Assignment 11 - Finite Difference Method

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Finite Difference Method

Theory :

a) **Dirichlet Boundary conditions**

$$y'' = p(x)\,y' + q(x)\,y + r(x) \quad , x \in [a,b]$$

s.t $\quad y(a) = \alpha, \quad y(b) = \beta$

$y_i \rightarrow$ exact value of the solⁿ at $x = x_i$

$w_i \rightarrow$ finite difference approximation to y_i.

$x_i \rightarrow$ grid points , $x_i \in [a,b]$

w_i is obtained by replacing derivatives by the finite difference formula.

Converts ODE into a system of discrete algebraic eqns. for $w_0, w_1 \cdots w_N$.

Grid

$x_i = a + ih$ where $h = \dfrac{b-a}{N}$

$h \rightarrow$ step size.

Finite diff. approximation

evaluate diff eq. at each interior grid point.

$$\{ y'' = p(x)\,y' + q(x)\,y + r(x) \} \Big|_{\substack{x = x_i \\ 1 \leq i \leq N-1}}$$

Replace derivatives by finite diff approximation

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+O(h^2)=p_i\frac{y_{i+1}-y_{i-1}}{2h}+q_iy_i+r_i+O(h^2)$$

$p_i, q_i, r_i \rightarrow p(x_i), q(x_i), r(x_i)$

Drop truncation errors and replace y's with w's.
for $i=1,2 \ldots N-1$.

$\{1 \le i \le N-1\}$ $$\frac{w_{i+1}-2w_i+w_{i-1}}{h^2}=p_i\frac{w_{i+1}-w_{i-1}}{2h}+q_iw_i+r_i \quad (1)$$

Two more eqs. come from the boundary conds

$$w_0=\alpha \quad (2)$$
$$w_N=\beta \quad (3)$$

These 3 equations constitute the finite diff method.

Multiply (1) by $-h^2$ and collect terms to get:

$$\left(-1-\frac{h}{2}p_i\right)w_{i-1}+\left(2+h^2q_i\right)w_i+\left(-1+\frac{h}{2}p_i\right)w_{i+1}=-h^2r_i$$

This gives us N+1 equations.
∴ The finite diff eqs. can be written in the form $AW=b$ where
$A \rightarrow (N+1)\times(N+1)$ tridiagonal matrix.

$$A = \begin{bmatrix} 1 & 0 & & & & & \\ l_1 & d_1 & u_1 & & & & \\ & l_2 & d_2 & u_2 & & & \\ & & \vdots & \vdots & \vdots & & \\ & & & l_{N-2} & d_{N-2} & u_{N-2} & \\ & & & & l_{N-1} & d_{N-1} & u_{N-1} \\ & & & & & 0 & 1 \end{bmatrix}$$

$$w = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-2} \\ w_{N-1} \\ w_N \end{bmatrix} \qquad b = \begin{bmatrix} \alpha \\ -h^2 r_1 \\ -h^2 r_2 \\ \vdots \\ -h^2 r_{N-1} \\ \beta \end{bmatrix}$$

Where $d_i = 2 + h^2 q_i$

$u_i = -1 + \frac{h}{2} p_i$

$l_i = -1 - \frac{h}{2} p_i$

It can be proved that above matrix has a unique solution.

$w = A^{-1} \cdot b$

we can find w (matrix) which is the finite difference approximation at each point.

Reference → A friendly introduction to Numerical Analysis (Brian Bradie)

Non-Dirichlet Boundary Conditions

The most general B.C. are the robin B.C.;

$$y'' = p(x)y' + q(x)y + r(x), \quad x \in [a,b]$$

S.t. $\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3$

$\beta_1 y(b) + \beta_2 y'(b) = \beta_3$

The grid remains the same.
Notations are the same.
We need N+1 eqs. for $w_0, w_1, \dots w_N$.
N-1 eqs. are obtained similarly as before:

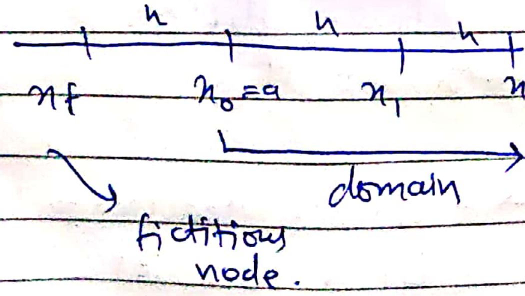
Template ← $\left(-1-\frac{h}{2}p_i\right)w_{i-1} + (2+h^2 q_i)w_i + \left(-1+\frac{h}{2}p_i\right)w_{i+1} = -h^2 r_i$

We introduce a fictitious node to the grid → w_f.

Applying the template at $x = x_0$,

$$\left(-1-\frac{h}{2}p_0\right)w_f + (2+h^2 q_0)w_0 + \left(-1+\frac{h}{2}p_0\right)w_1 = -h^2 r_0 \quad (1)$$

w_f is to be eliminated.



To remove w_f :

$$\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3 \Rightarrow \alpha_1 w_0 + \alpha_2 \frac{w_1 - w_f}{2h} = \alpha_3,$$

$$\Rightarrow w_f = w_1 - \frac{2h}{\alpha_2}(\alpha_3 - \alpha_1 w_0)$$

Putting this ①

$$\left[2 + h^2 q_0 - (2 + h p_0) h \frac{\alpha_1}{\alpha_2}\right] w_0 - 2w_1 = -h^2 r_0 - (2 + h p_0) h \frac{\alpha_3}{\alpha_2},$$

for newmann B.C $\alpha_1 = 0$.

$$\Rightarrow (2 + h^2 q_0) w_0 - 2w_1 = -h^2 r_0 - (2 + h p_0) h \alpha$$

where, $\alpha = \alpha_3 / \alpha_2$

Similarly at $x = b$,

$$-2w_{N-1} + \left[2 + h^2 q_N + (2 - h p_N) h \frac{\beta_1}{\beta_2}\right] w_N$$

$$= -h^2 r_N + (2 - h p_N) h \frac{\beta_3}{\beta_2}$$

for newmann B.C,

$$-2w_{N-1} + (2 + h^2 q_N) w_N = -h^2 r_N + (2 - h p_N) h \beta$$

where, $\beta = \beta_3 / \beta_2$

This gives us all $N+1$ eqs.

$W \rightarrow$ unknown matrix can be found by taking inverse of A and multiplying with b.

This will give us the finite diff. approximation.

b) $$-u'' + \pi^2 u = 2\pi^2 \sin(\pi x)$$

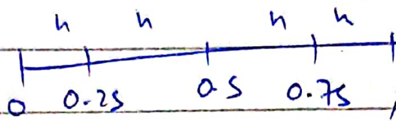
$$u(0) = u(1) = 0.$$

$p(x) = 0,$

$q(x) = \pi^2$

$r(x) = -2\pi^2 \sin(\pi x)$

$N = 4 \qquad x \in [0,1]$



$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{(1/4)^2} + O(h^2) + \pi^2 u_i = 2\pi^2 \sin\left(\frac{i\pi}{4}\right)$$

$$\Rightarrow -w_{i-1} + \left[2 + (\pi/4)^2\right] w_i - w_{i+1} = 2(\pi/4)^2 \sin(i\pi/4)$$

Writing out eqs. for $i = 1, 2, 3$, and using B.C we get:

$$\begin{bmatrix} 1 & 0 & & & \\ -1 & 2+(\pi/4)^2 & -1 & & \\ & -1 & 2+(\pi/4)^2 & -1 & \\ & & -1 & 2+(\pi/4)^2 & -1 \\ & & & 0 & 1 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2}(\pi/4)^2 \\ 2(\pi/4)^2 \\ \sqrt{2}(\pi/4)^2 \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} 0 & 0.725371 & 1.025830 & 0.725371 & 0 \end{bmatrix}^T$$

Comparing with actual solution,

$y_{exact} = \sin(\pi x)$

x_i	Approximate soln w_i	Exact soln u_i	Absolute error
0.00	0.0000	0.0000	0
0.25	0.725371	0.707107	0.018264
0.50	1.025830	1.0000	0.025830
0.75	0.725371	0.707107	0.018264
1.00	0.0000	0.0000	0

$$u'' + u = \sin(3x), \quad x \in [0, \pi/2]$$

$$u(0) + u'(0) = -1$$

$$u'(\pi/2) = 1$$

$N = 4$

$0 \quad \pi/8 \quad \pi/4 \quad 3\pi/8 \quad \pi/2$

$p(x) = 0, \quad q(x) = -1, \quad r(x) = \sin(3x).$

$p_i = 0; \quad q_i = -1; \quad r_i = \sin(3 i \pi/8).$

For Robic B.C at $x = 0$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = -1$

For neumann B.C at $x = \pi/2$, $\beta = 1$.

$$\begin{bmatrix} d-\pi/4 & -2 & & & \\ -1 & d & -1 & & \\ & -1 & d & -1 & \\ & & -1 & d & -1 \\ & & & -2 & d \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} \pi/4 \\ -(\frac{\pi}{8})^2 \sin(3\pi/8) \\ -(\frac{\pi}{8})^2 \sin(3\pi/4) \\ -(\pi/8)^2 \sin(9\pi/8) \\ (\pi/8)^2 + \pi/4 \end{bmatrix}$$

where $d = 2 - (\pi/8)^2$

$$w = \begin{bmatrix} -1.023672 & -0.935445 & -0.560486 & 0.0099517 & 0.519840 \end{bmatrix}^T$$

$$y_{exact} = \frac{3}{8}\sin(x) - \cos(x) - \frac{1}{8}\sin(3x)$$

x_i	Approx. sol^n w_i	Exact sol^n v_i	Absolute error.
0	-1.023672	-1.0000	0.023672
$\pi/8$	-0.935445	-0.895858	0.039587
$\pi/4$	-0.56048	-0.530330	0.030156
$3\pi/8$	0.0099517	0.0116068	0.001655
$\pi/2$	0.519840	0.5000	0.019840

Programming

```python
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import math as m
from scipy.stats import linregress


def finite_diff_method(a,b,alpha1,alpha2,alpha3,beta1,beta2,beta3,N,func_p,func_q,
    func_r):

    def func_ldu(x_arr):
        arr_l=np.array([])
        arr_d=np.array([])
        arr_u=np.array([])
        b_vec=np.array([])

        arr_l=-1-(h/2)*func_p(x_arr)
        arr_d=2+(h**2)*func_q(x_arr)
        arr_u=-1+(h/2)*func_p(x_arr)
        arr_r=func_r(x_arr)
        b_vec=(-1*(h**2)*arr_r)

        if alpha2==0:
            a11=1
            a12=0
            a_n1=1
            a_n2=0
            b1=(alpha3/alpha1)
            b_n1=(beta3/beta1)

        else:
            a11=arr_d[0]+((2*h*arr_l[0]*alpha1)/alpha2)
            a12=-2
            a_n1=arr_d[-1]-((2*h*arr_u[-1]*beta1)/beta2)
            a_n2=-2
            b1=(-1*(h**2))*arr_r[0]+((2*h*arr_l[0]*alpha3)/alpha2)
            b_n1=(-1*(h**2))*arr_r[-1]-((2*h*arr_u[-1]*beta3)/beta2)

        arr_d[0]=a11
        arr_d[-1]=a_n1
        arr_u[0]=a12
        arr_l[-1]=a_n2
        b_vec[0]=b1
        b_vec[-1]=b_n1

        return arr_l[1:],arr_d,arr_u[:-1],b_vec


    def form_tri_matrix(li,di,ui,x_arr):
        N=len(di)

        A_mat=np.zeros((N,N))

        for i in range(N):
            A_mat[i][i]=di[i]
            if i<N-1:
                A_mat[i][i+1]=ui[i]
                A_mat[i+1][i]=li[i]
        return A_mat


    def thomas_algo(A_mat,b_vec):
        N=len(b_vec)
        a=np.zeros((N))
        b=np.zeros((N))
```

```
        c=np.zeros((N))
        d=np.zeros((N))

        d=b_vec
        for i in range(N):
            b[i]=A_mat[i][i]
            if i<(N-1) :
                a[i+1]=A_mat[i+1][i]
                c[i]=A_mat[i][i+1]

        cp = np.zeros(N)
        dp = np.zeros(N)
        X = np.zeros(N)

        cp[0] = c[0]/b[0]
        dp[0] = d[0]/b[0]

        for i in np.arange(1,(N),1):
            dnum = b[i] - a[i]*cp[i-1]
            cp[i] = c[i]/dnum
            dp[i] = (d[i]-a[i]*dp[i-1])/dnum

        # Perform Back Substitution
        X[(N-1)] = dp[N-1]  # Obtain last xn

        for i in np.arange((N-2),-1,-1):  # use x[i+1] to obtain x[i]
            X[i] = (dp[i]) - (cp[i])*(X[i+1])

        return(X)


    h=(b-a)/(N+1)
    x_arr=np.linspace(a,b,N+2,float)
    li,di,ui,bv=func_ldu(x_arr)
    A_mat=form_tri_matrix(li,di,ui,x_arr)
    omega=thomas_algo(A_mat,bv)

    return x_arr,omega

def calc_rms_error(y_vec1,y_vec2):
    sum_ele=0
    for i in range(len(y_vec1)):
        ele=(y_vec1[i]-y_vec2[i])**2
        sum_ele=sum_ele+ele
    ans=m.sqrt(sum_ele/len(y_vec1))
    return ans


#Q1
def func_p1(x):
    ans_arr=np.zeros(len(x))
    for i in range(len(x)):
        ans_arr[i]=0
    return ans_arr

def func_q1(x):
    ans_arr=np.zeros(len(x))
    for i in range(len(x)):
        ans_arr[i]=(np.pi**2)
    return ans_arr

def func_r1(x):
    ans_arr=np.zeros(len(x))
    for i in range(len(x)):
        ans_arr[i]=-2*(np.pi**2)*np.sin(np.pi*x[i])
    return ans_arr
```

```python
def analytic_x1(x):
    return np.sin(np.pi*x)

x_vals11, approximation11 = finite_diff_method(0, 1, 1, 0, 0, 1, 0, 0, 3 ,func_p1,
    func_q1,func_r1)
y_anal11 = analytic_x1(x_vals11)
abs_error11 = np.abs(y_anal11 - approximation11)
err_mat11=np.column_stack((x_vals11,approximation11,y_anal11,abs_error11))
print("N=3")
data_11=pd.DataFrame(err_mat11,columns=["x_i","y_num","y_analytic","error"])
print(data_11)


x_vals12, approximation12 = finite_diff_method(0, 1, 1, 0, 0, 1, 0, 0, 8 ,func_p1,
    func_q1,func_r1)
y_anal12 = analytic_x1(x_vals12)
abs_error12 = np.abs(y_anal12 - approximation12)
err_mat12=np.column_stack((x_vals12,approximation12,y_anal12,abs_error12))
print("N=8")
data_12=pd.DataFrame(err_mat12,columns=["x_i","y_num","y_analytic","error"])
print(data_12)

N1 = []
max_abs_error = []
max_abs_error_ratio = [0]
rms_error = []
rms_error_ratio = [0]

for i in range(1, 7):
    N = 2**i
    x_vals1, approximation1 = finite_diff_method(0, 1, 1, 0, 0, 1, 0, 0, N ,func_p1
    , func_q1,func_r1)
    y_anal1 = analytic_x1(x_vals1)
    abs_error1 = np.abs(y_anal1 - approximation1)
    rms_error1 = calc_rms_error(y_anal1, approximation1)
    max_abs_error1 = np.max(abs_error1)
    plt.plot(x_vals1, approximation1, label = "N={}".format(N), linestyle='dashed')
    plt.scatter(x_vals1, approximation1, s = 10)

    N1.append(N)
    max_abs_error.append(max_abs_error1)
    rms_error.append(rms_error1)

x = np.linspace(0, 1, 100)
plt.plot(x, analytic_x1(x), label = 'Analytic Solution')
plt.title('Variation of solution with N')
plt.xlabel('x')
plt.ylabel('Solution(y)')
plt.legend()
plt.grid()
plt.show()

for i in range(0,5):
    ratio1 = max_abs_error[i]/max_abs_error[i+1]
    max_abs_error_ratio.append(ratio1)

    ratio2 = rms_error[i]/rms_error[i+1]
    rms_error_ratio.append(ratio2)


convergence_data1 =np.column_stack((N1,max_abs_error ,max_abs_error_ratio ,
    rms_error,rms_error_ratio))
convergence_table1=pd.DataFrame(convergence_data1,columns=["N","max_abs_error","
    Error Ratio","Rms Error","Error Ratio"])
print(convergence_table1)

plt.plot(N1, max_abs_error, label = 'Error')
```

```
plt.scatter(N1, max_abs_error)
plt.xscale('log')
plt.yscale('log')
plt.title('Log Plot')
plt.xlabel('N')
plt.ylabel('Max absolute error')
plt.legend()
plt.grid()
plt.show()

log_x=np.log10(N1)
log_y=np.log10(max_abs_error)
print("slope,intercept:",linregress(log_x,log_y)[0:2])


#Q2

def func_p2(x):
    ans_arr=np.zeros(len(x))
    for i in range(len(x)):
        ans_arr[i]=0
    return ans_arr

def func_q2(x):
    ans_arr=np.zeros(len(x))
    for i in range(len(x)):
        ans_arr[i]=-1
    return ans_arr

def func_r2(x):
    ans_arr=np.zeros(len(x))
    for i in range(len(x)):
        ans_arr[i]=np.sin(3*x[i])
    return ans_arr

def analytic_x2(x):
    return (3/8)*np.sin(x)-np.cos(x)-(1/8)*np.sin(3*x)


x_vals21, approximation21 = finite_diff_method(0, np.pi/2, 1, 1, -1, 0, 1, 1, 3 ,
    func_p2, func_q2,func_r2)
y_anal21 = analytic_x2(x_vals21)
abs_error21 = np.abs(y_anal21 - approximation21)
err_mat21=np.column_stack((x_vals21,approximation21,y_anal21,abs_error21))
print("N=3")
data_21=pd.DataFrame(err_mat21,columns=["x_i","y_num","y_analytic","error"])
print(data_21)


x_vals22, approximation22 = finite_diff_method(0, np.pi/2, 1, 1, -1, 0, 1, 1, 8 ,
    func_p2, func_q2,func_r2)
y_anal22 = analytic_x2(x_vals22)
abs_error22 = np.abs(y_anal22 - approximation22)
err_mat22=np.column_stack((x_vals22,approximation22,y_anal22,abs_error22))
print("N=8")
data_22=pd.DataFrame(err_mat22,columns=["x_i","y_num","y_analytic","error"])
print(data_22)

N2 = []
max_abs_error = []
max_abs_error_ratio = [0]
rms_error = []
rms_error_ratio = [0]

for i in range(1, 7):
    N = 2**i
    x_vals2, approximation2 = finite_diff_method(0, np.pi/2, 1, 1, -1, 0, 1, 1, N ,
```

```
    func_p2, func_q2,func_r2)
    y_anal2 = analytic_x2(x_vals2)
    abs_error2 = np.abs(y_anal2 - approximation2)
    rms_error2 = calc_rms_error(y_anal2, approximation2)
    max_abs_error2 = np.max(abs_error2)
    plt.plot(x_vals2, approximation2, label = "N={}".format(N), linestyle='dashed')
    plt.scatter(x_vals2, approximation2, s = 10)

    N2.append(N)
    max_abs_error.append(max_abs_error2)
    rms_error.append(rms_error2)

x = np.linspace(0, np.pi/2, 100)
plt.plot(x, analytic_x2(x), label = 'Analytic Solution')
plt.title('Variation of solution with N')
plt.xlabel('x')
plt.ylabel('Solution(y)')
plt.legend()
plt.grid()
plt.show()


for i in range(0,5):
    ratio1 = max_abs_error[i]/max_abs_error[i+1]
    max_abs_error_ratio.append(ratio1)

    ratio2 = rms_error[i]/rms_error[i+1]
    rms_error_ratio.append(ratio2)


convergence_data2 =np.column_stack((N2,max_abs_error ,max_abs_error_ratio ,
    rms_error,rms_error_ratio))
convergence_table2=pd.DataFrame(convergence_data2,columns=["N","max_abs_error","
    Error Ratio","Rms Error","Error Ratio"])
print(convergence_table2)

plt.plot(N2, max_abs_error, label = 'Error')
plt.scatter(N2, max_abs_error)
plt.xscale('log')
plt.yscale('log')
plt.title('Log Plot')
plt.xlabel('N')
plt.ylabel('Max absolute error')
plt.legend()
plt.grid()
plt.show()

log_x=np.log10(N2)
log_y=np.log10(max_abs_error)
print("slope,intercept:",linregress(log_x,log_y)[0:2])
```

Result and Discussion

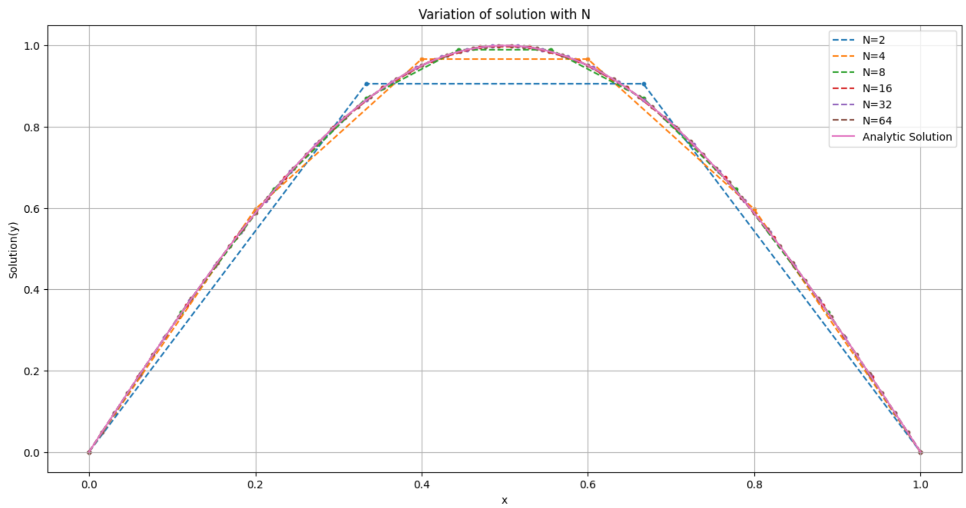


Figure 1: Q1 Variation with N

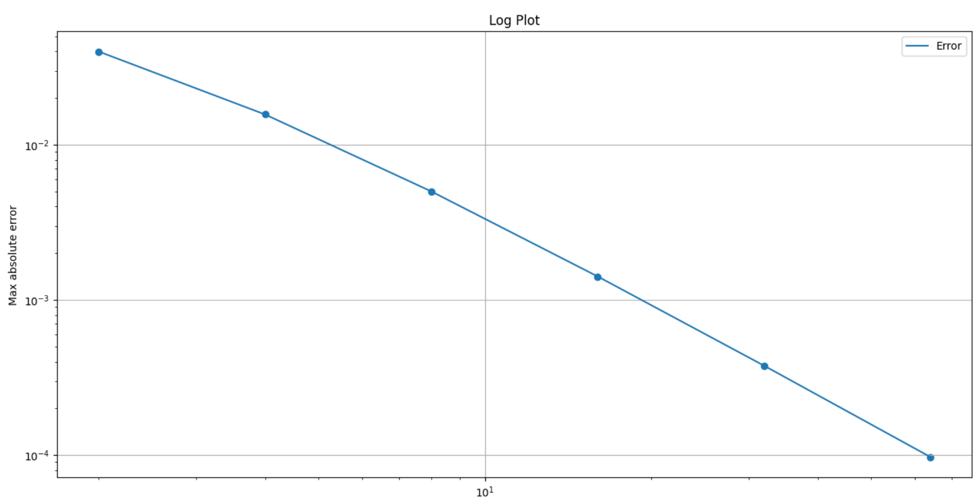


Figure 2: Q1 Log Plot

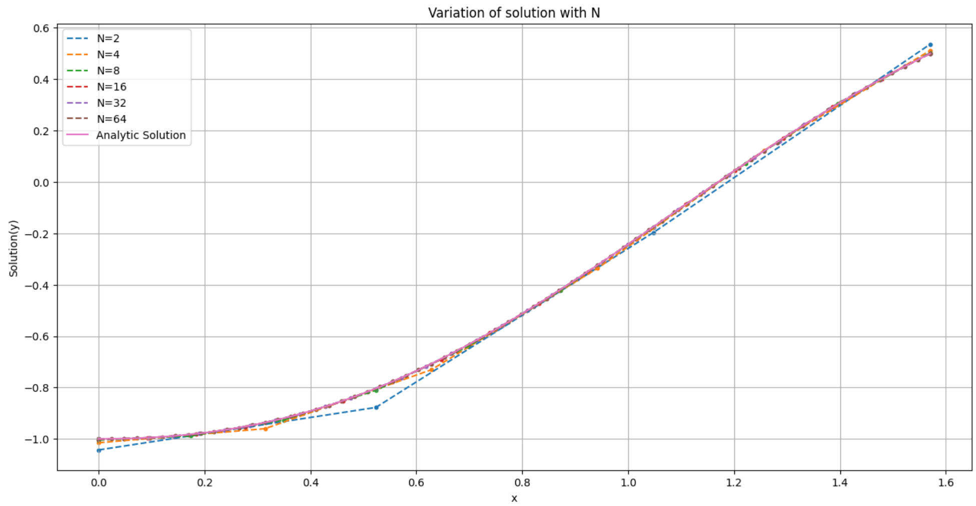


Figure 3: Q2 Variation with N

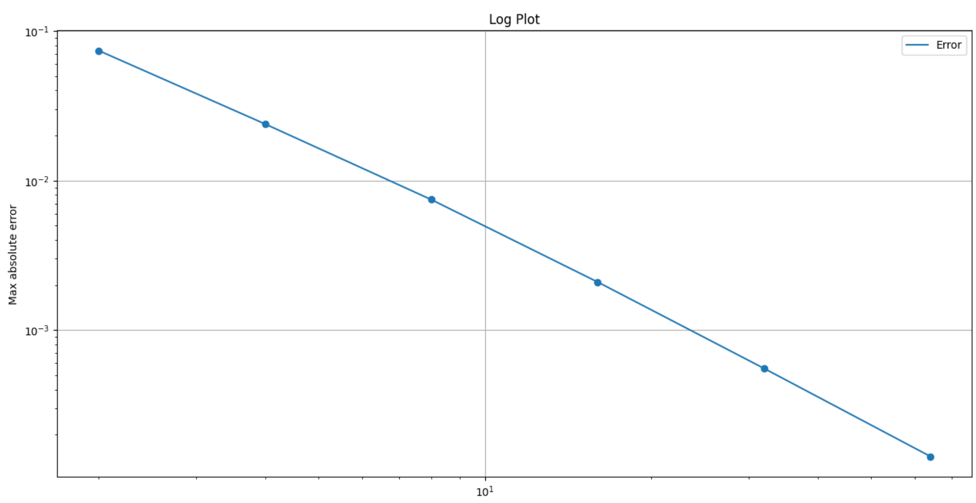


Figure 4: Q2 Log Plot

It can be seen that as the value of N increases the solution approaches the true solution.
From the Log error plot it can be seen that the error decreases contiously as the value N increases.

N=3

	x_i	y_num	y_analytic	error
0	0.00	0.000000	0.000000e+00	0.000000e+00
1	0.25	0.725371	7.071068e-01	1.826441e-02
2	0.50	1.025830	1.000000e+00	2.582978e-02
3	0.75	0.725371	7.071068e-01	1.826441e-02
4	1.00	0.000000	1.224647e-16	1.224647e-16

N=8

	x_i	y_num	y_analytic	error
0	0.000000	0.000000	0.000000e+00	0.000000e+00
1	0.111111	0.343758	3.420201e-01	1.738173e-03
2	0.222222	0.646054	6.427876e-01	3.266697e-03
3	0.333333	0.870427	8.660254e-01	4.401209e-03
4	0.444444	0.989813	9.848078e-01	5.004870e-03
5	0.555556	0.989813	9.848078e-01	5.004870e-03
6	0.666667	0.870427	8.660254e-01	4.401209e-03
7	0.777778	0.646054	6.427876e-01	3.266697e-03
8	0.888889	0.343758	3.420201e-01	1.738173e-03
9	1.000000	0.000000	1.224647e-16	1.224647e-16

Figure 5: Q1 solution for N=3,8

	N	max_abs_error	Error Ratio	Rms Error	Error Ratio
0	2.0	0.039911	0.000000	0.028221	0.000000
1	4.0	0.015695	2.542955	0.010652	2.649328
2	8.0	0.005005	3.135870	0.003409	3.124577
3	16.0	0.001417	3.531306	0.000978	3.485462
4	32.0	0.000377	3.757130	0.000263	3.717845
5	64.0	0.000097	3.876661	0.000068	3.851731

slope,intercept: (-1.7530398170271393, -0.7842493014240717)

Figure 6: Convergence of Q1 and Slope, Intercept

For higher values of N the solution becomes more accurate.
The slope and intercept come out to be the values shown above.
The error ratio slowly converges to 4 as predicted.

N=3

	x_i	y_num	y_analytic	error
0	0.000000	-1.023672	-1.000000	0.023672
1	0.392699	-0.935445	-0.895858	0.039587
2	0.785398	-0.560486	-0.530330	0.030156
3	1.178097	0.009952	0.011607	0.001655
4	1.570796	0.519840	0.500000	0.019840

N=8

	x_i	y_num	y_analytic	error
0	0.000000	-1.004579	-1.000000	0.004579
1	0.174533	-0.988479	-0.982190	0.006289
2	0.349066	-0.927038	-0.919688	0.007349
3	0.523599	-0.810976	-0.803525	0.007451
4	0.698132	-0.639750	-0.633252	0.006497
5	0.872665	-0.422654	-0.418021	0.004633
6	1.047198	-0.177453	-0.175240	0.002213
7	1.221730	0.073153	0.072865	0.000288
8	1.396263	0.306300	0.303908	0.002392
9	1.570796	0.503736	0.500000	0.003736

Figure 7: Q2 solution for N=3,8

	N	max_abs_error	Error Ratio	Rms Error	Error Ratio
0	2.0	0.073975	0.000000	0.047886	0.000000
1	4.0	0.023902	3.094950	0.016598	2.885063
2	8.0	0.007451	3.207900	0.005083	3.265184
3	16.0	0.002095	3.556265	0.001427	3.562139
4	32.0	0.000554	3.780787	0.000380	3.757640
5	64.0	0.000143	3.877566	0.000098	3.871931

slope,intercept: (-1.8057465051362185, -0.5408470673784687)

Figure 8: Convergence of Q2 and Slope, Intercept

For higher values of N the solution becomes more accurate.
The slope and intercept come out to be the values shown above.
The error ratio slowly converges to 4 as predicted.