James Munkres - Topology

Carter Hinsley's notes

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29 Local Compactness

Definition. A space X is *locally compact at x* if there is some compact subspace C of X that contains a nbhd of x. If X is locally compact at each of its points, X is said to be *locally compact.*

We ask: "Under what conditions is a space homeomorphic with a subspace of a compact Hausdorff space?"

Theorem 29.1. Let X be a space. X is locally compact Hausdorff iff there exists a space Y satisfying the following conditions:

- X is a subspace of Y.
- $Y - X$ comprises a single point.
- Y is a compact Hausdorff space.

Definition. If Y is a compact Hausdorff space and X is a proper subspace of Y dense in Y, then Y is said to be a *compactification* of X. If $Y - X$ equals a single point, then Y is called the *one-point compactification* of X.

Theorem 29.2. Let X be a Hausdorff space. Then X is locally compact iff given $x \in X$ and a nbhd U of x, there is a nbhd V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Corollary 29.3. Any open or closed subspace of a locally compact Hausdorff space is itself locally compact.

Corollary 29.4. A space X is homeomorphic to an open subspace of a compact Hausdorff space iff X is locally compact Hausdorff.

Exercises

1. Show that the rationals $\mathbb{Q}$ are not locally compact.

Proof. It will be sufficient to show that $\mathbb{Q}$ is not locally compact at 0; i.e., that there is no compact subspace of $\mathbb{Q}$ containing a neighborhood of 0.

Suppose for the sake of contradiction that there existed a compact subspace C of $\mathbb{Q}$ and an open nbhd U of 0 in C. Then there would be some open nbhd $(-\varepsilon, \varepsilon) \cap \mathbb{Q} \subseteq U$ of 0. Note that the inclusion $\mathbb{Q} \overset{\iota}{\hookrightarrow} \mathbb{R}$ is continuous; this map preserves compactness. Hence $\iota(C)$ is compact in $\mathbb{R}$. As $\mathbb{R}$ is Hausdorff, $\iota(C)$ is closed in $\mathbb{R}$. Thus $[-\varepsilon, \varepsilon] = \overline{(-\varepsilon, \varepsilon) \cap \mathbb{Q}} \subseteq \iota(C)$. Since $[-\varepsilon, \varepsilon]$ is uncountable while $\iota(C)$ is countable, we obtain a contradiction. Hence $\mathbb{Q}$ is not locally compact. $\square$