Your work on these assignments must be your own and follow the guidelines for academic integrity in the course syllabus. Any discussions with other students must be cited in your submission. If you have any questions, please ask the instructor.

Consult your email for the link and instructions on how to submit the assignment via Crowdmark. There is a deadline of **due 18 Apr 2022, 11:59pm**.

Students in MATH 4300 will be graded out of 70 **points**.

Students in MATH 7300 will be graded out of 75 **points**.

There are up to 77 points available.

[15] 1. (see lecture notes Ex 333) Show that the greatest distance arising from a set of n points in the plane is realized by at most n different pairs of points. Hint: Try induction and look at the topological graph whose edges are pairs of vertices at maximum distance.

[10] 2. (see lecture notes Ex 339) For each $n \geq 2$ show that the midpoints of the edges of a regular simplex in $\mathbb{E}^n$ form a 2-distance set (exactly 2 different distances are determined by pairs of points) with $\binom{n+1}{2} = \binom{n}{2} + n$ points.

[5] 3. (see lecture notes Ex 342) Show that the Petersen graph is a unit distance graph. Hint: Take one of the standard drawings and "twist" the centre.

[15] 4. (a) Show that if for $n \geq 5$, if $A_1, A_2, \ldots, A_{n+1} \subseteq \{1, 2, \ldots, n\}$ are distinct 3-sets ($|A_i| = 3$), then there exist i, j so that $|A_i \cap A_j| = 1$. Hint: Induction on n and consider the sets that contain two fixed integers (say 1 and 2).

[5] (b) Define a graph G_n with vertices $V_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \{0, 1\}, x_1 + \cdots + x_n = 3\}$ and $\mathbf{xy} \in E_n$ iff $||\mathbf{x} - \mathbf{y}||_2 = 2$. Show that $\chi(G_n) \geq \frac{(n-1)(n-2)}{6}$. (Hint: Use the result from graph theory that $\chi(G) \geq |V(G)|/\alpha(G)$, where $\alpha(G)$ is the independence number of G).

[2] (c) Show that $\chi(\mathbb{E}^n) \geq \frac{(n-1)(n-2)}{6}$.

[10] 5. (see Matoušek Exercise 5.7.1) Prove that the region $reg(p)$ of a point p in the Voronoi diagram of a finite point set $P \subseteq \mathbb{R}^d$ is unbounded if and only if p lies on the surface of $\operatorname{conv}(P)$.

[5] 6. (a) Show that 3 unit spheres in $\mathbb{R}^3$ can have at most 2 points in their intersection.

[10] (b) Let $U_3(n)$ be the maximum number of unit distances among n points in $\mathbb{R}^3$. Use the Kovári-Sós-Turán theorem (you need not prove it) to show that $U_3(n) \leq \left(\frac{1}{2} + o(1)\right) n^{5/3}$.

Total points: 77