Predator-Prey (Lotka-Volterra) Model

Lotka-Volterra Model: Classical model for interaction of predator and prey.

- Alfred Lotka (1920), an American biologist and actuary, published the mathematical predator-prey model and its cyclical nature.
- Lotka originated many useful theories of stable populations, including the logistic model.
- Vito Volterra (1925) proposed the same model to explain data from fish studies of his son-in-law Humberto D'Ancona on the fishing industry in Italy.
- The classical Lotka-Volterra predator-prey model for the dynamics of the populations of a predator and its prey species.

Predator-Prey (Lotka-Volterra) Model

Lotka-Volterra Model: Let $f(x, y)$ be the population of snowshoe hares and $g(x, y)$ be the population of lynx..

- Alfred Lotka (1920), an American biologist and actuary, published the mathematical predator-prey model and its cyclical nature.
- The rate of change in a population is equal to the net increase (births) into the population minus the net decrease (deaths) of the population.
- Predators have limitless appetite.
- During the process, the environment does not change in favour of one species and the genetic adaptation is sufficiently slow.
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- The growth model for the hare population is:

$$\frac{dx}{dt} = \alpha x - \beta xy$$

Predator-Prey (Lotka-Volterra) Model

Lotka-Volterra Model with $f(x, y)$ as the population of snowshoe hares and $g(x, y)$ as the population of lynx.

- Alfred Lotka (1920), an American biologist and actuary, published the mathematical predator-prey model and its cyclical nature.
- The primary growth for the lynx population depends on sufficient food for raising lynx kittens, which implies an adequate nutrients from predation on hares.
- This growth rate is similar to the death rate for the hare population with a different constant of proportionality, δxy.
- In the absence of hares, the lynx population declines in proportion to its own population, $-\gamma y$.
- The growth model for the hare population is:

$$\frac{dy}{dt} = \delta xy - \gamma y$$

Predator-Prey Model Analysis

Predator-Prey Model - Analysis: The model satisfies the system of $\boldsymbol{ODEs}$:

$$\frac{dx}{dt} = \alpha x - \beta x y$$

$$\frac{dy}{dt} = \delta x y - \gamma y$$

Population equilibrium occurs in the model when neither of the population levels is changing, i.e.

$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} = 0$$

- $y = 0, x = 0$ so $y_{eq} = 0, x_{eq} = 0$
- $\alpha - \beta y = 0 \implies y_{eq} = \frac{\alpha}{\beta}$ similarly $x_{eq} = \frac{\gamma}{\delta}$

PREDATOR-PREY MODEL ANALYSIS

Linear Analysis: The nonlinear model satisfies the system of $\boldsymbol{ODEs}$:

$$\frac{dy}{dt} = \delta xy - \gamma y = 0$$

$$\frac{dx}{dt} = \alpha x - \beta xy = 0$$

To linearise given equation convert it into this form

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} (x - x_{eq})' \\ (y - y_{eq})' \end{bmatrix} \approx J \begin{bmatrix} x - x_{eq} \\ y - y_{eq} \end{bmatrix}$$

Here J is jacobean of given matrix

$$J = \begin{bmatrix} \partial f / \partial x & \partial f / \partial y \\ \partial g / \partial x & \partial g / \partial y \end{bmatrix} = \begin{bmatrix} \alpha - \beta y_{eq} & -\beta x_{eq} \\ \delta y_{eq} & \delta x_{eq} - \gamma \end{bmatrix}$$

Predator-Prey Model Analysis

Linear Analysis at critical points(At x$_{eq}, y_{eq} = (0,0)$)

$$J = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} \alpha - \beta y_{eq} & -\beta x_{eq} \\ \delta y_{eq} & \delta x_{eq} - \gamma \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix}$$

This matrix (diagonal) has the eigenvalues and associated eigenvectors:

$\lambda_1 = \alpha, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$ and $\lambda_2 = -\gamma, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Thus, the equilibrium $(0,0)$ is a saddle node with solutions exponentially growing along the x-axis and decaying along the y-axis, so

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\alpha t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-\gamma t}$$

Second Critical Point, At $(\mathrm{x}_{eq}, y_{eq}) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{bmatrix} \alpha - \beta y_{eq} & -\beta x_{eq} \\ \delta y_{eq} & \delta x_{eq} - \gamma \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{bmatrix}$$

eigenvalues of this matrix are pure imaginary

$\lambda_1 = i\sqrt{\alpha\gamma} = +i\omega, \quad \lambda_2 = -i\sqrt{\alpha\gamma} = -i\omega$.

Thus, the equilibrium $\left(\frac{b_1}{b_2}, \frac{a_1}{a_2}\right)$ is a center, which suggests that the solution cycles around for the predator-prey model. The linear solution satisfies:

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = c_1 \begin{bmatrix} \cos(\omega t) \\ A\sin(\omega t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(\omega t) \\ -A\cos(\omega t) \end{bmatrix}$$

where $A = \frac{b_2}{a_2}\sqrt{\frac{a_1}{b_1}}$