Applications of Integration(Assignment 2b)

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1 Orthogonal Polynomials

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1- Theory

(a) What do you mean by orthogonal polynomials in an interval?

Ans. Orthogonal polynomials in a given interval are polynomials which are orthogonal/ perpendicular to each other (whose Dot product is zero)

$$\langle p, q \rangle = \int_a^b p(x)\, q(x)\, dx = 0$$

when p or $q \neq 0$

(b) Write down orthogonality condition for Legendre polynomials. - - - -

Ans. Orthogonal polynomials can be obtained by applying Gram Schmidt orthogonalisation process to the basis $\{1, x, x^2, \ldots\}$

Applying orthogonal Gram Schmidt orthogonalisation on orthogonal functions in the range $[-1, 1]$

$$P_0(x) = 1$$

$$P_1(x) = x - \frac{\langle x, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x \quad \left[\because \langle x, 1 \rangle = \int_{-1}^{1} x\, dx = 0\right] \quad \left[\langle 1, 1 \rangle = \int_{-1}^{1} 1\, dx = 2\right]$$

$$P_2(x) = x^2 - \frac{\langle x^2, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) - \frac{\langle x^2, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1(x) = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle x^2, x \rangle}{\langle x, x \rangle} x$$

$$\langle x^2, 1 \rangle = \int_{-1}^{1} x^2\, dx = \frac{x^3}{3} = \frac{1}{3}\left[x^3\right]_{-1}^{1} = \frac{1}{3}[1+1] = 2/3$$

$$\langle x^2, x \rangle = \int_{-1}^{1} x^3\, dx = \frac{x^4}{4} = \frac{1}{4}\left[x^4\right]_{-1}^{1} = 0$$

$$P_2(x) = x^2 - \frac{2/3}{2} - 0 = x^2 - 1/3$$

Here $P_0(x), P_1(x), P_2(x)$ polynomials are orthogonal to each other. These polynomials are Legendre polynomials.

1.1 Q(c)

$f(x) = C_0\mathrm{P}_0(x) + C_1\mathrm{P}_1(x) + C_2\mathrm{P}_2(x) + \ldots \quad -1 \le x \le 1$

$= \sum_{n=0}^{\infty} C_n \mathrm{P}_n(x)$

Multiplying both sides by $\mathbf{P}_m(\boldsymbol{x})\boldsymbol{dx}$ and integrating with respect to $\boldsymbol{x}$ from $\boldsymbol{x} = \mathbf{-1}$ to $\boldsymbol{x} = 1$ gives

$$\int_{-1}^{1} f(x)\mathrm{P}_m(x)dx = \sum_{n=0}^{\infty} C_n \int_{-1}^{1} \mathrm{P}_m(x)\mathrm{P}_n(x)dx$$

By means of the orthogonality property of the Legendre polynomials we can write

$$C_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\mathrm{P}_n(x)dx \quad n = 0, 1, 2, 3\ldots$$

Since $\mathbf{P}_n(\boldsymbol{x})$ is an even function of $\boldsymbol{x}$ when $\boldsymbol{n}$ is even, and an odd function when $\boldsymbol{n}$ is odd, it follows that if $\boldsymbol{f}(\boldsymbol{x})$ is an even function of $\boldsymbol{x}$ the coefficients $\boldsymbol{A_n}$ will vanish when $\boldsymbol{n}$ is odd; whereas if $f(x)$ is an odd function of x, the coefficients A_n will vanish when n is even.

Thus for and even function $f(x)$ we have

$$A_n = \begin{cases} 0 \\ (2n+1)\int_0^1 f(x)\mathrm{P}_n(x)dx \end{cases} \text{n is odd n is even}$$

whereas for an odd function $\boldsymbol{f}(\boldsymbol{x})$ we have

$$A_n = \begin{cases} (2n+1)\int_0^1 f(x)\mathrm{P}_n(x)dx \\ 0 \end{cases} \text{n is odd n is even}$$

1.2 Determine analytically

All terms in the expansion of the function

$$f(x) = 2 + 3x + 2x^4$$

$$f(x) = \sum_{n=0}^{\infty} \mathrm{Cn}\,\mathrm{Pn}(x)$$

$$f(x) = C_0P_0(x) + C_1P_1(x) + C_2P_2(x) + C_3P_3(x) +$$

we also know

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

$n = 0$

$$c_0 = \frac{1}{2}\int_{f}^{1} f(x)p_0(x)dx$$

$$\frac{1}{2}\int_{-1}^{1}\left(2 + 3x + 2x^4\right)$$

$$\frac{1}{2}\left[2x + \frac{3x^2}{2} + \frac{2x^5}{5}\right]_{-1}^{1}$$

$$\frac{1}{2}\left[2 + \frac{3}{2} + \frac{2}{5} + 2 - \frac{3}{12} + \frac{2}{5}\right]$$

$$\frac{1}{2}\left[4 + \frac{4}{5}\right] \Rightarrow \frac{1}{2}\left[\frac{20+4}{5}\right] = \frac{12}{5}$$

At $n = 1$

$$c_1 \Rightarrow \frac{3}{2}\int_{-1}^{1}\left(2+3x+2x^4\right)xdx$$

$$c_1 = \frac{3}{2}\int_{-1}^{1}\left(2x+3x^2+2x^5\right)dx$$

$$\frac{3}{2}\left[\frac{2x^2}{2}+\frac{3x^3}{3}+\frac{2x^6}{6}\right]_{-1}^{1}$$

$$\frac{3}{2}\left[x^2+x^3+\frac{x^6}{3}\right]_{-1}^{1}$$

$$\frac{3}{2}\left[\left(1+1+\frac{1}{3}-1-1-\frac{1}{3}\right)\right] \Rightarrow 3$$

At n=2

$$C_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

$$\frac{5}{2}\int_{-1}^{1}\left(2+3x+2x^4\right)\frac{1}{2}\left(3x^2-1\right)dx$$

$$\frac{5}{4}\int_{-1}^{1}\left(2x3x+2x^4\right)\left(3x^2-1\right)dx$$

$$\frac{5}{4}\int_{-1}^{1}\left(6x^2-2+9x^2-3x+6x^6-2x^4\right)dx$$

$$\frac{5}{4}\left[\frac{6x^3}{2}-2x+\frac{9}{4}x^4-\frac{5x^2}{2}+\frac{6x^1}{7}-\frac{2x^5}{5}\right]_{-1}^{1}$$

$$\frac{5}{4}\left[\frac{9}{4}-\frac{3}{2}+\frac{6}{7}-\frac{2}{5}-\frac{9}{4}+\frac{3}{2}+\frac{6}{7}-\frac{2}{5}\right]$$

$$\frac{5}{4}\left[-\frac{4}{5}\right] \Rightarrow -1$$

At $n = 3$

$$C_3 = \frac{7}{4}\int_{-1}^{1}\left(2+3x+2x^4\right)\left(5x^3-3x\right)dx$$

$$\frac{7}{4}\int_{-1}^{1}\left(10x^3-6x+15x^4-9x^2+10x^7-6x^5\right)dx$$

$$\Rightarrow \frac{7}{4}\left[\frac{10x^4}{4}-\frac{6}{2}x^2+\frac{15}{5}x^5-\frac{9x^3}{3}+\frac{10x^8}{8}-\frac{6x^6}{6}\right]_{-1}^{1}$$

$$\Rightarrow \frac{7}{4}\left[\frac{5}{2}-3+3-3+\frac{5}{4}-1-\left(\frac{5}{2}-3-3+3+\frac{5}{4}-1\right)\right]$$

$$\Rightarrow \frac{7}{4}\left[\frac{5}{2}+\frac{5}{4}-3-1-\frac{5}{2}+3-\frac{5}{4}+1\right] \Rightarrow 0$$

C_4

$$\Rightarrow \frac{9}{2}\times\frac{1}{8}\int_{-1}^{1}\left(2+3x+2x^4\right)\left(35x^4-30x^2+3\right)dx$$

$$= \frac{9}{16}\int_{-1}^{1}(70x^4 - 60x^3 + 6 + 105x^5 - 90x^3 + 9x + 70x^8 - 60x^6 + 6x^4)dx$$

$$= \frac{9}{16}\left[\frac{70}{5}x^5 - \frac{60}{3}x^3 + 6x + \frac{105}{6}x^6 - \frac{90}{4}x^4 + \frac{9}{2}x^2 + \frac{70}{6}x^9 - \frac{60}{7}x^7 + \frac{6}{5}x^5\right]_{-1}^{1}$$

$$\Rightarrow \frac{9}{16}\left[14x^5 - 20x^3 + 6x + \frac{105}{6}x^6 - \frac{45}{2}x^4 - \frac{9}{2}x^2 + \frac{35}{3}x^9 - \frac{60}{7}x^7 + \frac{6}{5}x^5\right]_{-1}^{1}$$

$$\Rightarrow \frac{9}{16}\left[14 - 20 + 6 + \frac{105}{6} - \frac{45}{2} + \frac{9}{2} + \frac{35}{3} - \frac{60}{7} + \frac{6}{5} + 14 - 20 + 6 - \frac{105}{6} + \frac{45}{2} - \frac{9}{2} + \frac{35}{3} - \frac{60}{7} + \frac{6}{5}\right]$$

$$\Rightarrow \frac{9}{16}\left[\frac{70}{3} - \frac{120}{7} + \frac{12}{5}\right] \Rightarrow \frac{9}{16}\left[\frac{250 + (-1800) + 252}{105}\right]$$

$$\Rightarrow \left(\frac{902}{105}\right) \times \frac{9}{16} = \frac{1353}{280}$$

$$\approx 4.83$$

$\therefore$ We can write $f(x) = 2 + 3x + 2x^4$ as

$$f(x) = \frac{12}{5}P_0(x) + 3P_1(x) + 1P_2(x) + 4.83P_4(x) + \ldots$$

1.3 First five terms in the expansion of $f(x) = \cos x \sin(x)$

As we know,

$$f(x) = \sum_{n=0}^{\infty} C_n \, \mathrm{Pn}(x)$$

$$C_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

At $n = 0$

$$c_0 \Rightarrow \frac{1}{2}\int_{-1}^{1}\cos x \sin x p_0(x)dx$$

$$\frac{1}{4}\int_{-1}^{1} 2\cos x \sin x dx$$

$$\Rightarrow \frac{1}{4}\int_{-1}^{1}\sin 2x dx \Rightarrow \frac{-1}{4}\left(\frac{\cos 2x}{2}\right)_{-1}^{1}$$

$$\Rightarrow -\frac{1}{4} \times \frac{1}{2}[\cos 2 - \cos 2] \Rightarrow 0$$

At $n = 1$

$$c_1 = \frac{3}{2}\int_{-1}^{1} x\cos x \sin x dx$$

$$\Rightarrow \frac{3}{4}\left[-\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x)\right]_{-1}^{1}$$

$$\Rightarrow \frac{3}{8}\left[-x\cos(2x) + \frac{1}{2}\sin 2x\right]_{-1}^{1}$$

$$\frac{3}{2}\left[-\cos 2 + \frac{1}{2}\sin 2 - \cos 2 + \frac{1}{2}\sin 2\right]$$

$$\frac{3}{x}[\sin 2-\sin 2]$$

$$\Rightarrow \frac{3}{8}[0.909297-2\times(-0.4161468)]$$

$$\Rightarrow \frac{3}{8}[0.909297+0.8322436]$$

$$\Rightarrow \frac{3}{8}[1.7415906]$$

$$=0.6531$$

At $n=2$

$$c_2=\frac{5}{4}\int_{-1}^{1}\cos x\sin x\left(3x^2-1\right)dx$$

$$C_2\Rightarrow \frac{5}{4}\left[\int_{-1}^{1}3x^2\sin x\cos x-\int_{-1}^{1}\cos x\sin x dx\right]$$

$$\Rightarrow \frac{5}{4}\left[\frac{3}{2}\left(-\frac{1}{2}x^2\cos 2x+1(2x\sin 2x+\cos 2x)\right)-\frac{1}{4}\cos 2x\right]_{-1}^{1}$$

$$\Rightarrow \frac{5}{4}\left[\frac{3}{2}\left(-\frac{1}{2}\cos 2+\frac{1}{4}(2\sin 2+\cos 2)-\left(-\frac{1}{2}\cos 2+\frac{1}{4}\quad\sin 2+\cos 2\right)\right)-\frac{1}{4}(\cos 2-\cos 2)\right]$$

$$\frac{5}{4}\times\frac{3}{2}[0]=0$$

At n = 3

$$c_3=\frac{1}{4}\times\frac{1}{2}\int_{2}^{1}\cos x\sin x\left(5x^3-3x\right)dx$$

$$\frac{7}{8}\left[\int_{-1}^{1}\sin 2x\left(5x^3-3x\right)dx\right]$$

$$\frac{7}{8}\left[\int_{-1}^{1}5x^3\sin 2x dx^-\int_{-1}^{1}3x\sin 2x dx\right]$$

$$\frac{7}{8}\left[\frac{5(4\cos 2+6\sin 2)}{8}-3\left(-\cos 2+\frac{1}{2}\sin 2\right)\right]$$

$$=\frac{7}{8}\left[\frac{5}{8}(4\times-0.04161468+6\times 0.909297)-3(-0.416148\quad+\frac{1}{2}\times 0.909297\right)\right]$$

$$\Rightarrow \frac{7}{8}\left[\frac{5}{8}\times(-9.6646+5.458)-3(-0.4161+0.4546)\right]$$

$$\Rightarrow \frac{7}{8}\left[\frac{5}{8}\times 3.7912-3(0.0385)\right]$$

$$\Rightarrow \frac{7}{8}[2.3695-0.1155]$$

$$=0.875\times 2.254\Rightarrow 1.97225$$

$n = 4$

$$c_4 = \frac{3}{2}\int_{-1}^{1} \sin x \cos x \left[\frac{1}{8}\left(35x^4 - 30x^2 + 3\right)\right] dx$$

$$\Rightarrow C_4 \Rightarrow \frac{9}{16}\int_{-1}^{1} \sin x \cos x \left[x^4 - 30x^2 + 3\right] dx$$

$$\tfrac{5}{4}\left[\tfrac{3}{2}\left(-\tfrac{1}{2}\cos 2 + \tfrac{\sin 2}{2} + \tfrac{\cos 2}{4} + \tfrac{1}{2}\cos 2 - \tfrac{1}{2}\sin 2 \quad -\tfrac{1}{4}\cos 2 - 0\right)\right]$$

$$\Rightarrow \frac{9}{16}\{35\int_{-1}^{1} \sin x \cos x x^4 dx - 30\int \sin x \cos x x^2 dx.$$

$$+ 3\int_{-1}^{1} \sin x \cos x dx$$

$$\Rightarrow \frac{9}{16}\{35(0) - 30(0) + 3(0)\} = 0$$

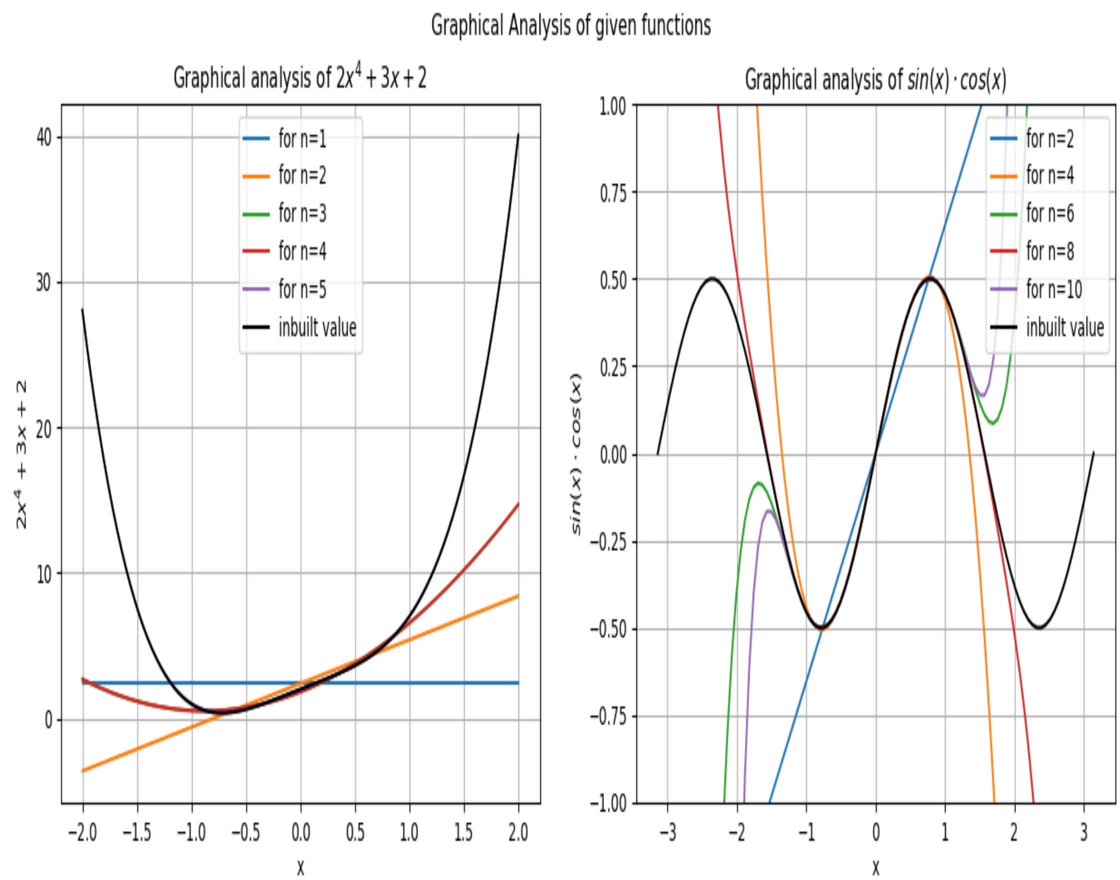
$$f(x) = C_0P_0(x) + C_1P_1(x) + C_2P_2(x) + C_3P_3(x)$$
$$+ C_4P_4(x)$$

$$f(x) = 0.6531p_1(x) + 1.97225p_3(x)$$

Programming

```
(base) hinton@hinton-VirtualBox:~$ /home/hinton/anaconda3/bin/python /home/hinton/Semester_4/MP3/Practical/module/ref.py
Non-zero terms in legendre series expansion of 2*x**4 + 3*x + 2 :
2.4*P0(x) + 3.0*P1(x) + 1.14*P2(x) + 0.46*P4(x)
Coefficients of Pn(x) have been rounded off to 2 decimal places

First 10 terms in legendre series expansion of sin(x)*cos(x):
0.0*P0(x) + 0.6531*P1(x) + 0.0*P2(x) + (-0.2125*P3(x)) + 0.0*P4(x) + 0.0145*P5(x) + -0.0*P6(x) + (-0.0004*P7(x)) + -0.0*P8(x) + 0.0001*P9(x)
Coefficients of Pn(x) have been rounded off to 4 decimal places
```


Graphical Analysis of given functions
Graphical analysis of $2x^4 + 3x + 2$
for n=1
for n=2
for n=3
for n=4
for n=5
inbuilt value
$2x^4 + 3x + 2$
X
Graphical analysis of $sin(x) \cdot cos(x)$
for n=2
for n=4
for n=6
for n=8
for n=10
inbuilt value
$sin(x) \cdot cos(x)$
X

2 Python program

```python

import math
import numpy as np
from scipy.special import eval_legendre
import matplotlib.pyplot as plt

def MySimp(a, b, n, f, Pnx, tbi,p):
  h = (b-a)/n
  x_arr = []
  for i in range(0,n+1):
    x_ele = a + i*h
    x_arr.append(x_ele)
  y_arr = []
  for i in range(0, n+1):
    y_ele = tbi(f,Pnx,p,x_arr[i])
    y_arr.append(y_ele)
  sum = 0
  for i in range(0, n+1):
    if i == 0 or i == n:
      sum = sum + y_arr[i]
    elif i % 2 == 0:
      sum = sum + 2*y_arr[i]
    elif i % 2 == 1:
      sum = sum + 4*y_arr[i]
  sum = sum*(h/3)
  return sum

def f2(x):
  return math.sin(x)*math.cos(x)

def f1(x):
  return 2*x**4 + 3*x + 2

def Pnx(n,x):
  return eval_legendre(n,x)

def tbi(f,Pnx,n,x):
  return f(x)*Pnx(n,x)

def expand(f,terms):
  cn = []
  for i in range(0,terms):
    coeff = ((2*i+1)/2)*MySimp(-1, 1, 100, f, Pnx, tbi, i)
    cn.append(coeff)
  return cn

terms1 = 5
Pnxs = ["P0(x)","P1(x)","P2(x)","P3(x)","P4(x)","P5(x)","P6(x)","P7(x)","P8(x)","P9
    (x)","P10(x)"]
arr1 = expand(f1,terms1)
print("Non-zero terms in legendre series expansion of 2*x**4 + 3*x + 2 : ")
for i in range(0,terms1):
  if round(arr1[i],2)!=0:
    if i<=3:
      print(f"{round(arr1[i],2)}*{Pnxs[i]}", end=" + ")
    else:
      print(f"{round(arr1[i],2)}*{Pnxs[i]}")
print("Coefficients of Pn(x) have been rounded off to 2 decimal places")

print("\nFirst 10 terms in legendre series expansion of sin(x)*cos(x): ")
terms2 = 10
arr2 = expand(f2,terms2)
for i in range(0,terms2):
  if round(arr2[i],4)<(-10e-15):
    if i<=8:
```

```
      print(f"({round(arr2[i],4)}*{Pnxs[i]})", end=" + ")
    else:
      print(f"({round(arr2[i],4)}*{Pnxs[i]})")
  else:
    if i<=8:
      print(f"{round(arr2[i],4)}*{Pnxs[i]}", end=" + ")
    else:
      print(f"{round(arr2[i],4)}*{Pnxs[i]}")
print("Coefficients of Pn(x) have been rounded off to 4 decimal places")
print()

fig1, (ax11, ax12) = plt.subplots(nrows=1, ncols=2, figsize=(15,5))
fig1.suptitle("Graphical Analysis of given functions")

ts = [1,2,3,4,5]
y_values1_1 = []
y_values1_2 = []
y_values1_3 = []
y_values1_4 = []
y_values1_5 = []
y_values1 = [y_values1_1,y_values1_2,y_values1_3,y_values1_4,y_values1_5]
err_arr = []

xs = np.linspace(-2,2,101)
py_val1_arr = []
for i in range(0,101):
  py_val1 = 2*xs[i]**4 + 3*xs[i] + 2
  py_val1_arr.append(py_val1)

for k in range(0,5):
  cn = expand(f1,ts[k])
  for i in range(0,101):
    expansion_value_at_xi = 0
    for j in range(0,ts[k]):
      expansion_value_at_xi = expansion_value_at_xi + cn[j]*Pnx(j,xs[i])
    y_values1[k].append(expansion_value_at_xi)

for i in range(0,5):
  ax11.plot(xs,y_values1[i], label=f"for n={i+1}")
ax11.plot(xs,py_val1_arr,label=f"inbuilt value", color='k')
ax11.grid()
ax11.set_xlabel("x")
ax11.set_ylabel(r'$2x^4 + 3x + 2$')
ax11.set_title(r'Graphical analysis of $2x^4 + 3x + 2$')
ax11.legend()

print()

ts = [2,4,6,8,10]
y_values2_1 = []
y_values2_2 = []
y_values2_3 = []
y_values2_4 = []
y_values2_5 = []
y_values2 = [y_values2_1,y_values2_2,y_values2_3,y_values2_4,y_values2_5]
err_arr = []

xs = np.linspace(-math.pi,math.pi,101)
py_val2_arr = []
for i in range(0,101):
  py_val2 = math.sin(xs[i])*math.cos(xs[i])
  py_val2_arr.append(py_val2)

for k in range(0,5):
  cn = expand(f2,ts[k])
  for i in range(0,101):
    expansion_value_at_xi = 0
```

```python
    for j in range(0,ts[k]):
      expansion_value_at_xi = expansion_value_at_xi + cn[j]*Pnx(j,xs[i])
    y_values2[k].append(expansion_value_at_xi)

for i in range(0,5):
  ax12.plot(xs,y_values2[i], label=f"for n={2*(i+1)}")
ax12.plot(xs,py_val2_arr,label=f"inbuilt value", color='k')
ax12.grid()
ax12.set_xlabel("x")
ax12.set_ylabel(r'$sin(x) \cdot cos(x)$')
ax12.set_ylim(-1,1)
ax12.set_title(r'Graphical analysis of $sin(x) \cdot cos(x)$')
ax12.legend()
fig1.savefig("graph.pdf")
```