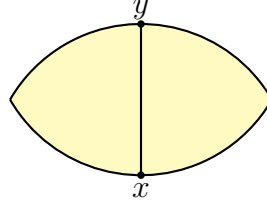
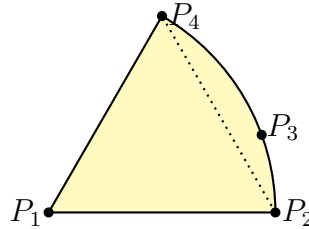


Q1: Up to rescaling, we can assume the greatest distance is 1. Let x, y be two points realizing distance 1. Then all other points must lie in the bigon, which is the intersection of two unit disk centered at x, y . If not, then the point outside the bigon would be at distance > 1 from x or y . Similarly, if x, y, z are three distinct points with pairwise distance 1, then all other points must lie in the Reuleaux triangle with vertices x, y, z .



Let us proceed by induction on n . The statement is clearly true for $n = 1, 2, 3$ as $\binom{n}{2} \leq n$ in these cases. Suppose the statement is true for any $n - 1$ points on the plane, $n \geq 4$. Let P_1, \dots, P_n be n distinct points on the plane with maximum distance 1, namely, $\text{diam}(\{P_1, \dots, P_n\}) = 1$. Let G be the graph with vertices P_1, \dots, P_n and edges pairs of vertices at distance 1. We want to show $|E(G)| \leq n$. If each vertex has most degree 2, then by the handshaking lemma, there are at most n edges, as required. If there exists some vertex has degree at least 3, then after reordering, we can assume this vertex is P_1 and three vertices connected to P_1 are P_2, P_3, P_4 . Note $\text{diam}(\{P_1, \dots, P_n\}) = 1$ and $\|P_1P_2\| = \|P_1P_4\| = 1$, so $\triangle P_1P_2P_4$ is acute. Reordering again if necessary, we can assume P_3 lies within the acute angle $\angle P_2P_1P_4$. Suppose a point P_i is at distance 1 from P_3 . Since $\|P_3P_i\| = \|P_1P_2\| = 1$, P_3P_i and P_1P_2 must intersect; if not, then some two points from P_1, P_2, P_3, P_i would be at distance > 1 . Similarly, P_3P_i and P_1P_4 must intersect. So P_3P_i intersects both P_1P_2 and P_1P_4 . It follows that $P_i = P_1$ is the only possibility. Plainly, P_1 is the only point at distance 1 from P_4 . With P_4 removed, we arrive at a subgraph with $n - 1$ vertices and one edge less from G . By induction hypothesis, the new graph has at most $n - 1$ edges. Hence, $|E(G)| \leq n$.



Q2: It suffices to prove the statement for the standard simplex $\Delta^n \subset \mathbb{R}^{n+1}$, since every regular simplex can be transformed into the standard simplex by an affine map after changing the ambient space dimension if necessary. Note Δ^n is the convex hull spanned the canonical basis of \mathbb{R}^{n+1} :

$$e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_{n+1} = (0, 0, \dots, 1).$$

Each e_i is a vertex of Δ^n and each $\text{conv}(\{e_i, e_j : i \neq j\})$ is an edge. So there are $\binom{n+1}{2}$ the midpoints of the edges of Δ^n and they are $\{\frac{1}{2}e_i + \frac{1}{2}e_j : 1 \leq i \neq j \leq n+1\}$. For any two distinct midpoints $\frac{1}{2}e_i + \frac{1}{2}e_j, \frac{1}{2}e_{i'} + \frac{1}{2}e_{j'}$, either i, j, i', j' are all distinct or $i = i', j \neq j'$ after reordering. In the former case,

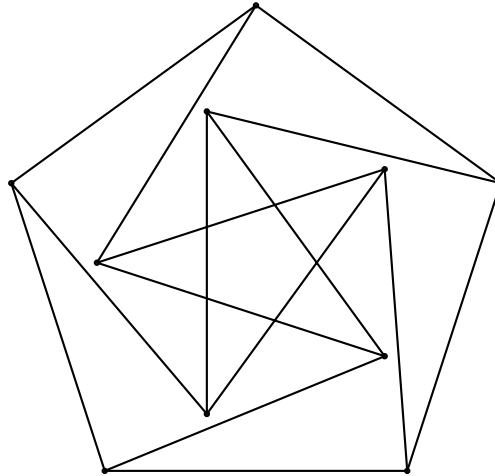
$$\text{dist}(\frac{1}{2}e_i + \frac{1}{2}e_j, \frac{1}{2}e_{i'} + \frac{1}{2}e_{j'}) = \|\frac{1}{2}e_i + \frac{1}{2}e_j - \frac{1}{2}e_{i'} - \frac{1}{2}e_{j'}\| = 1;$$

in the latter case

$$\text{dist}(\frac{1}{2}e_i + \frac{1}{2}e_j, \frac{1}{2}e_{i'} + \frac{1}{2}e_{j'}) = \|\frac{1}{2}e_i + \frac{1}{2}e_j - \frac{1}{2}e_{i'} - \frac{1}{2}e_{j'}\| = \|\frac{1}{2}e_j - \frac{1}{2}e_{j'}\| = \frac{\sqrt{2}}{2}.$$

So these midpoints form a 2-distance set.

Q3: Let $R = \sqrt{\frac{5+\sqrt{5}}{10}}$ and $r = \sqrt{\frac{5-\sqrt{5}}{10}}$. Draw the graph on the complex plane and the vertices are $Re^{\frac{\pi i}{10}}, Re^{\frac{5\pi i}{10}}, Re^{\frac{9\pi i}{10}}, Re^{\frac{13\pi i}{10}}, Re^{\frac{17\pi i}{10}}, re^{\frac{\pi i}{5}}, re^{\frac{3\pi i}{5}}, re^{\pi i}, re^{\frac{7\pi i}{5}}, re^{\frac{9\pi i}{5}}$.



Q4: (a) We prove something stronger:¹

Theorem 0.1 (Oddtown theorem). *If \mathcal{C} is a collection of subsets of $[n]$ such that each subset has an odd cardinality and all pairwise intersections have even cardinalities, then $|\mathcal{C}| \leq n$.*

In this problem's case, each subset has cardinality 3, which is odd, and the total number of subsets is $n + 1 > n$. It follows that there must be two sets A_i and A_j with $|A_i \cap A_j|$ odd, hence $|A_i \cap A_j| = 1$.

Proof of the oddtown theorem. For each subset $x \subset [n]$, identify x with a vector $f(x) = (x_1, \dots, x_n) \in \mathbb{F}_2^n$, where $x_i = 1$ if $i \in x$ and $x_i = 0$ if $i \notin x$. Indeed, f is a bijection from $2^{[n]}$ to \mathbb{F}_2^n . Now for $x, y \in \mathbb{F}_2^n$, $\langle x, y \rangle$ means the number (modulo 2) of coordinates that is 1 in both x and y . For each $c \in \mathcal{C}$, that c is of odd size is to say $\langle f(c), f(c) \rangle = 1$; for two different $c, d \in \mathcal{C}$, that $|c \cap d|$ is even is to say $\langle f(c), f(d) \rangle = 0$. Therefore, $f(\mathcal{C})$ is orthonormal and hence linearly independent. It follows that $|\mathcal{C}| = |f(\mathcal{C})| \leq \dim(\mathbb{F}_2^n) = n$. \square

(b) It is clear the statement is true for $n \leq 4$ as then $\frac{(n-1)(n-2)}{6} \leq 1$. Assume $n \geq 5$ from now on. Every point $x = (x_1, \dots, x_n) \in V_n$ can be identified with a 3-subset of $[n]$, which is $\{i \in [n] : x_i = 1\}$. Denote the identified set by $S(x)$ for each point x . (So S is the inverse function of f in part (a).) Note for any two points $x, y \in V_n$, $\|x - y\|_2 = 2$ if and only if x, y differs at exactly 4 coordinates if and only if $|S(x) \cap S(y)| = 1$. For each integer $i \in [n]$, there are $\binom{n-1}{4} \binom{4}{2} / 2$ such (unordered) pairs of points x, y that $S(x) \cap S(y) = \{i\}$. And so

$$|E_n| = 3n \binom{n-1}{4} \text{ and } |V_n| = \binom{n}{3}.$$

By part (a), for every $n + 1$ points of V_n , say a_1, \dots, a_{n+1} , there are i, j such that $|S(a_i) \cap S(a_j)| = 1$, that is, a_i and a_j are connected by an edge. So the independence number $\alpha(G) \leq n$. It follows that

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)} \geq \frac{\binom{n}{3}}{n} = \frac{(n-1)(n-2)}{6}$$

as required.

(c) It is clear the statement is true for $n \leq 4$ as then $\frac{(n-1)(n-2)}{6} \leq 1$. Assume $n \geq 5$. The graph described in part (b) can be drawn in \mathbb{E}^n . The problem is that each edge is of length 2. But this can be resolved by a rescaling $f : \mathbb{E}^n \rightarrow \mathbb{E}^n, v \mapsto \frac{1}{2}v$. Now there is an edge in if and only if x and y differs at 4 coordinates and hence $\|f(x) - f(y)\| = 1$. That is $f(G)$ is a unit distance graph in \mathbb{E}^n . Hence, $\chi(\mathbb{E}^d) \geq \chi(G) \geq \frac{(n-1)(n-2)}{6}$ as required.

¹Since I know a little bit about the sunflower lemma, it is not of surprise that I know this result as well.

Q5: Note that $B(p, \epsilon)$ is contained in $\text{reg}(p)$ for sufficiently small $\epsilon > 0$.

Suppose p lies on the surface of $\text{conv}(P)$. This implies that there is a hyperplane h such that $p \in h$ and $\text{conv}(P)$ lies entirely in one of the half-spaces determined by h . Assume the hyperplane is given by

$$h = \{x \in \mathbb{R}^d : \langle x, n \rangle = b\}$$

for some $n \in \mathbb{R}^d$ and $b \in \mathbb{R}$, and changing n to $-n$ if necessary, we can also assume that $\langle x, n \rangle \leq b$ for all $x \in \text{conv}(P)$. We claim that $p + tn \in \text{reg}(p)$ for all positive number t and hence $\text{reg}(p)$ is unbounded. To see the claim, we have the following inequality for all $q \in P$ and $t \geq 0$:

$$\begin{aligned} \|q - (p + tn)\|^2 - \|p - (p + tn)\|^2 &= \langle q - (p + tn), q - (p + tn) \rangle - \langle tn, tn \rangle \\ &= \langle (q - p) - tn, (q - p) - tn \rangle - \langle tn, tn \rangle \\ &= \langle q - p, q - p \rangle - 2\langle q - p, tn \rangle \\ &= \|q - p\|^2 + 2t\langle p, n \rangle - 2t\langle q, n \rangle \\ &= \|q - p\|^2 + 2tb - 2t\langle q, n \rangle \\ &\geq \|q - p\|^2 + 2tb - 2tb = \|q - p\|^2 \geq 0. \end{aligned}$$

It follows that $\|q - (p + tn)\| \geq \|p - (p + tn)\|$ for all $q \in P$ and so $p + tn \in \text{reg}(p)$ as claimed.

For the reverse direction, suppose $\text{reg}(p)$ is unbounded. As the intersection of half-spaces, $\text{reg}(p)$ is a convex polyhedron. Since $\text{reg}(p)$ is convex and unbounded, there exists a ray with initial point p lying entirely in $\text{reg}(p)$. Say the ray can be parametrized as $p + tn$ for some $n \in \mathbb{R}^d$ and parameter $t \in \mathbb{R}_{\geq 0}$. Set $b = \langle p, n \rangle$. We claim that the hyperplane

$$h = \{x \in \mathbb{R}^d : \langle x, n \rangle = b\}$$

intersects with $\text{conv}(P)$ at a face of $\text{conv}(P)$, or equivalently, $\text{conv}(P) \subset h_+ = \{x \in \mathbb{R}^d : \langle x, n \rangle \leq b\}$ and $\text{conv}(P) \cap h_+ \neq \emptyset$. It is clear $\text{conv}(P) \cap h_+ \neq \emptyset$ as $p \in h_+$ and hence p lies on surface of $\text{conv}(P)$ if the claim is true. To prove the claim, we note that if $P \subset h_+$, then $\text{conv}(P) \subset h_+$ as h_+ is convex, and so it suffices to show $P \subset h_+$. We suppose by contradiction that there is some $q \in P$ such that q is not in h_+ , that is, $\langle q, n \rangle > b$. Note $q \neq p$. Then by exactly the same calculation as above, we have

$$\|q - (p + tn)\|^2 - \|p - (p + tn)\|^2 = \|q - p\|^2 + 2tb - 2t\langle q, n \rangle = \|q - p\|^2 - 2t(\langle q, n \rangle - b).$$

Since $\langle q, n \rangle > b$, we have $\langle q, n \rangle - b > \epsilon > 0$ for some ϵ . Take some $t > \frac{\|q - p\|^2}{2\epsilon}$ and we have

$$\|q - (p + tn)\|^2 - \|p - (p + tn)\|^2 = \|q - p\|^2 - 2t(\langle q, n \rangle - b) < \|q - p\|^2 - 2 \frac{\|q - p\|^2}{2\epsilon} \epsilon = 0.$$

But this means $\|q - (p + tn)\| < \|p - (p + tn)\|$ and so $p + tn \notin \text{reg}(p)$, contradicting the fact the ray $\{p + tn : t \geq 0\}$ lies entirely in $\text{reg}(p)$.

Q6: (a). Denote by x_1, x_2, x_3 the centers of these three unit sphere. Then any point p in the intersection of these three spheres is at distance 1 from each x_i . In particular, since p is of the same distance from x_1 and x_2 , p lies on the plane bisecting and orthogonal to the line segment x_1x_2 . Similarly, p lies on the plane bisecting and orthogonal to the line segment x_1x_3 . The intersection of these two planes is a line and p must lie on this line. But there are at most two points on the line at distance 1 from x_1 . Hence, the intersection of three unit spheres consists of at most 2 points.

(b). Suppose we have n distinct points on \mathbb{R}^3 realizing the maximum number of unit distances. Let G be the graph such that the vertex set of G is these n points and two vertices are connected by an edge if and only if they are at unit distance from each other. Then G contains no subgraph isomorphic to $K_{3,3}$. Otherwise, there were three points in the intersection of three distinct unit spheres, violating part (a). So by the Kővári-Sós-Turán theorem, $|E(G)| = O(n^{2-1/3}) = O(n^{5/3})$.

To get the inequality in statement, we need a somewhat more precise estimate, cf. Theorem 17.2.5 on the lecture notes. Plainly, $U_3(n) = |E(G)| \leq ex(n, K_{3,3}) \leq (\frac{1}{2} + o(1))n^{5/3}$.