Mathematical Physics Lab Practicals

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Unique Paper Code: 32221301

Paper Title: Mathematical Physics Lab

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Programmes

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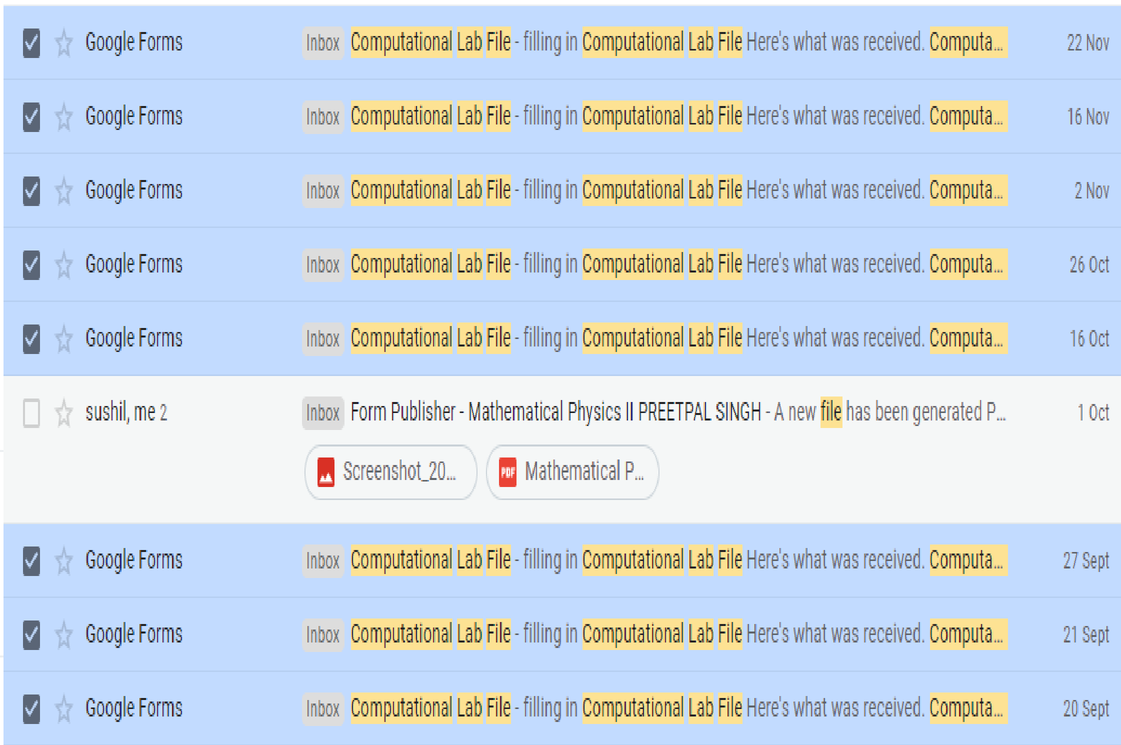


Figure 1: Submission of all Practicals

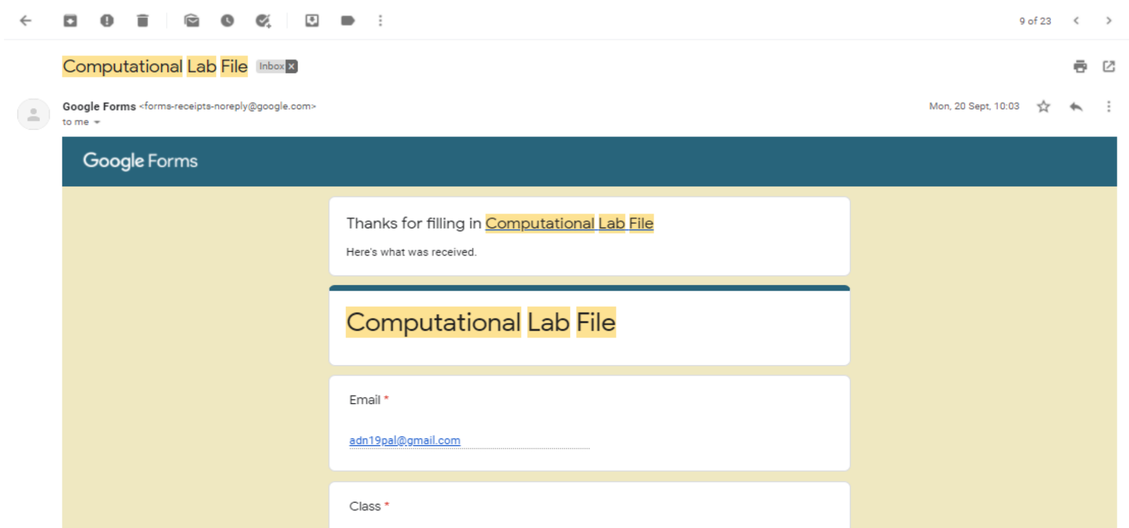


Figure 2: Trapezoidal and Simpson Method

1 Trapezoidal and Simpson Method

```
import numpy as np
import math
import matplotlib.pyplot as plt
import scipy.integrate as integrate
def trapeziodal(func,a,b,n):
    y=[]
    h=(b-a)/n
    for i in range(n+1):
        y.append(func(a+i*h))  #y at limit points
    trp=h*(func(a)+func(b))/2
    for j in range(1,len(y)-1):
        trp=trp+h*(y[j])
    return(trp)
def simpson(func,a,b,n):
    h=(b-a)/(2*n)
    simp=h*(func(a)+func(b))/3
    for i in range(1,2*n):
        if(i%2==0):
            simp=simp+2*h*func(a+i*h)/3
        elif(i%2==1):
            simp=simp+4*h*func(a+i*h)/3
    return(simp)


def graph(func,a,b,n):
    N=np.arange(1,n+1,1)    #Number of Subintervals
    H2=(b-a)/(2*N)
```

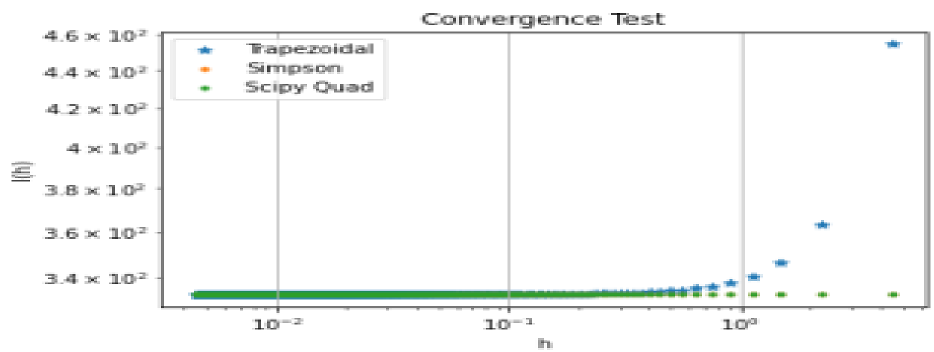
```
    It, Is, Iq =[], [], []
    for i in N:
        z=trapeziodal(func,a,b,i)
        It.append(z)
        z1=simpson(func,a,b,i)
        Is.append(z1)
        analytic = integrate.quad(func, a, b)
        Iq.append(analytic)
    plt.scatter(H2,It,label="Trapezoidal",marker="*")
    plt.scatter(H2,Is,label="Simpson",marker=".")
    plt.scatter(H2,Is,label="Scipy Quad",marker=".")
    plt.yscale("log")
    plt.xscale("log")
    plt.legend()
    plt.xlabel("h")
    plt.ylabel("I(h)")
    plt.title("Convergence Test")
    plt.grid(True)
    plt.show()


def Q2a():
    trp=trapeziodal(func,a,b,n)
    print("Integration Trapezoidal method",trp)

def Q2b():
    simp=simpson(func,a,b,n)
    print("Result by Simpson Method",float(simp))

def Q2c():

    analytic = integrate.quad(func, a, b)
    print("Analytic Solution = ", analytic[0])
    trap=trapeziodal(func,a,b,n)
    print("Solution by trapezoidal method: ",trap)
    err=abs(analytic[0]-trap)
    print("Truncation Error = ",float(err))

def Q2d():
    graph(func,a,b,n)


def Q3a():
    v=np.array([0.0, 0.5, 2.0, 4.05, 8.0, 12.5, 18.0, 24.5, 32.0, 40.5, 50.0])
    c=np.array([0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0])
    h=(c[-1]-c[0])/(len(c)-1)
    pwr_trap=h*(v[0]+v[-1])/2
    for j in range(1,len(v)-1):
        pwr_trap=pwr_trap+h*(v[j])
    print("Power using trapezoidal method: ", float(pwr_trap), "J")
    pwr_simp=h*(v[0]+v[-1])/3
    for k in range(1,(len(v)-1)):
        if(k%2==0):
            pwr_simp=pwr_simp+2*h*(v[k])/3
        elif(k%2==1):
            pwr_simp=pwr_simp+4*h*(v[k])/3
    print("Power using simpson method : ", float(pwr_simp), "J")
```

```python
def Q3b():
    h=(b-a)/n
    trp=trapeziodal(func,a,b,n)
    print("Integral is {:.8} using Trapezoidal Method".format(float(trp)))
    #Simpson
    simp=simpson(func,a,b,n)
    print("Integral is {:.8} using Simpson Method".format(float(simp)))
    graph(func,a,b,n)

if __name__ == "__main__":
    func=eval("lambda x:"+input("F: "))
    a=float(input("a = "))
    b=float(input("b = "))
    n=int(input("N = "))
    Q2a()
    Q2b()
    Q2c()
    Q2d()
    Q3a()
    Q3b()
```

Listing 1: Trapezoidal and Simpson Method

```
F: x**2
a = 1
b = 10
N = 1000
Integration Trapezoidal method 333.0001214999998

In [83]:  Q2b()

Result by Simpson Method 332.9999999999995

In [84]:  Q2c()

Analytic Solution =  333.0
Solution by trapezoidal method:  333.0001214999998
Truncation Error =  0.00012149999980692883

In [85]:  Q2d()
```

Figure 3: Trapezoidal and Simpson Method Output




```
[36]: Q3a()
```

```
Power using trapezoidal method:  16.705000000000002 J
Power using simpson method :  16.606666666666666 J
```

```
[37]: Q3b()
```

```
Integral is 333.00012 using Trapezoidal Method
Integral is 333.0 using Simpson Method
```

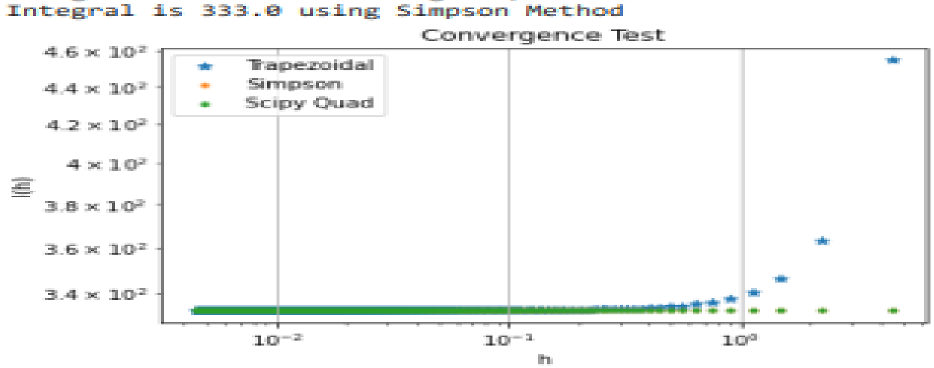


Figure 4: Trapezoidal and Simpson Method Output

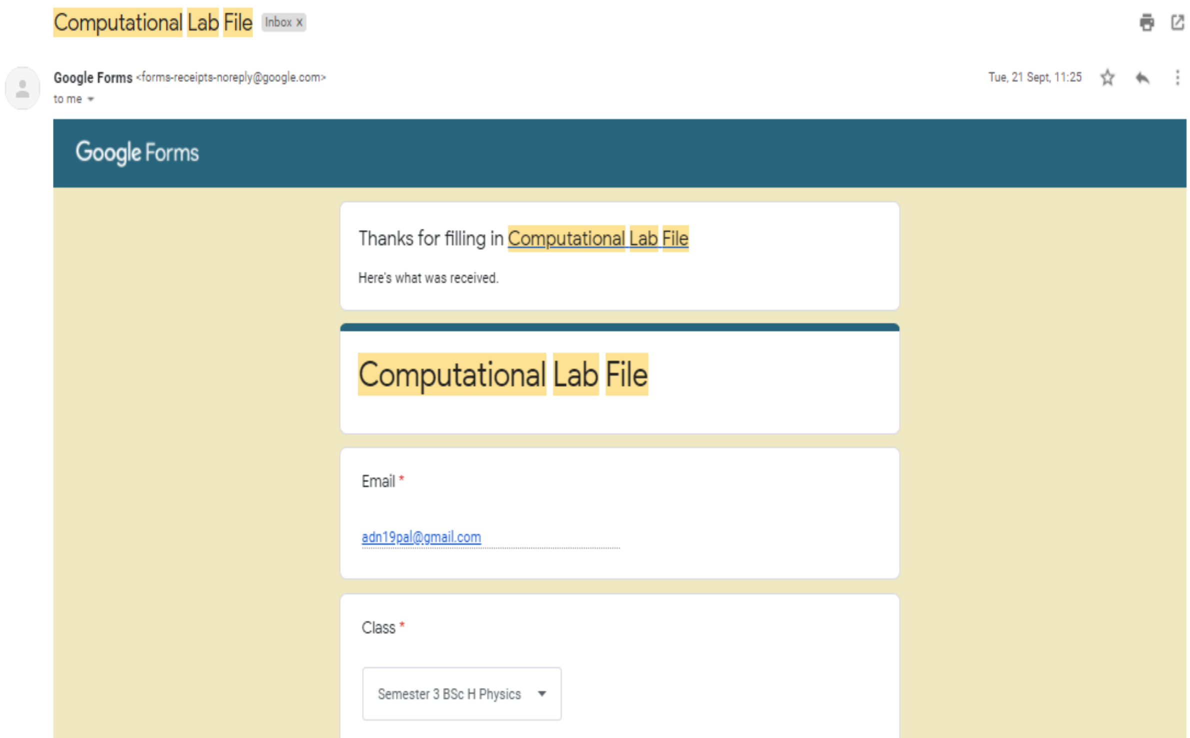


Figure 5: Legendre

2 Legendre Polynomial

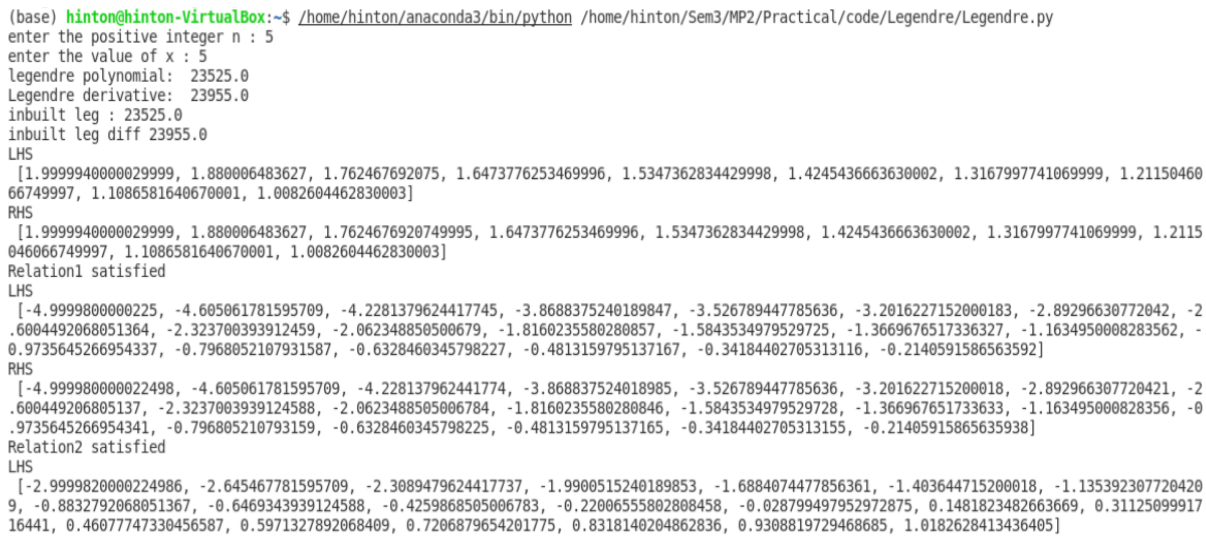
```
import matplotlib.pyplot as plt
import numpy as np
from scipy.special import eval_legendre,legendre
from scipy.integrate import quad
import math
n = float(input("enter the positive integer n : "))
x = float(input("enter the value of x : "))
def gm(n):
    if n == 1:
        return 1
    elif n == 0.5:
        return np.sqrt(np.pi)
    else :
        return (n-1)*gm(n-1)
def leg(n,x,m=0,p=0):
    if (n %2) == 0 :
        m = int(n/2)
    else :
```

```
        m = int((n-1)/2)
    for i in range(m+1):
        p+= (((-1)**i) * (gm(2*n-2*i+1))* (x**(n-2*i)))/((2**n) * gm(i+1) * gm(n-
    i+1) * gm(n-2*i+1))
    return p

def leg_dif(n,x,m=0,p=0):
    m = 0
    if (n %2) == 0 :
        m = int(n/2)
    else :
        m = int((n-1)/2)
    p = 0
    for i in range(m+1):
        p+= ((n-2*i)*((-1)**i) * (gm(2*n-2*i+1))* (x**(n-2*i-1)))/((2**n) * gm(i
    +1) * gm(n-i+1) * gm(n-2*i+1))
    return p

print("legendre polynomial: ",leg(n,x,m=0,p=0))
print("Legendre derivative: ",leg_dif(n,x,m=0,p=0))
print('inbuilt leg :',eval_legendre(n,x))
print('inbuilt leg diff',np.polyval(legendre(n).deriv(),x))
xdata = np.linspace(-0.999999,0.999999,100)
p0,p1,p2,p3,p0_diff,p1_diff,p2_diff,p3_diff,=[],[],[],[],[],[],[],[]


for i in xdata:
    p0.append(leg(0 ,i))
    p1.append(leg(1 ,i))
    p2.append(leg(2, i))
    p3.append(leg(3, i))
    p0_diff.append(leg_dif(0, i))
    p1_diff.append(leg_dif(1, i))
    p2_diff.append(leg_dif(2, i))
    p3_diff.append(leg_dif(3, i))
def writeintofile(file1,xdata,p0,p1,p2):
    with open(file1,'w') as file :
        for i in range(len(xdata)):
            file.write(str(xdata[i])+' , '+str(p0[i])+' , '+str(p1[i])+' , '+str(
    p2[i])+'\n')
writeintofile('leg00.dat', xdata, p0, p1, p2)
writeintofile('leg01.dat', xdata, p1_diff, p2_diff, p3_diff)
#plotting graph
plt.title("PLOT OF xdata VS Pn ",c= 'b')
plt.plot(xdata,p0)
plt.plot(xdata,p1)
plt.plot(xdata,p2)
plt.legend(['p0','p1','p2'],loc = 'best')
plt.xlabel('xdata')
plt.ylabel('Pn')
plt.grid()
plt.show()
plt.title("PLOT OF xdata VS DIFF Pn")
plt.plot(xdata,p1)
plt.plot(xdata,p0_diff)
plt.plot(xdata,p2_diff)
plt.legend(['p1','p0 diff','p2 diff'],loc = 'best')
```

```
plt.xlabel('xdata')
plt.ylabel('Pn diff')
plt.grid()
plt.show()


n,LHS,RHS=2,[],[]
for i in range(len(xdata)):
    LHS.append(n*p2[i])
    RHS.append(xdata[i]*p2_diff[i] - p1_diff[i])
print("LHS \n",LHS[0:10])
print("RHS \n", RHS[0:10])
if np.allclose(LHS,RHS):
        print("Relation1 satisfied")
else:
    print("Relation1 not satisfied")
writeintofile('leg02.dat', xdata, p2, p2_diff, p1_diff)

# relation 2
n,LHS,RHS=2,[],[]
for i in range(len(xdata)):
    LHS.append((2*n+1)*xdata[i]*p2[i])
    RHS.append((n+1)*p3[i]+n*p1[i])
print("LHS \n",LHS[0:20])
print("RHS \n", RHS[0:20])
if np.allclose(LHS,RHS):
        print("Relation2 satisfied")
else:
    print("Relation2 not satisfied")
writeintofile('leg03.dat', xdata, p2, p1, p3)
#Relation 3
n,LHS,RHS=3,[],[]
for i in range(len(xdata)):
    LHS.append(n*p3[i])
    RHS.append((2*n-1)*xdata[i]*p2[i] - (n-1)*p1[i])
print("LHS \n",LHS[0:20])
print("RHS \n", RHS[0:20])
if np.allclose(LHS,RHS):
        print("Relation3 satisfied")
else:
    print("Relation3 not satisfied")
writeintofile('leg04.dat', xdata, p3, p2, p1)

#orthogonality
A=[] ; B =[]
for n in range(3):
    for m in range(3):
        if n == m:
           A.append(2/(2*n+1))
        else:
           A.append(0)
        f=legendre(n)*legendre(m)
        inte , err = quad(f, -1, 1)
        B.append(inte)
RHS = np.array(B).reshape(3,3)
LHS = np.array(A).reshape(3,3)
print("RHS = ",RHS)
```

```python
print("LHS = ",LHS)
if np.allclose(LHS,RHS):
   print("Orthogonality verified")
else:
    print("Orthogonality not verified")
```

Listing 2: Legendre Polynomial



```
(base) hinton@hinton-VirtualBox:~$ /home/hinton/anaconda3/bin/python /home/hinton/Sem3/MP2/Practical/code/Legendre/Legendre.py
enter the positive integer n : 5
enter the value of x : 5
legendre polynomial:  23525.0
Legendre derivative:  23955.0
inbuilt leg : 23525.0
inbuilt leg diff 23955.0
LHS
 [1.9999940000029999, 1.880006483627, 1.762467692075, 1.6473776253469996, 1.5347362834429998, 1.4245436663630002, 1.3167997741069999, 1.21150460
66749997, 1.1086581640670001, 1.0082604462830003]
RHS
 [1.9999940000029999, 1.880006483627, 1.7624676920749995, 1.6473776253469996, 1.5347362834429998, 1.4245436663630002, 1.3167997741069999, 1.2115
046066749997, 1.1086581640670001, 1.0082604462830003]
Relation1 satisfied
LHS
 [-4.9999800000225, -4.605061781595709, -4.2281379624417745, -3.8688375240189847, -3.526789447785636, -3.2016227152000183, -2.89296630772042, -2
.6004492068051364, -2.323700393912459, -2.062348850500679, -1.8160235580280857, -1.5843534979529725, -1.3669676517336327, -1.1634950008283562, -
0.9735645266954337, -0.7968052107931587, -0.6328460345798227, -0.4813159795137167, -0.34184402705313116, -0.2140591586563592]
RHS
 [-4.999980000022498, -4.605061781595709, -4.228137962441774, -3.868837524018985, -3.526789447785636, -3.201622715200018, -2.892966307720421, -2
.600449206805137, -2.3237003939124588, -2.0623488505006784, -1.8160235580280846, -1.5843534979529728, -1.366967651733633, -1.163495000828356, -0
.9735645266954341, -0.796805210793159, -0.6328460345798225, -0.4813159795137165, -0.34184402705313155, -0.21405915865635938]
Relation2 satisfied
LHS
 [-2.9999820000224986, -2.645467781595709, -2.3089479624417737, -1.9900515240189853, -1.6884074477856361, -1.403644715200018, -1.135392307720420
9, -0.8832792068051367, -0.6469343939124588, -0.4259868505006783, -0.22006555802808458, -0.028799497952972875, 0.1481823482663669, 0.31125099917
16441, 0.46077747330456587, 0.5971327892068409, 0.7206879654201775, 0.8318140204862836, 0.9308819729468685, 1.0182628413436405]
```

Figure 6: Legendre

```
 [1.9999940000029999, 1.880006483627, 1.7624676920749995, 1.6473776253469996, 1.5347362834429998, 1.4245436663630002, 1.3167997741069999, 1.2115
046066749997, 1.1086581640670001, 1.0082604462830003]
Relation1 satisfied
LHS
 [-4.9999800000225, -4.605061781595709, -4.2281379624417745, -3.8688375240189847, -3.526789447785636, -3.2016227152000183, -2.89296630772042, -2
.6004492068051364, -2.323700393912459, -2.062348850500679, -1.8160235580280857, -1.5843534979529725, -1.3669676517336327, -1.1634950008283562, -
0.9735645266954337, -0.7968052107931587, -0.6328460345798227, -0.4813159795137167, -0.34184402705313116, -0.2140591586563592]
RHS
 [-4.999980000022498, -4.605061781595709, -4.228137962441774, -3.868837524018985, -3.526789447785636, -3.201622715200018, -2.892966307720421, -2
.600449206805137, -2.3237003939124588, -2.0623488505006784, -1.8160235580280846, -1.5843534979529728, -1.366967651733633, -1.163495000828356, -0
.973564526695434l, -0.796805210793159, -0.6328460345798225, -0.4813159795137165, -0.34184402705313155, -0.21405915865635938]
Relation2 satisfied
LHS
 [-2.9999820000224986, -2.645467781595709, -2.3089479624417737, -1.9900515240189853, -1.6884074477856361, -1.403644715200018, -1.135392307720420
9, -0.8832792068051367, -0.6469343939124588, -0.4259868505006783, -0.22006555802808458, -0.028799497952972875, 0.1481823482663669, 0.31125099917
16441, 0.46077747330456587, 0.5971327892068409, 0.7206879654201775, 0.8318140204862836, 0.9308819729468685, 1.0182628413436405]
RHS
 [-2.9999820000225004, -2.645467781595709, -2.3089479624417746, -1.9900515240189849, -1.6884074477856361, -1.4036447152000182, -1.13539230772042
02, -0.8832792068051365, -0.6469343939124592, -0.42598685050067875, -0.2200655580280857, -0.028799497952972652, 0.14818234826636734, 0.311250999
1716439, 0.4607774733045663, 0.5971327892068412, 0.7206879654201773, 0.8318140204862834, 0.9308819729468689, 1.0182628413436408]
Relation3 satisfied
/home/hinton/anaconda3/lib/python3.8/site-packages/numpy/lib/polynomial.py:1329: FutureWarning: In the future extra properties will not be copie
d across when constructing one poly1d from another
  other = poly1d(other)
RHS =  [[2.00000000e+00 0.00000000e+00 5.55111512e-17]
 [0.00000000e+00 6.66666667e-01 0.00000000e+00]
 [5.55111512e-17 0.00000000e+00 4.00000000e-01]]
LHS =  [[2.         0.         0.        ]
 [0.         0.66666667 0.        ]
 [0.         0.         0.4       ]]
Orthogonality verified
```

Figure 7: Legendre

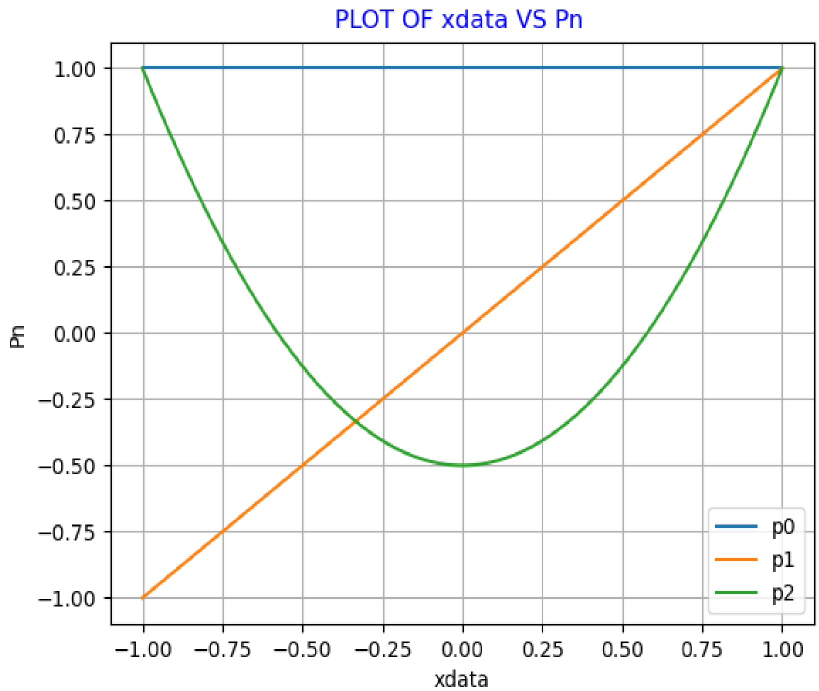


Figure 8: Legendre

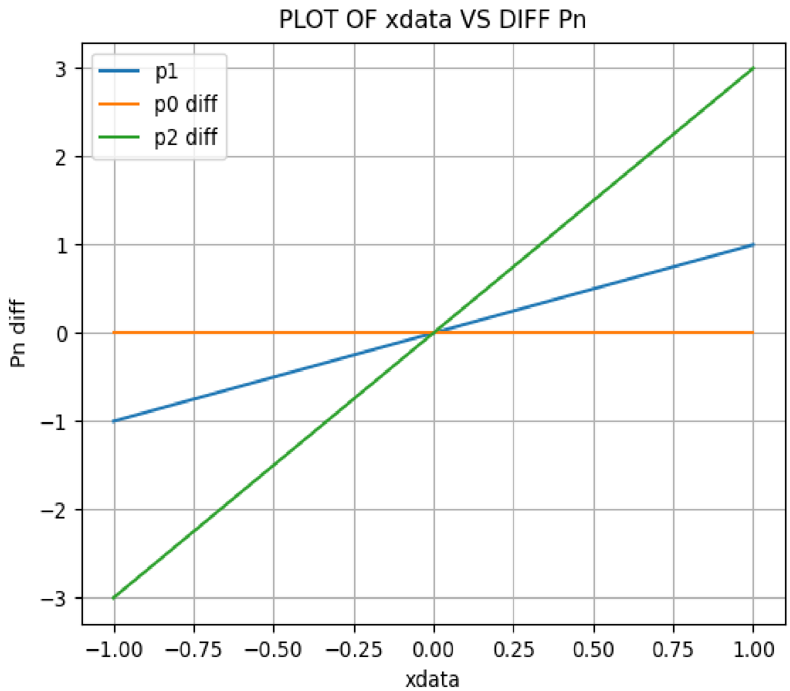


Figure 9: Legendre

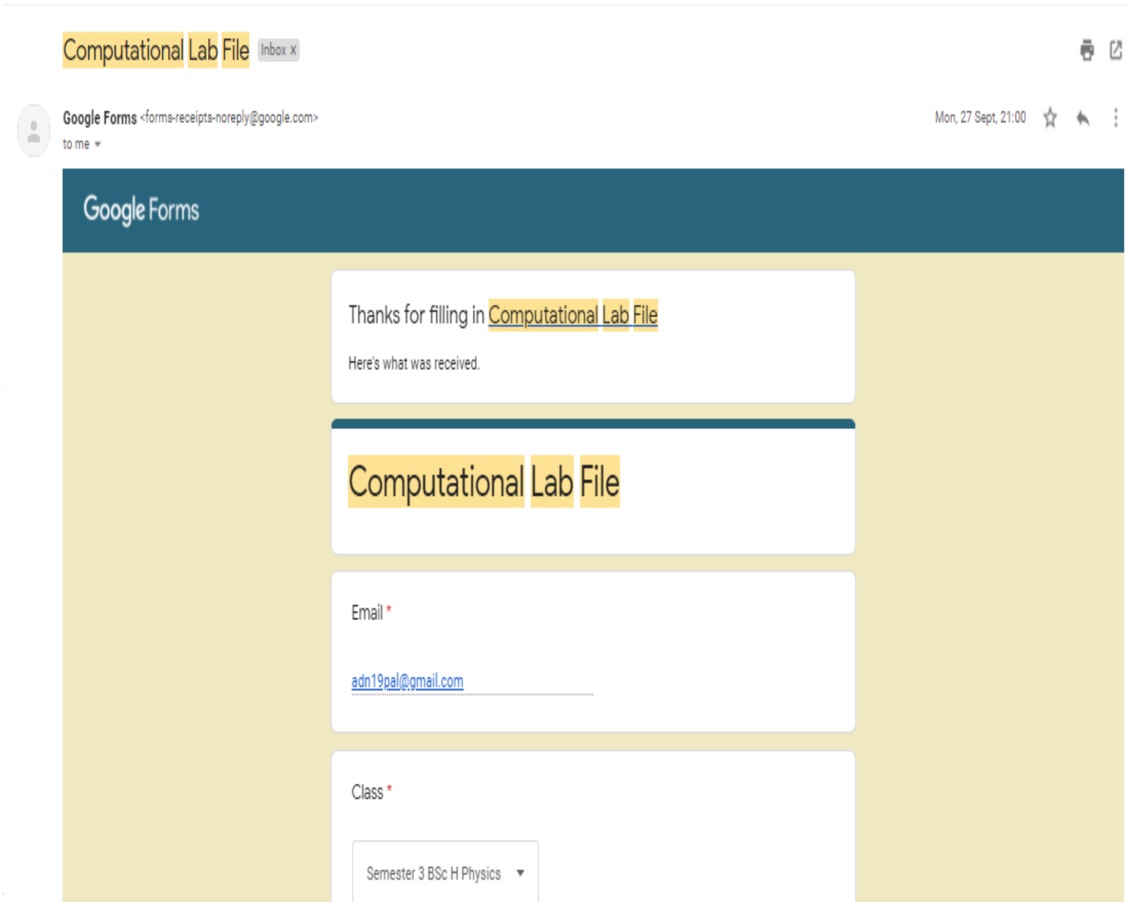


Figure 10: Lagrange Interpolation

3 Lagrange Interpolation

```
import numpy as np
from prettytable import PrettyTable
from scipy.interpolate import lagrange
import matplotlib.pyplot as plt
from sympy import Symbol
def Lagrange(x, L_x, L_y,y = 0):
  m = len(L_x)
  for i in range(0, m):
    prod_i = 1
    for j in range(m):
      if i==j:
        continue
```

```
      else:
        prod_i = prod_i*(x - L_x[j])/(L_x[i] - L_x[j])
    y = y + L_y[i]*prod_i
  return y
def Lagrange_inverse(y,L_x,L_y,x=0):
    return Lagrange(y,L_y,L_x,x=0)
def inbuilt(x, L_x, L_y):
    poly = lagrange(L_x, L_y)
    L=poly(x)
    return L

L1 = [0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, 2.2, 2.4, 2.6, 2.8,
     3.0]
L2 = [1.0, 0.99, 0.96, 0.91, 0.85, 0.76, 0.67, 0.57, 0.46, 0.34, 0.22, 0.11,
     0.00, -0.10, -0.18, -0.26]
I = [2.81, 3.24, 3.80, 4.30, 4.37, 5.29, 6.03]
V = [0.5, 1.2, 2.1, 2.9, 3.6, 4.5, 5.7]
mytable = PrettyTable(["Problem" , "inbuilt", "my function","Relative Error"])
mytable.add_rows([["3(a)(i)",inbuilt(2.3, L1, L2),Lagrange(2.3, L1, L2),
                  (inbuilt(2.3, L1, L2)-Lagrange(2.3, L1, L2))/inbuilt(2.3, L1,
     L2)],
                 ["3(a)(ii)",inbuilt(0.5, L2,L1),Lagrange(0.5, L2, L1,y = 0),
                  (inbuilt(0.5, L2,L1)-Lagrange(0.5, L1, L2,y = 0))/inbuilt(0.5,
      L2,L1)],
                 ["3(b)",inbuilt(2.4, V,I),Lagrange(2.4, V,I),
                  (inbuilt(2.4, V,I)-Lagrange(2.4, V,I))/inbuilt(2.4, V,I)]])
bes_inb,bes_inb_inv,lag_bes,lag_bes_inv,lin_inb,lin_inb_inv,lin_lag,lin_lag_inv=
     [],[],[],[],[],[],[],[]
for i in xrange:
  bes_inb.append(inbuilt(i, L1, L2))
  lag_bes.append(Lagrange(i, L1, L2))
x1range = np.linspace(0.0,6.0,100,float)
for i in x1range:
  lin_inb.append(inbuilt(i, V,I))
  lin_lag.append(Lagrange(i, V,I))
x2range = np.linspace(1.0,-0.3,100,float)
for i in x2range:
    bes_inb_inv.append(inbuilt(i, L2,L1))
    lag_bes_inv.append(Lagrange(i, L2, L1,y = 0))
x3range = np.linspace(2.0,6.5,100,float)
for i in x3range:
    lin_inb_inv.append(inbuilt(i, I,V))
    lin_lag_inv.append(Lagrange(i,I,V,y = 0))
fig,axs=plt.subplots(2,2,figsize=(15,8))
ax11,ax12,ax21,ax22=axs[0][0],axs[0][1],axs[1][0],axs[1][1]
ax11.plot(xrange,lag_bes,label="lagrange interpolation"),ax11.plot(xrange,bes_inb
     ,label="inbuilt")
ax11.scatter(L1,L2,label="Given Data Points"),ax11.scatter(2.3,Lagrange(2.3, L1,
     L2),label="Interpolation Point",c='red')
ax11.set_title("Bessel Interpolation"),ax11.set_ylabel("f(x)")

ax12.plot(x2range,lag_bes_inv,label="Inverse lagrange interpolation"),ax12.plot(
     x2range,bes_inb_inv,label="inbuilt")
ax12.scatter(L2,L1,label="Given Data Points"),ax12.set_ylabel("x")
ax12.set_ylim([-0.5,3.5]), ax12.set_title("Bessel Inverse Interpolation")
ax21.plot(x1range,lin_lag,label="lagrange interpolation"),ax21.plot(x1range,
     lin_inb,label="inbuilt")
```

```python
ax21.scatter(V,I,label="Given Data Points"),ax21.scatter(2.4,Lagrange(2.4, V, I),
     label="Interpolation Point",c='red')
ax21.set_title("V vs I Interpolation"),ax21.set_ylabel("V")
ax22.plot(x3range,lin_inb_inv,label="Inverse lagrange interpolation"),ax22.plot(
     x3range,lin_lag_inv,label="inbuilt")
ax22.scatter(I,V,label="Given Data Points"),ax22.set_title("I vs V Inverse
     Interpolation")
ax22.set_ylim([-10,10]),ax22.set_ylabel("I")
ax11.legend(),ax11.grid(True),ax12.legend(),ax12.grid(True),ax21.legend(),ax21.
     grid(True),ax22.legend(),ax22.grid(True)
plt.show()
```

Listing 3: Lagrange Interpolation

Problem	inbuilt	my function	Relative Error
3(a)(i)	0.05503416527935989	0.05503375768661485	7.406176562730459e-06
3(a)(ii)	1.5373769473726524	1.5373769470625294	0.397845435865116
3(b)	4.046204673660418	4.046204673660199	5.399921874158954e-14

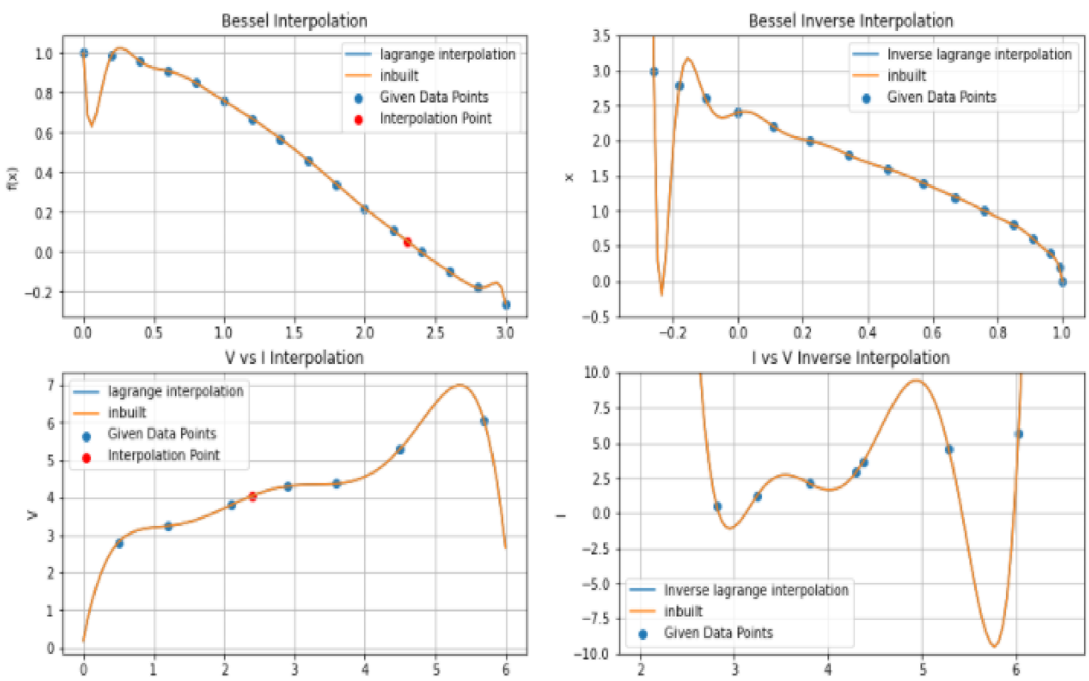


Figure 11: Lagrange Interpolation

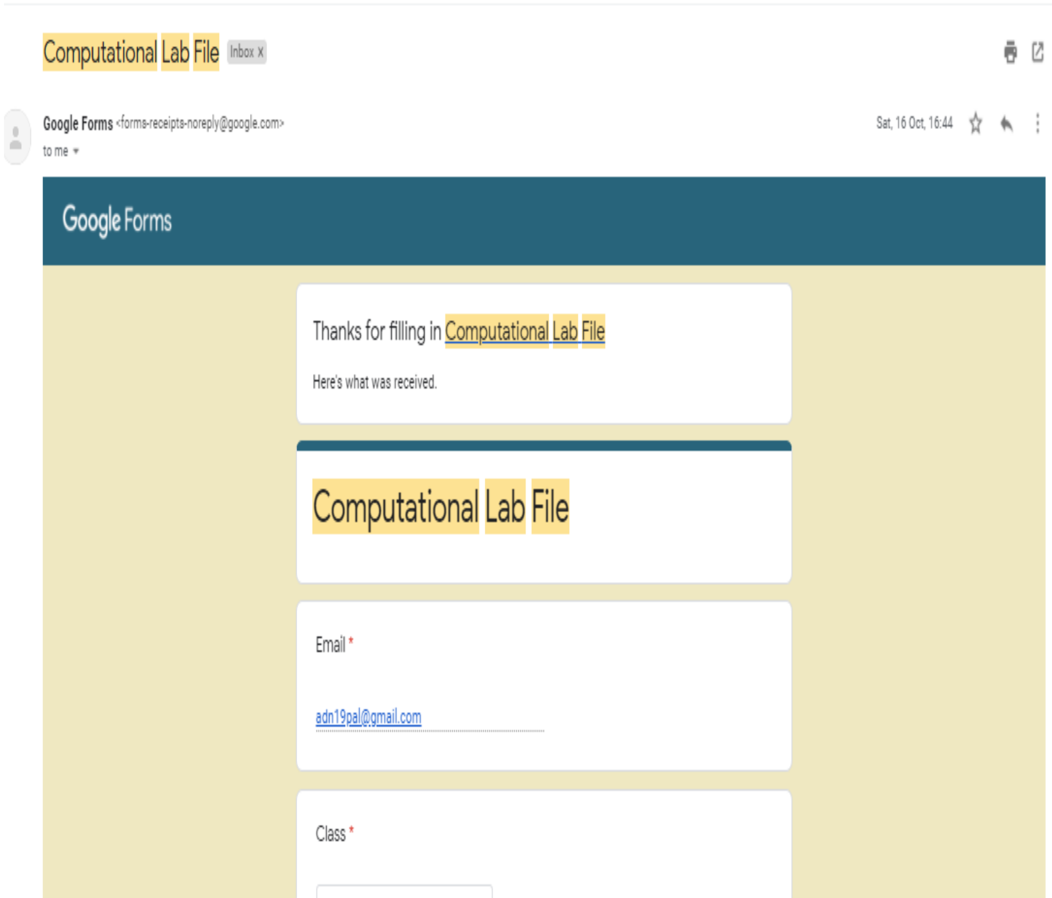


Figure 12: Radioactive Decay, RC Circuit and Stokes Law by Euler, RK2, RK4 Method

4 Radioactive Decay, RC Circuit and Stokes Law by Euler, RK2, RK4 Method

```
import numpy as np
import matplotlib.pyplot as plt
get_ipython().run_line_magic('matplotlib', 'inline')
import texttable as tt
tab = tt.Texttable()
def euler(f,a,b,n,yinit):
    h=(b-a)/(n)
    xs = a+np.arange(n)*h
    ys=np.zeros(n)
```

```
    y = yinit
    for j,x in enumerate(xs):
        ys[j]=y
        y+=h*f(x,y)
    return xs, ys
def rk2(f,a,b,n,yinit):
    h=(b-a)/(n)
    xs = a+np.arange(n)*h
    ys=np.zeros(n)
    y = yinit
    for j,x in enumerate(xs):
        ys[j]=y
        k0 = h*f(x,y)
        y+=h*f(x+h/2,y+k0/2)
    return xs, ys
def rk4(f,a,b,n,yinit):
    h=(b-a)/(n)
    xs = a+np.arange(n)*h
    ys=np.zeros(n)
    y = yinit
    for j,x in enumerate(xs):
        ys[j]=y
        k0 = h*f(x,y)
        k1 = h*f(x+h/2,y+k0/2)
        k2 = h*f(x+h/2,y+k1/2)
        k3 = h*f(x+h,y+k2)
        y+=(k0+2*k1+2*k2+k3)/6
    return xs, ys
def Analytic(yinit,a,b,h,tau):
    xs,ys=[],[]
    p=np.arange(a,b,h)
    for t in p:
        D=yinit*np.exp(-1*t/tau)
        ys.append(D)
        xs.append(t)
    return xs,ys
def graph(xs_1,ys_1,xs_2,ys_2,xs_3,ys_3,xs_4,ys_4,xs_5,ys_5,xs_6,ys_6,xs_7,ys_7,
     xs_8,ys_8,xs_9,ys_9,xs_10,ys_10,title):
    fig,axs=plt.subplots(3,2,figsize=(15,15))
    fig.suptitle(title, fontsize=30)
    ax11,ax12,ax21,ax22,ax31,ax32=axs[0][0],axs[0][1],axs[1][0],axs[1][1],axs
     [2][0],axs[2][1]
    ax11.plot(xs_1,ys_1,'^', color='green',label="euler"),ax11.plot(xs_4,ys_4,'-'
     , color='black',label="rk2")
    ax11.plot(xs_7,ys_7,'>',color='black',label="rk4"),ax11.plot(xs_7,ys_7,'*',
     color='brown',label="Analytic")
    ax11.set_title("Analytic v/s Euler v/s rk2 v/s rk4"),ax11.set_ylabel("N"),
     ax11.set_xlabel("Time")
    ax12.plot(xs_1,ys_1,'^', color='green',label="h ={}".format(xs_1[1]-xs_1[0]))
     ,ax12.plot(xs_2,ys_2,'-', color='black',label="h ={}".format(xs_2[1]-xs_2
     [0]))
    ax12.plot(xs_3,ys_3,'>',color='black',label="h ={}".format(xs_3[1]-xs_3[0])),
     ax12.plot(xs_7,ys_7,'*',color='brown',label="Analytic")
    ax12.set_title("Euler for different stepsize(h)"),ax12.set_ylabel("N"),ax12.
     set_xlabel("Time")
    ax21.plot(xs_4,ys_4,'^', color='green',label="h ={}".format(xs_4[1]-xs_4[0]))
     ,ax21.plot(xs_5,ys_5,'-', color='black',label="h ={}".format(xs_5[1]-xs_5
```

```
     [0]))
    ax21.plot(xs_6,ys_6,'>',color='black',label="h ={}".format(xs_6[1]-xs_6[0])),
     ax21.plot(xs_7,ys_7,'*',color='brown',label="Analytic")
    ax21.set_title("rk2 for different stepsize(h)"),ax21.set_ylabel("N"),ax21.
     set_xlabel("Time")
    ax22.plot(xs_7,ys_7,'^', color='green',label="h ={}".format(xs_7[1]-xs_7[0]))
     ,ax22.plot(xs_8,ys_8,'-', color='black',label="h ={}".format(xs_8[1]-xs_8
     [0]))
    ax22.plot(xs_9,ys_9,'>',color='black',label="h ={}".format(xs_9[1]-xs_9[0])),
     ax22.plot(xs_10,ys_10,'*',color='brown',label="Analytic")
    ax22.set_title("rk4 for different stepsize(h)"),ax22.set_ylabel("N"),ax22.
     set_xlabel("Time")
    ax31.plot(xs_1,(ys_10-ys_1)/ys_10,'.', color='green',label="euler"),ax31.plot
     (xs_1,(ys_10-ys_4)/ys_10,'.', color='black',label="rk2")
    ax31.plot(xs_1,(ys_10-ys_7)/ys_10,'.', color='red',label="rk4")
    ax31.set_title("Error Plot at h = 0.4"),ax31.set_ylabel("Absolute Error"),
     ax31.set_xlabel("Time")
    ax32.plot(xs_1,(ys_1-ys_4)/ys_1,'.', color='green',label="euler-rk2"),ax32.
     plot(xs_1,(ys_7-ys_4)/ys_7,'.', color='black',label="rk4-rk2")
    ax32.plot(xs_1,(ys_1-ys_7)/ys_1,'.', color='red',label="euler-rk4")
    ax32.set_title("Comparative Error Plot at h = 0.4"),ax32.set_ylabel("Absolute
      Error"),ax32.set_xlabel("Time")
    ax11.legend(),ax11.grid(True),ax12.legend(),ax12.grid(True),ax21.legend(),
     ax21.grid(True),ax22.legend(),ax22.grid(True)
    ax31.legend(),ax31.grid(True),ax32.legend(),ax32.grid(True)
    plt.show()
def q3_a(a,yinit,t_half):
    b = 5*t_half
    tau=t_half/np.log(2)
    h = t_half/10
    n = int((b-a)/h)
    decay = lambda x, y: -1*y/tau
    xs_1, ys_1 = euler(decay,a,b,n,yinit)
    xs_2, ys_2 = euler(decay,a,b,2*n,yinit)
    xs_3, ys_3 = euler(decay,a,b,4*n,yinit)
    xs_4, ys_4 = rk2(decay,a,b,n,yinit)
    xs_5, ys_5 = rk2(decay,a,b,2*n,yinit)
    xs_6, ys_6 = rk2(decay,a,b,4*n,yinit)
    xs_7, ys_7 = rk4(decay,a,b,n,yinit)
    xs_8, ys_8 = rk4(decay,a,b,2*n,yinit)
    xs_9, ys_9 = rk4(decay,a,b,4*n,yinit)
    xs_10, ys_10 = Analytic(yinit,a,b,h,tau)
    print("Radioactive Decay", "h =", h)
    headings_1 = ["t" ,"Analytic","euler","rk2","rk4","Ab_error euler","Ab_error
     rk2","Ab_error rk4"]
    tab.header(headings_1)
    for row in zip(xs_1,ys_10,ys_1,ys_4,ys_7,(ys_10-ys_1)/ys_10,(ys_10-ys_4)/
     ys_10,(ys_10-ys_7)/ys_10):
        tab.add_row(row)
        tab.set_max_width(0)
        tab.set_precision(6)
    s = tab.draw()
    print(s)
    tab.reset()

    graph(xs_1,ys_1,xs_2,ys_2,xs_3,ys_3,xs_4,ys_4,xs_5,ys_5,xs_6,ys_6,xs_7,ys_7,
     xs_8,ys_8,xs_9,ys_9,xs_10,ys_10,"Radioactive Decay")
```

```
def q3_b(a,yinit,R,C):
    b = 5*R*C
    tau=R*C
    h = tau/10
    n = int((b-a)/h)
    rc = lambda x, y: -1*y/tau
    xs_1, ys_1 = euler(rc,a,b,n,yinit)
    xs_2, ys_2 = euler(rc,a,b,2*n,yinit)
    xs_3, ys_3 = euler(rc,a,b,4*n,yinit)

    xs_4, ys_4 = rk2(rc,a,b,n,yinit)
    xs_5, ys_5 = rk2(rc,a,b,2*n,yinit)
    xs_6, ys_6 = rk2(rc,a,b,4*n,yinit)

    xs_7, ys_7 = rk4(rc,a,b,n,yinit)
    xs_8, ys_8 = rk4(rc,a,b,2*n,yinit)
    xs_9, ys_9 = rk4(rc,a,b,4*n,yinit)
    xs_10, ys_10 = Analytic(yinit,a,b,h,tau)
    print("RC Circuit", "h =", h)
    headings_1 = ["t" ,"Analytic","euler","rk2","rk4","$\delta$","Ab_error rk2","
     Ab_error rk4"]
    tab.header(headings_1)
    for row in zip(xs_1,ys_10,ys_1,ys_4,ys_7,(ys_10-ys_1)/ys_10,(ys_10-ys_4)/
     ys_10,(ys_10-ys_7)/ys_10):
        tab.add_row(row)
        tab.set_max_width(0)
        tab.set_precision(6)
    s = tab.draw()
    print(s)
    tab.reset()

    graph(xs_1,ys_1,xs_2,ys_2,xs_3,ys_3,xs_4,ys_4,xs_5,ys_5,xs_6,ys_6,xs_7,ys_7,
     xs_8,ys_8,xs_9,ys_9,xs_10,ys_10,"RC Circuit")

def q3_c(a,yinit,eta,rad,m):
    tau=m/((np.pi)*6*rad*eta)
    b = 5*tau
    h = tau/10
    n = int((b-a)/h)
    stokes = lambda x, y: -1*y/tau
    xs_1, ys_1 = euler(stokes,a,b,n,yinit)
    xs_2, ys_2 = euler(stokes,a,b,2*n,yinit)
    xs_3, ys_3 = euler(stokes,a,b,4*n,yinit)
    xs_4, ys_4 = rk2(stokes,a,b,n,yinit)
    xs_5, ys_5 = rk2(stokes,a,b,2*n,yinit)
    xs_6, ys_6 = rk2(stokes,a,b,4*n,yinit)
    xs_7, ys_7 = rk4(stokes,a,b,n,yinit)
    xs_8, ys_8 = rk4(stokes,a,b,2*n,yinit)
    xs_9, ys_9 = rk4(stokes,a,b,4*n,yinit)
    xs_10, ys_10 = Analytic(yinit,a,b,h,tau)
    xs_10.pop(int(xs_10[-1]))
    ys_10.pop(int(ys_10[-1]))
    print("Stokes Law", "h =", h)
```

```
    headings_1 = ["t" ,"Analytic","euler","rk2","rk4","Ab_error euler","Ab_error
     rk2","Ab_error rk4"]
    tab.header(headings_1)
    for row in zip(xs_1,ys_10,ys_1,ys_4,ys_7,(ys_10-ys_1)/ys_10,(ys_10-ys_4)/
     ys_10,(ys_10-ys_7)/ys_10):
        tab.add_row(row)
        tab.set_max_width(0)
        tab.set_precision(6)
    s = tab.draw()
    print(s)
    tab.reset()
    graph(xs_1,ys_1,xs_2,ys_2,xs_3,ys_3,xs_4,ys_4,xs_5,ys_5,xs_6,ys_6,xs_7,ys_7,
     xs_8,ys_8,xs_9,ys_9,xs_10,ys_10,"Stokes Law")
if __name__ == "__main__":
    q3_a(0,20000,4)
    q3_b(0,10,1e3,1e-6)
    q3_c(0,10,10,0.2,200)
```

Listing 4: Radioactive Decay, RC Circuit and Stokes Law by Euler, RK2, and RK4 Method

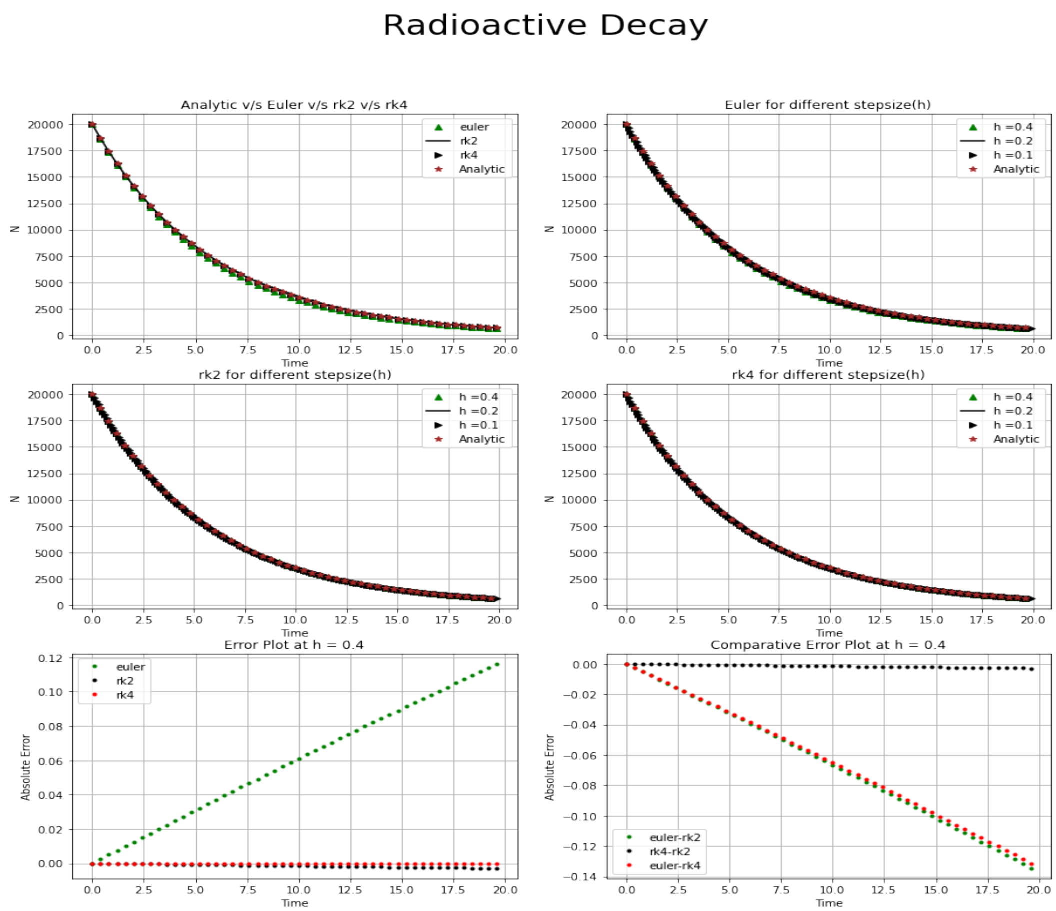


Figure 13: Radioactive Decay

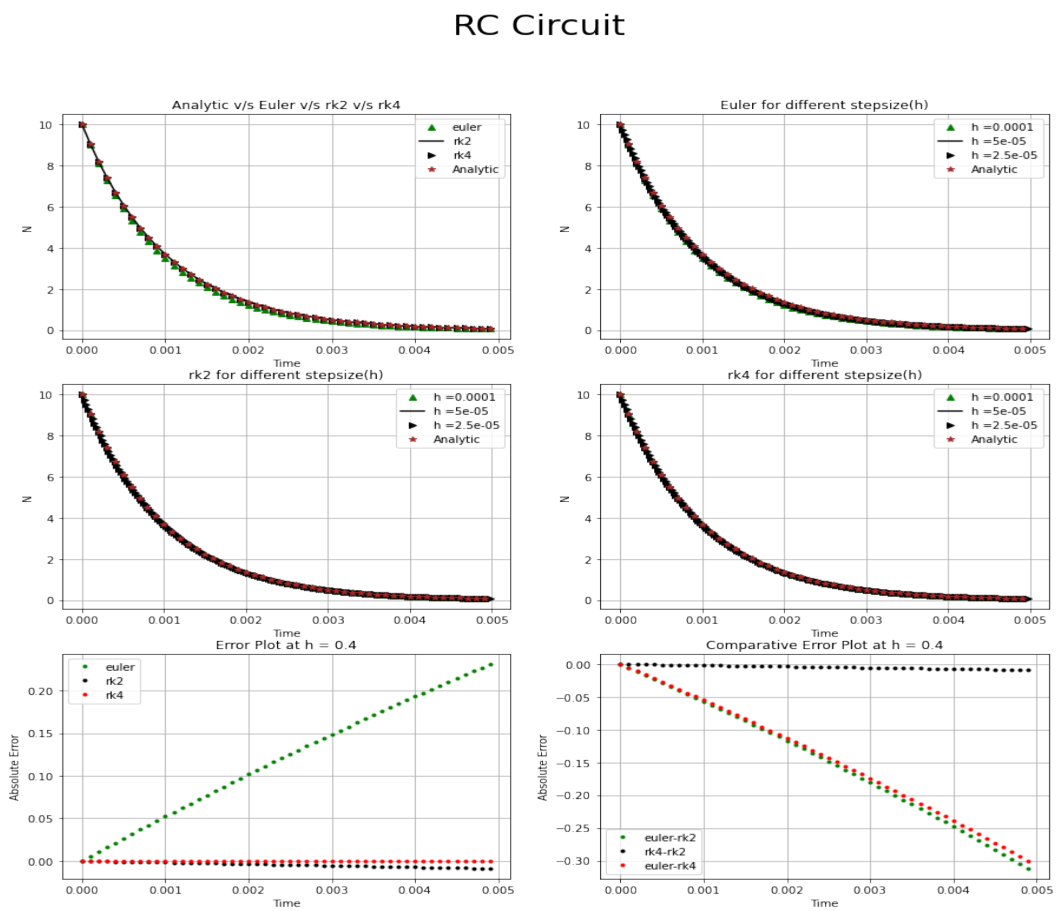


Figure 14: RC Circuit

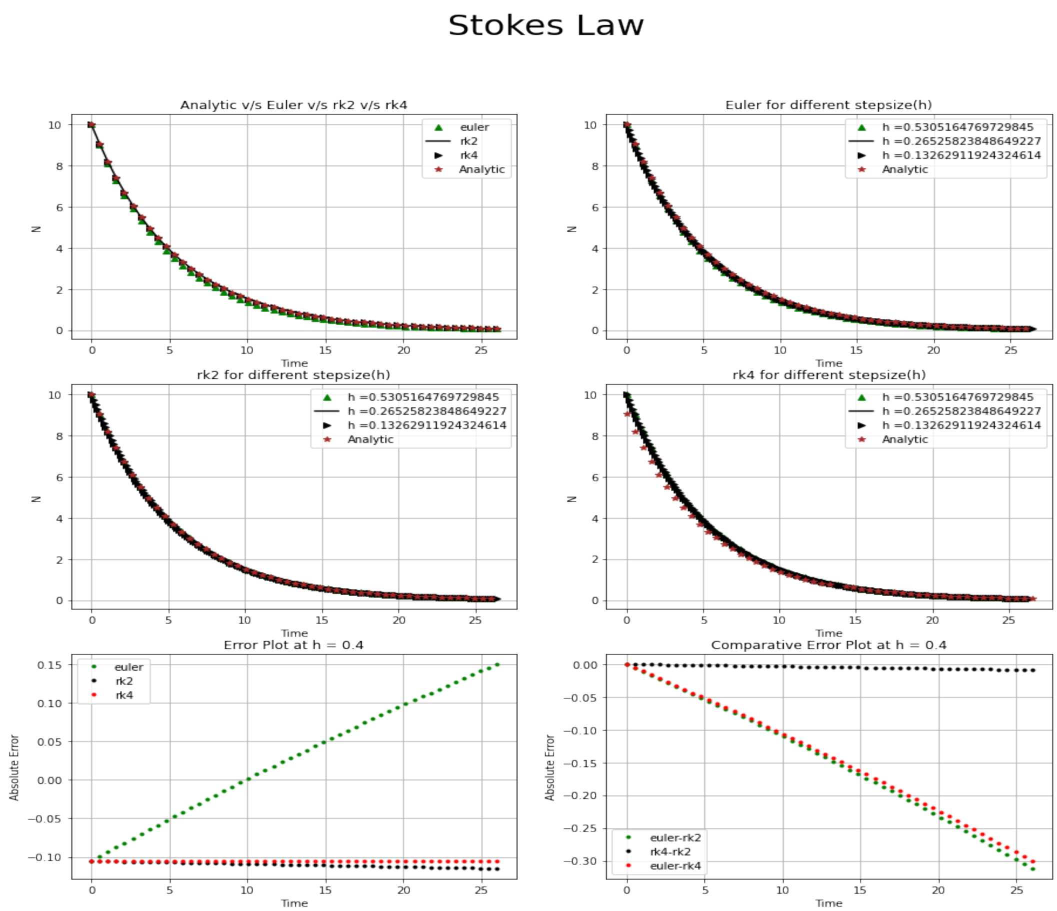


Figure 15: Stokes Law

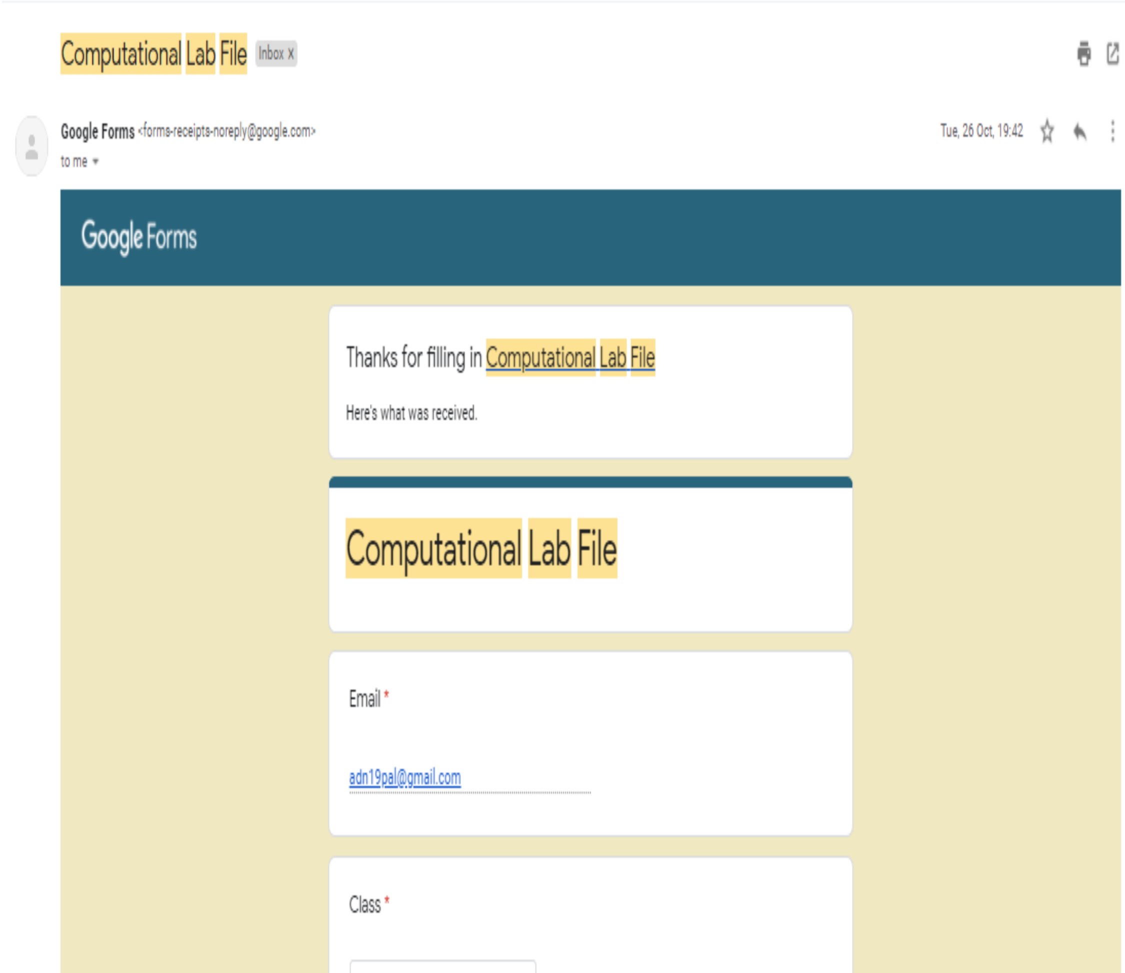


Figure 16: 2nd Order Coupled diffential equations using Euler, RK2, RK4 Method

5 2nd Order Coupled diffential equations using Euler, RK2, RK4 Method

```
import pandas as pd
#%matplotlib inline
import matplotlib.pyplot as plt
from scipy import integrate
import numpy as np
def simpe_harmonic(X, t, cons):  #simple harmonic oscillator
```

```
    k,m = cons
    x, y = X
    dotx = y
    dot2x = -k*x/m
    return np.array([dotx, dot2x])
def damped_harmonic(X, t, cons):  #damped oscillator
    k,m,b = cons
    x, y = X
    dotx = y
    doty = -k*x-b*dotx
    return np.array([dotx, doty])
def simple_pendulum(X, t, cons):  #Simple Pendulum
    g,L = cons
    x, y = X
    dotx = y
    doty = -g*x/L
    return np.array([dotx, doty])
def RK2(func, X0, t,cons):
    dt = t[1] - t[0]
    nt = len(t)
    X  = np.zeros([nt, len(X0)])
    X[0] = X0
    for i in range(nt-1):
        k1 =dt* func(X[i], t[i], cons)
        k2 = dt*func(X[i] +  k1, t[i] + dt, cons)
        X[i+1] = X[i] + (k1 +  k2 )/2
    return X, t


if __name__ == "__main__":
    k = 1; m = 1; cons=(k,m); t_p = 2*np.pi*np.sqrt(m/k); tmin=0; tmax=7*t_p; Nt
     = 1000; x0 = [2,0]
    t = np.linspace(tmin,tmax,Nt)
    Xrk2 = RK2(simpe_harmonic, x0, t, cons)
    x,y= Xrk2[0].T
    t=Xrk2[1]
    tdimless=t/t_p
    #Plotting

    plt.title('Simple Harmonic Oscillator', fontsize=20)
    plt.plot(tdimless,x,color='black',label="Displacement");plt.plot(tdimless,y,
     color='brown',label="Velocity")
    plt.grid(); plt.legend()
    plt.show()
#Damped Harmonic Oscillator
    k = 1; m = 1; t_p = 2*np.pi*np.sqrt(m/k); tmin=0; tmax=30*t_p; Nt = 1000; x0
     = [1,0]
    t = np.linspace(tmin,tmax,Nt)
    tdimless=t/t_p; dis=[];vel=[];time=[]
    ba=[0.15,2,5]
    for b in ba:
        cons=(m,k,b)
        Xrk2=RK2(damped_harmonic,x0,tdimless, cons)
        x,y= Xrk2[0].T
        tdimless=Xrk2[1]
        dis.append(x)
        vel.append(y)
```

```
    fig, axs = plt.subplots(3,figsize=(11,15))
    fig.suptitle('Damped Harmonic Oscillator', fontsize=20)
    axs[0].plot(tdimless,dis[0],label = "displacement")
    axs[0].set(xlabel="time ",title="Underdamped, b = 0.15")
    axs[0].plot(tdimless,vel[0],label = "velocity")
    axs[0].plot(tdimless,1/2*k*dis[0]**2)
    axs[0].grid(); axs[0].legend()
    axs[1].plot(tdimless,dis[1],label = "displacement")
    axs[1].set(xlabel="time ",title="Critically Damped, b = 2")
    axs[1].plot(tdimless,vel[1],label = "velocity")
    axs[1].plot(tdimless,1/2*k*dis[1]**2)
    axs[1].grid(); axs[1].legend()
    axs[2].plot(tdimless,dis[2],label = "displacement")
    axs[2].plot(tdimless,1/2*k*dis[2]**2)
    axs[2].set(xlabel="time ",title="Overdamped, b =5")
    axs[2].plot(tdimless,vel[2],label = "velocity")
    axs[2].grid(); axs[2].legend()
    plt.show()
#Simple pendulum
    g = 9.8; L = 1; cons=(g,L); t_p = 2*np.pi*np.sqrt(L/g); tmin=0; tmax=7*t_p;
     Nt = 1000; x0 = [2,0]
    t = np.linspace(tmin,tmax,Nt)
    Xrk2 = RK2(simple_pendulum, x0, t, cons)
    x,y= Xrk2[0].T
    t=Xrk2[1]
    tdimless=t/t_p
#Plotting
    plt.title('Simple Pendulum', fontsize=20)
    plt.plot(tdimless,x,color='black',label="Angular Displacement");plt.plot(
     tdimless,y,color='brown',label="Angular Velocity")
    plt.grid(); plt.legend()
    plt.show()
```

Listing 5: 2nd Order coupled diffential equations using Euler, RK2 and RK4 Method

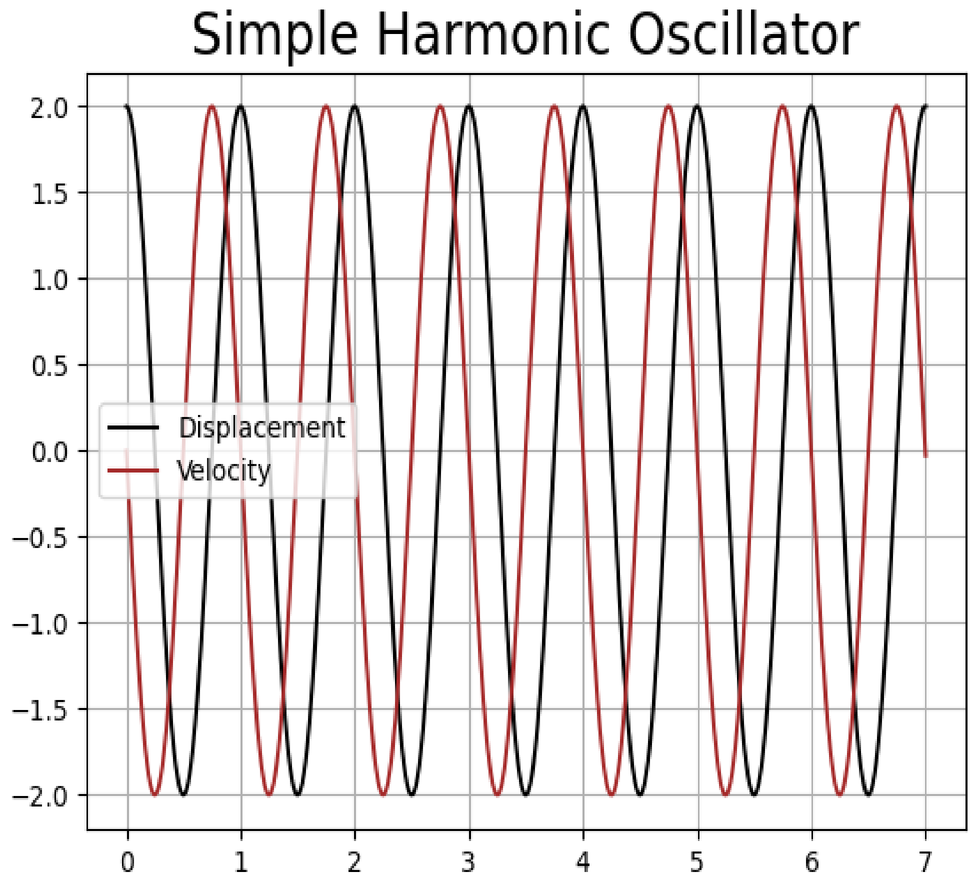


Figure 17: Simple Harmonic Oscillator

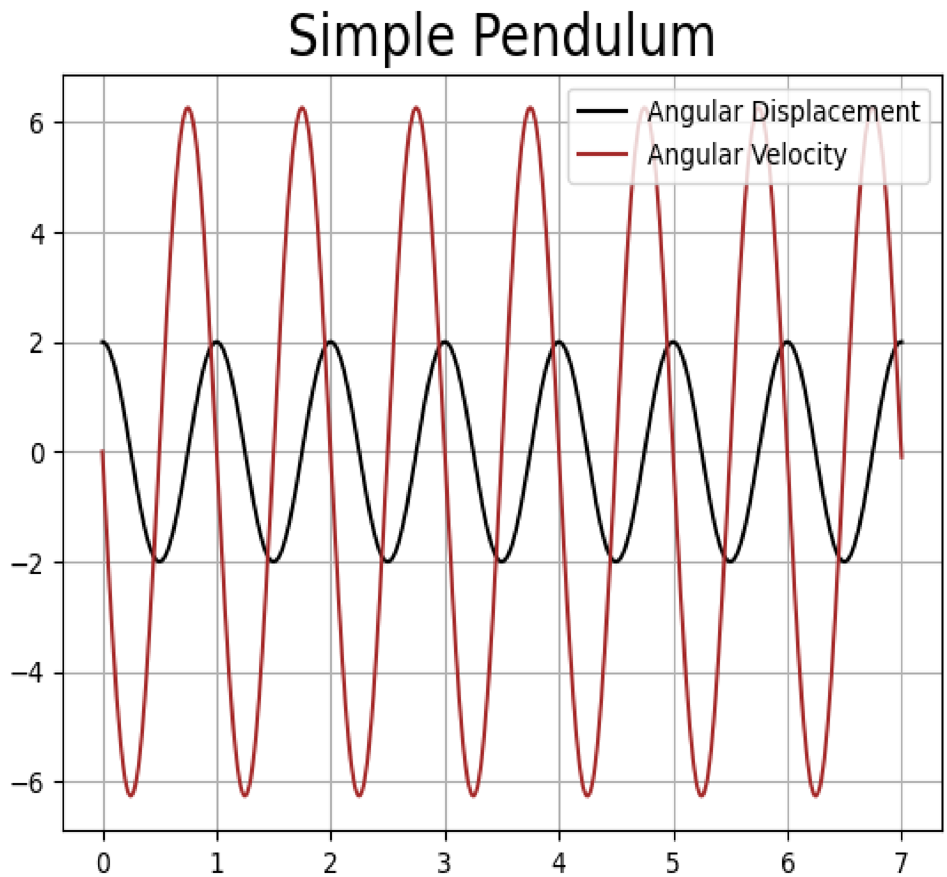


Figure 18: Simple Pendulum

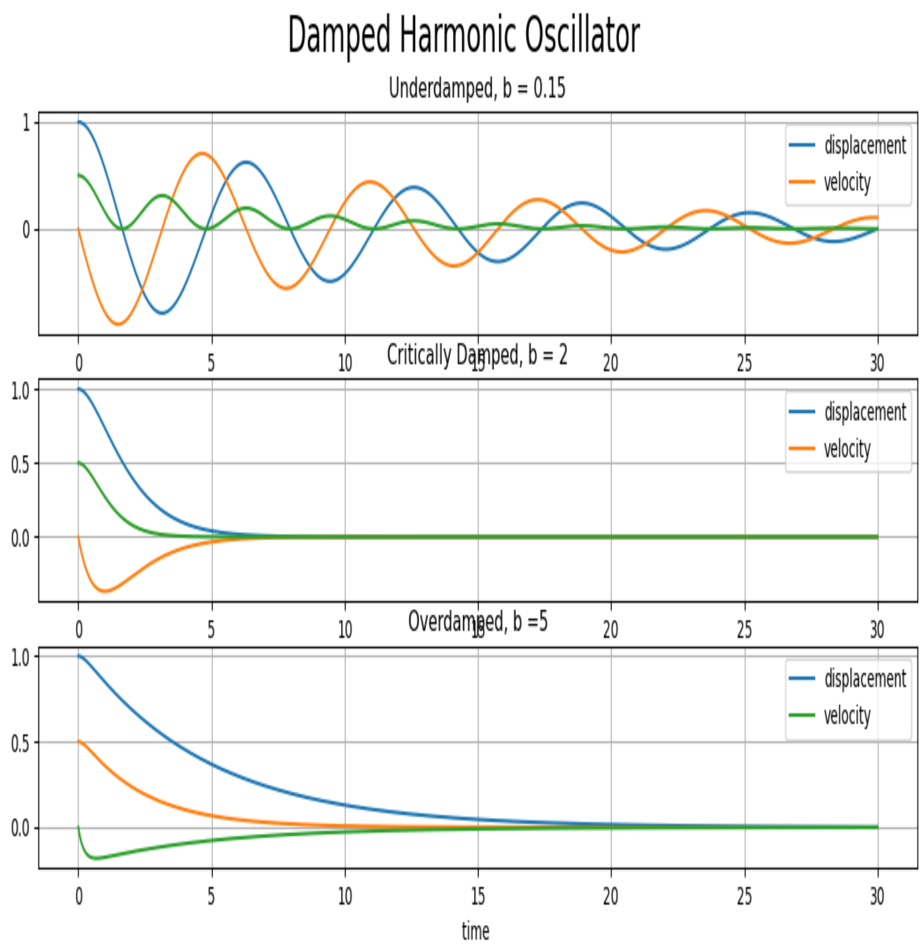


Figure 19: Damped Harmonic Oscillator

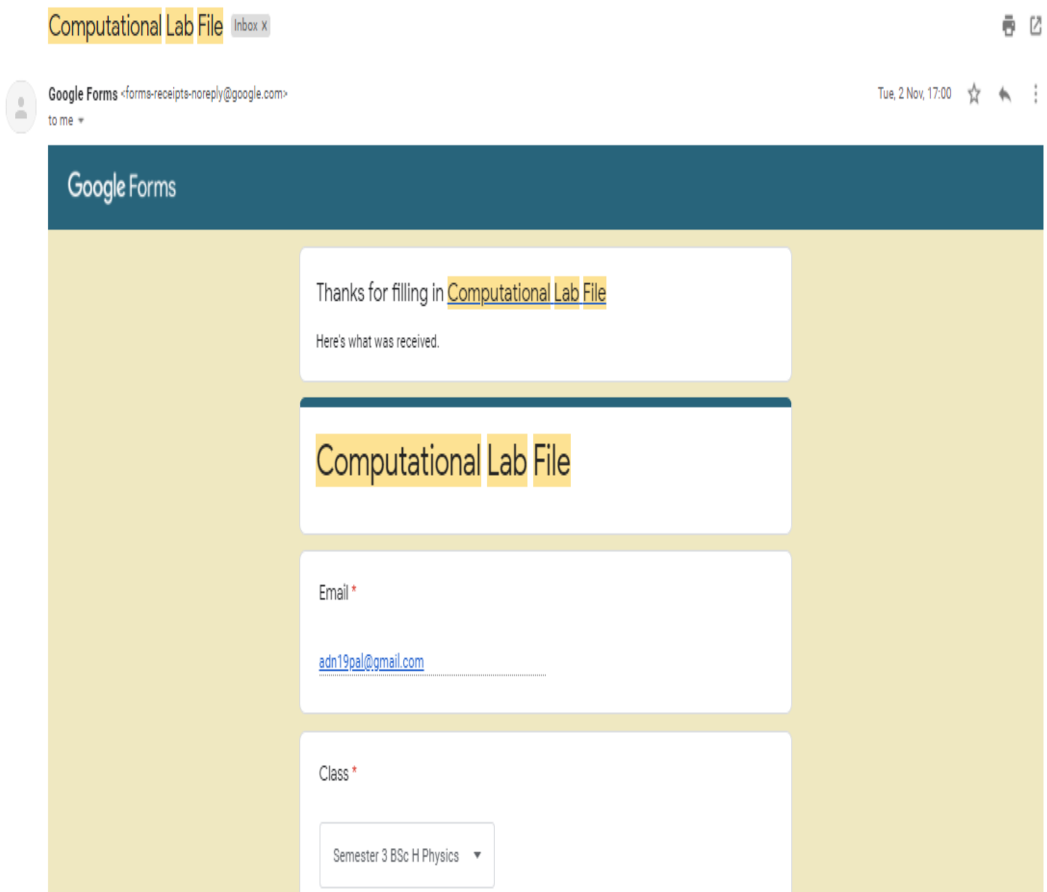


Figure 20: RK4 Method

6 RK4 Method for Simulataneous Differential Equations

```
import matplotlib.pyplot as plt
import numpy as np
import pandas as pd

def eqn(X, t):
    x, y = X
    dotx = y + x - x**3
    doty = -x
    return np.array([dotx, doty])
def RK4(func, X0, t):
```

```
    dt = t[1] - t[0]
    nt = len(t)
    X  = np.zeros([nt, len(X0)])
    X[0] = X0
    for i in range(nt-1):
        k1 = func(X[i], t[i])
        k2 = func(X[i] + dt/2. * k1, t[i] + dt/2.)
        k3 = func(X[i] + dt/2. * k2, t[i] + dt/2.)
        k4 = func(X[i] + dt    * k3, t[i] + dt)
        X[i+1] = X[i] + dt / 6. * (k1 + 2. * k2 + 2. * k3 + k4)
    return X

def graph(t,x,y,x1,y1,x2,y2,x3,y3,title):
    fig,axs=plt.subplots(2,2,figsize=(7,7))
    fig.suptitle(title, fontsize=30)
    ax11,ax12,ax21,ax22=axs[0][0],axs[0][1],axs[1][0],axs[1][1]

    ax11.scatter(t,x,color='black',label="x"),ax11.scatter(t,y,color='brown',
     label="y")
    ax11.set_title("x(0)=0 & y(0)=-1")
    ax12.scatter(t,x1,color='black',label="x"),ax12.scatter(t,y1,color='brown',
     label="y")
    ax12.set_title("x(0)=0 & y(0)=-2")
    ax21.scatter(t,x2,color='black',label="x"),ax21.scatter(t,y2,color='brown',
     label="y")
    ax21.set_title("x(0)=0 & y(0)=-3"),ax21.set_xlabel("Time")
    ax22.scatter(t,x3,color='black',label="x"),ax22.scatter(t,y3,color='brown',
     label="y")
    ax22.set_title("x(0)=0 & y(0)=-4"),ax22.set_xlabel("Time")

    ax11.legend(),ax11.grid(True),ax12.legend(),ax12.grid(True),ax21.legend(),
     ax21.grid(True),ax22.legend(),ax22.grid(True)
    plt.show()
def graph1(x,y,x1,y1,x2,y2,x3,y3,title):
    fig,axs=plt.subplots(2,2,figsize=(7,7))
    fig.suptitle(title, fontsize=30)
    ax11,ax12,ax21,ax22=axs[0][0],axs[0][1],axs[1][0],axs[1][1]
    ax11.scatter(x,y,color='black')
    ax11.set_title("1st condition");ax11.set_ylabel("y")
    ax12.scatter(x1,y1,color='black')
    ax12.set_title("2nd condition");
    ax21.scatter(x2,y2,color='black')
    ax21.set_title("3rd condition");ax21.set_xlabel("x");ax21.set_ylabel("y")
    ax22.scatter(x3,y3,color='black')
    ax22.set_title("4th condition");ax22.set_xlabel("x")
    ax11.legend(),ax11.grid(True),ax12.legend(),ax12.grid(True),ax21.legend(),
     ax21.grid(True),ax22.legend(),ax22.grid(True)
    plt.show()

if __name__ == "__main__":
    x0 = [0,0,0,0];y0 = [-1,-2,-3,-4];Nt = 100;tmax = 15
    t = np.linspace(0.,tmax, Nt)
    X0 = [x0[0], y0[0]]
    X1 = [x0[1], y0[1]]
    X2 = [x0[2], y0[2]]
    X3 = [x0[3], y0[3]]
    res = RK4(eqn, X0, t)
```

```
    res1 = RK4(eqn, X1, t)
    res2 = RK4(eqn, X2, t)
    res3 = RK4(eqn, X3, t)
    x, y = res.T;
    x1, y1 = res1.T
    x2, y2 = res2.T
    x3, y3 = res3.T
    graph(t,x,y,x1,y1,x2,y2,x3,y3,"RK4 Method")
    graph1(x,y,x1,y1,x2,y2,x3,y3,"x vs y")
    data = {"Time":t,"x_rk4":x2,"y_rk4":y2}
    print(pd.DataFrame(data))
```

Listing 6: RK4 Method for Simulataneous Differential Equations

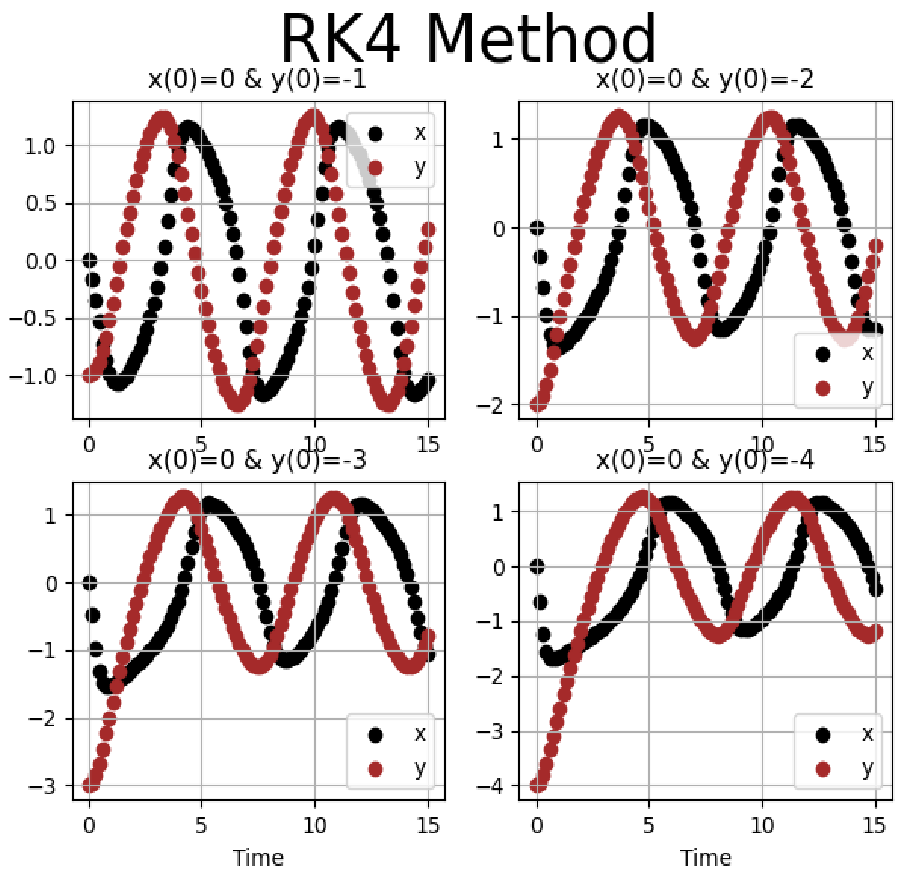


Figure 21: RK4 Method

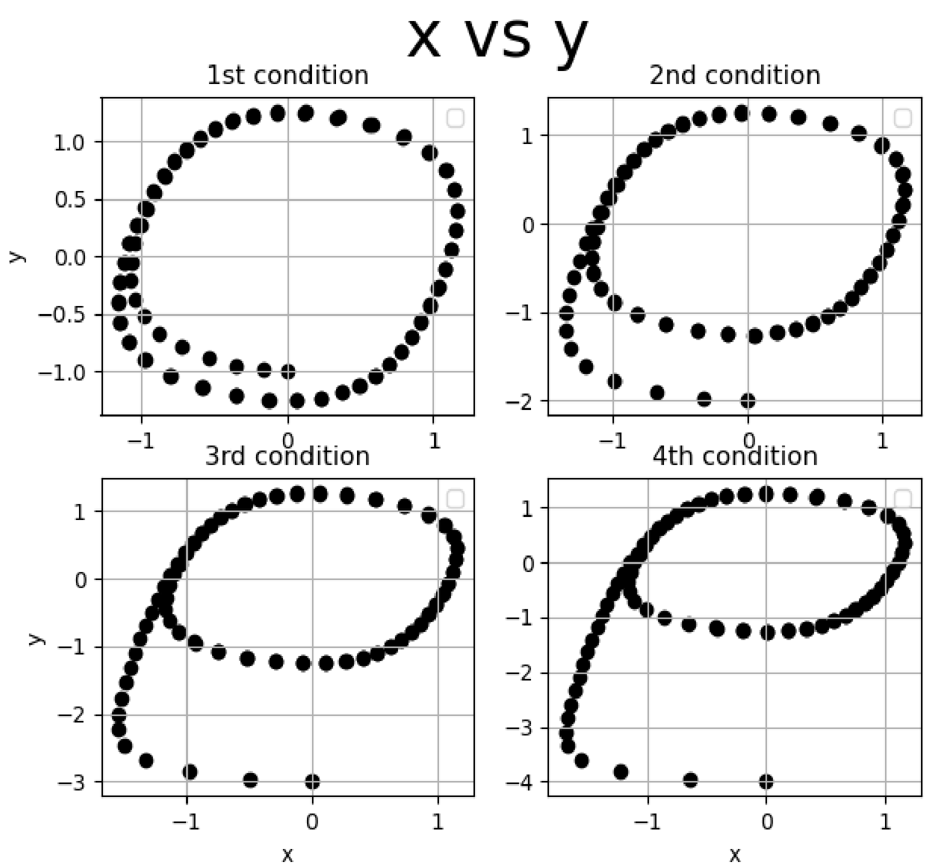


Figure 22: RK4 Method

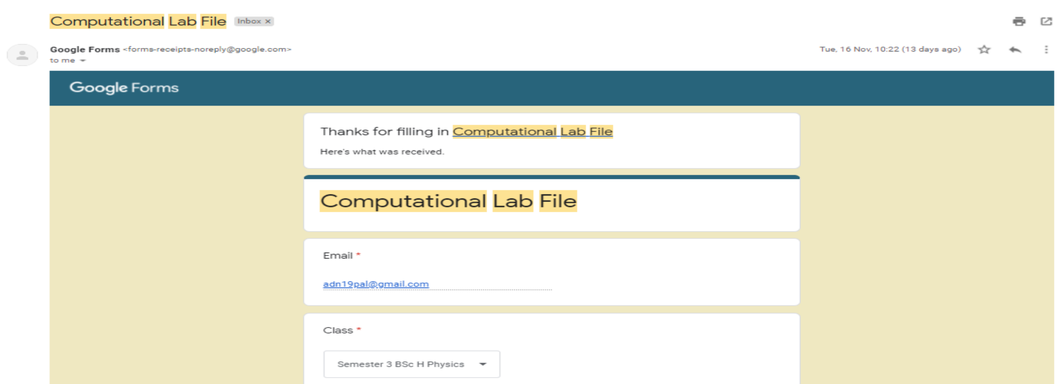


Figure 23: Gauss Elimination Method Output

7 Gauss Elimination Method

```
import numpy as np
def elimination(a,b):
    row, col = np.shape(a)
    if row == col:          #condition to check square matrix
        n = len(b)
        x  = np.zeros(n, float)
        for p in range(n-1):        #index the pivots
            if a[p,p] < 1e-12: #putting threshold condition to check pivot not to
 be zero
                for i in range(p+1,n):  #index elements under pivot
                    if np.fabs(a[i,p]) > np.fabs(a[p,p]):  #non zero rows under
 pivot element
                        a[[p,i]] = a[[i,p]]   #interchange rows and columns
                        b[[p,i]] = b[[i,p]]
                        break
            for i in range(p+1,n):
                if a[i,p] == 0:     #skip the row
                    continue
                factor = a[p,p]/a[i,p]
                for j in range(p,n):
                    a[i,j] = a[p,j] - a[i,j]*factor
                b[i] = b[p] - b[i]*factor     #elimination
        '''check rank of matrix to determine consistency'''
        rank_a = np.linalg.matrix_rank(a)
        aug_matrix = np.column_stack((a,b.T))
        aug_matrix_rank = np.linalg.matrix_rank(aug_matrix)
        if rank_a == aug_matrix_rank:
            if aug_matrix_rank == np.shape(b)[0]:
                print("System of equations has unique solution")
                print("Augmented Matrix \n" ,np.column_stack((a,b.T)))
                '''Back Substitution'''
                x[n-1] = b[n-1] / a[n-1,n-1]
                for i in range(n-2,-1, -1):
```

```python
                sum_ax = 0
                for j in range(i+1, n):
                    sum_ax += a[i, j]* x[j]
                x[i] = (b[i] - sum_ax) / a[i,i]

            print('The solution of the system: ')
            print(x)
            numpy_solution = np.linalg.solve(a, b)
            print("Solution by inbuilt function", numpy_solution)
        elif aug_matrix_rank < np.shape(b)[0]:
            print("system has infinitely many solutions")
    elif rank_a < aug_matrix_rank:
        print("System is inconsistent")
  else:
    print("Number of rows should be equal to number of columns")


if __name__ == "__main__":
    print("E.g. 1")
    a = np.array([[1,-2,1],
                [2,-5,4],
                [1,-4,6]], float)
    b = np.array([5,-3,10], float)
    elimination(a,b)

    print("E.g. 2")
    a = np.array([[1,-2,1,-1,1],
                [2,-5,4,1,-1],
                [1,-4,6,2,-1]], float)
    b = np.array([5,-3,10], float)
    elimination(a,b)
    print("E.g. 3")
    a = np.array([[1,-5,4],
                [1,-5,3],
                [2,-10,13]], float)
    b = np.array([3,6,5], float)
    elimination(a,b)

    print("E.g. 4")
    a = np.array([[12,-3],
                [-12,3]], float)
    b = np.array([6,-6], float)
    elimination(a,b)
```

Listing 7: Gauss Elimination

```
(base) hinton@hinton-VirtualBox:~$ /home/hinton/anaconda3/bin/python "/home/hinton/Sem3/MP2/Practical/code/Gauss Elimination/gauss_elim.py"
E.g. 1
System of equations has unique solution
Augmented Matrix
 [[ 1.   -2.    1.    5.  ]
 [ 0.    0.5  -1.    6.5 ]
 [ 0.    0.    0.25  7.75]]
The solution of the system:
[124.  75.  31.]
Solution by inbuilt function [124.  75.  31.]
E.g. 2
Number of rows should be equal to number of columns
E.g. 3
System is inconsistent
E.g. 4
system has infinitely many solutions
```

Figure 24: Gauss Elimination Method Output

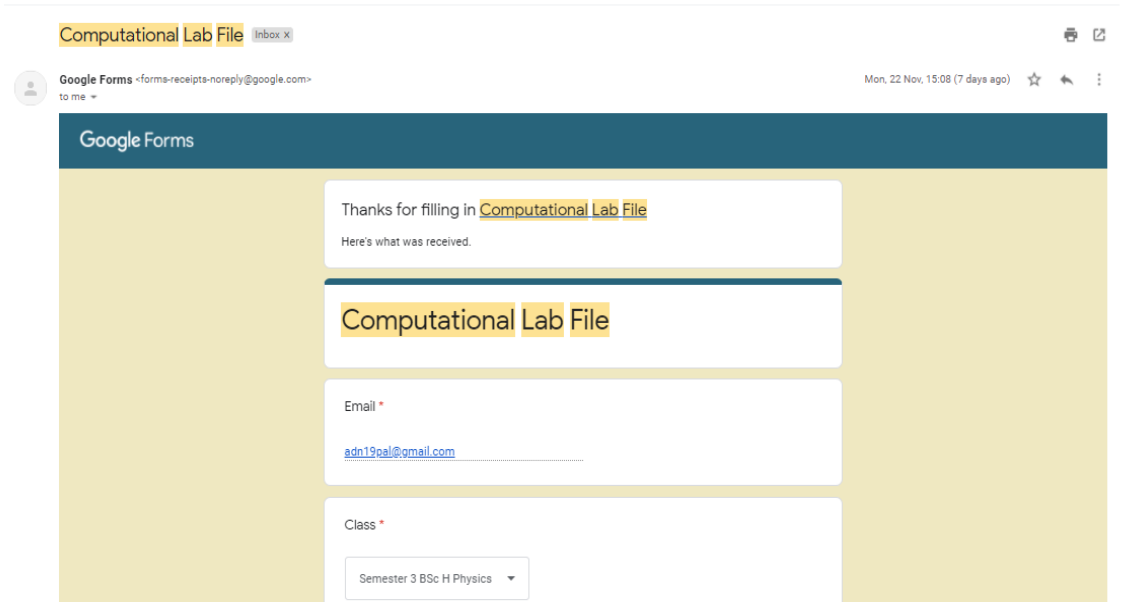


Figure 25: Gauss Seidel Method Output

8 Gauss Seidel Method

```
import numpy as np
from scipy.linalg import solve
def gaussseidel(A, B):
    row, col = np.shape(A)
    if row == col:
        n = 10000
        x = B/(np.diagonal(A))
        inbuilt = solve(A,B)
        for i in range(1, n):
            x_new = np.zeros_like(x)
            print(x)
            for i in range(row):
                x_new[i] = (B[i] - np.dot(A[i, :i], x_new[:i]) - np.dot(A[i, i + 1 :], x[i + 1 :])) / A[i, i]
            if np.allclose(x, x_new, rtol=1e-8):
                break
            else:
                x = x_new
        print("Solution: ",x)
        error = np.dot(A, x) - B
        print("Inbuilt Solution",solve(A,B))
        print("Error: ",error)
    else:
        print("Matrix is not square matrix")
```

```python
if __name__ == "__main__":
    A = np.array([[8, 3, -3], [-2, -8, 5], [3, 5, 10]])
    # initialize the RHS vector
    B = np.array([14,5,-8])
    # Find diagonal coefficients
    diag = np.diag(np.abs(A))
    # Find row sum without diagonal
    off_diag = np.sum(np.abs(A), axis=1) - diag
    if np.all(diag > off_diag):
        print('matrix is diagonally dominant')
    else:
        print('NOT diagonally dominant')
    gaussseidel(A,B)
```

Listing 8: Gauss Seidel Method

```
(base) hinton@hinton-VirtualBox:~$ /home/hinton/anaconda3/bin/python /home/hinton/Sem3/MP2/Practical/code/gaussseidel/gaussseidel.py
matrix is diagonally dominant
[ 1.75  -0.625 -0.8  ]
[ 1.684375   -1.54609375 -0.53226562]
[ 2.13018555 -1.4902124  -0.69394946]
[ 2.0485986  -1.57086806 -0.62914555]
[ 2.10314594 -1.54400245 -0.65894256]
[ 2.08189746 -1.55731346 -0.64591251]
[ 2.09177536 -1.55163916 -0.65171303]
[ 2.0874723  -1.55418872 -0.64914733]
[ 2.08939052 -1.55306471 -0.6502848 ]
[ 2.08854247 -1.55356362 -0.64978093]
[ 2.08891851 -1.55334271 -0.6500042 ]
[ 2.08875194 -1.55344061 -0.64990528]
[ 2.08882575 -1.55339724 -0.64994911]
[ 2.08879305 -1.55341645 -0.64992969]
[ 2.08880754 -1.55340794 -0.64993829]
[ 2.08880112 -1.55341171 -0.64993448]
[ 2.08880396 -1.55341004 -0.64993617]
[ 2.0888027  -1.55341078 -0.64993542]
[ 2.08880326 -1.55341045 -0.64993575]
[ 2.08880301 -1.5534106  -0.6499356 ]
[ 2.08880312 -1.55341053 -0.64993567]
[ 2.08880307 -1.55341056 -0.64993564]
Solution:  [ 2.08880307 -1.55341056 -0.64993564]
Inbuilt Solution [ 2.08880309 -1.55341055 -0.64993565]
Error:  [-1.72144553e-07  1.44201799e-07  8.88178420e-16]
```

Figure 26: Gauss Seidel Method Output