Linear Algebra cheat sheet

Vectors

dot product: $u * v = ||u|| * ||v|| * cos(\phi) = u_x v_x + u_y v_y$

cross product: $u \times v = \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix}$

norms:
$\|x\|_p := \sqrt[p]{\sum_{i=1}^n |x_i|^p}$
$\|x\|_1 := \sum_{i=1}^n |x_i| \quad \|x\|_\infty = \max_i |x_i|$

enclosed angle:

$$cos\phi = \frac{u * v}{||u|| * ||v||}$$

$$||u|| * ||v|| = \sqrt{(u_x^2 + u_y^2)(v_x^2 + v_y^2)}$$

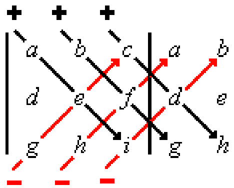
Matrices

basic operations

transpose: $[A^\mathrm{T}]_{ij} = [A]_{ji}$: "mirror over main diagonal"
conjungate transpose / adjugate: $A^* = (\overline{A})^\mathrm{T} = \overline{A^\mathrm{T}}$
"transpose and complex conjugate all entries"
(same as transpose for real matrices)
multiply: $A_{N \times M} * B_{R \times K} = M_{N \times K}$
invert: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
norm:
$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$, induced by vector p-norm
$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$
$\|A\|_1 = \max_j \sum_{i=1}^m |a_{ij}|$,
$\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$,
condition: $\mathrm{cond}(A) = \|A\| \cdot \|A^{-1}\|$

determinants

$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n A_{i,\sigma_i}$
For 3×3 matrices (Sarrus rule):



arithmetic rules:
$\det(A \cdot B) = \det(A) \cdot \det(B)$
$\det(A^{-1}) = \det(A)^{-1}$
$\det(rA) = r^n \det A$, for all $A^{n \times n}$ and scalars r

eigenvalues, eigenvectors, eigenspace

1. Calculate **eigenvalues** by solving $\det(A - \lambda I) = 0$
2. Any vector x that satisfies $(A - \lambda_i I)\, x = 0$ is **eigenvector** for λ_i.
3. $\mathrm{Eig}_A(\lambda_i) = \{x \in \mathbb{C}^n : (A - \lambda_i)x = 0\}$ is **eigenspace** for λ_i.

definiteness

defined on n×n square matrices:
$\forall \lambda \in \sigma(A)$.
$\lambda > 0 \iff$ positive-definite
$\lambda \geq 0 \iff$ positive-semidefinite
$\lambda < 0 \iff$ negative-definite
$\lambda \leq 0 \iff$ negative-semidefinite
if none true (positive and negative λ exist): indefinite
equivalent: eg. $x^T A x > 0 \iff$ positive-definite

rank

Let A be a matrix and $f(x) = Ax$.
$\mathrm{rank}(A) = \mathrm{rank}(f) = \dim(\mathrm{im}(f))$
= number of linearly independent column vectors of A
= number of non-zero rows in A after applying Gauss

kernel

$\mathrm{kern}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$ (the set of vectors mapping to 0)
For nonsingular A this has one element and $\dim(\mathrm{kern}(A)) = 0$ (?)

trace

defined on n×n square matrices: $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$
(sum of the elements on the main diagonal)

span

Let $v_1, \ldots, v_r$ be the column vectors of A. Then:
$\mathrm{span}(A) = \{\lambda_1 v_1 + \cdots + \lambda_r v_r \mid \lambda_1, \ldots, \lambda_r \in \mathbb{R}\}$

spectrum

$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda \text{ is eigenvalue of A}\}$

properties

square: $N \times N$
symmetric: $A = A^T$
diagonal: 0 except a_{kk}
$\Rightarrow$ implies triangular (eigenvalues on main diagonale)

orthogonal
: $A^T = A^{-1} \Rightarrow$ normal and diagonalizable

unitary
Complex analogy to orthogonal: A complex square matrix is unitary if all column vectors are orthonormal
$\Rightarrow$ diagonolizable
$\Rightarrow cond_2(A) = 1$
$\Rightarrow |det(A)| = 1$

nonsingular
$A^{n \times n}$ is nonsingular = invertible = regular iff:
There is a matrix $B := A^{-1}$ such that $AB = I = BA$
$det(A) \neq 0$
$Ax = b$ has exactly one solution for each b
The column vectors of A are linearly independent
$\mathrm{rank}(A) = n$
$f(x) = Ax$ is bijective (?)

$\Rightarrow det(A)^{-1} = det(A^{-1})$
$\Rightarrow (A^{-1})^{-1} = A$
$\Rightarrow (A^T)^{-1} = (A^{-1})^T$

diagonalizable
$A^{n \times n}$ can be diagonalized iff:
it has n linear independant eigenvectors
all eigenvalues are real and distinct
there is an invertible T, such that:

$$D := T^{-1} A T = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$A = T^{-1} D T \quad \textbf{and} \quad AT = TD$$

$\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A!
T can be created with eigenvectors of A and is nonsingular!

diagonally dominant matrix
$\forall i. |a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$
$\Rightarrow$ nonsingular

Hermitian
A square matrix A where $A^* = A$ (equal to its adjugate)
A real matrix is Hermitian iff symmetric
$\Rightarrow \Im(\det(A)) = 0$ (determinante is real)

triangular
A square matrix is right triangular (wlog n = 3):
$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$
$\Rightarrow$ Eigenvalues on main diagonale

idempotent
A square matrix A for which $AA = A$.
TODO: Use cases?

block matrices
Let B, C be submatrices, and A, D square submatrices. Then:
$\det \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(A)\det(D)$

minors