Galois Theory
Assignment 1

Questions will be marked both on correctness and clarity of presentation. If outside resources are used in completing the assignment, references are expected.

1. Give an example of an algebraic extension $F$ of $\mathbb{Q}$ such that $F$ is infinite dimensional over $\mathbb{Q}$, and $F \neq \overline{\mathbb{Q}}$.

2. Let $K$ be a finite extension of $F$ such that $[K : F]$ is relatively prime to 6. Show that for all $\alpha \in K$, $F(\alpha) = F(\alpha^3)$.

3. Give an example of fields $F \subset K \subset L$ such that $K/F$ is a normal extension, $L/K$ is a normal extension, but $L/F$ is not a normal extension. Prove the normality and non-normality of these extensions directly (i.e. without reference to the fundamental theorem of Galois theory or normal subgroups).

4. Show that every algebraic extension of a perfect field is perfect. Give an example showing that an extension of a perfect field may not be perfect if we drop the "algebraic" assumption (this last part might be hard, since it requires the construction of a clever transcendental extension).

5. Suppose that $F$ is a finite field, and $\overline{F}$ its algebraic closure. Prove that $[\overline{F} : F]$ is infinite.

6. Suppose that $E/K$ is algebraic, $F$ is an intermediate field and $E/K$ is a purely inseparable extension. Show that $E/F$ is a purely inseparable extension.

7. Prove that if $F = K(\alpha, \beta)$ where $\alpha, \beta$ are algebraic over $K$ and $\beta$ is separable over $K$, then $F$ is a simple extension of $K$.

8. Suppose that $E/K$ is algebraic, and let $F$ denote the set of all elements of $E$ that are purely inseparable over $K$. Show that $E$ is an inseparable extension of $K$, and that $E \cap K_s = K$, where $K_s$ is the set of all elements of $E$ that are separable over $K$.