3-Manifold Groups ABC

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PRELUDE

> Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine."
>
> — Michael Atiyah

PRELUDE

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Prelude

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In dimension 3, algebra almost “determines” geometry and topology.

Poincaré Conjecture (Perelman, Thurston, etc.)

Every 3-dimensional manifold which is closed, connected, and has trivial fundamental group, is homeomorphic to S^3.

Deficiency

Definition

Let $G = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_k \rangle$ be a finitely presented group. The deficiency of the presentation is defined to be $n - k$. The deficiency $\operatorname{def}(G)$ of G is the maximal deficiency of all possible presentations.

DEFICIENCY

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Some classical results from group (co)homology:

Given a finite presentation $1 \to R \to F_n \to G \to 1$ for G,

- $H_1(G) = G/[G, G] = F_n/[F_n, F_n]R$,
- $H_2(G) = (R \cap [F_n, F_n])/[R, F_n]$,
- $\operatorname{def}(G) \leqslant b_1(G) - b_2(G)$.

Deficiency

$$\begin{aligned}\operatorname{def}(G) &= \max\{\#\text{ generators } - \#\text{ relators in a finite presentation}\}\\ &\leqslant b_1(G) - b_2(G)\end{aligned}$$

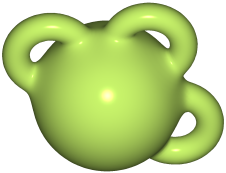
Examples

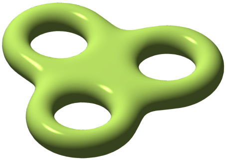
- $\operatorname{def}(F_n) = n$.
- $\operatorname{def}(\mathbb{Z}^2) = 1, \operatorname{def}(\mathbb{Z}^3) = 0$ and $\operatorname{def}(\mathbb{Z}^n) < 0$ for $n \geqslant 4$.
- $|G| < \infty \Longrightarrow \operatorname{def}(G) \leqslant 0$, for otherwise its abelianization is infinite.
- $\operatorname{def}(\pi_1(\Sigma_g)) = 2g - 1$ with the canonical presentation.
- The only finitely generated abelian groups with $\operatorname{def}(G) = 0$ are $\mathbb{Z}/n\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}^3$.

HEEGAARD SPLITTING

Handlebody $H_g = B^3$ with g copies $I \times D^2$ attached along $(\partial I) \times D^2 \to \partial B_3$

$$\partial H_g = \Sigma_g \quad \text{surface of genus } g$$





Heegaard Splitting

Definition

A Heegaard splitting of a closed 3-manifold M is a decomposition

$$M = H_1 \cup H_2$$

such that

- H_1, H_2 are handlebodies;
- $\partial H_1 = \partial H_2 = H_1 \cap H_2$.

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Example

- S^3 admits a Heegaard splitting with $H_1 = H_2 = B^3$ trivially.
- S^3 admits a Heegaard splitting with $H_1 = H_2 = S^1 \times D^2$.

Heegaard Splitting

Theorem (Moise, etc.)

A topological 3-manifold admits precisely one smooth structure (up to diffeomorphism) and precisely one piecewise structure (up to piecewise-linear homeomorphism).

In other words, every 3-manifold can be triangulated. And we always work with the PL or smooth category.

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Theorem

Every closed 3-manifold admits a Heegaard splitting.

Sketch of proof.

Triangulate M. H_1 is the closure of the regular neighbourhood of 1-skeleton and H_2 is the closure of the complement of H_1. $\square$

HEEGAARD SPLITTING

Apply the van Kampen theorem to $M = H_1 \cup H_2$:

$$\begin{aligned}\pi_1(M) &= \pi_1(H_1) *_{\pi_1(\Sigma_g)} \pi_1(H_2) \\ &= \langle x_1, \ldots x_g, y_1, \ldots, y_g \mid f_*(a_i) = g_*(a_i), i = 1, \ldots, 2g \rangle\end{aligned}$$

- $\pi_1(H_1) = \langle x_1, \ldots, x_g \rangle$,
- $\pi_1(H_2) = \langle y_1, \ldots, y_g \rangle$,
- $\pi_1(\Sigma_g) = \langle a_1, \ldots, a_{2g} \mid \prod_{i=1}^{g} [a_{2i-1}, a_{2i}] \rangle$,
- f_* is induced by $f : \Sigma_g \to \partial H_1$,
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Corollary

Let M be a closed 3-manifold. Then $\operatorname{def}(\pi_1(M)) \geqslant 0$.

Prime and Irreducible 3-manifolds

Connected sum

$$M_1 \# M_2 := (M_1 \setminus B_1) \cup_f (M_2 \setminus B_2)$$

- B_i = 3-balls
- f = an orientation-reversing homeomorphism between ∂B_1 and ∂B_2

Definition

A 3-manifold M is called prime if $M = M_1 \# M_2$ implies $M_1 = S^3$ or $M_2 = S^3$.

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Definition

A 3-manifold M is called irreducible if every embedded 2-sphere in M bounds a 3-ball in M.

$$\begin{aligned} \text{irreducible} &\implies \text{prime} \\ \text{prime} &\implies \text{irreducible or } S^1 \times S^2 \end{aligned}$$

Prime and Irreducible 3-manifolds

Example of irreducible 3-manifolds:

- S^3 (Alexander Theorem)
- Lens spaces $L(p,q)$
- Knot complements $\overline{S^3 - n(K)}$ and most of their Dehn fillings
- Surface bundles over S^1 not $S^1 \times S^2$, e.g. mapping torus
- Seifert manifolds, except $S^1 \times S^2$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$

Theorem (Epstein)

A 3-manifold M is Seifert fibered if and only if it is foliated by S^1.

One may interpret “foliated by S^1” as “a disjoint union of circles”.

PRIME DECOMPOSITION THEOREM

Prime Decomposition Theorem (Knerser, Milnor)

Every 3-manifold M with no spherical boundary components can be "uniquely" written as

$$M = M_1 \# \ldots \# M_n,$$

where M_i's are prime. In particular,

$$\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_n).$$

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Corollary

Let M be closed. If $\pi_1(M)$ is finite, then M is irreducible.

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Kneser's Conjecture (Stallings)

If $\pi_1(M) = G_1 * G_2$, then there is a connected sum decomposition $M = M_1 \# M_2$ with $\pi_1(M_i) = G_i$. In particular, M is not irreducible.

Rigidity

In dimension 3, algebra almost “determines” geometry and topology.

Theorem (Waldhausen, Scott, etc)

Let M_1, M_2 be closed 3-manifolds, not lens spaces. If M_1 is prime and $\pi_1(M_1) = \pi_1(M_2)$, then $M_1 = M_2$.

One may compare it with the Mostow rigidity theorem.

Theorem (Mostow, Prasad, Marden)

If two hyperbolic 3-manifolds with finite volume have isomorphic fundamental groups, they are isometric.

This theorem implies in particular that the geometry of finite-volume hyperbolic 3-manifolds is determined by their topology.

Sphere Theorem

Here is a powerful tool.

Sphere Theorem (Papakyriakopoulos)

Let M be a 3-manifold. If $\pi_2(M)$ is not trivial, then there is an embedded 2-sphere $S^2 \hookrightarrow M$ representing a non-trivial element in $\pi_2(M)$. In particular, M is reducible.

With some work, we see the sphere theorem implies the following.

Theorem

Let M be a 3-manifold and $\widetilde{M} \to M$ be a covering. Then M is irreducible if and only if $\widetilde{M}$ is irreducible.

Sphere Theorem

Corollary

If M is an irreducible 3-manifold, then

- $\pi_2(M)$ *is trivial;*
- $\pi_1(M)$ *is finite if and only if* $\pi_3(M)$ *is non-trivial;*
- $\pi_1(M)$ *is infinite if and only if* $\pi_3(M)$ *is trivial; in this case,* $\pi_1(M)$ *is torsion-free and M is aspherical, and if M is closed, M is* $K(\pi, 1)$.

Sketch of proof.

The universal cover $\widetilde{M}$ has the same higher homotopy groups π_n as M for $n \geqslant 2$. Also note that $\pi_1(M)$ is infinite if and only if $\widetilde{M}$ is non-compact. The conclusions follow from the Hurewicz theorem applied to $\widetilde{M}$. $\square$

Deficiency of 3-manifold Groups

Corollary

If M is closed and irreducible, then $\operatorname{def}(\pi_1(M)) = 0$.

Proof.

It remains to see $\operatorname{def}(\pi_1(M)) \leqslant 0$. It is clearly true if $\pi_1(M)$ is finite. If $\pi_1(M)$ is infinite, then the previous corollary tells us that M is the classifying space of $\pi_1(M)$. Hence, $H_k(\pi_1(M)) = H_k(M)$ and so $b_2(\pi_1(M)) = b_2(M) = b_1(M) = b_1(\pi_1(M))$ by duality theorems. □

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Corollary

The possible abelian fundamental groups of a closed 3-manifold are $\pi_1(S^3) = 1, \pi_1(S^1 \times S^2) = \mathbb{Z}, \pi_1(T^3) = \mathbb{Z}^3$ *and* $\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$.

Incompressible Surface

Recall that to study an arbitrary 3-manifold, we cut it into irreducible pieces along spheres. It turns out irreducible 3-manifolds can be further decomposed.

Definition

Let $S \subset M$ be a properly embedded surface, i.e. $\partial S \subset \partial M$. We say S is <u>incompressible</u> if the map $\pi_1(S) \to \pi_1(M)$ induced by inclusion is injective and S does not bound a 3-ball.

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Loop Theorem (Papakyriakopoulos)

Let M be a 3-manifold and $F \subset \partial M$ is a boundary component. If the induced homomorphism $\pi_1(F) \to \pi_1(M)$ is not injective, then there is a proper embedding $g : (D^2, \partial D^2) \to (M, \partial M)$ such that $g(\partial D^2)$ represents a non-trivial element in $\ker(\pi_1(F) \to \pi_1(M))$.

Loop Theorem

Corollary

Let M be a 3-manifold. There exist compact 3-manifolds $N_1, \ldots, N_m$ whose boundary components are incompressible and a free group F such that $\pi_1(M) = \pi_1(N_1) * \cdots * \pi_1(N_m) * F$.

Sketch of proof.

Cut N along the disks obtained by the Loop Theorem. $\square$

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Corollary

The fundamental group $\pi_1(M)$ is infinite cyclic if and only if $M = S^1 \times S^2$ or $M = S^1 \times D^2$.

JSJ Decomposition

JSJ Decomposition Theorem (Jaco-Shalen, Johannson)

Let M be an irreducible 3-manifold with empty or toroidal boundary. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \ldots, T_m$ such that each component of M cut along $T_1 \cup \cdots \cup T_m$ is atoroidal or Seifert fibered. Any such collection of tori with a minimal number of components is unique up to isotopy.

Recall that a 3-manifold is Seifert fibered if it is foliated by circles. And a 3-manifold N is atoroidal if any map $T \to N$ from a torus T to N which induces a monomorphism $\pi_1(T) \to \pi_1(N)$ can be homotoped into the boundary of N. We can say something more about the atoroidal piece.

Geometrization Theorem

We say a 3-manifold is <u>spherical</u> (resp. <u>hyperbolic</u>), if it admits a complete metric of constant curvature $+1$ (resp. -1).

Elliptization Theorem

Every closed 3-manifold with finite fundamental group is spherical.

Recall that a closed 3-manifold M with finite fundamental group has universal cover S^3. Hence $\pi_1(M)$ is a finite subgroup of $\mathrm{SO}(4)$.

Hyperbolization Theorem

Let N be an irreducible 3-manifold with empty or toroidal boundary. Suppose that N is atoroidal and not homeomorphic to $S^1 \times D^2, T^2 \times I$, or $K \tilde{\times} I$. If $\pi_1(N)$ is infinite, then N is hyperbolic.

Geometrization Theorem

Geometrization Theorem

Let M be an irreducible 3-manifold with empty or toroidal boundary. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \ldots, T_m$ in M such that each component of M cut along $T_1 \cup \cdots \cup T_m$ is hyperbolic or Seifert fibered. Furthermore, any such collection of tori with a minimal number of components is unique up to isotopy.

Proof.

This is a direct consequence of the JSJ-Decomposition Theorem with the Elliptization Theorem and the Hyperbolization Theorem and the aforementioned facts that spherical 3-manifolds as well as $S^1 \times D^2, T^2 \times I$, and $K \tilde{\times} I$ are Seifert fibered, and that hyperbolic 3-manifolds are atoroidal. $\square$

This theorem can be further developed into the Thurston's geometrization conjecture (now a theorem).

Geometrization Theorem

There are many consequences (and/or side products) of the Thurston's geometrization conjecture. Here is one of them.

Theorem

Let M be an irreducible 3-manifold with empty or toroidal boundary. Suppose $\pi_1(M) = A \times B$ where A is infinite and B is non-trivial. Then $M = S^1 \times \Sigma$ where Σ is a surface.

Note that a free product of non-trivial groups is never a direct product.

Coherence

A group is called <u>coherent</u> if each of its finitely generated subgroups is finitely presented.

For example, $F_2 \times F_2$ is not coherent. For $n \neq 3$, $\mathrm{SL}(n, \mathbb{Z})$ is known to be coherent; for $n = 3$, it is currently unknown.

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Compact Core Theorem (Scott)

If Y is a 3-manifold such that $\pi_1(Y)$ is finitely generated, then Y has a compact submanifold M such that the induced map $\pi_1(M) \to \pi_1(Y)$ is an isomorphism.

Corollary

All 3-manifold groups are coherent.

Linearity

Let R be a commutative ring with unity. We say that G is linear over R if there exists an embedding $G \to \mathrm{GL}(n,R)$ for some n.

An irreducible 3-manifold M with empty or toroidal boundary

- is Seifert fibered;
  - fundamental groups of Seifert fibered manifolds are linear over $\mathbb{Z}$.
- is hyperbolic;
  - $\pi_1(M)$ admits a faithful representation $\pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$, which lifts to a faithful representation $\pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$.
- admits an incompressible torus.

Conjecture (Thurston)

All 3-manifold groups are linear.

Residually Finiteness

A group G is called residually finite if for every $g \in G \setminus \{1\}$, there is a finite group H and a homomorphism $f : G \to H$ such that $f(g)$ is non-trivial.

The Baumslag-Solitar group $BS(2,3)$ is not residually finite. The infinite dihedral group is not residually finite either.

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Theorem (Mal'cev-Selberg)

Finitely generated linear groups are residually finite.

Theorem (Hempel)

All 3-manifold group are residually finite.