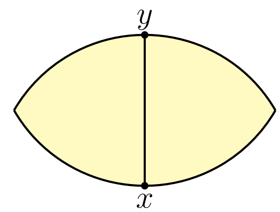
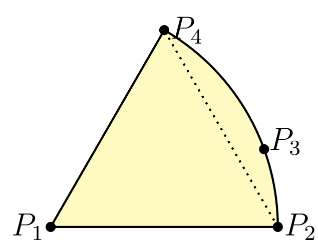
Q1: Up to rescaling, we can assume the greatest distance is 1. Let x, y be two points realizing distance 1. Then all other points must lies in the bigon, which is the intersection of two unit disk centered at x, y. If not, then the point outside the bigon would have distance > 1 from x or y. Similarly, if x, y, z are three distinct points with pairwise distance 1, then all other points must lies in the Reuleaux triangle with vertices x, y, z.



Let us proceed by induction on n. The statement is clearly true for $n = 1, 2, 3$ as $\binom{n}{2} \leq n$ in these cases. Suppose the statement is true for any $n-1$ points on the plane. Let $P_1, \ldots, P_n$ be n distinct points on the place with maximum distance 1, namely, $\operatorname{diam}(\{P_1, \ldots, P_n\}) = 1$. Let G be the graph with vertices $P_1, \ldots, P_n$ and edges pairs of vertices at distance 1. We want to show $|E(G)| \leq n$. If each vertex has most degree 2, then by the handshaking lemma, there are at most n edges, as required. If there exists some vertex has degree at least 3, then after reordering, we can assume this vertex is P_1 and three vertices connected to P_1 are P_2, P_3, P_4. Note $\operatorname{diam}(\{P_1, \ldots, P_n\}) = 1$ and $||P_1P_2|| = ||P_1P_4|| = 1$, so $\triangle P_1P_2P_4$ is acute. Reordering again if necessary, we can assume P_3 lies within the acute angle $\angle P_2P_1P_4$. Suppose a point P_i is at distance 1 from P_3. Since $||P_3P_i|| = ||P_1P_2|| = 1$, P_3P_i and P_1P_2 must intersect; if not, then some two points from P_1, P_2, P_3, P_i would be at distance > 1. Similarly, P_3P_i and P_1P_4 must intersect. So P_3P_i intersects both P_1P_2 and P_1P_4. It follows that $P_i = P_1$ is the only possibility. Plainly, P_1 is the only point at distance 1 from P_4. With P_4 removed, we arrive at a subgraph with $n-1$ vertices and one edge less from G. By induction hypothesis, the new graph has at most $n-1$ edges. Hence, $|E(G)| \leq n$.



Q2: It suffices to prove the statement for the standard simplex $\Delta^n \subset \mathbb{R}^{n+1}$, since every regular simplex can be transformed into the standard simplex by an affine map. Note Δ^n is the convex hull spanned the canonical basis of $\mathbb{R}^{n+1}$:

$$e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_{n+1} = (0, 0, \ldots, 1).$$

Each e_i is a vertex of Δ^n and each $\mathrm{conv}(\{e_i, e_j : i \neq j\})$ is an edge. So there are $\binom{n+1}{2}$ the midpoints of the edges of Δ^n and they are $\{\frac{1}{2}e_i + \frac{1}{2}e_j : 1 \leq i \neq j \leq n+1\}$. For any two distinct midpoints $\frac{1}{2}e_i + \frac{1}{2}e_j, \frac{1}{2}e_{i'} + \frac{1}{2}e_{j'}$, either $i \neq i', j \neq j'$ or $i = i', j \neq j'$ after reordering. In the former case,

$$\mathrm{dist}(\frac{1}{2}e_i + \frac{1}{2}e_j, \frac{1}{2}e_{i'} + \frac{1}{2}e_{j'}) = ||\frac{1}{2}e_i + \frac{1}{2}e_j - \frac{1}{2}e_{i'} - \frac{1}{2}e_{j'}|| = 1;$$

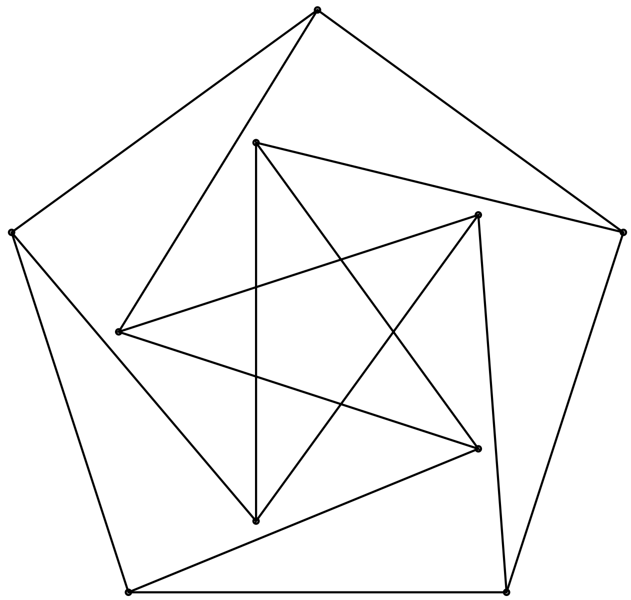
in the latter case

$$\mathrm{dist}(\frac{1}{2}e_i + \frac{1}{2}e_j, \frac{1}{2}e_{i'} + \frac{1}{2}e_{j'}) = ||\frac{1}{2}e_i + \frac{1}{2}e_j - \frac{1}{2}e_{i'} - \frac{1}{2}e_{j'}|| = ||\frac{1}{2}e_j - \frac{1}{2}e_{j'}|| = \frac{\sqrt{2}}{2}.$$

So these midpoints form a 2-distance set.

Q3: Let $R = \sqrt{\frac{5+\sqrt{5}}{10}}$ and $r = \sqrt{\frac{5-\sqrt{5}}{10}}$. Draw the graph on the complex plane and the vertices are

$$Re^{\frac{\pi i}{10}}, Re^{\frac{5\pi i}{10}}, Re^{\frac{9\pi i}{10}}, Re^{\frac{13\pi i}{10}}, Re^{\frac{17\pi i}{10}}, re^{\frac{\pi i}{5}}, re^{\frac{3\pi i}{5}}, re^{\pi i}, re^{\frac{7\pi i}{5}}, re^{\frac{9\pi i}{5}}.$$



Q4: (a)

(b) It is clear the statement is true for $n \leq 4$ as then $\frac{(n-1)(n-2)}{6} \leq 1$. Assume $n \geq 5$ from now on. Every point $x = (x_1, \dots, x_n) \in V_n$ can be identified with a 3-subset of $[n]$, which is $\{i \in [n] : x_i = 1\}$. Denote the identified set by $S(x)$ for each point x. Note for any two points $x, y \in V_n$, $||x - y||_2 = 2$ if and only if x, y differs at exactly 4 coordinates if and only if $|S(x) \cap S(y)| = 1$. For each integer $i \in [n]$, there are $\binom{n-1}{4}\binom{4}{2}/2$ such (unordered) pairs of points x, y that $S(x) \cap S(y) = \{i\}$. And so

$$|E_n| = 3n\binom{n-1}{4} \text{ and } |V_n| = \binom{n}{3}.$$

By part (a), for every $n+1$ points of V_n, say $a_1, \dots, a_{n+1}$, there are i, j such that $|S(a_i) \cap S(a_j)| = 1$, that is, a_i and a_j are connected by an edge. So the independence number $\alpha(G) \leq n$. But then

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)} \geq \frac{\binom{n}{3}}{n} = \frac{(n-1)(n-2)}{6}$$

as required.

(c) It is clear the statement is true for $n \leq 4$ as then $\frac{(n-1)(n-2)}{6} \leq 1$. Assume $n \geq 5$. The graph described in part (b) can be draw in $\mathbb{E}^n$. The problem is that each edge is of length 2. But this can be resolved by a rescaling $f : \mathbb{E}^n \to \mathbb{E}^n, v \mapsto \frac{1}{2}v$. Now there is an edge in if and only x and y differs at 4 coordinates and hence $||f(x) - f(y)|| = 1$. That is $f(G)$ is a unit distance graph in $\mathbb{E}^n$. Hence, $\chi(\mathbb{E}^d) \geq \chi(G) \geq \frac{(n-1)(n-2)}{6}$ as required.

Q5: Note that $B(p, \epsilon)$ is contained in $reg(p)$ for sufficiently small $\epsilon > 0$.

Suppose p lies on the surface of $\text{conv}(P)$. This implies that there is a hyperplane h such that $p \in h$ and $\text{conv}(P)$ lies entirely in one of the half-spaces determined by h. Assume the hyperplane is given by

$$h = \{x \in \mathbb{R}^d : \langle x, n \rangle = b\}$$

for some $n \in \mathbb{R}^d$ and $b \in \mathbb{R}$, and changing n to $-n$ if necessary, we can also assume that $\langle x, n \rangle \leq b$ for all $x \in \text{conv}(P)$. We claim that $p + tn \in reg(p)$ for all positive number t and hence $reg(p)$ is unbounded. To see the claim, we have the following inequality for all $q \in P$ and $t \geq 0$:

$$\begin{aligned}
||q - (p + tn)||^2 - ||p - (p + tn)||^2 &= \langle q - (p + tn), q - (p + tn) \rangle - \langle tn, tn \rangle \\
&= \langle (q - p) - tn), (q - p) - tn) \rangle - \langle tn, tn \rangle \\
&= \langle q - p, q - p \rangle - 2\langle q - p, tn \rangle \\
&= ||q - p||^2 + 2t\langle p, n \rangle - 2t\langle q, n \rangle \\
&= ||q - p||^2 + 2tb - 2t\langle q, n \rangle \\
&\geq ||q - p||^2 + 2tb - 2tb = ||q - p||^2 \geq 0.
\end{aligned}$$

It follows that $||q - (p + tn)|| \geq ||p - (p + tn)||$ for all $p \in P$ and so $p + tn \in reg(p)$ as claimed.

Now suppose $reg(p)$ is unbounded. As the intersection of half-spaces, $reg(p)$ is a convex polyhedron. Since $reg(p)$ is convex and unbound, there exists a ray with initial point p lying entirely in $reg(p)$. Say the ray can be parametrized as $p + tn$ for some $n \in \mathbb{R}^d$ and parameter $t \in \mathbb{R}_{\geq 0}$. Set $b = \langle p, n \rangle$. We claim that the hyperplane

$$h = \{x \in \mathbb{R}^d : \langle x, n \rangle = b\}$$

intersects with $\text{conv}(P)$ at a face of $\text{conv}(P)$, or equivalently, $\text{conv}(P) \subset h_+ = \{x \in \mathbb{R}^d : \langle x, n \rangle \leq b\}$ and $\text{conv}(P) \cap h_+ \neq \emptyset$. It is clear $\text{conv}(P) \cap h_+ \neq \emptyset$ as $p \in h_+$. To prove the claim, we note that if $P \subset h_+$, then $\text{conv}(P) \subset h_+$ as h_+ is convex, and so it suffices to show $P \subset h_+$. We suppose by contradiction that there is some $q \in P$ such that q is not in h_+, that is, $\langle q, n \rangle > b$. Note $q \neq p$. Then by exactly the same calculation as above, we have

$$||q - (p + tn)||^2 - ||p - (p + tn)||^2 = ||q - p||^2 + 2tb - 2t\langle q, n \rangle = ||q - p||^2 - 2t(\langle q, n \rangle - b).$$

Since $\langle q, n \rangle > b$, we have $\langle q, n \rangle - b > \epsilon > 0$ for some ϵ. Take some $t > \frac{||q-p||^2}{2\epsilon}$ and we have

$$||q - (p + tn)||^2 - ||p - (p + tn)||^2 = ||q - p||^2 - 2t(\langle q, n \rangle - b) < ||q - p||^2 - 2\frac{||q - p||^2}{2\epsilon}\epsilon = 0.$$

But this means $||q - (p + tn)|| < ||p - (p + tn)||$ and so $p + tn \notin reg(p)$, contradicting the fact the ray $\{b + tn : t \geq 0\}$ lies entirely in $reg(p)$

Q6: (a). Denote by x_1, x_2, x_3 the centers of these three unit sphere. Then any point p in the intersection of these spheres is at distance 1 from each x_i. In particular, since p is of the same distance from x_1 and x_2, p lies on the plane bisecting and orthogonal to the line segment x_1x_2. Similarly, p lies on the plane bisecting and orthogonal to the line segment x_1x_3. The intersection of these two planes is a line and p must lie on this line. But there are at most two points on the line is at distance 1 from x_1. Hence, the intersection of three unit spheres consists of at most 2 points.

(b). Suppose we have n distinct points on $\mathbb{R}^3$ realizing the maximum number of unit distances. Let G be the graph such that the vertex set of G is these n points and two vertices are connected by an edge if and only if they are of unit distance from each other. Then G contains no subgraph isomorphic to $K_{3,3}$. Otherwise, there were three points in the intersection of three distinct unit spheres, violating part (a). So by the Kövári-Sós-Turán theorem, $|E(G)| = O(n^{2-1/3}) = O(n^{5/3})$.

To get the inequality in statement, we need a somewhat more precise estimate, *cf.* Theorem 17.2.5 on the lecture notes. Plainly, $U_3(n) = |E(G)| \leq ex(n, K_{3,3}) \leq (\frac{1}{2} + o(1))n^{5/3}$.