James Munkres - Topology

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29 Local Compactness

Definition. A space X is *locally compact at x* if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said to be *locally compact.*

We ask: "Under what conditions is a space homeomorphic with a subspace of a compact Hausdorff space?"

Theorem 29.1. Let X be a space. X is locally compact Hausdorff iff there exists a space Y satisfying the following conditions:

- X is a subspace of Y.
- $Y - X$ comprises a single point.
- Y is a compact Hausdorff space.

Definition. If Y is a compact Hausdorff space and X is a proper subspace of Y dense in Y, then Y is said to be a *compactification* of X. If $Y - X$ equals a single point, then Y is called the *one-point compactification* of X.

Theorem 29.2. Let X be a Hausdorff space. Then X is locally compact iff given $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Corollary 29.3. Any open or closed subspace of a locally compact Hausdorff space is itself locally compact.

Corollary 29.4. A space X is homeomorphic to an open subspace of a compact Hausdorff space iff X is locally compact Hausdorff.

Exercises

(1) Show that the rationals $\mathbb{Q}$ are not locally compact.

Proof. It will be sufficient to show that $\mathbb{Q}$ is not locally compact at 0; i.e., that there is no compact subspace of $\mathbb{Q}$ containing a neighborhood of 0.

Suppose for the sake of contradiction that there existed a compact subspace C of $\mathbb{Q}$ and an open neighborhood U of 0 in C. Then there would be some open neighborhood $(-\varepsilon, \varepsilon) \cap \mathbb{Q} \subseteq U$ of 0. Note that the inclusion $\mathbb{Q} \overset{\iota}{\hookrightarrow} \mathbb{R}$ is continuous; this map preserves compactness. Hence $\iota(C)$ is compact in $\mathbb{R}$. As $\mathbb{R}$ is Hausdorff, $\iota(C)$ is closed in $\mathbb{R}$. Thus $[-\varepsilon, \varepsilon] = \overline{(-\varepsilon, \varepsilon) \cap \mathbb{Q}} \subseteq \iota(C)$. Since $[-\varepsilon, \varepsilon]$ is uncountable while $\iota(C)$ is countable, we obtain a contradiction. Hence $\mathbb{Q}$ is not locally compact. $\square$

Supplementary Exercises: Nets

Nets are generalizations of sequences that characterize limit points, continuous functions, and compact sets not only in metric topologies, but in *all* topologies.

Definition. A *directed set* J is a poset wherein for each pair $\alpha, \beta \in J$, there exists an element $\gamma \in J$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

(1) Show that the following are directed sets:

- Any simply ordered set, under the relation $\leq$.

  Proof. We take $\preceq$ to be $\leq$. Let S be a simply (totally) ordered set with elements x, y. Then, as S is simply ordered, either $x \leq y$ or $y \leq x$. By reflexivity, $x \leq x$ and $y \leq y$ in either case, so y or x serves as γ respectively. $\square$

- The collection of all subsets of a set S, partially ordered by inclusion.

  Proof. We are speaking of the power set $\mathcal{P}(S)$. Let $U, V \in \mathcal{P}(S)$. Then there are inclusions $U \hookrightarrow \mathcal{P}(S)$ and $V \hookrightarrow \mathcal{P}(S)$. As the identity map $\mathcal{P}(S) \hookrightarrow \mathcal{P}(S)$ is an inclusion, $\mathcal{P}(S)$ serves as γ. $\square$

- A collection $\mathscr{A}$ of subsets of S that is closed under finite intersections, partially ordered by reverse inclusion.

  Proof. We use $\preceq$ to denote $\hookleftarrow$. Let $U, V \in \mathscr{A}$. As $U \cap V \in \mathscr{A}$, we have $U \preceq U \cap V, V \preceq U \cap V$. $\square$

- The collection of all closed subsets of a space X, partially ordered by inclusion.

  Proof. X itself is closed; hence for all U, V closed subsets of X, $U \preceq X$ and $V \preceq X$. $\square$

(2) A subset K of J is said to be *cofinal* in J if for each $\alpha \in J$, there exists $\beta \in K$ such that $\alpha \preceq \beta$. Show that if J is a directed set and K is cofinal in J, then K is a directed set.

Proof. We must show that for every pair of elements $x, y \in K$, there exists an element $z \in K$ such that $x \preceq z$ and $y \preceq z$. Because $K \subseteq J$, we know that $x, y \in J$. Consequently, there exists $\beta \in J$ such that $x \preceq \beta$ and $y \preceq \beta$. As K is cofinal in J, there exists $\gamma \in K$ such that $\beta \preceq \gamma$. By transitivity, we have $x \preceq \gamma$ and $y \preceq \gamma$; that is, K is a directed set. $\square$

(3) Let X be a topological space. A *net* in X is a function f from a directed set J into X. If $\alpha \in J$, we usually denote $f(\alpha)$ by x_α. We denote the net f itself by the symbol $(x_\alpha)_{\alpha \in J}$, or merely by (x_α) if the index set is understood.

The net (x_α) is said to *converge* to the point $x \in X$ (written $x_\alpha \to x$) if for each neighborhood U of x, there exists $\alpha \in J$ such that $\alpha \preceq \beta \implies x_\beta \in U$. Show that these definitions reduce to familiar ones when $J = \mathbb{Z}^+$.

Proof. Of course, $\mathbb{Z}^+$ is a directed set (Z^+ is a totally ordered set). We say that the sequence $(x_n)_{n \in \mathbb{Z}^+}$ converges to the point $x \in X$ if for each neighborhood U of x, there exists $M \in \mathbb{Z}^+$ such that $M \leq n \implies x_n \in U$. This is the usual definition, compatible with the net definition. $\square$

(4) Suppose that $(x_\alpha)_{\alpha \in J} \to x \in X$ and $(y_\alpha)_{\alpha \in J} \to y \in Y$. Show that $(x_\alpha \times y_\alpha) \to x \times y \in X \times Y$.

Proof. As $(x_\alpha) \to x$ and $(y_\alpha) \to y$, we may say that for any neighborhoods U and V of x and y respectively, there exist $\alpha, \beta \in J$ such that $\alpha \preceq \gamma \implies x_\gamma \in U$ and $\beta \preceq \delta \implies x_\delta \in V$.

We would like to show that for any neighborhood $U \times V$ of $x \times y$, there's a $\eta \in J$ such that whenever $\eta \preceq \alpha$, we have $x_\alpha \times y_\alpha \in U \times V$. Because U and V are themselves open neighborhoods of x and y respectively, we may use the $\alpha, \beta \in J$ from the convergence of the corresponding nets. As J is a directed set, there exists $\phi \in J$ such that $\alpha \preceq \phi$ and $\beta \preceq \phi$. By transitivity of the relation $\preceq$, we have $\phi \preceq \psi \implies x_\psi \in U, y_\psi \in V$. It follows that $\phi \preceq \psi \implies x_\psi \times y_\psi \in U \times V$. Hence $(x_\alpha \times y_\alpha) \to x \times y \in X \times Y$. $\square$

(5) Show that if X is Hausdorff, a net in X converges to at most one point.

Proof. Suppose a net $(x_\alpha)_{\alpha \in J}$ in X converges to two points $x, x' \in X$. We must show that $x = x'$. Convergence means that we may take any neighborhood U of x or U' of x' and find some $\alpha \in J$ such that $\alpha \preceq \beta$ implies $x_\beta \in U$ and $x_\beta \in U'$. Because X is Hausdorff, if $x \neq x'$ we may take disjoint neighborhoods U, U' and obtain a contradiction. Hence $x = x'$. $\square$

(6) **Theorem.** Let $A \in X$. Then $x \in \overline{A}$ iff there is a net of points of A converging to x.

Proof. The case wherein $x \in A$ is trivial, so we consider only when $x \notin A$.

($\implies$) Assuming $x \in \overline{A}$, every deleted neighborhood of x contains another point y of A. We would like to construct a net $(x_\alpha)_{\alpha \in J}$ so that we can select any neighborhood U of x in X and find $\alpha \in J$ so that $\alpha \preceq \beta \implies x_\beta \in U \cap X$. As A is a subspace of X, we may take any two neighborhoods U, V of x in X and we will have $x \in U \cap V$; it follows that the set of "neighborhoods" of x in A endowed with the relation of reverse inclusion form a directed set, denoted J. We construct a net $f : J \to A$ by letting $f(U) = y$ be that element of which we are guaranteed existence by x's status as a limit point of A. The convergence of this net to x is automatic.

($\impliedby$) Let $(x_\alpha)_{x \in J}$ be a net of points of A converging to x; that is, for any neighborhood $U \subseteq X$ of x there exists some $\alpha \in J$ such that $\alpha \preceq \beta \implies x_\beta \in U$. As $\alpha \preceq \alpha$ so that $x_\alpha \in A$, we see that every neighborhood of x contains at least one point of A. Hence $x \in \overline{A}$. $\square$

(7) **Theorem.** Let $f : X \to Y$. Then f is continuous iff for every convergent net $(x_\alpha) \to x$ in X, the net $(f(x_\alpha))$ converges to $f(x)$.

(8) Let $f : J \to X$ be a net in X; let $f(\alpha) = x_\alpha$. If K is a directed set and $g : K \to J$ is a function such that

- $i \preceq j \implies g(i) \preceq g(j)$,
- $g(K)$ is cofinal in J,

then the composite function $f \circ g : K \to X$ is called a *subnet* of (x_α). Show that if the net (x_α) converges to x, so does any subnet.

(9) Let $(x_\alpha)_{\alpha \in J}$ be a net in X. x is an *accumulation point* of the net (x_α) if for each neighborhood U of x, the set of those α for which $x_\alpha \in U$ is cofinal in J.

Lemma. The net (x_α) has the point x as an accumulation point iff some subnet of (x_α) converges to x.

(10) **Theorem.** X is compact iff every net in X has a convergent subnet.

(11) **Corollary.** Let G be a topological group; let A and B be subsets of G. If A is closed in G and B is compact, then $A \cdot B$ is closed in G.

(12) Check that the preceding exercises remain correct if condition (2) is omitted from the definition of *directed set.*