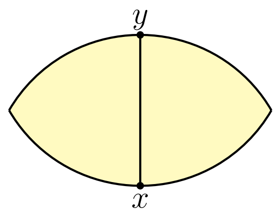
Q1: Up to rescaling, we can assume the greatest distance is 1. Let x, y be two points realizing distance 1. Then all other points must lies in the bigon, which is the intersection of two unit disk centered at x, y. If not, then the point outside the bigon would have distance > 1 from x or y. Similarly, if x, y, z are three distinct points with pairwise distance 1, then all other points must lies in the Reuleaux triangle with vertices x, y, z.



Let us proceed by induction on n. The statement is clearly true for $n = 1, 2, 3$ as $\binom{n}{2} \leq 3$ in these cases.

Q2: It suffices to prove the statement for the standard simplex $\Delta^n \mathbb{R}^{n+1}$, since every regular simplex can be mapped to the standard simplex by an affine map. Note Δ^n is the convex hull spanned the canonical basis

$$e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_{n+1} = (0, 0, \dots, 1).$$

Each e_i is a vertex of Δ^n and each $\mathrm{conv}(\{e_i, e_j : i \neq j\})$ is an edge. So there are $\binom{n+1}{2}$ the midpoints of the edges of Δ^n and they are of the form $\frac{1}{2}e_i + \frac{1}{2}e_j, i \neq j$. For any two midpoints $\frac{1}{2}e_i + \frac{1}{2}e_j, \frac{1}{2}e_{i'} + \frac{1}{2}e_{j'}$, either $\{i, j\} \neq \{i', j'\}$ or $i = i', j \neq j'$ after reordering. In the former case,

$$\mathrm{dist}(\frac{1}{2}e_i + \frac{1}{2}e_j, \frac{1}{2}e_{i'} + \frac{1}{2}e_{j'}) = ||\frac{1}{2}e_i + \frac{1}{2}e_j - \frac{1}{2}e_{i'} - \frac{1}{2}e_{j'}|| = 1;$$

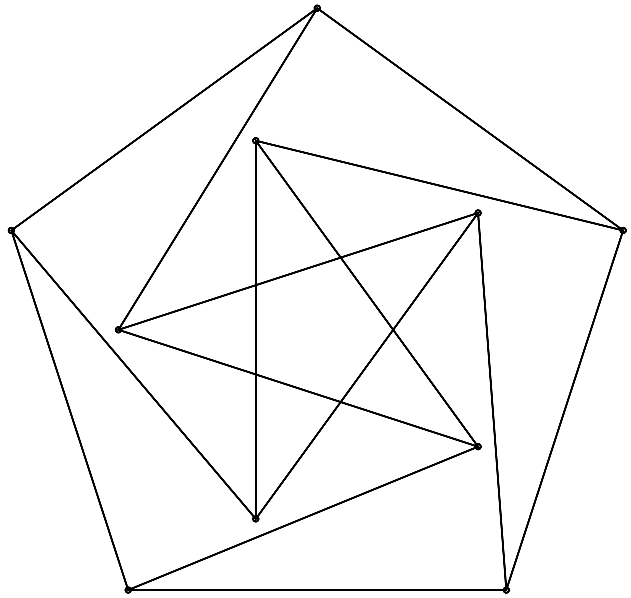
in the latter case

$$\mathrm{dist}(\frac{1}{2}e_i + \frac{1}{2}e_j, \frac{1}{2}e_{i'} + \frac{1}{2}e_{j'}) = ||\frac{1}{2}e_i + \frac{1}{2}e_j - \frac{1}{2}e_{i'} - \frac{1}{2}e_{j'}|| = ||\frac{1}{2}e_j - \frac{1}{2}e_{j'}|| = \frac{\sqrt{2}}{2}.$$

So these midpoints form a 2-distance set.

Q3: Let $R = \sqrt{\frac{5+\sqrt{5}}{10}}$ and $r = \sqrt{\frac{5-\sqrt{5}}{10}}$. Draw the graph on the complex plane and the vertices are

$$Re^{\frac{\pi i}{10}}, Re^{\frac{5\pi i}{10}}, Re^{\frac{9\pi i}{10}}, Re^{\frac{13\pi i}{10}}, Re^{\frac{17\pi i}{10}}, re^{\frac{\pi i}{5}}, re^{\frac{3\pi i}{5}}, re^{\pi i}, re^{\frac{7\pi i}{5}}, re^{\frac{9\pi i}{5}}.$$



Q4:

Q5: Note that $B(p, \epsilon)$ is contained in $reg(p)$ for sufficiently small $\epsilon > 0$. Suppose $reg(p) \subset \mathbb{R}^d$ is bounded. As the intersection of half-spaces containing an ϵ-ball, $reg(p)$ is a convex polyhedron and has at least $d+1$ vertices. Let V be the vertex set of $reg(p)$. By the Carathéodory's theorem, p can be written as a convex combination of some $v_1, v_2 \ldots, v_{d+1} \in V$, say

$$p = \sum_{i=1}^{d+1} \lambda_i v_i, \text{ where each } \lambda_i \geq 0 \text{ and } \sum_{i=1}^{d+1} \lambda_i = 1.$$

As a vertex in the Voronoi diagram, each vertex is the intersection of d hyperplanes bisecting and orthogonal to the line segments connecting some two points in P. Say v_i is the intersection of the hyperplanes bisecting and orthogonal to the line segments joining p to $x_{i,1}, x_{i,2}, \ldots, x_{i,d}$. Then, $\mathrm{dist}(v_i, p) = \mathrm{dist}(v_i, x_{i,j})$ for all $j = 1, \ldots, d$.

Now suppose $reg(p) \subset \mathbb{R}^d$ is unbounded.

Q6: (a). Denote by x_1, x_2, x_3 the centers of these three unit sphere. Then any point p in the intersection of these spheres is at distance 1 from each x_i. In particular, since p is of the same distance from x_1 and x_2, p lies on the plane bisecting and orthogonal to the line segment x_1x_2. Similarly, p lies on the plane bisecting and orthogonal to the line segment x_1x_3. The intersection of these two planes is a line and p must lie on this line. But there are at most two points on the line is of distance 1 from x_1. Hence, the intersection of three unit spheres consists of at most 2 points.

(b). Suppose we have n distinct points on $\mathbb{R}^3$ realizing the maximum number of unit distances. Let G be the graph such that the vertex set of G is these n points and two vertices are connected by an edge if and only if they are of unit distance from each other. Then G contains no subgraph isomorphic to $K_{3,3}$. Otherwise, there were three points in the intersection of three distinct unit spheres, violating part (a). So by the Kövári-Sós-Turán theorem, $|E(G)| = O(n^{2-1/3}) = O(n^{5/3})$.

To get the inequality in statement, we need a somewhat more precise estimate, *cf.* Theorem 17.2.5 on the lecture notes. Plainly, $U_3(n) \leq |E(G)| \leq ex(n, K_{3,3}) \leq (\frac{1}{2} + o(1))n^{5/3}$.