On the Jones Polynomial

constructing link invariants via braids & von Neumann algebras

A Monica Queen Thesis

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Abstract

This expository essay is aimed at introducing the Jones polynomial. We will see the encapsulation of the Jones polynomial, which will involve topics in functional analysis and geometrical topology; making this essay an interdisciplinary area of mathematics. The presentation is based on a lot of different sources of material (check references), but we will mainly be giving an account on Jones' papers ([25],[26],[27],[28],[29]) and Kauffman's papers ([33],[34],[35]). A background in undergraduate *Linear Algebra*, *Linear Analysis*, *Abstract Algebra*, *Commutative Algebra*, & *Topology* is essential. It would also be useful if the reader has a background in *Knot Theory* & *Operator Algebras*.

Ars longa, vita brevis.

This essay is dedicated to my grandmother **Grace**, who passed away during the start of writing this.

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A Introduction

The objective of this M.Sc. thesis, as are most, is to prove to the university of satisfactory scholastic work. However, the author accepts that such egotistical objectives will not keep the writer, not to mention the peruser, engaged. Accordingly, the genuine motivation behind this thesis is to introduce the writer's discoveries and thoughts insightfully and to impart them to other people. Subsequently, ideally, this will add to the assemblage of information regarding the matter.

Thus the aims of this essay are two-fold. The first is to give a solid introduction on knot theory and give the combinatorial method of the Jones polynomial (via Kauffman's bracket [33] and skein relation [15, 20, 26]). This gives us a good base on what the Jones polynomial is, and the importance of the polynomial invariant. For instance, it was until the Jones polynomial was created that the trefoil knot was proven to be distinct from its reflection. The second aim is to give a brief explanation on how Vaughan Jones originally came to the realisation that there was a link between the von Neumann algebras he was working on and the construction of a most interesting link invariant. This is given by first realising that all links can be represented by closed braids; and by defining a trace on the presentation from the braid group $\mathcal{B}_n$ to the quotient algebra of $\mathcal{B}_n$ called the Temperley-Lieb algebras $\mathcal{TL}_n$. This method is based on Jones' original construction in 1984 [25, 26].

Purely going on the notion of physical space, the ether, and atoms as vortex knots, the mathematical theory of knots was formulated by physicists Peter Tait, William Thomson, and James Maxwell in the late 1800s [40, 60]. While their motivation for studying knots and links (i.e., atoms being knotted in ether) was widely discredited in 1887 by Michelson and Morley [46], where they proved that the ether does not exist, the amount of contributions made by them was vast enough. It was until after the inexistence of the ether was proven that mathematicians started studying knots. And in the time of writing, knot theory has major applications linked to chemistry and biology, such as knots in DNA and proteins. For a nice read of introductory knot theory; see Adams' book [2].

To put it simply, knot theory, is the study of distinguishing links (disjoint union of closed smooth curves in three-dimensional space) up to ambient isotopy (intuitively, this means that we can move the link around without allowing it to pass by itself or cutting it). Distinguishing links is usually done using link invariants (algebraic objects that do not change under ambient isotopy). There are many link invariants such as the coloring invariant and the crossing number invariant - these are easy to visualise, however they are not good enough. The crossing number invariant, for instance, is difficult to determine since its definition states that it has to be over the minimal link diagram (so this is summed over the minimum diagram of all possible diagrams - i.e., infinity). The **Jones polynomial**, however, is the most *interesting* and one of the most *powerful* knot invariants that are linked to many areas of mathematics and science. Vaughan Jones constructed his polynomial invariant in 1984 [26]. For that, he won the Fields medal in 1990 and gave his speech wearing his home country's (New Zealand) rugby jersey: a historical day any knot theorist can tell you about.

Jones, however, did not make the connection on his own. In fact, while Jones was giving a lecture on von Neumann algebras in Geneva in 1982, Didier Hatt-Arnold, a graduate student, was the one who raised the similarities between the algebra he was working on and the braid group [16]. This then motivated Jones to explore possible representations of the braid group into his algebra he was working on. Finally leading him to the construction of the marvellous link invariant two years later [26].

The fascination of this link invariant, from a knot theory point of view, is that it could detect mirror links. Since 1928, the only known polynomial link invariant was the Alexander polynomial [6], and while profound in its own way, detecting a link's mirror was not one of its strengths. However, the main fascination with this link invariant is the fact that it came from a completely different field of mathematics. This lead to other link invariants that arose from areas such as category theory.

While, at first, it may have seemed as a surprising result from the subject of von Neumann algebras,

it became apparent that this subject has many amazing facts that are linked with other subjects such as Topological Quantum Field Theory. However, recalling that the mathematical theory of knots was formulated by physicsts, it should not be surprising when physics saves the day again. As Jones said,

> "God may or may not play dice, but She sure loves a von Neumann algebra." [30]

Despite the ease of defining the Jones polynomial in Chapter C via the Kauffman bracket and the skein relation, we should mention that it is generally known that the Jones polynomial of a link is NP-hard [24]. However, when calculating the Jones polynomial of a braid representation of a link, it can be computed in polynomial time [48]. We will not discuss computation in this paper, but we see in the final chapter that it is in fact a lot easier to calculate the Jones polynomial via its original construction than the more visual and diagrammatic ones. This is not to say that the Kauffman bracket is not useful, in fact the reason it is liked more so today than the original construction is because of its astounding correlation into category and representation theory; see Khovanov's paper [39].

Outline

In Chapter A, we will start with a brief overview on knot theory which includes all the essential information we need for the remainder of the essay.

In Chapter C, we will give two important definitions of the Jones polynomial: via the Kauffman bracket [33] and the Skein relation version, which John Conway conjured in the 1960s for the Alexander polynomial [15].

The next chapter (Chapter D) is on braid theory. We start with the diagrammatic definition of braids and see how it correlates with the Artin braid group, which is the algebraic definition. We also see a correlation between the symmetric group and the braid group.

In Chapter E, we relate the previous two chapters. We will associate to every knot a braid, and prove Alexander's theorem, which provides a pseudo-converse by showing that every braid closure is equivalent to a link. We conclude with an overview of Markov's moves and Markov's theorem. This gives an equivalence relation that will help us in making a one-to-one correspondence with braids and knots. However, two braids do not have to be braid equivalent in order to be Markov equivalent. Markov's theorem states that the braid closures are Reidemeister equivalent if and only if the braids are Markov equivalent. Thus the one-to-one correspondence will quotient the braid group with the Markov equivalence class.

The second part of the thesis focuses on more of Jones' work: we begin by defining the necessary analytical preliminaries in Chapter F and discuss the Temperley-Lieb algebras in algebraic and diagrammatic settings in Chapter G.

With the foundations set, we now have the necessary background in order to conclude the essay (Chapter H), i.e., constructing the Jones polynomial via the original method.

B   Knots & links

We begin with a brief overview of knot theory. We discuss all preliminary definitions and results needed to define the Jones polynomial and motivate our chosen thesis in the coming chapters.

This chapter is based on the presentations given in [55], [3], [4], [13], [15], [16], [20], [26], [28], [29], [33], [34], [35], [37], [38], [41], [43], [44], and [49].

Most of our discussion is based on that in the author's exposition on knot theory [55, Ch. 1].

B.I   Defining a knot and link

Definition B.1 (knot)**.** A *knot* is a smooth embedding of a circle $\mathbb{S}^1$ into $\mathbb{R}^3$ or $\mathbb{S}^3$.*

Here, a smooth *embedding* means a smooth map which is injective and whose differential is nowhere zero. Pictorially, Kauffman & Lomonaco Jr. [37, p. 2] drew the embedded circles in $\mathbb{R}^3$ as the following.
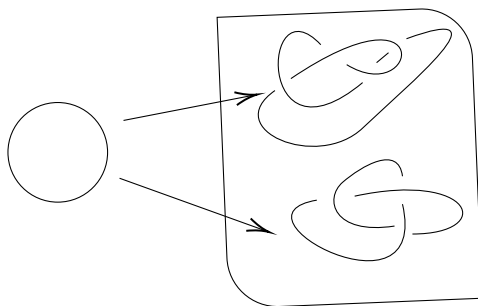

Figure B-i: knot definition with embedded circles in $\mathbb{R}^3$

In the above figure, the two embeddings are knots. It is easy to see that they are not the *same*. However, we will need to define what knots being "the same" means before claiming such (see next section).

Proposition B.2. *The definition of a knot is equivalent to the following:*

- *A knot is a subset of $\mathbb{R}^3$ that consists of piecewise linear simple closed curves.*

- *A knot is a subset of $\mathbb{R}^3$, which is diffeomorphic to $\mathbb{S}^1$.*

Here, *piecewise linear* means that the curve can be written as a finite union of line segments such that the segments are disjoint from one another, except when consecutive, i.e.,

$$K = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_n, a_0]$$

Here, *diffeomorphism* means that the map and its inverse are both smooth. *Homeomorphism*, on the other hand, is when the map and its inverse are both continuous.†

We can define a knot via homeomorphism (i.e., a subset of $\mathbb{R}^3$ which is homeomorphic to a circle). However, this requires us to define *tame knots* which are knots that can be defined smoothly or piecewise-linearly; and *wild knots* which are knots that are not tame.

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*$\mathbb{S}^3$ is essentially $\mathbb{R}^3$ with its one point compactification. We will mostly use $\mathbb{R}^3$.

†So this means that all diffeomorphisms are homeomorphisms, but not the other way around; because, differentiability implies continuity, but continuity does not imply differentiability.

To avoid such complications, we will always assume smoothness.

Example B.3. The most basic knot is the unknot, written as 0_1, which is drawn below. We also have the trefoil 3_1 and the trefoil's reflection $\overline{3_1}$. We call 3_1 the left-handed trefoil, whereas we call its reflection $\overline{3_1}$, the right-handed trefoil (fact: the right-handed trefoil and its reflection are two distinct knots; see Proposition C.14).
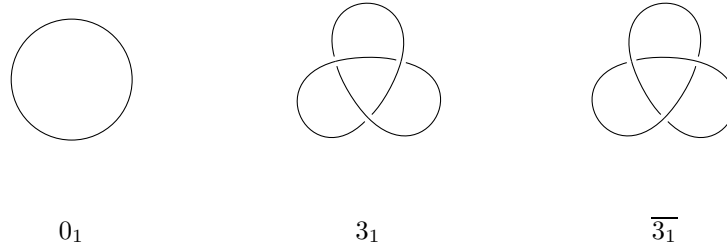

$$0_1 \qquad\qquad 3_1 \qquad\qquad \overline{3_1}$$

Figure B-ii: examples of knots

Example B.4. The following is not a knot, since the differential at $(*)$ is undefined. However, for the sake of completion, we mention that since this is a subset of $\mathbb{R}^3$ and is homeomorphic to a circle, we can see that it is an example of a *wild knot*.
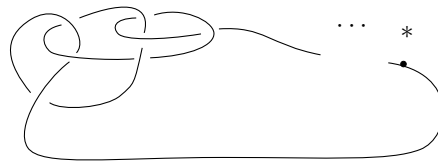
Figure B-iii: wild knot

Such knots can have pathological behaviour, hence why we mentioned that we will always assume smoothness throughout this paper.

Definition B.5 (link)**.** A *link* of m components is a smooth embedding of n disjoint circles into $\mathbb{R}^3$. Thus, a knot is just a link with one component.

Example B.6. An unlink is a finite number of disjointed unknots. We usually write an m-unlink as $\bigcirc^m$ (where m denotes the number of disjointed unknots). The figure to the right is an example of an unlink $\bigcirc^2$.

Hopf Link   Unlink of two components

Figure B-iv: examples of links

Remark. Note that it is not always the case that a theorem for knots works for links as well. However, for the purposes of this paper, we will use them interchangeably, unless stated otherwise.

B.II Link equivalency

We now discuss link equivalency. Essentially, we need two links to be equivalent if we can go from one link to the other without forming any self-intersections or breaking the definition of a link. So, we begin by explaining why homotopy equivalency is too weak for links.

Any two knots are *homotopy equivalent*, i.e., for knots $K_1 \colon S^1 \to \mathbb{R}^3$ and $K_2 \colon S^1 \to \mathbb{R}^3$, there exists a continuous function $H \colon \mathbb{R}^1 \times [0,1] \to \mathbb{R}^3$ such that $H(x,0) = K_1(x)$ and $H(x,1) = K_2(x)$. So, under homotopy, all knots would be trivial by pulling on the ends until the "knotted" part disappears (also known as the gentleman's knot - see Figure B-v).
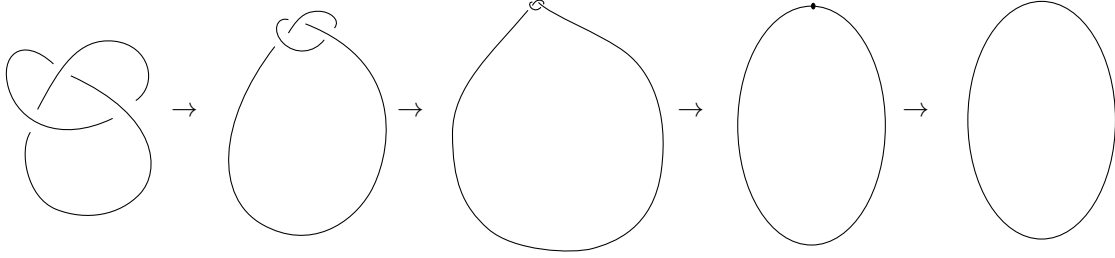

Figure B-v: the gentleman's knot

So, our definition of link equivalence needs to be stronger than that of homotopy.

Definition B.7 (ambient isotopy)**.** Two maps $A, B \subseteq \mathbb{R}^3$ are said to be *ambient isotopic* if there exists a continuous map $H \colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that the map $h_t \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $h_t(x) = H(x,t)$ is a diffeomorphism for all $t \in [0,1]$ with h_0 equal to the identity map, and $h_1(A) = B$. We call the map H, the *ambient isotopy* map between A and B.

Remark. In Definition B.7, ambient isotopy is essentially homotopy where it is a smooth family of diffeomorphisms.

Definition B.8 (link equivalence)**.** Two links are *equivalent* if they are ambient isotopic. When two links L_1 and L_2 are equivalent, we write $L_1 \sim L_2$.

Remark. Fisher [19] proved that any orientation-preserving[‡]diffeomorphism f of $\mathbb{R}^n$ is smoothly ambient isotopic to the identity. This allows us to characterise ambient isotopies as orientation-preserving diffeomorphisms; in particular, links are equivalent if there exists an orientation $-$ preserving piecewise linear diffeomorphism $h \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(L_1) = L_2$, where L_1 and L_2 are links.

Definition B.9 (link type)**.** Any two links which are equivalent have the same *link type*: the equivalence class of links. Similarly, we call the equivalence class of knots *knot type*.

Proposition B.10. *Ambient isotopy is an equivalence relation. (So, link equivalence is an equivalence relation.)*

Proof. Let L_1, L_2, and L_3 be any links.

For reflexivity, this is obvious.

[‡]For those with a background in differential geometry, a diffeomorphism between two oriented manifolds M and N is said to be *orientation-preserving* if the pullback of the orientation on N is the same, up to equivalence, of the orientation on M [61, Ch. 6].

For symmetry, we suppose $L_1 \sim L_2$. Then, we have a function h_t such that $h_0(L_1) = L_1$ and $h_1(L_1) = L_2$ where the map and its inverse are smooth. It follows that $h_0^{-1}(L_2) = L_2$ and $h_1^{-1}(L_2) = L_1$, which means $L_2 \sim L_1$.

Finally, for transitivity, suppose $L_1 \sim L_2$ and $L_2 \sim L_3$. Then, we have two functions h_t and f_t such that h_t is the same as h_t from the symmetry above; and $f_0(L_2) = L_2$ and $f_1(L_2) = L_3$. So, it then follows that $f_t \circ h_t(x) = f_t(h_t(x))$ is also a diffeomorphism (map compositions are smooth if both maps are smooth), so then $f_0(h_0(L_1)) = f_0(L_1) = L_1$ and $f_1(h_1(L_1)) = f_1(L_2) = L_3$. Thus, $L_1 \sim L_3$. ∎

Thus we have the following definition of an unlink and unknot.

> **Definition B.11** (unlink)**.** The *unknot* 0_1 is a knot that is equivalent to a circle. So, the *m-unlink* $\bigcirc^m$ is a link that is equivalent to m disjointed unknots.

B.III   Link diagrams and Reidemeister's theorem

In this section, we discuss elementary methods of projecting a link onto a two-dimensional plane. The reason for doing so is because it would not be reasonable for us to work in $\mathbb{R}^3$, as it is harder to imagine three-dimensional objects in our head. Before projecting the link, we first need to define the most elementary method of finding an equivalent link. This elementary method is called the triangle move. Here is a precise definition.

> **Definition B.12** (triangle-move)**.** Consider a link L in $\mathbb{R}^3$, we find a planar triangle in $\mathbb{R}^3$ that intersects L in exactly one of the edges of the triangle. We then delete that edge from L, and replace it by the other two edges of the triangle. The inverse process of this move is done by replacing the two edges of the triangle with one (literally the opposite). We call this the *triangle-move* and denote it by Δ and its inverse process by Δ^{-1}, see the figure below.
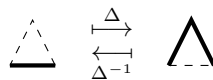

Figure B-vi: triangle-move

Two links are equivalent if and only if they differ by a finite sequence of the triangle-moves or the inverse of the moves. A detailed proof can be found in [13, pp. 6–8, Proposition 1.10].

Remark. It is easy to show that the Δ-move induces an equivalence relation. (It is closely analogous to the proof from Proposition B.10.)

Now we can define the *link projection map* $\rho \colon \mathbb{R}^3 \to \mathbb{R}^2$. As explained in [41, pp. 2–4], this is done by:

– small triangle-moves such that each line segment of the link projects to a line in $\mathbb{R}^2$,

– that the projections of two such segments intersect in at most one point which for disjoint segments is not an end point, and

– that no point belongs to the projections of three segments.

Given such a situation, the image of the link in $\mathbb{R}^2$ together with "over/under" information (this refers to the relative heights above $\mathbb{R}^2$ of the two inverse images of a crossing, which is always indicated in diagrams by breaks in the under-passing segments) at the crossings is called a *link diagram* of the given link. So, the only types of crossings in a link diagram that are allowed are essentially shown in Figure B-vii. And the crossings that are not allowed are triple points, tangents, and vertical edges (shown in Figure B-viii).
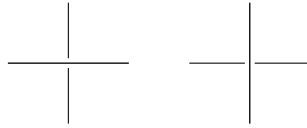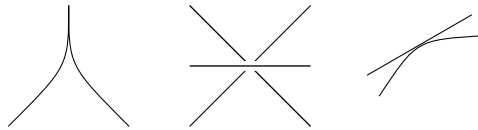
Figure B-vii: crossings that are allowed

Figure B-viii: vertical edge; triple point; tangent
crossings that are not allowed

Definition B.13 (link universe)**.** A *link universe* is a union of closed piece-wise linear curves in the plane such that there are finitely many self-intersections and that all self-intersections are transverse (i.e., at each crossing, there are exactly four half-curves with that crossing as an end point, so that none of them are tangent to another).

A *link diagram* is essentially a link universe with a choice, i.e., the points of self-intersection are equipped with height information with the lower strand being indicated with a break in the curve. If the link diagram has one component (i.e., a projection of a knot), then we call it a *knot diagram*.

Example B.14. A crossing in a link universe would then have two crossing choices in a link diagram.


link universe   crossing choices
                                 link diagrams
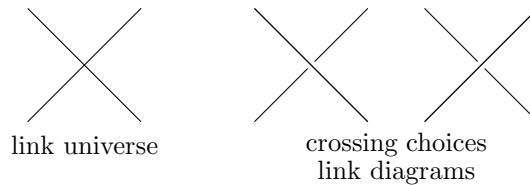
If links L_1 and L_2 are equivalent, then they are related by a sequence of triangle moves, as previously mentioned. After moving all the vertices of all the triangles by a very small amount, it can be assumed that the projections of no three of the vertices lie on the line in $\mathbb{R}^2$ and the projections of no three edges pass through a single point. So then each triangle projects to a triangle.

The following example shows that using triangle moves is dreary and time consuming. This motivates the usefulness of the upcoming theorem.

Example B.15. Consider the following piecewise linear link.
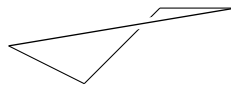
It is easy to see that this is equivalent to the unknot by a twist. However, with the triangle moves, we first find a planar triangle that intersects our link in one of the edges of the triangle. The intersected edge is thickened for clarity.

And we can now replace the thick edge with the two dashed edges on our triangle.

Thus, we see that this is equivalent to the unknot.
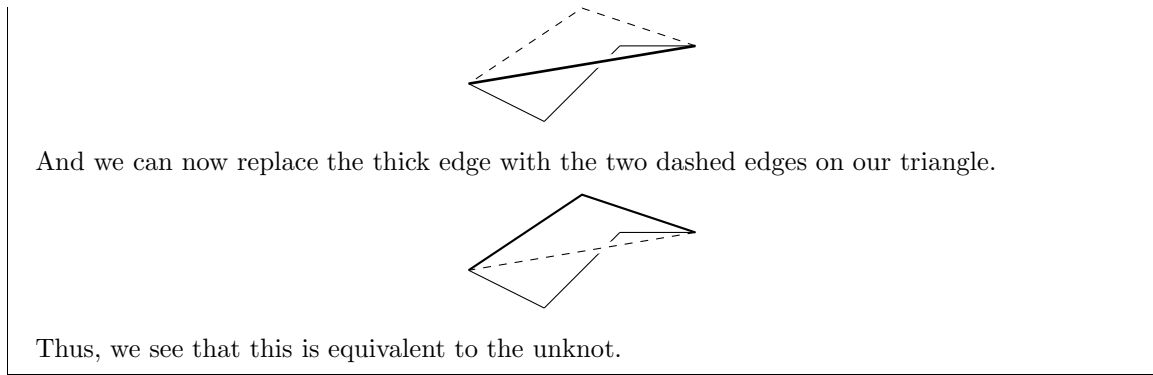
Thankfully, there is an easier method of showing link equivalency with Reidemeister's theorem. The theorem says that two link diagrams are equivalent if and only if they are related by a finite sequence of three moves (see Figure B-ix). The threorem also says that every link has a link diagram. This statement is quite fascinating as it simplifies the triangle moves. Here is a precise statement of the theorem.

Theorem B.16 (Reidemeister's theorem [56])**.**

1. *Any link L is equivalent to a link which has a link diagram.*

2. *Two links L_1, L_2 with diagrams D_1, D_2 are equivalent if and only if their diagrams are related by a finite sequence of Reidemeister moves and planar isotopies.*

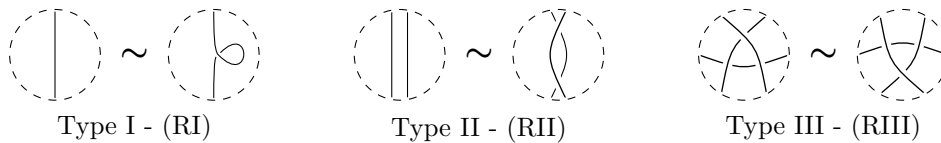Type I - (RI)   Type II - (RII)   Type III - (RIII)

Figure B-ix: Reidemeister moves

Remark. The figures above show regions of the diagram, when the move is made the rest of the diagram is left unchanged (hence why we put dashed lines: to remind ourselves that we should only think of the enclosed region for the moment).

We note that some texts include planar isotopy as a Type O - (RO) move. Intuitively, this means moving the knot around by stretching or compressing.

The diagrams are said to be *regularly isotopic* if a Type I move is not used. So the theorem shows that there is a bijective correspondence between the set of equivalence classes, i.e., (piecewise linear links in $\mathbb{R}^3$ with Δ-move equivalence) $\leftrightarrow$ (piecewise linear link diagrams in $\mathbb{R}^2$ with R-move equivalence).

Proof (sketch) of Theorem B.16. We show that any triangle-move can be reduced to a sequence of Reidemeister moves and planar isotopies. There are two cases: first being a planar isotopy - a regular triangle-move (Figure B-vi); the second is when the strands of the link come in contact with our chosen triangle which gives the three Reidemeister moves as shown in Figure B-x.
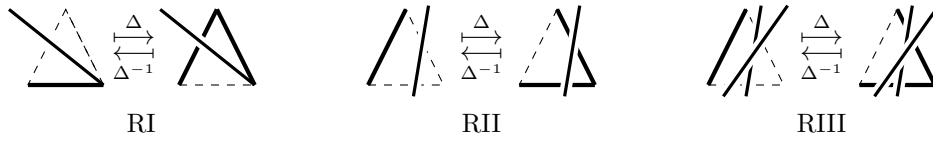
Figure B-x: Triangle moves $\leftrightarrow$ Reidemeister moves

∎

Our strategy for proving that a function is a link invariant (see B.V), is to show that it preserves the Reidemeister moves.

B.IV   Orientation

We now discuss orientations, and see how the figure eight knot is equivalent to its reflection.

An *orientation* for a link is a choice of direction along each of the components. This is indicated on the corresponding link diagram by adding an arrow to each component. The components of an n-component link can be oriented in 2^n ways.

Definition B.17 (sign). The *sign* $\epsilon(C) = \pm 1$ of an oriented crossing C is given by the figure below.

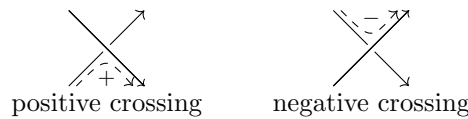positive crossing   negative crossing

Figure B-xi: sign of crossing

Remark. Figure B-xi can be thought of as moving along the orientation of the under-strand until we reach a crossing where we need to "jump" to the over strand; if we go to the right at the "jump" then the sign is positive, and if we go to the left then it is negative.

Definition B.18. The *writhe* $\omega(D)$ of a link diagram D is the sum of all the crossing signs, i.e. $\omega(D) = \sum_{C \text{ a crossing in } D} \epsilon(C)$.

Example B.19. The following shows an example of finding the writhe of the left-handed trefoil, Hopf link, and the unlink (with a twist).

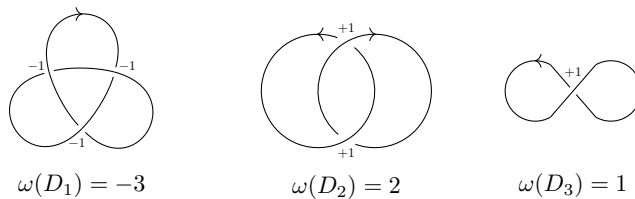$$\omega(D_1) = -3 \qquad \omega(D_2) = 2 \qquad \omega(D_3) = 1$$

Figure B-xii: writhe example

Definition B.20 (reverse). The *reverse* of a link is done by reversing the orientations of all the components in its corresponding diagram. For a link L, we denote the reverse by rL.

Remark. It is easy to see that the writhe of the diagrams in Example B-xii remains the same when reversing the orientation.

We can generalise this remark. By looking at a single crossing, we can see that changing all orientations of all the components in a link does not change the sign. So the signs of the following are the same.

$$\epsilon \left(\times\!\!\!\times \right) = +1 = \epsilon \left(\times\!\!\!\times \right)$$

Similarly,

$$\epsilon \left(\times\!\!\!\times \right) = -1 = \epsilon \left(\times\!\!\!\times \right)$$

Thus, we have $\omega(L) = \omega(rL)$. And so this means the writhe of an unoriented knot diagram is well defined.

> **Definition B.21** (reflection)**.** The *reflection* of a link is done by replacing all over-strands with under-strands at each crossing and vice versa of the original link's link diagram. For link L, we say $\overline{L}$ is the reflection of L.

By the end of the next chapter, we will show that the trefoil and its reflection are in fact two distinct knots. For now, however, we will show that both the figure eight knot and its reflection are equivalent.

Proposition B.22. *The figure eight knot 4_1 is equivalent to its reflection $\overline{4_1}$.*

Proof. We show this using the Reidemeister moves.
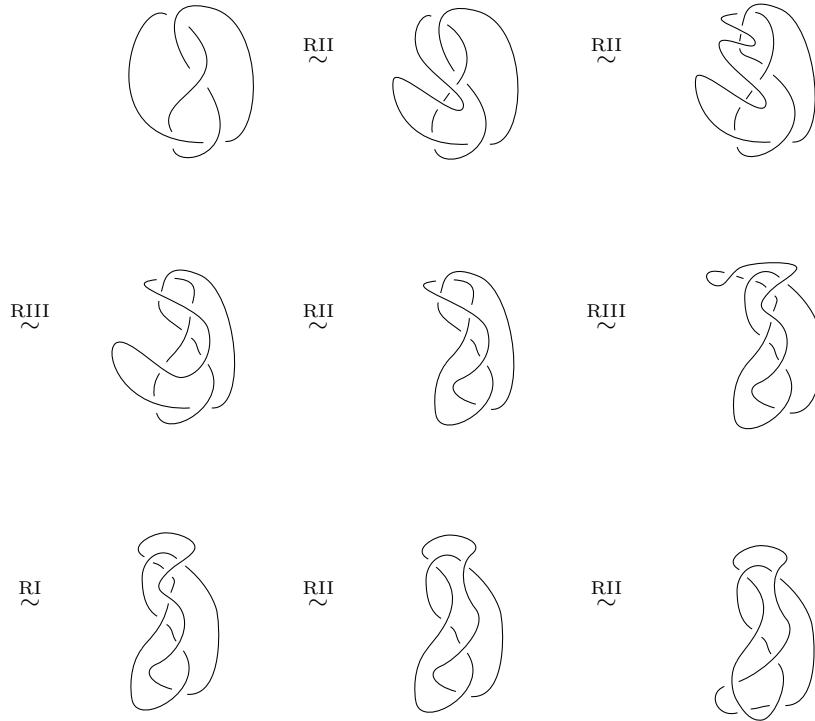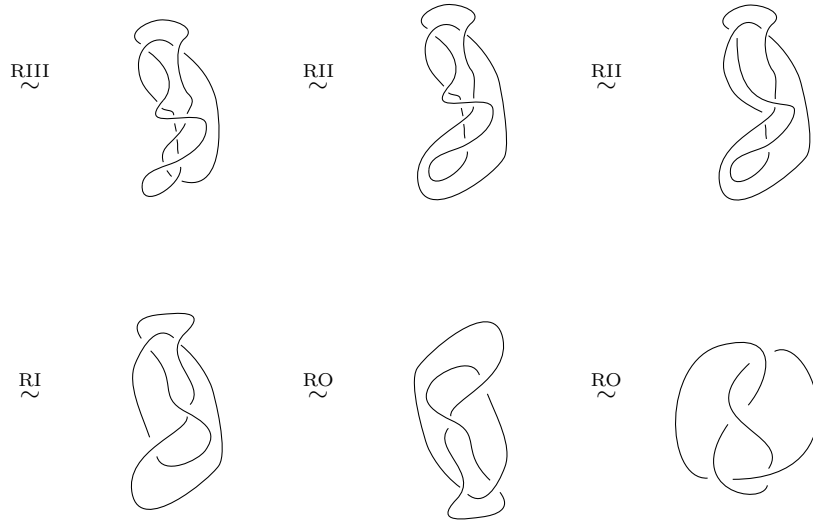
There exists some knots that have both its reverse and mirror to be distinct to its original knot. One such example is the knot $K = 9_{32}$. In fact, $K, rK, \overline{K}, \overline{rK}$ are all distinct oriented knots ([41, p. 4]). However, the reverse of a knot is *often* equivalent to its original knot.

Proposition B.23. *The reflection of a link diagram negates the writhe of the original link diagram.*

Proof sketch. We look at a single crossing and see what happens to the sign when the crossing is reflected,

$$\epsilon \left(\times\!\!\!\times \right) = -\epsilon \left(\times\!\!\!\times \right).$$

∎

Proposition B.24. *The writhe of a link preserves under regular isotopy, but not under ambient isotopy.*

Proof. A function that is preserved under regular isotopy but not under ambient isotopy, means that we need to show the writhe does not change under RII and RIII moves, but does change under RI moves.

$$\text{RI}: \quad \omega \left(\text{\scriptsize⟲} \right) = \omega \left(\| \right) + 1$$

$$\omega \left(\text{\scriptsize⟳} \right) = \omega \left(\| \right) - 1$$

For RII, given any orientation, we can see

$$\omega \left(\text{\scriptsize⟋⟍} \right) = \omega \left(\text{\scriptsize⟋⟍} \right) = \omega \left(\text{\scriptsize⟋⟍} \right) = \omega \left(\text{\scriptsize⟋⟍} \right)$$
$$= \omega \left(\text{\scriptsize≍} \right) = 0, \text{ for the given orientation.}$$

And it should be clear that it is also true for its reflection, i.e., $\omega\left(\smile\frown\right) = 0 = \omega\left(\asymp\right)$, given any orientation. Now RIII is analogous to what we did for RII. So the details are left for the reader, i.e., given any orientation we have,

$$\omega\left(\text{⤬} \right) = 0 = \omega\left(\text{⤬} \right)$$

∎

B.V   Invariants

By only considering the Reidemeister moves, it is difficult to distinguish links. This is because there is no method that can tell us whether we have not made the right number of Reidemeister moves yet. So the best way to distinguish links is by using invariants, which are algebraic ways of characterising these topological objects.

> **Definition B.25** (invariant). An *invariant* f is a function
>
> $$f\colon \{\text{equivalence classes of topological objects}\} \to S$$
>
> for some set S, with the property $f(A) = f(B)$ whenever A and B are equivalent.

So a *link invariant* is just an invariant function $f\colon \{\text{link type}\} \to S$ for some set S.

Thus, to prove that two links are in fact distinct (the main problem of knot theory), we need to show that they produce different outputs in a link invariant.

Remark. Invariant functions can only tell the objects apart - they can not show equivalency.

> **Definition B.26** (crossing number). The *crossing number* is a link invariant, such that it is defined as the minimal number of crossings needed for a diagram of the link L, where the minimum is taken over the infinity of possible diagrams of a link. We write $c(L)$ for the link L.

For low crossing numbers, we can find the knots by tabulating all knot diagrams; see the following result. In general, however, it is difficult to know which knots have the same knot type. This is solved by associating a knot to an invariant.

Proposition B.27. *There are no knots with crossing number one or two.*

Proof. We first draw the link universe with one component and one crossing and then show all the possible choices. The strategy here is to draw a link universe with the number of crossings we need without having to create more than one component.

| link universe | link diagram
unknot | link diagram
unknot |
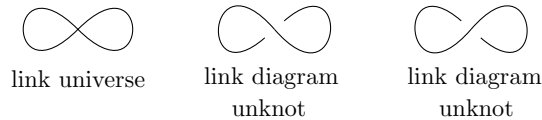
We clearly see that the first and second choices are unknots (via RI). Therefore, there are no knots with crossing number one. We leave the details for crossing number two to the reader, as it follows the same argument.   ∎

Definition B.28 (unlinking number)**.** The *unlinking number* is another link invariant. It is defined to be the minimum number of crossing changes (from "over" to "under" or vice versa) needed to change link L to the unlink, where the minimum is taken over all possible sets of crossing changes in all possible diagrams of L. We write this as $u(L)$.

So, like the crossing number invariant, the unlinking number is very hard to calculate.

Definition B.29 (complexity [57])**.** We define the *complexity* of a link $\kappa(L)$ to be the minimal ordered pair of integers $(c(L), u(L))$, where $c(L)$ is the crossing number of link L and $u(L)$ is the unlinking number of L. We order complexities *lexocographically*, i.e., $(a, b) < (c, d)$ if and only if $(a < c)$ or $(a = c$ and $b < d)$.

Example B.30. The most basic complexity, is that of the unknot, i.e., $\kappa\,(\bigcirc) = (0, 0)$.

For the following trefoil,
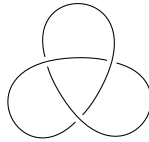
We can see that it has complexity $(3, 1)$, by changing one of its crossings as shown below,
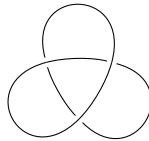
For the figure-eight knot 4_1, it is not hard to see that it has complexity $(4, 1)$ by also changing one of its crossings.

So, $\kappa(0_1) < \kappa(3_1) < \kappa(4_1)$.

Lemma B.31. *Every link L with a corresponding diagram D has an unlinking number.*

Proof sketch. We begin by choosing a starting point on all of the components on the link diagram. Following a clockwise orientation, we follow each component of the link diagram from the starting point. When a crossing appears, if it is an overpass, we change it to an underpass, otherwise we keep it as it is. Once we are done, we have an unlink. This shows us that the unlinking number $u(L) \leq c(L)$. ∎

There are many other link invariants, but in this paper we will mainly be focusing on the Jones polynomial and the Kauffman bracket.

C   The Kauffman bracket construction

We will now define the Jones polynomial. This will be based on the Kauffman bracket [33, 34, 35]. We will then show that this admits a skein relation [15, 20, 26]. We will not go deep into the theory of skein relations. The interested reader is directed to [15] and [20].

Our discussion is based on that in Kauffman's paper [33], Jones' paper [26], Jones' book [28], Conway's paper [15], and HOMFLY-PT's paper [20, 53].

Remark. We note that HOMFLY-PT is an acronym for the eight people who wrote the two papers, namely, Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter. The PT are the other two who wrote a different paper on the same subject: Pyztycki and Traczyk.

Notation [15, p. 330]. Formally, when we look at an enclosed region of a link such that there are four distinct points (NW, NE, SE, SW), then we call the enclosed region a *tangle*.
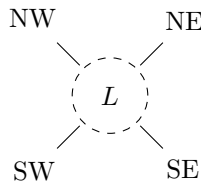

Figure C-i: tangle

We call it an *elementary 2-tangle* if the tangle consists of exactly one crossing or none. Such as the following.
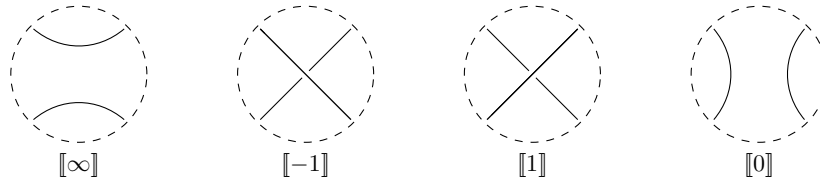

Figure C-ii: elementary 2-tangles

These elementary 2-tangles are usually recognised as the skein relation of a link (see §C.III). For more details on tangles, the interested reader is directed to Conway's paper [15].

Definition C.1 (Laurent polynomial)**.** A *Laurent polynomial* is an expression of the form $p = \sum_{k=-N}^{N} a_k A^k$ for some natural number N and some list of coefficients $a_{-N}, \ldots, a_N \in \mathbb{Z}$.

The aim of the Kauffman bracket is to define an easier, more combinatorial approach to the Jones polynomial.

C.I   'states model' and the Kauffman bracket polynomial

Our first definition of the Jones polynomial will require us to define the Kauffman bracket. We will begin by defining a state of a link diagram (as was presented in Kauffman's paper [34]).

Given a non-oriented link diagram, we notice that each crossing is divided into four regions. By convention, we label the regions of each crossing A and B, by starting from the over-strand in a counterclockwise fashion. Both possible crossings are labelled below.
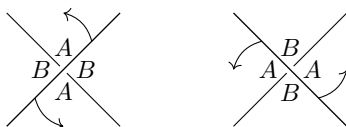
Definition C.2 (splitting marker)**.** Given a non-oriented link diagram, a *splitting marker* is defined as a marker that essentially smooths the crossing. We colour the splitting marker in red in the figure below, and show how we can get two different split diagrams for any crossing.
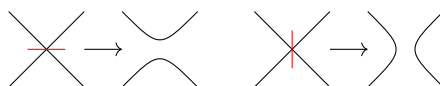
Figure C-iii: splitting marker

Definition C.3 (state)**.** A *state* of a link L is a choice of the splitting marker for each crossing in L. So, by convention, we define an A-smoothing and a B-smoothing as the following.
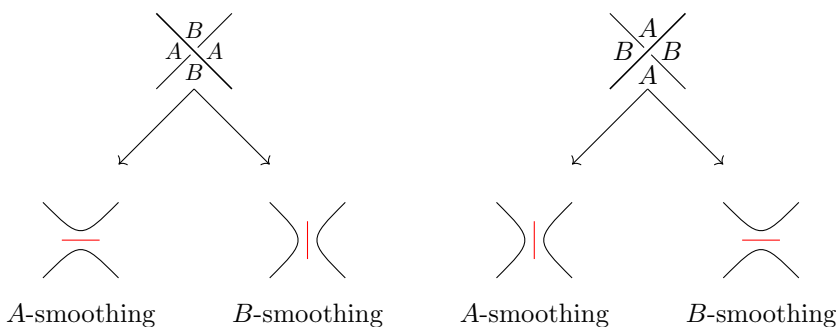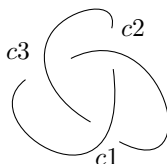
A-smoothing $\quad$ B-smoothing $\quad$ A-smoothing $\quad$ B-smoothing

Figure C-iv: states

So for an n crossing diagram L we have a state s is a bijective map $s \mapsto \{A, B\}^n$. This means we have 2^n states for an n crossing diagram.

Notation. When we smooth out a crossing on a diagram D, we call the A-smoothed diagram $D_{\{A\}}$, and we call the B-smoothed diagram $D_{\{B\}}$. The subscripted set represents the smoothings of the respected crossings on D. For example, if we have two crossings c_1, c_2 on a diagram D, then the four states would be $D_{\{A,A\}}$, $D_{\{A,B\}}$, $D_{\{B,A\}}$, and $D_{\{B,B\}}$.

Remark. Since these are tangles, we can instead define the splitting of a single crossing. We immediately get the other crossings, by rotating by $\pi/2$. For instance, if we define $[\![-1]\!] \mapsto \{[\![\infty]\!]_A, [\![0]\!]_B\}$, then by rotating all the tangles and keeping A and B in place, we get $[\![1]\!] \mapsto \{[\![0]\!]_A, [\![\infty]\!]_B\}$.

Example C.4. Consider the trefoil knot diagram D drawn below. To find all the states of D, we need to resolve all three crossings c_1, c_2, and c_3 to find all possible 2^3 states.

So, the first two possible diagrams are obtained by considering crossing c_1 and finding the A-smoothing and B-smoothing.
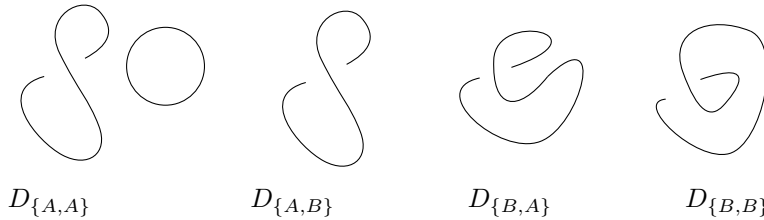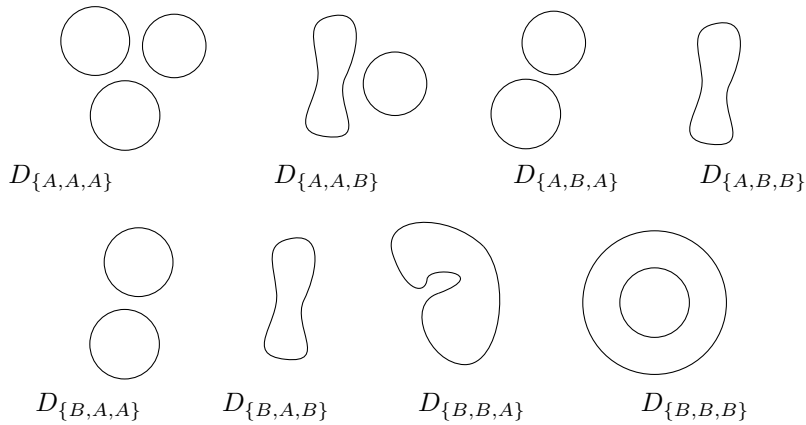
$$D_{\{A\}} \qquad\qquad D_{\{B\}}$$
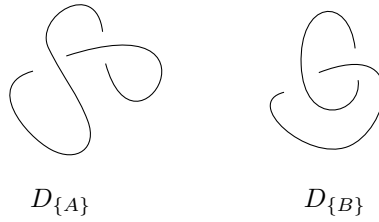
The next possible diagrams are obtained by resolving the second crossing c_2 of $D_{\{A\}}$ and $D_{\{B\}}$.

$$D_{\{A,A\}} \qquad D_{\{A,B\}} \qquad D_{\{B,A\}} \qquad D_{\{B,B\}}$$

And so the 8 states are thus obtained by resolving the last crossing c_3 of $D_{\{A,A\}}$, $D_{\{A,B\}}$, $D_{\{B,A\}}$, and $D_{\{B,B\}}$.

$$D_{\{A,A,A\}} \qquad D_{\{A,A,B\}} \qquad D_{\{A,B,A\}} \qquad D_{\{A,B,B\}}$$

$$D_{\{B,A,A\}} \qquad D_{\{B,A,B\}} \qquad D_{\{B,B,A\}} \qquad D_{\{B,B,B\}}$$

Definition C.5 (Kauffman bracket)**.** The *Kauffman bracket* is a function,

$$\langle D \rangle \colon \{\text{unoriented link diagrams D}\} \to \mathbb{Z}[A, B, d],$$

that is an invariant under RII and RIII, and is uniquely defined by the following properties,

(i) $\langle \bigcirc \rangle = 1$,

(ii) $\langle D \cup \bigcirc \rangle = d\langle D \rangle$,

(iii) $\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle$,

(iv) $\langle \times \rangle = A \langle)(\rangle + B \langle \asymp \rangle$

The bracket is well defined with states and splitting markers. After labelling the regions, when we split the diagram, we multiply the bracket of the split diagram with the region the splitting marker

is coinciding with.

$$\left\langle \vcenter{\hbox{⧓}} \right\rangle = A \left\langle \asymp \right\rangle + B \left\langle \,)(\, \right\rangle$$

Remark. In tangle notation, (iii) would be $\langle \llbracket -1 \rrbracket \rangle = A \langle \llbracket \infty \rrbracket_A \rangle + B \langle \llbracket 0 \rrbracket_B \rangle$ and by rotating, we get (iv). Thus, defining both is not necessary, but is convenient.

Proposition C.6. *The Kauffman bracket is well-defined, i.e., we have the Kauffman bracket defined by its state-sum*

$$\langle D \rangle = \sum_{\text{states } s \text{ of } D} A^{|a(s)|} B^{|b(s)|} d^{|s|-1},$$

where $a(s)$ is the number of A-smoothings in the state s, $b(s)$ is the number of B-smoothings in the state s, and $|s|$ is the number of components in the state s.

Proof. We show the uniqueness of the polynomial bracket by induction on the number of crossings in the link diagram.

For a diagram with no crossings, we clearly have $\langle \bigcirc \rangle = 1$, since the only state would be the unknot itself. Now if the link diagram D is empty, then we need $\langle D \cup \bigcirc \rangle = 1$, which is clearly satisfied.

If the link diagram D is not empty, we see that any state s of D with $\bigcirc$ is the same as the state of $D \cup \bigcirc$. We note $|a(s \cup \bigcirc)| = |a(s)|$ and $|b(s \cup \bigcirc)| = |b(s)|$, since they have the same crossings and so the smoothings will be the same (except for the extra circle which will not affect $a(s)$ or $b(s)$). So

$$\left\langle D \cup \bigcirc \right\rangle = \sum_s A^{|a(s)|} B^{|b(s)|} d^{|s|+1-1}$$

$$= \sum_s A^{|a(s)|} B^{|b(s)|} d^{|s|-1} d,$$

which is clearly $d \langle D \rangle$. Thus for an m-unlink (no crossings), we have,

$$\left\langle \bigcirc^m \right\rangle = d^{m-1}.$$

We now suppose that the Kauffman bracket is in fact unique for all link diagrams with less than $c(D)$ crossings (inductive hypothesis). Thus, we want to show that for any arbitrary link diagram D, we have $\langle D \rangle = A \langle D_A \rangle + B \langle D_B \rangle$. And by the inductive hypothesis,

$$A\langle D_A \rangle + B\langle D_B \rangle = A \sum_{s \in D_A} A^{|a(s)|} B^{|b(s)|} d^{|s|-1} + B \sum_{s \in D_B} A^{|b(S)|} B^{|b(s)|} d^{|s|-1}$$

$$= \sum_{s \in D_A} A^{|a(s)|+1} B^{|b(s)|} d^{|s|-1} + \sum_{s \in D_B} A^{|a(s)|} B^{|b(s)|+1} d^{|s|-1}$$

And since $\sum_{s \in D_A} A^{|a(s)|+1} B^{|b(s)|} d^{|s|-1}$ is the same as $\llbracket 1 \rrbracket \mapsto \{ \llbracket 0 \rrbracket_A, \llbracket \infty \rrbracket_B \}$ and, similarly, for the other sum we have it being $\llbracket -1 \rrbracket \mapsto \{ \llbracket \infty \rrbracket_A, \llbracket 0 \rrbracket_B \}$. Adding the states of $\llbracket 1 \rrbracket$ with the states of $\llbracket -1 \rrbracket$ gives us the states of D. Thus, we have

$$= \sum_{s \in D} A^{|a(s)|} B^{|b(s)|} d^{|s|-1} = \langle D \rangle$$

And we are done. ∎

Proposition C.7. *The Kauffman bracket is invariant under regular isotopy if we have $B = A^{-1}$ and $d = -A^2 - A^{-2}$.*

Proof. For RII,

$$\langle \asymp \rangle = A \left\langle \times \right\rangle + B \left\langle \right\rangle$$

$$= A \left(A \left\langle \right|\left\rangle + B \langle \frown \rangle \right) + B \left(A \left\langle \right\rangle + B \left\langle \right|\left\rangle \right)$$

$$= A^2 \left\langle \right|\left\rangle + B^2 \left\langle \right|\left\rangle + AB \langle \frown \rangle + ABd \left\langle \right|\left\rangle$$

$$= (A^2 + B^2 + ABd) \left\langle \right|\left\rangle + AB \langle \frown \rangle$$

We need $(A^2 + B^2 + ABd) \left\langle \right|\left\rangle + AB \langle \frown \rangle = \langle \frown \rangle$, so $AB = 1$ and $A^2 + B^2 + ABd = 0$. And we are done. As for RIII, we have

$$\left\langle \right\rangle = A \langle \rangle + B \left\langle \right\rangle$$

By using the fact that the bracket is invariant under RII, we find,

$$= A \langle \smile \rangle + B \left\langle \right\rangle$$

$$= A \langle \rangle + B \left\langle \right\rangle$$

$$= \left\langle \right\rangle$$

Thus, the bracket is invariant under regular isotopy. ∎

And by the uniqueness of the Kauffman bracket, we then have the following state sum, where B and d are replaced with A^{-1} and $(-A^2 - A^{-2})$, respectively, we have

$$\langle D \rangle = \sum_{\text{states } s \text{ of } D} A^{\alpha(s)} (-A^2 - A^{-2})^{|\text{components of state } s| - 1}$$

where $\alpha(s) = |a(s)| + |b(s)|$.

Example C.8. Consider the left-handed trefoil (the trefoil with the negative writhe). We will calculate its bracket polynomial using its inductive and state-sum definition. We begin with the inductive definition.

$$\left\langle \right\rangle = A \left\langle \right\rangle + A^{-1} \left\langle \right\rangle$$

$$= A \left(A \left\langle \right\rangle + A^{-1} \langle \rangle \right) + A^{-1} \left(A \langle \rangle + A^{-1} \left\langle \right\rangle \right)$$

$$= A^2 (-A^2 - A^{-2}) \langle \rangle + A \langle \bigcirc\bigcirc \rangle + A^{-1} \langle \bigcirc \rangle$$

$$+ A^{-2} \left(A \langle \bigcirc \rangle + A^{-1} \langle \bigcirc\bigcirc \rangle \right) + A \langle \bigcirc\bigcirc \rangle + A^{-1} \langle \bigcirc \rangle$$

$$= (-A^4 - 1)(A \langle \bigcirc\bigcirc \rangle + A^{-1} \langle \bigcirc \rangle) + 2A \langle \bigcirc\bigcirc \rangle + 3A^{-1}$$

$$+ A^{-3} \langle \bigcirc\bigcirc \rangle$$

$$= (-A^5 + A^{-3} + A)(-A^2 - A^{-2}) \langle \bigcirc \rangle - A^3 + 2A^{-1}$$

$$= -A^{-5} - A^3 + A^7$$

We now use the state-sum definition.

state		$\alpha(s)$	$|\text{components of state}|$
$\{A, A, A\}$		3	3
$\{A, A, B\}$		1	2
$\{A, B, A\}$		1	2
$\{A, B, B\}$		-1	1
$\{B, A, A\}$		1	2
$\{B, A, B\}$		-1	1
$\{B, B, A\}$		-1	1
$\{B, B, B\}$		-3	2

Thus, the polynomial becomes

$$\langle D \rangle = \sum_{\text{states } s \text{ of } D} A^{\alpha(s)}(-A^2 - A^{-2})^{|\text{components of state } s|-1}$$
$$= A^3 d^2 + Ad + Ad + A^{-1} + Ad + A^{-1} + A^{-1} + A^{-3}d$$
$$= d(A^3 d + 3A + A^{-3}) + 3A^{-1}$$
$$= (-A^2 - A^{-2})(-A^5 - A + 3A + A^{-3}) + 3A^{-1}$$
$$= A^7 + A^3 - 2A^3 - 2A^{-1} - A^{-1} - A^{-5} + 3A^{-1} = A^7 - A^3 - A^{-5}$$

Let us see what happens to the Kauffman bracket when we apply the Reidemeister moves.

Lemma C.9. *The Kauffman bracket is not invariant under RI.*

Proof.

$$\langle \smile\!\!\!\circ \rangle = A \langle \smile \rangle + A^{-1} \langle \overline{\circ} \rangle$$
$$= A \langle \smile \rangle + A^{-1}(-A^2 - A^{-2}) \langle \smile \rangle = -A^{-3} \langle \smile \rangle$$

and

$$\langle \smile\!\!\!\circ \rangle = A \langle \overline{\circ} \rangle + A^{-1} \langle \smile \rangle$$
$$= -A^3 \langle \smile \rangle$$

So, the Kauffman bracket is not invariant under RI.   ∎

Example C.10. We can use the facts we proved in Lemma C.9 for a faster calculation. For instance, let us see the trefoil again.

$$\left\langle \text{⟨trefoil⟩} \right\rangle = A \left\langle \text{⟨diagram⟩} \right\rangle + A^{-1} \left\langle \text{⟨diagram⟩} \right\rangle$$

we now apply two RI moves and multiply $(-A^3)^2 = A^6$. So, we would get

$$= A^7 + A^{-1} \left\langle \text{⟨diagram⟩} \right\rangle = A^7 + A^{-1} \left(A \langle \text{⟨diagram⟩} \rangle + A^{-1} \left\langle \text{⟨diagram⟩} \right\rangle \right)$$

we then apply an RI move again to both and find,

$$= A^7 + A^{-1}(A(-A^3) + A^{-1}(-A^{-3})) = A^7 - A^3 - A^{-5}$$

Hence we have simplified the calculation from Example C.8 by threefold.

C.II   Kauffman's bracket polynomial definition of the Jones polynomial

We have already seen from Lemma C.9 that the bracket is not invariant under RI. We now discuss the method of 'normalising' the bracket with respect to its writhe in order to make it invariant under ambient isotopy. Here is a precise definition.

Definition C.11 (The Jones polynomial - via Kauffman bracket)**.** The *Jones polynomial* is the function

$$V(L) \colon \{\text{oriented links L with corresponding diagrams D}\} \to \mathbb{Z}[t^{-1/2}, t^{1/2}],$$

given by $V(L) = ((-A)^{-3\omega(D)}\langle|D|\rangle)_{t^{1/2}=A^{-2}}$, where $|D|$ is the link diagram D without orientation.

Proposition C.12. *The Jones polynomial is a link invariant.*

Proof. We check what happens to the Jones polynomial under the three Reidemeister moves (ambient isotopy). We already know that the bracket polynomial is invariant under regular isotopy from Proposition C.7, and we also know from Proposition B.24 that the writhe is invariant under regular isotopy. Thus, we only need to show that the Jones polynomial is invariant under RI.

Now remember from Lemma C.9, if we apply an RI move to the diagram, then we need to multiply by a factor of $-A^{\pm 3}$ depending on its writhe.

We now consider the positive twist, i.e.,

$$\omega\left(\;\smile\!\circ\;\right) = +1 \qquad \text{and} \qquad \left\langle\;\smile\!\circ\;\right\rangle = -A^3\left\langle\;\smile\;\right\rangle$$

So,

$$\begin{aligned}
V\left(\;\smile\!\circ\;\right) &= (-A)^{-3\omega\left(\smile\!\circ\right)}\left\langle\;\smile\!\circ\;\right\rangle \\
&= (-A)^{-3(\omega(\smile)+1)}\left(-A^3\right)\left\langle\;\smile\;\right\rangle \\
&= (-A)^{-3\omega(\smile)}\left\langle\;\smile\;\right\rangle = V\left(\;\smile\;\right)
\end{aligned}$$

And similarly,

$$\begin{aligned}
V\left(\;\circ\!\smile\;\right) &= (-A)^{-3\omega\left(\circ\!\smile\right)}\left\langle\;\circ\!\smile\;\right\rangle \\
&= (-A)^{-3\omega(\smile)+3}(-A^{-3})\left\langle\;\smile\;\right\rangle \\
&= (-A)^{-3\omega(\smile)}\left\langle\;\smile\;\right\rangle = V\left(\;\smile\;\right)
\end{aligned}$$

Thus, it is also invariant under RI, making it a link invariant under ambient isotopy. ∎

Example C.13. Let us calculate the Jones polynomial for the trefoil from our examples above,

$$V(\overline{3_1}) = -A^{-3(-3)}(-A^{-5} - A^3 + A^7) = A^4 + A^{12} - A^{16} = -t^{-4} + t^{-3} + t^{-1}$$

**Proposition C.14.** *There exists two distinct knots with crossing number three.*

Proof sketch. This also follows an argument analogous to the one in Proposition B.27. We begin by drawing all possible link universes with three crossings such that

1. there are no twists (since an RI can undo it thus leaving it with a lower crossing number)

2. if there is a loop, it has to be connected to the whole of the knot, since we are trying to draw a link universe of one component.

So, we have the following link universe.
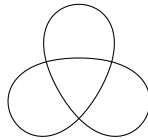
By tabulating all 2^3 knot diagrams, we find that there are only in fact two that are not the unknot.
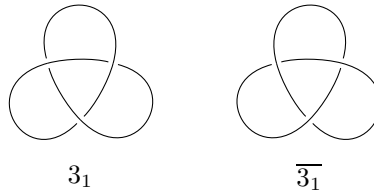
So, we only need to show that $V(3_1) \neq V(\overline{3_1})$.

We have already calculated $V(\overline{3_1})$ form Example C.13. And for the sake of not repeating onesself, we will soon see from Example H.4 that we have $V(3_1) = A^{-4} + A^{-12} - A^{-16}$. Thus, $V(3_1) \neq V(\overline{3_1})$, and so $3_1 \not\sim \overline{3_1}$. ∎

So, now we know that the trefoil and its reflection are not equivalent knots. And so we have the following result.

Proposition C.15. *The Jones polynomial of a mirror link $\overline{L}$ of link L has the polynomial $V(\overline{L}) = (V(L))_{t=t^{-1}}$.*

Proof. We know that the writhe of the reflection of the link L is the negation of the writhe of L. As for the Kauffman bracket, it is not hard to see that $A = B$ when looking at the reflection, since, $\langle \rangle = A \langle \rangle \langle \rangle + B \langle \rangle$, whereas $\langle \rangle = A \langle \rangle + B \langle \rangle \langle \rangle$. Thus, $\langle \overline{D} \rangle = (\langle D \rangle)_{A=A^{-1}}$. And so, $V(\overline{D}) = (-A^{3\omega(D)}(\langle D \rangle))_{A=A^{-1}} = V(D)_{A=A^{-1}} = V(D)_{t=t^{-1}}$ ∎

C.III The skein relation definition of the Jones polynomial

In this section, we will begin by exploring the Jones polynomial defined by its skein relation and we will also see the Kauffman bracket polynomial and how its normalisation (with respect to the writhe) is equivalent to the Jones polynomial.

The *skein relation* was shown by John Conway in the 1960's [15]. He defined the Alexander polynomial via the skein relation (now sometimes known as the Alexander-Conway polynomial). So what is a skein relation?

Definition C.16 (skein relation). The *skein relation* is defined as the relation between three link diagrams which we call L_+, L_- and L_0. These diagrams are the same, except in the neighbourhood of a single crossing. See the below figure.
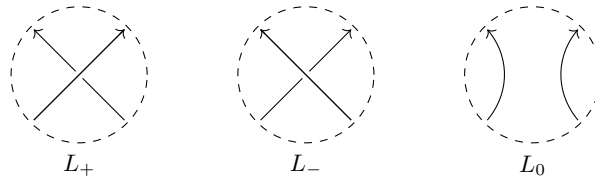

Figure C-v: skein relation

In other words, they are the three elementary 2-tangles $[\![1]\!]$, $[\![-1]\!]$ and $[\![0]\!]$.

To define an invariant function that admits a skein relation, we have to notice that the skein relation is related to the unknotting number, in that when we have a skein relation, the complexity of the other two link diagrams are strictly less than that of the original link diagram.

We now define an invariant function P that admits a skein relation, and is unique. We will not go too deep into this theory. The interested reader is directed to [20] and [41].

Definition C.17 (skein relation invariant polynomial)**.** It is said that a link invariant $P\colon \{\text{Oriented links in } \mathbb{R}^3\} \to A$, where A is a commutative ring, admits a *skein relation invariant* if

(i) $P(0_1) = P(\bigcirc) = 1$ and

(ii) there exists three invertible elements a_+, a_- and a_0 in A, such that whenever we have three links L_+, L_- and L_0 that are skein related, then $a_+ P(L_+) + a_- P(L_-) + a_0 P(L_0) = 0$.

The following theorem says that if the skein relation invariant polynomial is well defined (i.e., it is a link invariant), then it is uniquely determined by a_+, a_- and a_0.

Theorem C.18. *If the skein relation invariant polynomial P defined above is in fact an invariant, then it is uniquely determined by a_+, a_- and a_0.*

The proof of the uniqueness of the skein relation invariant is closely analogous to the uniqueness of the Jones polynomial via skein theory, Proposition C.20, and so is left as an exercise for the reader. The existence proof of this polynomial can be found in [41].
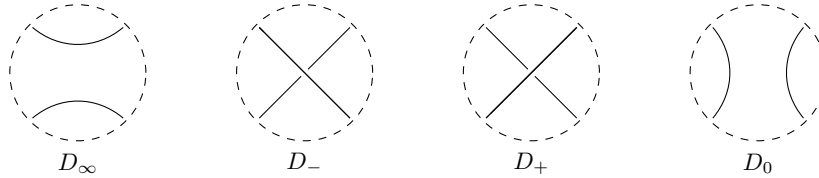
Proposition C.19 (skein relation invariant is homogeneous [57, p. 72])**.** *The skein relation invariant is homogeneous in a_+, a_-, and a_0, i.e., all of the terms have a total degree of zero of the variables.*

Proposition C.20 (The Jones polynomial - skein theory)**.** *The Jones polynomial invariant $V(L)$ admits a skein relation such that it is a function $V\colon \{\text{Oriented links in } \mathbb{R}^3\} \to \mathbb{Z}[t^{-1/2}, t^{1/2}]$ such that $V(0_1) = V(\bigcirc) = 1$, and whenever three oriented links L_+, L_-, L_0 are skein related, then*

$$t^{-1} V(L_+) - t V(L_-) + (t^{-1/2} - t^{1/2}) V(L_0) = 0.$$

And it is uniquely determined by those two conditions.

Proof. We first show that the Jones polynomial admits a skein relation, then we will show that it can be uniquely determined by the skein relation conditions. The first property $V(\bigcirc) = 1$ is trivial. For the second property we need to remove D_∞ terms found in the Kauffman bracket. So consider the diagrams below.

$$D_\infty \qquad\qquad D_- \qquad\qquad D_+ \qquad\qquad D_0$$

So we have $\langle D_- \rangle = \langle \times \rangle = A \langle D_\infty \rangle + A^{-1} \langle D_0 \rangle$ and $\langle D_+ \rangle = \langle \times \rangle = A \langle D_0 \rangle + A^{-1} \langle D_\infty \rangle$. And so noting the substitution $t^{1/2} = A^{-2}$, we have

$$A^4 (-A)^{-3\omega(D_+)} \langle D_+ \rangle - A^{-4}(-A)^{-3\omega(D_-)} \langle D_- \rangle + (A^2 - A^{-2})(-A)^{-3\omega(D_0)} \langle D_0 \rangle$$

Now $\omega(D_0) + 1 = \omega(D_+)$ and $\omega(D_-) = \omega(D_0) - 1$. So the above formula becomes

$$(-A)^{-3\omega(D_0)}(-A \langle D_+ \rangle + A^{-1} \langle D_- \rangle + (A^2 - A^{-2})\langle D_0 \rangle)$$
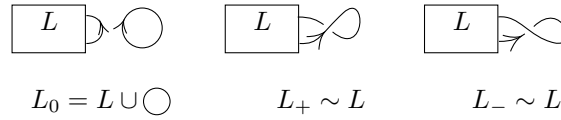
Now we substitute the relations we wrote above $\langle D_- \rangle$ and $\langle D_+ \rangle$, so

$$-A(A \langle D_0 \rangle + A^{-1} \langle D_\infty \rangle) + A^{-1}(A \langle D_\infty \rangle + A^{-1} \langle D_0 \rangle) + (A^2 - A^{-2}) \langle D_0 \rangle = 0.$$

So we are done.

We now show the uniqueness by induction on the number of crossings n on an arbitrary link L.

We note that we already know $V(\bigcirc) = 1$. And by considering any arbitrary link L, we will now see that for an m component unlink $V(\bigcirc^m) = (t^{1/2} + t^{-1/2})^{m-1}$.

$$L_0 = L \cup \bigcirc \qquad\qquad L_+ \sim L \qquad\qquad L_- \sim L$$

By the skein relation definition, we have $t^{-1}V(L) - tV(L) + (t^{-1/2} - t^{1/2})V(L \cup \bigcirc) = 0$, and thus

$$V\left(L \cup \bigcirc\right) = \frac{t - t^{-1}}{t^{-1/2} - t^{1/2}}V(L) = -(t^{1/2} + t^{-1/2})V(L).$$

So by induction on m, we find $V(L \cup \bigcirc^m) = (-t^{1/2} - t^{-1/2})^m V(L)$. Now we know that the link L_0 will have one less crossing than L_+ and L_-. So, without loss of generality, if our oriented link L is L_+, then we have complexity $(c(L_+), u(L_+))$. Since $c(L_0) < c(L_+)$, we have $\kappa(L_0) < \kappa(L_+)$. Now $c(L_-) = c(L_+)$, but $u(L_-) = c(L_+) - 1$. So we have the complexities of the two links L_- and L_0, strictly less than the complexity of L_+. Now, by induction on the complexity, we can see that the Jones polynomial is in fact unique.   ∎

Remark. It can be seen that by substituting a_+ for t^{-1}, a_- for $-t$, and a_0 for $t^{-1/2} - t^{1/2}$, the skein relation invariant P becomes the Jones polynomial V.

We have shown in the above proof what the Jones polynomial of the disjoint union of a link and the m-unlink is, so we shall write it as the following corollary.

Corollary C.21. *The disjoint union of a link L and the m-unlink is given by $V\left(L \cup \bigcirc^m\right) = (-t^{1/2} - t^{-1/2})^m V(L)$.*

We then have the following theorem, which we will prove at the end of the essay using nicer results with braid representations.

Theorem C.22. *The two disjoint links L_1 and L_2 has the following Jones polynomial,*

$$V(L_1 \cup L_2) = (-t^{1/2} - t^{-1/2})V(L_1)V(L_2)$$

Example C.23. Let us calculate the Jones polynomial of the figure eight knot 4_1 drawn below via skein theory. We let the knot 4_1 be the knot L_+ where we have circled our chosen neighbourhood.

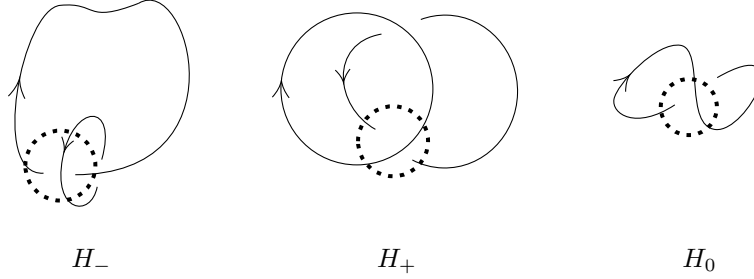$$L_+ \qquad\qquad\qquad L_- \qquad\qquad\qquad L_0$$
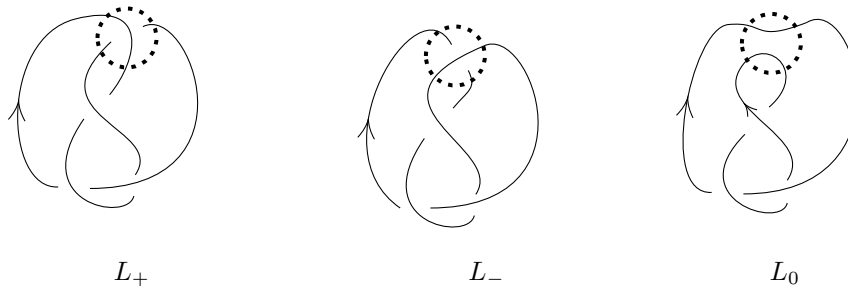
Note that the knot L_- is Reidemeiser equivalent to the unknot 0_1, so $V(L_-) = 1$. Whereas L_0

is Reidemeister equivalent to the Hopf link with the given orientations. So, by the definition of the Jones polynomial, we have

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0$$
$$t^{-1}V(4_1) - t + (t^{-1/2} - t^{1/2})V(L_0) = 0 \qquad (*)$$
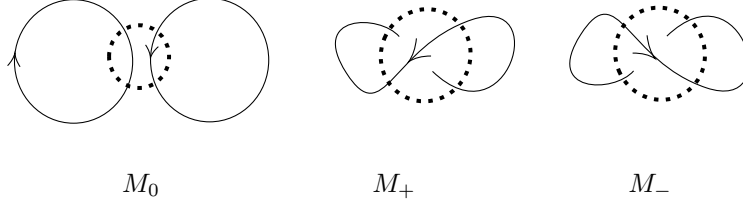
Now, we need to find what $V(L_0)$ is. So, we let L_0 be the Hopf link H_- (since they are Reidemeister equivalent).

$$H_- \qquad\qquad\qquad H_+ \qquad\qquad\qquad H_0$$

We note that the knot $L_0 \sim_R 0_1$. So, from the definition of the Jones polynomial we get,

$$t^{-1}V(H_+) - tV(H_-) + (t^{-1/2} - t^{1/2})V(H_0) = 0 \qquad (**)$$

and so we need to calculate the Jones polynomial of the unlink: $V(L_+)$. We let L_+ be M_0 since they are R-equivalent.

$$M_0 \qquad\qquad\qquad M_+ \qquad\qquad\qquad M_-$$

Note that the knots M_+ and M_- are R-equivalent to 0_1, so we have $V(M_+) = V(M_-) = 1$. So then using the skein theory definition of the Jones polynomial, we have $t^{-1} - t + (t^{-1/2} - t^{1/2})V(M_0) = 0$. And so

$$V(M_0) = \frac{t - t^{-1}}{t^{-1/2} - t^{1/2}}$$
$$= \frac{(t^{-1/2} - t^{1/2})(-t^{1/2} - t^{-1/2})}{(t^{-1/2} - t^{1/2})} = -t^{1/2} - t^{-1/2}$$

Now noting that we have $H_+ \sim_R M_0$, we consider Equation (**),

$$t^{-1}(-t^{1/2} - t^{-1/2}) - tV(H_-) + t^{-1/2} - t^{1/2} = 0$$
$$V(H_-) = t^{-1}(-t^{-3/2} - t^{1/2})$$
$$= -t^{-5/2} - t^{-1/2}$$

Now we can calculate the Jones polynomial of our figure eight knot. We consider Equation (*),

$$V(4_1) = t(t - (t^{-1/2} - t^{1/2})V(H_-))$$

$$= t^2 - (t^{1/2} - t3/2)(-t^{-5/2} - t^{-1/2})$$
$$= t^2 + t^{-2} + 1 - t^{-1} - t$$

∎

Remark. The Alexander polynomial [6] also admits a skein relation [15]. From the skein relation invariant polynomial, we can get the Alexander polynomial by substituting $a_+ = 1$, $a_- = -1$ and $a_0 = t^{-1/2} - t^{1/2}$ [20]. That is,

$$\nabla(L_+) - \nabla(L_-) + (t^{-1/2} - t^{1/2})\nabla(L_0) = 0.$$

While the skein relation invariant polynomial determines both Alexander's and Jones' polynomial, there are no other relations between them, for instance

– $V(4_1) = V(11_{388})$, but $\nabla(4_1) \neq \nabla(11_{388})$.

– $V(11_{388}) = V(\overline{11_{388}})$ and $\nabla(11_{388}) = \nabla(\overline{11_{388}})$, but $P(11_{388}) \neq P(\overline{11_{388}})$.

– $\nabla(11_{471}) = 1 = \nabla(0_1)$, but $V(11_{471}) \neq 1$.

On another note, we notice that for any knot K, $V(K)_{t=-1} = \nabla(K)_{t=-1}$. This is the determinant of K.

Remark. The Jones polynomial is as strong as the 3-colouring invariant polynomial is weak.

The following conjecture is still open. Khovanov homology [39], however, does detect the unknot.

Conjecture C.24 (Jones [26])**.** *Does there exist a non-trivial knot K such that $V(K) = 1$?*

From the author's exposition on knot theory [55], it was shown that the Alexander polynomial can be defined using a method known as "colouring", and thus we have the following conjecture.

Conjecture C.25. *Can we define the Jones polynomial via colouring?*

While these two definitions were amongst the last to be created with respect to its history, they are amongst the easiest since it is more of a combinatorial calculation, requiring little to no knowledge of more results.

The next chapters will introduce some results and notation we need, to define the original definition of the Jones polynomial. It is in the author's opinion that the original definition of the Jones polynomial (presented in the final chapter), is by far the easiest in terms of both comprehension and calculation.

D The braid group $\mathcal{B}_n$

Thus far, we only know of topological methods for studying links, which motivates the next two chapters where we will learn algebraic methods for studying links. This is because braids can be defined both algebraically and geometrically. So, the aim of the next two chapters will be to discuss the connection between braids and links (Alexander's and Markov's theorem), in hopes of constructing a link invariant. In fact, the original construction of the Jones polynomial [26] (see Chapter H) utilises these algebraic methods.

We begin with introducing the notion of braids, which happen to be closely related to links with the exception that it cannot be tangled with itself (and is not closed). Braid equivalence, similarly to link equivalence, is defined with ambient isotopy. Thus, allowing us to define invariant functions on braids.

The aim of this chapter is to form the braid group from a geometrical perspective onto an algebraic presentation. This is done by defining a multiplication (braid concatenation) on the set of braids $\mathcal{B}_n$, thus giving us the group structure we need. By defining a homomorphism from the braid group onto the symmetric group, we find that the kernel is an interesting group called the pure braid group.

This chapter is based on the presentations given in [3], [4], [10], [12], [13], [16], [17], [18], [27], [28], [29], [32], [37], [38], [43], [44], [49], [58], and [65].

D.I   Braids

Informally, braids consist of a top and a bottom bar, with strings tied to each bar, such that the strings always go downwards. There exists a permutation from the n points on the top bar to the n points on the bottom bar. Recall that a *permutation* of a set S is a bijective map $f \colon S \to S$. Here is a precise definition.

Definition D.1 (braid)**.** An *n-braid* is a set of n injective (non-intersecting) smooth paths (called strands) in $\mathbb{R}^2 \times [0, 1]$ connecting the points on the top bar with the points on the bottom bar (i.e., there exists a permutation from the n points on the top bar to the n points on the bottom bar). Such that the projection of each of the paths onto the $\mathbb{R}^2$ plane (shadow - over/under strands) is a diffeomorphism (i.e., no self-tangles: the strings never have a horizontal tangent vector). An example of a 3-braid can be seen below in Figure D-i.
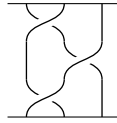
Figure D-i: Example of a 3-braid

Remark. Note that the end points of the braids are fixed. And a strand is always going down. So equivalency is essentially ambient isotopy while fixing the end points.

Definition D.2 (braid equivalence)**.** Similarly to links, two braids are *equivalent* if they are *ambient isotopic*.

Similarly to links and knots, we call the equivalence class of braids, *braid type*.

Proposition D.3. *Braid equivalence in the above definition is an equivalence relation.*

Proof. It is obviously reflexive. It is symmetric since if braid α can be transformed into braid β, then by looking over the 'moves' of the transformation, we can just go backwards. Finally, it is transitive since if braid α can be transformed into braid β and then to braid γ, by looking at the 'moves' of the two transformations, we can concatenate them and form one so that we can get a transformation from α to γ. ∎

We can, just as with links, perform elementary moves, the Δ-moves (Definition B.12), on braids to show that two braids are equivalent. So by fixing the end points of the braids, we can move the strings however we would like, ensuring that we still satisfy the requirements of a braid and without breaking the strands.

Example D.4. The RII move on a braid via the triangle move:
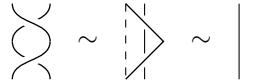
It is easy to see the RIII move on a braid via the triangle move, and is left as an exercise for the reader.

Remark. The bars on the braids are usually drawn to emphasise that the diagram represents a braid. So this is not always necessary.

Now that we know we can perform the RII and RIII moves on a braid, it is typical to wonder if we can perform an RI move. Obviously we cannot since by definition, the strand cannot self-tangle. However, in the next chapter, we will see that when relating braids to knots (via closure), there exists a fascinating move that mimics the RI move when closing the braid.

Definition D.5 (braid permutation). Suppose we have an n-braid α, with A_1 connected to $A'_{1'}$, A_2 connected to $A'_{2'}$, ..., A_n connected to $A'_{n'}$. Then, we can assign to α a permutation,

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ 1' & 2' & \cdots & n' \end{pmatrix}.$$

This is called the *braid permutation*.

The *trivial n-braid*, which we get by connecting A_1 to A'_1, A_2 to A'_2, ..., A_n to A'_n corresponds to the identity permutation, $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$.
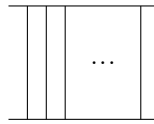
Figure D-ii: the trivial n-braid

The braid permutation for Figure D-i is

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 3).$$

Proposition D.6. *The braid permutation is invariant under ambient isotopy on braids.*

Proof. If two braids are equivalent, their braid permutations are equal (this is clear since by definition, in order to transform one braid into another, we have to keep the end points fixed). Thus, this is a braid invariant. ∎

D.II   Representations of groups

This section will give a brief on representations of groups.

Definition D.7 (representation). A *representation* of a finite group G on an n-dimensional $\mathbb{F}$-vector space V is a group homomorphism $\phi\colon G \to \mathrm{GL}_n(V)$.

Given a group G that acts on set $X = \{x_1, x_2, \ldots, x_n\}$. We call the vector space $\mathbb{F}(X)$ a *permutation representation* and consists of linear combinations.

So, recall that the group algebra $\mathbb{C}[S_n]$ is generated by $a_i = (i \ i+1)$ for $i \leq n-1$. And has the following relations.

(i) $a_i^2 = 1$

(ii) $a_i a_j = a_j a_i$ if $|i - j| > 1$

(iii) $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$

This is called the *presentation* of $\mathbb{C}[S_n]$.

D.III   The braid group

In this section we will give the set of braids a group structure by defining a multiplication. This is useful because we can use algebraic instead of geometrical structures.

We let $\mathcal{B}_n$ be the set of the equivalence classes of all n-braids.

Definition D.8 (braid multiplication). *Braid multiplication* of n-braids is defined by concatenation of the braids by putting one braid on top of the other, i.e., $\mathcal{B}_n \times \mathcal{B}_n \to \mathcal{B}_n$. By convention, $\alpha\beta$ means α on top of β.
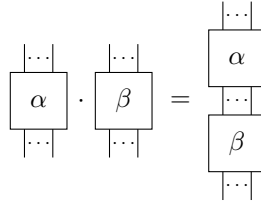
It is typical to ask if (how) we can multiply an n-braid with an m-braid, for $n \neq m$. In fact we will need to do this in the next chapter when relating a braid to a knot. It turns out that there exists a natural inclusion from $\mathcal{B}_n$ to $\mathcal{B}_{n+1}$ by adding a vertical string to the right of the braid. So, we have the inclusions,

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$$

This is quite similar to the Temperley-Lieb algebra planar diagram relations in Chapter G. For the sake of completeness and less ambiguity, we shall define precise definitions we can use.

Definition D.9 (braid inclusion). The *braid inclusion map* $\mathrm{incl}_n \colon \mathcal{B}_n \to \mathcal{B}_{n+1}$ is defined by,
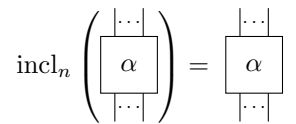
Figure D-iii: braid inclusion

So, we can multiply an n-braid with an m-braid for $n < m$, means that we have to use the inclusion map on the n-braid recursively until $n = m$. We have the following example.

Example D.10. Let $\alpha \in \mathcal{B}_n$ and $\varrho \in \mathcal{B}_{n+3}$, so to multiply them we need to perform the braid inclusion map on α thrice, i.e.,

$$\mathrm{incl}_n(\alpha) \in \mathcal{B}_{n+1} \Rightarrow \mathrm{incl}_{n+1}(\mathrm{incl}_n(\alpha)) \in \mathcal{B}_{n+2} \Rightarrow \mathrm{incl}_{n+2}(\mathrm{incl}_{n+1}(\mathrm{incl}_n(\alpha))) \in \mathcal{B}_{n+3}$$

visually we have,

$$\boxed{\alpha} \cdot \boxed{\varrho} = \mathrm{incl}_{n+2} \circ \mathrm{incl}_{n+1} \circ \mathrm{incl}_n \left(\boxed{\alpha} \right) \cdot \boxed{\varrho}$$

$$= \boxed{\alpha} \cdot \boxed{\varrho} = \boxed{\alpha} \\ \boxed{\varrho}$$
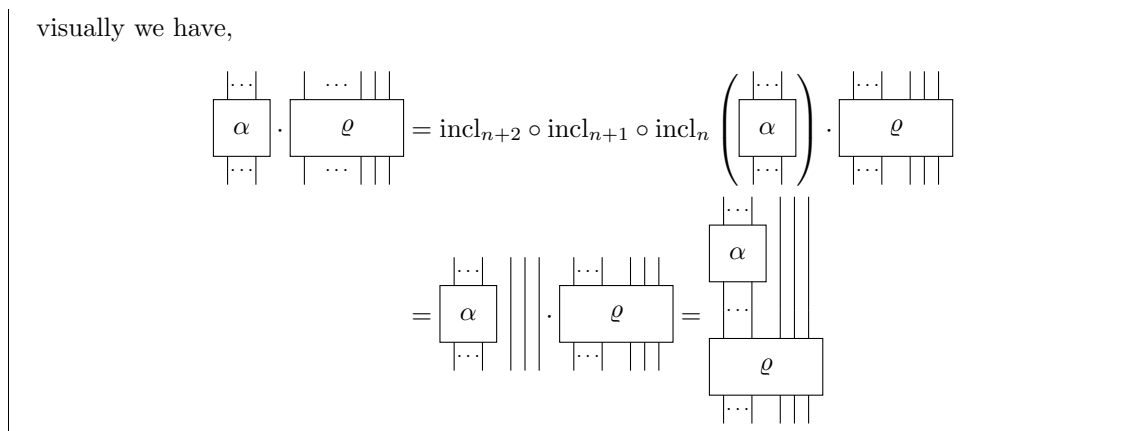
In general, the n-braid $\alpha\beta$ is not equivalent to $\beta\alpha$. However, braid multiplication is associative, i.e., $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ (see proof of Theorem D.12).
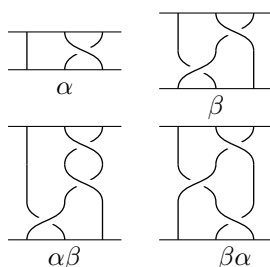

Figure D-iv: ([49, Exercise 10.2.1, p. 202]) braid multiplication is not commutative, i.e. $\alpha\beta$ is not equivalent to $\beta\alpha$ (their braid permutations are different, hence they are not equivalent)

The braid identity Ξ is the trivial braid such that $\Xi\alpha = \alpha = \alpha\Xi$.

Figure D-v: trivial braid

Finally, the inverse of a braid is the reflection of a braid, such that $\alpha^{-1} \cdot \alpha = \Xi = \alpha \cdot \alpha^{-1}$.

$$\left(\boxed{\alpha} \right)^{-1} = \boxed{\wp}$$
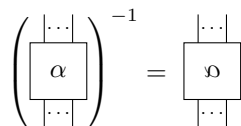
Figure D-vi: braid inverse

and so,

$$\alpha \cdot \alpha^{-1} = \boxed{\alpha} \cdots \boxed{\mho} = \| \cdots \| = \Xi$$

Example D.11. The following shows an example of multiplying a braid by its inverse.
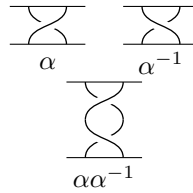
Figure D-vii: braid inverse example

The braid $\alpha\alpha^{-1}$ is clearly equivalent to the trivial braid after an RII move.

We are done. Let us summarize the braid group with the following statement.

Theorem D.12 (the geometrical braid group)**.** *The set of all n-braids $\mathcal{B}_n$ forms a group under braid composition (concatenation) defined below. Given two n-braids α and β,*

$$\boxed{\alpha} \;\; \odot \;\; \boxed{\beta} \;\; = \;\; \boxed{\genfrac{}{}{0pt}{}{\alpha}{\beta}} \;\; = \;\; \boxed{\alpha\beta}$$

Proof. Let α, β, γ be n-braids. Composing two n-braids, by definition, will give a new n-braid. So we have closure: $\odot \colon \mathcal{B}_n \odot \mathcal{B}_n \to \mathcal{B}_n$. We also have associativity, i.e. $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ (see figure below).

$$\boxed{\genfrac{}{}{0pt}{}{\alpha}{\beta\gamma}} \;\; = \;\; \boxed{\genfrac{}{}{0pt}{}{\alpha}{\genfrac{}{}{0pt}{}{\beta}{\gamma}}} \;\; = \;\; \boxed{\genfrac{}{}{0pt}{}{\alpha\beta}{\gamma}}$$

We have an identity Ξ which is the trivial n-braid. So, $\alpha\Xi = \alpha = \Xi\alpha$. The figure below shows $\alpha\Xi = \alpha$ (from this, it then follows that $\Xi\alpha = \alpha$).
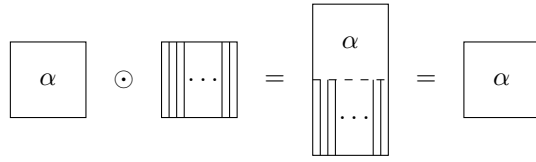
Finally, we have inverses. By definition, there exists an n-braid, called α^{-1} for the sake of argument, where the composition of this with another n-braid, say α, would produce a braid which is equivalent to the trivial braid by elementary triangle moves. So, $\alpha\alpha^{-1} = \Xi = \alpha^{-1}\alpha$. The inverse braid is the mirror image of the braid. ∎

So, similarly to how there exists a natural inclusion of the set $\mathcal{B}_n$, the braid group $\mathcal{B}_n$ is embedded as a subgroup of $\mathcal{B}_{n+1}$ (with the braid inclusion map).

There exists a representation of the group by writing all the braids as words on some generators. The generators σ_i is defined by connecting A'_i to A_{i+1} and A'_{i+1} to A_i and is seen in Figure D-viii.
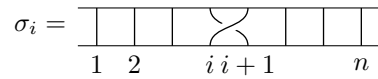
Figure D-viii: braid generator σ_i

And the inverse, σ_i^{-1} is just
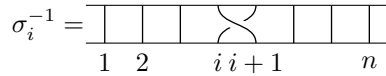
These generators are also known as *elementary transpositions*.

We can use these generators to express any element in the braid group.
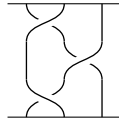
Figure D-ix: $\sigma_1\sigma_2\sigma_1^{-1}$

We can also express any element in the braid group using these generators. We can do this by dividing our braid diagram with lines parallel to the bars. The rule is that we can only have one crossing in each divided rectangle, if there are more crossings, then we can just shift one crossing downwards or upwards. See below figure.
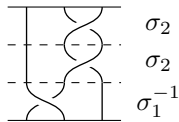
Figure D-x: $\sigma_2\sigma_2\sigma_1^{-1}$

Thus the braid group $\mathcal{B}_n$ can be described by those generators and three relations. It is usually referred to the Artin braid group, since Artin introduced it [8, 9, 12].

Definition D.13 (Artin's braid group $\mathcal{B}_n$ [8, 9])**.** The *Artin braid group* $\mathcal{B}_n$ is the group generated by σ_i for $i = 1, \ldots, n-1$,

$$\sigma_i = \quad \text{(diagram)}$$

and has the following relations

(i) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| \geq 2$

(ii) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$

(iii) $\sigma_i \sigma_i^{-1} = \Xi$, for $i = 1, \ldots n-1$

Notice that the first relation just means that we can shift the crossings if they are "separated."
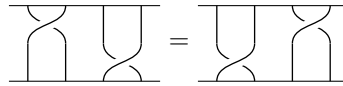
Figure D-xi: $\sigma_i \sigma_j = \sigma_j \sigma_i$, if $|i - j| \geq 2$

The second relation, here, is essentially the Redemeister 3 move.
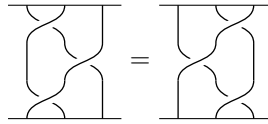
Figure D-xii: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

The third relation is just the Redeimeister 2 move.
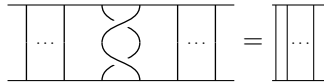
Figure D-xiii: $\sigma_i \sigma_i^{-1} = \Xi$

Example D.14. $\mathcal{B}_1$ is generated by σ_1
$\mathcal{B}_2$ is generated by σ_1, σ_2 and has the relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$
$\mathcal{B}_3$ is generated by $\sigma_1, \sigma_2, \sigma_3$ and has (in addition to the relations in $\mathcal{B}_2$) $\sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3$ and $\sigma_1 \sigma_3 = \sigma_3 \sigma_1$.

Remark. The braid group $\mathcal{B}_2$ is isomorphic to $\mathbb{Z}$.

We have already seen the braid permutation from Definition D.5. So it is not hard to see that there exists a natural homomorphism from the Artin braid group $\mathcal{B}_n$ into the symmetric group S_n.

Corollary D.15. *There is a group homomorphism φ, from $\mathcal{B}_n$ onto the symmetric group S_n. We have the exact sequence*

$$1 \to P\mathcal{B}_n \to \mathcal{B}_n \to S_n \to 1$$

where the kernel of the homomorphism is known as the pure braid group $P\mathcal{B}_n$.

So the end points of elements in the pure braid group are not permuted. We will give one example. The interested reader is directed to [12].

Example D.16. An example of an element in the pure braid group $P\mathcal{B}_n$ is shown below,

$$\sigma_2\sigma_1\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$$

The generators of the pure braid group are given by

$$\mathrm{pb}_{i,j} = (\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})\sigma_i^2(\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}),$$

for $i < j$. And so, visually, we have

$$\mathrm{pb}_{i,j} = $$
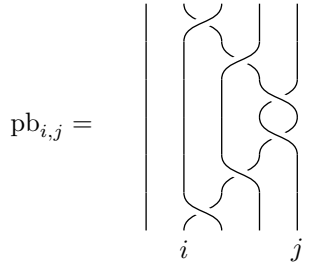
Figure D-xiv: pure braid generator

Now we have an easier, purely algebraic, method of showing that two braids are equivalent.

Proposition D.17 ([49, Exercise 10.2.4, p. 207])**.** *The following relations hold for any $\mathcal{B}_n$ for $n \geq 3$.*

$$\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2^{-1}\sigma_1\sigma_2 \ \text{and} \ \sigma_2\sigma_1^{-1}\sigma_2^{-1} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1$$

Proof.

$$\begin{aligned}
\sigma_1\sigma_2\sigma_1^{-1} &= \sigma_2^{-1}\sigma_2\sigma_1\sigma_2\sigma_1^{-1} \\
&= \sigma_2^{-1}\sigma_1\sigma_2\sigma_1\sigma_1^{-1} = \sigma_2^{-1}\sigma_1\sigma_2
\end{aligned}$$

$$\begin{aligned}
\sigma_2\sigma_1^{-1}\sigma_2^{-1} &= \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1 \\
&= \sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1 = \sigma_1^{-1}\sigma_2^{-1}\sigma_1
\end{aligned}$$

These will look like

$$\sigma_1\sigma_2\sigma_1^{-1} \qquad \sigma_2^{-1}\sigma_1\sigma_2$$

And

$$\sigma_2\sigma_1^{-1}\sigma_2^{-1} \qquad \sigma_1^{-1}\sigma_2^{-1}\sigma_1$$

Which are clearly just the Reidemeister type 3 move.   ∎

To motivate our next example, we will very briefly discuss the conjugacy problem. Our discussion is based on that in Birman's survey [12, Ch. 5, p. 60]. The *conjugacy problem* asks for an algorithm to decide whether $\{a\} = \{a'\}$, where $\{a\}, \{a'\}$ are the conjugacy classes of the elements $a, a' \in \mathcal{B}_n$. A solution was first given in 1965 by Garside [22]. While, there does exist a finite set of conjugates of a representation of an arbitrary element $a \in \mathcal{B}_n$, it is exponential in both n and $|a|$. This makes it difficult for us to understand the combinatorial solution sometimes.

Example D.18. The following two braids are known to be equivalent. So we will show that these two braids are equivalent by using the braid group relations and generators. In other words, we are finding a solution that gives us an algorithm to go from the first braid to the second braid (the conjugacy problem). The reader is advised to attempt it.

$$\sigma_1^{-1}\sigma_2^{-1}\sigma_3^2\sigma_2\sigma_1\sigma_3^{-1}\sigma_1\sigma_2\sigma_3^{-1} \qquad\qquad \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3^2\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_3^{-1}\sigma_2\sigma_3^{-1}$$
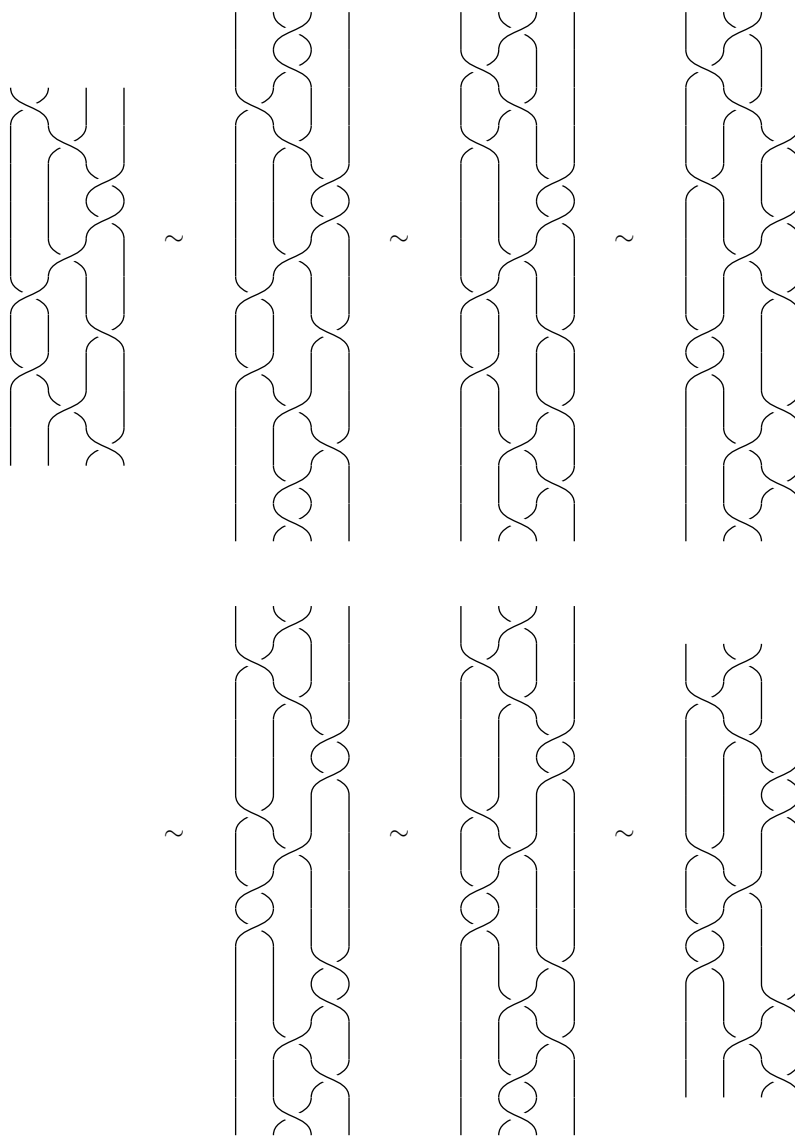
Proof.

$$
\begin{aligned}
\sigma_1^{-1}\sigma_2^{-1}\sigma_3^2\sigma_2\sigma_1\sigma_3^{-1}\sigma_1\sigma_2\sigma_3^{-1} &= \sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^2\sigma_2\sigma_1\sigma_3^{-1}\sigma_1\sigma_2\sigma_3^{-1}\sigma_2\sigma_2^{-1} \\
&= \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^2\sigma_2\sigma_1\sigma_3^{-1}\sigma_1\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1} \\
&= \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_3\sigma_2\sigma_3^{-1}\sigma_1^2\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1}
\end{aligned}
$$

$$= \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3^2\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_3^{-2}\sigma_2\sigma_3^{-1}\sigma_2^{-1}$$
$$= \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3^2\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2\sigma_2^{-1}$$
$$= \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3^2\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_3^{-1}\sigma_2\sigma_3^{-1}$$

∎

We will show each relation diagrammatically.

E   The link between links & braids

We will now see how to associate links with braids and vice versa.

This chapter is based on the presentations given in [2], [3], [4], [10], [12], [13], [16], [17], [18], [21], [26], [27], [28], [29], [33], [34], [38], [44], [49], [58], [59], and [65].

However, mainly our discussions will be based on that in Birman's book [10] and Birman's survey [12].

> **Definition E.1** (braid closure)**.** Consider any n-braid α. We define the *closure* of the braid α by connecting the parallel arcs on the right from the top bar to the bottom bar by starting with the far right, and we denote it as $\widehat{\alpha}$.
>
>
>
> Figure E-i: braid closure
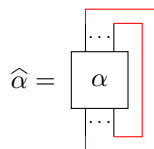
> **Example E.2.** The closure of the braid $\sigma_1\sigma_2\sigma_1$ is shown below and is Reidemeister equivalent to a link (it should be fairly easy to recognise which link).
>
>
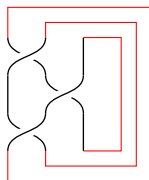
The braid closure is a regular diagram of a link. We can assign an orientation by going from the top to the bottom each time we add an arc, i.e.,
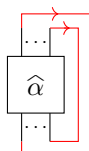
Thinking of the braid closure as a map from braid types to link types. It is typical to question whether this map is injective or surjective. We can easily show that it is in fact not injective (for instance, $\widehat{\sigma_1\sigma_2\sigma_1^{-1}} = \widehat{\sigma_2}$, but $\sigma_1\sigma_2\sigma_1^{-1} \not\sim \sigma_2$). Fascinatingly enough, it is, however, surjective. This was first shown in 1923 by Alexander [5].

Theorem E.3 (Alexander's theorem [5])**.** *Every link is ambient isotopic to a braid closure.*

Before we begin to sketch the proof. We will need to define Seifert circles.

> **Definition E.4** (Seifert circles)**.** Let D be an oriented diagram for the oriented link L and let D' be modified as shown in Figure E-ii.
>
>
>
> Figure E-ii: Seifert's algorithm
>
> Diagram D' is the same as D except in a small neighbourhood of each crossing where the crossing has been removed in the only way compatible with the orientation. This D' is just a disjoint

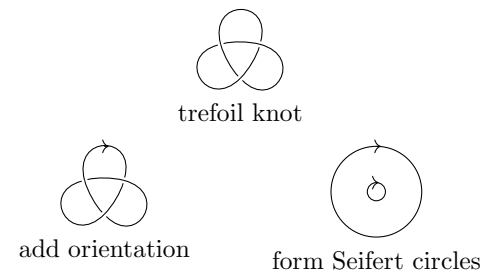union of oriented simple closed curves in $\mathbb{R}^2$, called *Seifert circles*.

Example E.5.

Figure E-iii: trefoil knot Seifert circles

Definition E.6 (braided around)**.** We say that a link is *braided around* a point if the Seifert circles of the link are nested and coherent (i.e., they all orient clockwise around the point).

We have the following equivalent definition for a link being braided around a point.

Proposition E.7. *A link is braided around a point if each edge of the link is visible from the point and is oriented clockwise.*

Remark. It is in the author's opinion that checking Seifert circles is easier at times. It also motivates Vogel's proof of Alexander's theorem in 1990 [62] which was based on Yamada's proof in 1987 [66] using Seifert circles and "infinity moves". Vogel's proof turns out to terminate quicker than the original proof given by Alexander. The interested reader is directed to [12].

Thus it is easy to see that the only three knots in our knot table from the Appendix that are already braided around a point are 3_1, 5_1, and 7_1.

Proof sketch of Theorem E.3. We will show that there is a method that can show that any link can be braided around a point. Consider a link diagram L and a point O on the plane P of the diagram (the point should not touch the link). If there exists a point on the plane such that the link diagram is braided around the point, then we "cut" the diagram along that point to get the braid.
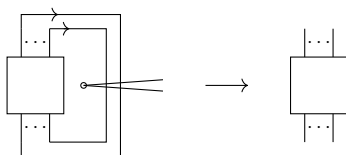

Figure E-iv: cutting a braided around link

Now consider an arbitrary link diagram. We need to construct this, so that it will be braided around some point. We need to first set a point O on the plane P. If there happens to be an edge, say AB, that is oriented counterclockwise, then we apply the triangle method by finding some point C on P, such that ABC contains O, by replacing AB by AC and BC. Now AC and BC are both oriented clockwise: the orientation we were looking for. We continue doing this until we get a link diagram that is braided around our point O.
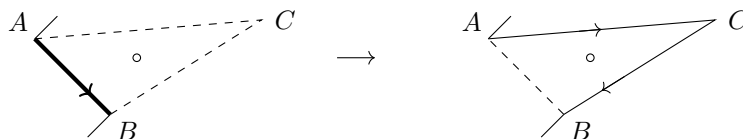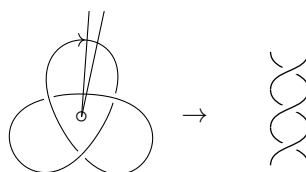
Figure E-v: Alexander's method

We know that this method will eventually terminate given any point O, since with every triangle move we make, the number of counterclockwise edges decreases by one, thus eventually giving us a link that is braided around a point O. ∎

Example E.8. We have already mentioned above that the trefoil is already braided around. So, we have the following diagram of the left-handed trefoil.
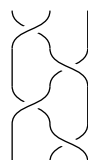

After cutting it, we see that the braid representation is given by σ_1^{-3}. It is easy to see that the right-handed trefoil's braid representation is σ_1^3.

Now let us find the braid representation of the following figure-eight knot using Alexander's method.
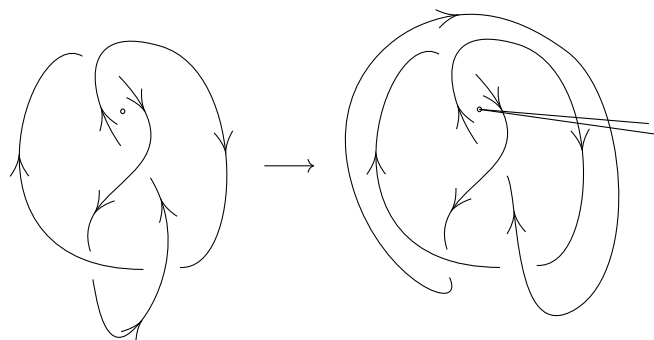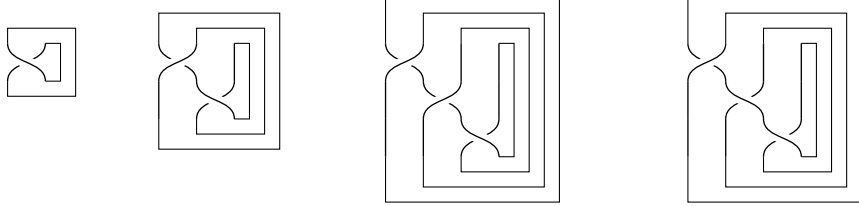
After cutting the diagram from our center point, we find the braid representation to be $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$,

---

If two braids are equivalent, it is obvious that their closures are also equivalent. However, as we have mentioned above, the map of the braid closure is non-injective. The following are more examples showing equivalent braid closures but non-equivalent braids.

Example E.9. Let us show that the closure of the 2-braid σ_1^{-1}, the 3-braid $\sigma_1\sigma_2^{-1}$, and the 4-braids $\sigma_1\sigma_2\sigma_3^{-1}$ and $\sigma_1\sigma_2^{-1}\sigma_3^{-1}$ are all equivalent to the trivial knot.

They are all equivalent to the trivial knot via a finite number of RI moves.

We now discuss how to form a bijective braid closure map. In 1935, Markov found three moves on the braid group, that when related by them, their closures are link equivalent [45]. Four years later, in 1939, Weinberg reduced the three moves that Markov found, to two moves [63]. These moves are called Markov's moves.

Definition E.10 (Markov's moves). Let $\mathcal{B}_\infty = \bigcup_{k \geq 1} \mathcal{B}_k$, i.e. the infinite union of the braid groups. There are two operations, called *Markov's moves*, which may be performed in $\mathcal{B}_\infty$.

M1 If $\beta \in \mathcal{B}_n$, then $\beta \xrightarrow{\text{M1}} \gamma\beta\gamma^{-1}$, for some $\gamma \in \mathcal{B}_n$. The braid $\gamma\beta\gamma^{-1}$ is said to be the *conjugate* of β.
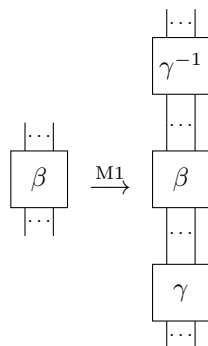

Figure E-vi: Markov M1 move

It is easy to see that taking the closure after an M1 move gives the same link, since γ and γ^{-1} both *cancel* out.

M2 If $\beta \in \mathcal{B}_n$, then $\beta \xrightarrow{\text{M2}} \text{incl}_n(\beta)\sigma_n$ or $\beta \xrightarrow{\text{M2}} \text{incl}_n(\beta)\sigma_n^{-1}$, for the generator $\sigma_n \in \mathcal{B}_{n+1}$. We call the braid $\text{incl}_n(\beta)\sigma_n^{\pm 1}$ the *stabilisation* of β.
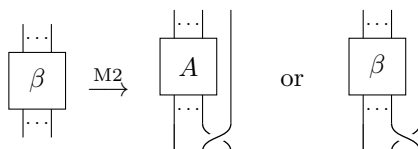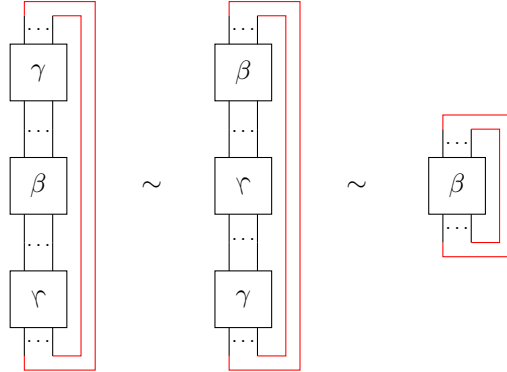

Figure E-vii: Markov M2 move

It is easy to see that taking the closure after an M2 move produces the same link since the move mimics an RI move.

We say that the two braids $\alpha, \beta \in \mathcal{B}_\infty$ are *Markov equivalent* if you can transform one into the other by a finite sequence of the above moves (and their inverses). We denote it as $\alpha \sim_M \beta$.

For clarity, the closure after an M1 move will look like,

Similarly, the closure after an M2 move will look like,

Theorem E.11 (Markov's theorem [45])**.** *Let K_1, K_2 be two oriented links which can be formed from the braids β_1, β_2, respectively. Then $K_1 \sim K_2 \Leftrightarrow \beta_1 \sim_M \beta_2$.*

For a proof of this; see Birman's survey [12, pp. 18–26] or Birman's original paper [10].

E.I Motivation

Since we have reached the end of the first part, we will summarise and thus motivate the second part with all of what we learned here.

Alexander's theorem E.3 says that all links have a braid representation, and Markov's theorem E.11 says that two links are equivalent if and only if their braid representations are Markov equivalent (i.e., they can be transformed by M1 and M2 moves). So, then we call the *Markov braid type* the infinite union of the braid group $\mathcal{B}_\infty$ that is quotiented by the two Markov moves (Markov equivalence). And so we have,

$$\{\text{link type}\} \simeq \{\text{Markov braid type}\}$$

So if we prove that a function is invariant under the Markov moves, then by Markov's theorem we have that it is a link invariant. And this will be the strategy for the upcoming original construction of the Jones polynomial (See Chapter H).

More precisely, for $a \in \mathcal{B}_n$ and any $i < n$, we have $\widehat{a} = \widehat{\sigma_i a \sigma_i^{-1}}$ and $\widehat{a} = \widehat{\text{incl}_n(a)\sigma_n^{\pm 1}}$. So our strategy of constructing the Jones polynomial will be to define a tracial function on the Temperley-Lieb algebra. Where the Temperley-Lieb algebra is seen as a quotient algebra of the group algebra $\mathbb{C}[\mathcal{B}_n]$. There exists other similar constructions, however for the purposes of this essay we will use the Temperley-Lieb algebras since it has a very comprehensible diagrammatic definition. To see other similar constructions; a very nice read is [27]. The following chapter will briefly touch on the work that Jones did in 1983 [25]. We just want to see the correlation he made between the von Neumann algebras and constructing a link invariant.

F   von Neumann algebras

The aim of this chapter is to give the essential knowledge in order to understand the construction and encapsulation of the Jones polynomial. So this chapter will give a brief overview on operator algebras.

The interested reader is directed to [25].

This chapter is based on the presentations given in [14], [23], [25], [28], [31], [50], [51], [52], [59], and [64].

F.I   Background

We shall first review some background we will need. The discussion in this section is mainly based on that in the author's exposition on analysis & its foundations [54].

A *norm* over a vector space V over $\mathbb{C}$ is a map $\|\cdot\| : V \to \mathbb{R}$ that satisfies

   (i) $\|v\| \geq 0$, with equality if and only if $v = 0$ for all $v \in V$

   (ii) $\|\lambda v\| = |\lambda| \|v\|$ for all $v \in V$ and $\lambda \in \mathbb{C}$

   (iii) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$

A *normed vector space* is a vector space V over $\mathbb{C}$ with a norm $\|\cdot\|$. We say that a normed vector space is a *Banach space* if it is complete (i.e., any Cauchy sequence converges). We call a map $\varphi : V \to \mathbb{C}$ a *linear functional* if it satisfies $\varphi(\lambda v + \mu w) = \lambda\varphi(v) + \mu\varphi(w)$, for all $\lambda, \mu \in \mathbb{C}$ and $v, w \in V$. We say the linear functional is *bounded* if $\|\varphi\| = \sup_{\|x\| \leq 1} |\varphi(x)| < \infty$. The *dual* of the normed vector space V, is the normed vector space of all bounded linear functionals $\varphi : V \to \mathbb{C}$, and we denote it as V^*.

Now, we call a map $T : V \to V$ *linear* if it satisfies $T(\lambda v + \mu w) = \lambda T(v) + \mu T(w)$ for all $v, w \in \mathbb{C}$ and $v, w \in V$. We call the linear map *bounded* if $\|T\| = \sup_{\|v\| \leq 1} \|T(v)\| < \infty$. We let $\mathbb{B}(V)$ be the normed vector space of all bounded linear maps $T : V \to V$.

A *Hilbert space* is essentially a complete inner product on a vector space V over $\mathbb{C}$, i.e.,

$$\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{C}$$

satisfying the following for all $x, y, z \in V$ and $\lambda \in \mathbb{C}$,

   (i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

   (ii) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$

   (iii) $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$

Notice that if we let the norm $\|x\| = \sqrt{\langle x, x \rangle}$ for any $x \in V$, then we can see $|\langle x, y \rangle| \leq \|x\| \|y\|$ for any $x, y \in V$. This inequality is known as the *Cauchy-Shwarz inequality* (it is easy to show that this is indeed satisfied). So, in fact, an inner product induces a norm.

Notation. Note that the notation used for the inner product $\langle \cdot, \cdot \rangle$ should not be confused with the Kauffman bracket $\langle \cdot \rangle$ from the previous chapters. In fact, all notations used in this chapter are self-contained, unless stated otherwise.

The *adjoint* of an operator $T : \mathcal{H} \to \mathcal{H}'$ is the unique linear map $T^* : \mathcal{H} \to \mathcal{H}$ such that for all $x \in \mathcal{H}'$ and $y \in \mathcal{H}$ we have $\langle T^*x, y \rangle = \langle x, Ty \rangle$.

An *algebra* is a vector space V over $\mathbb{C}$ if there is a multiplication operation $V \times V \to V$, given by $(u, v) \mapsto uv$ such that for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{C}$, we have $u(v+w) = uv+uw$, $(u+v)w = uw+vw$ and $(\lambda u)(\mu v) = (\lambda\mu)(uv)$. We call an algebra V over $\mathbb{C}$ a *normed algebra* if for all $u, v \in V$ we

have $\|uv\| \le \|u\|\|v\|$. We call a normed algebra a *Banach algebra* if it is a Banach space. If V is a normed algebra with a multiplicative identity, then we call V *unital*. Given V is a an algebra, and a vector subspace W, then we call B a *subalgebra* if $v, w \in W$ implies $vw \in W$.

And if V is an algebra that is commutative as a ring, then it is a *commutative algebra*.

We shall assume that our algebras are unital.

We define the *spectrum* of x, denoted as $\sigma(x)$, in a Banach algebra V as the set of complex numbers λ such that $x - \lambda I$ is not invertible, i.e., $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda I \text{ is not invertible}\}$.

Proposition F.1. *The spectrum of an element in a Banach algebra A is a non-empty compact subset of $\mathbb{C}$.*

Proof sketch. First start with the *Murray-von Neumann criterion*, which states that for x in a Banach algebra A, if $\|x\| < 1$ then $1 - x$ is invertible. It then follows that the spectrum of x is in fact closed. Then it follows from Heine-Borel, $\sigma(x)$ is compact since it is closed and bounded.

To show that it is non-empty, we need to define the resolvent set of x $\rho(x)$. This is the complement of the spectrum $\rho(x) = \mathbb{C} \setminus \sigma(x)$. And we also define the resolvent map of x for $\lambda \in \rho(x)$ as $R_\lambda(x) = (\lambda - x)^{-1}$.

Then it is easy to see that as $\lambda \to \infty$ we have $\|R_\lambda(x)\| \to 0$. It is also easy to see that for $\lambda, \mu \in \rho(x)$, we have $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$.

There is an analogue to Louiville's theorem for complex functions, so it then follows that the spectrum is non-empty. ∎

F.II C^*-algebras

Definition F.2 (**-algebra*). A **-algebra* is an algebra A with an involution map $* : x \mapsto x^*$ which for $x, y \in A$ and $\lambda \in \mathbb{C}$ satisfies

(i) $(xy)^* = y^* x^*$

(ii) $(x + y)^* = x^* + y^*$

(iii) $\lambda^* = \bar{\lambda}$

(iv) $x^{**} = x$

We say a **-algebra is an *abstract C^*-algebra* if it is a Banach **-algebra where the norm satisfies the C^*-*identity* $\|x\|^2 = \|x^* x\|$.

The completion of a Banach algebra with an involution is an abstract C^*-algebra.

Example F.3. The complex numbers $\mathbb{C}$ forms a C^*-algebra, by defining the involution $\lambda^* = \overline{\lambda}$ and where the norm is $\|\lambda\| = |\lambda|$.

We say that a **-algebra is a *concrete C^*-algebra* if it is a closed subalgebra of $\mathbb{B}(\mathcal{H})$.

Example F.4. So the set of bounded operators on a Hilbert space $\mathcal{H}$, $\mathbb{B}(\mathcal{H})$, is one example of a concrete C^*-algebra.

Given a C^*-algebra A, if A has a unit $1 \in A$, i.e., an identity element. Then it can easily be shown $1^* = 1$. This can be seen by $1 = (1^*)^* = (1^*1)^* = (1^*(1^*)^*) = 1^*1 = 1^*$.

Proposition F.5. *Given a* C^**-algebra* A*, the involution is continuous.*

Proof. Let $x_n \in A$ such that $x_n \to x \in A$. So, $\|x_n^* - x^*\| = \|(x_n - x)^*\|$. Now we want to show $\|(x_n - x)^*\| = \|x_n - x\|$. So $\|x_n - x\|^2 = \|(x_n - x)^*(x_n - x)\| \leq \|(x_n - x)^*\|\|x_n - x\|$, so if $x_n - x \neq 0$ then $\|x_n - x\| \leq \|(x_n - x)^*\|$ (otherwise if $x_n - x = 0$, then we are done). So, $\|(x_n - x)^*\| \leq \|((x_n - x)^*)^*\| = \|x_n - x\|$. So, then, $\|(x_n - x)^*\| = \|x_n - x\|$.
So, we now have $\|x_n^* - x^*\| = \|x_n - x\| \to 0$ since $x_n \to x$. Therefore, the involution is continuous. ∎

Proposition F.6. *If* A *is a* C^**-algebra, then* $\sigma(x^*) = \sigma(x)$ *for all* $x \in A$.

Given a C^*-algebra A, then we have

  – $x \in A$ is *normal* if $xx^* = x^*x$.

  – $y \in A$ is *Hermitian* if $x = x^*$.

  – $u \in A$ is *unitary* if $uu^* = u^*u = 1$.

  – $p \in A$ is a *projection* if $p = p^* = p^2$.

The following result is left as an exercise to the reader.

Proposition F.7. *If* x *is an element of a* C^**-algebra* A*. Then* x^*x *is Hermitian.*

Definition F.8. Given C^*-algebras A and B, a **-homomorphism* is an algebra homomorphism $\alpha : A \to B$ such that $\alpha(x^*) = \alpha(x)^*$ for all $x \in A$. A *-homomorphism, α, between unital C^*-algebras is itself *unital* if $\alpha(1) = 1$.

Theorem F.9. *Any* **-homomorphism is continuous, with closed image, and is bounded with norm at most one. Any injective* **-homomorphism is an isometry.*

So, the norm on a C^*-algebra is determined by the rest of the algebraic structure.

A *-homomorphism is an *isomorphism* of C^*-algebras if it is bijective.

Proposition F.10. *The inverse of a bijective* **-homomorphism is also a* **-homomorphism.*

F.III   Three topologies on $\mathbb{B}(\mathcal{H})$

Definition F.11 (WOT, SOT, & norm topology)**.**

norm The *norm topology* on $\mathbb{B}(\mathcal{H})$ is given by the norm $\|T\| = \sup_{\|v\| \leq 1} \|T(v)\|$.

WOT The *weak operator topology* on $\mathbb{B}(\mathcal{H})$ is given by

$$\{b \in \mathbb{B}(\mathcal{H}) : |\langle ax, y \rangle - \langle bx, y \rangle| < \epsilon \text{ where } a \in \mathbb{B}(\mathcal{H}), x, y \in \mathcal{H} \text{ and } \epsilon > 0\}$$

SOT The *strong operator topology* (SOT) on $\mathbb{B}(\mathcal{H})$ is given by

$$\{b \in \mathbb{B}(\mathcal{H}) : \|ax - bx\| < \epsilon \text{ where } a \in \mathbb{B}(\mathcal{H}), x \in \mathcal{H} \text{ and } \epsilon > 0\}$$

So, we then have that the norm topology contains the strong and weak operator topologies, i.e., WOT $\prec$ SOT $\prec$ the norm topology. And so for any $A \subset \mathbb{B}(\mathcal{H})$, we have $\overline{A}^{\|\cdot\|} \subset \overline{A}^{\text{SOT}} \subset \overline{A}^{\text{WOT}}$.

F.IV   von Neumann algebras

Definition F.12 (commutant). We define the *commutant* of $A \subseteq \mathbb{B}(\mathcal{H})$ as

$$A' = \{y \in \mathbb{B}(\mathcal{H}) : xy = yx \text{ for all } x \in A\},$$

and the *bicommutant* by $A'' = (A')'$.

The following theorem is known as the *bicommutant theorem*.

Theorem F.13. *Let $M \subset \mathbb{B}(\mathcal{H})$ be a unital *-subalgebra. Then the following three are equivalent.*

(i) *M is closed in WOT*

(ii) *M is closed in SOT*

(iii) *$M = M'' = (M')'$*

Definition F.14 (von Neumann algebra). A unital *-subalgebra $M \subseteq \mathbb{B}(\mathcal{H})$ is a *von Neumann algebra* if it satisfies any of the properties in the above theorem.

It is clear that any von Neumann algebra is a C^*-algebra.

Example F.15. The set of bounded linear maps on the Hilbert space $\mathbb{B}(\mathcal{H})$ is a von Neumann algebra. We also have $M_n(\mathbb{C})$ as a von Neumann algebra.

We note that von Neumann algebras come in pairs M, M'.

F.V   Factors

Definition F.16 (factor). We call a von Neumann algebra M a *factor* if $M \cap M' = \mathbb{C} \cdot 1_{\mathbb{B}(\mathcal{H})}$ (i.e., the center is trivial).

Example F.17. The set of bounded linear maps on the Hilbert space $\mathbb{B}(\mathcal{H})$ is thus also a factor.

The only commutative factor is $\mathbb{C}$.

Given M is a von Neumann algebra, then

– let $e, f \in M$ be two projections, then e and f are *equivalent* if there exists $u \in M$ such that $u^*u = e$ and $uu^* = f$. We write $e \sim f$ if they are equivalent.

– let $e, f \in M$ be two projections, then if there exists $u \in M$ such that $u^*u = e$ and $u^*uf = u^*u$. We write this as $e \preceq f$.

– we say a projection $p \in M$ such that $p \neq 0$ is *minimal* if for all projections $a \in M$ we have that if $a \leq p$ then either $a = 0$ or $a = p$.

– we say a projection $e \in M$ is *finite* if we have that for all projections $f \in M$, if $e \sim f \leq e$ then $f = e$.

The types of factors $M \subseteq \mathbb{B}(\mathcal{H})$:

– *type* I *factor* if there exists a minimal projection in M.

– we call the factor $M \subseteq \mathbb{B}(\mathcal{H})$ a *type* II *factor* if there exists a finite projection in M and there does not exist a minimal projection in M.

– given $M \subseteq \mathbb{B}(\mathcal{H})$ is a type II factor and $1_{\mathbb{B}(\mathcal{H})}$ is finite, then M is a *type* II$_1$ *factor*.

– if $1_{\mathbb{B}(\mathcal{H})}$ is infinite and there exists finite projections, then it is said to be a *type* II_∞ *factor*.

– finally, we have that the factor $M \subseteq \mathbb{B}(\mathcal{H})$ is a *type* III *factor* if there exists no finite projections in M.

We will only discuss type II_1 factors in this paper. So to simplify, a factor is of type II_1 if all projections are finite (i.e., $1_{\mathbb{B}(\mathcal{H})}$ is finite) and there does not exist a minimal projection.

F.VI   Trace

We call a linear functional f on a von Neumann algebra M,

– *positive* if $f(x^*x) \geq 0$ for all $x \in M$,

– *faithful* if when we have $f(x^*x) = 0$, then $x = 0$,

– *state* if $f(1) = 1$, and

– *tracial* if $f(xy) = f(yx)$ for all $x, y \in M$.

Type II_1 factors M admit a unique faithful tracial state $\mathrm{tr}\colon M \to \mathbb{C}$, so

(i) $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in M$

(ii) $\mathrm{tr}(1) = 1$

(iii) $\mathrm{tr}(xx^*) > 0$ for all $x \in M$ where $x \neq 0$.

We call a linear functional $\mathrm{tr}\colon M \to \mathbb{C}$ a *trace* if $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in M$.

Given M is a finite factor, now we can say that two projections $e, f \in M$ are equivalent if $\mathrm{tr}(e) = \mathrm{tr}(f)$.

F.VII   The index theorem

We define a Hilbert space $L^2(M, \mathrm{tr}_M)$ to be the completion of M with respect to the inner product $\langle a, b \rangle = \mathrm{tr}_M(b^*a)$.

Definition F.18 (Jones index). Given N is a subfactor of M, we define the *Jones index* of N as $[M : N] = \dim_N L^2(M, \mathrm{tr}_M)$.

Theorem F.19 (index theorem). *Given N is a II_1 subfactor of M, then we have*

$$[M : N] \in \left\{ 4\cos^2(\pi/(n+2)) : n \geq 1 \right\} \cup [4, \infty).$$

Jones proved this using the basic construction explained in the following section.

F.VIII   Jones' basic construction

In this section we will discuss Jones' basic construction for type II_1 subfactors. This discussion is majorly based on that of Jones' paper [25], Penneys' lecture notes [52], and Speicher's lecture notes [59].

Theorem F.20 (GNS - Gelfand-Naimark-Segal). *Every abstract C^*-algebra is isometrically *-isomorphic to a concrete C^*-algebra.*

Theorem F.21 (GNS construction). *Given a unital C^*-algebra A and a state $f\colon A \to \mathbb{C}$. There exists a Hilbert space $\mathcal{H}'$, a *-homomorphism $\pi\colon A \to \mathbb{B}(\mathcal{H}')$, and a unit vector $h \in \mathcal{H}'$ such that for all $a \in A$ we have $f(a) = \langle \pi(a)h, h \rangle$.*

We have the GNS representation of a factor M on $L^2(M, \text{tr}_M)$, which is given by $\langle x, y \rangle = \text{tr}_M(y^*x)$.

Since the entire chapter was filled with definition and results, it would be reasonable to summarise what it is exactly we need in order to discuss Jones' basic construction.

So a von Neumann algebra A is a unital $*$-subalgebra of $\mathbb{B}(\mathcal{H})$ such that $A = A''$; see Theorem F.13. A factor M is a von Neumann algebra such that the center is trivial (i.e., $M \cap M' = \mathbb{C} \cdot 1_{\mathbb{B}(\mathcal{H})}$). A factor is of type II_1 if it is of infinite dimension and it admits a unique tracial state. We then define a Hilbert space $L^2(M, \text{tr}_M)$ to be the completion of M with respect to the inner product $\langle a, b \rangle = \text{tr}_M(b^*a)$. So, M acts on $L^2(M, \text{tr}_M)$ by both left and right multiplication. So, for a projection $p \in M$, we have the representation $L^2(M, \text{tr}_M)p$. We then have that for II_1 factors $N \subset M$, we have $[M : N] = \dim_N(L^2(M, \text{tr}_M))$.

We have the conjugate-linear unitary functional $J \colon L^2(M, \text{tr}_M) \to L^2(M, \text{tr}_M)$ given by $x \mapsto x^*$. Then we have $JM'J = M$.

Now consider a subfactor $N \subset M$, and the Hilbert space $L^2(M, \text{tr}_M)$. We see that are already five II_1 factors, namely: M, M', N, N', and JNJ.

Proposition F.22 ([25, pp. 7,8]). *Given finite von Neumann algebras $N \subset M$ with a faithful normal trace tr_N, with $1_N = 1_M$. Then there exists a normal linear map $E_N \colon M \to N$ defined by $\text{tr}_N(E_N(x)y) = \text{tr}_N(xy)$ for $x \in M$ and $y \in N$, and satisfies the following properties*

  (i) $E_N(axb) = aE_N(x)b$ for $x \in M$ and $a, b \in N$

  (ii) $E_N(x^*) = E_N(x)^*$ for all $x \in M$

  (iii) $E_N(x^*)E_N(x) \leq E_N(x^*x)$ for all $x \in M$

  (iv) $E_N(x^*x) = 0$ implies $x = 0$ for all $x \in M$

The above proposition essentially says that E_N is a trace-preserving, faithful and positive linear map. We call E_N the *conditional expectation* of M to N.

Proposition F.23 ([59, Proposition 5.4]). *Given II_1 factors $N \subset M$, there exists the conditional expectation $E_N \colon M \to N$, and an orthogonal projection $e_N \colon L^2(M, \text{tr}_M) \to L^2(N, \text{tr}_N)$ such that*

  (i) it is uniquely determined by $e_N x e_N = E_N(x)e_N$ for $x \in M$

  (ii) $xe_N = e_N x$ if and only if $x \in N$

  (iii) $N = M \cap \{e_N\}'$

Basic construction. Given II_1 factors $N \subset M$. Let $e_N \colon L^2(M, \text{tr}_M) \to L^2(N, \text{tr}_N)$ be an orthogonal projection. Then we have $N = M \cap \{e_N\}'$. So, a direct corollary to this is that we have the von Neumann algebra $\langle M, e_N \rangle$ on $L^2(M, \text{tr}_M)$ which is defined by $(M \cup \{e_N\})'' = JN'J$. Thus we also have $[M : N] = \text{tr}(e_N)^{-1} = [\langle M, e_N \rangle : M]$, where the trace is defined on $\langle M, e_N \rangle$.

And so analogously to how for a von Neumann algebra M there exists a von Neumann algebra M'; The basic construction tells us that for a subfactor $N \subset M$ there exists subfactors JNJ and $JN'J$ on $L^2(M, \text{tr}_M)$.

Thus, by iterating the basic construction, we find a tower M_i of II_1 factors with $M_0 = N$, $M_1 = M$ and $M_{i+1} = \langle M_i, e_{M_{i-1}} \rangle$.

$$N \subset M \overset{e_M}{\subset} M_2 \overset{e_{M_2}}{\subset} \cdots \overset{e_{M_n}}{\subset} \langle M_n, e_{M_n} \rangle = M_{n+1} \subset \cdots$$

For better readability, we let $e_i = e_{M_i}$, and call the sequence of (e_i) the *Jones projections*.

Theorem F.24. *The Jones projections (e_i) satisfy the following properties.*

(i) $[M:N] e_i e_{i\pm 1} e_i = e_i$

(ii) $e_i e_j = e_j e_i$ *if* $|i - j| \geq 2$

(iii) $e_i^2 = e_i^* = e_i$

(iv) $[M:N] tr_{n+1}(x e_n) = tr_n(x)$ *where* x *is a word on* $1, e_1, e_2, \ldots, e_{n-1}$

These relations are similar to those of the braid group and the Markov moves. This is the similarity that Didier Hatt-Arnold pointed out to Vaughan Jones in 1982 (see Introduction A). The reader should instantly notice that the Jones projections (e_i) satisfy the properties of the Temperley-Lieb algebras from Definition G.5 given $d^2 = [M:N]$. For instance, for part (i) we have,

$$[M:N]e_n e_{n-1} e_n = [M:N]E_n(e_{n-1})e_n$$
$$= [M:N]\mathrm{tr}_{n+1}(e_{n-1})e_n$$
$$= [M:N][M:N]^{-1}e_n = e_n$$

G The Temperley-Lieb algebras

The Temperley-Lieb algebra was first introduced in 1971 by Lieb and Temperley [42]. In this chapter, we will see the relation between the von Neumann algebra that we constructed in the previous chapter (via the Jones' projections) and the Temperley-Lieb algebra. This will help our understanding since it can be explained in a diagrammatic setting.

Our discussion will mainly depend on that found in [52].

G.I Quantum integers

Definition G.1. A *qunatum integer* $[n]_q$ for $n \in \mathbb{Z}$ and $q \in \{Q \cup -Q\}$ where $Q := \{e^{i\theta} : \theta \in (0, \pi/2)\} \cup [1, \infty)$, is defined as $\dfrac{q^n - q^{-n}}{q - q^{-1}}$.

A quantum integer is essentially a deformation of an integer by q.

Example G.2. So the quantum integer $[1]_q$ is 1, while $[2]_q = q + q^{-1}$.

Proposition G.3. *We can double a quantum integer,* $[2]_q[n]_q = [n+1]_q + [n-1]_q$.

Proof.

$$[2]_q[n]_q = \frac{(q + q^{-1})(q^n - q^{-n})}{q - q^{-1}}$$
$$= \frac{q^{n+1} - q^{1-n} + q^{n-1} - q^{-n-1}}{q - q^{-1}}$$
$$= \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} + \frac{q^{n-1} - q^{-(n-1)}}{q - q^{-1}}$$

And so the result then follows. ∎

Proposition G.4. *Given* $m \geq a$, *we have* $[m-a]_q = [m]_q[a+1]_q - [m+1]_q[a]_q$.

Proof.

$$
\begin{aligned}
[m-a]_q &= \frac{q^{m-a} - q^{a-m}}{q - q^{-1}} \\
&= \frac{(q^{m-a} - q^{a-m})(q - q^{-1})}{(q - q^{-1})(q - q^{-1})} \\
&= \frac{q^{m-a+1} - q^{m-a-1} - q^{a-m+1} + q^{a-m-1}}{q^2 + q^{-2}} \\
&= \frac{q^{m+a+1} - q^{m+a+1} + q^{-m-a-1} - q^{-a-m-1} + q^{m-a+1} - q^{m-a-1} - q^{a-m+1} + q^{a-m-1}}{q^2 + q^{-2}} \\
&= \frac{q^{m+(a+1)} - q^{m-(a+1)} - q^{(a+1)-m} + q^{-m-(a+1)}}{q^2 + q^{-2}} \\
&\quad - \frac{q^{a+(m+1)} - q^{-a+(m+1)} - q^{a-(m+1)} + q^{-a-(m+1)}}{q^2 + q^{-2}} \\
&= \left(\frac{q^m - q^{-m}}{q - q^{-1}} \right) \frac{q^{a+1} - q^{-(a+1)}}{q - q^{-1}} - \left(\frac{q^{m+1} - q^{-(m+1)}}{q - q^{-1}} \right) \frac{q^a - q^{-a}}{q - q^{-1}} \\
&= [m]_q [a+1]_q - [m+1]_q [a]_q
\end{aligned}
$$

■

G.II Defining the Temperley-Lieb algebra TL_n

Definition G.5. For $d \in \{Q \cup -Q\}$, $TL_n(d)$ is the associative unital (i.e., has an identity) $*$-algebra with generators $e_1, e_2, \ldots, e_{n-1}$ and relations

(i) $e_i^2 = e_i = e_i^*$

(ii) $e_i e_j = e_j e_i$ for $|i - j| \geq 2$

(iii) $e_i e_{i \pm 1} e_i = d^{-2} e_i$.

So we can reduce each word in the algebra with the relations.

Example G.6. So, TL_1 is generated by 1, TL_2 is generated by 1 and e_1, TL_3 is generated by 1, e_1, e_2, $e_1 e_2$ and $e_2 e_1$, and so on.

Definition G.7. The n-th *Catalan number* c_n is given by $\dfrac{1}{n+1} \dbinom{2n}{n}$.

In our above example, we saw $\dim(TL_1) = 1 = c_1$, $\dim(TL_2) = 2 = c_2$ and $\dim(TL_3) = 5 = c_3$. So, we have the following result.

Proposition G.8. TL_n *is finite dimensional.*

Sketch of Proof. Essentially, since this is a finitely generated group. We have any word in the algebra can be written as

$$
(e_{i_1} e_{i_1 - 1} \cdots e_{k_1})(e_{i_2} e_{i_2 - 1} \cdots e_{k_2}) \cdots (e_{i_p} e_{i_p - 1} \cdots e_{k_p})
$$

for $1 \leq i_1 < i_2 < \cdots < i_p$ and for $1 \leq k_1 < k_2 < \cdots < k_p$. Thus the dimension is less than or equal to the nth Catalan number.

$$
\dim(TL_n) \leq c_n
$$

■

It is easy to realise that this is essentially the type II_1 algebra formed from the Jones projections (e_i) in the previous chapter.

G.III   Diagrammatic definition of the Temperley-Lieb algebra gTL_n

We now discuss the diagrammatic definition of the Temperley-Lieb algebras. This will help us in the construction of the Jones polynomial in the next chapter.

> **Definition G.9.** For $d \in \{Q \cup -Q\}$, we call the algebra $gTL_n(d)$ the *Temperley-Lieb algebra*, whose basis is the set of non-intersecting string diagrams on a rectangle with n boundary points on the top and bottom. The multiplication is defined by concatenation $gTL_n \times gTL_n \to gTL_n$ (analogously to braid concatenation) and is shown below.
>
>
>
> If a closed loop appears in the diagram, we can remove it by multiplying the diagram by the weight of the closed loop (which is d).
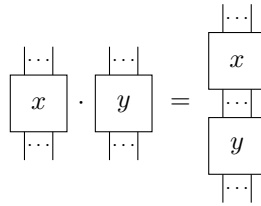
We have that two elements in gTL_n are equivalent if the strings are ambient isotopic. So by holding all the boundary points in place, we can move the strings around, given that they do not intersect or break the definition.

Example G.10. An example of an element in gTL_5 would be the following

---
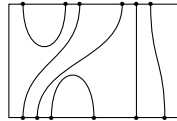
Example G.11. The elements of gTL_1 is just a diagram with a single string. The elements of gTL_2 is the following
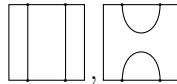
We also have the following elements of gTL_3
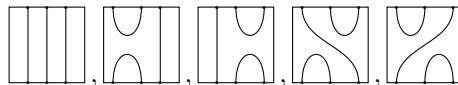
So, we have the generators for gTL_n,
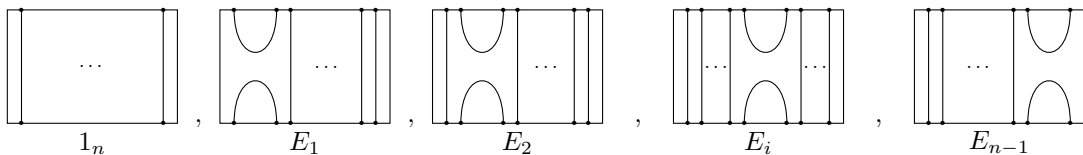
Figure G-i: the gTL_n generators

To see the relations, we begin by multiplying E_i^2,
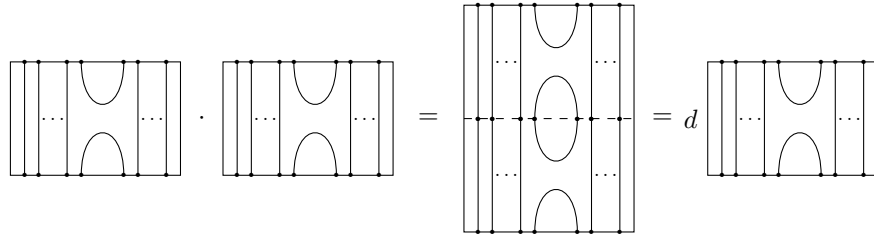
Figure G-ii: $E_i^2 = dE_i$

Therefore, we have $E_i^2 = dE_i$.

We also have $E_i E_j = E_j E_i$ where $|i - j| > 1$,
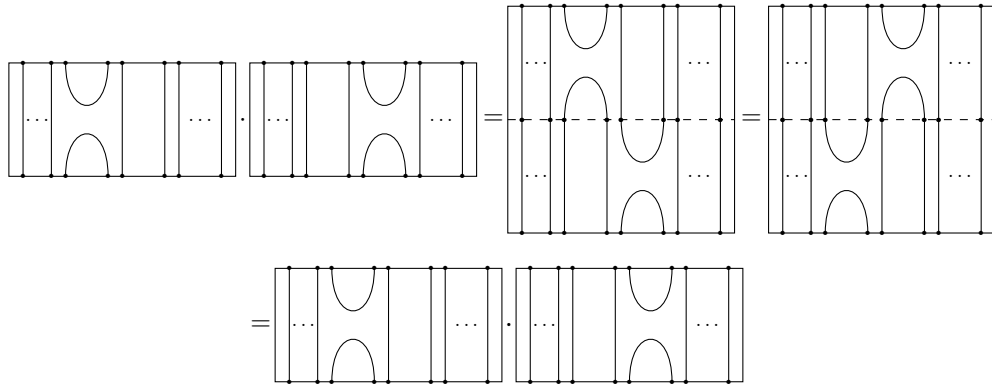
Figure G-iii: $E_i E_j = E_j E_i$ where $|i - j| > 1$

Our final relation is $E_i E_{i+1} E_i = E_i$.
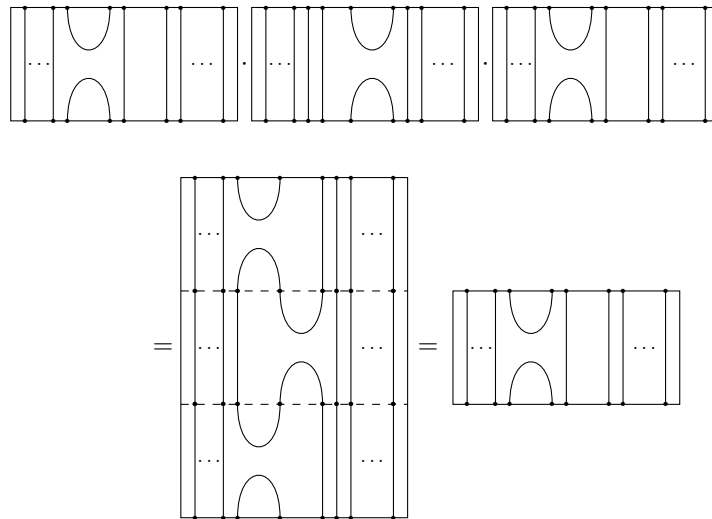
Figure G-iv: $E_i E_{i+1} E_i = E_i$

We leave the reader to check $E_i E_{i-1} E_i = E_i$.

The involution * of a diagram would be the reflection about the horizontal line. Thus, $E_i = E_i^*$. However, for non-generators we have,
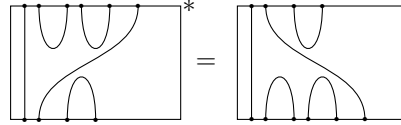
Figure G-v: involution example on gTL_n

Thus the relations for our generators in gTL_n are

  (i) $E_i^2 = dE_i = dE_i^*$

  (ii) $E_i E_j = E_j E_i$ for $|i - j| \geq 2$

  (iii) $E_i E_{i\pm 1} E_i = E_i$

These relations are closely related to that of TL_n. One can define a homomorphism by mapping e_i to $d^{-1} E_i$. In fact, gTL_n is *-isomorphic to TL_n.

Theorem G.12. *There exists a *-isomorphism from TL_n to gTL_n.*

So from this point forth, we denote $\mathcal{TL}_n$ as our Temperley-Lieb algebra, where we will use the diagrammatic definition. So we have the generators E_i for $i = 1, 2, \ldots, n - 1$ and the relations above.

G.IV   Linear operations on $\mathcal{TL}_n$

We now discuss some sub-planar algebra on $\mathcal{TL}_n$ which will make things easier to comprehend.

Definition G.13. Consider the following diagram in $\mathcal{TL}_n$,

$$x = \boxed{\;x\;}$$
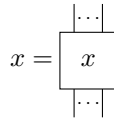
we then have the following linear operations on $\mathcal{TL}_n$

  (i) The *inclusion* $i_n \colon \mathcal{TL}_n \to \mathcal{TL}_{n+1}$ is a unital, injective *-homomorphism. We essentially add a string to the right of the diagram.

$$i_n(x) = \boxed{\;x\;}\,\big|$$
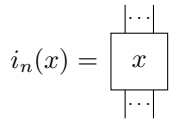
Figure G-vi: inclusion on $\mathcal{TL}_n$

  (ii) The *conditional expectation* $\mathcal{E}_{n+1} \colon \mathcal{TL}_{n+1} \to \mathcal{TL}_n$ is a surjective *-map of $\mathbb{C}$-vector spaces. We essentially close the last string on the right.

$$\mathcal{E}_{n+1}(x) = \boxed{x}$$

Figure G-vii: conditional expectation $\mathcal{E}_{n+1}$ on $\mathcal{TL}_{n+1}$

(iii) The *trace* $\mathrm{tr}_n \colon \mathcal{TL}_n \to \mathcal{TL}_0$ is a linear *-map of $\mathbb{C}$-vector spaces. Essentially, it is the closure of the diagram.

$$\mathrm{tr}_n(x) = \boxed{x}$$

Figure G-viii: trace tr_n on $\mathcal{TL}_n$

(iv) The *identity* $\mathrm{id}_n \colon \mathcal{TL}_n \to \mathcal{TL}_n$ which is given by

$$\mathrm{id}_n(x) = \boxed{x}$$

It is typical to wonder whether the conditional expectation on $\mathcal{TL}_n$ defined above is related to the conditional expectation on factors in the previous chapter. This would make sense as to why we started with von Neumann algebras before defining the Temperley-Lieb algebras. The following proposition will answer our question.

Proposition G.14. *There exists the following relations,*

(i) $\mathcal{E}_{n+1} \circ i_n = d \cdot \mathrm{id}_n$

(ii) $\mathrm{tr}_{n+1} = \mathrm{tr}_n \circ \mathcal{E}_{n+1}$

(iii) $(i_n \circ i_{n-1} \circ \mathcal{E}_n(x))E_n = E_n i_n(x)E_n$ *for all $x \in \mathcal{TL}_n$*

(iv) $\mathrm{tr}_n(xy) = \mathrm{tr}_n(yx)$ *for all $x, y \in \mathcal{TL}_n$*

(v) $\mathrm{tr}_{n+1}(i_n(x) \cdot E_n) = \mathrm{tr}_n(x)$ *for all $x \in \mathcal{TL}_n$*

(vi) $\mathrm{tr}_n(\mathcal{E}_{n+1}(x) \cdot y) = \mathrm{tr}_{n+1}(x \cdot i_n(y))$ *for all $x \in \mathcal{TL}_{n+1}$ and $y \in \mathcal{TL}_n$*

(vii) $\mathrm{tr}_n(1_n) = d^n$

Proof.

(i) $\mathcal{E}_{n+1} \circ i_n \colon \mathcal{TL}_n \to \mathcal{TL}_n$,

$$\mathcal{E}_{n+1} \circ i_n \left(\boxed{x} \right) = \mathcal{E}_{n+1} \left(\boxed{x} \right) = \boxed{x} = d \cdot \mathrm{id}_n \left(\boxed{x} \right)$$

(ii) $\mathrm{tr}_{n+1}\colon \mathcal{TL}_{n+1} \to \mathcal{TL}_0$,

$$\mathrm{tr}_{n+1}\left(\boxed{x}\right) = \boxed{x} = \mathrm{tr}_n\left(\boxed{x}\right) = \mathrm{tr}_n \circ \mathcal{E}_{n+1}\left(\boxed{x}\right)$$

(iii) We first have $i_n \circ i_{n-1} \circ \mathcal{E}_n(x)$,
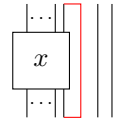
and so, multiplying by E_n gives
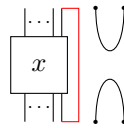
This can be seen to be the same as $E_n i_n(x) E_n$,
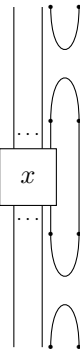
Therefore, we can see that for any $x \in \mathcal{TL}_n$, we have $E_n i_n(x) E_n = (i_n \circ i_{n-1} \circ \mathcal{E}_n(x)) E_n$.

(iv) $tr_n(xy)\colon \mathcal{TL}_n \times \mathcal{TL}_n \to \mathcal{TL}_0$,

$$\mathrm{tr}_n\left(\boxed{x} \cdot \boxed{y}\right) = \mathrm{tr}_n\left(\begin{array}{c}\boxed{x}\\ \boxed{y}\end{array}\right) = \begin{array}{c}\boxed{x}\\ \boxed{y}\end{array}$$

By moving the x diagram around in a clockwise fashion, we get,

$$= \begin{array}{c}\boxed{y}\\ \boxed{x}\end{array} = \mathrm{tr}_n(yx)$$

(v) By (ii), we know $\text{tr}_{n+1} = \text{tr}_n \circ \mathcal{E}_{n+1}$. And we also know $E_n \in \mathcal{TL}_{n+1}$, so $\text{tr}_{n+1}(i_n(x) \cdot E_n) = \text{tr}_n(\mathcal{E}_{n+1}(i_n(x) \cdot E_n))$. And so,

And by straightening the last string, it is easy to see that we get,

(vi) From the above property, we know $\text{tr}_n(\mathcal{E}_{n+1}(x) \cdot y) = \text{tr}_{n+1}(i_n(\mathcal{E}_{n+1}(x) \cdot y) \cdot E_n)$. So, we now need to show $i_n(\mathcal{E}_{n+1}(x) \cdot y) \cdot E_n = x \cdot i_n(y)$.

And so $i_n(\mathcal{E}_{n+1}(x) \cdot y) \cdot E_n$ becomes

Now from (i), we know $\text{tr}_{n+1} = \text{tr}_n \circ \mathcal{E}_{n+1}$, so

Now, $\mathcal{E}_{n+1}(x \cdot i_n(y))$ is

$$
\mathcal{E}_{n+1}\left(\boxed{x} \cdot \boxed{y} \right) = \mathcal{E}_{n+1}\left(\begin{matrix} x \\ y \end{matrix} \right) = \begin{matrix} x \\ y \end{matrix}
$$

Thus $\mathrm{tr}_n(\mathcal{E}_{n+1}(x) \cdot y) = \mathrm{tr}_{n+1}(x \cdot i_n(y))$.

(vii) And finally,

$$
\mathrm{tr}_n\left(\Big| \cdots \Big| \right) = \Big| \cdots \Big| = d^n
$$

∎

Remark. Now comparing the above relations and the relations found in Propositions F.22 and F.23, we find that they are related. For instance, in factors $N \subset M$, we know $\mathrm{tr}_M(E_N(x)) = \mathrm{tr}_N(x)$ for all $x \in N$ where $E_N\colon N \to M$. This is the same as (ii) in the above proposition. This means that the conditional expectation on factors is related to the conditional expectation on $\mathcal{TL}_n$, and the Jones projections are related to the generators on $\mathcal{TL}_n$. But to relate one to the other, we must be extra careful with the inclusion map on $\mathcal{TL}_n$. For instance, part (iii) above is similar to the relation in part (i) of Proposition F.23.

The trace on $\mathcal{TL}_n$ is called the *Markov trace*, and is unique up to a scalar (the scalar being d^n).

Remark. To get a normal trace, i.e., $\mathrm{tr}_n(1) = 1$. We need to define the trace as the number of loops minus n. So, then it is clear that we would get: $\mathrm{tr}_n(1) = 1$; $\mathrm{tr}_n(xy) = \mathrm{tr}_n(yx)$; and $\mathrm{tr}_{n+1}(xE_n) = d^{-1}\mathrm{tr}_n(x)$ for all $x, y \in \mathcal{TL}_n$. We will not use the normal trace, since we will normalise our Jones polynomial in the following chapter. So this is only an interesting remark and nothing else.

G.V   The Jones-Wenzl idempotents

In this section we will discuss the Jones-Wenzl idempotents. We say an element is an idempotent if when multiplied by itself it equals itself. So any identity is itself an idempotent. We now let $[n]_q = [n]$, for better readability.

Proposition G.15. *Suppose there exists $f^{(n)} \in \mathcal{TL}_n$ such that $f^{(n)} \neq 0$, $(f^{(n)})^2 = f^{(n)}$ and for all $i = 1, \ldots, n-1$ we have $E_i f^{(n)} = f^{(n)} E_i$. Then $f^{(n)}$ is unique for each $\mathcal{TL}_n$.*

Proof. Let $f_1^{(n)}, f^{(n)} \in \mathcal{TL}_n$ satisfy all properties and $f_1^{(n)} \neq f^{(n)}$. So by writing them as a linear combination of the generators $\{1, E_1, \ldots, E_{n-1}\}$ we have

$$
f^{(n)} = x + \sum_i y_i E_i \qquad f_1^{(n)} = x' + \sum_i y_i' E_i
$$

It then follows that $x = x' = 1$. Since both are idempotents we then have,

$$
f^{(n)} = f^{(n)} f_1^{(n)}
$$

$$= \left(1 + \sum_i y_i E_i\right) f_1^{(n)}$$

Now $\left(\sum_i y_i E_i\right) f_1^{(n)} = 0$, and so the result then follows, i.e., $f^{(n)} = f_1^{(n)}$. ∎

> **Definition G.16** (Jones-Wenzl idempotents [52, p. 30])**.** The *Jones-Wenzl idempotents* are $f^{(n)} \in \mathcal{TL}_n$ are defined inductively such that
>
> (i) $f^{(0)} \in \mathcal{TL}_0$. is the empty diagram
>
> (ii) $f^{(1)} \in \mathcal{TL}_1$ is the single strand
>
> (iii) if $[n+1] \neq 0$, then
>
> $$f^{(n+1)} = i_n\left(f^{(n)}\right) - \frac{[n]}{[n+1]} i_n\left(f^{(n)}\right) E_n i_n\left(f^{(n)}\right).$$

To understand this better, we will need to use the diagrams. So, $f^{(n+1)}$ can be rewritten as,

$$\text{(JW)}$$

> **Example G.17.**
>
>

Now we consider mapping $q \in \{Q \cup -Q\}$ to $[2]$, and find that the map is bijective, thus our map uniquely determines q (see [52]). So we let $d = [2]$.

Proposition G.18 ([52, p. 31])**.** *Suppose $n \geq 0$ and $[1], \ldots, [n], [n+1] \neq 0$ so $f^{(0)}, \ldots, f^{(n+1)}$ are well-defined. Then, $f^{(n+1)}$ satisfies the following.*

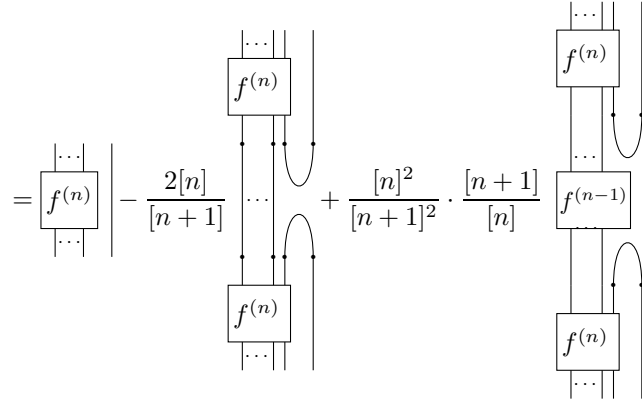(i) $f^{(n+1)} = \left(f^{(n+1)}\right)^* = \left(f^{(n+1)}\right)^2$

(ii) $\mathcal{E}_{n+1}\left(f^{(n+1)}\right) = \dfrac{[n+2]}{[n+1]} f^{(n)}$

(iii) $\left(i_n\left(f^{(n)}\right)\right) f^{(n+1)} = f^{(n+1)}\left(i_n\left(f^{(n)}\right)\right) = f^{(n+1)}$

(iv) $tr_{n+1}\left(f_{(n+1)}\right) = [n+2]$

(v) $E_j f^{(n+1)} = f^{(n+1)} E_j = 0$ *for all* $j = \{1, \ldots, n\}$

Proof. We show this by induction on n. This is obvious for $n = 0$. So suppose this is true for any n. Then, we need to show that it is also true for $n+1$.

(i)

$$\left(f^{(n+1)}\right)^2 = \left(\boxed{f^{(n)}} - \frac{[n]}{[n+1]} \ \parbox{2cm}{\centering $f^{(n)}$ \\ $f^{(n)}$} \right)^2$$

$$= \boxed{f^{(n)}\ f^{(n)}} - \frac{[n]}{[n+1]} - \frac{[n]}{[n+1]} + \frac{[n]^2}{[n+1]^2}$$

And by the inductive hypothesis, we have $\left(f^{(n)}\right)^2 = f^{(n)}$ and $\mathcal{E}_n\left(f^{(n)}\right) = \frac{[n+1]}{[n]} f^{(n-1)}$

And again by the inductive hypothesis, we know $f^{(n)}\left(i_{n-1}\left(f^{(n-1)}\right)\right) = \left(i_{n-1}\left(f^{(n-1)}\right)\right) f^{(n)} = f^{(n)}$. And so by stretching the E_n loops, it is easy to see that we get the same diagram as the second term. Equivalently we can use Proposition G.14 to see (our last diagram) $i_n\left(f^{(n)}\right) E_n f^{(n-1)} E_n i_n\left(f^{(n)}\right)$ is equivalent to $i_n\left(f^{(n)}\right) E_n i_n\left(f^{(n)}\right)$. Thus by further simplifying, we find that it is in fact idempotent, i.e., $\left(f^{(n+1)}\right)^2 = f^{(n+1)}$.

(ii) We will now use the results from Proposition G.14. So,

$$\mathcal{E}_{n+1}\left(f^{(n+1)}\right) = \mathcal{E}_{n+1}\left(i_n\left(f^{(n)}\right) - \frac{[n]}{[n+1]} i_n\left(f^{(n)}\right) E_n i_n\left(f^{(n)}\right)\right)$$

$$= [2]\mathrm{id}_n\left(f^{(n)}\right) + \frac{[n]}{[n+1]} f^{(n)} f^{(n)}$$

$$= [2]f^{(n)} + \frac{[n]}{[n+1]} f^{(n)}$$

and using the result "doubling the quantum integer" from Proposition G.3, we find $[2][n+1] = [n+2] + [n]$, and so the result becomes $\frac{[n+2]}{[n+1]} f^{(n)}$.

(iii) We use the results from Proposition G.14,

$$f^{(n+1)}\left(i_n\left(f^{(n)}\right)\right) = \left(i_n\left(f^{(n)}\right) - \frac{[n]}{[n+1]} i_n\left(f^{(n)}\right) E_n i_n\left(f^{(n)}\right)\right)\left(i_n\left(f^{(n)}\right)\right)$$

$$= \left(i_n\left(f^{(n)}\right)\right)\left(i_n\left(f^{(n)}\right)\right) - \frac{[n]}{[n+1]} i_n\left(f^{(n)}\right) E_n i_n\left(f^{(n)}\right) i_n\left(f^{(n)}\right)$$

$$= i_n\left(f^{(n)}\right) - \frac{[n]}{[n+1]} i_n\left(f^{(n)}\right) E_n i_n\left(f^{(n)}\right)$$

$$= f^{(n+1)}$$

(iv) Now from (ii), we have $\mathcal{E}_{n+1}\left(f^{(n+1)}\right) = \frac{[n+2]}{[n+1]} f^{(n)}$. So, then since $\mathrm{tr}_n \circ \mathcal{E}_{n+1} = \mathrm{tr}_{n+1}$, we have

$$\mathrm{tr}_{n+1}\left(f^{(n+1)}\right) = \mathrm{tr}_n \circ \mathcal{E}_{n+1}\left(f^{(n+1)}\right) = \frac{[n+2]}{[n+1]} \mathrm{tr}_n\left(f^{(n)}\right)$$

and by the inductive hypothesis, we have $\mathrm{tr}_n\left(f^{(n)}\right) = [n+1]$, and so

$$\mathrm{tr}_{n+1}\left(f^{(n+1)}\right) = [n+2]$$

(v) Finally, by the inductive hypothesis, for any $j = 1, \ldots n - 1$ we have $E_j f^{(n+1)} = f^{(n+1)} E_j$. So let $j = n$, then

$$E_n f^{(n+1)} = E_n i_n \left(f^{(n)} \right) - \frac{[n]}{[n+1]} E_n i_n \left(f^{(n)} \right) E_n i_n \left(f^{(n)} \right)$$

So, from Proposition G.14, we have $E_n i_n \left(f^{(n)} \right) E_n = \left(i_n \circ i_{n-1} \circ \mathcal{E}_n \left(f^{(n)} \right) \right) E_n$. And from (ii) we know $\mathcal{E}_n \left(f^{(n)} \right) = \frac{[n+1]}{[n]} f^{(n-1)}$, so then

$$E_n f^{(n+1)} = E_n i_n \left(f^{(n)} \right) - \frac{[n]}{[n+1]} \left(i_n \circ i_{n-1} \circ \mathcal{E}_n \left(f^{(n)} \right) \right) E_n i_n \left(f^{(n)} \right)$$

$$= E_n i_n \left(f^{(n)} \right) - \left(i_n \circ i_{n-1} \left(f^{(n-1)} \right) \right) E_n i_n \left(f^{(n)} \right)$$

Now we have $i_n \circ i_{n-1} \left(f^{(n-1)} \right) E_n i_n \left(f^{(n)} \right)$ is equivalent to $E_n i_n \left(f^{(n)} \right)$, and so the result then follows.

∎

Theorem G.19 ([52, p. 31]). *Suppose $tr_j(y^* x)$ is positive and semi-definite (i.e., $tr_j(x^* x) \geq 0$ and $tr_j(x^* x) = 0$ if and only if $x^* x = 0$) for all $j \geq 0$, $x, y \in \mathcal{TL}_n$. Then*

$$q + q^{-1} \in \{2\cos(\pi/n) : n \geq 3\} \cup [2, \infty)$$

Proof. We show this by contradiction. We will assume $q + q^{-1} \notin \{2\cos(\pi/n) : n \geq 3\} \cup [2, \infty)$. Using our Jones-Wenzl idempotents, we want to show that $tr_j((f^{(w)})^* f^{(w)}) = [w+1] < 0$ for some w.

If we let $q = e^{i\theta}$ for some $\theta \in (0, \pi/2)$ such that $\theta \neq \pi/n$ for some $\pi \geq 3$, then by recalling $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$, we have

$$[j] = \frac{(e^{i\theta})^j - (e^{i\theta})^{-j}}{(e^{i\theta}) - (e^{-i\theta})} = \frac{\sin(j\theta)}{\sin(\theta)}$$

Now by letting $k \geq 2$ be minimal such that $\theta > \pi/(k+1)$, we have

$$\frac{\pi}{k} > \theta > \frac{\pi}{k+1}$$

then since $\sin(\theta), \sin(2\theta), \ldots, \sin(k\theta) > 0$ but we have $\sin((k+1)\theta) < 0$ so then $[1], [2], \ldots, [k] > 0$, but $[k+1] < 0$.

So since $k \geq 2$ and $[1], [2], \ldots, [k] \neq 0$ we have our idempotents $f^{(0)}, \ldots, f^{(k)}$ are well-defined. Thus, we get our contradiction by taking the trace $tr_k \left((f^{(k)})^* f^{(k)} \right) = [k+1] < 0$. ∎

H   Jones' original construction

We will now construct our Jones polynomial using braids and the Temperley-Lieb algebra. This discussion is based on that in [1], [2], [3], [4], [12], [16], [17], [26], [28], [29], [33], [34], [35], [37], [38], [44], [49], [52], and [59].

To get invariance under conjugation (the Markov M1 move), we need to define a trace. To define a trace, we need a representation of $\mathcal{B}_n$ in a type II$_1$ factor since it is not finite (recall the motivation section §E.I).

Proposition H.1. *The homomorphism $\phi_n \colon \mathbb{C}[\mathcal{B}_n] \to \mathcal{TL}_n$ which is defined by*

$$\phi_n(\sigma_i) = A(1_n) + A^{-1}E_i \quad \text{and} \quad \phi_n(\sigma_i^{-1}) = A^{-1}(1_n) + AE_i,$$

is a representation of $\mathcal{B}_n$ (i.e., the homomorphism preserves the structure of $\mathcal{B}_n$), where $d = ([2]_q)_{-A^2=q}$.

Proof. If $|i - j| \geq 2$, then

$$
\begin{aligned}
\phi_n(\sigma_i\sigma_j) &= \phi_n(\sigma_i)\phi_n(\sigma_j) \\
&= (A(1_n) + A^{-1}E_i)(A(1_n) + A^{-1}E_j) \\
&= A^2 + E_j + E_i + A^{-2}E_iE_j \\
&= (A(1_n) + A^{-1}E_j)(A(1_n) + A^{-1}E_i) \\
&= \phi_n(\sigma_j)\phi_n(\sigma_i) \\
&= \phi_n(\sigma_j\sigma_i)
\end{aligned}
$$

since if $|i = j| > 1$, then $E_iE_j = E_jE_i$.

$$
\begin{aligned}
\phi_n(\sigma_i\sigma_{i+1}\sigma_i) &= \phi_n(\sigma_i)\phi_n(\sigma_{i+1})\phi_n(\sigma_i) \\
&= (A(1_n) + A^{-1}E_i)(A(1_n) + A^{-1}E_{i+1})(A(1_n) + A^{-1}E_i) \\
&= (A^2 + E_{i+1} + E_i + A^{-2}E_iE_{i+1})(A(1_n) + A^{-1}E_i) \\
&= A^3 + 2AE_i + AE_{i+1} + A^{-1}E_{i+1}E_i + A^{-1}E_i^2 + A^{-1}E_iE_{i+1} + A^{-3}E_iE_{i+1}E_i \\
&= A^3 + 2AE_i + AE_{i+1} + A^{-1}E_{i+1}E_i + A^{-1}dE_i + A^{-1}E_iE_{i+1} + A^{-3}E_i \\
&= A^3 + E_i(2A + A^{-1}d + A^{-3}) + AE_{i+1} + A^{-1}E_{i+1}E_i + A^{-1}E_iE_{i+1}
\end{aligned}
$$

Note that

$$\phi_n(\sigma_{i+1}\sigma_i\sigma_{i+1}) = A^3 + E_{i+1}(2A + A^{-1}d + A^{-3}) + AE_i + A^{-1}E_iE_{i+1} + A^{-1}E_{i+1}E_i$$

So, we need $d = -A^2 - A^{-2} = ([2]_q)_{-A^2=q}$.

Now

$$
\begin{aligned}
\phi_n(\sigma_i)\phi_n(\sigma_i^{-1}) &= (A(1_n) + A^{-1}E_i)(A^{-1}(1_n) + AE_i) \\
&= 1_n + A^2E_i + A^{-2}E_i + E_i^2 \\
&= 1_n + E_i(A^2 + A^{-2}) + dE_i = 1_n
\end{aligned}
$$

∎

The homomorphism ϕ_n defined above can be thought of as resolving crossings in braids to produce $\mathcal{TL}_n$ diagrams, i.e.,

$$\phi_n(\sigma_i) = \phi_n\left(\rtimes\right) = A\,\asymp + A^{-1}\,\smile$$

and

$$\phi_n(\sigma_i^{-1}) = \phi_n\left(\ltimes\right) = A\,\smile + A^{-1}\,\asymp$$

We need to take a composition of the maps ϕ_n and the Markov trace tr_n to map from $\mathbb{C}[\mathcal{B}_n] \to \mathbb{C}(q)$. Now let us explore the map $\mathrm{tr}_n \circ \phi_n$.

Proposition H.2. *$\mathrm{tr}_n \circ \phi_n(\rtimes) = -A^3d$, while $\mathrm{tr}_n \circ \phi_n(\ltimes) = -A^{-3}d$.*

Proof.

$$\mathrm{tr}_n \circ \phi_n \left(\bignozzle \right) = A \bigcirc\bigcirc + A^{-1} \bigcirc$$
$$= Ad^2 + A^{-1}d = A(d)^2 - A^{-1}d$$
$$= A(-A^{-2} - A^2)^2 + A^{-1}(-A^{-2} - A^2)$$
$$= A^5 + A = -A^3 d$$

And

$$\mathrm{tr}_n \circ \phi_n \left(\bignozzle \right) = A \bigcirc + A^{-1} \bigcirc\bigcirc$$
$$= Ad + A^{-1}d^2$$
$$= A^{-5} + A^{-1} = -A^{-3}d$$

■

Now $d^{-1}\mathrm{tr}_n \circ \phi_n$ is in fact the braid analog of the Kauffman bracket. This can easily be seen with the state sum solution. The states of a braid closure is given by resolving the crossings in the braid, then taking the trace. Thus

$$\langle \hat{b} \rangle = d^{-1}\mathrm{tr}_n \circ \phi_n(b) = \sum_{\text{states } s} A^{\alpha(s)} d^{|s|-1}$$

Thus we can then define the Jones polynomial to be the braid analog of the Kauffman bracket multiplied by $(-A^{-3})^{\omega(\hat{b})}$.

However, we want to show that it is indeed a link invariant without the Kauffman bracket. So we have the identity map of a braid $\mathrm{id}_n : \mathcal{B}_n \to \mathcal{B}_n$. Note that we have the following relations,

(i) $\phi_{n+1} \circ \mathrm{incl}_n(\alpha) = i_n \circ \phi_n(\alpha)$, where $\alpha \in \mathcal{B}_n$

(ii) $\mathrm{id}_n \circ \phi_n(\alpha) = \phi_n \circ \mathrm{id}_n(\alpha)$, where $\alpha \in \mathcal{B}_n$

This is true since, for instance, in the first relation, the inclusion of a braid would not change the $\mathcal{TL}_n$ representation. We draw the first relation,

$$\phi_{n+1} \circ \mathrm{incl}_n(b) = \boxed{\phi_n(b)} = i_n \circ \phi_n(b)$$

So we can thus use the relations from Proposition G.14.

Since any braid word $\alpha \in \mathcal{B}_n$ can be written with braid generators, we have the *writhe* defined on braids as the sum of the exponents of the generators.

$$\omega(\sigma_1^{i_1} \sigma_2^{i_2} \cdots \sigma_{n-1}^{i_j}) = \sum_z i_z$$

Equivalently, the writhe could also be defined as the writhe of the closure of the braid since an orientation is given with the braid closure.

Theorem H.3 (The original Jones polynomial)**.** *For a braid $b \in \mathcal{B}_n$. We can define the Jones polynomial link invariant as the following*

$$V(L) = V(\hat{b}) = \frac{(-A^{-3})^{\omega(b)}\, tr_n \circ \phi_n(b)}{d}.$$

Proof. We will prove this in two ways for the sake of argument and not necessity.

We will first show that this is in fact an invariant under the Markov moves, which in turn shows that this is invariant under the Reidemeister moves, making it a link invariant. And the second proof will show that this is obviously equivalent to our definitions of the Jones polynomial (Definition C.11 & Proposition C.20).

Invariance under Markov moves:

M1 Given any braid $\gamma \in \mathcal{B}_n$, we have $\omega(\gamma) = -\omega(\gamma^{-1})$. So $\omega(\gamma b \gamma^{-1}) = \omega(\gamma) + \omega(b) + \omega(\gamma^{-1}) = \omega(b)$. So,

$$
\begin{aligned}
V(\widehat{\gamma b \gamma^{-1}}) &= \frac{(-A^{-3})^{\omega(\gamma b \gamma^{-1})} \mathrm{tr}_n \circ \phi_n(\gamma b \gamma^{-1})}{d} \\
&= \frac{(-A^{-3})^{\omega(b)} \mathrm{tr}_n(\phi_n(\gamma) \phi_n(b) \phi_n(\gamma)^{-1})}{d} \\
&= \frac{(-A^{-3})^{\omega(b)} \mathrm{tr}_n(\phi_n(\gamma)^{-1} \phi_n(\gamma) \phi_n(b))}{d} = \frac{(-A^{-3})^{\omega(b)} \mathrm{tr}_n(\phi_n(b))}{d} = V(\hat{b})
\end{aligned}
$$

M2 Now given any braid $\sigma_n \in \mathcal{B}_{n+1}$, we have $\omega(\sigma_n) = 1$, and $\phi_{n+1}(\sigma_n) = A1_{n+1} + A^{-1}E_n$. We also note $\omega(\mathrm{incl}_n(b)) = \omega(b)$, so then

$$
\begin{aligned}
V(\widehat{\mathrm{incl}_n(b)\sigma_n}) &= \frac{(-A^{-3})^{\omega(b\sigma_n)} \mathrm{tr}_{n+1} \circ \phi_{n+1}(\mathrm{incl}_n(b)\sigma_n)}{d} \\
&= \frac{(-A^{-3})^{\omega(b)+1} \mathrm{tr}_{n+1}(\phi_{n+1}(\mathrm{incl}_n(b))(A + A^{-1}E_n))}{d}
\end{aligned}
$$

Recall the above relation $\phi_{n+1} \circ \mathrm{incl}_n = i_n \circ \phi_n$. We also know from Proposition G.14 that the Markov trace $\mathrm{tr}_{n+1} = \mathrm{tr}_n \circ \mathcal{E}_{n+1}$. So, then $\mathrm{tr}_n \circ \mathcal{E}_{n+1}(A \cdot i_n(\phi_n(b)) + A^{-1} \cdot i_n(\phi_n(b)) \cdot E_n)$. And again from Proposition G.14, we know $\mathrm{tr}_n \circ \mathcal{E}_{n+1} \circ i_n = d \cdot \mathrm{tr}_n \circ \mathrm{id}_n$, and $\mathrm{tr}_n(\mathcal{E}_{n+1}(i_n(\phi_n(b)) \cdot E_n)) = \mathrm{tr}_n(\phi_n(b))$. So,

$$
\begin{aligned}
&= \frac{(-A^{-3})^{\omega(b)+1} \mathrm{tr}_n(\phi_n(b))(Ad + A^{-1})}{d} \\
&= \frac{(-A^{-3})^{\omega(b)+1} \mathrm{tr}_n(\phi_n(b))(-A^{-3})}{d} \\
&= \frac{(-A^{-3})^{\omega(b)} \mathrm{tr}_n(\phi_n(b))}{d} = V(\hat{b})
\end{aligned}
$$

Analogously to the above, we find that under M2, where the braid becomes $b\sigma_n^{-1}$, the Jones polynomial $V(\widehat{b\sigma_n^{-1}}) = V(\hat{b})$.

Thus, we are done.

Skein relation:

Let us now ensure that this satisfies the skein relation definition. Consider single crossings in the braid closure,

$$
\begin{aligned}
A^4 V \left(\vcenter{\hbox{\includegraphics{}}} \right) - A^{-4} V \left(\vcenter{\hbox{\includegraphics{}}} \right) + (A^2 - A^{-2}) V \left(\vcenter{\hbox{\includegraphics{}}} \right) &= A^4 \frac{(-A^{-3})(-A^3 d)}{d} \\
&\quad - A^{-4} \frac{(-A^{-3})^{-1}(-A^{-3}d)}{d} + (A^2 - A^{-2})\frac{d^2}{d} \\
&= A^4 - A^{-4} + (A^2 - A^{-2})(-A^{-2} - A^2) = 0.
\end{aligned}
$$

This completes the proof. ∎

Example H.4. For example, recall that the closure of σ_1^3 is equivalent to the right-handed trefoil knot, while σ_1^{-3} is equivalent to the left-handed trefoil knot.

We can see then, that

$$
\begin{aligned}
V(3_1) &= V(\widehat{\sigma_1^3}) \\
&= \frac{(-A^{-3})^3 \mathrm{tr}_2 \circ \phi_2(\sigma_1^3)}{d} = \frac{-A^{-9}\mathrm{tr}_2(\phi_2(\sigma_1)^3)}{d} \\
&= \frac{-A^{-9}\mathrm{tr}_2(A + A^{-1}E_1)^3}{d} \\
&= \frac{-A^{-9}\mathrm{tr}_2(A^3 + 3AE_1 + 3A^{-1}E_1^2 + A^{-3}E_1^3)}{d} \\
&= \frac{-A^{-9}(A^3 d^2 + 3Ad + 3A^{-1}d^2 + A^{-3}d^3)}{d} \\
&= -A^{-6}d - 3A^{-8} - 3A^{-10}d - A^{-12}d^2 \\
&= (-A^2 - A^{-2})(-A^{-6} - 3A^{-10} - A^{-12}(-A^2 - A^{-2})) - 3A^{-8} \\
&= A^{-4} + A^{-12} - A^{-16}
\end{aligned}
$$

It is clear that the left-handed trefoil $V(\overline{3_1}) = V(\widehat{\sigma_1^{-3}})$, and the bracket polynomial would be $\phi_2(\sigma_1^{-1})^3 = (AE_1 + A^{-1})^3 = A^3 E_1^3 + 3AE_1^2 + 3A^{-1}E_1 + A^{-3}$, and the Markov trace of the bracket polynomial becomes $A^3 d^3 + 3Ad^2 + 3A^{-1}d + A^{-3}d^2$. And finally we have the Jones polynomial, $V(\overline{3_1}) = -A^9(d(A^3 d + 3A + A^{-3}) + 3A^{-1}$, which further simplified turns out to be $A^4 + A^{12} - A^{16}$.

To conclude, we see that the easiest to define (i.e., via the Kauffman bracket) is not necessarily the easiest to work with, while the most difficult to define (i.e., the original construction) is perhaps the easiest to work with.

H.I   Proving Theorem C.22

In this section we will prove $V(L_1 \cup L_2) = dV(L_1)V(L_2)$ and $V(L_1 \# L_2) = V(L_1)V(L_2)$, where $\#$ is the connected sum of the links. We will prove these results using braid representations instead of the skein relation and Kauffman bracket (where they can be derived easily). The reason we do this with braid representations is so that we can ensure comprehension.

We saw in Chapter D, that the braid inclusion map essentially adds a strand to the right. So now we define the braid left-inclusion map lfincl_n. The reason for this is to define the tensor product of two braids.

Definition H.5 (braid left-inclusion). The *braid left-inclusion map* $\mathrm{lfincl}_n \colon \mathcal{B}_n \to \mathcal{B}_{n+1}$ for $n \geq 1$ is defined by,

Figure H-i: left inclusion lfincl_n on $\mathcal{B}_n$

We similarly define the *left-inclusion* on $\mathcal{TL}_n$, i.e., $li\colon \mathcal{TL}_n \to \mathcal{TL}_{n+1}$.

Definition H.6 (braid tensor product). Let $\alpha \in \mathcal{B}_n$ and $\beta \in \mathcal{B}_m$. We define the *tensor product* $\otimes\colon \mathcal{B}_n \times \mathcal{B}_m \to \mathcal{B}_{n+m}$ by $\alpha \otimes \beta \mapsto \mathrm{incl}_{n\otimes m}(\alpha) \cdot \mathrm{lfincl}_{m\otimes n}(\beta)$, where the inclusion tensor product of the two integers is given by $\mathrm{incl}_{n\otimes m} = \mathrm{incl}_{n+m-1} \circ \mathrm{incl}_{n+m-2} \circ \cdots \circ \mathrm{incl}_n$ (the left inclusion is defined analogously). And so we thus have.
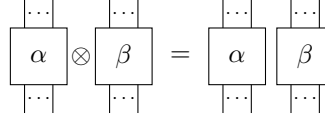
Figure H-ii: braid tensor product

So it is easy to see that the closure of the tensor product of two braids is essentially two disjoint links. For instance, let the two links have the following braid representations, $L_1 = \widehat{\alpha}$ and $L_2 = \widehat{\beta}$, so

$$L_1 \cup L_2 = \widehat{\alpha} \cup \widehat{\beta} = \widehat{\alpha \otimes \beta}$$

Remark. It is easy to see that the union of two closure braids is well-defined, i.e., $\widehat{\alpha \otimes \beta} \sim \widehat{\beta \otimes \alpha}$.

We then analogously define the *tensor product* on $\mathcal{TL}_n$, i.e., $\otimes\colon \mathcal{TL}_n \times \mathcal{TL}_m \to \mathcal{TL}_{n+m}$,
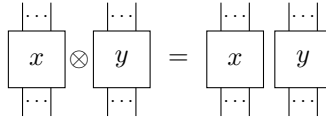
We have the relation $\phi_{n+m}(\alpha \otimes \beta) = \phi_n(\alpha) \otimes \phi_m(\beta)$, where $\alpha \in \mathcal{B}_n$ and $\beta \in \mathcal{B}_m$. This obviously says something along the lines of, "if two braids are disjoint, then so are their Temperley-Lieb representations".

Lemma H.7. *Let $\alpha \in \mathcal{B}_n$ and $\beta \in \mathcal{B}_m$, then we have $tr_{n+m} \circ \phi_{n+m}(\alpha \otimes \beta) = tr_n \circ \phi_n(\alpha) \cdot tr_m \circ \phi_m(\beta)$.*

Proof sketch. The diagrammatic sketch of this is pretty straight forward. Essentially, we need to show that the trace of disjoint $\mathcal{TL}$ diagrams are independent of each other. Note the relation above, $\phi_{n+m}(\alpha \otimes \beta) = \phi_n(\alpha) \otimes \phi_m(\beta)$.
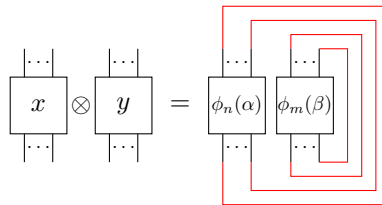
Which is clearly the same as $\mathrm{tr}_n \circ \phi_n(\alpha) \cdot \mathrm{tr}_m \circ \phi_m(\beta)$. ∎

Proof of Theorem C.22. Since $L_1 \cup L_2 = \widehat{\alpha \otimes \beta}$ for $\alpha \in \mathcal{B}_n$ and $\beta \in \mathcal{B}_m$, we then have

$$V\left(\widehat{\alpha \otimes \beta}\right) = \frac{(-A^{-3})^{\omega(\alpha)+\omega(\beta)}\mathrm{tr}_{n+m} \circ \phi_{n+m}(\alpha \otimes \beta)}{d}$$

$$= \frac{(-A^{-3})^{\omega(\alpha)+\omega(\beta)}\mathrm{tr}_n \circ \phi_n(\alpha) \cdot \mathrm{tr}_m \phi_m(\beta)}{d}$$

And so it is then clear that multiplying by dd^{-1} will give us the following,

$$V(L_1 \cup L_2) = dV(L_1)V(L_2)$$

And recall $d = [2]_{q=-A^2=-t^{-1/2}}$, and so the result follows.   ■

Definition H.8 (connected sum of links). $L_1 \# L_2$ is the *connected sum* of the links, and is shown below
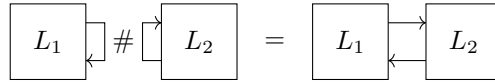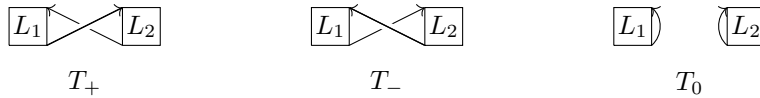

Figure H-iii: connected sum of links $\#$

It is easy to see that this is well-defined.

An almost direct corollary one can make is the following.

Corollary H.9. *For links L_1, L_2, we have $V(L_1 \# L_2) = V(L_1)V(L_2)$.*

Proof. We "glue" our link diagrams such that $T_+ \sim T_- \sim L_1 \# L_2$.

It is clear that we have $T_0 \sim L_1 \cup L_2$, so then

$$V(L_1 \# L_2)(t^{-1} - t) + d(t^{-1/2} - t^{1/2})V(L_1)V(L_2) = 0$$

And so $V(L_1 \# L_2) = V(L_1)V(L_2)$.   ■

We now show this result using braid representations. We first need to define what the connected sum of braids is.

Definition H.10 (braid connected sum). If $L_1 = \widehat{\alpha}$ and $L_2 = \widehat{\beta}$ for $\alpha \in \mathcal{B}_n$ and $\beta \in \mathcal{B}_m$. We have $L_1 \# L_2 = \widehat{\alpha \clubsuit \beta}$, where $\clubsuit \colon \mathcal{B}_n \times \mathcal{B}_m \to \mathcal{B}_{n+m-1}$ is given by

$$\alpha \clubsuit \beta = \mathrm{incl}_{n \otimes (m-1)}(\alpha) \cdot \mathrm{lfincl}_{m \otimes (n-1)}(\beta)$$
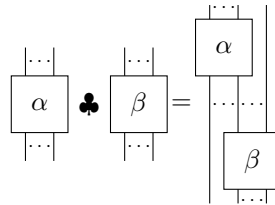
Figure H-iv: braid connected sum $\clubsuit$

This is well-defined, as the closure of $\alpha \clubsuit \beta$ can easily be seen to be ambient isotopic to the closure of $\beta \clubsuit \alpha$.

So now we have the following theorem.

Theorem H.11. *For $\alpha \in \mathcal{B}_n$ and $\beta \in \mathcal{B}_m$ we have,*

$$V(\widehat{\alpha \clubsuit \beta}) = V(\widehat{\alpha})V(\widehat{\beta})$$

To prove this, we begin with a lemma.

Lemma H.12. *The braid closure of $\alpha \clubsuit \beta$ is equivalent to the braid closure of $\alpha \heartsuit \beta$ where $\heartsuit \colon \mathcal{B}_n \times \mathcal{B}_m \to \mathcal{B}_{n+m}$ is given by $\alpha \heartsuit \beta = \sigma_n \cdot (\alpha \otimes \beta)$, where $\sigma_n \in \mathcal{B}_{n+m}$, $\alpha \in \mathcal{B}_n$ and $\beta \in \mathcal{B}_m$,*
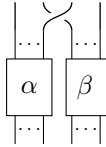
Proof. This is obviously well defined. We will show this diagrammatically to ensure that the braid closures are in fact ambient isotopic.

∎
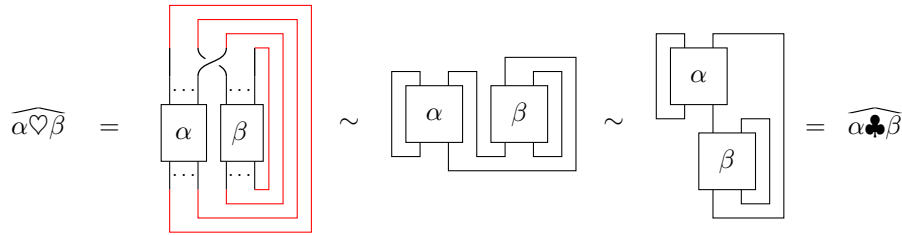
Proof sketch of Theorem H.11. By the lemma above, we know that finding the Jones polynomial of the closure of $\alpha \heartsuit \beta$ is enough.

So we show this by induction on m. For $m = 1$, β is the one strand (and so $V(\widehat{\beta}) = 1$). So $\alpha \otimes \beta = \mathrm{incl}_n(\alpha)$, and we know $\sigma_n \mathrm{incl}_n(\alpha)$ is Markov equivalent to α. So the result then follows.

Now suppose that this is true for $\alpha \in \mathcal{B}_n$ and $\beta \in \mathcal{B}_{m-1}$. So for $\alpha \in \mathcal{B}_n$ and $\beta \in \mathcal{B}_m$ we have

$$\phi_{n+m}(\sigma_n \cdot (\alpha \otimes \beta)) = (A + A^{-1}E_n)\phi_{n+m}(\alpha \otimes \beta)$$

And so we now take the trace of $E_n \phi_{n+m}(\alpha \otimes \beta)$ and find
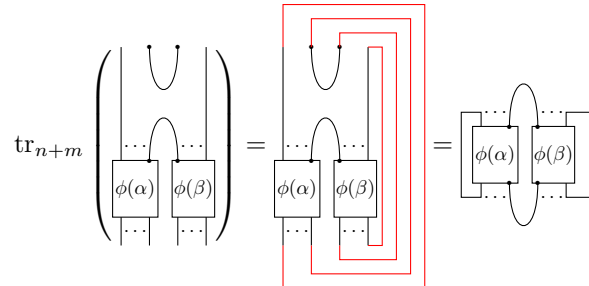
Now we will see all the different possibilities connecting E_n, by only looking at the connection between the four points. The first possibility, for instance, gives
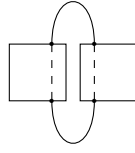
So, the trace of $\phi_{n+m}(\alpha \otimes \beta)$ without E_n would give d^2 (multiplied by the rest of the diagram), whereas $E_n\phi_{n+m}(\alpha \otimes \beta)$ gives d (multiplied by the rest of the diagram). The second and third possibilities are the same, but with opposite choices. So without loss of generality, we have,
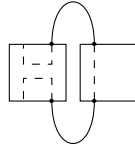
This clearly becomes a single strand. So the trace would give d. Whereas, the trace of $\phi_{n+m}(\alpha \otimes \beta)$ without E_n would give d^2. We also have,
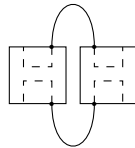
Similarly to the previous possibilities, we have the trace of $E_n\phi_{n+m}(\alpha \otimes \beta)$ giving d, and d^2 for the trace of $\phi_{n+m}(\alpha \otimes \beta)$ without E_n. We then have two other possibilities (i.e., both strands in a diagram are going down/up and all strands are going down/up) - the reader is advised to try them.

We see that it will always give us $\mathrm{tr}_{n+m}(E_n\phi_{n+m}(\alpha \otimes \beta)) = d^{-1}\mathrm{tr}_{n+m} \circ \phi_{n+m}(\alpha \otimes \beta)$. So then we have

$$\mathrm{tr}_{n+m}(E_n\phi_{n+m}(\alpha \otimes \beta)) = d^{-1} \cdot \mathrm{tr}_{n+m} \circ \phi_{n+m}(\alpha \otimes \beta)$$

The result then follows,

$$V(\widehat{\alpha\heartsuit\beta}) = \frac{(-A^{-3})^{\omega(\alpha)+\omega(\beta)+1}\mathrm{tr}_{n+m} \circ \phi_{n+m}(\alpha \otimes \beta)(A + A^{-1}d^{-1})}{d}$$

$$= \frac{(-A^{-3})^{\omega(\alpha)+\omega(\beta)}\mathrm{tr}_{n+m} \circ \phi_{n+m}(\alpha \otimes \beta)}{d^2} = V(\widehat{\alpha})V(\widehat{\beta})$$

∎

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