Your work on these assignments must be your own and follow the guidelines for academic integrity in the course syllabus. Any discussions with other students must be cited in your submission. If you have any questions, please ask the instructor.

Consult your email for the link and instructions on how to submit the assignment via Crowdmark. There is a deadline of **due 4 Mar 2022, 11:59pm**.

Students in MATH 4300 will be graded out of 70 **points**.
Students in MATH 7300 will be graded out of 80 **points**.
There are up to 85 points available.

Some extra definitions needed for this problem set:

For a set $X \subseteq \mathbb{R}^d$, define the *dual*, denoted X^* to be

$$X^* = \{\mathbf{y} \in \mathbb{R}^d \mid \text{ for all } \mathbf{x} \in X,\ \mathbf{x} \bullet \mathbf{y} \leq 1\}.$$

[10] 1. A polytope P is said to satisfy Gale's evenness condition with respect to a linear ordering $v_1 < v_2 < \cdots < v_n$ of $V(P)$ if for any $X \subseteq V(P)$, if $\operatorname{conv}(X)$ is a facet of P, then any two distinct vertices in $V(P) \setminus X$ are separated by an even number of vertices in X. (See Definition 6.2.3 of the lecture notes).

Let P be the hypercube in $\mathbb{R}^3$: $V(P) = \{0,1\}^3$. Find an ordering of $V(P)$ for which P satisfies Gale's evenness condition.

[10] 2. Prove that for any k with $1 \leq k \leq d$ and any k finite sets $A_1, A_2, \ldots, A_k \subseteq \mathbb{R}^d$, there exists a $(k-1)$-flat (an affine space of dimension $k-1$) such that for every hyperplane containing it, for every $i \in \{1, 2, \ldots, k\}$, each of the closed half-spaces contain at least $\frac{1}{d+1}|A_i|$ elements of A_i. (The case $k = 1$ was the Centerpoint Theorem).

[10] 3. Let $C = \{\mathbf{x} \in \mathbb{R}^d \mid |x_1| + \cdots + |x_d| \leq 1\}$. Show that C^* is the d-dimensional hypercube $\{\mathbf{x} \in \mathbb{R}^d \mid \max |x_i| \leq 1\}$.

[15] 4. Let X be a polytope containing the origin. Show that $(X^*)^* = X$.

[10] 5. Show that a d-dimensional simplex in $\mathbb{R}^d$ can be expressed as the intersection of $d+1$ half-spaces.

[10] 6. Verify the description of the faces of the hypercube we discussed in class by giving the supporting hyperplanes (and explaining why they are supporting hyperplanes).

7. For $d \geq 1$, let $V \subseteq \mathbb{R}^{d+1}$ be the set of all points whose coordinates are permutations of the set $\{1, 2, \ldots, d+1\}$. The polytope $P = \operatorname{conv}(V)$ is called the *permutahedron*.

[5] (a) Verify that P is a d-dimensional polytope.

[5] (b) Verify that P has $(d+1)!$ vertices.

[10] (c) Show that the edges of P are induced by pairs of vertices where the permutations (given in their coordinates) differ by a transposition of two adjacent coordinates. Can you give a combinatorial description of the vertex set of a k-face? (No need to give a detailed proof in the general case).

Total points: 85