B.Sc. (H) Physics Semester 3 | **Computational Lab - MPII**
Sept. 20, 2021 | **Time Limit : 4 Hrs**

Name ____________________ **Roll No.** ____________________

The following rules apply:

- **Each step carries weight.**
- **Step should be commented.**
- **Answers must be supported.**

Question:	1	2	3	Total
Points:	0	7	3	10
Score:				

Experiment 3
Interpolation of Data : Polynomial Functions
Lagrange Interpolation

The idea is to describe the discrete data in terms of continuous functions, find a curve that exactly fits a given set of data. The Lagrange method tries to fit a sample of $n+1$ data (x_i, y_i) with a polynomial $p(x)$ (theoretical or empirical model) of degree $m = n$

$$p(a_0, ..., a_n, x) \equiv y(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{(n-1)} + a_nx^n$$

We thus interpolate the value of y for some x. We get a set of $n+2$ equations

$$y = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{(n-1)} + a_nx^n$$

$$y_i = a_0 + a_1x_i + a_2x_i^2 + ... + a_{n-1}x_i^{(n-1)} + a_nx_i^n \quad \forall \quad i = [0, n]$$

Writing it in a compact form we get

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \\ 1 & x & x^2 & \cdots & x^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \\ y \end{bmatrix} \longrightarrow \begin{vmatrix} y_0 & 1 & x_0 & x_0^2 & \cdots & x_0^n \\ y_1 & 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ y_n & 1 & x_n & x_n^2 & \cdots & x_n^n \\ y & 1 & x & x^2 & \cdots & x^n \end{vmatrix} = 0$$

The augmented matrix then yields

$$y_0C_{11} + y_1C_{21} + ... + y_nC_{(n+1)1} + yC_{(n+2)1} = 0$$

where C_{ij} are the cofactors.

$$y = -y_0\frac{C_{11}}{C_{(n+2)1}} - y_1\frac{C_{21}}{C_{(n+2)1}} - ... - y_n\frac{C_{(n+1)1}}{C_{(n+2)1}}$$

For a non-singular square matrix A we can find a unique inverse Z (often named A^{-1}) such that $AZ = I = ZA$. A procedure inspired by Cramer's rule for finding the inverse is

$$A^{-1} = \frac{A_{adj}}{|A|} \quad where \quad A_{adj} = C'$$

the adjoint A_{adj} being determined from the transpose of the cofactor C. The cofactor itself is determined from the minor matrices M_{ij} where $c_{ij} = (-1)^{i+j}a_{ij}|M_{ij}|$

1. **CONCEPT**

   (a) (0 points) **(Function Fit)**
   Suppose the given data set contains $n+1$ elements (x_i, y_i). The idea is to look for a linear combination of exactly $n+1$ set of polynomial (basis) functions $p_k(x) = x^k$ for all $k = 0$ to n that can best describe the given data. In nutshell, for all $i = [0, n]$

   $$y_i = \sum_{k=0}^{n} \alpha_k x_i^k$$

   where coefficients α_k decides the unique linear combination is required.

   (b) (0 points) **(Interpolation)**
   Now for any arbitrary x, the estimate y can be obtained by the relation

   $$y(x) = \sum_{k=0}^{n} \alpha_k x^k$$

2. **METHOD/CODE**

   (a) (3 points) **(Lagrange Interpolation Basis Function)**
   Given an $n+1$ data set (x_i, y_i) for $i = 0, ..., n$; the basis set of Lagrange fundamental polynomials $L_i(x) \forall\ i = 0$ to n is found to be

   $$L_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x - x_k)}{(x_i - x_k)}$$

   and the Lagrange interpolating polynomial is

   $$p_n(x) = y_0 L_0(x) + y_1 L_1(x) + ... + y_n L_n(x)$$

   (b) (3 points) **(Inverse Lagrange Interpolation)**
   Given an $n+1$ data set (x_i, y_i) for $i = 0, ..., n$; the basis set of of Lagrange fundamental polynomials $L_i(y) \forall\ i = 0$ to n is found to be

   $$L_i(y) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(y - y_k)}{(y_i - y_k)}$$

   and the inverse Lagrange interpolating polynomial is

   $$q_n(y) = x_0 L_0(y) + x_1 L_1(y) + ... + x_n L_n(y)$$

   (c) (1 point) **(Compare with inbuilt Python command)** The Lagrange Interpolation and Inverse Lagrange Interpolation polynomials.

3. **APPLICATIONS**

   (a) (2 points) **(Bessel Function)**
   Consider a set of data for the Bessel function $J_0(\beta)$:

β	0.00	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
$J_0(\beta)$	1.0	0.99	0.96	0.91	0.85	0.76	0.67	0.57	0.46	0.34	0.22	0.11	0.00	-0.10	-0.18	-0.26

   Find out the value of the Bessel function at $\beta = 2.3$.
   Also find out the value of β for which the Bessel function $J_0(\beta) = 0.5$.

   (b) (1 point) **(Linear Interpolation)**
   Consider a set of measurements of photodetector voltage as a function of incident laser intensity:

I	2.81	3.24	3.80	4.30	4.37	5.29	6.03
V	0.5	1.2	2.1	2.9	3.6	4.5	5.7

   Find out the value of incident laser intensity if the detected photodetector voltage is 2.4.