**B.Sc. (H) Physics Semester 3** — **Computational Lab - MPII**
**Oct. 5, 2021** — **Time Limit : 4 Hrs**

**Name** ______________________ **Roll No.** ______________________

The following rules apply:

- **Each step carries weight.**
- **Step should be commented.**
- **Answers must be supported.**

| Question: | 1 | 2 | 3 | Total |
|---|---|---|---|---|
| Points: | 0 | 4 | 6 | 10 |
| Score: | | | | |

# Experiment 4
## Differential Equation : First Order
## Euler Method

*In a variety of problems in physics ODE govern the underlying natural phenomena. There are two sides to "solving" differential equations numerically: 1. the operation of discretization and 2. the mathematics of convergence. Here we discuss the idea of discretization. We start with the first-order ODE in one dimension as our model, together with a boundary condition:*

$$\frac{dx}{dt} = f(t, x) \quad given \quad f(a) = f_a$$

*where $t$ is some independent variable (could be position, time, etc.), $x$ a function of $t$ and $f(t, x)$ is the given slope function of $t, x$. We must provide the value of $x(t)$ at some point to produce a unique solution, and we have taken $t = a$ with $f(a)$ given. The goal of a numerical solution to the above is to generate a set of values approximating $x(t)$ at some definite locations guided by the ODE itself (i.e. not just a random set of values at random locations).*
*From Taylor series expansion, any (well-behaved) function can be expanded close to a known point $t$ by*

$$f(t+\epsilon) = f(t) + \epsilon f'(t) + \frac{1}{2}\epsilon^2 f''(t) + O(\epsilon^3)$$

*for small $\epsilon$. This expansion says, among other things, that if you know the value of a function and all its derivatives at a point, then you can find its value at nearby points. As a corollary, if you know the value of a function and some of its derivatives at a point, you can find an approximation to the value of the function at nearby points.*
*We project these continuous functions $x(t)$ onto a grid, we might space the grid points equally $t_n = n\delta t$ for a provided $\delta t$, to get $f_n$. Finally, our numerical approximation (which will be compared with $f_n$ ) is denoted $\tilde{f}_n$.*

1. **CONCEPT**

   (a) (0 points) **(Differential Equation -** ***First Order*****)**
   In many problems, the direct functional relation between the dependent variable $x$ and the independent variable $t$ is not known. However, the rate of change in $x$ with respect to $t$ is known and is given by a function $f(t, x)$. The idea is to deduce the function $x(t)$ from the rate equation

   $$\frac{dx}{dt} = f(t, x(t))$$

   (b) (0 points) **(Initial Value Problem)**
   Any $DE$ represent a family of curves/surfaces, however most problems pertains to picking out a particular curve which satisfies the given conditions. In nutshell, we seek a solution $x(t)$ which satisfies the initial conditions

   $$x(t_0) = x_0$$

2. **METHOD/CODE**

   (a) (2 points) **(Euler Method - Forward)**
   The derivative is approximated as $x' = \{x(t+h) - x(t)\}/h$ and the $ODE$ is represented by

   $$\frac{x(t+h) - x(t)}{h} = f(t, x(t)) \quad \Rightarrow \quad x(t+h) = x(t) + hf(t, x(t))$$

   The domain $(a, b)$ is populated with $N$ equispaced nodal points to get $(a, t_1, ..., t_N, b)$ representative points where $t_i = a + ih$ and $h = (b-a)/(N+1)$. The corresponding $x_i$ is evaluated using the above recursive relation to yield $(x_0, x_1, ..., x_N, x_{N+1})$.

   (b) (2 points) **(Runge Kutta - Classical)**
   With two values of slopes, the classical second order Runge-Kutta method (RK2) is summarized as follows,

   $$k_1 = hf(t_n, x_n)$$
   $$k_2 = hf(t_n + h, x_n + k_1)$$
   $$x_{n+1} = x_n + (k_1 + k_2)/2$$

   The domain $(a, b)$ is populated with $N$ equispaced nodal points to get $(a, t_1, ..., t_N, b)$ representative points where $t_i = a + ih$ and $h = (b-a)/(N+1)$. The corresponding $t_i$ is evaluated using the above recursive relation to yield $(x_0, x_1, ..., x_N, x_{N+1})$.

3. **APPLICATIONS**

   (a) (2 points) **(Radioactive Decay)**
   The time $t$ and the population $N$ bears the relation $\frac{dN}{dt} = -\lambda N$.
   Note $f(t, N) = -N/\tau$ where $\tau = 1/\lambda$.
   Given $N_0 = 20,000$ and $t_{1/2} = 4\ yrs$.

   (b) (2 points) **(*RC* Circuit)**
   The time $t$ and the voltage $V$ bears the relation $dV/dt = -V/\tau$ where $\tau = RC$.
   Note $f(t, V) = -V/\tau$.
   Given $V_0 = 10\ V$, $R = 1\ k\Omega$ and $C = 1\ \mu F$.

   (c) (2 points) **(Stokes' Law)**
   The time $t$ and the speed $v$ bears the relation $mdv/dt = -6\pi\eta av$ where all variables have their usual meaning.
   Note $f(t, v) = -v/\tau$ where $\tau = m/6\pi\eta a$.
   Given $\eta = \ Ns/m$, $a = \ m$ and $m = \ gm$.

Plot the time evolution of the respective solutions and highlight the main features of the solution. Make and display the tabulated output with time.