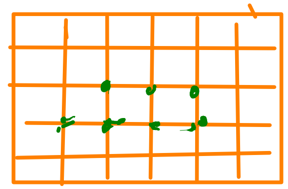
BVP - FInite Difference Method

05.04.2022

The linear and nonlinear Shooting methods for boundary-value problems can present problems of instability.

The finite difference methods have better stability characteristics, but they generally require more computation to obtain a specified accuracy.

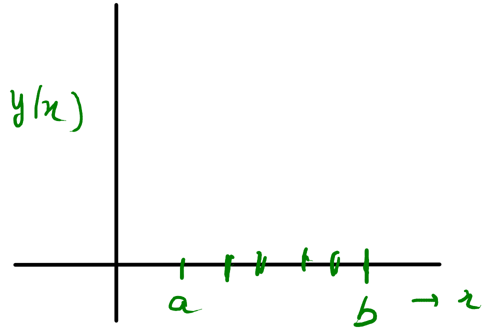
The finite difference methods replace each of the derivatives in the differential equation with an appropriate difference-quotient approximation



The particular difference quotient and step size h are chosen to maintain a specified order of truncation error.

However, h cannot be chosen too small because of the general instability of the derivative approximations.

$$y'' = f(x, y, y') \qquad a < x < b$$



Basic Strategy

-- Discretize the continuous solution domain into a discrete finite difference grid.

-- Approximate the exact derivatives in the boundary-value ODE by algebraic finite difference approximations (FDA).

-- Substitute the FDAs into the ODE to obtain an algebraic finite difference equation (FDE) for each internal grid point.

-- Solve the resulting system of algebraic FDEs
 - for linear ODE it is a system of linear equations
 - for non-linear ODE it is a system of non-linear equations

Linear BVP with Dirichlet BC

Consider the linear second-order boundary-value problem

$$y'' = p(x)\, y + q(x)\, y' + r(x) \qquad a \leq x \leq b \qquad (1)$$

$$y(a) = \alpha, \qquad y(b) = \beta$$

Discretisation

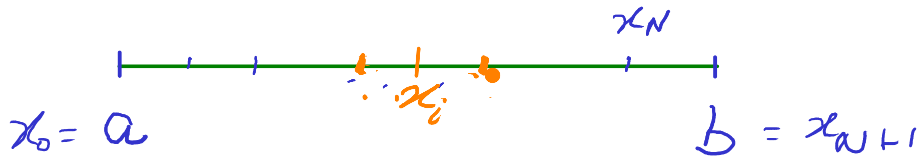
Divide the interval [a, b] into (N +1) equal subintervals with endpoints at the mesh points

$$x_i = a + ih, \quad i = 0, 1, \ldots, N+1, \quad \text{where} \quad h = \frac{b-a}{N+1}$$

with $x_0 = a$ & $x_{N+1} = b$

There are N number of internal grid points and 2 boundary points.

Choosing the step size h in this manner converts the BVP into a system of linear equations and facilitates the application of a matrix algorithm



At each of the interior grid points, x_i for i = 1, 2, . . . , N, the differential equation is approximated as

$$y''(x_i) = p(x_i)\, y(x_i) + q(x_i)\, y'(x_i) + r(x_i) \qquad (2)$$

Expanding $y(x)$ in a Taylor series about x_i, and evaluating its value at x_{i-1} & x_{i+1}

The series is valid in the region $[x_{i-1}, x_{i+1}]$

It is assumed that $y \in C^4[x_{i-1}, x_{i+1}]$

$$\cdots + \frac{h^n}{n!} y^{(n)}(x_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi)$$

$$y(x_{i+1}) = y(x_i + h)$$

$$= y(x_i) + h\, y'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{6} y'''(x_i) + \frac{h^4}{24} y^{(4)}(\xi_i^+)$$

for some $\xi_i^+ \in (x_i, x_{i+1})$ and (3)

$$y(x_{i-1}) = y(x_i - h)$$

$$= y(x_i) - h\, y'(x_i) + \frac{h^2}{2} y''(x_i) - \frac{h^3}{6} y'''(x_i) + \frac{h^4}{24} y^{(4)}(\xi_i^-)$$

for some $\xi_i^- \in (x_{i-1}, x_i)$ (4)

Adding

$$y(x_{i+1}) + y(x_{i-1}) = 2y(x_i) + h^2 y''(x_i) + \underline{\frac{h^4}{24}\left[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-)\right]}$$

Solving for $y''(x_i)$

$$y''(x_i) = \frac{1}{h^2}\left[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right] - \frac{h^2}{24}\left[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-)\right].$$

The Intermediate Value Theorem can be used to simplify the error term

$$\boxed{y''(x_i) = \frac{1}{h^2}\left[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right] - \frac{h^2}{12} y^{(4)}(\xi_i)}$$

for some $\xi_i \in (x_{i-1}, x_{i+1})$ (5)

This is called the centered-difference formula for $y''(x_i)$

A centered-difference formula for $y'(x_0)$

$$\boxed{y'(x_i) = \frac{1}{2h}\left[y(x_{i+1}) - y(x_{i-1})\right] - \frac{h^2}{2}y'''(\eta_i)} \quad (6)$$

for some $\eta_i \in (x_{i-1}, x_{i+1})$

Define $y_i = y(x_i)$ $p(x_i) = p_i$, $q(x_i) = q_i$

$r(x_i) = r_i$

Using (5) and (6) in (2)

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = p_i y_i + q_i\left(\frac{y_{i+1} - y_i}{2h}\right) + r_i$$

$$- \frac{h^2}{12}\left[2q_i y'''(\eta_i) - y^{(4)}(\xi_i)\right]$$

This results in a Finite-Difference method with truncation error of order $O(h^2)$

w_i - approx sol^n at $x = x_i$ → y_{num}

y_i - exact sol^n at $x = x_i$ y_{exact}

$$w_0 = y_0 = \alpha \quad , \quad w_{N+1} = y_{N+1} = \beta$$

$$\left(\frac{-w_{i+1} + 2w_i - w_{i-1}}{h^2}\right) + q_i \left(\frac{w_{i+1} - w_{i-1}}{2h}\right) + p_i w_i = -r_i$$

$i = 1, \ldots, N$

$$\boxed{-\left(1 + \frac{h}{2} q_i\right) w_{i-1} + \left(2 + h^2 p_i\right) w_i - \left(1 - \frac{h}{2} q_i\right) w_{i+1} = -h^2 r_i}$$

$i = 1$

$$-\left(1+\frac{h}{2}q_1\right)w_0 + \left(2+h^2 p_1\right)w_1 - \left(1-\frac{h}{2}q_i\right)w_2 = -h^2 r_1$$

α.

$$\left(2+h^2 p_1\right) w_1 - \left(1-\frac{h}{2}q_1\right)w_2 = \left(1+\frac{h}{2}q_1\right)\alpha - h^2 r_1$$

$i = 2, \ldots, N-1$

$$-\left(1+\frac{h}{2}q_i\right)w_{i-1} + \left(2+h^2 q_i\right) w_i - \left(1-\frac{h}{2}q_i\right)w_{i+1} = -h^2 r_i$$

$i = N$

$$-\left(1+\frac{h}{2}q_N\right)w_{N-1} + \left(2+h^2 q_N\right)w_N = -h^2 r_N + \left(1-\frac{h}{2}q_N\right)\beta$$

$$AW = B$$

$$A = \begin{bmatrix} d_1 & u_1 & 0 & & & & 0 \\ l_2 & d_2 & u_2 & & & & \\ & l_3 & d_3 & u_3 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & l_{N-2} & d_{N-2} & u_{N-2} & \\ & 0 & & & l_{N-1} & d_{N-1} & u_{N-1} \\ & & & & 0 & l_N & d_N \end{bmatrix}$$

$$N \times N$$

$$d_i = 2 + h^2 p_i$$

$$u_i = -1 + \frac{h}{2} q_i$$

$$l_i = -1 - \frac{h}{2} q_i$$

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ \\ w_{N-2} \\ w_{N-1} \\ w_N \end{bmatrix}, \quad B = \begin{bmatrix} -h^2 r_1 + \left(1 + \frac{h}{2} q_1\right)\alpha \\ -h^2 r_2 \\ \vdots \\ -h^2 r_{N-2} \\ -h^2 r_{N-1} \\ -h^2 r_N + \left(1 - \frac{h}{2} q_N\right)\beta \end{bmatrix}$$

If p, q, r are continuous on $[a,b]$

If $p(x) \geq 0$ on $[a,b]$

continuity of q over $[a,b]$ implies

that $\exists$ a constant L s.t.

$$|q(x)| \leq L \text{ on } [a,b]$$

If h is chosen s.t. $h < \frac{2}{L}$. then

for each i, $-1 < \frac{hq_i}{2} < 1$.

$-1 - \frac{h}{2}q_i$ and $-1 + \frac{h}{2}q_i$ are always $-$ve

$$\therefore \left| -1 - \frac{h}{2}q_i \right| = 1 + \frac{h}{2}q_i$$

& $\left|-1+\frac{h}{2}q_i\right| = 1-\frac{h}{2}q_i$

from 2nd to (N-1) th row of matrix A

$$\underbrace{\left|-1-\frac{h}{2}q_i\right| + \left|-1+\frac{h}{2}q_i\right|}_{\text{off diagonal terms}} = 2 \le \left|2+h^2p_i\right|$$

-- Diagonally Dominant Matrix

The $n \times n$ matrix A is said to be diagnally dominant when $|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$ holds for each $i = 1, 2, \ldots, n$

If $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ then strictly diagonally dominant

Algorithm/ Pseudocode

define functions

Input $-$ $a, b, \alpha, \beta, \underbrace{p, q, r}$

Output $-$ $w_1, w_2, \dots w_N$ $\quad | w_0 = \alpha, \quad w_{N+1} = \beta |$

(1) $h = \dfrac{b-a}{N+1}$

Define the arrays

$x_i = a + i * h$

$d_1 =$ $\quad u_1 =$

$l_N =$ $\quad d_N =$

(2) for $i = 2, \dots, N-1$

$x_i =$, $\quad l_i =$

$u_i =$ $\quad d_i =$

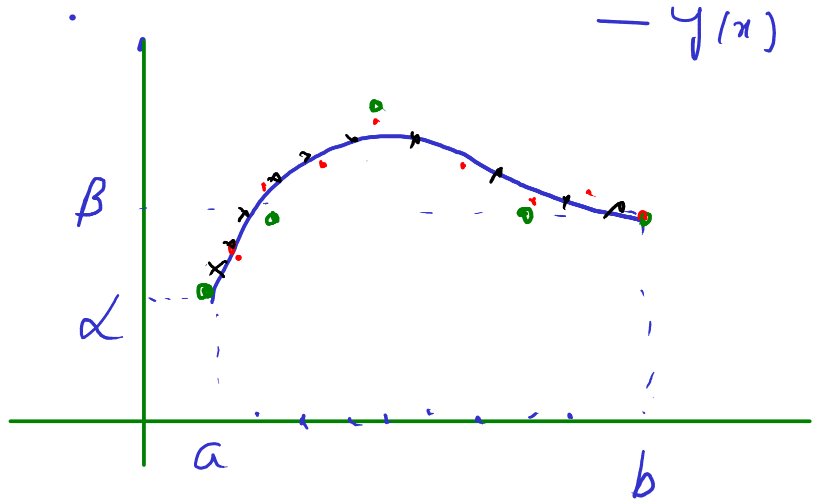
(3) call tridiag $-$ o/p is w_i for $i = 1, \dots N$

(4) output (x_i, w_i) $\quad i = 0, \dots N+1$.

(5) plot (x_i, w_i) and (x_i, y_i)

$\downarrow$ point $\downarrow$ for diff N-values

$\hookrightarrow$ curve



(6) repeat for next α

$N = 4, 8, 16..$

$\downarrow$ print

x_i

Table 1

x_i	w_i	$y(x_i)$	$\lvert w_i - y(x_i) \rvert$

$$\boxed{\begin{gathered} -y'' + \pi^2 y = 2\pi^2 \sin(\pi x) \\ y(0) = y(1) = 0 \end{gathered}}$$

$N = 8$

x_i $\quad w_i$ $\quad y_i$ $\quad E_i = |w_i - y_i|$ $\quad \max(E)$

$\vdots$

i

$$(\text{rms error})^2 = \frac{1}{N}\sum_{i=1}^{N}(w_i - y_i)^2$$

N	max absolute error	Error Ratio	rms error	error ratio
4	e_1	—		
8	e_2	e_1/e_2		
16	e_3	e_2/e_3		
32		$\vdots$ $\}$ 4		
64				
128				

Non-Dirichlet Boundary Conditions

Neumann B.C. $\qquad y'(a) = \alpha, \quad y'(b) = \beta$

Robin $\qquad \alpha_1 y(a) + \underline{\alpha_2} y'(a) = \alpha_3$

$\qquad \beta_1 y(b) + \underline{\beta_2} y'(b) = \beta_3$

$$y''(x) = p(x)\, y(x) + q(x)\, y'(x) + r(x)$$

$x \in [a, b]$

<u>Assume</u>

$\alpha_2 \neq 0$ & $\beta_2 \neq 0$ for Non-dirichlet BC

$$a = x_0, \; x_1, \; \ldots, \; x_N = b$$

$$x_i = a + ih \qquad h = \frac{b-a}{N}$$

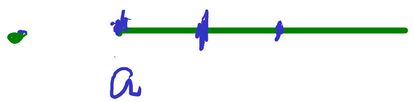
need to find $w_0, w_1, \ldots, w_N$ $\longrightarrow$ $N+1$ variable

$\Rightarrow$ $N+1$ no of linear eq^n

$$-\left(1 + \frac{h}{2} q_i\right) w_{i-1} + \left(2 + h^2 p_i\right) w_i - \left(1 - \frac{h}{2} q_i\right) w_{i+1} = -h^2 r_i$$

Consider boundary $x = a$

$$\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3$$



① Take forward derivate at a

$$y'(a) \simeq \frac{y(a+h) - y(a)}{h} + O(h)$$

→ would increase the error

→ this will maintain tridiagonal structure of problem

② To maintain same accuracy we need to approx $y'(a)$ by a forward diff with truncation error of $O(h^2)$.

F

$$y'(a) = \frac{-3y(a) + 4y(a+h) - y(a+2h)}{2h} + O(h^2)$$

→ destroys tridiagonal structure

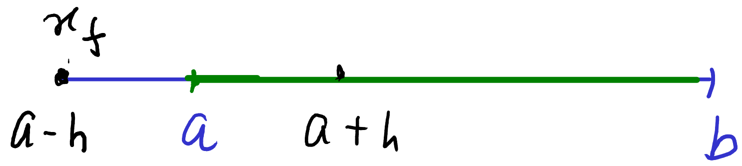
③ To have both

– truncation error of $O(h^2)$

– tridiagonal system

introduce a fictitious grid point

$$x_{-1} = x_f$$



computational template $\quad w_f = w(a-h)$

$i = 0$

$$\left(-1 - \frac{h}{2} q_0\right) w_f + \left(2 + h^2 p_0\right) w_0 + \left(-1 + \frac{h}{2} q_0\right) w_1 = -h^2 r_0$$

B.C. at $x = a$ $\qquad \alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3$

$$\longrightarrow \quad \alpha_1 w_0 + \alpha_2 \frac{w_1 - w_f}{2h} = \alpha_3$$

$$\omega_f = \omega_1 - \frac{2h}{\alpha_2}(\alpha_3 - \alpha_1 \omega_0)$$

$$\boxed{\left[2 + h^2 p_0 - (2 + h q_0) h \frac{\alpha_1}{\alpha_2}\right] \omega_0 - 2\omega_1 = -h^2 r_0 - (2 + h q_0) h \frac{\alpha_3}{\alpha_2}}$$

For Neumann B.C. $\alpha_1 = 0$, & B.C. $y'(a) = \alpha = \alpha_3/\alpha_2$

$$(2 + h^2 p_0)\,\omega_0 - 2\omega_1 = -h^2 r_0 - (2 + h q_0) h \alpha$$

Similarly at $x = b$.

$$-2\omega_{N-1} + \left[2 + h^2 p_N + (2 - h q_N) h \frac{\beta_1}{\beta_2}\right] \omega_N$$

$$= -h^2 r_N + (2 - h q_N) h \frac{\beta_3}{\beta_2}$$

$N+1$ no of linear eq^{n}s in

$\omega_0, \omega_1, \ldots, \omega_N$.

General matrix formulation for linear BVP with linear BC

$$\underline{AW = B}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & & & & & \\ l_1 & d_1 & u_1 & & & & \\ & l_2 & d_2 & u_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & l_{N-1} & d_{N-1} & u_{N-1} \\ & & & & & a_{N+1,N} & a_{N+1,N+1} \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ -h^2 r_1 \\ -h^2 r_2 \\ \vdots \\ -h^2 r_{N-1} \\ b_{N+1} \end{bmatrix} \quad ; \quad W = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \\ w_N \\ w_{N+1} \end{bmatrix}$$

$$d_i = 2 + h^2 p_i \, , \quad u_i = -1 + \frac{h}{2} q_i \, , \quad l_i = -1 - \frac{h}{2} q_i$$

$$a_{11} = \begin{cases} 1 & \text{Dirichlet BC at } x = a \\ d_0 & \text{Neumann BC at } x = a \\ d_0 + 2h\, l_0 \alpha_1 / \alpha_2 & \text{Robin B.C. at } x = a \end{cases}$$

$$a_{12} = \begin{cases} 0 & \text{Dirichlet B.C. at } x = a \\ -2 & \text{otherwise} \end{cases}$$

$$a_{N+1,\,N+1} = \begin{cases} 1 & \text{D B C at } x = b \\ d_N & \text{Neumann BC at } x = b \\ d_N - 2h\,u_N\,\beta_1/\beta_2 & \text{Robin BC at } x = b \end{cases}$$

$$a_{N+1,\,N} = \begin{cases} 0 & \text{D B C} \\ -2 & \text{otherwise} \end{cases}$$

$$b_1 = \begin{cases} \alpha & \text{DBC at } x=a \quad y(a)=\alpha \\ -h^2 r_0 + 2h\, l_0\, \alpha & \text{NBC} \quad y'(a)=\alpha \\ -h^2 r_0 + 2h\, l_0\, \alpha_3/\alpha_2 & \text{RBC} \end{cases}$$

$$b_{N+1} = \begin{cases} \beta & \text{DBC at } x=b \\ -h^2 r_N - 2h\, u_N\, \beta & \text{NBC} \\ -h^2 r_N - 2h\, u_N\, \beta_3/\beta_2 & \text{RBC} \end{cases}$$

<u>DBC</u> $\quad y(a)=\alpha, \quad y(b)=\beta$

<u>NBC</u> $\quad y'(a)=\alpha, \quad y'(b)=\beta$

<u>RBC</u> $\quad \alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3, \quad \beta_1 y(b) + \beta_2 y'(b) = \beta_3$

$$y'' + y = \sin(3x) \qquad x \in [0, \tfrac{\pi}{2}]$$

$$y(0) + y'(0) = -1 \qquad y'(\tfrac{\pi}{2}) = 1$$

$$y_{exact} = \frac{3}{8}\sin x - \cos x - \frac{1}{8}\sin(3x)$$

$$-y'' + \pi^2 y = 2\pi^2 \sin(\pi x) \qquad 0 \le x \le 1$$

$$y(0) = y(1) = 0$$

$$y_{exact} = \sin(\pi x)$$

Solving Tridiagonal System — Crout factorisation for symmetric tridiagonal

$A = LU$

$n \times n$ system

$$a_{11} x_1 + a_{12} x_2 \qquad\qquad = a_{1,n+1}$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \qquad\qquad = a_{2,n+1}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\ldots a_{n-1,n-2} x_{n-2} + a_{n-1,n-1} x_{n-1} + a_{n-1,n} x_n = a_{n-1,n+1}$$

$$a_{n,n-1} x_{n-1} + a_{nn} x_n = a_{n,n+1}$$

1. Set $l_{11} = a_{11}$

$u_{12} = a_{12}/l_{11}$

$z_1 = a_{1,n+1}/l_{11}$

2. For $i = 2, \ldots, n-1$ set $l_{i,i-1} = a_{i,i-1}$ — ith row of L

$l_{ii} = a_{ii} - l_{i,i-1} u_{i-1,i}$

$u_{i,i+1} = a_{i,i+1}/l_{ii}$ — $(i+1)$th column of U

$$z_i = (a_{i,n+1} - l_{i,i} z_{i-1}) / l_{ii}$$

3 Set $l_{n,n-1} = a_{n,n-1}$ nth row of L

$$l_{nn} = a_{nn} - l_{n,n-1} u_{n-1,n}$$

$$z_n = (a_{n,n+1} - l_{n,n-1} z_{n-1}) / l_{nn}$$

4 $x_n = z_n$

5 For $i = n-1, \ldots, 1$, $x_i = z_i - u_{i,j+1} x_{i+1}$

6 output $x_1, \ldots x_n$