Lotka Volterra Model

Team 18

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- Studied Lotka Volterra Predator Prey Model
- Linear analysis of differential equations to understand dynamics of this model.
- Used Scipy's inbuilt Odeint, Euler, RK2, RK4 Numerical Methods to integrate differential equations.
- Compared results of Euler, RK2, RK4 method with Scipy's more efficient Odeint Numerical Method and discussed limitations of methods and oscillatory function's effect on Numerical Predictions.
- Plotted Population v/s Time graph and Phase Portrait Graph to confirm oscillator behaviour
- Plotted Surface plot to show maximum point at equilibrium points.

The model satisfies the system of ODEs:

$$\frac{dx}{dt} = x(\alpha - \beta y)$$

$$\frac{dy}{dt} = -y(\gamma - \delta x)$$

where,

- x is the number of prey.
- y is the number of some predator.
- $\frac{dy}{dt}$ and $\frac{dx}{dt}$ represent the growth rates of the two populations over time
- t represents time.
- α, β, γ and δ are parameters describing the interaction of the two species.

Predator-Prey (Lotka-Volterra) Model
Lynx and Hare

Figure: **Lynx and Hare: Specialized tightly linked predator and prey relationship.**

Predator-Prey (Lotka-Volterra) Model

Hudson Bay Company Data

Year	Hares *(1000)	Lynx *(1000)
1900	30	4
1901	47.2	6.1
1902	70.2	9.8
1903	77.4	35.2
1904	36.3	59.4
1905	20.6	41.7
1906	18.1	19
1907	21.4	13
1908	22	8.3
1909	25.4	9.1
1910	27.1	7.4

Year	Hares *(1000)	Lynx *(1000)
1911	40.3	8
1912	57	12.3
1913	76.6	19.5
1914	52.3	45.7
1915	19.5	51.1
1916	11.2	29.7
1917	7.6	15.8
1918	14	9.7
1919	16.2	10.1
1920	24.7	8.6

FIGURE: Data of Lynx and Hare population from 1900-1920 observed by Hudson Bay Company (Canada).

How Lotka Volterra Predator-Prey Model evolves with time

- Alfred Lotka (1920), an American biologist and actuary, published the mathematical predator-prey model and its cyclical nature.
- Lotka originated many useful theories of stable populations, including the logistic model.
- Vito Volterra (1925) proposed the same model to explain data from fish studies of his son-in-law Humberto D'Ancona on the fishing industry in Italy.
- The classical Lotka-Volterra predator-prey model for the dynamics of the populations of a predator and its prey species.

To keep the model simple, some assumptions are made that would be unrealistic in most of the predator-prey situations in real world. Specifically, it is assumed that

1. Only two species exist : Predator and Prey.
2. The prey population finds ample food at all times and are born and then die through predation or inherent death.
3. Predators have limitless appetite and they are born and their birth rate is positively affected by the rate of predation, and they die naturally.
4. The food supply of the predator population depends entirely on the size of the prey population.
5. The rate of change of population is proportional to its size.
6. During the process, the environment does not change in favour of one species and the genetic adaptation is sufficiently slow.

Let $f(x, y)$ be the population of snowshoe hares.

- The growth model for the hare population is:

$$\frac{dx}{dt} = \alpha x - \beta xy = f(x, y)$$

- The rate of change in a population is equal to the net increase (births) into the population minus the net decrease (deaths) of the population.
- The rate of predation upon the prey is assumed to be proportional to the rate at which the predators and the prey meet; this is represented above by βxy.
- If either x or y is zero then there can be no predation.

Let $g(x, y)$ be the the population of lynx.

- The growth model for the hare population is:

$$\frac{dy}{dt} = \delta xy - \gamma y = g(x, y)$$

- The primary growth for the lynx population depends on sufficient food for raising lynx kittens, which implies an adequate nutrients from predation on hares.
- This growth rate is similar to the death rate for the hare population with a different constant of proportionality, δxy.
- In the absence of hares, the lynx population declines in proportion to its own population, $-\gamma y$.

Predator-Prey Model - Analysis: The model satisfies the system of $ODEs$:

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

Population equilibrium occurs in the model when neither of the population levels is changing, i.e.

$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} = 0$$

- $y = 0, x = 0$ so $y_{eq} = 0, x_{eq} = 0$
- $\alpha - \beta y = 0 \implies y_{eq} = \frac{\alpha}{\beta}$ similarily $x_{eq} = \frac{\gamma}{\delta}$

Linear Analysis: The nonlinear model satisfies the system of $ODEs$:

$$\frac{dy}{dt} = \delta xy - \gamma y = 0$$

$$\frac{dx}{dt} = \alpha x - \beta xy = 0$$

To linearise given equation convert it into this form

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} (x - x_{eq})' \\ (y - y_{eq})' \end{bmatrix} \approx J \begin{bmatrix} x - x_{eq} \\ y - y_{eq} \end{bmatrix}$$

Here J is jacobian of given matrix

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} \alpha - \beta y_{eq} & -\beta x_{eq} \\ \delta y_{eq} & \delta x_{eq} - \gamma \end{bmatrix}$$

Linear Analysis at critical points(At $\mathrm{x}_{eq}, y_{eq} = (0,0)$)

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} \alpha - \beta y_{eq} & -\beta x_{eq} \\ \delta y_{eq} & \delta x_{eq} - \gamma \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix}$$

This matrix (diagonal) has the eigenvalues and associated eigenvectors:

$$\lambda_1 = \alpha, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \lambda_2 = -\gamma, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, the equilibrium $(0,0)$ is a saddle node with solutions exponentially growing along the x-axis and decaying along the y-axis, so

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\alpha t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-\gamma t}$$

Second Critical Point, At $(\mathrm{x}_{eq}, y_{eq}) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{bmatrix} \alpha - \beta y_{eq} & -\beta x_{eq} \\ \delta y_{eq} & \delta x_{eq} - \gamma \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{bmatrix}$$

eigenvalues of this matrix are pure imaginary
$\lambda_1 = i\sqrt{\alpha\gamma} = +i\omega, \quad \lambda_2 = -i\sqrt{\alpha\gamma} = -i\omega$.
Thus, the equilibrium $\left(\frac{b_1}{b_2}, \frac{a_1}{a_2}\right)$ is a center, which suggests that the solution cycles around for the predator-prey model. The linear solution satisfies:

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = c_1 \begin{bmatrix} \cos(\omega t) \\ A\sin(\omega t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(\omega t) \\ -A\cos(\omega t) \end{bmatrix}$$

where $A = \frac{b_2}{a_2}\sqrt{\frac{a_1}{b_1}}$

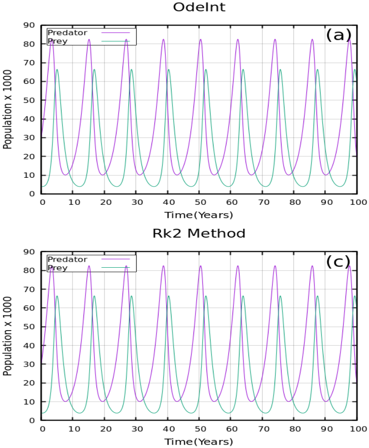

FIGURE: Time v/s Population **Euler Method:** 100000 data points; **RK2 Method:** 10000 data points and **RK4 Method:** 1000 data points

Predator-Prey (Lotka-Volterra) Model

Phase Portrait Plot

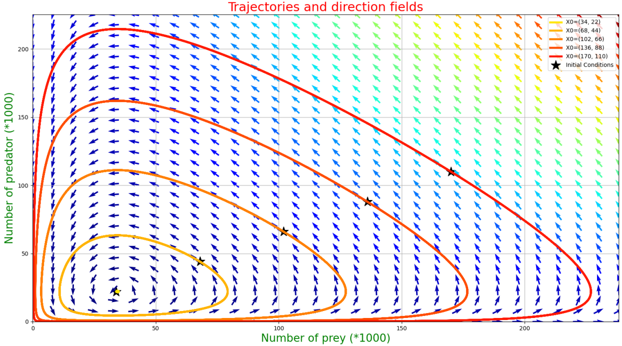

Figure: **Trajectories and Direction fields:** Phase-space plot for the predator prey problem for various initial conditions of the prey and predator population. The contours describe solutions of the system determined by their initial data, and since they are closed curves, the solutions are periodic oscillations.

Looking at the results of the simulation we can see that as the population of the prey begins the rise, the number of predators also begins to rise till the point at which predators kill off the prey faster than they can reproduce. Then the numbers begin to fall for the prey which thus causes a lack of food for the predators which numbers also begin to decline. The solution to this simulation is periodic meaning that the cycle will continue till infinity with the rise and fall of both populations. This looks very similar to the solution of simple harmonic motion, such as an un-damped spring-mass system except for the addition of a secondary plot