B.Sc. (H) Physics Semester 3 | **Computational Lab MP-II**
October 25, 2021 | **Time Limit : 4 Hrs**

Name ______________________ **Roll No.** ______________________

The following rules apply:

- **Each step carries weight.**
- **Step should be commented.**
- **Answers must be supported.**

Question:	1	2	3	Total
Points:	0	6	4	10
Score:				

Experiment 6
Differential Equation : Second Order Runge Kutta Method

In a variety of problems in physics ODE govern the underlying natural phenomena. There are two sides to "solving" differential equations numerically: 1. the operation of discretization and 2. the mathematics of convergence. Here we discuss the idea of discretization with the second-order ODE in one dimension as our model, together with a boundary condition:

$$\frac{d^2y}{dx^2} = f(x,y) \quad \textit{given} \quad y(a) = y_a \quad \textit{and} \quad y'(a) = v_a$$

where x is some independent variable (could be position, time, etc.), y a function of x and $f(x,y)$ is the given slope function of x, y. We must provide the value of $y(x)$ at some point to produce a unique solution, and we have taken $x = a$ with $y(a)$ given. In many cases, we are also given $y'(a)$. The goal of a numerical solution to the above is to generate a set of values approximating $y(x)$ at some definite locations guided by the ODE itself (i.e. not just a random set of values at random locations). From Taylor series expansion, any (well-behaved) function can be expanded close to a known point x by

$$y(x+\epsilon) = y(x) + \epsilon y'(x) + \frac{1}{2}\epsilon^2 y''(x) + O(\epsilon^3)$$

for small ϵ. This expansion says, among other things, that if you know the value of a function and all its derivatives at a point, then you can find its value at nearby points. As a corollary, if you know the value of a function and some of its derivatives at a point, you can find an approximation to the value of the function at nearby points.
We project these continuous functions $y(x)$ onto a grid, we might space the grid points equally $x_n = n\delta x$ for a provided δx, to get y_n. Finally, our numerical approximation (which will be compared with y_n) is denoted $\tilde{y}_n$.

1. **CONCEPT**

   (a) (0 points) **(Differential Equation -** ***Second Order*****)**
   In many problems, the direct functional relation between the dependent variable y and the independent variable x is not known. However, the second rate of change in y with respect to x is known and is given by a function $f(x, y)$. The idea is to deduce the function $y(x)$ from the second rate equation

   $$\frac{d^2y}{dx^2} = f(x, y(x))$$

   (b) (0 points) **(Initial Value Problem)**
   Any DE represent a family of curves/surfaces, however most problems pertain to picking out a particular curve which satisfies the given conditions. In nutshell, we seek a solution $y(x)$ which satisfies the initial conditions

   $$y(x_0) = y_0 \quad \& \quad y'(x_0) = v_0$$

2. **METHOD/CODE**

   (a) (4 points) **(Runge Kutta Method - Order 2)**
   In another form of order 2 Runge-Kutta, the incremental solution is represented by

   $$y(x+h) = y(x) + k_2 \quad \text{where} \quad k_1 = h * f(x, y) \;\; \& \;\; k_2 = h * f\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$$

   The domain (a, b) is populated with N equispaced nodal points to get $(a, x_1, ..., x_N, b)$ representative points where $x_i = a + ih$ and $h = (b - a)/(N + 1)$. The corresponding y_i is evaluated using the above recursive relation to yield $(y_0, y_1, ..., y_N, y_{N+1})$.

   (b) (2 points) **(Reducing Order 2 to Order 1)**
   The second order is reduced to the first order as

   $$\frac{dy}{dx} = z \quad \text{whereof} \quad \frac{dz}{dx} = f(x, y(x))$$

3. **APPLICATIONS** With all variables having their usual meanings

   (a) (1 point) **(Simple Harmonic Oscillator)**
   The time t and the displacement x bears the relation

   $$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

   for a simple harmonic oscillator.

   (b) (1 point) **(Damped Harmonic Oscillator)**
   The time t and the displacement x bears the relation

   $$\frac{d^2x}{dt^2} = -\frac{b}{m}\frac{dx}{dt} - \frac{k}{m}x$$

   for a damped oscillator.

   (c) (1 point) **(Simple Pendulum)**
   The time t and the displacement θ bears the relation

   $$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

   .

   (d) (1 point) **(Coupled Pendulum)**
   Consider two equal masses each of mass m,coupled via a spring of spring constant k. The time t and the displacement $x_{A \text{ or } B}$ bears the relation

   $$\frac{d^2x_A}{dt^2} = -\left(\omega_0^2 + \frac{k}{m}\right)x_A \qquad , \qquad \frac{d^2x_B}{dt^2} = -\left(\omega_0^2 + \frac{k}{m}\right)x_B$$

Plot the time evolution of the respective solutions and highlight the main features of the solution. Make and display the tabulated output with time.
Explain the values you choose in the class.