A Layman's Introduction to Knots and Jones Polynomials

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Nov 2023

Definitions

Knot = A piece-wise linear (or smooth) embedding of S^1 into $\mathbb{R}^3$ or S^3

Link = A p.l. (or smooth) embedding of disjoint circles into $\mathbb{R}^3$ or S^3

DEFINITIONS

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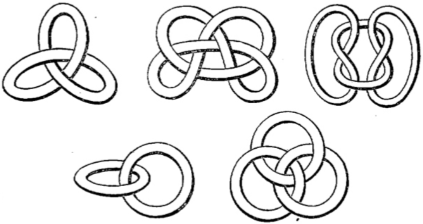


Figure: Illustrations of knots and links, including a trefoil knot, top left, in an 1869 paper by Lord Kelvin on his knotted vortex theory of atoms.

KNOT EQUIVALENCE

Two knots are equivalent if one knot can be pushed about smoothly, without intersecting itself, to coincide with another knot.

unknot = the boundary of a simplicial disk

Reidemeister Moves

Theorem (Reidemeister 1927)

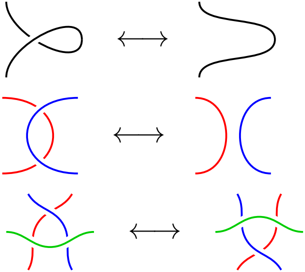
Two knots are equivalent if and only if all their diagrams are connected by a finite sequence of Reidemeister moves of Type I, II or III.

In this case, we also say their diagrams are equivalent.

I.

II.

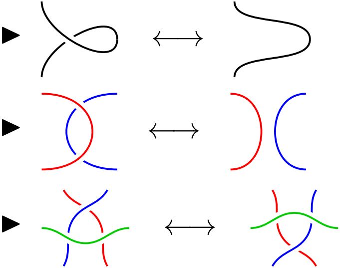
III.



Reidemeister Moves

Remark

The following moves can be seen (exercise) to be consequences of the three types of Reidemeister move.



Recognition Problem

Given two knots/knot diagrams, determining the (non-)equivalence of two knots.

Unkotting Problem

Given a knot (diagrams), determining whether it is the unknot.

- Both problems are NP.
- $n =$ the sum of crossing numbers of two diagrams; an upper bound on the number of Reidemeister moves is $2^{2^{2^{\cdots^{n}}}}$, where where the height of the tower of 2s is $10^{1,000,000n}$ (Coward & Lackenby 2014).
- $c =$ the sum of crossing numbers of a knot diagram; an upper bound on the number on the number of Reidemeister moves required to arrive at the standard unknot is $(236c)^{11}$ (Lackenby 2015).

Knot Complements

Knot:	$K \subset S^3$
Regular neighbourhood:	$n(K)$
Knot complement:	$\overline{S^3 - n(K)}$

$$K_1 = K_2 \quad \Longrightarrow \quad \overline{S^3 - n(K_1)} = \overline{S^3 - n(K_2)}$$

$$\text{Invariant of } \overline{S^3 - n(K)} \quad \Longrightarrow \quad \text{Invariant of } K$$

Topological and Geometricl Invariants

- $\pi_1(K) = \pi_1(\overline{S^3 - n(K)})$, the knot group of K
- Hyperbolic volume of $\overline{S^3 - n(K)}$

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- Hyperbolic volume of $\overline{S^3 - n(K)}$

Fact: The only knot with infinite cyclic knot group is the unknot.

Knot Complements

Theorem (Gordon and Luecke 1989)

If K_1 and K_2 are unoriented knots in S^3 and there is an orientation preserving homeomorphism between their complements, then K_1 and K_2 are equivalent (as unoriented knots).

Theorem (Whitten 1987)

If K_1 and K_2 are prime knots in S^3 with isomorphic knot groups, then their complements are homeomorphic.

Theorem (Waldhausen 1966)

If there exists an isomorphism between two knot groups sending longitude to longitude and meridian to meridian, then these two knots are equivalent.

Diagrammatic Invariant

Idea

Knots are equivalent $\iff$ Knot diagrams are equivalent

Diagrammatic invariants: invariants that respect Reidemeister moves

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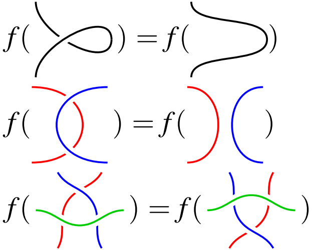
Diagrammatic invariants: invariants that respect Reidemeister moves

A diagrammatic polynomial invariant:

Knot:	$K \subset S^3$
Knot polynomial:	$f(K) \in \mathbb{Z}[t]$ or $\mathbb{Z}[t, t^{-1}]$



Kauffman Bracket

Definition

The Kauffman bracket is a function from unoriented link diagrams in the oriented plane to Laurent polynomials $\mathbb{Z}[A, A^{-1}]$. It maps a diagram D to $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ and is characterized by

1. $\left\langle \bigcirc \right\rangle = 1$;
2. $\left\langle L \cup \bigcirc \right\rangle = (-A^2 - A^{-2})\langle L \rangle$;
3. 

Or if you tilt your head $\frac{\pi}{2}$,

3'. 

Kauffman Bracket: Examples

$$\left\langle \bigcirc\bigcirc \right\rangle = (-A^2 - A^{-2}) \left\langle \bigcirc \right\rangle = -A^2 - A^{-2}$$

So it does not respect Reidemeister moves of Type I.

Writhe of a Knot/Link

Mirror

HOPF LINK

Left Trefoil Knot

Right Trefoil Knot

Mirror Image

Theorem

The Jones polynomial of the mirror image $\bar{L}$ of an oriented link L is the conjugate under $t \leftrightarrow t^{-1}$ of the polynomial of L.

Mirror Image

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The Jones polynomial of the mirror image $\bar{L}$ of an oriented link L is the conjugate under $t \leftrightarrow t^{-1}$ of the polynomial of L.

Proof.

asd

Fail for palindromes

Connected Sum

Fail for palindromes

Yet Another Approach

Skein Relation

Yet Another Another Approach

Skein Relation

Yet More Approaches

Skein Relation

CONJECTURE

Coloured Jones Polynomials

Whatever ite means

Conjectures

AJ Conjecture
Volume Conjecture