \right\rangle = A \left\langle \asymp \right\rangle + A^{-1} \left\langle)(\right\rangle$.

Applying this rules repetitively, we get a Laurent polynomial.

KAUFFMAN BRACKET

1. $\left\langle \bigcirc \right\rangle = 1$;
2. $\left\langle L \cup \bigcirc \right\rangle = (-A^2 - A^{-2})\langle L \rangle$;
3. ⟨  ⟩ $= A$ ⟨  ⟩ $+ A^{-1}$ ⟨  ⟩.

An easy calculation with the unlink:

$$\begin{aligned} \left\langle \bigcirc\bigcirc \right\rangle &= (-A^2 - A^{-2}) \left\langle \bigcirc \right\rangle \\ &= -A^2 - A^{-2} \end{aligned}$$

1. $\left\langle \bigcirc \right\rangle = 1;$
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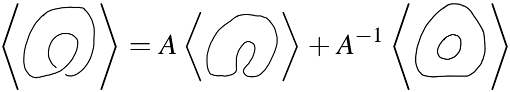
3. $\left\langle \times \right\rangle = A \left\langle \asymp \right\rangle + A^{-1} \left\langle)(\right\rangle.$

Recall that

$$\left\langle \bigcirc\bigcirc \right\rangle = -A^2 - A^{-2}.$$

Another example:



$$= A \cdot 1 + A^{-1}(-A^2 - A^{-2})$$
$$= -A^{-3}$$

KAUFFMAN BRACKET

$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$$= A \cdot 1 + A^{-1}(-A^2 - A^{-2})$$
$$= -A^{-3}$$

Good News

The Kauffman bracket respects Reidemeister moves of Type II and III.

Bad News

The Kauffman bracket does not respect Reidemeister moves of Type I.

Indeed, Type I gives an extra $-A^3$ factor and Type I' $-A^{-3}$.

WRITHE

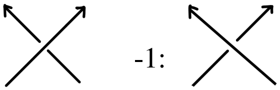
Oriented Link

A link with a choice of orientation for each complement

Definition

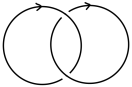
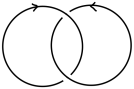
The writhe $w(D)$ of a diagram D of an oriented link is the sum of signs of the crossings of D, where each crossing has sign $+1$ or -1 according to the following:

+1: -1:



Property of Writhe

- $w(D)$ does not change if D is changed under Reidemeister moves of Type II or III.
- $w(D)$ does change by $+1$ or -1 if D is changed under a Reidemeister move of Type I.
- The writhe of a knot diagram does <u>not</u> depend on the choice of orientation.

$w($  $) = +2;$ $w($  $) = -2$

$w($  $) = -3$

Jones Polynomial

Theorem

Let D be a diagram of an oriented link L. Then the expression

$$(-A)^{-3w(D)}\langle D\rangle$$

is an invariant of the oriented link L.

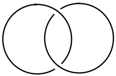
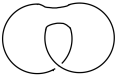
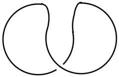
Definition-Theorem

The Jones polynomial $V(L)$ of an oriented link L is the Laurent polynomial in $t^{1/2}$ with integral coefficients, defined by

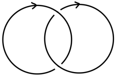
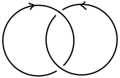
$$V(L) = \left((-A)^{-3w(D)}\langle D\rangle\right)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{1/2}, t^{-1/2}],$$

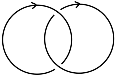
where D is any oriented diagram for L.

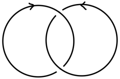
Hopf Link

$\Bigg\langle$  $\Bigg\rangle = A \Bigg\langle$  $\Bigg\rangle + A^{-1} \Bigg\langle$  $\Bigg\rangle$

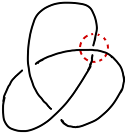
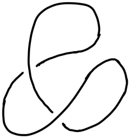
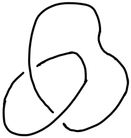
$$= A(-A^3) + A^{-1}(-A^{-3}) = -A^4 - A^{-4}$$

$w($  $) = 2; \quad w($  $) = -2$

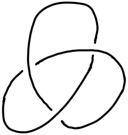
$V($  $) = \left(-A^{-2} - A^{-10}\right)_{t^{1/2}=A^{-2}} = -t^{1/2} - t^{5/2}$

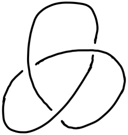
$V($  $) = \left(-A^{10} - A^{2}\right)_{t^{1/2}=A^{-2}} = -t^{-5/2} - t^{-1/2}$

Left-handed Trefoil Knot

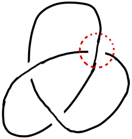
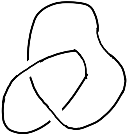
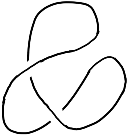
$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$$= A(-A^3)^2 + A^{-1}(-A^{-4} - A^4) = A^7 - A^3 - A^{-5}$$

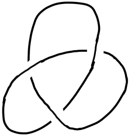
$w\Big($  $\Big) = -3$

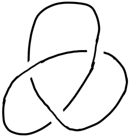
$V\Big($  $\Big) = \left(-A^{16} + A^{12} + A^4\right)_{t^{1/2}=A^{-2}} = -t^{-4} + t^{-3} + t^{-1}$

Right-handed Trefoil Knot

$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$$= A(-A^4 - A^{-4}) + A^{-1}(-A^{-3})^2 = A^{-7} - A^{-3} - A^5$$

$w($  $) = 3$

$V($  $) = \left(-A^{-16} + A^{-12} + A^{-4}\right)_{t^{1/2}=A^{-2}} = -t^4 + t^3 + t^1$

Mirror Image

Mirror Image

Put a mirror aside a knot and the image of the knot is known as the mirror image of the knot, or mathematically, the knot obtained by a refection in a plane.

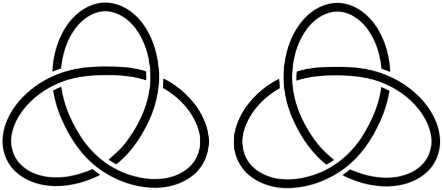


Figure: The right-handed and left-handed trefoil knots are not equivalent

Properties of Jones Polynomials

Theorem

The Jones polynomial of the mirror image $\bar{L}$ of an oriented link L is the conjugate under $t \leftrightarrow t^{-1}$ of the polynomial of L.

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The Jones polynomial of the mirror image $\bar{L}$ of an oriented link L is the conjugate under $t \leftrightarrow t^{-1}$ of the polynomial of L.

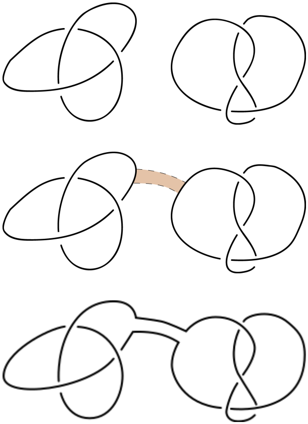
Proof.

The mirror image negates the writhe of any oriented diagram by exchange the positive and negative crossings. The mirror effect on the Kauffman bracket is that A is replaced by A^{-1}.

$$V(\text{left-handed trefoil knot}) = -t^{-4} + t^{-3} + t^{-1}$$
$$V(\text{right-handed trefoil knot}) = -t^{4} + t^{3} + t^{1}$$

Connected Sum

Connected sum of (oriented) knots $K_1 \# K_2$:



Properties of Jones Polynomials

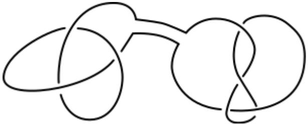
Theorem

Let K_1, K_2 be two (oriented) knots. Then we have

$$V(K_1 \# K_2) = V(K_1)V(K_2).$$

Sketch of proof

Consider a calculation of the polynomial of $K_1 \# K_2$ and operate firstly on the crossings of just one summand.



Yet Another Approach

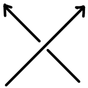
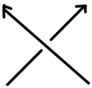
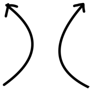
Definiton-Theorem

The Jones polynomial invariant is a function from the set of oriented links in S^3 to $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ such that

I. $V(\bigcirc) = 1$;

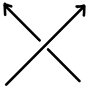
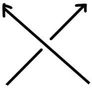
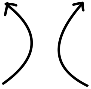
II. whenever three oriented links L_+, L_- and L_0 are the same, except in the neighbourhood of a point where they are shown below, then

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0.$$

L_+:  L_-:  L_0: 

Skein Relation

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0$$

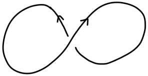
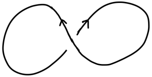
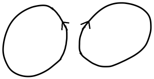
L_+:  L_-:  L_0: 

Skein Relation

In general, a skein relationship requires three link diagrams that are identical except at one crossing. To recursively define a knot (link) polynomial, a function F is fixed and for any triple of diagrams and their polynomials labelled as above,

$$F(L_+, L_-, L_0) = 0.$$

UNLINK

L_+:  L_-:  L_0: 

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0$$

But L_+ and L_- are both the unknot, so $V(L_+) = V(L_-) = 1$.

$$\begin{aligned} V(L_0) &= -\frac{t^{-1} - t}{t^{-1/2} - t^{1/2}} \\ &= -t^{-1/2} - t^{1/2} \end{aligned}$$

YET ANOTHER APPROACH

- Label crossings in a knot diagram D as $1, \dots, n$.
- A <u>state</u> for D is a function $s : \{1, \dots, n\} \to \{+1, -1\}$.
- There are 2^n states.
- Let sD be the diagram by replacing each crossings according to

  for the i^{th} crossing. Note that sD is then an unlink.
- $|sD|$ = number of complements in the unlink sD.

State Sum

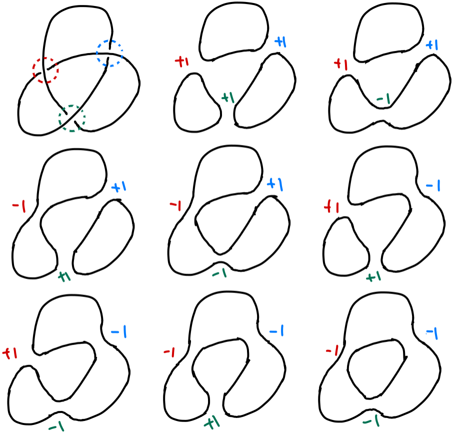
Definition-Theorem (Kauffman 1986)

Let D be a link diagram (of an oriented link L) with n crossings. Then the Kauffman bracket of D is given by

$$\langle D \rangle = \sum_{s} (A^{\sum_{i=1}^{n} s(i)} (-A^{-2} - A^{2})^{|sD|-1})$$

where the summation is over all states $s : \{1, \ldots, n\} \to \{+1, -1\}$. Therefore, the Jones polynomial of L is given by

$$V(L) = \left((-A)^{-3w(D)} \langle \sum_{s} (A^{\sum_{i=1}^{n} s(i)} (-A^{-2} - A^{2})^{|sD|-1}) \rangle \right)_{t^{1/2}=A^{-2}}.$$



$$\langle \text{left-handed trefoil knot} \rangle = 3A^{-1} + (A^{-3} + 3A)(-A^{-2} - A^2) + A^3(-A^{-2} - A^2)^2$$
$$= A^7 - A^3 - A^{-5}$$

Open Problem

Facts:

- There are nontrivial knots $K_1 \neq K_2$ with $V(K_1) = V(K_2)$.
- There are links with the same Jones polynomial as unlinks.

Conjecture

The only knot with Jones polynomial $V(K) = 1$ is the unknot.

The conjecture has been confirmed for several families of knots, including alternating knots and knots up to 23 crossings (as of 2019).

Coloured Jones Polynomials

None of the three methods discussed in this talk was Jones' original approach, where his formulation of the polynomial came from his study of operator algebras. His work is generalized to vast families of invariants, called quantum invariants.

- The coloured Jones polynomials is an infinite sequence of Laurent polynomials $\{J_{K,n}(t)\}_n$, encoding the Jones polynomial of K and these of the links K^s that are the parallels of K.
- Formulae for $J_{K,n}(t)$ come from representation theory of $SU(2)$ by decomposition of tensor product of representations.
- $J_{K,n}(t)$ can be calculated from any knot diagram via processes such as skein theory, state sums, R-matrices, and ...

Coloured Jones Polynomials

Fot this talk:

- $J_{K,1}(t) = 1$,
- $J_{K,2}(t) = V(K)$ - original Jones polynomial,
- $J_{K,3}(t) = V(K^2) - 1$,
- $J_{K,4}(t) = V(K^3) - 2V(K)$,
- ...

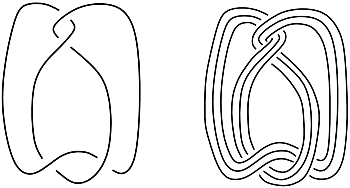


Figure: Figure-8 knot and its parallels

Conjecture

Volume Conjecture

Let K be a hyperbolic knot. Then

$$\mathrm{vol}(K) = \lim_{N \to \infty} \frac{2\pi \log |\langle K \rangle_N|}{N},$$

where

$$\langle K \rangle_N = \lim_{q \to e^{2\pi i/N}} \frac{J_{K,N}(q)}{J_{\text{unknot},N}(q)}.$$

Note that this conjecture relates quantum invariants of knots to the hyperbolic geometry of knot complements.