To clarify, 0 in my solutions can mean the real number 0 or the zero vector or the origin point in the Euclidean space. One might easily judge the meaning from the context. I will not distinguish a point and the vector from 0 to it either.

Q1: We choose the order

$$000 < 010 < 110 < 100 < 101 < 111 < 011 < 001,$$

where each binary is a vertex of the hypercube with the obvious identification, for example $000 = (0, 0, 0)$. Note the Hamming distance for adjacent vertices at above ordering is 1. We know that the 6 facets of P are of the form $\{0 * *\}, \{1 * *\}, \{*0*\}, \{*1*\}, \{* * 0\}$ and $\{* * 1\}$, *cf.* Q6 if you want. Then it is routine to check the Gale's evenness condition for each facet, where the vertices of each facet are in bold font:

(1) Facet $\{0 * *\}$: **000** $<$ **010** $< 110 < 100 < 101 < 111 <$ **011** $<$ **001**;
(2) Facet $\{1 * *\}$: $000 < 010 <$ **110** $<$ **100** $<$ **101** $<$ **111** $< 011 < 001$;
(3) Facet $\{*0*\}$: **000** $< 010 < 110 <$ **100** $<$ **101** $< 111 < 011 <$ **001**;
(4) Facet $\{*1*\}$: $000 <$ **010** $<$ **110** $< 100 < 101 <$ **111** $<$ **011** < 001;
(5) Facet $\{* * 0\}$: **000** $<$ **010** $<$ **110** $<$ **100** $< 101 < 111 < 011 < 001$;
(6) Facet $\{* * 1\}$: $000 < 010 < 110 < 100 <$ **101** $<$ **111** $<$ **011** $<$ **001**.

Q2: By the Centerpoint theorem, we know there is a center point x_i for each set A_i. Now if $x_1, \ldots, x_k$ are affinely independent, then there is a unique $(k-1)$-flat containing all x_i; if not, then the $(k-1)$-flat is not unique, but we can still find one. Denoting by P the/a $(k-1)$-flat containing all x_i, we claim P is the desired affine space. Let S be any hyperplane containing P. Then in particular, S contains all x_i. By the definition of a centerpoint, each of the closed half-space determined by S contains $\frac{1}{d+1}|A_i|$ points from A_i as it contains x_i. We are done.

Q3: Take $Y = (y_1, \dots, y_d) \in C^*$. Then by definition, we have $Y \cdot X \leq 1$ for any $X \in C$. In particular, take

$$X_1^+ = (1, 0, \dots, 0), X_2^+ = (0, 1, \dots, 0), \dots, X_d^+ = (0, 0, \dots, 1)$$

and

$$X_1^- = (-1, 0, \dots, 0), X_2^- = (0, -1, \dots, 0), \dots, X_d^- = (0, 0, \dots, -1),$$

from C. Then for each i, we have

$$Y \cdot X_i^\pm = \pm y_i \leq 1,$$

that is, $-1 \leq y_i \leq 1$. So $Y \in \{\mathbf{x} \in \mathbb{R}^d : \max |x_i| \leq 1\}$. Therefore, $C^* \subset \{\mathbf{x} \in \mathbb{R}^d : \max |x_i| \leq 1\}$.

To see the reverse inclusion, take $Y = (y_1, \dots, y_d) \in \{\mathbf{x} \in \mathbb{R}^d : \max |x_i| \leq 1\}$. Then $-1 \leq y_i \leq 1$ for each i. For any $X = (x_1, \dots, x_d) \in C$, we have

$$\begin{aligned}
Y \cdot X &= y_1 x_1 + y_2 x_2 + \cdots + y_d x_d \\
&\leq |y_1 x_1 + y_2 x_2 + \cdots + y_d x_d| \\
&\leq |y_1 x_1| + |y_2 x_2| + \cdots + |y_d x_d| \\
&= |y_1||x_1| + |y_2||x_2| + \cdots + |y_d||x_d| \\
&\leq |x_1| + |x_2| + \cdots + |x_d| \leq 1.
\end{aligned}$$

This says $Y \in C^*$ and so $\{\mathbf{x} \in \mathbb{R}^d : \max |x_i| \leq 1\} \subset C^*$.

Therefore, $C^* = \{\mathbf{x} \in \mathbb{R}^d : \max |x_i| \leq 1\}$.

Q4: Pick any two $f, g \in X^*$. Then for all $x \in X$ and $\lambda \in [0,1]$, we have

$$x \cdot (\lambda f + (1-\lambda)g) = \lambda x \cdot f + (1-\lambda) x \cdot g \leq \lambda + (1-\lambda) = 1.$$

This says X^* is convex whatever X is. And $(X^*)^*$ is also convex since it is a dual.

Take $x \in X$. Then for any $y \in X^*$, we have $x \cdot y \leq 1$ by the definition of X^*. And so $x \in (X^*)^*$ and it follows that $X \subset (X^*)^*$.

To see the reverse inclusion, we show that if $x \notin X$, then $x \notin (X^*)^*$. Note X is closed and convex with $0 \in X$. If $x \notin X$, then by the separation theorem, there is a hyperplane h separating X and $\{x\}$. To be more precise, there is some $b \in \mathbb{R}^d$ and a constant $c \in \mathbb{R}$ such that $h = \{z \in \mathbb{R}^d : b \cdot z = c\}$ and X and x are contained in the different open half-spaces determined h. Since $0 \in X$, we must have $c \neq 0$. Setting $b' = b/c$, we see that $h = \{z \in \mathbb{R}^d : b' \cdot z = 1\}$. Moreover, as $0 \in X$, $X \subset \{z \in \mathbb{R}^d : b' \cdot z < 1\}$ and $x \in \{z \in \mathbb{R}^d : b' \cdot z > 1\}$. The first inclusion means that $b' \in X^*$ and the second $x \notin (X^*)^*$.

Therefore, $X = (X^*)^*$ for a polytope X with $0 \in X$.

Q5: Say the d-simplex Δ in $\mathbb{R}^d$ is constructed from $d+1$ affinely independent points $x_1, \ldots, x_{d+1}$, namely, $d+1$ points in general positions, and so

$$\Delta = \{\sum_{i=1}^{d+1} \lambda_i x_i : \text{each } \lambda_i \geq 0 \text{ and } \sum_{i=1}^{d+1} \lambda_i = 1\} = \mathrm{conv}(\{x_1, \ldots, x_{d+1}\}).$$

We aim to find $d+1$ half-spaces such that Δ is their intersection. Note for each i, the set $\{x_j : j \neq i\}$ is also affinely independent and hence it determines a hyperplane

$$P_i = \{ \sum_{j=1, j\neq i}^{d+1} \lambda_j x_j : \sum_{j=1, j\neq i}^{d+1} \lambda_j = 1\}.$$

Observe that the point x_i cannot lie in P_i; otherwise, $x_1, \ldots, x_{d+1}$ are not affinely independent. Hence, x_i must lie in one of the closed half-spaces determined by P_i and let us call it P_i^+, indeed,

$$P_i^+ = t(x_i - x_{i+1}) + P_i = \{t(x_i - x_{i+1}) + \sum_{j=1, j\neq i}^{d+1} \lambda_j x_j : \sum_{j=1, j\neq i}^{d+1} \lambda_j = 1, t \geq 0\},$$

where $i+1$ is taken module $d+1$ and the formula has the meaning that it is the addition of the plane P_i with some vector of the same direction as the vector $x_i - x_{i+1}$. We can write it in a nicer way

$$P_i^+ = \{\sum_{j=1}^{d+1} \lambda_j x_j : \lambda_i \geq 0 \text{ and } \sum_{j=1}^{d+1} \lambda_j = 1\}.$$

Then P_i^+'s are the desired half-spaces. Indeed, we can see

$$\begin{aligned}
\cap_{i=1}^{d+1} P_i^+ &= \cap_{i=1}^{d+1} \{\sum_{j=1}^{d+1} \lambda_j x_j : \lambda_i \geq 0 \text{ and } \sum_{j=1}^{d+1} \lambda_j = 1\} \\
&= \{\sum_{j=1}^{d+1} \lambda_j x_j : \lambda_1, \ldots, \lambda_{d+1} \geq 0 \text{ and } \sum_{j=1}^{d+1} \lambda_j = 1\} \\
&= \Delta.
\end{aligned}$$

Q6: We can write the hypercube as $P = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1\}$. The vertices have each coordinate either 0 or 1, for example, $(0, 1, 0)$ is a vertex. We write for short each vertex as a binary number, for example, we write $000 = (0, 0, 0)$. Then then the vertices are $000, 001, 010, 011, 100, 101, 110, 111$.

The obvious d-faces of hypercube are then the hypercube itself when $d = 3$ and the vertices when $d = 0$. Now for $d = 2$, we claim the the 6 facets of P are of the form $\{0{*}{*}\}, \{1{*}{*}\}, \{{*}0{*}\}, \{{*}1{*}\}, \{{*}{*}0\}$ and $\{* * 1\}$, where $*$ is either 0 or 1 and we take the convex hull of these four points. More precisely, by the notation $\{0 * *\}$, we mean

$$\mathrm{conv}(\{(0, y, z) : y, z = 0, 1\}),$$

which is easily seen to be a square, and similar for other notations. Now associated with $\{0 * *\}, \{1 * *\}$ are the hyperplanes $x = 0$ and $x = 1$. By definition, all vertices of $\{0 * *\}$ (resp. $\{1 * *\}$) lies on the plane $x = 0$ (resp. $x = 1$) and the intersection $P \cap \{x = 0\}$ (resp. $P \cap \{x = 1\}$) is precisely $\{(x, y, z) \in P : x = 0\}$ (resp. $\{(x, y, z) \in P : x = 1\}$) and is indeed the convex hull spanned by vertices $\{0 * *\}$ (resp. $\{1 * *\}$). And P lies at one of the half spaces determined by $x = 0$ (resp. $x = 1$), namely, the half-space $x \geq 0$ (resp. $x \leq 1$). Hence $\{0 * *\}$ and $\{1 * *\}$ are facets. The same argument applies for other facets $\{{*}0{*}\}, \{{*}1{*}\}, \{* * 0\}, \{* * 1\}$. For $d = 1$, they are also the 1-face of the facets, and they are the line segments joining two vertices with hamming distance 1. If you insist I shall find the supporting plane associated to each edge, let me find one for you. Consider the line segment joining $(0, 0, 0)$ and $(1, 0, 0)$ and the plane $y + z = 0$. Since $(0, 0, 0)$ and $(1, 0, 0)$ lies one the plane so the line segment, and indeed $P \cap \{y + z = 0\} = \{(x, y, z) \in P : y = z = 0\}$ and P lies entirely in the half-space $y + z \geq 0$. The same story happens for other edges. And as for vertices, 0-faces, an example of supporting plane is $x + y + z = 0$ where $P \cap \{x + y + z = 0\} = 000$ and P lies in $x + y + z \geq 0$.

Q7: (a): As the convex hull of a finite set, it is clearly a polytope. It is left to show the dimension of the convex hull is d.

For any $Y = (y_1, \dots, y_{d+1}) \in V$, note that $\sum_{i=1}^{d+1} y_i = 1 + 2 + \cdots + (d+1) = \frac{(d+1)(d+2)}{2}$, that is, all points Y lies on the hyperplane $\sum_{i=1}^{d+1} x_i = \frac{(d+1)(d+2)}{2}$. As the hyperplane is convex, it contains $\mathrm{conv}(V)$. So $\dim(\mathrm{conv}(V)) \leq d$.

Now it suffices to find $d+1$ points from V that are affinely independent. We claim

$$\begin{aligned}
X_1 &= (1, 2, 3 \dots, d+1), \\
X_2 &= (2, 1, 3, \dots, d+1), \\
X_3 &= (1, 3, 2, \dots, d+1), \\
&\dots \\
X_{d+1} &= (1, 2, 3, \dots, d+1, d),
\end{aligned}$$

where X_i is obtained by applying the transposition $(i-1 \quad i)$ on $(1, 2, \dots, d+1)$ for $i \neq 1$, are the desired points. To see that, we have

$$X_i - X_1 = (0, \dots, 0, 1, -1, 0, \dots, 0),$$

that is, $(i-1)$-th coordinate is -1 and i-th coordinate is 1 and the other coordinates are 0. Consider $X_2 - X_1, X_3 - X_1, \dots, X_{d+1} - X_1$. If

$$\sum_{i=2}^{d+1} \lambda_i (X_i - X_1) = \lambda_2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \lambda_{d+1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix} = 0,$$

then reading coordinate-wisely we have

$$\lambda_2 = 0, -\lambda_2 + \lambda_3 = 0, \dots, -\lambda_{i-1} + \lambda_i = 0, \dots, -\lambda_d + \lambda_{d+1} = 0, -\lambda_{d+1} = 0,$$

where the only solution is $\lambda_2 = \cdots = \lambda_{d+1} = 0$. Hence, $X_2 - X_1, X_3 - X_1, \dots, X_{d+1} - X_1$ are linearly independent and so $X_1, \dots, X_{d+1}$ are affinely independent.

Remark: If one has some knowledge with the fast Fourier transform, in particular, the circulant matrix (https://en.wikipedia.org/wiki/Circulant_matrix), then he/she can see immediately that the rotations of $(1, 2, \dots, d+1)$ are *linearly* independent and concludes the convex hull is of dimension $\geq d$.

(b): We need to show every point in V is an extremal point. Fix $Y \in V$. Now after re-indexing the coordinates, we can assume $Y = (1, 2, \dots, d+1)$. Suppose that $Y \in \mathrm{conv}(V \backslash \{Y\})$, then by Carathéodory's theorem, we can find $X_1, \dots, X_{d+1} \in V \backslash \{Y\}$ such that Y is a convex combination of these $d+1$ points, namely, we can find $\lambda_i \geq 0$ with $\sum_{i=1}^{d+1} \lambda_i = 1$ such that

$$Y = \lambda_1 X_1 + \cdots + \lambda_{d+1} X_{d+1}. \tag{1}$$

Write $X_i = (x_1^i, x_2^i, \dots, x_{d+1}^i)$. Now reading coordinate-wisely of Equation 1, we have

$$\lambda_1 x_j^1 + \lambda_2 x_j^2 + \cdots + \lambda_{d+1} x_j^{d+1} = j,$$

for each $j = 1, \dots, d+1$. When $j = 1$, since each $x_1^i \geq 1$, we have

$$1 = \lambda_1 x_1^1 + \lambda_2 x_1^2 + \cdots + \lambda_{d+1} x_1^{d+1} \geq \lambda_1 + \lambda_2 + \cdots + \lambda_{d+1} = 1,$$

and the equality is obtained if and only if $x_1^1 = x_1^2 = \cdots = x_1^{d+1} = 1$. Since $x_1^i, x_2^i, \dots, x_{d+1}^i$ is a permutation of $1, 2, \dots, d+1$ for each i, we must have $x_2^i, x_3^i \dots, x_{d+1}^i$ is then a permutation of $2, 3, \dots, d+1$ for each i. Then similar argument applies when $j = 2$, where we have

$$2 = \lambda_1 x_2^1 + \lambda_2 x_2^2 + \cdots + \lambda_{d+1} x_2^{d+1} \geq \lambda_1 2 + \lambda_2 2 + \cdots + \lambda_{d+1} 2 = (\sum_{i=1}^{d+1} \lambda_i) 2 = 2,$$

and the equality is obtained if and only if $x_2^1 = x_2^2 = \cdots = x_2^{d+1} = 2$. Repeating the argument, we have

$$\begin{aligned} x_1^1 &= x_1^2 = \cdots = x_1^{d+1} = 1, \\ x_2^1 &= x_2^2 = \cdots = x_2^{d+1} = 2, \\ &\dots \\ x_{d+1}^1 &= x_{d+1}^2 = \cdots = x_{d+1}^{d+1} = d+1. \end{aligned}$$

This means $Y = X_1 = \cdots = X_{d+1}$, contradicting to the fact that X_i's are taken from $V \backslash \{Y\}$. Hence Y is an extremal point and so a vertex.

(c): Recall that a proper face of a polytope P is of the form $P \cap h$, where h is a hyperplane such that P is fully contained in one of the closed half-spaces determined by h. In other words, $X_1, \dots, X_k \in V$ forms a face if and only if there is a linear functional $\langle f, \_ \rangle$ for some $f \in \mathbb{R}^{d+1}$ such

(1) $\langle f, \_ \rangle = c$ determines the hyperplane h for some constant c;
(2) the solutions for $\langle f, x \rangle = c, x \in V$ are $X_1, \dots, X_k$;
(3) for any other point $y \in V - \{X_1, \dots, X_k\}$, we have $\langle f, y \rangle < c$.

Write $f = (f_1, \dots, f_{d+1}) \in \mathbb{R}^{d+1}$. Take $X = (x_1, \dots, x_{d+1}) \in V$, then $\langle f, X \rangle = \sum_{i=1}^{d+1} x_i f_i$. Denote by $v(n)$ the index of the coordinate of X such that $x_{v(n)} = n$. Then X maximizes $\langle f, \_ \rangle$ if and only if

$$f_{v(1)} \leq \cdots \leq f_{v(d+1)}.$$

If f has all different coordinates, then the above sorted sequence is unique, and so there is only a single vertex that maximizes $\langle f, \_ \rangle$ — this gives an alternative proof for part (b).

Now to get an edge, we need exactly two ways to sort the coordinates of f in non-decreasing order. Say $f_k = f_l$ but no other coordinates are equal to each other, so there are $(d+1) - 1 = d$ distinct values from the coordinates of f. Then we can sort them in the two ways

$$\cdots < f_k \leq f_l < \dots \text{ or } \cdots < f_l \leq f_k < \dots.$$

Then the two vertices of the edge have the same coordinates, but differ from each other at where k and l appears in them and they must appear adjacent to each other, say $k = v(i-1)$ and $l = v(i)$. In short, they differ by a transposition $(i-1\ i)$.

To get a k-face, we can argue in exactly the same way. We need there to be $(d+1) - k$ distinct values from the coordinates of f. In other words, we need to partition $[d+1] = \{1, 2, \dots, d+1\}$ into $(d+1) - k$ subsets, where each subset is nonempty and consists of consecutive integers, and then the vertices of a k-face are precisely a coset of the subgroup consisting of the permutations that fixes each subset set-wisely. For example, take $d = 3$. If we partition $[4] = \{1\} \cup \{2, 3, 4\}$ then a possible 2-face is the convex hull determined by

$$(1,2,3,4), (1,2,4,3), (1,3,2,4), (1,3,4,2), (1,4,2,3), (1,4,3,2);$$

if we partition $[4] = \{1, 2\} \cup \{3, 4\}$, then a possible 2-face is the square determined by

$$(1,2,3,4), (2,1,3,4), (1,2,4,3), (2,1,4,3).$$

And here (i, j, k, l) is understood as $\begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix}$.