Dirac Delta Function

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Theory

Dirac Delta Distribution

The Dirac delta can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

The Dirac delta is not a function because it doesn't give finite value at $x = 0$. At $x = 0$ it shows a peak, which is ∞ while calculating analytically but computationally it's some big value which can vary for different distributions.

Representations of Dirac Delta function $\delta(x)$ as a limit of Sequence of Functions

Limit of sequence of Lorentzian

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon} \frac{1}{1 + \frac{x^2}{\epsilon^2}}$$

Limit of sequence of sinc functions

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{\sin\left(\frac{x}{\varepsilon}\right)}{\pi x}$$

Limit of sequence of Exponential functions

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} e^{-\frac{|x|}{\varepsilon}}$$

Limit of sequence of Gaussian functions

This is the normalized Gaussian distribution function. The area under the curve is unity and the peak value $\frac{1}{\varepsilon\sqrt{\pi}}$

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon\sqrt{\pi}} e^{-\frac{x^2}{\varepsilon^2}}$$

Limit of sequence of Inverse Cosh Square functions

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \frac{1}{\cosh^2\left(\frac{x}{\varepsilon}\right)}$$

Properties of Dirac Delta Function

$$\delta(x - a) = \begin{cases} +\infty, & x = a \\ 0, & x \neq a \end{cases}$$

$$\int_{a-\epsilon}^{a+\epsilon} \delta(x) dx = 1$$

$$\int_{a-\epsilon}^{a+\epsilon} f(x)\delta(x - a) dx = f(a)$$

$3 - D$ form of Dirac Delta Function

$$\int_{\text{allspace}} \delta^3 (\vec{r} - \vec{a}) d\tau = \delta(x - a_1)\, \delta(y - a_2)\, \delta(z - a_3)$$

where r $\rightarrow$ (x, y, z) and a $\rightarrow$ (a_1, a_2, a_3)

$$\int_{\text{allspace}} \delta^3(\vec{r}-\vec{a})d\tau = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta\left(x-a_1\right)\delta\left(y-a_2\right)\delta\left(z-a_3\right)dxdydz = 1$$

Thus our generalised equation becomes

$$\int_{\text{allspace}} f\left(\vec{r}\right)\delta^3(\vec{r}-\vec{a})d\tau = f(\vec{a})$$

Evaluate

- $$\int_{-\infty}^{\infty} \delta(x-2)(x+1)^2dx$$

  Comparing the given equation with,

  $$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a)$$

  Here, $f(x) = (x+1)^2$ and $a = 2$, putting $x = 2$ in $f(x)$

  $$\int_{-\infty}^{+\infty} (x+1)^2\delta(x-2)dx = f(2)$$

  $$f(2) = (2+1)^2 = 9$$

  $$\int_{-\infty}^{+\infty} (x+1)^2\delta(x-2)dx = 9$$

- $$\int_{-\infty}^{\infty} 9x^2\delta(3x+1)dx$$

  $$\int_{-\infty}^{+\infty} 9x^2\delta[3(x+1/3)]dx$$

  Here, $f(x) = 9x^2$ and $a = -\frac{1}{3}$

  $$\int_{-\infty}^{+\infty} f(x)\delta(kx)dx = \frac{1}{|k|}\int_{-\infty}^{+\infty} f(x)\delta(x)dx$$

  $$= \frac{1}{3}\int_{-\infty}^{+\infty} 9x^2\delta\left(x+\frac{1}{3}\right)dx$$

  $$= \frac{1}{3}\int_{-\infty}^{+\infty} 9\left(\frac{1}{9}\right) = \frac{1}{3}$$

- $$\int_{-\infty}^{\infty} 5\mathrm{e}^{t^2}\cos(t)\delta(t-3)\mathrm{d}t$$

  Comparing it with

  $$\int_{-\infty}^{+\infty} f(x)\delta(x-a)dx = f(a)$$

  Here, $f(t) = 5e^{t^2}\cos t$ and $a = 3$

  $$\int_{-\infty}^{+\infty} 5e^{t^2}\cos(t)\delta(t-3)dt = 5e^9\cos 3$$

Programming

```python
import numpy as np
import matplotlib.pyplot as plt
from IntegrationModule import *
from prettytable import PrettyTable

#First Representaition
def Representation1(x, epsilon, a = 0):
    return epsilon/(np.pi*((x-a)**2+epsilon**2))

#Second Representaition
def Representation2(x, epsilon, a = 0):
    return np.exp(-(x-a)**2/(2*epsilon))/(np.sqrt(2*np.pi*epsilon))


#Plots
x = np.linspace(-5, 5, 1000)


fig, (ax1, ax2) = plt.subplots(nrows = 1, ncols = 2)
for i in range(1, 6):
    ax1.plot(x, Representation1(x, 0.4/(2**i), a = 2), label = f"n = {i}")

ax1.set_xlim(1,3)
ax1.set_xlabel('x')
ax1.set_ylabel(r'$\phi_{\epsilon}(x)$')
ax1.set_title(r'$\phi_{\epsilon}(x)=\frac{\epsilon}{\pi((x-a)^2+\epsilon^2)}$ at a = 2')
ax1.legend()
ax1.grid()

for i in range(1, 6):
    ax2.plot(x, Representation1(x, 0.4/(2**i), a = -2), label = f"n = {i}")

ax2.set_xlim(-3,-1)
ax2.set_xlabel('x')
ax2.set_ylabel(r'$\phi_{\epsilon}(x)$')
ax2.set_title(r'$\phi_{\epsilon}(x)=\frac{\epsilon}{\pi((x-a)^2+\epsilon^2)}$ at a = -2')
ax2.legend()
ax2.grid()


fig, (ax3, ax4) = plt.subplots(nrows = 1, ncols = 2)
for i in range(1, 6):
    ax3.plot(x, Representation2(x, 0.4/(2**i), a = 2), label = f"n = {i}")

ax3.set_xlim(0,4)
ax3.set_xlabel('x')
ax3.set_ylabel(r'$\phi_{\epsilon}(x)$')
ax3.set_title(r'$\phi_{\epsilon}(x)=\frac{e^{\frac{-(x-a)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}}$ at a = 2')
ax3.legend()
ax3.grid()

for i in range(1, 6):
    ax4.plot(x, Representation2(x, 0.4/(2**i), a = -2), label = f"n = {i}")

ax4.set_xlim(-4,0)
ax4.set_xlabel('x')
ax4.set_ylabel(r'$\phi_{\epsilon}(x)$')
ax4.set_title(r'$\phi_{\epsilon}(x)=\frac{e^{\frac{-(x-a)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}}$ at a = -2')
ax4.legend()
ax4.grid()

```

```
plt.show()

#Integral 1
epsilon_list = [0.4/(2**1), 0.4/(2**2), 0.4/(2**3), 0.4/(2**4), 0.4/(2**5)]

i_val= []

legendre_11 = []
for i in range(1, 6):
    int = MyLegQuadrature(lambda x: Representation1(x, 0.4/2**i), -10, 10, 100)
    legendre_11.append(int)
    i_val.append(i)

hermite_11 = []
for i in range(1, 6):
    int = MyHermiteQuad(lambda x: Representation1(x, 0.4/2**i), 100)
    hermite_11.append(int)

simpson_11 = []
for i in range(1,6):
    int = My_Simp(lambda x: Representation1(x, 0.4/2**i), -10, 10, 10000)
    simpson_11.append(int)




legendre_12 = []
for i in range(1, 6):
    int = MyLegQuadrature(lambda x: Representation2(x, 0.4/2**i), -10, 10, 100)
    legendre_12.append(int)


hermite_12 = []
for i in range(1, 6):
    int = MyHermiteQuad(lambda x: Representation2(x, 0.4/2**i), 100)
    hermite_12.append(int)

simpson_12 = []
for i in range(1,6):
    int = My_Simp(lambda x: Representation2(x, 0.4/2**i), -10, 10, 10000)
    simpson_12.append(int)


table_11 = PrettyTable()
table_11.title = 'Integral I - First Representation'

table_11.field_names = ["n", "Legendre", "Hermite", "Simpson"]
for i in range(0,5):
    table_11.add_row([i_val[i], legendre_11[i], hermite_11[i], simpson_11[i]])

table_12 = PrettyTable()

table_12.field_names = ["n", "Legendre", "Hermite", "Simpson"]
table_12.title = 'Integral I - Second Representation'

for i in range(0,5):
    table_12.add_row([i_val[i], legendre_12[i], hermite_12[i], simpson_12[i]])



#Integral 2

legendre_21 = []
for i in range(1, 6):
    int = MyLegQuadrature(lambda x: Representation1(x, 0.4/2**i)*(x+1)**2, -10, 10,
```

```
     100)
    legendre_21.append(int)

hermite_21 = []
for i in range(1, 6):
    int = MyHermiteQuad(lambda x: Representation1(x, 0.4/2**i)*(x+1)**2, 100)
    hermite_21.append(int)

simpson_21 = []
for i in range(1,6):
    int = My_Simp(lambda x: Representation1(x, 0.4/2**i)*(x+1)**2, -10, 10, 10000)
    simpson_21.append(int)

legendre_22 = []
for i in range(1, 6):
    int = MyLegQuadrature(lambda x: Representation2(x, 0.4/2**i)*(x+1)**2, -10, 10,
     100)
    legendre_22.append(int)

hermite_22 = []
for i in range(1, 6):
    int = MyHermiteQuad(lambda x: Representation2(x, 0.4/2**i)*(x+1)**2, 100)
    hermite_22.append(int)

simpson_22 = []
for i in range(1,6):
    int = My_Simp(lambda x: Representation2(x, 0.4/2**i)*(x+1)**2, -10, 10, 10000)
    simpson_22.append(int)

table_21 = PrettyTable()
table_21.title = 'Integral II - First Representation'

table_21.field_names = ["n", "Legendre", "Hermite", "Simpson"]
for i in range(0,5):
    table_21.add_row([i_val[i], legendre_21[i], hermite_21[i], simpson_21[i]])

table_22 = PrettyTable()

table_22.field_names = ["n", "Legendre", "Hermite", "Simpson"]
table_22.title = 'Integral II - Second Representation'

for i in range(0,5):
    table_22.add_row([i_val[i], legendre_22[i], hermite_22[i], simpson_22[i]])



#Integral 3

legendre_31 = []
for i in range(1, 6):
    int = MyLegQuadrature(lambda x: Representation1(3*x+1, 0.4/2**i)*9*x**2, -10,
    10, 100)
    legendre_31.append(int)

hermite_31 = []
for i in range(1, 6):
    int = MyHermiteQuad(lambda x: Representation1(3*x+1, 0.4/2**i)*9*x**2, 100)
    hermite_31.append(int)

simpson_31 = []
for i in range(1,6):
    int = My_Simp(lambda x: Representation1(3*x+1, 0.4/2**i)*9*x**2, -10, 10,
    10000)
    simpson_31.append(int)


legendre_32 = []
```

```python
for i in range(1, 6):
    int = MyLegQuadrature(lambda x: Representation2(3*x+1, 0.4/2**i)*9*x**2, -10,
    10, 100)
    legendre_32.append(int)

hermite_32 = []
for i in range(1, 6):
    int = MyHermiteQuad(lambda x: Representation2(3*x+1, 0.4/2**i)*9*x**2, 100)
    hermite_32.append(int)

simpson_32 = []
for i in range(1,6):
    int = My_Simp(lambda x: Representation2(3*x+1, 0.4/2**i)*9*x**2, -10, 10,
    10000)
    simpson_32.append(int)

table_31 = PrettyTable()
table_31.title = 'Integral III - First Representation'

table_31.field_names = ["n", "Legendre", "Hermite", "Simpson"]
for i in range(0,5):
    table_31.add_row([i_val[i], legendre_31[i], hermite_31[i], simpson_31[i]])

table_32 = PrettyTable()

table_32.field_names = ["n", "Legendre", "Hermite", "Simpson"]
table_32.title = 'Integral III - Second Representation'

for i in range(0,5):
    table_32.add_row([i_val[i], legendre_32[i], hermite_32[i], simpson_32[i]])


print(table_11)
print(table_12)

print(table_21)
print(table_22)

print(table_31)
print(table_32)
```

Discussion

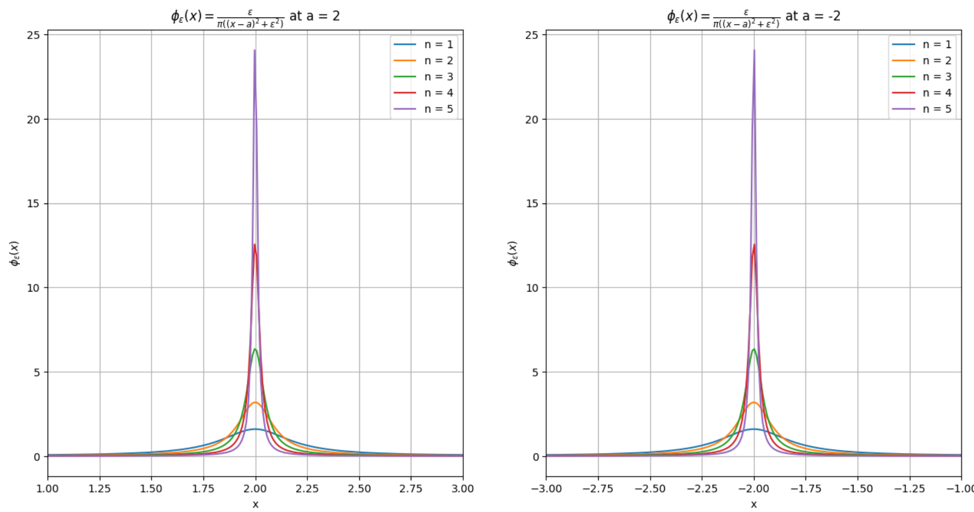

Figure 1: Lorentzian

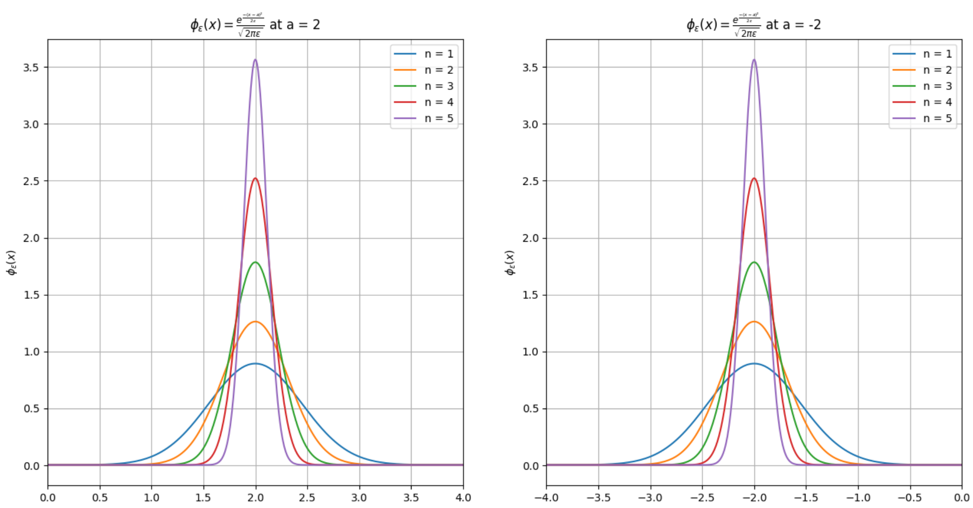

Figure 2: Gaussian

It can be seen that for both Lorentzian and Gaussian functions, as n increases (epsilon decreases) the function goes on to approach the Dirac Delta function.

Integral I - First Representation

n	Legendre	Hermite	Simpson
1	0.9872693017980531	0.8018755921285674	0.9872693017980544
2	0.993634014470185	0.784474525647807	0.9936340144701845
3	0.9968169276635842	0.5549866551529189	0.9968169276635938
4	0.9984084538847899	0.31229750554688646	0.9984084538847978
5	0.9992042256989998	0.1613081306992545	0.9992042256990022

Figure 3: Integral 1 - Representation 1

Integral I - Second Representation

n	Legendre	Hermite	Simpson
1	1.0000000000000009	0.8451542547285166	0.9999999999999996
2	0.9999999999999998	0.9128709291752731	0.9999999999999981
3	0.9999999999999998	0.9534625681436216	0.9999999999999994
4	0.9999999999999999	0.9757648930556437	1.0
5	0.9999999999999986	0.9730658074536955	0.9999999999999974

Figure 4: Integral 1 - Representation 2

Integral II - First Representation

n	Legendre	Hermite	Simpson
1	2.2210180744612917	0.8826384851529757	2.221018074461295
2	1.6203174466930597	0.8330487387461047	1.6203174466930648
3	1.3126347715282154	0.5818086676924241	1.3126347715282196
4	1.1569393916930066	0.32620705919461346	1.1569393916930149
5	1.0786255715846804	0.16833529609817977	1.0786255715846804

Figure 5: Integral 2 - Representation 1

Integral II - Second Representation			
n	Legendre	Hermite	Simpson
1	1.2000000000000006	0.9658905768325902	1.1999999999999922
2	1.0999999999999992	0.9889435066065675	1.099999999999993
3	1.0499999999999983	0.9968018111167517	1.0499999999999934
4	1.0249999999999986	0.9990591292702894	1.0249999999999977
5	1.0124999999999986	0.9866866879255048	1.012499999999999

Figure 6: Integral 2 - Representation 2

Integral III - First Representation			
n	Legendre	Hermite	Simpson
1	1.5890495882880704	0.44233279445088286	1.589049588288069
2	0.9645034951046733	0.6943119169425515	0.9645034951046771
3	0.6497490851350587	1.286286185638971	0.6497490851350555
4	0.4917492029743003	2.499664831398374	0.49174920976463693
5	0.4125321149422492	4.801439309606865	0.41259353284373357

Figure 7: Integral 3 - Representation 1

Integral III - Second Representation			
n	Legendre	Hermite	Simpson
1	0.3999999999999997	0.32488790511963017	0.39999999999999836
2	0.3666666666666614	0.31582443333214194	0.366666666666662
3	0.35	0.3626104860417059	0.3499999999999997
4	0.341666666666666	0.4981003543056717	0.341666666666665
5	0.33750000000000013	0.7040529115387347	0.33750000000000036

Figure 8: Integral 3 - Representation 2

It can be seen that Gauss Legendre and Simpson 1/3 method give accurate results, but Gauss Hermite fails to give a proper result in most of the cases.
Values of integrals remain same throughout the different values of n as the area under the curve does not depend on the epsilon taken.