Your work on these assignments must be your own and follow the guidelines for academic integrity in the course syllabus. Any discussions with other students must be cited in your submission. If you have any questions, please ask the instructor.

Consult your email for the link and instructions on how to submit the assignment via Crowdmark. There is a deadline of **due 14 Feb 2022, 11:59pm**.

Students in MATH 4300 will be graded out of 80 **points**.

Students in MATH 7300 will be graded out of 100 **points**.

There are up to 105 points available.

1. An function $f : \mathbb{R}^d \to \mathbb{R}^k$ is called an *affine transformation* iff there exists a $k \times d$ matrix B and vector $\mathbf{c} \in \mathbb{R}^k$ so that $f(\mathbf{x}) = B\mathbf{x} + \mathbf{c}$. In the following, assume f is a fixed affine transformation.

   [5] (a) Prove that if $C \subseteq \mathbb{R}^d$ is a convex set, then $f(C)$ as well.

   [5] (b) Let $A \subseteq \mathbb{R}^d$ be such that $f(A)$ is convex. Is it necessarily true that A is convex?

   [5] (c) For an arbitrary set $X \subseteq \mathbb{R}^d$, show that $\mathrm{conv}(f(X)) = f(\mathrm{conv}(X))$.

[15] 2. For any set $Y \subseteq \mathbb{R}^d$, let $||\cdot||$ be the usual Euclidean norm on $\mathbb{R}^d$ and define $\mathrm{diam}(Y) = \sup\{||\mathbf{x} - \mathbf{y}|| : \mathbf{x}, \mathbf{y} \in Y\}$. Show that for any $X \subseteq \mathbb{R}^d$,

$$\mathrm{diam}(X) = \mathrm{diam}(\mathrm{conv}(X)).$$

   You may assume that X is a finite set.

[15] 3. (a) Let $K \subseteq \mathbb{R}^d$ be a convex set. Suppose that $n \geq d+1$ and $C_1, C_2, \ldots, C_n \subseteq \mathbb{R}^d$ are convex sets with the property that the intersection of every $d+1$ of them contains a translated copy of K. Prove that $\cap_{i=1}^n C_i$ contains a translated copy of K also.

[10] (b) Find an example of 4 convex sets in $\mathbb{R}^2$ with the property that the intersection of every 3 of them contains a line segment of length 1, but the intersection of all 4 contains no line segment of length 1. Does this contradict part (a)?

[10] 4. (a) Prove that if $X \subseteq \mathbb{R}^2$ with $|X| = 3$ and $\mathrm{diam}(X) \leq 1$, then there is a disc of radius $\frac{1}{\sqrt{3}}$ that contains X. Results in Section 1.7 of the course textbook may be helpful and can be used without proof (please cite, though).

[10] (b) Prove that if $X \subseteq \mathbb{R}^2$ is any finite set with $\mathrm{diam}(X) \leq 1$, there is a disc of radius $\frac{1}{\sqrt{3}}$ that contains X.

[15] 5. Prove that if $A \subseteq \mathbb{R}^d$ with $|A| = d+2$ is in general position (no $d+1$ points of A are affinely dependent), then there is a unique $\mathbf{x} \in \mathbb{R}^d$ so that if A is partitioned $A = A_1 \cup A_2$ ($A_1 \cap A_2 = \emptyset$), then either $\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2) = \emptyset$ or $\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2) = \{\mathbf{x}\}$.

[15] 6. Let $R_1, R_2, \ldots, R_k$ be rectangles in $\mathbb{R}^2$ whose sides are parallel to the coordinate axes ($R_i = [a_1, a_2] \times [b_1, b_2]$). If every pair of rectangles have non-empty intersection, show that $\cap_{i=1}^k R_i \neq \emptyset$.

Total points: 105