Assignment 1

Q1: Consider the field extension $F = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \cdots)$. Clearly, $F/\mathbb{Q}$ is an algebraic extension and $F \subsetneq \overline{\mathbb{Q}}$ since i is not in F. Given any positive integer n, we see that $\mathbb{Q}(\sqrt[n+1]{2}) \subset F$. The minimal polynomial of $\sqrt[n+1]{2}$ is $x^{n+1} - 2 \in \mathbb{Q}[x]$ — the irreducibility is given by Eisenstein's Criterion taking prime $p = 2$. Hence

$$[F : \mathbb{Q}] \geq [\mathbb{Q}(\sqrt[n+1]{2}) : \mathbb{Q}] = n + 1 > n.$$

Therefore, F is infinite dimensional over $\mathbb{Q}$.

Q2: Suppose that $F(\alpha) \neq F(\alpha^3)$. Clearly, $F(\alpha)/F(\alpha^3)$ is a finite extension. We see that α is a root of the polynomial $x^3 - \alpha^3 \in F(\alpha^3)[x]$, then the minimal polynomial of α over $F(\alpha^3)$ divides $x^3 - \alpha^3$. But α is not in $F(\alpha^3)$, so the minimal polynomial of α have degree 2 or 3 and hence $[F(\alpha) : F(\alpha^3)] = 2$ or 3. But then,

$$\begin{aligned}[K : F] &= [K : F(\alpha)][F(\alpha) : F(\alpha^3)][F(\alpha^3) : F] \\ &= 2[K : F(\alpha)][F(\alpha^3) : F] \text{ or } 3[K : F(\alpha)][F(\alpha^3) : F].\end{aligned}$$

But this contradicts to the assumption $[K : F]$ is relatively prime to 6. Hence we must have $F(\alpha) = F(\alpha^3)$.

Q3: Consider $F = \mathbb{Q}, K = \mathbb{Q}(\sqrt{2}), L = \mathbb{Q}(\sqrt[4]{2})$. We insist real roots so they are subfields of $\mathbb{R}$. We claim that L/K and K/F are normal but L/F is not.

The minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}$ is clearly $x^2 - 2 \in \mathbb{Q}[x]$, whose roots are $\pm\sqrt{2}$. But K contains both $\pm\sqrt{2}$ and so is the splitting field of $x^2 - 2$. Therefore, K/F is normal. Similarly, the minimal polynomial of $\sqrt[4]{2}$ over $\mathbb{Q}(\sqrt{2})$ is $x^2 - \sqrt{2}$, whose roots are $\pm\sqrt[4]{2}$ both lying in $\mathbb{Q}(\sqrt[4]{2})$. Hence L is the splitting field $x^2 - \sqrt{2}$ and so L/K is normal.

On the other hand, the minimal polynomial of $\sqrt[4]{2}$ over $\mathbb{Q}$ is $x^4 - 2$ — the irreducible is checked by Eisenstein's Criterion with prime $p = 2$. But the roots are then

$$\sqrt[4]{2}, -\sqrt[4]{2}, i\sqrt[4]{2}, -i\sqrt[4]{2}.$$

But $i\sqrt[4]{2}$ is not in E, then $x^4 - 2$ cannot fact completely in $\mathbb{Q}(\sqrt[4]{2})[x]$. So L/F is not normal.

Q4: Let F be perfect and E/F an algebraic extension. Let $f(x) \in E[x]$ be an irreducible polynomial. Assume α is a root of f in an algebraic closure $\overline{F}$ — note that $\overline{F}$ is also an algebraic closure of E hence we definitely can find such a root in $\overline{F}$. Let $f'(x) \in F[x]$ be the minimal polynomial of α over F. Since F is perfect, f' has no repeated root. Regarding f' as a polynomial in $E[x]$, we see f divides f' and hence has no repeat root as well. Therefore, f is separable and E is perfect.

As an counter example, the function field $\mathbb{F}_2(t)$ is not perfect — $\mathbb{F}_2(t^{1/2})/\mathbb{F}_2(t)$ is not a separable extension. The main difference is that we cannot find a root of a polynomial, say $X^2 - t \in \mathbb{F}_2(t)[X]$, in $\overline{\mathbb{F}_2}$.

Q5: Assume $[\overline{F} : F]$ is finite. Then $\overline{F}$ is also finite and has $|F|^{[\overline{F}:F]}$ elements. Consider the polynomial

$$f(x) = 1 + \prod_{a \in \overline{F}} (x - a) \in \overline{F}[x].$$

This is a well define polynomial since it is a product of finitely many terms. But $f(\alpha) = 1$ for every element $\alpha \in \overline{F}$, in other words, f has no root over $\overline{F}$. This contradicts to the definition of $\overline{F}$.

Q6: Take any element $\alpha \in E$. The minimal polynomial f_K of α over K is purely inseparable, namely, has one root only. But the minimal polynomial f_F of α over F divides f_K (by treating f_K as a polynomial in $F[x]$) and hence has one root only as well. In other words, f_F is purely inseparable and so E/F is purely inseparable.

Q7:

Q8: